Assignment 8

Solutions

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 4.1, 4.2 in Bartle–Sherbert.

(1) (Modified 4.1.1) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)

(a) [1pt] $|x^3 - 1| < 1/2$, $c = 1$. (*Hint:* $x^3 - 1 = (x - 1)(x^2 + x + 1)$.)

$\triangleright$ Note that $|x^3 - 1| = |x - 1| \cdot |x^2 + x + 1|$. So, for $0 < x < 2$, we have $|x^3 - 1| \leq |x - 1| \cdot 7$. Pick $\delta = (1/2)/7$. **Since this δ is** < 1, for all x such that $0 < |x - 1| < \delta$ **we have** $0 < x < 2$, so $|x^3 - 1| \leq 7|x - 1| < 7 \cdot (1/2)/7 = 1/2$.

(b) [1pt] $|x^3 - 1| < 10^{-3}$, $c = 1$.

$\triangleright$ Pick $\delta < 10^{-3}/7$, e.g. $\delta = 10^{-4} < 1$. Then $|x^3 - 1| \leq 7|x - 1| < 7 \cdot (10^{-4}) < 10^{-3}$.

(c) [1pt] $|x^3 - 1| < \frac{1}{10^{-3}}$, $c = 1$.

$\triangleright$ As in two previous items, pick $\delta < \min\{1, 10^3/7\} = 1$. Then $|x^3 - 1| \leq |x|^3 + 1 < 2^3 + 1 = 9 < 10^3$.

(Note that for $\delta = 10^3/7$, x may be > 2, so the bound $|x^3 - 1| \leq |x - 1| \cdot 7$ is false for such δ. That's why we need to take $\min\{1, 10^3/7\}$.)

(d) [1pt] $|x^2 \cos x^3 - 0| < 0.00001$, $c = 0$.

$\triangleright$ Since $-1 \leq \cos\theta \leq 1$ for any $\theta \in \mathbb{R}$, we have $|x^2 \cos x^3 - 0| \leq |x^2|$. Therefore, for $\delta = 0.001$ we have that if $0 < |x - 0| < \delta$, then $|x^2 \cos x^3 - 0| \leq |x^2| < 0.001^2 < 0.00001$.

(2) [3pt] (Modified 4.1.9) *Use the ε–δ definition of limit* to show that

(a) $\lim\limits_{x \to 2} \frac{1}{1-x} = -1$,

$\triangleright$ $\frac{1}{1-x} - (-1) = \frac{2-x}{1-x} = \frac{2-x}{-1+(2-x)}$. For $|2 - x| < 1/2$, we have $|-1 + (2 - x)| \geq 1/2$, so

$$\left| \frac{2 - x}{-1 + (2 - x)} \right| \leq 2|2 - x|.$$

So, given $\varepsilon > 0$, it suffices to take $\delta = \min\{1/2, \varepsilon/2\}$.

(b) $\lim\limits_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

$\triangleright$ $\frac{x}{1+x} - \frac{1}{2} = \frac{x-1}{2(x+1)}$. For $x > 0$ (guaranteed by, for example, $|x - 1| < 1$), we have $|2(x + 1)| \geq 2$, so

$$\left| \frac{x - 1}{2(x + 1)} \right| \leq |x - 1|/2.$$

So, given $\varepsilon > 0$, it suffices to take $\delta = \min\{1, 2\varepsilon\}$.

(3) [2pt] (4.1.11) Show that the following limits do not exist:

(a) $\lim_{x\to 0}(x + \operatorname{sgn} x)$,

▷ *Solution directly from definition of limit.* Suppose $\lim_{x\to 0}(x+\operatorname{sgn}(x)) = L$. Take $\varepsilon_0 = 1/2$. Then for whichever $\delta > 0$ we take, there is an x_0 such that $0 < x_0 < \delta$ in case $L < 0$, or $-\delta < x_0 < 0$ in case $L \geq 0$. For definiteness, consider $L < 0$ (other case is similar).
Then $x_0 + \operatorname{sgn}(x_0) = 1 + x_0 > 1$, while $L + \varepsilon_0 \leq 0 + 1/2 = 1/2$. So $|L - (x_0 + \operatorname{sgn}(x_0))| \geq 1/2$, which contradicts $\lim_{x\to 0}(x + \operatorname{sgn}(x)) = L$.

▷ *Solution using sequential criterion.* Consider $x_n = (-1)^n/n \to 0$ $(n \to \infty)$. Then $\lim(x_n + \operatorname{sgn} x_n) = \lim((-1)^n/n + (-1)^n)$ and the latter limit does not exist. By sequential criterion, the limit in question does not exist.

(b) $\lim_{x\to 0}\sin(1/x^2)$.

