Assignment 9

There are total 22 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 5.1, 5.2, and partially 5.3 in Bartle–Sherbert.

(1) [2pt] (5.1.7+) (Local separation from zero) Let $A \subseteq \mathbb{R}$, $c \in A$, $f : A \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that for any $\alpha \in \mathbb{R}$ such that $0 < \alpha < f(c)$, there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c) \cap A$, then $f(x) > \alpha$.

(2) [3pt] (5.1.13) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for $x \in \mathbb{Q}$ and $g(x) = x + 3$ for $x \notin \mathbb{Q}$. Find all points at which g is continuous.

(3) [2pt] (Exercise 5.2.5) Let g be defined on $\mathbb{R}$ and by $g(1) = 0$, and $g(x) = 2$ if $x \neq 1$, and let $f(x) = x+1$ for all $x \in \mathbb{R}$. Show that $\lim\limits_{x\to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Composition of Continuous Functions Theorem (Theorem 5.2.6)?

(4) [3pt] (5.2.6) Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. suppose that $\lim\limits_{x\to c} f = b$ and that g is continuous at b. Show that $\lim\limits_{x\to c} g(f(x)) = g(b)$.
(*Hint:* (Re)define f to be b at c, apply composition of continuous functions.)
NOTE. This statement says that lim and a *continuous* function can be swapped: $\lim\limits_{x\to c} g(f(x)) = g(\lim\limits_{x\to c} f(x))$. The previous exercise shows that continuity of g is essential.

(5) [2pt] (5.2.7) Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

(6) (a) [2pt] (Exercise 5.1.12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Show that $f(x) = 0$ at every point $x \in \mathbb{R}$.
(b) [2pt] (Exercise 5.2.8) Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. (*Hint:* Consider $f - g$.)

(7) [2pt] (5.3.1) Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous on I function such that $f(x) > 0$ for all $x \in I$. Prove that there is a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

(8) [4pt] (5.3.13) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to -\infty} f = \lim\limits_{x\to +\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or a minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained. (*Hint:* Pick M large enough and inspect how f behaves on the interval $[-M, M]$, and on $\mathbb{R} \setminus [-M, M]$.)