Homework 6

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Task 1

Task 2

Let R and S be equivalence relations on a set A.

Theorem. *The relation $R \cup S$ is an equivalence relation on A.*

Proof. Let us disprove this statement by first assuming the theorem holds and then providing a counterexample. Suppose the theorem holds true and $A = \{x, y, z\}$. Let $R = \{(x, x), (y, y), (z, z), (x, y), (y, x)\}$ and $S = \{(x, x), (y, y), (z, z), (y, z), (z, y)\}$.

It follows that $R \cup S = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y)\}$ But for $R \cup S$, $(x, y), (y, z) \in R \cup S$ does not mean $(x, z) \in R \cup S$. Therefore the relation is not transitive. Since the relation is not transitive, the relation $R \cup S$ is not an equivalence relation on A. $\square$

Theorem. *The relation $R \cap S$ is an equivalence relation on A.*

Proof. We can prove $R \cap S$ to be an equivalence relation by showing it is reflexive, symmetric, and transitive. For reflexivity, let $x \in A$. We have $(x, x) \in R$ because R is reflexive and $(x, x) \in S$ because S is also reflexive. Therefore it holds that $(x, x) \in R \cap S$. For symmetry, let $(x, y) \in R \cap S$. Then, $(x, y) \in R$ and $(x, y) \in S$. By the symmetry of R, it holds that $(y, x) \in R$ and by the symmetry of S, it holds that $(y, x) \in S$. Therefore it holds that $(y, x) \in R \cap S$. For transitivity, let $(x, y), (y, z) \in R \cap S$. By the transitivity of R and S, it holds that $(x, z) \in R$ and $(x, z) \in S$. Therefore $(x, z) \in R \cap S$. Thus we can conclude the relation $R \cap S$ is an equivalence relation on A. $\square$