Homework 2

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Problem 1

Theorem. *Let $n \in \mathbb{Z}$. Prove that $2n^2 + n$ is odd if and only if $cos(\frac{n\pi}{2})$ is even.*

Proof. Since the range of the cosine function is [-1, 1], the only even solution for cos(x) is 0. We know that cos(x) = 0 if and only if $x = \frac{\pi}{2} + k\pi$, where k is an integer. In order for $cos(\frac{n\pi}{2})$ to be even, we must find n such that $\frac{n\pi}{2} = \frac{\pi}{2} + k\pi$. This occurs for all n, where n is not divisible by 2, that is, all odd n.

Looking at $2n^2 + n$, we notice that $2n^2$ will always be even, since it contains a factor of 2. If n is even, then if the additional n term is odd, $2n^2 + n$ will be odd, since an odd integer added to an even integer is odd[1]. Thus, $2n^2 + n$ will be odd if and only if n is odd.

Since the condition for both propositions to be true is that n is an odd number, satisfying this condition would satisfy both propositions, and even values of n would satisfy neither proposition, Thus, the biconditional implication holds. □

Problem 2

Theorem. *Let $x, y \in \mathbb{Z}$. Prove that if xy is odd, then $x^2 + y^2$ is even.*

Proof. We can examine if xy is even or odd based on 3 cases.

Case 1: x and y are both even.
If x and y are both even, we can then express $x = 2k$, for some integer k, and $y = 2m$, for some integer m. Then, $xy = (2k)(2m) = 2(km)$. Since xy contains a factor of 2, it must be even.

Case 2: x and y are opposite parity. (x is even and y is odd)
If x and y are opposite parity, we can then express $x = 2k$, for some integer k, and $y = 2m + 1$, for some integer m. Then, $xy = (2k)(2m + 1) = (4km + 2k) = 2(2km + k)$. Since xy contains a factor of 2, it must be even.

[1] Let a and b be integers of opposite parity. We can then express $a = 2m$ for some integer m and $b = 2n + 1$ for some integer n. Then $a + b = 2m + 2n + 1 = 2(m + n) + 1$. Thus the sum of two integers with opposite parity is odd.

Case 3: x and y are both odd.
If x and y are both odd, we can then express $x = 2k + 1$, for some integer k, and $y = 2m + 1$, for some integer m. Then $xy = (2k + 1)(2m + 1) = (4km + 2k + 2m + 1) = 2(2km + k + m) + 1$. Since xy can be expressed in the form of $2a + 1$, where a is an integer, xy must be odd.

Based on the 3 cases, to satisfy the proposition xy is odd, x and y must both be odd.

Now we examine the expression x^2+y^2 for some odd x and some odd y. We can represent $x = 2k+1$, for some integer k, and $y = 2m + 1$, for some integer m. Then $x^2 + y^2 = (2k + 1)^2 + (2m + 1)^2 = (4k^2 + 4k + 1) + (4m^2 + 4m + 1) = 2(2k^2 + m^2 + 2k + 2m + 1)$. Since the expression $x^2 + y^2$ contains a factor of 2, it must be even.

$\square$

Problem 3

Theorem. *Prove that there is a student S in MA 240 for whom the following is true: If S eats a chocolate on Valentine's Day, then* everyone *in MA 240 will eat a chocolate on Valentine's Day.*

Proof. We can prove this statement by 2 cases.

Case 1: Student S does not eat a chocolate on Valentine's Day.
In this case, since one student S does not eat a chocolate, the proposition that *everyone* in MA 240 eats a chocolate will fail. This is the vacuous case, and since the initial proposition is false, the implication must hold true.

Case 2: Student S eats a chocolate on Valentine's Day.
In this case, knowing nothing about any other students in the class, it is possible that the second proposition will hold true. If the second proposition holds true and every other student in the class eats a chocolate, then the implication holds. However, if someone else in the class does not eat a chocolate, this other student will be the existing student S' for which the implication is vacuously true as shown in the first case.

It follows that for every combination of students eating chocolate on Valentine's Day, this statement will hold true, whether there exists a student who does not eat chocolate and the implication holds vacuously, or everyone in MA 240 eats chocolate. $\square$