MA 232 - Linear Algebra

Homework 4 - Solutions

Problem 1 [20pts] Find the line $y = C + Dx$ that best fits the data $(x, y) = \{(-2,4), (-1,2), (0,-1), (1,0), (2,0)\}$.

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = b$$

We solve $A^T A \hat{x} = A^T b$

$$A^T A = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

Hence $\hat{C} = 1$, $\hat{D} = -1$ and $y = 1 - x$ is the best fitting line

Problem 2 [20pts] Use the Gram-Schmidt method to find orthonormal vectors A, B, C from $a = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

We will find A, B, C orthogonal and then normalize them.

$$A = a \ , \quad B = b - \frac{A^T b}{A^T A} \cdot A \ , \quad C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

$$A^T b = -1 \ , \quad A^T c = 0 \ , \quad B^T c = -1 \ , \quad B^T B = \frac{3}{2}$$

$$B = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} \ , \quad C = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

$$\frac{A}{\|A\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \ , \quad \frac{B}{\|B\|} = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$$

$$\frac{C}{\|C\|} = \frac{\sqrt{3}}{2} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

Problem 3 [20pts] Suppose Q_1, Q_2 are square $n \times n$ matrices that are orthonormal. Show that their product Q_1Q_2 is an orthonormal square matrix.

A matrix is orthonormal iff

$$Q^TQ = \mathbb{I}$$

We have by hypothesis that

$$Q_1^TQ_1 = \mathbb{I} \quad \text{and} \quad Q_2^TQ_2 = \mathbb{I}$$

Hence $(Q_1Q_2)^T(Q_1Q_2) =$

$$= Q_2^T \underbrace{Q_1^TQ_1}_{\mathbb{I}} Q_2 = Q_2^T \mathbb{I} Q_2 =$$

$$= Q_2^TQ_2 = \mathbb{I}$$

Thus Q_1Q_2 is orthonormal.

Problem 4 [20pts] Let A, B, C, D be 2×2 matrices. Does the following equality always hold? (If yes prove why, if not find a counterexample)

$$det(\begin{bmatrix} \text{A} & \text{B} \\ \text{C} & \text{D} \end{bmatrix}) = det(A) \cdot det(D) - det(C) \cdot det(B)$$

Consider the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$ Hence $\det(M) = 4$

But $\det A \det D - \det C \det B = 0$

Problem 5 [20pts] Reduce $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ to U and find the determinant of A as a product of pivots.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $\det(A) = 1 \cdot 1 \cdot 1 = 1$