Homework 3

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Problem 1

Theorem. *The contestant has a red dot on their forehead.*

Proof. We can prove this theorem through proof by contradiction.
First let us denote this contestant A and the others B and C. Now suppose not, that is, suppose contestant A has a blue dot on their forehead. Then, with all three contestants raising their hands, let us consider the perspective of contestant B. Contestant B notices that contestant A, with a blue dot, is raising their hand, and that contestant C, with a red dot, is also raising their hand. With the knowledge that contestant A has a blue dot, contestant C must be raising their hand to signify contestant B themselves has a red dot; then, contestant B can easily conclude that they have a red dot, and indicate this to the gameshow host, as could contestant C. However, neither contestant made any such easy conclusions. Everyone's hand is raised, but no one can easily conclude the color of the dot on their forehead because none of them see a blue dot from their own perspective. Thus, the assumption has been contradicted, and it holds true that contestant A must instead have a red dot. □

Problem 2

Theorem. *A real number is irrational if and only if it is a different distance from every rational number.*

Proof. Since this is a bi-conditional statement, we must prove both 1) if a real number is irrational, then it has a different distance from every rational number and 2) if a number has a different distance from every rational number, then it is irrational.

Let us begin with proving the first statement as mentioned above.

1) If a real number is irrational, then it has a different distance from every rational number. We can prove this statement through proving the contrapositive of the statement: If a real number has the same distance from two different rational numbers, then the number can not be irrational.

Suppose that a rational number x is the same distance from two distinct rational numbers a and b. This would imply that $|x - a| = |x - b|$. Then it follows

$$
\begin{aligned}
|x-a|^2 &= |x-b|^2 \\
x^2 - 2xa + a^2 &= x^2 - 2xb + b^2 \\
-2xa + 2xb &= -a^2 + b^2 \\
2x(b-a) &= b^2 - a^2 \\
2x(b-a) &= (b-a)(b+a) \\
2x &= b+a \\
x &= \frac{b+a}{2}
\end{aligned}
$$

Since the sum and product of two rationals is always rational, we can conclude that x is rational as well. Thus the statement holds true.

2) If a number has a different distance from every rational number, then it is irrational. We can prove this statement through proving the contrapositive.

The contrapositive of the statement is if a number is rational, then it has the same distance from every rational number.

Suppose any rational number y is a different distance from every rational number. Consider a rational number y. Then both $y + \frac{1}{2}$ and $y - \frac{1}{2}$ are rationals, as the sum and difference between two rationals is also rational. Since we are assuming that y is a different distance from every rational number, it must hold that

$$
\begin{aligned}
|y + \frac{1}{2}| &\neq |y - \frac{1}{2}| \\
|y - (y + \frac{1}{2})| &\neq |y - (y - \frac{1}{2})| \\
|-\frac{1}{2}| &\neq |\frac{1}{2}| \\
\frac{1}{2} &\neq \frac{1}{2}
\end{aligned}
$$

Since the contrapositive holds true, the implication must also hold true. Therefore if a number has a different distance from every rational number, then it is irrational.

Having addressed the implication in both directions, it follows that a real number is irrational if and only if it is a different distance from every rational number. $\square$