MA 230 Final Exam Review

1. Given the matrix:

$$A = \begin{bmatrix} .7 & .2 & .1 \\ .2 & .7 & .1 \\ .1 & .1 & .8 \end{bmatrix}$$

   (a) Calculate the determinant and trace of A.

   (b) Find the eigenvalues and eigenvectors of A. Verify that the product of the eigenvalues is the determinant and the sum of the eigenvalues is the trace.

   Note that the determinant of any matrix B is always the product of its eigenvalues and the trace of B is always the sum of its eigenvalues.

   (c) Is A positive definite, negative definite, or indefinite?

Solution:

(a)

$$\det A = \begin{vmatrix} .7 & .2 & .1 \\ .2 & .7 & .1 \\ .1 & .1 & .8 \end{vmatrix} = .7\begin{vmatrix} .7 & .1 \\ .1 & .8 \end{vmatrix} - .2\begin{vmatrix} .2 & .1 \\ .1 & .8 \end{vmatrix} + .1\begin{vmatrix} .2 & .7 \\ .1 & .1 \end{vmatrix}$$
$$= .7(.55) - .2(.15) - .1(.05) = .35$$

$$\mathrm{Tr}A = .7 + .7 + .8 = 2.2$$

(b)

$$\det A - \lambda I_3 = \begin{vmatrix} .7-\lambda & .2 & .1 \\ .2 & .7-\lambda & .1 \\ .1 & .1 & .8-\lambda \end{vmatrix}$$
$$= (.7-\lambda)\begin{vmatrix} .7-\lambda & .1 \\ .1 & .8-\lambda \end{vmatrix} - .2\begin{vmatrix} .2 & .1 \\ .1 & .8-\lambda \end{vmatrix} + .1\begin{vmatrix} .2 & .7-\lambda \\ .1 & .1 \end{vmatrix}$$
$$= (.7-\lambda)(.55 - 1.5\lambda + \lambda^2) - .2(.15 - .2\lambda) + .1(-.05 - .1\lambda)$$
$$= -\lambda^3 + 2.2\lambda^2 - 1.55\lambda + 0.35$$
$$= -(\lambda - 1)(\lambda - .5)(\lambda - .7)$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = .5, \lambda_3 = .7$. The product of the eigenvalues is $1 \times .5 \times .7 = .35$ and the sum of the eigenvalues is $1 + .5 + .7 = 2.2$ which matches with the calculations above.

For $\lambda_1 = 1$:

$$\begin{bmatrix} -.3 & .2 & .1 \\ .2 & -.3 & .1 \\ .1 & .1 - .2 & \end{bmatrix} \begin{bmatrix} v_{1a} \\ v_{1b} \\ v_{1c} \end{bmatrix} = \vec{0} \implies \begin{bmatrix} -.3v_{1a} & .2v_{1b} & .1v_{1c} \\ .2v_{1a} & -.3v_{1b} & .1v_{1c} \\ .1v_{1a} & .1v_{1b} & -.2v_{1c} \end{bmatrix} = \vec{0}$$

The sum of all the columns is zero, so $v_{1a} = v_{1b} = v_{1c}$. The eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = .5$:

$$\begin{bmatrix} .2 & .2 & .1 \\ .2 & .2 & .1 \\ .1 & .1 & .3 \end{bmatrix} \begin{bmatrix} v_{2a} \\ v_{2b} \\ v_{2c} \end{bmatrix} = \vec{0} \implies \begin{bmatrix} .2v_{2a} & .2v_{2b} & .1v_{2c} \\ .2v_{2a} & .2v_{2b} & .1v_{2c} \\ .1v_{2a} & .1v_{2b} & .3v_{2c} \end{bmatrix} = \vec{0}$$

The first two columns are identical, so $v_{2a} = -v_{2b}$ and $v_{2c} = 0$. The eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda_3 = .7$:

$$\begin{bmatrix} 0 & .2 & .1 \\ .2 & 0 & .1 \\ .1 & .1 & .1 \end{bmatrix} \begin{bmatrix} v_{3a} \\ v_{3b} \\ v_{3c} \end{bmatrix} = \vec{0} \implies \begin{bmatrix} 0 & .2v_{3b} & .1v_{3c} \\ .2v_{3a} & 0 & .1v_{3c} \\ .1v_{3a} & .1v_{3b} & .1v_{3c} \end{bmatrix} = \vec{0}$$

The sum of the first two columns is twice the third column, so $v_{3a} + v_{3b} = 2v_{3c}$. The eigenvector is $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

(c) A is positive definite because all of the eigenvalues are positive.

2. Find the two points on the circle $x^2 + y^2 = 25$ that minimize and maximize $f(x, y) = 6x - 8y$.

Solution: First setup the Lagrangian, calculate the gradient, set it equal to zero, and simplify.

$$L(x, y, \lambda) = 6x - 8y - \lambda(x^2 + y^2 - 25)$$

$$\nabla L = \begin{bmatrix} 6 - 2\lambda x \\ -8 - 2\lambda y \\ x^2 + y^2 - 25 \end{bmatrix} = \vec{0} \implies \begin{bmatrix} \lambda x \\ \lambda y \\ x^2 + y^2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 25 \end{bmatrix}$$

Solving for x and y in the first two partial derivatives and plugging that into the third yields the following.

$$\frac{9}{\lambda^2} + \frac{16}{\lambda^2} = 25 \implies \lambda = \pm 1$$

Using these two values for λ, the critical points are $(3, -4)$ and $(-3, 4)$. Note that the region of potential solutions are on the circle centered at the origin with radius 5. Because this is a closed and bounded region, by the Extreme Value Theorem, a minimum and maximum must exist. This means one of the critical points is a minimum and the other is a maximum. Note that $f(3, -4) = 50 > f(-3, 4) = -50$, so $(3, -4)$ is the point that maximizes f and $(-3, 4)$ is the point that minimizes f.