MA 441, Fall 2021

Assignment 5.

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.2–3.3 in Bartle–Sherbert.

1. Basic properties of limit. Squeeze theorem

(1) Find limits using Squeeze Theorem:
  (a) [2pt] $\lim\limits_{n\to\infty} \frac{n^2+2015n(\sin n+3\cos n^7)-1}{2n^2-\cos(3n^2+1)}$,
  (b) [2pt] $\lim\limits_{n\to\infty} \sqrt{n^2+\cos(2014n+1)} - \sqrt{n^2-\sin(n^3-1)}$. (*Hint:* Once you get rid of sin and cos by Squeeze Theorem, multiply and divide by the conjugate, $\sqrt{} + \sqrt{}$.)

(2) (a) [3pt] (Example 3.1.11d) Prove that $n^{1/n} \to 1$ $(n \to \infty)$.
  (b) [2pt] (3.2.14a) Use Squeeze theorem to find limit of the sequence (n^{1/n^2}).

(3) [2pt] (3.2.8) Find a mistake in the following argument.
"Find $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n$ as shown below:

$$\begin{aligned}
\lim_{n\to\infty}(1+\tfrac{1}{n})^n &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n}) \\
&= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot\lim_{n\to\infty}(1+\tfrac{1}{n})\cdots\lim_{n\to\infty}(1+\tfrac{1}{n}) \\
&= \left(\lim_{n\to\infty}(1+\tfrac{1}{n})\right)^n = 1^n = 1.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = 1$."

Comment. To reiterate, the argument above is erroneous and the obtained value of the limit is wrong, too. The limit is actually equal to e, as shown in class.

2. Monotone Convergence Theorem

(4) [3pt] (3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone, hence convergent. Find the limit.

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(5) [2pt] Find a mistake in the following argument:

"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Show that $\lim(x_n) = 0.5$. Indeed, let $\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\lim(x_{n+1}) = \lim(1 - x_n)$$
$$\lim(x_{n+1}) = 1 - \lim(x_n)$$
$$x = 1 - x,$$

so $x = 0.5$"

(6) [3pt] (3.3.11) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

(7) (a) [2pt] (Exercise 3.3.12) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

(b) [2pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare[1] y_n to x_n.)

[1] *Compare* here does not mean "write a short essay about how y_n is the same as x_n but with K", but rather determine which is greater.