MA 230 Exam 2 - Solutions

Form A

April 6, 2023

- There are 5 problems, worth a total of 100 points.
- Showcase your work: providing just the answer will result in a minimum of points.
- Remember to pledge your work.
- There is no calculator permitted on this exam.
- You are allowed one page of notes (both sides).

For instructor's use only

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Name: ____

Pledge: ____

Problem 1. Let $f(x, y) = 3x^2 + 2y^2 + 2x - 5y + 4$.

(a) Find the directional derivative at point $(1, -1)$ in the direction $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

(Leave your answer as a fraction if needed.)

(b) Determine whether $f(x, y)$ is strictly convex or strictly concave.

Solution:

(a) The unit vector that represents the direction is $\vec{u} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \end{bmatrix}^\top$ and the gradient is $\nabla f(x, y) = \begin{bmatrix} 6x + 2 \\ 4y - 5 \end{bmatrix}$.

$$D_{\vec{u}} = \nabla f(1, -1) \cdot \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \end{bmatrix} = -\frac{68}{13}$$

(b) The only critical point is $(x, y) = (-\frac{1}{3}, \frac{5}{4})$. The Hessian is $\nabla^2 f(x, y) = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$. It is clear that $f_{xx} > 0$ and $\det \nabla^2 f(x, y) > 0$ for all ordered pairs (x, y), so the point $(-\frac{1}{3}, \frac{5}{4})$ is a local maximum.

Problem 2. Find the quadratic approximation around $(0,0)$ for $f(x,y) = \ln(x^2+y^2+1)$.

Solution:

$$\begin{aligned}
f(0,0) &= 0 \\
\frac{\partial f}{\partial x} &= \frac{2x}{2x^2+y^2+1} = f_x(0,0) = 0 \\
\frac{\partial f}{\partial y} &= \frac{2y}{2x^2+y^2+1} = f_y(0,0) = 0 \\
\frac{\partial^2 f}{\partial x^2} &= \frac{2(x^2+y^2+1) - 2x(2x)}{(x^2+y^2+1)^2} = f_{xx}(0,0) = 2 \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{2x(x^2+y^2+1) - 2x(2y)}{(x^2+y^2+1)^2} = f_{xy}(0,0) = 0 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{2(x^2+y^2+1) - 2y(2y)}{(x^2+y^2+1)^2} = f_{yy}(0,0) = 2
\end{aligned}$$

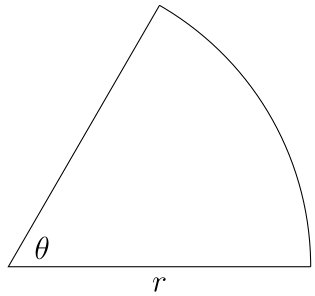
The Quadratic Approximation is given by

$$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

Therefore,

$$\ln(x^2+y^2+1) \approx x^2+y^2$$

Problem 3. The length r of the radius of a circular sector is increasing at a rate of 5 meters per second. The angle θ of a circular sector is decreasing at a rate of $\frac{1}{2}$ radians per second. The area of the circular sector as a function of r and θ is $A(r, \theta) = \frac{\theta}{2}r^2$. A diagram of a circular sector is shown below.



(a) Is the diagram above convex? Briefly explain why.

(b) Find $\frac{\partial A}{\partial r}$ and $\frac{\partial A}{\partial \theta}$ in terms of r and θ. Express their respective units.

(c) How fast is the area of the circular sector changing, in meters squared per second, when $r = 12$ meters and $\theta = \frac{\pi}{6}$ radians?

Solution:

(a) The diagram above is convex. This is because if any two points in the sector are chosen, all points on the line drawn between them will also be in the sector.

(b) $\frac{\partial A}{\partial r} = \theta r$ meters (meters squared per meter also suffices) and $\frac{\partial A}{\partial \theta} = \frac{r^2}{2}$ meters per radian.

(c)

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial A}{\partial r}\frac{dr}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} \\ &= (\theta r)(5) + \left(\frac{r^2}{2}\right)\left(-\frac{1}{2}\right)\end{aligned}$$

With $r = 12$ and $\theta = \frac{\pi}{6}$, we have $\frac{\partial A}{\partial t} = 10\pi - 36$

Problem 4. Let $f(x, y) = (x + y, x - y)$ and $g(x, y) = (x^2 + y^2, x^2 - y^2)$.

(a) Find J_f, the Jacobian (the determinant of the Jacobian matrix) of f.

(b) Find J_g, the Jacobian (the determinant of the Jacobian matrix) of g.

(c) Which of the Jacobians are never zero?

Solution:

(a) Let $f_1 = x + y$ and $f_2 = x - y$.

$$J_f = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

(b) Let $g_1 = x^2 + y^2$ and $g_2 = x^2 - y^2$.

$$J_g = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy$$

(c) It is clear that J_f is never zero for any value of x and y.

Problem 5. Let $f(x) = |x|$

(a) Use the formula for convexity to show $f(x)$ is convex.

(b) What is another, simpler way to show $f(x)$ is convex?

(c) Show that $f(x)$ is not strictly convex.

Solution:

(a) The formula for convexity is $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for $x_1, x_2 \in \mathrm{dom} f$ and $\lambda \in [0,1]$. Then plugging in the problem conditions and utilizing the triangle inequality:

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= |\lambda x_1 + (1-\lambda)x_2| \\ &\leq |\lambda x_1| + |(1-\lambda)x_2| \\ &= \lambda|x_1| + (1-\lambda)|x_2| \\ &= \lambda f(x_1) + (1-\lambda)f(x_2) \end{aligned}$$

Therefore $f(x) = |x|$ is convex.

(b) Drawing the epigraph suffices.

(c) The formula for strict convexity is $f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2)$ for $x_1, x_2 \in \mathrm{dom} f$ and $\lambda \in [0,1]$. If $x_1 = 0$, then the strict inequality doesn't hold, hence not being strictly convex. Note that using the epigraph does **not** hold for strict convexity.