MA 441, Fall 2021

Assignment 5.

Solutions.

There are total 23 points in this assignment. 20 points is considered 100%. If you go over 20 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

This assignment covers sections 3.2–3.3 in Bartle–Sherbert.

1. Basic properties of limit. Squeeze theorem

(1) Find limits using Squeeze Theorem:

(a) [2pt] $\lim_{n\to\infty} \frac{n^2+2015n(\sin n+3\cos n^7)-1}{2n^2-\cos(3n^2+1)}$,

$\triangleright$ Note that by properties of sin and cos, and by triangle inequality,

$-4 \le \sin n + 3\cos n^7 \le 4 \quad \text{and} \quad -1 \le \cos(3n^2+1) \le 1,$

so

$$\frac{n^2 - 4\cdot 2015n - 1}{2n^2+1} \le \frac{n^2 + 2015n(\sin n + 3\cos n^7) - 1}{2n^2 - \cos(3n^2+1)} \le \frac{n^2 + 4\cdot 2015n - 1}{2n^2-1}.$$

Note that

$$\lim_{n\to\infty} \frac{n^2 - 4\cdot 2015n - 1}{2n^2+1} = \lim_{n\to\infty} \frac{1 - 4\cdot 2015/n - 1/n^2}{2 + 1/n^2} = 1/2.$$

Similarly, the limit on RHS is also 1/2. Therefore by Squeeze theorem, the original limit exists and is equal to 1/2.

(b) [2pt] $\lim_{n\to\infty} \sqrt{n^2+\cos(2014n+1)} - \sqrt{n^2-\sin(n^3-1)}$. (*Hint:* Once you get rid of sin and cos by Squeeze Theorem, multiply and divide by the conjugate, $\sqrt{} + \sqrt{}$.)

$\triangleright$ Note that

$$\sqrt{n^2-1} - \sqrt{n^2+1} \le \sqrt{n^2+\cos(2014n+1)} - \sqrt{n^2-\sin(n^3-1)} \le \sqrt{n^2+1} - \sqrt{n^2-1}.$$

Further, note that

$$\lim_{n\to\infty} \sqrt{n^2-1} - \sqrt{n^2+1} = \lim_{n\to\infty} \frac{\left(\sqrt{n^2-1} - \sqrt{n^2+1}\right)\left(\sqrt{n^2-1} + \sqrt{n^2+1}\right)}{\sqrt{n^2-1} + \sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{-2}{\sqrt{n^2-1} + \sqrt{n^2+1}} = \lim_{n\to\infty} \frac{-2/n}{\sqrt{1-1/n^2} + \sqrt{1+1/n^2}} = \frac{0}{2} = 0.$$

Similarly, the limit of RHS is also 0. By Squeeze theorem, the limit of the original sequence exists and is equal to 0.

(2) (a) [3pt] (Example 3.1.11d) Prove that $n^{1/n} \to 1$ $(n\to\infty)$.

$\triangleright$ (The below proof follows textbook.) Let $n^{1/n} = 1 + k_n$. (Note that $k_n \ge 0$.) We have $(1+k_n)^n = n$. By Binomial Theorem,

$$n = (1+k_n)^n \ge 1 + nk_n + \frac{n(n-1)}{2}k_n^2 \ge 1 + \frac{n(n-1)}{2}k_n^2,$$

so $k_n^2 \geq 2/n$ for $n \geq 2$. Let $\varepsilon > 0$ be given. Put $K > 2/\varepsilon^2$ and $K \geq 2$. Then if $n > K$, we get

$$|n^{1/n} - 1| = k_n \leq \sqrt{2/n} < \sqrt{2/K} < \sqrt{2/(2/\varepsilon^2)} = \varepsilon.$$

(b) [2pt] (3.2.14a) Use Squeeze theorem to find limit of the sequence (n^{1/n^2}).

$\triangleright$ Note that $1 \leq n^{1/n^2} \leq n^{1/n}$, and $\lim\limits_{n\to\infty} 1 = \lim\limits_{n\to\infty} 1^{1/n} = 1$, so by squeeze theorem, $\lim\limits_{n\to\infty} n^{1/n^2} = 1$ as well.

(3) [2pt] (3.2.8) Find a mistake in the following argument.
"Find $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n$ as shown below:

$$\begin{aligned}\lim_{n\to\infty}(1+\tfrac{1}{n})^n &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n})\\ &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot\lim_{n\to\infty}(1+\tfrac{1}{n})\cdots\lim_{n\to\infty}(1+\tfrac{1}{n})\\ &= \left(\lim_{n\to\infty}(1+\tfrac{1}{n})\right)^n = 1^n = 1.\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = 1$."
Comment. To reiterate, the argument above is erroneous and the obtained value of the limit is wrong, too. The limit is actually equal to e, as shown in class.

$\triangleright$ Conider the second equality

$$\lim_{n\to\infty}(1+\tfrac{1}{n})\cdot(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n}) = \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot\lim_{n\to\infty}(1+\tfrac{1}{n})\cdots\lim_{n\to\infty}(1+\tfrac{1}{n}).$$

In the right hand side, the number of factors is undefined (as it depends on n) so the equality does not make sense.

