Assignment 11

Solutions

There are total 18 points in this assignment. 16 points is considered 100%. If you go over 16 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 6.1, 6.2 in Bartle–Sherbert.

1. Basic properties of the derivative

(1) [2pt] (∼6.1.4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for x rational, $f(x) = 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (Hint: Use the limit of ratio definition of derivative.)

$\triangleright$ Note that $0 \le |f(x) - f(0)| \le |x|^3$ for any $x \in \mathbb{R}$. Therefore,

$$0 \le |f(x) - f(0)/(x - 0)| \le |x|^2.$$

By squeeze theorem for limits, since $|x|^2 \to 0$ as $x \to 0$,

$$\frac{|f(x) - f(0)|}{|x - 0|} \to 0 \text{ as } x \to 0,$$

so

$$\frac{f(x) - f(0)}{x - 0} \to 0 \text{ as } x \to 0$$

too (by another application of squeeze theorem: $-|g| \le g \le |g|$). By definition of derivative, $f'(0) = 0$.

(2) [3pt] *(This is problem 7 of HW10. It was not properly stated there. If you did that problem by saying "f, g are not even defined at 0", which is technically correct, do this problem now. If you already did it assuming $f, g = 0$ at 0, I'll take your solution from HW10.)*

Using the "limit of ratio" definition of the derivative, establish whether the following functions are differentiable at 0. In the case of positive answer, find the derivative at 0.

(a) $f(x) = \begin{cases} x\sin(1/x) & x \ne 0, \\ 0 & x = 0. \end{cases}$

(b) $g(x) = \begin{cases} x^2\sin(1/x) & x \ne 0, \\ 0 & x = 0. \end{cases}$

$\triangleright$ See Problem 7 of HW10. (By the way, note that the assignment $f(0) = 0$ and $g(0) = 0$ is the only possible one for the functions to have a chance of being differentiable since it is the only choice that makes f, g continuous.)

(3) [2pt] (6.1.10) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2\sin(1/x^2)$ for $x \ne 0$, and $h(0) = 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative h' is not bounded on the interval $[-1, 1]$.

$\triangleright$ For $x \ne 0$ we have

$$f'(x) = (x^2\sin(1/x^2))' = 2x\sin(1/x^2) + x^2 \cdot \frac{-2}{x^3}\cos(1/x^2) =$$

$$= 2x\sin(1/x^2) - \frac{2}{x}\cos(1/x^2),$$

so f is differentiable at every $x \ne 0$. At $x = 0$, we have

$$f'(0) = \lim_{x\to 0} \frac{x^2\sin(1/x^2)}{x} = \lim_{x\to 0} x\sin(1/x^2) = 0.$$

Finally, note that at $x = -1/\sqrt{2\pi n}$, $f'(x) = 0 + 2\sqrt{2\pi n} \times 1 = 2\sqrt{2\pi n}$. So for any $M > 0$ there is an $x \in [-1, 1]$ such that $|f'(x)| > M$ (suffices to take $n > M^2$), which means f' is not bounded on $[-1, 1]$.

(4) [2pt] ($\sim$6.1.14) Given that the function $h(x) = x^3+2016x+1$, $x \in \mathbb{R}$, has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at the points y corresponding to $x = 0, 1, -1$.

$\triangleright$ We have $h(0) = 1$, $h(1) = 2018$, $h(-1) = -2016$. We also have $h'(x) = 3x^2 + 2016 \neq 0$ at $0, 1, -1$. So by the inverse function theorem we get

$$(h^{-1})'(1) = 1/h'(0) = 1/2016, \quad (h^{-1})'(2018) = 1/h'(1) = 1/2019,$$
$$(h^{-1})'(-2016) = 1/h'(-1) = 1/2019.$$

(5) [2pt] (6.1.16) Given that the restriction of the tangent function tan to $I = (-\pi/2, \pi/2)$ is strictly increasing and $\tan(I) = \mathbb{R}$, let $\arctan : \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of tan to I. Show that arctan is differentiable on $\mathbb{R}$ and $(\arctan y)' = (1 + y^2)^{-1}$ for $y \in \mathbb{R}$.

$\triangleright$ Note that $\tan' x = (\cos / \sin)'(x) = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = 1/\cos^2 x \neq 0$ on I. So the inverse function theorem applies and we get

$$\arctan' y = \frac{1}{1/\cos^2 x},$$

Where x is such that $\tan x = y$. Recall that $\cos^2 x + \sin^2 x = 1$, so $1 + \tan^2 x = 1/\cos^2 x$. We get in the above formula:

$$\arctan' y = \frac{1}{1/\cos^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}.$$

2. Mean Value Theorem

(6) [2pt] (6.2.6) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (Hint: Apply the Mean Value theorem to sine on the interval $[x, y]$.)

$\triangleright$ Assume $x > y$ (if $x = y$ statement is immediate, if $x < y$, the argument is similar). Apply the Mean Value Theorem to sin on $[x, y]$: there is $c \in (x, y)$ such that $\sin x - \sin y = \cos c \cdot (x - y)$. Then

$$|\sin x - \sin y| = |\cos c| \cdot |x - y| \leq |x - y|.$$

(7) [3pt] (Example 6.2.10(c)) (Bernoulli's inequality) Let $\alpha \in \mathbb{R}$, $\alpha > 1$. Prove that

$$(1 + x)^\alpha \geq 1 + \alpha x \text{ for all } x > -1.$$

(*Hint:* Apply the Mean Value theorem to $(1 + x)^\alpha$ on $[0, x]$.)

$\triangleright$ First consider the case $x > 0$. We have by the MVT on $[0, x]$:

$$(1 + x)^\alpha - (1 + 0)^\alpha = \alpha(1 + c)^{\alpha-1}(x - 0), \quad c \in (0, x).$$

Since $c > 0$ and $\alpha - 1 > 0$, we have $(1 + c)^{\alpha-1} \geq 1$, so

$$(1 + x)^\alpha - 1 = \alpha(1 + c)^{\alpha-1}(x - 0) > \alpha x,$$

as required.

The case $x = 0$ is immediate.

For the case $-1 < x < 0$, apply MVT to $(1 + x)^\alpha$ on $[x, 0]$:

$$(1 + x)^\alpha - (1 + 0)^\alpha = \alpha(1 + c)^{\alpha-1}(x - 0), \quad c \in (x, 0).$$

Now $0 < 1 + c < 1$, so $(1 + c)^{\alpha-1} > 1$. Since $x < 0$, we therefore get

$$(1 + x)^\alpha - 1 = \alpha(1 + c)^{\alpha-1}(x - 0) > \alpha x,$$

as required.

(8) [2pt] (6.2.17) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$, and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$. (*Hint:* Use the Mean Value Theorem.)

$\triangleright$ Notice that for the function $h = f - g$ we have $h(0) = 0$ and $h' \leq 0$, so h is decreasing, which means $h(x) \leq h(0) = 0$ for all $x \geq 0$, that is, $f(x) \leq g(x)$ for all $x \geq 0$.

$\triangleright$ We can also argue directly by MVT. Consider $h = f - g$. By MVT on $[0, x]$ we have, for some $c \in (0, x)$, that

$$f(x) - g(x) = h(x) - h(0) = h'(c)(x - 0),$$

which is ≤ 0 for all $x \geq 0$, since $h' \leq 0$.