Assignment 7

Solutions

This homework assignment is a self-study. It is issued earlier than usual (you have 12 days to do it instead of one week). There will be no new HW on Oct 13.

In terms of course grade, this is treated as a usual homework.

There are total 30 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% for this homework and it will count towards your course grade (not over 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers Section 3.6 in Bartle–Sherbert.

1. Properly divergent sequences

In the exercises below we look at sequences that "go to infinity".

Definition. Let (x_n) be a sequence of real numbers. We say that (x_n) *tends to* (diverges to) $+\infty$, and write $\lim(x_n) = +\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number K such that if $n > K$, then $x_n > \alpha$.

Another notation is $x_n \to +\infty$ $(n \to \infty)$.

(1) [2pt] Give an analogous definition of a sequence that tends to $-\infty$.

$\triangleright$ We say that (x_n) *tends to* (diverges to) $-\infty$, and write $\lim(x_n) = -\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number K such that if $n > K$, then $x_n < \alpha$.

Definition. We say that (x_n) is *properly divergent* in case we have either $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$.

(2) ($\sim$ Example 3.6.2) For the following sequences determine whether they are properly divergent.

(a) [1pt] $x_n = n$.

$\triangleright$ Show that $\lim(x_n) = +\infty$. Let $\alpha \in \mathbb{R}$ be given. Put N to be any natural number $> \alpha$. Then $x_n = n > N > \alpha$.

(b) [1pt] $x_n = n^2$.

$\triangleright$ Show that $\lim(x_n) = +\infty$. Let $\alpha \in \mathbb{R}$ be given. Put N to be any natural number $> \alpha$. Then $x_n = n^2 > n > N > \alpha$.

(c) [1pt] $x_n = (-1)^n n$.

$\triangleright$ Show that that (x_n) is not properly divergent. Indeed, let $\alpha = 0$. Then for any $N \in \mathbb{N}$, one of numbers x_{N+1}, x_{N+2} is $< 0 = \alpha$, so the sequence does not converge to $+\infty$, and the other is $> 0 = \alpha$, so the sequence does not converge to $-\infty$.

(d) [2pt] $x_n = c^n$, where c is a given real number. (*Hint:* Note that the answer depends on c. For $c > 1$, use Bernoulli's inequality.)

$\triangleright$ If $c > 1$, then $c = 1+d$, $d > 0$. We have $c^n = (1+d)^n \geq 1+dn \to +\infty$ since, given $\alpha \in \mathbb{R}$, we can put N to be any natural number $> (\alpha-1)/d$, which gives $x^n \geq 1 + nd > 1 + Nd > \alpha$.

If $-1 \leq c \leq 1$, then the sequence (c^n) is bounded and therefore not properly divergent.

If $c < -1$, then it is not properly divergent since it alternates sign, by the same argument as in (2c).

(3) [3pt]

(a) Suppose (x_n) is properly divergent. Show that (x_n) is ultimately nonzero and that $\lim(1/x_n) = 0$.

$\triangleright$ Assume $\lim(x_n) = +\infty$ (the case of $-\infty$ is similar). Put $\alpha = 0$. By definition, there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$, we have $x_n > \alpha = 0$, which precisely means that the sequence is ultimately positive. Further, let $\varepsilon > 0$ be given. Put $\alpha = 1/\varepsilon$. Then there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $x_n > \alpha$, i.e., $1/x_n > 1/\alpha = \varepsilon$ for all $n > N = \max\{N_0, N_1\}$.

(b) Suppose $\lim(y_n) = 0$ and $y_n > 0$ for all sufficiently large n. $\lim(1/y_n) = +\infty$.

$\triangleright$ We will show $\lim(1/y_n) = +\infty$ by definition. Let $\alpha > 0$ be given (if $\alpha \leq 0$, take $N = 1$). Consider $\varepsilon = 1/\alpha > 0$. Since $\lim(y_n) = 0$, there is $N \in \mathbb{N}$ such that $0 < y_n < \varepsilon = 1/\alpha$ for all $n > N$, i.e. $1/y_n > \alpha$, as required.

