MA 230 Exam 1 - Solutions

Form A

February 23, 2023

- There are 5 problems, worth a total of 100 points.
- Showcase your work: providing just the answer will result in a minimum of points.
- Remember to pledge your work.
- There is no calculator permitted on this exam.
- You are allowed one page of notes (both sides).

For instructor's use only

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Name: ____________________

Pledge: ____________________

Problem 1. The following vectors are given:

$$a_1 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}.$$

(a) Determine if all 4 vectors are linearly independent.

(b) Determine if a_1, a_2, a_3 are linearly independent.

(c) What is the determinant of the matrix whose rows are the vectors a_1, a_2, a_3?

(d) Let the matrix whose columns are the vectors a_1, a_2, a_3 be denoted as A. What is the determinant of A^{-1}?

Solution:

(a) Since we have 4 vectors in $\mathbb{R}^3$, it follows that one vector can be made as a linear combination of the other 3. Therefore the 4 vectors are linearly dependent.

(b)

$$\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{pmatrix} \overset{R_2 \leftrightarrow R_1}{\simeq} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{pmatrix} \overset{R_3 \leftarrow R_3 - 5R_1}{\simeq} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix} \overset{R_1 \leftarrow R_1 - 2R_2}{\simeq}$$

$$\begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix} \overset{R_3 \leftarrow R_3 + 2R_2}{\simeq} \begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{pmatrix} \overset{R_3 \leftarrow \frac{1}{13}R_3}{\simeq} \begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Since we have 3 pivots and a 3×3 matrix, it follows that the 3 vectors are linearly independent.

(c) If the rows of a matrix (denote this matrix as X) are vectors a_1, a_2, and a_3, then the transpose of the matrix must be the columns a_1, a_2, and a_3. We know $\det(X) = 0 \cdot (2 \cdot 0 - 8 \cdot (-1)) - 1 \cdot (1 \cdot 0 - 5 \cdot (-1)) + 4 \cdot (1 \cdot 8 - 2 \cdot 5) = -13$, and $\det(X) = \det(X^{\mathrm{T}})$. Therefore the determinant of the matrix whose rows are the vectors a_1, a_2, and a_3 is -13.

(d) The determinant of A^{-1} is equal to $\frac{1}{\det(A)}$. Therefore, $\det(A^{-1}) = -\frac{1}{13}$.

Problem 2. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be linearly independent vectors in $\mathbb{R}^n$. Is there a value of $t \in \mathbb{R}$, for which the three vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + t\mathbf{b} + \mathbf{c}$, and $\mathbf{b} + \mathbf{c}$ are linearly dependent?

Solution:

Set up the coefficient matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

When $\det(A) = 0$, then the three vectors are linearly dependent.

$\det(A) = 1 \cdot (t \cdot 1 - 1 \cdot 1) - 1 \cdot (1 \cdot 1 - 1 \cdot 0) + 0 \cdot (1 \cdot 1 - t \cdot 0) = t - 2$. Therefore when $t = 2$, the vectors are linearly dependent.

Problem 3. Is given matrix diagonalizable? If so, what are the corresponding matrices P and D?

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Solution:

Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

Then, we have

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - (3)(2) = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0,$$

so that $\lambda_1 = -1$ and $\lambda_2 = 4$.

For $\lambda_1 = -1$,

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ 2x_1 + 3x_2 &= 0 \end{aligned}$$

Let $x_2 = 2$. Then $x_1 = -3$. Therefore we have $v_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

For $\lambda_2 = 4$,

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}$$

$$\begin{aligned} -3x_1 + 3x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned}$$

Let $x_2 = 1$. Then $x_1 = 1$. Therefore we have $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad P = (\mathbf{x_1 x_2}) = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

Problem 4. Prove that a square matrix and its transpose have the same characteristic polynomial, and therefore the same set of eigenvalues.

Solution:

Let A be a square matrix. So the characteristic polynomial of A is

$$P(\lambda) = |A - \lambda I|$$

We know that the determinant of a square matrix and its transpose are equal. That is, for any square matrix X

$$|X| = |X^{\mathrm{T}}| \tag{1}$$

Now the characteristic polynomial of A^{T} is

$$P^{\mathrm{T}}(\lambda) = |A^{\mathrm{T}} - \lambda I|$$

Note that

$$\begin{aligned}
P^{\mathrm{T}}(\lambda) &= |A^{\mathrm{T}} - \lambda I| \\
&= |A^{\mathrm{T}} - \lambda I^{\mathrm{T}}| && \text{Since } I^{\mathrm{T}} = I \\
&= |(A - \lambda I)^{\mathrm{T}}| && \text{Since } (A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} \\
&= |A - \lambda I| && \text{Using (1)} \\
&= P(\lambda)
\end{aligned}$$

It follows that a square matrix and its transpose have the same characteristic polynomial, and since eigenvalues are the roots of the characteristic polynomial, it then holds that both A and A^{T} must have the same set of eigenvalues.

Problem 5. Let $f(x, y) = 8x^2 + 4(c + 2)xy + 4cy^2$.

(a) Create the corresponding quadratic matrix.

(b) What value of c makes the matrix in (a) positive semi-definite?

(c) Explain why if c is not equal to the value in (b), then the matrix is indefinite.

(d) What is the smallest eigenvalue of the matrix when plugging in the value of c computed from (b)?

Solution:

(a) The corresponding quadratic matrix is

$$\begin{pmatrix} 8 & 2(c+2) \\ 2(c+2) & 4c \end{pmatrix}$$

(b) Using Sylvestor's Criteria, the first entry (D_1) is positive, so we need to find when the determinant (D_2) of the matrix, which is $32c - 4(c + 2)^2$, is equal to 0. Solving for c, we have $-4c^2 + 16c - 16 = -(4 - 2c)^2$. When $c = 2$, the determinant is 0.

(c) If $c \neq 2$, then (D_2) is negative. If the leading principal minors are opposite signs, then matrix is indefinite.

(d) Since the matrix in (b) is positive semi-definite, it follows that one eigenvalue must be positive and the second eigenvalue must be 0. Since 0 is less than any positive integer, it follows 0 is the smallest eigenvalue.