Homework 7

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Problem 1

Theorem. *If two sets A and B have the same cardinality, then for any set C, the sets $A \times C$ and $B \times C$ have the same cardinality as well.*

Proof. Suppose $|A|$ and $|B|$. This implies that there exists a bijection $f : A \to B$, as well as $g : B \to A$ that maps the elements from each set in a one-to-one relation. We can then map the set C to itself, through mapping each element in C to itself, and define the bijection to be $h : C \to C$. The Cartesian Product defines $A \times C = \{(a, c) \mid a \in A \text{ and } c \in C\}$, and $B \times C = \{(b, c) \mid b \in B \text{ and } c \in C\}$. We can then define an injection $i : A \times C \mapsto B \times C$ by $(a, c) \mapsto (f(a), h(c))$. Since we have an injection defined, it follows that $|A \times C| = |B \times C|$. $\square$

Problem 2

Theorem. *A set A is countable if and only if there exists a surjection $f : \mathbb{N} \to A$.*

Proof. Proof of Implication: If A is countable then there exists a surjection $f : \mathbb{N} \to A$.

Case 1: A is finite.

If A is finite, there exists a bijection f of some set $\mathbb{N}_n$ onto A and we can define F on $\mathbb{N}$ by

$$F(x) = \begin{cases} f(x) & \text{for } x = 1, \dots , n \\ f(n) & \text{for } x > n \end{cases}$$

Thus F is a surjection of $\mathbb{N}$ onto A.

Case 2: A is countably infinite.

If A is countably infinite, then there exists a bijection of F of $\mathbb{N}$ onto A, which is also a surjection of $\mathbb{N}$ onto A.

Proof of Converse: If there exists a surjection $f : \mathbb{N} \to A$, then A is countable.

Suppose that $g : \mathbb{N} \to A$ is a surjection. Let us define $h : A \to \mathbb{N}$ by $h(a) =$ the smallest $n \in \mathbb{N}$

such that $g(n) = a$, which the Well-Ordering Principle guarantees the existence of such n. Then h is injective because if $h(a_1) = h(a_2)$, then $g(h(a_1)) = a_1$ and $g(h(a_2)) = a_2$, which implies that $a_1 = a_2$. Since $\mathbb{N}$ is countably infinite and $h(A) \subseteq \mathbb{N}$, it follows that $h(A)$ is countable. Since h is a bijection from A to $h(A)$, A is countable. □

Problem 3

Theorem. *The infinite strip* $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R} \text{ and } 0 \leq y \leq 1\}$ *and the cartesian plane* $\mathbb{R}^2$ *have the same cardinality.*

Proof. The infinite strip S can be considered the Cartesian product of $\mathbb{R}$ and $Y = [0, 1]$, and the Cartesian plane is the Cartesian product $\mathbb{R} \times \mathbb{R}$. In Problem 1, we proved that if two sets A and B have the same cardinality, then for any set C, the sets $A \times C$ and $B \times C$ have the same cardinality as well. Thus, we can prove this statement by showing $Y = [0, 1]$ has the same cardinality as $\mathbb{R}$ (consider set C in this case is $\mathbb{R}$, and that $\mathbb{R} \times Y$ is equivalent to $Y \times \mathbb{R}$ in cardinality).

Proof: $Y = [0, 1]$ has the same cardinality as $\mathbb{R}$: Given the Schröder-Bernstein theorem, we can prove this by showing an injection in both directions. The injection $f : [0, 1] \to \mathbb{R}$ is trivial, simply let f map each real number in $[0, 1]$ to itself in $\mathbb{R}$. Then, to show the injection $g : \mathbb{R} \to [0, 1]$, define, for $x \in \mathbb{R}$,

$$g(x) = \frac{1}{\pi} \cdot \arctan(x) + \frac{1}{2}.$$

This maps an injection from $\mathbb{R}$ to $[0, 1]$. Thus, $Y = [0, 1]$ has the same cardinality as $\mathbb{R}$, and the statement holds. □