MA 441, Fall 2021

# Assignment 7

There are total 30 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% for this homework and it will count towards your course grade (not over 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers Section 3.6 in Bartle–Sherbert.

## 1. Properly divergent sequences

In the exercises below we look at sequences that "go to infinity".

Definition. Let $(x_n)$ be a sequence of real numbers. We say that $(x_n)$ *tends to* (diverges to) $+\infty$, and write $\lim(x_n) = +\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number $K$ such that if $n > K$, then $x_n > \alpha$.

Another notation is $x_n \to +\infty$ $(n \to \infty)$.

(1) [2pt] Give an analogous definition of a sequence that tends to $-\infty$.

Definition. We say that $(x_n)$ is *properly divergent* in case we have either $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$.

(2) ($\sim$ Example 3.6.2) For the following sequences determine whether they are properly divergent.
  (a) [1pt] $x_n = n$.
  (b) [1pt] $x_n = n^2$.
  (c) [1pt] $x_n = (-1)^n n$.
  (d) [2pt] $x_n = c^n$, where $c$ is a given real number. (*Hint:* Note that the answer depends on $c$. For $c > 1$, use Bernoulli's inequality.)

(3) [3pt]
  (a) Suppose $(x_n)$ is properly divergent. Show that $(x_n)$ is ultimately nonzero and that $\lim(1/x_n) = 0$.
  (b) Suppose $\lim(y_n) = 0$ and $y_n > 0$ for all sufficiently large $n$. $\lim(1/y_n) = +\infty$.

(4) [2pt] (Theorem 3.6.4) Prove that following theorem (we may call it the squeeze theorem for properly divergent sequences).
Let $(x_n)$ and $(y_n)$ be two sequences of real numbers and suppose that

$$x_n \le y_n \text{ for all } n \in \mathbb{N}.$$

If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.
If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.

(5) [2pt] (Exercise 3.6.1) Show that if $(x_n)$ is an unbounded sequence, then it has a properly divergent subsequence.

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(6) [3pt] (~Exercise 3.6.2) Give examples of sequences $(x_n)$ and $(y_n)$ that tend to $+\infty$ and:
  (a) $\lim(x_n/y_n) = 0$,
  (b) $\lim(x_n/y_n) = 10$,
  (c) $\lim(x_n/y_n) = +\infty$,
  (d) $\lim(x_n/y_n)$ does not exist as either a real number or infinity.

(7) [3pt] (Theorem 3.6.5) Prove that following theorem.
Let $(x_n)$ and $(y_n)$ be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}$, $L > 0$, we have
$$\lim(x_n/y_n) = L.$$
Then $\lim(x_n) = +\infty$ if and only if $\lim(y_n) = +\infty$.
(*Hint:* Show that for $n$ large enough, $\frac{1}{2}L < x_n/y_n < \frac{3}{2}L$. Then use the theorem in Prob. 4.)

(8) [3pt]
  (a) Suppose that $\lim(a_n) = L \neq 0$ and $\lim(b_n) = +\infty$. Show that $(a_n b_n)$ is properly divergent. (*Hint:* Use the above theorem. Don't forget that you need two positive sequences to use it.)
  (b) Suppose that $\lim(a_n) = L \neq 0$, $\lim(b_n) = 0$, and $b_n > 0$ for all sufficiently large $n$. Use the previous item and Problem 3b to show that $(a_n/b_n)$ is properly divergent.

(9) [2pt] (Exercise 3.6.6) Let $(x_n)$ be properly divergent and let $(y_n)$ be such that $\lim(x_n y_n)$ exists as a real number. Show that $(y_n)$ converges to 0.

(10) (~ Exercise 3.6.8, 10) For the following sequences determine whether they are properly divergent.
  (a) [2pt] $x_n = \sqrt{n^2 - 1}/\sqrt{n + 100}$. (*Hint:* Use the theorem in Prob. 7.)
  (b) [1pt] $x_n = \sin\sqrt{n}$.
  (c) [2pt] $x_n$ if it is given that $\lim(x_n/n) = L$, where $L > 0$. (*Hint:* Don't forget that to use the previous theorem, you have to explain why $x_n > 0$ first.)