MA 441, Fall 2021

Assignment 4.

There are total 34 points in this assignment. 31 points is considered 100%. If you go over 31 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.1, 3.2 in Bartle–Sherbert.

1. Intervals in $\mathbb{R}$

(1) In this exercise you have to deliver specific inequalities from the definition of a convergent sequence. In each case below, find a number $K \in \mathbb{N}$ such that the corresponding inequality holds for all $n \geq K$. Give a *specific natural number* as your answer, for example $K = 1000$, or $K = 2 \cdot 10^7$, or $K = 139$, etc. (Not necessarily the smallest possible.)

You can (but you are discouraged to) use a calculator if you want to. However, 1) this problem can be done without using a calculator, 2) even if you do use one, your answers still should easily verifiable without it.

(a) [1pt] $\left|\frac{890534890.6451}{n}\right| < 0.00019011$

$\triangleright$ Note that $890534890.6451 < 10^{20}$ (you don't even need to count digits), and $0.00019011 > 10^{-10}$. Take $K = 10^{20}/10^{-10} = 10^{30}$. Then if $n > K$, we have

$$\left|\frac{890534890.6451}{n}\right| < \frac{10^{20}}{n} < \frac{10^{20}}{K} = \frac{10^{20}}{10^{30}} < 10^{-10} < 0.00019011.$$

(b) [1pt] $\left|\frac{100-n}{n} - (-1)\right| < 0.0054352$,

$\triangleright \left|\frac{100-n}{n} - (-1)\right| = \left|\frac{100}{n}\right|$. So for any $n > K = 10000$, $\left|\frac{100-n}{n} - (-1)\right| = \frac{100}{n} < 0.01 < 0.054352$.

(c) [2pt] $\left|\frac{200^{10}n+10^{100}}{n^2-10^{200}}\right| < 0.1$,

$\triangleright$ Let $K = \max\{100 \cdot 200^{10}, 10 \cdot 2 \cdot 10^{200}\}$. (It is not hard to see that in fact the latter number is greater, but we don't care. In fact, we can just take their sum $10 \cdot 200^{10} + 10 \cdot 2 \cdot 10^{200}$ and not think which is bigger.)

Then for any $n > K$, the numerator is estimated above

$$200^{10}n + 10^{100} < 0.01Kn + 0.01K < 0.01Kn + 0.01Kn = 0.02Kn < 0.02n^2.$$

On the other hand, the denominator is estimated below

$$n^2 - 10^{200} > n^2 - K > n^2 - n > 0.5n^2.$$

Then the ratio

$$\left|\frac{200^{10}n + 10^{100}}{n^2 - 10^{200}}\right| < \frac{0.02n^2}{0.5n^2} < 0.1$$

(d) [2pt] $\left|\frac{\cos(863n)}{\log n}\right| < 0.032432$.

$\triangleright$ For any $n \in \mathbb{N}$, $|\cos(863n)| \leq 1$. Also, for $n > 10^{100}$, $\log n > 100$, so $1/\log n < 0.01$. Therefore, for $n > 10^{100}$,

$$\left|\frac{\cos(863n)}{\log n}\right| < 0.01 < 0.032432.$$

(2) Reminder. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* $\varepsilon_0 > 0$ that demonstrates that (x_n) does not converge to x.

(a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

$\triangleright$ For $\varepsilon_0 = 0.01$, *none* of terms lie in $(1 - \varepsilon_0, 1 + \varepsilon_0)$. In particular, for any K there is $n > K$ (any works) s.t. $|1 - x_n| \geq \varepsilon_0$.

(b) [2pt] $x_n = 1/n$, $x = 1/2021$.

$\triangleright$ For $\varepsilon_0 = 1/2016 - 1/2017 > 0$, terms x_n with $n > 2016$ do not lie in $(1/2016 - \varepsilon_0, 1/2016 + \varepsilon_0)$. In particular, for any K there is $n > K$ (any $n > 2016$ works) s.t. $|1/2016 - x_n| \geq \varepsilon_0$.

