# Homework #4 - Lectures 8 and 9: Option pricing in continuous time, BSM model, Greeks

FE-620

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**Problem 4.1** (14.16 in Hull)
Assume that the stock price $S_t$ follows the geometric Brownian motion process

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t \,. \tag{1}$$

What is the process followed by:

(a) $y = 2S$

(b) $y = S^2$

(c) $y = \sin(S)$

(d) $y = \frac{1}{S} e^{r(T-t)}$

**Solution.** Application of Ito's lemma gives

$$dy = \mu y dt + \sigma y dz \tag{2}$$
$$dy = (2\mu + \sigma^2) y dt + 2\sigma y dz \tag{3}$$
$$dy = (\mu \arcsin y_t \sqrt{1 - y_t^2} - \frac{1}{2}\sigma^2 y_t \arcsin^2 y_t) dt + \sigma \arcsin y_t \sqrt{1 - y_t^2} dz \tag{4}$$
$$dy = (-r - \mu + \sigma^2) y dt - \sigma y dz \tag{5}$$

**Problem 4.2** (Evaluating the Black-Scholes formula)
Consider an European option on a non-dividend-paying stock. Currently the stock price is \$34, the option exercise price is \$33, the risk-free interest rate is 5.5%, the volatility is 25% per annum, and the time to maturity of the option is 6 months.

Using the Black-Scholes European option price formula, evaluate the price of this option, assuming that:

(a) it is an European call option

(b) it is an European put option

(c) Verify that put-call parity holds

**Solution.**

i) Application of the Black-Scholes formula gives the European call option price

$$C = 3.39661 \tag{6}$$

ii) The price of the put option is

$$P = 1.50148 \tag{7}$$

iii) Put-call parity is

$$C - P = S_0 - e^{-rT}K = 1.89514 \tag{8}$$

**Problem 4.3**(variation on 15.28 in Hull)
The closing prices of Tesla stock for the month of February 2022 are given in the table below.

Estimate the annualized stock price volatility. What is the standard error of your estimate?

```
> head(price,30)
           TSLA.Close
2022-02-01     931.25
2022-02-02     905.66
2022-02-03     891.14
2022-02-04     923.32
2022-02-07     907.34
2022-02-08     922.00
2022-02-09     932.00
2022-02-10     904.55
2022-02-11     860.00
2022-02-14     875.76
2022-02-15     922.43
2022-02-16     923.39
2022-02-17     876.35
2022-02-18     856.98
2022-02-22     821.53
2022-02-23     764.04
2022-02-24     800.77
2022-02-25     809.87
2022-02-28     870.43
```

**Solution.** The annualized volatility of the TSLA stock obtained from the closing prices is obtained from the standard deviation of the log-returns $u_i$ by multiplication with $\sqrt{252}$

$$\sigma_{ann} = \sqrt{252} sd(u_i) = 61.50\% \tag{9}$$

The error of the annualized volatility is estimated using the formula from Hull:

$$\delta\sigma_{ann} = \frac{\sigma_{ann}}{\sqrt{2n}} = 9.98\% \tag{10}$$

where $n = 19$ is the number of price observations.

**Problem 4.4**(Variation on 19.26 in Hull)
Consider a 1-year European call option on a stock when the stock price is \$28, the strike price is \$28, the risk-free rate is 2.0%, and the volatility is 25% per annum. Compute the price, delta, gamma and vega of the option under the Black-Scholes-Merton model.

i) Verify that delta is correct by changing the stock price to \$28.1 and recomputing the option price.

ii) Verify that gamma is correct by recomputing the delta for the situation where the stock price is \$28.1.

**Solution.** Use the table in Hull (shown below from slides of Lecture 9) to compute the exact Greeks. Note that $N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$ is the derivative of the normal cumulative distribution function $N(x)$.

The results are shown in the table below.

The exact result is compared with the finite difference approximations for Delta (shown in red)

$$\Delta \simeq \frac{C(S_0 + 0.1) - C(S_0)}{0.1} \tag{11}$$

and for Gamma (shown in blue)

$$\Gamma \simeq \frac{\Delta(S_0 + 0.1) - \Delta(S_0)}{0.1} \tag{12}$$

They are reasonably close.