(4) [2pt] (Theorem 3.6.4) Prove that following theorem (we may call it the squeeze theorem for properly divergent sequences).
Let (x_n) and (y_n) be two sequences of real numbers and suppose that

$$x_n \leq y_n \text{ for all } n \in \mathbb{N}.$$

If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.

$\triangleright$ Let $\lim(x_n) = +\infty$. Show that $\lim(y_n) = +\infty$. Let $\alpha \in \mathbb{R}$ be given. Since $\lim(x_n) = +\infty$, there is $N \in \mathbb{N}$ such that for all $n > N$, $x_n > \alpha$. But then $y_n \geq x_n > \alpha$, so $\lim(y_n) = +\infty$.

Argument in the second case is similar.

(5) [2pt] (Exercise 3.6.1) Show that if (x_n) is an unbounded sequence, then it has a properly divergent subsequence.

$\triangleright$ For each $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $|x_{n_k}| > k$ (if there isn't, then (x_n) is bounded by $\max\{k, |x_1|, \ldots, |x_{n_{k-1}}|\}$). Consider the subsequence (x_{n_k}). It either has infinitely many positive terms or infinitely many negative terms (maybe both, but not neither since there are infinitely many in total). Then the corresponding subsequence tends to $+\infty$ or $-\infty$.

(6) [3pt] ($\sim$Exercise 3.6.2) Give examples of sequences (x_n) and (y_n) that tend to $+\infty$ and:

(a) $\lim(x_n/y_n) = 0$,

$\triangleright$ $x_n = n$, $y_n = n^2$.

(b) $\lim(x_n/y_n) = 10$,

$\triangleright$ $x_n = 10n$, $y_n = n$.

(c) $\lim(x_n/y_n) = +\infty$,

$\triangleright$ $x_n = n^2$, $y_n = n$.

(d) $\lim(x_n/y_n)$ does not exist as either a real number or infinity.

$\triangleright$ $x_n = (n)$, $y_n = (1, 2^2, 3, 4^2, 5, 6^2, \ldots)$, that is, y_n alternates between n and n^2. Then $(x_{2n}/y_{2n}) \to 0$, while $(x_{2n+1}/y_{2n+1}) \to 1$. Then

(x_n/y_n) is neither convergent (by divergence criteria), nor properly divergent (since it is bounded).

(7) [3pt] (Theorem 3.6.5) Prove that following theorem.
Let (x_n) and (y_n) be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}$, $L > 0$, we have

$$\lim(x_n/y_n) = L.$$

Then $\lim(x_n) = +\infty$ if and only if $\lim(y_n) = +\infty$.
(*Hint:* Show that for n large enough, $\frac{1}{2}L < x_n/y_n < \frac{3}{2}L$. Then use the theorem in Prob. 4.)

$\triangleright$ Since $\lim(x_n/y_n) = L$, there is $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n/y_n - L| < L/2$. Then $\frac{1}{2}L < x_n/y_n < \frac{3}{2}L$, and, since $y_n > 0$,

$$\tfrac{1}{2}Ly_n < x_n < \tfrac{3}{2}Ly_n.$$

Now let $\lim(y_n) = +\infty$. Then it immediately follows from the definition that $(\frac{1}{2}Ly_n) \to +\infty$. By (4), $\lim(x_n) = +\infty$ since ultimately $x_n > \frac{1}{2}Ly_n$.
Let $\lim(x_n) = +\infty$. Then, similarly to the above, we ultimately have $y_n > \frac{2}{3L}x_n \to +\infty$ $(n \to \infty)$.

(8) [3pt]
(a) Suppose that $\lim(a_n) = L \neq 0$ and $\lim(b_n) = +\infty$. Show that (a_nb_n) is properly divergent. (*Hint:* Use the above theorem. Don't forget that you need two positive sequences to use it.)

