Q1

(a) define $(\bar{u}(u,v), \bar{v}(u,v)) = \bar{x}^{-1} y(u,v)$

since $\bar{x}^{-1} y$ is a diffeomorphism, $\bar{u}$ and $\bar{v}$ are the unique such functions. Apply x to both sides

$$x(\bar{u}(u,v), \bar{v}(u,v)) = x \bar{x}^{-1} y(u,v)$$
$$= y(u,v)$$

as desired

(b) $\Rightarrow$ $y_u = x_{\bar{u}} \dfrac{\partial \bar{u}}{\partial u} + x_{\bar{v}} \dfrac{\partial \bar{v}}{\partial u}$ by the chain rule

$y_v = x_{\bar{u}} \dfrac{\partial \bar{u}}{\partial v} + x_{\bar{v}} \dfrac{\partial \bar{v}}{\partial v}$

(c) $\Rightarrow$ $y_u \times y_v = \left(x_{\bar{u}} \dfrac{\partial \bar{u}}{\partial u} + x_{\bar{v}} \dfrac{\partial \bar{v}}{\partial u}\right) \times \left(x_{\bar{u}} \dfrac{\partial \bar{u}}{\partial v} + x_{\bar{v}} \dfrac{\partial \bar{v}}{\partial v}\right)$

by distributivity & $x_{\bar{u}} \times x_{\bar{u}} = \vec{0}$, $x_{\bar{v}} \times x_{\bar{v}} = \vec{0}$

$$= \left(\frac{\partial \bar{u}}{\partial u} \cdot \frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{v}}{\partial u} \frac{\partial \bar{u}}{\partial v}\right) x_{\bar{u}} \times x_{\bar{v}}$$

$$= \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{u}}{\partial v} \\ \frac{\partial \bar{v}}{\partial u} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix} x_{\bar{u}} \times x_{\bar{v}}$$

$$= |J| \; x_{\bar{u}} \times x_{\bar{v}}$$

Q2

(a) let $x = (x_1, x_2, x_3)$

$\Rightarrow x_*(u_1) = (u_1[x_1], u_1[x_2], u_1[x_3])$

$= \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)$

$= \vec{x_u}$

same for $x_*(u_2)$.

(b) let $P \in M$ and $x(u_0, v_0) = P$

Define $\alpha(t) = x(u_0 + t, v_0)$

$\Rightarrow \frac{d\,\alpha(t)}{dt}\Big|_{t=0} = x_u(u_0, v_0)$

$\Rightarrow x_u[f](P) = \frac{d}{dt} f(x(u_0 + t, v_0))\Big|_{t=0}$

$= \frac{\partial}{\partial u} f(x(u,v))\Big|_{(u_0, v_0)}$

similar for $x_v[f]$

Q3

$$\alpha = x(\sqrt{2}t, e^t)$$

$$\Rightarrow \alpha'(t) = \frac{d}{dt}\left(x(\sqrt{2}t, e^t)\right)$$

$$= x_u(\sqrt{2}t, e^t) \cdot \frac{d\sqrt{2}t}{dt} + x_v(\sqrt{2}t, e^t) \cdot \frac{de^t}{dt}$$

$$= \sqrt{2}\, x_u(\sqrt{2}t, e^t) + e^t\, x_v(\sqrt{2}t, e^t)$$

(b)

$$x(u,v) = v(\cos u, \sin u, 1)$$

- $x_u = v(-\sin u, \cos u, 0) \Rightarrow x_u(\sqrt{2}t, e^t) = e^t(-\sin\sqrt{2}t, \cos\sqrt{2}t, 0)$

$$\Rightarrow \|x_u(\sqrt{2}t, e^t)\| = e^t$$

- $x_v = (\cos u, \sin u, 1) \Rightarrow x_v(\sqrt{2}t, e^t) = (\cos\sqrt{2}t, \sin\sqrt{2}t, 1)$

$$\Rightarrow \|x_v(\sqrt{2}t, e^t)\| = \sqrt{2}$$

$$\Rightarrow \alpha' \cdot \left(\frac{x_u}{\|x_u\|}\right) = (\sqrt{2}\, x_u + e^t\, x_v) \cdot \frac{x_u}{\|x_u\|}$$

$$= \sqrt{2}\frac{\|x_u\|^2}{\|x_u\|} + e^t \frac{x_v \cdot x_u}{\|x_u\|}$$

$$= \sqrt{2}e^t + 0 = \sqrt{2}e^t$$

$$\Rightarrow \alpha' \cdot \left(\frac{x_v}{\|x_v\|}\right) = (\sqrt{2}\, x_u + e^t\, x_v) \cdot \left(\frac{x_v}{\|x_v\|}\right)$$

$$= \sqrt{2}\frac{x_u \cdot x_v}{\|x_v\|} + e^t \frac{\|x_v\|^2}{\|x_v\|}$$

$$= 0 + e^t\sqrt{2} = \sqrt{2}e^t$$

Q4

(a) $u = t \Rightarrow du = dt$, $v = t^2 \Rightarrow dv = 2t\,dt$

$$\int_\alpha \phi = \int_{-1}^{1} (t^2)^2\,dt + 2(t)(t^2)\,2t\,dt$$

$$= \frac{5t^5}{5}\Big]_{-1}^{1} = 2$$

(b) want f, s.t. $f_u = v^2$, $f_v = 2uv$

$$\Rightarrow f = \int v^2\,du = uv^2 + g(v)$$

$$\Rightarrow 2uv = f_v = 2uv + g'(v)$$

$$\Rightarrow g'(v) = k \text{ a constant}$$

$$\Rightarrow f(u,v) = uv^2 + K$$

$\Rightarrow$ By F.T.L.I

$$\int_\alpha \phi = f(\alpha(1)) - f(\alpha(-1))$$

$$= (1)(1)^2 - (-1)(1)^2 = 2$$