(3) (3.1.6cd) Use the definition of limit of a sequence to establish the following limits.

(a) [2pt] $\lim\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$.

$\triangleright$ Note that

$$\left|\frac{3n+1}{2n+5} - 3/2\right| = \frac{13}{4n+10} \leq \frac{13}{4n}.$$

Let $\varepsilon > 0$ be given. Put $K > 13/4\varepsilon$. Then for any $n > K$, we have

$$\left|\frac{3n+1}{2n+5} - 3/2\right| \leq \frac{13}{4n} < \frac{13}{4K} < \frac{13}{4 \cdot 13/4\varepsilon} = \varepsilon.$$

(b) [2pt] $\lim\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

$\triangleright$ Note that

$$\left|\frac{n^2-1}{2n^2+3} - 1/2\right| = \frac{5}{4n^2+6} \leq \frac{5}{4n^2} \leq \frac{5}{4n} \leq \frac{5}{n}.$$

Let $\varepsilon > 0$ be given. Put $K > 5/\varepsilon$. For any $n > K$, we have

$$\left|\frac{n^2-1}{2n^2+3} - 1/2\right| \leq \frac{5}{n} < \frac{5}{K} < \varepsilon.$$

(4) (3.1.8) Let (x_n) be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$.

(a) [2pt] Use definition of limit to prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

$\triangleright$ Follows immediately from definition of limit since $||x_n| - 0| = |x_n| = |x_n - 0|$.

(b) [2pt] Use definition of limit to prove that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$.

$\triangleright$ Three cases: $x < 0$, or $x = 0$, or $x > 0$.
1) Case $x = 0$ is covered in previous item.
2) Case $x > 0$. In the definition of $x_n \to x$ as $n \to \infty$, put $\varepsilon = x/2$. This provides $K = K(x/2)$ such that for $n > K$, $|x_n - x| < x/2$, so $x_n > x/2 > 0$. (In other words: if limit of x_n is greater than 0, then eventually all x_n are also greater than 0.) Then for $n > K$, $|x_n| = x_n$ and $|x| = x$, so $|x_n| \to |x|$.
3) Case $x < 0$. Similarly to previous case, we get that eventually $|x_n| = -x_n$, so $|x_n| = -x_n \to -x = |x|$.

$\triangleright$ Another proof follows immediately from the triangle inequality: $||x_n| - |x|| \le |x_n - x|$, so if $|x_n - x| < \varepsilon$, then $||x_n| - |x|| < \varepsilon$ as well.

(c) [2pt] Give an example to show that the convergence of $(|x_n|)$ does not imply the convergence of (x_n).

$\triangleright$ $x_n = (-1)^n$. Then $|x_n| = 1$ and $|x_n| \to 1$, but (x_n) does not converge.

(5) [3pt] (Exercise 3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit, "$\lim XY = \lim X \cdot \lim Y$") *cannot* be used.

$\triangleright$ Let (b_n) be bounded $|b_n| < M$. Let $\varepsilon > 0$ be given. Then since (a_n) converges, there is $K = K(\varepsilon/M)$ such that $|a_n| < \varepsilon/M$ for all $n > K$. Then for all $n > K$, $|a_n b_n| < M \cdot \varepsilon/M = \varepsilon$. By definition of limit, $\lim(a_n b_n) = 0$.
We cannot directly use limit of a product since (b_n) may be divergent.

(6) (a) [2pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.

$\triangleright$ Let $\varepsilon > 0$ be given. Since X and Y converge, there is K_1 such that $|x - x_n| < \varepsilon/2$ for all $n > K_1$ and K_2 such that $|y - y_n| < \varepsilon/2$ for all $n > K_2$. Then for $n > K = \max\{K_1, K_2\}$,

$$|(x-y) - (x_n - y_n)| = |(x - x_n) - (y - y_n)| \le$$
$$|x - x_n| + |y - y_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By definition of limit, $X - Y \to x - y$ as $n \to \infty$.
NOTE. By triangle inequality $|a - b| \le |a| + |b|$, not $|a| - |b|$. The inequality $|a - b| \le |a| - |b|$ is generally wrong. (Exercise: give a short explanation why it can't possibly be always true.)

