Ma 410 - Differential Geometry **Hw-7** **Due: November 2, 2022 by 11:59 pm**

Name (Printed):

Pledge and Sign:

Upload solutions to Grade Scope by the due date. Assign solution pages to corresponding problems. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. (a) [2 pts.] Prove Corollary 3.4 from 4.3, that is, if $\mathbf{x}$ and $\mathbf{y}$ are overlapping patches in M, then there exist unique differentiable functions $\bar{u}$ and $\bar{v}$ such that:

$$\mathbf{y}(u,v) = \mathbf{x}(\bar{u}(u,v), \bar{v}(u,v))$$

(b) [4 pts.] Prove the chain rule:

$$\mathbf{y}_u = \frac{\partial \bar{u}}{\partial u}\mathbf{x}_u + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_v, \quad \mathbf{y}_v = \frac{\partial \bar{u}}{\partial v}\mathbf{x}_u + \frac{\partial \bar{v}}{\partial v}\mathbf{x}_v$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated on $\bar{u}$ and $\bar{v}$.

(c) [2 pts.] Deduce that $\mathbf{y}_u \times \mathbf{y}_v = J\mathbf{x}_u \times \mathbf{x}_v$, where J is the Jacobian of the mapping $\mathbf{x}^{-1}\mathbf{y} : D \to \mathbb{R}^2$

2. Let $\mathbf{x}$ be a patch in M.

(a) [4 pts.] If $\mathbf{x}_*$ is the tangent map of $\mathbf{x}$, show that: $\mathbf{x}_*(U_1) = \mathbf{x}_u, \mathbf{x}_*(U_2) = \mathbf{x}_v$, where $\{U_1, U_2\}$ is the natural frame field on $\mathbb{R}^2$.

(b) [4 pts.] If f is a differentiable function on M, prove

$$\mathbf{x}_u[f] = \frac{\partial}{\partial u}(f(x)), \text{ and}, \mathbf{x}_v[f] = \frac{\partial}{\partial v}(f(x))$$

3. Let C be the circular cone parameterized by $\mathbf{x}(u,v) = (v\cos u, v\sin u, v)$. If α is the curve $\mathbf{x}(\sqrt{2}t, e^t)$,

(a) [4 pts.] Express α' in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$

(b) [4 pts.] Show that at each point of α, the velocity α' bisects the angle between $\mathbf{x}_u$ and $\mathbf{x}_v$. [Hint: verify that:

$$\alpha' \cdot \left(\frac{\mathbf{x_u}}{\|\mathbf{x}_u\|}\right) = \alpha' \cdot \left(\frac{\mathbf{x_v}}{\|\mathbf{x}_v\|}\right)$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated on $(\sqrt{2}t, e^t)$].

4. Let $\alpha : [-1,1] \to \mathbb{R}^2$ be the curve segment $\alpha(t) = (t, t^2)$.

(a) [4 pts.] If $\phi = v^2 du + 2uvdv$, compute $\displaystyle\int_\alpha \phi$

(b) [4 pts.] Find a function f such that $df = \phi$ and verify the Fundamental theorem of Line Integrals in that case.