## *Greek Letters for European Options on an Asset that Provides a Yield at Rate q*

| *Greek Letter* | *Call Option* | *Put Option* |
|---|---|---|
| Delta | $e^{-qT}N(d_1)$ | $e^{-qT}[N(d_1)-1]$ |
| Gamma | $\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$ | $\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$ |
| Theta | $-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$ | $-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $+qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$ |
| Vega | $S_0\sqrt{T}N'(d_1)e^{-qT}$ | $S_0\sqrt{T}N'(d_1)e^{-qT}$ |
| Rho | $KTe^{-rT}N(d_2)$ | $-KTe^{-rT}N(-d_2)$ |

| **Problem 4.4** | | | | | |
|---|---|---|---|---|---|
| | | **Call price** | **Delta** | **Gamma** | **Vega** |
| **S0** | 28 | 3.04376 | 0.58121 | 0.05581 | 10.93811 |
| **S0 + 0.1** | 28.1 | 3.10216 | | | |
| **Delta(FD)** | | 0.58400 | | | |
| | | | | | |
| | | **Delta** | | | |
| **S0** | 28 | 0.58121 | | | |
| **S0 + 0.1** | 28.1 | 0.58678 | | | |
| **Gamma(FD)** | | 0.05562 | | | |

| | |
|---|---|
| Net profit analysis (per share) | |
| **Sell Delta hedge, get ST*n(shares)** | 494.27 |
| **Hedging cost** | -538.30 |
| **Net profit/loss of hedging strategy** | -44.03 |

Figure 1: Balance of profit for the Delta hedge on the TSLA option.

**Problem 4.5** (Delta hedging an option position)
A trader sells a 1-year maturity European call option on TSLA stock on 1-Feb-2022, with strike \$932.00, and Delta hedges it by taking a long position in the stock. The trader adjusts the hedge dynamically each day of the month of February, assuming the Black-Scholes model with implied volatility $\sigma = 60\%$ for the computation of Delta.

The trader unwinds the hedge (sells the stock position) after the market close on 28-Feb-2022. What is the profit/loss of the hedging strategy? Assume zero interest rate.

*Hint.* The problem can be solved by adapting the Delta hedging spreadsheet. Suppose we sell 100 options.

On 1-Feb-2022 the option has $\Delta = 0.6174$ and the trader starts the hedge by going long 60.93 shares for each 100 options. This is rebalanced daily until 28-Feb-2022 when the trader holds 56.78 shares.

The total cost of setting up the hedge and rebalancing it is \$53,830.00

At the end of the 28-Feb-2022 trading day the hedge is unwound by selling these shares for \$49,427. This means that the net PnL of the hedge is -\$4,403 (or equivalently -\$44.03 per option).

The trader realized a loss.

| | Black-Scholes pricing on each day | | | | | |
|---|---|---|---|---|---|---|
| Date | S(t) | T-t | d1 | d2 | Call | Delta |
| 2/1/22 | 931.25 | 1.00 | 0.298658258 | -0.3013417 | 219.32 | 0.6174 |
| 2/2/22 | 905.66 | 1.00 | 0.251527658 | -0.3472807 | 203.33 | 0.5993 |
| 2/3/22 | 891.14 | 0.99 | 0.223790085 | -0.3738242 | 194.29 | 0.5885 |
| 2/4/22 | 923.32 | 0.99 | 0.282520367 | -0.3138975 | 213.18 | 0.6112 |
| 2/7/22 | 907.34 | 0.98 | 0.25255792 | -0.3426611 | 203.08 | 0.5997 |
| 2/8/22 | 922 | 0.98 | 0.278848515 | -0.3151693 | 211.53 | 0.6098 |
| 2/9/22 | 932 | 0.98 | 0.296407056 | -0.2964071 | 217.23 | 0.6165 |
| 2/10/22 | 904.55 | 0.97 | 0.245271824 | -0.3463362 | 200.15 | 0.5969 |
| 2/11/22 | 860 | 0.97 | 0.159019965 | -0.4313794 | 173.88 | 0.5632 |
| 2/14/22 | 875.76 | 0.96 | 0.18895605 | -0.4002323 | 182.44 | 0.5749 |
| 2/15/22 | 922.43 | 0.96 | 0.276433342 | -0.3115414 | 209.65 | 0.608[illegible] |
| 2/16/22 | 923.39 | 0.96 | 0.27756169 | -0.309197 | 209.80 | 0.6093 |
| 2/17/22 | 876.35 | 0.95 | 0.187623908 | -0.3979161 | 181.53 | 0.5744 |
| 2/18/22 | 856.98 | 0.95 | 0.148312424 | -0.4356983 | 170.03 | 0.5590 |
| 2/22/22 | 821.53 | 0.94 | 0.075177504 | -0.5079177 | 150.42 | 0.5300 |
| 2/23/22 | 764.04 | 0.94 | -0.05057317 | -0.6324421 | 120.98 | 0.4798 |
| 2/24/22 | 800.77 | 0.94 | 0.028954907 | -0.5516851 | 138.81 | 0.5115 |
| 2/25/22 | 809.87 | 0.93 | 0.047286269 | -0.5321223 | 143.10 | 0.5189 |
| 2/28/22 | 870.43 | 0.93 | 0.170878153 | -0.4072963 | 175.62 | 0.5678 |

Figure 2: Details for the Delta hedge on the TSLA option.