Assignment 2.

There are total 20 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (but not over 115%).

This assignment covers sections 2.1, 2.2, 2.3 in Bartle–Sherbert.

(1) [2pt] Prove that there does not exist a rational number r such that $r^2 = 7$. (*Hint:* Go similarly to the proof about $r^2 = 2$, but use divisibility by 7 instead of divisibility by 2.)

(2) [5pt]
  (a) ($\sim$2.1.8a) Let x, y be rational numbers. Prove that $xy, x-y$ are rational numbers. (*Hint:* Start by writing $x = \frac{m}{n}$, $y = \frac{k}{l}$, where $m, n, k, l \in \mathbb{Z}$.)
  (b) (2.1.8b) Let x be a rational number, y an irrational number. Prove that $x + y$ is irrational. Prove that if, additionally, $x \neq 0$, then xy is irrational.
  (c) Let x, y be irrational numbers. Is it true that $x+y$ is always irrational? Is it true that $x + y$ is always rational? (*Hint:* No and No.)
  (d) Same question about xy.

(3) [4pt] (2.1.9) Let $K = \{s + t\sqrt{2} \mid s, t \in \mathbb{Q}\}$. Show that K satisfies the following:
  (a) If $x_1, x_2 \in K$ then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.
  (b) If $x \neq 0$ and $x \in K$ then $1/x \in K$. (*Hint:* Get rid of irrationality in the denominator.)

  Comment. In other words, K is a subfield of $\mathbb{R}$.

(4) [2pt] (2.2.4) Let $a, x \in \mathbb{R}$ and $\varepsilon > 0$. Show that $|x - a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$. (*Hint:* Don't forget that "A if and only if B" means "(if A then B) AND (if B then A)".)

(5) [3pt]
  (a) Let $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.
  (b) (2.3.4) Let $T = \{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$. Find $\inf T$ and $\sup T$. (*Hint:* If you are not sure what's going on, try to draw this set to get an idea.)

(6) (a) [2pt] (Part of 2.3.11) Let $S \subset \mathbb{R}$ be a bounded above set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \leq \sup S$. (*Hint:* Follow the definition.)
  (b) [2pt] (2.3.10) Show that if A and B are bounded above nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded above set and $\sup A \cup B = \sup\{\sup A, \sup B\}$. (*Hint:* Follow the definition.)