Homework 9

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Problem 1

Theorem. *K_n has an Euler circuit if and only if n is odd.*

Proof. We can prove this statement through proving the implication and then the converse.

Proof of Implication: If K_n has an Euler circuit, then n is odd.
A graph K_n will have n vertices with $n-1$ edges for each vertex, so each vertex would have a degree of $n-1$. We also know that a graph has an Euler circuit if and only if the degree of every vertex is even. That is, $n-1$ must be even for K_n to contain an Euler circuit. If $n-1$ is even, then it follows that n must be odd.

Proof of Converse: If n is odd, then K_n has an Euler circuit.
Suppose n is odd. We know a graph K_n will have n vertices with $n-1$ edges for each vertex, so each vertex would have a degree of $n-1$. If n is odd then each vertex would have an even degree, which satisfies the definition of an Euler circuit. Therefore if n is odd, then K_n has an Euler circuit. $\square$

Problem 2

Theorem. *In a planar graph (without multi-edges and loops) $F \leq \frac{2}{3}E$, where F is is the number of faces, and E is the number of edges.*

Proof. Suppose we have a planar graph G without multi-edges and loops. Let d be the degree of a vertex v. The degree d of a vertex v is the number of times it appears as an endpoint of an edge. Since each edge has two endpoints, d counts each edge twice, hence it follows that the sum of degrees is equal to twice the number of edges, which we can express as $2E$. We also notice that each face is enclosed by at least 3 edges, thus the total number of edges must be greater than or equal to $3F$. Putting this together we get,

$$\begin{aligned} 3F &\leq 2E \\ F &\leq \frac{2}{3}E. \end{aligned}$$

$\square$

Problem 3

Theorem. *K_5 is not planar.*

Proof. We can use proof by contradiction. Assume that K_5 is planar. Then the graph must satisfy Euler's formula for planar graphs. We know K_5 has 5 vertices and 10 edges, so we get

$$\begin{aligned} 5 - 10 + F &= 2 \\ -5 + F &= 2 \\ F &= 7. \end{aligned}$$

This tells us that if the graph is drawn without any edges crossing, then there would be $F = 7$ faces.

Looking at the amount of edges that surround each face, we notice that each face must be surrounded by at least 3 edges. From problem 2, we know that $3F \leq 2E$, where F is is the number of faces, and E is the number of edges. However, this is not possible since we know that $F = 7$ and $E = 10$, and $21 \not\leq 20$. This is a contradiction. Therefore K_5 is not planar. $\square$