MA 441, Fall 2021

Assignment 3.

There are total 18 points in this assignment. 16 points is considered 100%. If you go over 16 points, you will get over 100% (but not over 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.3–2.5 in Bartle–Sherbert.

(1) [3pt] (2.3.4) let S be a a nonempty bounded[1] set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as \mid s \in S\}$. Prove that $\inf(aS) = a \inf S$, $\sup(aS) = a \sup S$.

(2) (a) [2pt] (2.4.7) Let A and B be nonempty bounded subsets of $\mathbb{R}$. Show that $A + B = \{a + b : \ a \in A, b \in B\}$ is a bounded set. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

(b) [2pt] Find $\sup\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\sup\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$. (*Hint:* for the last two questions, use the previous item **??**.)

(c) [2pt] For A, B as in item **??**, show that $AB = \{ab : \ a \in A, b \in B\}$ is a bounded set. Is it true that always $\sup AB = \sup A \cdot \sup B$?

(3) [2pt] (2.4.19) If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (In other words, the set $\{ru \mid r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.) (*Hint:* Divide the required inequality by u..)

(4) [3pt] Find the intersection $\bigcap\limits_{n=1}^{\infty} I_n$ in the following cases. (Just the answer suffices in this exercise.)

(a) $I_n = [1 - \frac{1}{n}, 1]$ for each $n \in \mathbb{N}$.
(b) $I_n = (1 - \frac{1}{n}, 1)$ for each $n \in \mathbb{N}$.
(c) $I_n = [n, \infty)$ for each $n \in \mathbb{N}$.
(d) $I_n = [3 - \frac{1}{n^2}, 5 + \frac{3}{n}]$ for each $n \in \mathbb{N}$.
(e) $I_n = [0, 1]$ for each $n \in \mathbb{N}$.

(5) [4pt] (2.5.10) Let $I_1 = [a_1, b_1] \supseteq I_2 = [a_2, b_2] \supseteq I_3 = [a_3, b_3] \supseteq \dots$ be an infinite nested system of closed intervals. Let $u = \sup\{a_n | n \in \mathbb{N}\}$ and $v = \inf\{b_n | n \in \mathbb{N}\}$. Prove that

$$[u, v] = \bigcap_{n=1}^{\infty} I_n.$$

(*Hint:* This is a set equality. To prove an equality $A = B$ of sets A and B, you have to do to things: show that every element of A is also an element of B, and that every element of B is also an element of A. That is, two inclusions $A \subseteq B$ and $B \subseteq A$..)

[1] Which means bounded above and below.