Assignment 3.

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% (but not over 115%) for this homework and it will count towards your course grade.

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.3–2.5 in Bartle–Sherbert.

(1) [3pt] (2.3.4) let S be a a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as \mid s \in S\}$. Prove that $\inf(aS) = a \inf S$, $\sup(aS) = a \sup S$.

$\triangleright$ Since S is bounded, it has sup and inf. Denote $u = \sup S$. Then, since $a > 0$, $au \geq as$ for all $s \in S$, i.e., au is an upper bound for aS. To show that it's the least upper bound, assume v is any upper bound for aS. Then $v \geq as$, that is $v/a \geq s$ for all $s \in S$, that is v/a is an upper bound for S and therefore $v/a \geq \sup S = u$, so $v \geq ua$.

Argument for $\inf(aS)$ is similar.

(2) (a) [2pt] (2.4.7) Let A and B be nonempty bounded[1] subsets of $\mathbb{R}$. Show that $A + B = \{a + b : \ a \in A, b \in B\}$ is a bounded set. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

$\triangleright$ Every $c \in A + B$ is of the form $c = a + b$, $a \in A$, $b \in B$. So $c = a + b \leq \sup A + \sup B$. Therefore, $\sup A + \sup B$ is an upper bound of $A + B$. Similarly, $\inf A + \inf B$ is a lower bound of $A + B$. Show that $\sup A + \sup B$ it is the least upper bound.

Let $\varepsilon > 0$ be given. There is $a \in A$ s.t. $a > \sup A - \varepsilon/2$, $b \in B$ s.t. $a > \sup B - \varepsilon/2$. Then $a + b \in A + B$ and $a + b > \sup A + \sup B - \varepsilon$.

Argument for inf is similar.

$\triangleright$ There is also a way to deal with $A + B$ without ε's.

First of all, since $\sup A \geq a$ for all $a \in A$ and $\sup B \geq b$ for all $b \in B$, we have $\sup A + \sup B \geq a + b$ for all $a \in A, b \in B$, i.e. $\sup A + \sup B$ is an upper bound of $A + B$.

Second, let u be any upper bound of $A + B$, $u \geq a + b$ for all $a \in A, b \in B$. Fixing $b \in B$ in this inequality we see that u is an upper bound of the set $A + b$, so by in-class statement, $u \geq b + \sup A$ for all $b \in B$. Now, u is an upper bound of the set $B + \sup A$, so $u \geq \sup(B + \sup A) = \sup B + \sup A$, so $\sup A + \sup B$ is the least upper bound of $A + B$.

Argument for inf is similar.

(b) [2pt] Find $\sup\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\sup\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$. (*Hint:* for the last two questions, use the previous item 2a.)

$\triangleright$ $\sup\{\frac{1}{n} : \ n \in \mathbb{N}\} = 1$ by definition of sup.

$\inf\{\frac{1}{n} : \ n \in \mathbb{N}\} = 0$ by Archimedean property.

Note that then $\sup\{-\frac{1}{n} : \ n \in \mathbb{N}\} = 0$ and $\inf\{-\frac{1}{n} : \ n \in \mathbb{N}\} = -1$

Then by item 2a,

$$\sup\left\{\tfrac{1}{n} - \tfrac{1}{m} : \ m, n \in \mathbb{N}\right\} =$$

[1] Which means bounded above and below.

$$\sup\left\{\tfrac{1}{n}:\ n\in\mathbb{N}\right\}+\sup\left\{-\tfrac{1}{m}:\ m\in\mathbb{N}\right\}=1+0=1.$$

Similarly, $\inf\{\frac{1}{n}-\frac{1}{m}:\ m,n\in\mathbb{N}\}=-1+0=-1$.

(c) [2pt] For A,B as in item 2a, show that $AB=\{ab:\ a\in A, b\in B\}$ is a bounded set. Is it true that always $\sup AB=\sup A\cdot\sup B$?