$\triangleright$ Assume $L > 0$ (the case $L < 0$ is similar). Then both (a_n) and (b_n) are ultimately positive (by setting in the definitions $\varepsilon = L/2$ and $\alpha = 0$, respectively). For simplicity of notation, assume they are both positive (formally, we pass to their positive tails, which does not change any of the involved limits).
Now apply the preceding theorem to $x_n = a_nb_n$ and $y_n = b_n$. We have $(x_n/y_n) = (a_n) \to L$, so (a_nb_n) is properly divergent since (b_n) is.

$\triangleright$ Another solution is directly by the definition. (It ends up repeating the proof of the theorem we used in the first solution.)
Assume $L > 0$ (the case $L < 0$ is similar). Since $L/2 > 0$, there is $N_0 \in \mathbb{N}$ such that $|a_n - L| < L/2$ for all $n > N_0$. In particular, $a_n > L/2$ for such n.
Show that $(a_nb_n) \to +\infty$ by the definition. Let $\alpha > 0$ be given (if $\alpha \leq 0$, set $N = N_0$ in the definition). Since $(b_n) \to +\infty$, there is $N_1 \in \mathbb{N}$ such that $b_n > \alpha/(L/2)$ for all $n > N_1$. Then for all $n > N = \max\{N_0, N_1\}$ we have

$$a_nb_n > \tfrac{L}{2} \cdot \tfrac{\alpha}{L/2} = \alpha,$$

as required. (We didn't really need to restrict to positive α, but multiplying inequalities is easier to keep track of if everything is positive.)

(b) Suppose that $\lim(a_n) = L \neq 0$, $\lim(b_n) = 0$, and $b_n > 0$ for all sufficiently large n. Use the previous item and Problem 3b to show that (a_n/b_n) is properly divergent.

$\triangleright$ By Problem 3b, $(1/b_n) \to +\infty$. Then the previous item applies to (a_n) and $(1/b_n)$, which gives $\lim(a_n/b_n) = +\infty$.

(9) [2pt] (Exercise 3.6.6) Let (x_n) be properly divergent and let (y_n) be such that $\lim(x_n y_n)$ exists as a real number. Show that (y_n) converges to 0.

$\triangleright$ Suppose not. Then there is an $\varepsilon_0 > 0$ and a subsequence (y_{n_k}) such that $|y_{n_k}| > \varepsilon_0$ for all $k \in \mathbb{N}$. Then for the the subsequence $(x_{n_k} y_{n_k})$ of $(x_n y_n)$ we have $|x_{n_k} y_{n_k}| > \varepsilon_0 \cdot |x_{n_k}| \to +\infty (k \to \infty)$. Thus, $(x_{n_k} y_{n_k})$ is unbounded, so $(x_n y_n)$ cannot converge to a real number.

$\triangleright$ Another solution is to notice that $(1/x_n) \to 0$ by Problem 3a, so $\lim(y_n) = \lim(x_n y_n \cdot \frac{1}{x_n}) = \lim(x_n y_n) \lim(1/x_n) = 0 \cdot 0 = 0$.

(10) ($\sim$ Exercise 3.6.8, 10) For the following sequences determine whether they are properly divergent.

(a) [2pt] $x_n = \sqrt{n^2 - 1}/\sqrt{n + 100}$. (*Hint:* Use the theorem in Prob. 7.)

$\triangleright$ Note that $\lim(x_n/\sqrt{n}) = 1 > 0$, so by Problem 7, $\lim(x_n) = +\infty$ since $\lim \sqrt{n} = +\infty$.

(b) [1pt] $x_n = \sin \sqrt{n}$.

$\triangleright$ $-1 \leq \sin \sqrt{n} \leq 1$, so the sequence is bounded and therefore not properly divergent. (It is also not convergent, but that (a) was not asked and (b) requires proof, which is not one-line.)

(c) [2pt] x_n if it is given that $\lim(x_n/n) = L$, where $L > 0$. (*Hint:* Don't forget that to use the previous theorem, you have to explain why $x_n > 0$ first.)

$\triangleright$ Since $\lim(x_n/n) = L > 0$, the sequence (x_n/n) and therefore (x_n) is ultimately positive. By the theorem in Problem 7, $\lim(x_n) = +\infty$ since $\lim(n) = +\infty$.