Homework 5

Arjun Koshal

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Problem 1

Theorem. *Given any two x and y such that $x < y$, there exists an irrational number z such that $x < z < y$.*

Proof. To prove this theorem, we will use the following 3 statements.
1) (Denisty Theorem) Given any two x and y such that $x < y$, there exists an rational number z such that $x < z < y$
2) $\sqrt{2}$ is irrational, as we have proved before.
3) The product between an irrational and rational number (that is non-zero) will always be irrational, as we have proved before.

Let us consider $x\left(\frac{1}{\sqrt{2}}\right)$ and $y\left(\frac{1}{\sqrt{2}}\right)$. It then follows from 1) that there exists a rational number $r \neq 0$ such that

$$x\left(\frac{1}{\sqrt{2}}\right) < r < y\left(\frac{1}{\sqrt{2}}\right)$$

Then it follows that

$$x < r\sqrt{2} < y$$

It then follows from statements 2) and 3) that if we set $z = r\sqrt{2}$, then z is an irrational number. Thus the theorem holds true. □

Problem 2

Theorem. *Let $S \subset \{1, 2, ..., 1000\}$ be a set of 100 natural numbers; there exist distinct nonempty subsets $X, Y \subset S$ such that the sum of the elements of X equals the sum of the elements of Y.*

Proof. Given some set S of 100 natural numbers, there exist $2^{100} - 1$ distinct subsets excluding the empty subset for both subsets X and Y. The sum of the elements in any such set must be at least 1, as the empty set is not included, and the sum of the elements in any such set must be at most $901 + 902 + ... + 1000$, which is 95050.

Since there are more possible set combinations than possible sums ($2^{100} - 1 > 2^{20} > 10^6 > 95050$), we conclude by the pigeon hole theorem that there exist distinct nonempty subsets $X, Y \subset S$ such that the sum of the elements of X equals the sum of the elements of Y. □