(b) [2pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.

$\triangleright$ Follows immediately from previous item and $Y = (X + Y) - X$.

(c) [2pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

$\triangleright$ For example, $x_n = y_n = (-1)^n$.

(7) [5pt] Determine the following limits (or establish they do not exist):

(a) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000n+100000}$,

$\triangleright$

$$\frac{2n^2-1}{1000n+100000} = \frac{\frac{1}{n^2}(2n^2-1)}{\frac{1}{n^2}(1000n+100000)}$$
$$= \frac{2-1/n^2}{1000+100000/n}\cdot n \geq \frac{1}{1000}\cdot n.$$

The latter is not bounded above, so the sequence is not bounded, therefore divergent. The limit does not exist.

$\triangleright$ Alternatively, we can assume that limit exists, and obtain that then $\lim(n)$ also exists:

$$\lim_{n\to\infty} \frac{2n^2-1}{1000n+100000} = \lim_{n\to\infty} \frac{\frac{1}{n^2}(2n^2-1)}{\frac{1}{n^2}(1000n+100000)} =$$
$$= \lim_{n\to\infty} \frac{2-1/n^2}{1000+100000/n}\cdot n = \lim_{n\to\infty} \frac{2-0}{1000+0}\cdot n,$$

which does not exist.

Note. Strictly speaking, in the second solution we are operating with limits that do not exist, i.e. with meaningless values. However, such solutions are usually acceptable because it is implied that we argue by contradiction: assuming that the original limit exists, we show by the above equalities that that $\lim(n)$ exists, obtaining a contradiction.

(b) $\lim\limits_{n\to\infty} \frac{2\sqrt{n^2+1}-10}{1000n+100000}$,

$\triangleright$

$$\lim_{n\to\infty} \frac{2\sqrt{n^2+1}-10}{1000n+100000} = \lim_{n\to\infty} \frac{\frac{1}{n}(2\sqrt{n^2+1}-10)}{\frac{1}{n^2}(1000n+100000)} =$$
$$\lim_{n\to\infty} \frac{2\sqrt{1+1/n^2}-10/n}{1000+100000/n} = \lim_{n\to\infty} \frac{2-0}{1000+0} = 1/500.$$

Here we used that $\sqrt{1+1/n^2} \to 1$ $(n\to\infty)$. (And we use a similar limit in the next item.) I was taking it for granted when grading, but it is not hard to show this limit by definition. Note that $\sqrt{1+1/n^2} \leq 1+1/n^2$, so $|\sqrt{1+1/n^2}-1| \leq 1/n^2 \leq 1/n$. So for a given $\varepsilon > 0$, we have the inequality required in the definition of limit whenever $n > 1/\varepsilon$.

(c) $\lim\limits_{n\to\infty} \frac{2n^2-1}{0.1\sqrt[5]{n^{11}+12}-10000}$.

$\triangleright$

$$\lim_{n\to\infty} \frac{2n^2-1}{1000\sqrt[5]{n^{11}+12}-100000} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[5]{n^{11}}}(2n^2-1)}{\frac{1}{\sqrt[5]{n^{11}}}(1000\sqrt[5]{n^{11}+12}-100000)} =$$
$$\lim_{n\to\infty} \frac{\frac{2}{\sqrt[5]{n}}-\frac{1}{\sqrt[5]{n^{11}}}}{1000\sqrt[5]{1+12/n^{11}}-100000/\sqrt[5]{n^{11}}} = \lim_{n\to\infty} \frac{0-0}{1000\cdot 1-0} = 0,$$