MA 232 - Linear Algebra

Homework 4 (Solutions)

Problem 1 [20pts] Find the eigenvalues and eigenvectors of the following matrices $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ and A^2. Compare their eigenvalues.

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = (\lambda+3)(\lambda-2)$$

Hence $\lambda = 2, -3$. We find $V(2)$ ("the eigenspace of 2")

$$A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow (A-2I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to find the nullspace of $A-2I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$

$$\sim \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \rightsquigarrow -3x+3y=0 \rightsquigarrow x=y \rightsquigarrow V(2) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

Likewise $V(-3) = N(A+3I) = \left\langle \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\rangle$

For A^2 we see $Ax = \lambda x \Rightarrow A\cdot(Ax) = \lambda Ax \Rightarrow A^2x = \lambda^2 x$

Hence $\lambda = 9, 4$, i.e. the eigenvalues of A squared and the corresponding eigenvectors stay the same

Problem 2 [20pts] Show that A and its transpose A^T have the same eigenvalues. Find an example that shows that they don't have the same eigenvectors.

$$\det(A-\lambda I) = \det(A-\lambda I)^T = \det(A^T-\lambda I)$$

Recall 1) $\det A = \det A^T$ for any A

2) $(A+B)^T = A^T + B^T$

<u>Counterexample</u>: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$

Eigenvalue $\lambda = 1$ $\quad V(1) = N(A-I) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$

$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ Eigenvalue $\lambda = 1$ $\quad V(1) = N(A^T - I) = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$

Problem 3 [20pts] Diagonalize the following matrices in the form $S\Lambda S^{-1}$.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$\det(A-\lambda I) = \lambda^2 - 4\lambda + 3$, Hence $\lambda = 1, 3$

$V(1) = N(A-I) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$

$V(3) = N(A-3I) = \langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle$

$$A = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{S} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{S^{-1}}$$

$\det(B-\lambda I) = \lambda^2 - 4\lambda$, Hence $\lambda = 0, 4$

$V(4) = N(A-4I) = \langle \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \rangle$

$V(0) = N(A) = \langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rangle$

$$B = \underbrace{\begin{bmatrix} 1/3 & -1 \\ 1 & 1 \end{bmatrix}}_{S} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 3/4 & 3/4 \\ -3/4 & 1/4 \end{bmatrix}}_{S^{-1}}$$

Problem 4 [20pts] Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ -\lambda & 0 \end{vmatrix} - \lambda \begin{vmatrix} 2-\lambda & 2 \\ 2 & -\lambda \end{vmatrix}$$

$$= 4\lambda - \lambda(\lambda^2 - 2\lambda - 4)$$

$$= -\lambda(\lambda^2 - 2\lambda - 8) = -\lambda(\lambda+2)(\lambda-4)$$

$$\lambda = 0, -2, 4$$

$$V(0) = N(A) = \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle \quad \text{unit } \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$V(-2) = N(A+2I) = \left\langle \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \quad \text{unit } \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$V(4) = N(A-4I) = \left\langle \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\rangle \quad \text{unit } \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Problem 5 [20pts] Test to see if $R^T R$ is positive definite in each case.

$$R = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

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1) $R^T R = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\det(R^T R) = 2\begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & 5 \\ 3 & 4 \end{vmatrix}$

$= 2 \cdot 9 - 3 \cdot 3 + 3(-3) = 0$. Hence, at least one eigenvalue is 0 and $R^T R$ is NOT positive definite.

2) $R^T R = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 6 & 5 \\ 0 & \frac{11}{6} \end{bmatrix}$ Positive Definite

Positive pivots hence positive eigenvalues

3) $R^T R = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$ Positive Definite