(a) For the cylinder the unit normal is
$(\cos u, \sin u, 0)$

$\Rightarrow$ let $e_1 = (0, 0, 1)$

by 1.3 $S_p(e_1) = 0\, e_1 \Rightarrow e_1$ is an eigenvector of S_p with the eigenvalue $k_1 = 0$

let $e_2 = (-\sin u, \cos u, 0)$

by 1.3 $S_p(e_2) = -\frac{1}{r} e_2 \Rightarrow e_2$ is an eigenvector of S_p with the eigenvalue $k_2 = -\frac{1}{r}$

$\Rightarrow$ principal directions $e_1 = (0,0,1)$, $e_2 (-\sin u, \cos u, 0)$
" curvatures $k_1 = 0$, $k_2 = -\frac{1}{r}$

at every point

(b) For the saddle surface $z = xy$ at $(0,0,0)$
and take $u_1 = (1,0,0)$, $u_2 (0,1,0)$.
let $e_1 = \frac{1}{\sqrt{2}}(u_1 + u_2)$ as in Example 1.3

$S(e_1) = \frac{1}{\sqrt{2}}[S(u_1) + S(u_2)] = \frac{1}{\sqrt{2}}[u_2 + u_1]$
$= e_1$

$\Rightarrow e_1$ is an eigenvector with eigenvalue 1

let $e_2 = \frac{1}{\sqrt{2}}(u_1 - u_2)$

by Example 1.3
$\Rightarrow S(e_2) = \frac{1}{\sqrt{2}}[S(u_1) - S(u_2)] = \frac{1}{\sqrt{2}}(u_2 - u_1)$

$= -e_2$

$\Rightarrow e_2$ is an eigenvector with eigenvalue -1

$\Rightarrow$ principal directions $e_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $e_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$
principal curvatures $k_1 = 1$, $k_2 = -1$
$\Big\}$ at $(0,0,0)$

At an umbilic point the shape operator S has

$K = \det \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k^2 \geq 0$. Hence, if $K < 0$ umbilics are impossible.

If $K = 0 \Rightarrow$ Both pricipal curvatures are 0, so every normal curvature is 0. therefore the surface is flat at that point.

(a) $x_u = (1, 0, f_u)$

$x_v = (0, 1, f_v)$

$E = x_u \cdot x_u = 1 + f_u^2$

$F = x_u \cdot x_v = f_u f_v$

$G = x_v \cdot x_v = 1 + f_v^2$

$U = (-f_u, -f_v, 1) / \sqrt{1 + f_u^2 + f_v^2}$

note:

$EG - F^2 = 1 + f_u^2 + f_v^2 + f_u^2 f_v^2 - (f_u f_v)^2$

$= 1 + f_u^2 + f_v^2$

$\Rightarrow U = (-f_u, -f_v, 1) / \sqrt{EG - F^2}$

$= (-f_u, -f_v, 1) / w$

$L = x_{uu} \cdot U = (0, 0, f_{uu}) \cdot \vec{v} = \dfrac{f_{uu}}{w}$

$M = x_{uv} \cdot U = (0, 0, f_{uv}) \cdot \vec{v} = \dfrac{f_{uv}}{w}$

$N = x_{vv} \cdot U = (0, 0, f_{vv}) \cdot \vec{v} = \dfrac{f_{vv}}{w}$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\dfrac{f_{uu} f_{vv} - f_{uv}^2}{w^2}}{w^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{w^4}$$

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{(1 + f_v^2)\dfrac{f_{uu}}{w} + (1 + f_u^2)\dfrac{f_{vv}}{w} - 2\dfrac{f_u f_v f_{uv}}{w}}{2w^2}$$

$$= \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2w^3}$$

(b) flat $\Leftrightarrow K = 0 \Leftrightarrow \dfrac{f_{uu} f_{vv} - f_{uv}^2}{w^4} = 0 \Leftrightarrow f_{uu} f_{vv} - f_{uv}^2 = 0$

minimal $\Leftrightarrow H = 0 \Leftrightarrow (1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv} = 0$