MA 441, Fall 2021

Assignment 4.

There are total 34 points in this assignment. 31 points is considered 100%. If you go over 31 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.1, 3.2 in Bartle–Sherbert.

(1) In this exercise you have to deliver specific inequalities from the definition of a convergent sequence. In each case below, find a number $K \in \mathbb{N}$ such that the corresponding inequality holds for all $n \geq K$. Give a *specific natural number* as your answer, for example $K = 1000$, or $K = 2 \cdot 10^7$, or $K = 139$, etc. (Not necessarily the smallest possible.)

You can (but you are discouraged to) use a calculator if you want to. However, 1) this problem can be done without using a calculator, 2) even if you do use one, your answers still should easily verifiable without it.

(a) [1pt] $\left|\frac{890534890.6451}{n}\right| < 0.00019011$

(b) [1pt] $\left|\frac{100-n}{n} - (-1)\right| < 0.0054352$,

(c) [2pt] $\left|\frac{200^{10}n+10^{100}}{n^2-10^{200}}\right| < 0.1$,

(d) [2pt] $\left|\frac{\cos(863n)}{\log n}\right| < 0.032432$.

(2) Reminder. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* $\varepsilon_0 > 0$ that demonstrates that (x_n) does not converge to x.

(a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

(b) [2pt] $x_n = 1/n$, $x = 1/2021$.

(3) (3.1.6cd) Use the definition of limit of a sequence to establish the following limits.

(a) [2pt] $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$.

(b) [2pt] $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

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(4) (3.1.8) Let (x_n) be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$.

(a) [2pt] Use definition of limit to prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

(b) [2pt] Use definition of limit to prove that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$.

(c) [2pt] Give an example to show that the convergence of $(|x_n|)$ does not imply the convergence of (x_n).

(5) [3pt] (Exercise 3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit, "$\lim XY = \lim X \cdot \lim Y$") *cannot* be used.

(6) (a) [2pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.

(b) [2pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.

(c) [2pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

(7) [5pt] Determine the following limits (or establish they do not exist):

(a) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000n+100000}$,

(b) $\lim\limits_{n\to\infty} \frac{2\sqrt{n^2+1}-10}{1000n+100000}$,

(c) $\lim\limits_{n\to\infty} \frac{2n^2-1}{0.1\sqrt[5]{n^{11}+12}-10000}$.