Assignment 1.

Part 1. Solutions.

There are total 23 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (but not over 115%).

Make sure your solutions contain full proofs. Bare answers will not earn you much.

This assignment covers section 1.1, 1.3 in Bartle–Sherbert.

(1) (1.1.10) Let $f(x) = 1/x^2$, $x \neq 0$, $x \in \mathbb{R}$.

(a) [1pt] Determine the image $f(E)$ where $E = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$.

$\triangleright$ Recall that for *positive* numbers $a^2 \leq b^2$ if and only if $a \leq b$, and $a \leq b$ if and only if $1/b \leq 1/a$. So we get that as x runs through $1 \leq x \leq 2$, the value $1/x^2$ runs through $1/2^2 \leq 1/x^2 \leq 1/1^2$, so $f(E) = \{y \in \mathbb{R} \mid \frac{1}{4} \leq y \leq 1\}$.

(b) [1pt] Determine the inverse image $f^{-1}(G)$ where $G = \{x \in \mathbb{R} \mid 1 \leq x \leq 4\}$.

$\triangleright$ To find the inverse image of G, solve the inequality $1 \leq 1/y^2 \leq 4$:

$$1 \leq 1/y^2 \leq 4 \Leftrightarrow 1 \leq 1/|y| \leq 2 \Leftrightarrow \frac{1}{2} \leq |y| \leq 1.$$

Hence, the answer is $f^{-1}(G) = \{y \in \mathbb{R} \mid -1 \leq y \leq -\frac{1}{2} \text{ or } \frac{1}{2} \leq y \leq 1\}$.

(2) [5pt] (1.1.22+) Let $f : A \to B$ and $g : B \to C$.

(a) Show that if $g \circ f$ is injective, then f is injective. Give an example that shows that g need not be injective.

$\triangleright$ If f is non-injective, that is there is $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, i.e. $g \circ f$ is non-injective.
On the other hand, injectiveness of $g \circ f$ does not imply injectiveness of g. For example, for $f : \mathbb{N} \to \mathbb{Z}$, $f(n) = n$ for all $n \in \mathbb{N}$, and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$ for all $n \in \mathbb{Z}$, the composition $g \circ f$ is injective.

(b) Show that if $g \circ f$ is surjective, then g is surjective. Give an example that shows that f need not be surjective.

$\triangleright$ Assume g is not surjective, that is there is z in range of g such that $g(y)$ is never $= z$. But then $g(f(x))$ is never $= z$, too.
As an example for the second claim, one can use $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = |n|$ for all $n \in \mathbb{Z}$, and $g : \mathbb{Z} \to \mathbb{N}$, $g(n) = |n| + 1$ for all $n \in \mathbb{Z}$. $g(f)$ is surjective, but f is not.

(3) [2pt] (1.3.5) Exhibit (define explicitly) a bijection from N to the set of all odd integers greater than 2016.

$\triangleright$ $n \mapsto 2n + 2015$.

(4) Exhibit (define explicitly) a bijection between

(a) [2pt] $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$,

$\triangleright$ Map $\mathbb{Z} \to \mathbb{Z} \setminus \{0\}$ as follows. For $n < 0$, $n \to n$. For $n \geq 0$, $n \to n + 1$.

(b) [3pt, optional[1]] $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$.

[1]That is, not included in the denominator of the grade.

$\triangleright$ Map $\mathbb{Q} \to \mathbb{Q} \setminus \{0\}$ as follows. For $q \in \mathbb{Q}$, $r \notin \mathbb{Z}$, $r \to r$. For $r \in \mathbb{Z}$, use the previous mapping.