Assignment 12.

Solutions

There are total 34 points in this assignment. 30 points is considered 100%. If you go over 30 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers section 6.4 in Bartle–Sherbert.

(1) For a given function f and a point x_0, find Taylor's polynomials $P_2(x)$, $P_5(x)$, $P_{2016}(x)$ of $f(x)$ at x_0.

(a) [2pt] $f(x) = \sin x$ at $x_0 = \pi/2$. Compare to cos at 0.

$\triangleright$ $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)} = -\cos x$, $f^{(4)}(x) = \sin x$. Plugging in $x = \pi/2$, we get

$$P_2(x) = P_3(x) = 1 - (x - \pi/2)^2/2,$$

$$P_4(x) = P_5 = 1 - (x - \pi/2)^2/2 + (x - \pi/2)^4/24.$$

$2016 = 4k$, so

$$P_{2016}(x) = P_{2017}(x) =$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \ldots - \frac{(x-\pi/2)^{2014}}{2014!} + \frac{(x-\pi/2)^{2016}}{2016!}.$$

Denoting $y = x - \pi/2$, get Taylor polynomials at 0 for $\cos y = \sin x$.

(b) [2pt] $f(x) = \cos x$ at $x_0 = -\pi/2$. Compare to sin at 0.

$\triangleright$ $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f^{(3)}(x) = \sin x$, $f^{(4)}(x) = \cos x$. Plugging in $x = -\pi/2$, we have

$$P_2(x) = P_1(x) = (x + \pi/2),$$

$$P_4(x) = P_3(x) = (x + \pi/2) - (x + \pi/2)^3/6,$$

$$P_5(x) = (x + \pi/2) - (x + \pi/2)^3/6 + (x + \pi/2)^5/120.$$

Since $2016 = 4k$, we get

$$P_{2016}(x) = P_{2015}(x) =$$

$$= (x + \pi/2) - (x + \pi/2)^3/6 + \ldots - (x + \pi/2)^{2015}/2015!.$$

(c) [2pt] $f(x) = x^3$ at $x_0 = 2$. Compare $P_3(x), P_5(x), P_{2016}(x)$ to $f(x)$.

$\triangleright$ $f'(x) = 3x^2$, $f''(x) = 6x$, $f^{(3)}(x) = 6$, $f^{(4)}(x) = 0$. Plugging in $x = 2$, we have

$$P_2(x) = 8 + 12(x - 2) + 6(x - 2)^2,$$

$$P_3(x) = P_4(x) = P_5(x) = P_{2016}(x) =$$

$$= 8 + 12(x - 2) + 6(x - 2)^2 + (x - 2)^3.$$

Note that $R_3 = f^{(4)}(c)(x-2)^4/24 = 0$, so $P_3(x) = P_4(x) = P_{2015}(x) = x^3$. This can also be seen by directly expanding brackets in expression for P_3 obtained above.

(d) [2pt] $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.

▷ $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2!}{(1-x)^3}$, $f^{(3)}(x) = \frac{3!}{(1-x)^4}$, $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$. Plugging in $x = 0$, we have

$$P_k(x) = 1 + x + x^2 + \ldots + x^k.$$

(e) [2pt] $f(x) = \frac{1}{x}$ at $x_0 = 1$. Compare to the previous item.

▷ $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2!}{x^3}$, $f^{(3)}(x) = \frac{-3!}{x^4}$, $f^{(k)}(x) = (-1)^k \frac{k!}{x^{k+1}}$. Plugging in $x = 1$, we have

$$P_k(x) = 1 - (x-1) + (x-1)^2 + \ldots + (-1)^k (x-1)^k.$$

Similarity to previous item comes from the fact that, denoting $h(x) = 1/(1-x)$, $g(x) = 1/x$, we have

$$g(x) = g(x-1+1) = g(y+1) = h(-y), \text{ where } y = x - 1.$$

(You can take for granted that $(\sin x)' = \cos x, (\cos x)' = -\sin x$.)

(2) [3pt] (Part of exercise 6.4.7) If $x > 0$, show that

$$\left| \sqrt[4]{1+x} - \left(1 + \frac{1}{4}x - \frac{3}{32}x^2\right) \right| \leq \frac{7}{128}x^3.$$

(*Hint:* Apply Taylor's Theorem to $f(x) = \sqrt[4]{1+x}$ with $n = 2$.)

