Assignment 8

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 4.1, 4.2 in Bartle–Sherbert.

(1) (Modified 4.1.1) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)
  (a) [1pt] $|x^3 - 1| < 1/2$, $c = 1$. (*Hint:* $x^3 - 1 = (x - 1)(x^2 + x + 1)$.)
  (b) [1pt] $|x^3 - 1| < 10^{-3}$, $c = 1$.
  (c) [1pt] $|x^3 - 1| < \frac{1}{10^{-3}}$, $c = 1$.
  (d) [1pt] $|x^2 \cos x^3 - 0| < 0.00001$, $c = 0$.

(2) [3pt] (Modified 4.1.9) *Use the ε–δ definition of limit* to show that
  (a) $\lim\limits_{x\to 2} \frac{1}{1-x} = -1$,
  (b) $\lim\limits_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$.

(3) [2pt] (4.1.11) Show that the following limits do not exist:
  (a) $\lim\limits_{x\to 0}(x + \operatorname{sgn} x)$,
  (b) $\lim\limits_{x\to 0} \sin(1/x^2)$.

(4) [3pt] (4.1.15) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.
  (a) Show that f has limit at $x = 0$ (*Hint*: you can use the ε–δ definition directly, or the sequential criterion and squeeze theorem).
  (b) Prove that if $c \neq 0$, then f does not have limit at c. (*Hint*: you can use sequential criterion.)

(5) [2pt] (Theorem 4.2.4 for difference) *Using ε–δ definition,* prove that limit of functions preserves difference. That is, prove the following:
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A, and f, g be functions on A to $\mathbb{R}$. If $\lim\limits_{x\to c} f = L$, and $\lim\limits_{x\to c} g = M$, then $\lim\limits_{x\to c} f - g = L - M$.

(6) [2pt] Using arithmetic properties of limit, find the following limits.
  (a) $\lim\limits_{x\to 1} \frac{x^{100}+2}{x^{100}-2}$.
  (b) $\lim\limits_{x\to 1} \frac{2x^2-x-1}{x^2-3x+2}$. (*Hint:* Denominator turns to 0 at $x = 1$, but you can cancel out $(x - 1)$.)
  (c) $\lim\limits_{x\to 0} \frac{(x+1)^{20}-1}{x}$.
  (d) $\lim\limits_{x\to c} \frac{(x-c+1)^2-1}{x-c}$.

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(7) (a) [2pt] (4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim\limits_{x\to c} g = 0$. Prove that $\lim\limits_{x\to c} fg = 0$.

Explain why Theorem 4.2.4 (Arithmetic Properties of Limit) cannot be used.

(b) [1pt] ($\sim$4.2.11b) Determine whether $\lim_{x\to 0} x\cos(1/x^2)$ exists in $\mathbb{R}$.

(8) [4pt] (4.2.15) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) = \sqrt{f(x)}$. If $\lim_{x\to c} f$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$. (*Hint:* $a^2 - b^2 = (a-b)(a+b)$. Another hint: you will probably have to consider cases $\lim_{x\to c} f = 0$ and $\lim_{x\to c} f \neq 0$ separately.)