$\triangleright$ A is bounded if and only if there is a real number $M>0$ s.t. $|a|<M$ for all $a\in A$.
B is bounded if and only if there is a real number $L>0$ s.t. $|b|<L$ for all $b\in B$.
Then for all $a\in A, b\in B$, $|ab|<ML$. Therefore AB is bounded.
Answer to second question is generally No. For example for $A=B=\{-1,0\}$ we have $\sup A=\sup B=0$, but $\sup AB=1$.
(If both A and B only have positive elements then the answer is Yes, but you were not asked to prove this.)

(3) [2pt] (2.4.19) If $u>0$ is any real number and $x<y$, show that there exists a rational number r such that $x<ru<y$. (In other words, the set $\{ru \mid u\in\mathbb{Q}\}$ is dense in $\mathbb{R}$.) (*Hint:* Divide the required inequality by u..)

$\triangleright$ Since $u>0$, we have $\frac{x}{u}<\frac{x}{u}$. By Density theorem, there is $r\in\mathbb{Q}$ such that $\frac{x}{u}<r<\frac{x}{u}$, that is $x<ru<y$, exactly as required.

(4) [3pt] Find the intersection $\bigcap_{n=1}^{\infty} I_n$ in the following cases. (Only the answer suffices in this exercise.)

(a) $I_n=[1-\frac{1}{n},1]$ for each $n\in\mathbb{N}$.

$\triangleright$ $\bigcap_{n=1}^{\infty}[1-\frac{1}{n},1]=\{1\}$.

(b) $I_n=(1-\frac{1}{n},1)$ for each $n\in\mathbb{N}$.

$\triangleright$ $\bigcap_{n=1}^{\infty}(1-\frac{1}{n},1)=\emptyset$.

(c) $I_n=[n,\infty)$ for each $n\in\mathbb{N}$.

$\triangleright$ $\bigcap_{n=1}^{\infty}[n,\infty)=\emptyset$.

(d) $I_n=[3-\frac{1}{n^2},5+\frac{3}{n}]$ for each $n\in\mathbb{N}$.

$\triangleright$ $\bigcap_{n=1}^{\infty}[3-\frac{1}{n^2},5+\frac{3}{n}]=[3,5]$.

(e) $I_n=[0,1]$ for each $n\in\mathbb{N}$.

$\triangleright$ $\bigcap_{n=1}^{\infty}[0,1]=[0,1]$.

(5) [4pt] (2.5.10) Let $I_1=[a_1,b_1]\supseteq I_2=[a_2,b_2]\supseteq I_3=[a_3,b_3]\supseteq\dots$ be an infinite nested system of closed intervals. Let $u=\sup\{a_n|n\in\mathbb{N}\}$ and $v=\inf\{b_n|n\in\mathbb{N}\}$. Prove that

$$[u,v]=\bigcap_{n=1}^{\infty} I_n.$$

(*Hint:* This is a set equality. To prove an equality $A=B$ of sets A and B, you have to do to things: show that every element of A is also an element of B, and that every element of B is also an element of A. That is, two inclusions $A \subseteq B$ and $B \subseteq A$..)

$\triangleright$ Note that it was proved in class that $u \in \bigcap_{n=1}^{\infty} I_n$. (To recall, this is because $a_n \leq u \leq b_n$ for every n. Indeed, u is $\geq$ than every a_n because u is an upper bound for $\{a_n\}$. Further, $u \leq$ every b_n because every b_n is an upper bound for $\{a_n\}$, therefore $b_n \geq$ the exact upper bound, u.) By a similar argument, $v \in \bigcap_{n=1}^{\infty} I_n$ as well.

Let $x \in [u, v]$. Then, since for each $n \in \mathbb{N}$, $u, v \in I_n$, x also belongs to I_n. So $x \in \bigcap_{n=1}^{\infty} I_n$.

For the reverse inclusion, let $x \in \bigcap_{n=1}^{\infty} I_n$. Then $a_n \leq x \leq b_n$ for each $n \in \mathbb{N}$, so $u \leq x \leq v$ since x is an upper bound for $\{a_n\}$ and a lower bound for $\{b_n\}$, therefore not smaller than the exact upper bound u for $\{a_n\}$ and not greater than the exact lower bound for $\{b_n\}$, v (we could just say "by the choice of u, v"). So $x \in [u, v]$.