MA 441, Fall 2021

Assignment 6.

Part 1

Solutions

There are total 21 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.3–3.5 in Bartle–Sherbert.

(1) (3.3.13abd) Establish the convergence and find the limits of the following sequences.

(a) [1pt] $((1 + 1/n)^{n+1})$,

$\triangleright$ $\lim((1+1/n)^{n+1}) = \lim((1+1/n)^n)(1+1/n) = e \cdot 1 = e$. (**Note** that here and in the following items a "backwards" argument is implied, i.e. we *first* say that limits of $((1 + 1/n)^n)$ and $(1 + 1/n)$ exist and *then* conclude that the limit of $((1 + 1/n)^{n+1})$ exists and is equal to the product of the corresponding limits.)

(b) [1pt] $((1 + 1/n)^{-2n})$,

$\triangleright$ $\lim((1 + 1/n)^{2n}) = \lim((1 + 1/n)^n)((1 + 1/n)^n) = e \cdot e = e^2$.
Then $\lim((1 + 1/n)^{-2n}) = \lim(((1 + 1/n)^{2n})^{-1}) = (e^2)^{-1} = e^{-2}$.

(c) [1pt] $((1 - 1/n)^n)$.

$\triangleright$ $\lim((1-1/n)^n) = \lim((1+1/(n-1))^{-n}) = \lim(((1+1/(n-1))^n)^{-1}) = e^{-1}$.

(2) [2pt] (3.4.1) Give an example of an unbounded sequence that has a convergent subsequence.

$\triangleright$ $(0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots) = ((1 + (-1)^n) \cdot n/4)$.

(3) [3pt] (3.4.14) Let (x_n) be a bounded sequence and let

$$s = \sup\{x_n : n \in \mathbb{N}\}.$$

Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to x.

$\triangleright$ Denote $X = (x_n)$. Pick x_{n_1} with $s - x_{n_1} < 1$. Since s is the exact upper bound of (x_n) and $x_n \neq s$, there is a n_2 with

$$s - x_{n_2} < \min\left\{\tfrac{1}{2}, s - x_1, s - x_2, \ldots, x - x_{n_1}\right\}.$$

Then $n_2 > n_1$ and $s - x_{n_2} < 1/2$. Proceed by induction: having found $n_k > n_{k-1} > \ldots n_2 > n_1$, put n_{k+1} to be such that

$$s - x_{n_{k+1}} < \min\left\{\frac{1}{k+1}, s - x_1, s - x_2, \ldots, x - x_{n_k}\right\}.$$

Then $n_{k+1} > n_k$ and $s - x_{n_{k+1}} < \frac{1}{k+1}$, i.e. $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ form a subsequence of (x_n) and $\lim\limits_{k\to\infty}(x_{n_k}) = s$.

(4) (a) [3pt] (3.4.9) Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.

$\triangleright$ Suppose it is not true that $\lim X = 0$. That means that there is an $\varepsilon_0 > 0$ and a subsequence (x_{n_k}) of X such that $|x_{n_k} - 0| \geq \varepsilon_0$. Then (x_{n_k}) cannot have a subsequence converging to 0, which is a contradiction to condition of the problem.
(**Note** that 0 is unimportant here. The important part is that it's the same number for all subsequences.)

(b) [2pt] Suppose that every subsequence of $X = (x_n)$ has a converging subsequence. Is it true that in this case, X must converge?

$\triangleright$ Generally speaking, no. For example, $x_n = (-1)^n$. In this sequence, every subsequence has a subsequence $(1, 1, 1, \ldots)$ or $(-1, -1, -1, \ldots)$.