▷ By Taylor's Theorem for $\sqrt[3]{1+x}$ at 0 and $n = 2$, we have

$$\sqrt[4]{1+x} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}(1+c)^{-11/4}x^3,$$

where c lies between 0 and x, so $c \geq 0$ and $0 < (1+c)^{-11/4} \leq 1$. Therefore,

$$\left| \sqrt[4]{1+x} - \left(1 + \frac{1}{4}x - \frac{3}{32}x^2\right) \right| = \left| \frac{7}{128}x^3(1+c)^{-11/4} \right| \leq \frac{7}{128}x^3.$$

(3) (a) [2pt] Suppose $A \in \mathbb{R}$. Show that $\lim\limits_{n\to\infty} \frac{A^n}{n!} = 0$.

Hint: take tail of this sequence that starts with $m > 2|A|$ and represent

$$\frac{A^n}{n!} = \frac{A^m}{m!} \cdot \frac{A^{n-m}}{(m+1)\cdots n}.$$

▷ Since $\lim(|x_n|) = 0$ implies $\lim(x_n) = 0$, considering $|A|$ instead of A, we may assume $A \geq 0$. Let n_0 be a fixed natural number bigger than $2A$. For $n > n_0$, split A^n as $A^{n_0} \cdot A^{n-n_0}$. Also split $n! = n_0! \cdot (n_0+1)(n_0+2) \cdot \ldots \cdot (n-1)n \geq n_0! \cdot n_0^{n-n_0}$. Then

$$\frac{A^n}{n!} = \frac{A^{n_0} \cdot A^{n-n_0}}{n_0! \cdot (n_0+1)(n_0+2) \cdot \ldots \cdot (n-1)n} \leq$$

$$\frac{A^{n_0}}{n_0!} \cdot \frac{A^{n-n_0}}{n_0^{n-n_0}} =$$

$$\frac{A^{n_0}}{n_0!} \cdot \left(\frac{A}{n_0}\right)^{n-n_0} \leq$$

$$\frac{A^{n_0}}{n_0!} \cdot \left(\frac{1}{2}\right)^{n-n_0} \to 0$$

as $n \to \infty$, because $\frac{A^{n_0}}{n_0!}$ is a constant and $\left(\frac{1}{2}\right)^{n-n_0} \to 0$ as $n \to \infty$.

(b) [2pt] (6.4.8) If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

$\triangleright$ Note that $f^{(n)} = e^x$. In Taylor's Theorem, we have

$$R_n = e^c(x - x_0)^{n+1}/(n+1)!.$$

Note that c depends on x, x_0 and n, but in any case c is between x and x_0, so $|c| \le \max\{|x|, |x_0|\} = C$ (C is a constant with respect to n). Then we have

$$R_n = e^c(x - x_0)^{n+1}/(n+1)! \le e^C \frac{(x - x_0)^{n+1}}{(n+1)!} \to 0$$

as $n \to \infty$ by the previous problem.

(c) [2pt] (6.4.9) If $g(x) = \cos x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

$\triangleright$ Note that $|f^{(n)}| \le 1$. In Taylor's Theorem, we have

$$R_n = f^{(n)}(c)(x - x_0)^{n+1}/(n+1)!$$

Note that c depends on x, x_0 and n, but in any case $|f^{(n)}(c)| \le 1$, so

$$R_n = f^{(n)}(c)(x - x_0)^{n+1}/(n+1)! \le (x - x_0)^{n+1}/(n+1)! \to 0$$

as $n \to \infty$ by 3a.

(4) [4pt] (Part of exercise 6.4.11) If $x > 0$ and $n \in \mathbb{N}$, show that

$$\left|\ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}\right| < \frac{x^{n+1}}{n+1}.$$

(*Hint:* Apply Taylor's Theorem to $f(x) = \ln(1+x)$.)

$\triangleright$ By Taylor's theorem, the left hand side of the above inequality is $|R_n(x)|$,

$$R_n(x) = \frac{(\ln(1+x))^{(n+1)}\big|_{x=c}}{(n+1)!} x^{n+1}$$

for some point $c \in (0, x)$. Since $(\ln(1+x))^{(n+1)} = \frac{(-1)^{n+2}n!}{(1+x)^{n+1}}$, we get that

$$|R_n(x)| = \frac{1}{(n+1)(1+c)^{n+1}} x^{n+1} < \frac{1}{n+1} x^{n+1},$$

since $c > 0$, as required.

