Homework 11

Arjun Koshal

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Problem 1

Theorem. *The function $f(x) = 1 - 5x$ is continuous at $x = 2$.*

Proof. Let $\varepsilon > 0$ be given. We must show that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x-c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

$$\begin{aligned} |f(x) - f(2)| &= |1 - 5x - 1 + 5 \cdot 2| \\ &= |10 - 5x| \\ &= 5|x - 2| \end{aligned}$$

Let $\delta = \frac{\varepsilon}{5}$. Then if $|x - 2| < \delta = \frac{\varepsilon}{5}$, then $5|x - 2| < \varepsilon$. Since for all $\varepsilon > 0$, there exists a $\delta = \frac{\varepsilon}{5}$ such that $|x - 2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$, the function $f(x) = 1 - 5x$ is continuous at $x = 2$. □

Problem 2

Theorem. *Every nonempty set $A \subseteq \mathbb{R}$ that has a lower bound has a greatest lower bound,* $\inf(A)$.

Proof. Let $A \subseteq \mathbb{R}$ be a non-empty set with a lower bound. Then there exists $x \in \mathbb{R}$ such that $x \leq a$ for all $a \in A$. This implies that $-x \geq -a$ for all $a \in A$. Hence $-x$ is an upper bound for the set $-A = \{-a : a \in A\}$. By the Least Upper Bound Principle, $-A$ has a supremum, b.

Suppose $-b$ is the infimum of A. Since b is an upper bound for $-A$, it follows $b \geq -a$ for all $a \in A$, thus $-b \leq a$ for all $a \in A$. This implies $-b$ is a lower bound for A. Let c be any other lower bound for A. Similarly to x, $-c$ is an upper bound for $-A$. Since b is the least upper bound for $-A$, it follows that $b \leq -c$, thus $-b \geq c$. Therefore $-b$ is the greatest lower bound, and $-b = \inf(A)$ □

Problem 3

Theorem. *Define $f : \mathbb{R} \to \mathbb{R}$ as*

$$f(x) = \begin{cases} \frac{1}{n} & \textit{if } x = \frac{m}{n} \textit{ with } m,\ n \textit{ in lowest terms,} \\ 0 & \textit{otherwise} \end{cases}.$$

Then $f(x)$ is continuous if and only if x is irrational.

Proof. Proof of Implication: Suppose there exists a rational $c \in \mathbb{R}$. Then $f(c) > 0$. Since irrationals are dense in $\mathbb{R}$ by the corollary of the density of rationals theorem, there exists a sequence $t_1, t_2, t_3, ..., \in \mathbb{R}$ of irrationals such that $t_n \to c$, as $n \to \infty$.

Therefore, $\lim_{n\to\infty} t_n = c$; however, $\lim_{n\to\infty} f(t_n) \neq f(c)$, since $\lim_{n\to\infty} f(t_n) = 0$. This does not satisfy the epsilon-delta definition of continuity, since for all $\varepsilon > 0$, there does not exist a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Thus if the function is continuous, x must be irrational.

Proof of Converse: Suppose there exists an irrational $c \in \mathbb{R}$. Let $\varepsilon > 0$ and an integer $N > \frac{1}{\varepsilon}$. For any fixed positive integer $n \leq N$, the rationals in $\mathbb{R}$, with the denominator equivalent to n are in the form of $\frac{m}{n}$, where $1 \leq m \leq n$, so there are only n choices for m. Thus there are only a finite number of rationals in $\mathbb{R}$ that have a denominator less than N. Therefore there exists a $\delta > 0$ such that $(c - \delta, c + \delta) \subset \mathbb{R}$, where the interval contains no rationals. If $x \in (c - \delta, c + \delta)$ is rational, then $f(x) = 0$ and if $x \in (c - \delta, c + \delta)$ is irrational, then $f(x) = \frac{1}{n}$.

For x is rational, then $f(x) = 0 < \varepsilon$ and for x is irrational, then $0 < f(x) \leq \frac{1}{N} < \varepsilon$. In both cases it follows that $f(x) < \varepsilon$. Since $f(c) > 0$, it follows then that $|f(x) - f(c)| < \varepsilon$. For all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Therefore if x is irrational, then the function is continuous. $\square$