▷ *Solution directly from definition of limit.* Suppose $\lim_{x\to 0}\sin(1/x^2) = L$. Take $\varepsilon_0 = 1/2$. Then for any $\delta > 0$, there is $x \in (-\delta, \delta)$ of the form $\left(\sqrt{\pi/2 + 2\pi n}\right)^{-1}$, since the corresponding sequence converges to 0. Similarly, there is $y \in (-\delta, \delta)$ of the form $y = \left(\sqrt{-\pi/2 + 2\pi n}\right)^{-1}$. For these x and y we get $\sin(1/x^2) = 1$, $\sin(1/x^2) = -1$, so $1, -1 \in (L - \varepsilon_0, L + \varepsilon_0)$. Therefore, $1 - (-1) < 2\varepsilon_0 = 1$, so $2 < 1$, which is a contradiction.

▷ *Solution using sequential criterion.* Consider

$$x_n = \left(\sqrt{(-1)^n\pi/2 + 2\pi n}\right)^{-1} \to 0$$

as $n \to \infty$. Then $\sin(1/x_n^2) = (-1)^n$, so $\lim(\sin(1/x_n^2))$ does not exist. By sequential criterion, the limit in question does not exist.

(4) [3pt] (4.1.15) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has limit at $x = 0$ (*Hint*: you can use the ε–δ definition directly, or the sequential criterion and squeeze theorem).

▷ *Solution using Squeeze theorem for sequences.* Let (x_n) be a sequence that converges to 0, $x_n \neq 0$. Then $f(x_n) = 0$ or $= x_n$, so $-|x_n| \leq f(x_n) \leq |x_n|$. Since $|x_n| \to 0$ and $-|x_n| \to 0$ $(n \to \infty)$, by Squeeze theorem for sequences, $(f(x_n)) \to 0 = f(0)$, so f is continuous at 0.

▷ *Solution using the ε–δ definition of limit.* Let $\varepsilon > 0$ be given. Put $\delta = \varepsilon$. Then if $|x - 0| < \delta$, we have $|f(x) - f(0)|$ is either $|0 - 0| < \varepsilon$, or $|x - 0| < \delta = \varepsilon$. In either case, $|f(x) - f(0)| < \varepsilon$. By definition of continuity, f is continuous at 0.

▷ *Solution using Squeeze theorem for functions.* Note that for any x, $-|x| \leq f(x) \leq |x|$. Since

$$\lim_{x\to 0}|x| = \lim_{x\to 0}(-|x|) = 0,$$

by squeeze theorem, $\lim_{x\to 0} f(x)$ exists and is equal to 0.

(b) Prove that if $c \neq 0$, then f does not have limit at c. (*Hint*: you can use sequential criterion.)

▷ By density theorem, there is a sequence of rational numbers $x_n \to c$ as $n \to \infty$, $x_n \neq c$; and a sequence of irrational numbers $y_n \to c$ as $n \to \infty$, $y_n \neq c$. Then $\lim(x_n) = c$ and $\lim(y_n) = 0$.

Since $c \neq 0$, the sequence

$$z_n = x_n \qquad \text{for even } n,$$
$$z_n = y_n \qquad \text{for odd } n$$

is divergent, so $\lim_{x\to 0} f(x)$ does not exist.

(5) [2pt] (Theorem 4.2.4 for difference) *Using ε–δ definition,* prove that limit of functions preserves difference. That is, prove the following:
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A, and f, g be functions on A to $\mathbb{R}$. If $\lim_{x\to c} f = L$, and $\lim_{x\to c} g = M$, then $\lim_{x\to c} f - g = L - M$.

$\triangleright$ Suppose $\varepsilon > 0$ is given. Since $\lim_{x\to c} f = L$, there is $\delta_1 > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Since $\lim_{x\to c} g = M$, there is $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Then for the $\delta = \min\{\delta_1, \delta_2\}$, we have that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - g(x) - (L - M)| = |f(x) - L - (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < 2\varepsilon/2 = \varepsilon$.

(6) [2pt] Using arithmetic properties of limit, find the following limits.

(a) $\lim_{x\to 1} \frac{x^{100}+2}{x^{100}-2}$.

$\triangleright$ Note that $\lim_{x\to 1} x = 1$, so $\lim_{x\to 1} x^{100} = 1^{100} = 1$, and $\lim_{x\to 1} x^{100} \pm 2 = 1 \pm 2$. Denominator is nonzero, so $\lim_{x\to 1} \frac{x^{100}+2}{x^{100}-2} = \frac{3}{-1} = -3$.

(b) $\lim_{x\to 1} \frac{2x^2-x-1}{x^2-3x+2}$. (*Hint:* Denominator turns to 0 at $x = 1$, but you can cancel out $(x - 1)$.)