Alternatively, we can say that the mistake is that it is not specified to which particular sequences the rule about product of sequences is applied.

2. Monotone Convergence Theorem

(4) [3pt] (3.3.2) Let $x_1 > 1$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone, hence convergent. Find the limit.

$\triangleright$ Note that $x_n > 1$ implies $x_{n+1} = 2 - 1/x_n > 2 - 1/1 = 1$. Since $x_1 > 1$, we have $x_n > 1$ for all $n \in \mathbb{N}$.

But then $x_{n+1} - x_n = 2 - 1/x_n - x_n = -\frac{1}{x_n}(x_n^2 - 2x_n + 1) = -\frac{1}{x_n}(x_n - 1)^2 \leq 0$, so the sequence is decreasing (and bounded above by x_1, below by 1).

By monotone convergence theorem, there is $\lim(x_n) = x$. Taking limits of both sides of the equality $x_{n+1} = 2 - 1/x_n$, we get $x = 2 - 1/x$ (note that $x \geq 1 > 0$ since $x_n > 1$, so we can use the rule about division), so $x = 1$.

(5) [2pt] Find a mistake in the following argument:

"Let (x_n) be a sequence given by $x_1 = 1$, $x_{n+1} = 1 - x_n$. In other words, $(x_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Show that $\lim(x_n) = 0.5$. Indeed, let

$\lim(x_n) = x$. Apply limit to both sides of equality $x_{n+1} = 1 - x_n$:

$$\begin{aligned} \lim(x_{n+1}) &= \lim(1 - x_n) \\ \lim(x_{n+1}) &= 1 - \lim(x_n) \\ x &= 1 - x, \end{aligned}$$

so $x = 0.5$"

$\triangleright$ Equalities involving limits $\lim(x_{n+1})$ and $\lim(x_n)$ don't make sense until we prove that the limits exist. (Further, we cannot use that $\lim(A + B) = \lim A + \lim B$, unless we prove that limits $\lim A$ and $\lim B$ exist.)

(6) [3pt] (3.3.11) Establish convergence or divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbb{N}.$$

$\triangleright$ We try to use monotone convergence theorem. First find $y_{n+1} - y_n$ to see if (y_n) is monotone:

$$\begin{aligned} y_{n+1} - y_n &= \\ \left(\tfrac{1}{n+2} + \tfrac{1}{n+3} + \cdots + \tfrac{1}{2n} + \tfrac{1}{2n+1} + \tfrac{1}{2(n+1)}\right) &- \\ -\left(\tfrac{1}{n+1} + \tfrac{1}{n+2} + \tfrac{1}{n+3} + \cdots + \tfrac{1}{2n}\right) &= \\ = -\frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2(n+1)} &\geq \\ \geq -\frac{1}{n+1} + \frac{1}{2(n+1)} + \frac{1}{2(n+1)} &= 0. \end{aligned}$$

So $y_{n+1} - y_n \geq 0$ for all $n \in \mathbb{N}$, so (y_n) is increasing.
Second, show that y_n is bounded.

$$\begin{aligned} 0 \leq y_n &= \tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{2n} \leq \\ &\leq \tfrac{1}{n+1} + \tfrac{1}{n+1} + \cdots + \tfrac{1}{n+1} = \\ &= n \cdot \tfrac{1}{n+1} < 1. \end{aligned}$$

By monotone convergence theorem, (y_n) converges.
(**Note:** we didn't prove that $\lim(y_n) = 1$. Moreover, it is not true.)

(7) (a) [2pt] (Exercise 3.3.12) Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$, $n \in \mathbb{N}$. Prove that that (x_n) converges. (*Hint*: for $k \geq 2$, $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$.)

$\triangleright$ $x_n = x_{n-1} + 1/n^2$ is an increasing sequence (therefore, bounded below). Show that it's bounded above.

$$\begin{aligned} x_n &= \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \cdots + \tfrac{1}{n^2} \leq \\ &\leq 1 + \tfrac{1}{2\cdot 1} + \tfrac{1}{3\cdot 2} + \cdots + \tfrac{1}{n(n-1)} = \\ &= 1 + \left(\tfrac{1}{1} - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \left(\tfrac{1}{n-1} - \tfrac{1}{n}\right) = \\ &= 1 + 1 - 1/n < \\ &< 2. \end{aligned}$$

By monotone convergence theorem, (x_n) converges.
(By the way, $\lim(x_n)$ is **not** 2. In fact, $\lim(x_n) = \pi^2/6 < 1.7$, but that's a totally different story.)

(b) [2pt] Let K be a natural number $K \geq 2$. Let $y_n = \frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \cdots + \frac{1}{n^K}$, $n \in \mathbb{N}$. Prove that that (y_n) converges. (*Hint*: compare y_n to x_n.)

$\triangleright$ (y_n) is an increasing sequence and $0 < y_n \leq x_n \leq 2$, so (y_n) is bounded. Therefore by monotone convergence theorem, (y_n) converges.