Assignment 12.

There are total 34 points in this assignment. 30 points is considered 100%. If you go over 30 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much. This assignment covers section 6.4 in Bartle–Sherbert.

(1) For a given function f and a point x_0, find Taylor's polynomials $P_2(x)$, $P_5(x)$, $P_{2016}(x)$ of $f(x)$ at x_0.
  (a) [2pt] $f(x) = \sin x$ at $x_0 = \pi/2$. Compare to cos at 0.
  (b) [2pt] $f(x) = \cos x$ at $x_0 = -\pi/2$. Compare to sin at 0.
  (c) [2pt] $f(x) = x^3$ at $x_0 = 2$. Compare $P_3(x), P_5(x), P_{2016}(x)$ to $f(x)$.
  (d) [2pt] $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.
  (e) [2pt] $f(x) = \frac{1}{x}$ at $x_0 = 1$. Compare to the previous item.
  (You can take for granted that $(\sin x)' = \cos x, (\cos x)' = -\sin x$.)

(2) [3pt] (Part of exercise 6.4.7) If $x > 0$, show that
$$\left|\sqrt[4]{1+x} - \left(1 + \frac{1}{4}x - \frac{3}{32}x^2\right)\right| \leq \frac{7}{128}x^3.$$
(*Hint:* Apply Taylor's Theorem to $f(x) = \sqrt[4]{1+x}$ with $n = 2$.)

(3) (a) [2pt] Suppose $A \in \mathbb{R}$. Show that $\lim\limits_{n\to\infty} \frac{A^n}{n!} = 0$.
  Hint: take tail of this sequence that starts with $m > 2|A|$ and represent
$$\frac{A^n}{n!} = \frac{A^m}{m!} \cdot \frac{A^{n-m}}{(m+1)\cdots n}.$$
  (b) [2pt] (6.4.8) If $f(x) = e^x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.
  (c) [2pt] (6.4.9) If $g(x) = \cos x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$, for each fixed x_0 and x.

(4) [4pt] (Part of exercise 6.4.11) If $x > 0$ and $n \in \mathbb{N}$, show that
$$\left|\ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}\right| < \frac{x^{n+1}}{n+1}.$$
(*Hint:* Apply Taylor's Theorem to $f(x) = \ln(1+x)$.)

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(5) [5pt] (6.4.14+) Use nth derivative test to determine whether or not $x = 0$ is a point of relative extremum of the following functions. If it is, specify whether it is a point maximum or minimum.

(a) $f(x) = x^n$, $n \in \mathbb{N}$,

(b) $f(x) = \sin x - \tan x$,

(c) $f(x) = \cos x - 1 + \frac{1}{2}x^2$.

(6) [3pt] Let $a \in \mathbb{R}$ be a constant s.t. $a > 0$. Consider the function $f(x) = x^2 - a$. Find the recursive relation provided by Newton's Method. Write out first 4 terms of the corresponding sequence if $x_1 = 1$ and $a = 2$. Check on a calculator how small $|x_4 - \sqrt{2}|$ is.

COMMENT. You should have gotten $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. We already have seen that this sequence converges to $\sqrt{a}$ when we covered the Monotone Convergence Theorem. Now we finally know where this sequence actually comes from.

(7) [3pt] Apply Newton's Method to find a recursive relation for approximating $\sqrt[3]{2}$. For the interval $I = [1, 2]$, find M, m, and K in the statement of Newton's Method. Find an interval $I^* \subseteq [1, 2]$ s.t. the convergence of a sequence given by the above relation and any $x_1 \in I^*$ is guaranteed. (*Hint:* $f(x) = x^3 - 2$.)