$\triangleright$ $\lim_{x\to 1} \frac{2x^2-x-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(2x+1)(x-1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{2x+1}{x-2}$. Now the limit can be computed similarly to the previous item, $\lim_{x\to 1} \frac{2x+1}{x-2} = \frac{2\cdot 1+1}{1-2} = -3$.

(c) $\lim_{x\to 0} \frac{(x+1)^{20}-1}{x}$.

$\triangleright$ Note that by binomial formula $(x+1)^{20}-1 = 1+20x+x^2p(x)$, where $p(x)$ is a polynomial in x. So, $\frac{(x+1)^{20}-1}{x} = (1 + 20x + x^2p(x) - 1)/x = 20 + xp(x)$. We have $\lim_{x\to 0} \frac{(x+1)^{20}-1}{x} = \lim_{x\to 0} 20 + xp(x)$. The latter limit by arithmetic properties of limits is $\lim_{x\to 0} 20+xp(x) = 20+0\cdot p(0) = 20$.

(d) $\lim_{x\to c} \frac{(x-c+1)^2-1}{x-c}$.

$\triangleright$ Note that $\frac{(x-c+1)^2-1}{x-c} = \frac{(x-c)^2+2(x-c)+1-1}{x-c} = x - c + 2$, so $\lim_{x\to c} \frac{(x-c+1)^2-1}{x-c} = \lim_{x\to c} x - c + 2 = c - c + 2 = 2$.

(7) (a) [2pt] (4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x\to c} g = 0$. Prove that $\lim_{x\to c} fg = 0$.
Explain why Theorem 4.2.4 (Arithmetic Properties of Limit) cannot be used.

$\triangleright$ We show this by definition of a limit. Let f be bounded by $M > 0$ on a δ_0-neighborhood of c, i.e. $|f(x)| < M$ for all $x \in A$, $|x - c| < \delta_0$. Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g = 0$, there is $\delta_1 > 0$ such that $|g(x)| < \varepsilon/M$ for all $x \in A$, $0 < |x - c| < \delta_1$. Then put $\delta = \min\{\delta_0, \delta_1\}$, and get that for all $x \in A$, $0 < |x - c| < \delta$,

$$|f(x)g(x)| < M \cdot \varepsilon/M = \varepsilon.$$

So $\lim_{x\to c} fg = 0$ by definition of a limit.

$\triangleright$ Another way is to use the squeeze theorem: note that for δ_0 and M as above, we have

$$0 \le |fg| \le Mg$$

on a δ_0-neighborhood of c. So by squeeze theorem, $\lim_{x\to c} |fg| = 0$. Then since $-|fg| \le fg \le |fg|$, by another application of the squeeze theorem, we get $\lim_{x\to c} fg = 0$.

Theorem 4.2.4 does not apply because f may have no limit at c.

(b) [1pt] ($\sim$4.2.11b) Determine whether $\lim_{x\to 0} x\cos(1/x^2)$ exists in $\mathbb{R}$.

$\triangleright$ Note that $\lim_{x\to 0} x = 0$ and $|\cos(1/x^2)| \le 1$, so by the problem above $\lim_{x\to 0} x\cos(1/x^2) = 0$. (We could also use the Squeeze Theorem directly.)

(8) [4pt] (4.2.15) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose $f(x) \ge 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) = \sqrt{f(x)}$. If $\lim_{x\to c} f$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$. (*Hint:* $a^2 - b^2 = (a-b)(a+b)$. Another hint: you will probably have to consider cases $\lim_{x\to c} f = 0$ and $\lim_{x\to c} f \ne 0$ separately.)

$\triangleright$ Let $\lim_{x\to c} f = L \ge 0$ (since $f \ge 0$). If $L \ne 0$, note that

$$0 \le \left|\sqrt{f} - \sqrt{L}\right| = \tfrac{|f-L|}{\sqrt{f}+\sqrt{L}} \le \tfrac{|f-L|}{\sqrt{L}},$$

so by squeeze theorem $\lim_{x\to c} \left|\sqrt{f} - \sqrt{L}\right| = 0$, i.e. $\lim_{x\to c} \sqrt{f} = \sqrt{L}$.

If $L = 0$, show by definition that $\lim_{x\to c} \sqrt{f} = 0$. Indeed, since $\lim_{x\to c} f = 0$, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| = f(x) < \varepsilon^2$ whenever $x \in A$, $0 < |x - c| < \delta$. But then for the same values of x we have $\sqrt{f(x)} < \varepsilon$, so $\lim_{x\to c} \sqrt{f} = 0$ by definition of a limit.