Homework #3 - Lectures 6 and 7

FE-620 Fall 2022

Due 28-Oct-2022

Problem 3.1
Assume that a stock price is $S_0 = \$100$, the stock does not pay any dividends, and the risk-free interest rate is 5% per annum (with continuous compounding).

1. Compute the lower bound for the price of a 3-month European call option with strike price $K = \$92$
2. Compute the lower bound on the price of a 3-month European put option with strike price $K = \$105$.
3. Assume that the price of a 3-month European call option with strike price \$95 is \$8.05. What is the price of the European put option on the same stock with the same strike and maturity?

Solution. 1) from lecture 6 we have the lower bound on the price of an European call option on a non-dividend paying stock

$$c \geq S_0 - e^{-rT} K \tag{1}$$

Substituting numerical values gives

$$c \geq 100 - e^{-0.05 \cdot 0.25} 92 = 9.143\,. \tag{2}$$

2) use the lower bound on the price of an European put option on a non-dividend paying stock

$$p \geq e^{-rT} K - S_0 \tag{3}$$

Substituting numerical values gives that the put price must be larger than

$$p \geq e^{-0.05 \cdot 0.25} 105 - 100 = 3.696\,. \tag{4}$$

3) We use the call-put parity relation

$$c(K,T) + e^{-rT}K = p(K,T) + S_0 \tag{5}$$

Using this equation we have the put option price in terms of the known call option price

$$p(K,T) = c(K,T) + e^{-rT}K - S_0 = 8.05 + e^{-0.05 \cdot 0.25} 95 - 100 = 1.870\,. \tag{6}$$

Problem 3.2
Suppose that c_1, c_2 and c_3 are the prices of European call options with strike prices K_1, K_2 and K_3, respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All option have the same maturity. Show that

$$c_2 \leq 0.5(c_1 + c_3)$$

This condition must be satisfied by all European option prices with the same maturity, and is known as the *butterfly arbitrage condition*.

Hint: Consider a portfolio that is long one option with strike K_1, long one option with strike K_3, and short two options with strike price K_2. Plot the payoff of this portfolio as a function of the stock price at maturity $S(T)$, and convince yourself that it is always positive.

Solution. Construct a portfolio as indicated, consisting of one long option with strike K_1, one long option with strike K_3, and short two options with strike price K_2.

The value of this portfolio at maturity is

$$V(T) = (S_T - K_1)^+ + (S_T - K_3)^+ - 2(S_T - K_2)^+ \tag{7}$$

This is a strictly positive function of S_T. For example, taking $K_i = (90, 100, 110)$ the plot of the payoff $V(T)$ vs S_T is shown in Figure 1.

Therefore the value of the portfolio must be positive at any other time, including time zero (at the present time). This gives that the present value of the portfolio $V(0)$ must satisfy

$$0 \leq V(0) = c_1 + c_3 - 2c_2 \tag{8}$$

or equivalently

$$c_2 \leq \frac{1}{2}(c_1 + c_3) \tag{9}$$

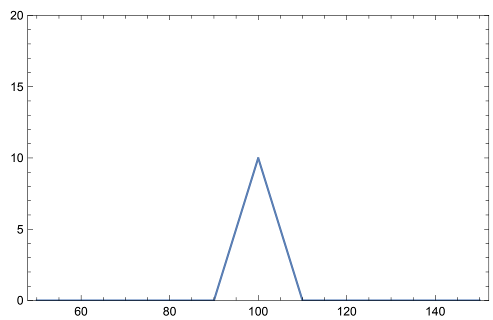

Figure 1: Payoff of the butterfly spread with $K_i = (90, 100, 110)$.

Problem 3.3

The current price of a non-dividend paying tech stock is \$100 and its volatility is 25%. The risk-free rate is 5%.

Using a one-time step binomial tree, compute the forward price for a forward contract with maturity $T = 1Y$.

Solution. Using risk-neutral pricing, the option price is computed as the discounted expectation of the payoff $\max(S(T) - K, 0)$.

The tree multipliers are

$$u = e^{\sigma\sqrt{\Delta T}} = e^{0.55\cdot\sqrt{0.25}} = 1.316\,, \quad d = 1/u = 0.760 \tag{10}$$

We get

$$c(K,T) = e^{-rT}[p \cdot \max(S_0u - 100, 0) + (1-p) \cdot \max(S_0d - 100, 0)] \tag{11}$$

where

$$p = \frac{e^{rT} - d}{u - d} = 0.455 \tag{12}$$

is the risk-neutral probability of the up price state.

Substituting into the pricing formula we get the price of the European call option

$$c(K,T) = e^{-0.052\times 0.25} \times 0.455 \times (100u - 100) = 14.225\,. \tag{13}$$

Problem 3.4
A stock price is currently \$50. It is known that at the end of 6 months it will be either \$55 or \$42. The risk-free rate of interest with continuous compounding is 5% per annum. Calculate the value of a 6-month European call option on the stock with an exercise price $K = \$50$.

Solution. Proceed as in problem 3.3, except that now the up and down factors are

$$u = \frac{62}{60} = 1.0333\,, \quad d = \frac{55}{60} = 0.9167\,. \tag{14}$$

The probability for the up state is

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times 0.5} - 55/60}{62/60 - 55/60} = 0.931 \tag{15}$$

We get

$$c(K,T) = e^{-rT}[p \cdot \max(62-60, 0) + (1-p) \cdot \max(55-60, 0)] = 1.81656 \tag{16}$$

Problem 3.5
The price of a non-dividend paying stock is \$100. Use a three-step tree to value:

(a) a 6-month American call option with strike price \$100

(b) a 6-month American put option with strike price \$100.

The volatility is 35% and the risk-free rate for all maturities is 5% with continuous compounding.

Solution. The solution in Excel is shown in the table below. The binomial tree has 3 steps, each spanning $\Delta T = 2$ months.

The American call price is $C = 11.819$ and the American put price is $P = 9.559$.

			Tree for the underlying asset			
S0	100		100	115.360	133.079	153.520
sigma	0.35			86.685	100.000	115.360
r	0.05				75.143	86.685
Delta T	0.1667		K	100.000		65.138
		Payoff tree (call)	**11.819**	20.372	33.909	53.520
u	1.154			3.679	7.518	15.360
d	0.867				0.000	0.000
p	0.494					0.000
1-p	0.506		K	100		
		Payoff tree (put)	**9.559**	3.359	0.000	0.000
DiscFactor	0.992			15.758	6.688	0.000
					24.857	13.315
Put-Call parity	2.260	0.000				34.862

Figure 2: Three-step American option pricing implemented in Excel.