Homework 1: Asymptotic Notations

Arjun Koshal

September 27, 2022

Problem 1

```
for i = 0 to A.length - 1 do // runs n times with cost: c1
  if A[i] == A[i+1] then // runs n-1 times with cost: c2
    return 1 // runs 1 time with cost: c3
  end if
end for
return 0
```

Best Case: The first two elements in the array are duplicates. $T(n) = (c_1 + c_2 + c_3) = b$, for some constant b. Therefore, $T(n) = O(1)$ is the best case.

Worst Case: There would be no duplicates within the array. $T(n) = c_1(n) + c_2(n-1) + c_3 + c_4 = (c_1 + c_2)n - c_2 + c_3 + c_4 = an + b$, for some constants a and b. Therefore, $T(n) = O(n)$ is the worst case.

Problem 2

```
i = NIL // runs 1 time with cost c1
for j = 1 to A.length do // runs n-1 times with cost c2
  if A[j] = v then // runs n-2 times with cost c3
    i = j // runs 1 time with cost c4
    return i // runs 1 time with cost c5
  end if
end for
return i // runs 1 time with cost c6
```

Best Case: The first element j, satisfies $A[j] = v$. $T(n) = c_1 + c_2 + c_3 + c_4 + c_5 = b$, for some constant b. Therefore, $T(n) = O(1)$ is the best case.

Worst Case: There would be no element j such that $A[j] = v$. $T(n) = c_1 + c_2(n-1) + c_3(n-2) + c_4 + c_5 + c_6 = c_1 + (c_2 + c_3)n - c_2 - 2c_3 + c_6 = (c_2 + c_3)n + (c_1 - c_2 - 2c_3 + c_6) = an + b$, for some constants a and b. Therefore, $T(n) = O(n)$ is the worst case.

Problem 3

$$f(n) = n^4 + 10n^2 + 5$$
$$cn^4 \geq n^4 + 10n^2 + 5$$

We can see that the smallest integral value for the equation above to hold true must be $c = 2$.

$$2n^4 \geq n^4 + 10n^2 + 5$$
$$512 \geq 256 + 160 + 5$$
$$n_0 = 4$$

Problem 4

$$f(n) = 3n^3 - 2n$$
$$c_2n^3 \geq 3n^3 - 2n$$

We can see that the tightest integral value for the equation above to hold true must be $c_2 = 3$.

$$c_1n^3 \leq 3n^3 - 2n$$

We can see that the tightest integral value for the equation above to hold true must be $c_1 = 2$.

$$3n^3 \geq 3n^3 - 2n$$
$$2n^3 \leq 3n^3 - 2n$$

It then follows that the tightest integral value that satisfies both the lower and upper bounds is $n_0 = 2$.

Problem 5

Theorem 1. $3n - 4 \in \Omega(n^2)$

Proof. Let us disprove this statement using proof by contradiction. Suppose that $3n - 4 \in \Omega(n^2)$. Then, it follows

$$3n - 4 \geq cn^2$$

Because $4n > 4$, we can express the equation as

$$3n - 4n \geq cn^2$$
$$-n \geq cn^2$$
$$-\frac{1}{c} \geq n$$

Since $c > 0$, it is impossible for $-\frac{1}{c} \geq n$ to hold true. It follows then that $3n - 4 \in \Omega(n^2)$ is false. □

Problem 6

Function 1: $\Theta(n\lg n)$
Function 2: $\Theta(\sqrt[3]{n})$
Function 3: $\Theta(n^3)$
Function 4: $\Theta(n)$
Function 5: $\Theta(n)$

Problem 7

a. $\Theta(n^2)$ because there are 2 nested for loops that iterate linearly.

b. $\Theta(n)$ because the first for loop is not bounded by a variable, so there is a constant number of operations in the outer for loop, whereas he inner for loop iterates linearly.

c. $\Theta(m\lg n)$ because the outer loop iterates linearly and the inner loop iterates lg times.

d. $\Theta(m^2)$ because the outer for loop is bounded by $m * m$, which is greater than the inner for loop that is bounded by m.