Assignment 9

Solutions

There are total 22 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 5.1, 5.2, and partially 5.3 in Bartle–Sherbert.

(1) [2pt] (5.1.7+) (Local separation from zero) Let $A \subseteq \mathbb{R}$, $c \in A$, $f : A \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that for any $\alpha \in \mathbb{R}$ such that $0 < \alpha < f(c)$, there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c) \cap A$, then $f(x) > \alpha$.

$\triangleright$ Put $\varepsilon = f(x) - \alpha > 0$ in the definition of continuity of f at c, get the required δ.

(2) [3pt] (5.1.13) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for $x \in \mathbb{Q}$ and $g(x) = x + 3$ for $x \notin \mathbb{Q}$. Find all points at which g is continuous.

$\triangleright$ Note that for each $c \in \mathbb{R}$, there is a sequence (x_n) in $\mathbb{Q}$ s.t. $(x_n) \to c$, and a sequence (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ s.t. $(y_n) \to c$. Then $\lim(g(x_n)) = 2c$ and $\lim(g(y_n)) = c + 3$, so by the sequential criterion, if $2c \neq c + 3$, then g is not continuous at c. $2c = c + 3$ if and only if $c = 3$, so g is not continuous at all points $c \neq 3$.

This leaves us to check whether g is continuous at $c = 3$. Note that at $c = 3$, $|g(x) - g(c)| \leq \max\{|2x - 2c|, |x + 3 - (c + 3)|\}$, because $|g(x) - g(c)|$ equals to one of the latter two numbers. So, given $\varepsilon > 0$, put $\delta = \varepsilon/2$, get that for all $|x - c| < \delta$, we have $|g(x) - g(c)| < \max\{2\delta, \delta\} = \varepsilon$, so $g(x)$ is continuous at $c = 3$ by the definition of continuity. (Instead of the definition, we could also use the squeeze theorem for a limit of a function.)

(3) [2pt] (Exercise 5.2.5) Let g be defined on $\mathbb{R}$ and by $g(1) = 0$, and $g(x) = 2$ if $x \neq 1$, and let $f(x) = x+1$ for all $x \in \mathbb{R}$. Show that $\lim\limits_{x\to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Composition of Continuous Functions Theorem (Theorem 5.2.6)?

$\triangleright$ The composition is given by

$$(g \circ f)(x) = g(f(x)) = g(x+1) = \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } x \neq 0. \end{cases}$$

Then $g(f(0)) = 0$ and $\lim\limits_{x\to 0}(g(f(x)) = \lim_{x\to 0} 2 = 2$.

This does not contradict the composition of continuous functions theorem (this theorem would assert that those two values must be the same), since the function g is not continuous at the point $f(0)$.

(4) [3pt] (5.2.6) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. suppose that $\lim\limits_{x\to c} f = b$ and that g is continuous at b. Show that $\lim\limits_{x\to c} g(f(x)) = g(b)$.

(*Hint:* (Re)define f to be b at c, apply composition of continuous functions.)

Note. This statement says that lim and a *continuous* function can be swapped: $\lim\limits_{x\to c} g(f(x)) = g(\lim\limits_{x\to c} f(x))$. The previous exercise shows that continuity of g is essential.

$\triangleright$ Redefine f at c to be equal to $b = \lim_{x\to c} f(x)$. With an abuse of notation, we denote the new function by the same letter f. Then f is continuous at c and g is continuous at b, so $g \circ f$ is continuous at c, i.e. $\lim\limits_{x\to c} g(f(x)) = g(b)$, as required.

$\triangleright$ Another solution can be done using the Sequential Criteria for limit and for continuity. Let (x_n) be a sequence converging to c, $x_n \neq c$. Then by the sequential criterion for limit of a function, $(f(x_n)) \to b$. But then by the sequential criterion for continuity, $g(f(x_n)) \to g(b)$. Therefore, by the sequential criterion for limit of a function, $\lim_{x\to c} g(f(x)) = g(b)$.

$\triangleright$ Finally, you can use the ε–δ definitions of limits and continuity to get a solution similar to the one above (or to the proof of composition of continuous functions theorem).

(5) [2pt] (5.2.7) Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

$\triangleright$ For example, f which is 1 at rational points and -1 at irrational points.

(6) (a) [2pt] (Exercise 5.1.12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Show that $f(x) = 0$ at every point $x \in \mathbb{R}$.
$\triangleright$ By the sequential criterion, for any $c \in \mathbb{R}$ if $\lim\limits_{x\to c} f$ exists, it is equal to 0 (we used Density theorem to get a sequence of rational numbers (r_n) that approaches c). Since f is continuous all such limits exist and are equal to value of f at respective points: $\lim\limits_{x\to c} f = f(c)$. Then as we showed, for all $c \in \mathbb{R}$, $\lim\limits_{x\to c} f = 0$, therefore for all $c \in \mathbb{R}$, $f(c) = 0$.

(b) [2pt] (Exercise 5.2.8) Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (*Hint:* Consider $f - g$.)
$\triangleright$ Apply item (a) to the function $f - g$.
(**Note** that there is no point in repeating the argument, you can just use item (a) to conclude $f - g = 0$, so $f = g$.)

(7) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous on I function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

$\triangleright$ Note that by the Maximum-Minimum theorem, f attains its minimum on I at some point $c \in I$. Then put $\alpha = f(c) > 0$. Then since c is a point of absolute minimum, $f(x) \geq f(c) = \alpha > 0$ for all $x \in I$.

(8) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f = \lim\limits_{x\to+\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained. (*Hint:* Pick M large enough and inspect how f behaves on the interval $[-M, M]$, and on $\mathbb{R} \setminus [-M, M]$.)

$\triangleright$ Assume for definiteness that there is $a \in \mathbb{R}$ such that $f(a) > 0$. Since $\lim\limits_{x\to\infty} f = 0$, there is $M_1 > 0$ such that $|f(x)| < f(a)$ on (M_1, ∞). Since $\lim\limits_{x\to-\infty} f = 0$, there $M_2 > 0$ such that $|f(x)| < f(a)$ on $(-\infty, -M_2)$. Note that in particular, these conditions mean that a is neither in $(-\infty, -M_2)$, nor in (M_1, ∞), because if it were, we would have $|f(a)| < f(a)$.

By the Maximum-Minimum theorem, f attains its maximum on $[-M_2, M_1]$ at some point c. Note that the value $f(x)$ of that maximum is at least $f(a)$ because, as mentioned above, $a \in [-M_2, M_1]$. That means $f(c) \geq f(x)$ for any $x \in \mathbb{R}$, i.e. c is a point of absolute maximum of f.

(If we only have points with $f(a) < 0$, we can repeat the argument with minimum instead of maximum.)

Note that both maximum and minimum need not be achieved. For example $f(x) = 1/(x^2 + 1)$ is continuous on $\mathbb{R}$, goes to 0 at $\pm\infty$, and is > 0 for all x, so it does not attain its infimum.