(5) [5pt] (6.4.14+) Use nth derivative test to determine whether or not $x = 0$ is a point of relative extremum of the following functions. If it is, specify whether it is a point maximum or minimum.

(a) $f(x) = x^n$, $n \in \mathbb{N}$,

$\triangleright$ We see that $f'(0), \ldots, f^{(n-1)}(0) = 0$, and $f^{(n)}(0) = n! > 0$. So if n is odd, by nth derivative test 0 is not a point of relative extremum. If n is even, by nth derivative test 0 is a point of relative minimum.

COMMENT. The result was, of course, obvious anyway, so this was purely an exercise in applying the test.

(b) $f(x) = \sin x - \tan x$,

$\triangleright$ We have

$$f'(x) = (\sin x - \tan x)' = \cos x - \frac{1}{\cos^2 x},$$

which is 0 at 0. Compute second derivative:

$$f''(x) = \left(\cos x - \frac{1}{\cos^2 x}\right)' = -\sin x - 2\frac{\sin x}{\cos^3 x},$$

which is still 0 at 0. Compute third derivative:

$$f^{(3)}(x) = \left(-\sin x - 2\frac{\sin x}{\cos^3 x}\right)' = -\cos x - 2\frac{\cos^4 x + 3\sin^2 x \cos^2 x}{\cos^6 x},$$

so $f^{(3)}(0) = -3 \neq 0$. Therefore, since $n = 3$ is odd, by nth derivative test, $f(x)$ has no relative extremum at 0.

(c) $f(x) = \cos x - 1 + \frac{1}{2}x^2$.

$\triangleright$ $f'(0) = f''(0) = f^{(3)}(0) = 0$, $f^{(4)}(0) = 1 \neq 0$, so $n = 4$ in nth derivative test. Since 4 is an even number and $f^{(4)}(0) > 0$, $x = 0$ is a point of local minimum.

(6) [3pt] Let $a \in \mathbb{R}$ be a constant s.t. $a > 0$. Consider the function $f(x) = x^2 - a$. Find the recursive relation provided by Newton's Method. Write out first 4 terms of the corresponding sequence if $x_1 = 1$ and $a = 2$. Check on a calculator how small $|x_4 - \sqrt{2}|$ is.

Comment. You should have gotten $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. We already have seen that this sequence converges to $\sqrt{a}$ when we covered the Monotone Convergence Theorem. Now we finally know where this sequence actually comes from.

$\triangleright$ We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

With $x_1 = 1$ and $a = 2$, we have

$$x_2 = \frac{3}{2},\ x_3 = 17/12,\ x_4 = 577/408.$$

On a calculator we can see that $|x_4 - \sqrt{2}| = 0.000002123901\ldots < 2.2 \cdot 10^{-6}$.

(7) [3pt] Apply Newton's Method to find a recursive relation for approximating $\sqrt[3]{2}$. For the interval $I = [1, 2]$, find M, m, and K in the statement of Newton's Method. Find an interval $I^* \subseteq [1, 2]$ s.t. the convergence of a sequence given by the above relation and any $x_1 \in I^*$ is guaranteed. (*Hint:* $f(x) = x^3 - 2$.)

$\triangleright$ Consider $f(x) = x^3 - 2$. On $[1, 2]$ we have $|f'(x)| = |3x^2| \geq 3 = m$, $|f''(x)| = |6x| \leq 18 = M$. That gives us $K = M/2m = 3$ in the statement of Newton's Method.

Now compose the recursive relation:

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2x_n}{3} + \frac{2}{3x_n^2}.$$

An interval I^* on which convergence is guaranteed can be given as

$$(\sqrt[3]{2} - 1/K, \sqrt[3]{2} + 1/K) \cap [1, 2] = [1, \sqrt[3]{2} + 1/3).$$

(In particular, since $(7/6)^3 < 2$, the convergence is guaranteed by Newton's Method Theorem on $[1, 7/6 + 1/3] = [1, 1.5]$.)