MA 441, Fall 2021

Assignment 2.

Solutions

There are total 20 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (but not over 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.1, 2.2, 2.3 in Bartle–Sherbert.

(1) [2pt] Prove that there does not exist a rational number r such that $r^2 = 7$. (*Hint:* Go similarly to the proof about $r^2 = 2$, but use divisibility by 7 instead of divisibility by 2.)

$\triangleright$ Suppose there is $r = a/b$ with $r^2 = 7$ and $\gcd(a, b) = 1$. Then $a^2 = 7b^2$. Since the right hand side is a multiple of 7, the left hand side must be a multiple of 7, too. But then a is a multiple of 7, i.e. $a = 7c$, so $49c^2 = 7b^2$ and $7c^2 = b^2$. Repeating argument, conclude that $b = 7d$. This contradicts the assumption that a, b are coprime.

(2) [5pt]

(a) ($\sim$2.1.8a) Let x, y be rational numbers. Prove that $xy, x-y$ are rational numbers. (*Hint:* Start by writing $x = \frac{m}{n}$, $y = \frac{k}{l}$, where $m, n, k, l \in \mathbb{Z}$.)

$\triangleright$ Let $x = \frac{m}{n}$, $y = \frac{k}{l}$, where $m, k \in \mathbb{Z},\ n, l \in \mathbb{N}$. Then $xy = \frac{mk}{nl}$, $x-y = \frac{ml-nk}{nl}$ ($nl \neq 0$ since $n \neq 0, l \neq 0$) are both rational.

(b) (2.1.8b) Let x be a rational number, y an irrational number. Prove that $x + y$ is irrational. Prove that if, additionally, $x \neq 0$, then xy is irrational.

$\triangleright$ Follows from previous item and fact that $y = (x+y)-x,\ y = (xy)/x$.

(c) Let x, y be irrational numbers. Is it true that $x+y$ is always irrational? Is it true that $x + y$ is always rational? (*Hint:* No and No.)

$\triangleright$ Answer to both questions is No. Examples:
$\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$,
$\sqrt{2} + \sqrt{2} = 2\sqrt{2} \notin \mathbb{Q}$.

(d) Same question about xy.

$\triangleright$ Answer to both questions is No. Examples:
$\sqrt{2} \cdot (\sqrt{2}) = 2 \in \mathbb{Q}$,
$\sqrt{2} \cdot \sqrt{3} = \sqrt{6} \notin \mathbb{Q}$.
Note that the latter relies on $\sqrt{3}, \sqrt{6}$ being irrational numbers. To avoid that and only use irrationality of $\sqrt{2}$, which was proven in lectures, we can use $\sqrt{2} \cdot \big(\sqrt{2} - 1\big) = 2 - \sqrt{2}$.

(3) [4pt] (2.1.9) Let $K = \{s + t\sqrt{2} \mid s, t \in \mathbb{Q}\}$. Show that K satisfies the following:

(a) If $x_1, x_2 \in K$ then $x_1 + x_2 \in K$ and $x_1x_2 \in K$.

$\triangleright$ Let $x_1 = s_1 + t_1\sqrt{2}$, $x_2 = s_2 + t_2\sqrt{2}$. Then

$$x_1 + x_2 = (s_1 + s_2) + (t_1 + t_2)\sqrt{2} \in K,$$

and

$$x_1x_2 = (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) = (s_1s_2 + 2t_1t_2) + (s_1t_2 + s_2t_1)\sqrt{2} \in K.$$

(b) If $x \neq 0$ and $x \in K$ then $1/x \in K$. (*Hint:* Get rid of irrationality in the denominator.)

$\triangleright$ Let $x = s + t\sqrt{2}$. Since $\sqrt{2} \notin Q$, the number $s - t\sqrt{2}$ is nonzero, and we can write

$$\frac{1}{x} = \frac{1}{s+t\sqrt{2}} = \frac{s-t\sqrt{2}}{(s+t\sqrt{2})(s-t\sqrt{2})} = \frac{s-t\sqrt{2}}{s^2-2t^2} = \frac{s}{s^2-2t^2} + \frac{-t}{s^2-2t^2}\sqrt{2} \in K.$$

Comment. In other words, K is a subfield of $\mathbb{R}$.

(4) [2pt] (2.2.4) Let $a, x \in \mathbb{R}$ and $\varepsilon > 0$. Show that $|x-a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$. (*Hint:* Don't forget that "A if and only if B" means "(if A then B) AND (if B then A)".)

$\triangleright$

$$\begin{gathered} |x-a| < \varepsilon \iff \\ -\varepsilon < x - a < \varepsilon \iff \\ a - \varepsilon < x - a + a < \varepsilon + a, \end{gathered}$$

as required.

(5) [3pt]

(a) Let $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

$\triangleright$ $\inf\{1 - \frac{1}{n} :\ n \in \mathbb{N}\} = 1 - 1 = 0$ by definition of sup.
$\sup\{1 - \frac{1}{n} :\ n \in \mathbb{N}\} = 1$ by Archimedean property.

(b) (2.3.4) Let $T = \{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$. Find $\inf T$ and $\sup T$. (*Hint:* If you are not sure what's going on, try to draw this set to get an idea.)

$\triangleright$ Note that for all $n \in \mathbb{N}$, we have

$$\frac{1}{2} = 1 - \tfrac{(-1)^2}{2} \leq 1 - \tfrac{(-1)^n}{n} \leq 1 - \tfrac{(-1)^1}{1} = 2.$$

Since $1/2$ and 2 are members of the set T, by definition of sup and inf we get $\sup T = 2$, $\inf T = 1/2$.

(6) (a) [2pt] (Part of 2.3.11) Let $S \subset \mathbb{R}$ be a bounded above set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \leq \sup S$. (*Hint:* Follow the definition.)

$\triangleright$ For every $s \in S' \subseteq S$, $s \leq \sup S$. Therefore $\sup S$ is *an* upper bound for S', so $\sup S' \leq \sup S$.

(b) [2pt] (2.3.10) Show that if A and B are bounded above nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded above set and $\sup A \cup B = \sup\{\sup A, \sup B\}$. (*Hint:* Follow the definition.)

$\triangleright$ Since for every $x \in A \cup B$, $x \in A$ or $x \in B$, it follows that $\sup\{\sup A, \sup B\}$ is indeed an upper bound of $A \cup B$, so $\sup A \cup B \leq \sup\{\sup A, \sup B\}$.
Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by previous item $\sup A \cup B \geq \sup A$ and $\sup A \cup B \geq \sup B$. Therefore, $\sup A \cup B = \sup\{\sup A, \sup B\}$.