MA 232 - Linear Algebra

Homework 6 (due November 27)

Problem 1 [15 pts]
Find the parabola $C + Dt + Et^2$ that fits best the following set of data: $b = 0, 0, 1, 0, 0$, at the times $t = -2, -1, 0, 1, 2$.

$$\left.\begin{aligned} C + D(-2) + E(-2)^2 &= 0 \\ C + D(-1) + E(-1)^2 &= 0 \\ C + D(0) + E(0)^2 &= 1 \\ C + D(1) + D(1)^2 &= 0 \\ C + D(2) + D(2)^2 &= 0 \end{aligned}\right\}$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

Doesn't have a solution

Least Square Approximation

$$A^TA = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence $\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$\hat{C} = 34/70$

$\hat{D} = 0$

$\hat{E} = -10/70$

Problem 2 [15 pts]
Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2 span the column space of
$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$

We first find an orthonormal basis for $C(A)$

$B = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $B^TB = 9$

$c = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ $\quad B^Tc = -9$

$$C = c - \frac{B^Tc}{B^TB} B = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

We now consider $d = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$ which is independent (check)

$B^Td = 9$, $C^Td = 18$, $C^TC = 9$

$$D = d - \frac{B^Td}{B^TB} B - \frac{C^Td}{C^TC} C = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix}$$

$q_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ $\quad q_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\quad q_3 = \frac{1}{6} \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix}$

Problem 3 [5 pts]
Find the determinants of U, U^{-1} (when it exists), U^2 for:

$$U = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}, U = \begin{bmatrix} \text{a} & \text{b} \\ 0 & \text{d} \end{bmatrix}$$

1) $\det U = 1\ 2\ 3 = 6$, $\det U^2 = \det U \det U = 36$

$$\det U^{-1} = \frac{1}{\det U} = \frac{1}{6}$$

2) $\det U = ad$, $\det U^2 = (ad)^2$, $\det U^{-1}$ exists iff $ad \neq 0$ in case it exists is $\frac{1}{ad}$

Problem 4 [5 pts]
Show that if A is not invertible, then AB is not invertible.

A matrix A is invertible iff $\det A \neq 0$

$\det(AB) = \det A \det B$

Hence, since $\det A = 0$, $\det(AB) = 0$ and consequently AB is not invertible.

Problem 5 [15 pts]
Find whether the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

$\det(A-\lambda\mathbb{I}) = -(\lambda-2)^2(\lambda-1)$, $\lambda = 2, 2, 1$

For $\lambda = 2$, $V(2) = N(A-2\mathbb{I}) = \left\langle \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

For $\lambda = 1$, $V(1) = N(A-\mathbb{I}) = \left\langle \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix} \right\rangle$

Hence $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & -1/3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 3 & -6 & -5 \\ -3 & 6 & 6 \end{bmatrix}$

Remark: If $V(2)$ had dimension 1, i.e. it was generated by a single vector, then A would not be diagonalizable.

Problem 6 [15 pts]
Orthogonally diagonalize the matrix A, i.e. $A = PDP^T$, where P is orthogonal.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = (\lambda - 2)^2(\lambda - 8)$$

$$V(2) = N(A - 2I) = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \underset{\text{Gram-Schmidt}}{\overset{\text{orthonormal}}{\rightsquigarrow}} \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\rangle$$

$$V(8) = N(A - 8I) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \overset{\text{unit}}{\rightsquigarrow} \left\langle \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

$$A = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{=P} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 8 \end{bmatrix} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{=P^T}$$

Problem 7 [15 pts]

Consider the matrix $A = \begin{bmatrix} 2 & \text{b} \\ 1 & 0 \end{bmatrix}$. Find a value of b that makes:

- $A = QDQ^T$ possible, i.e. orthogonal diagonalization possible.
- $A = SDS^{-1}$ impossible.
- A^{-1} impossible.

1) Every symmetric matrix is orthogonally diagonalizable

Hence, for $b=1$ $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ is orthogonally diagonalizable

2) $\det(A - \lambda I) = \lambda^2 - 2\lambda - b$. For $b = -1$

we get $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$V(1) = N(A - I) = N\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$

only one vector in the eigenspace Hence, non-diagonalizable

3) For $b=0$, $\det(A) = 0$, hence A is not invertible

Problem 8 [15 pts]
Find the Cholesky factor of A (Recall the Cholesky factor C must be upper triangular with positive diagonal entries and such that $A = C^T C$).

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$

$A \sim \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$. Hence $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 9 & & \\ & 1 & \\ & & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & & \\ & 1 & \\ & & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Check $C^T C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$