Q1

(a) ϕ is closed $\Rightarrow d\phi = 0$

$$\Rightarrow \int_{\partial\alpha} \phi \underset{\uparrow}{=} \int_{\alpha} d\phi = 0$$

By Stokes' Thm

(b) ϕ is exact $\Rightarrow \exists$ a function f s.t. $df = \phi$

$\Rightarrow$ By assumption α is closed, i.e. the terminal point Q, is the same as the initial point P, $P = Q$.

$$\int_{\alpha} \phi = \int_{\alpha} df = f(Q) - f(P) = 0$$

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Q2

(a) $$d\psi = \frac{\partial}{\partial u}\frac{u}{u^2+v^2}\,du\,dv - \frac{\partial}{\partial v}\frac{v}{u^2+v^2}\,dv\,du$$

$$= \frac{u^2+v^2-2u^2}{(u^2+v^2)^2}\,du\,dv - \frac{u^2+v^2-2v^2}{(u^2+v^2)^2}\,dv\,du$$

$$= \frac{u^2+v^2-2u^2+u^2+v^2-2v^2}{(u^2+v^2)^2}\,du\,dv = 0$$

If it is exact $\int_\alpha \psi = 0$, where

$\alpha: u^2+v^2=1 \Rightarrow$ set $u=\cos t$, $v=\sin t$

$du=-\sin t\,dt$, $dv=\cos t\,dt$

$$\Rightarrow \int_\alpha \psi = \int_0^{2\pi} \frac{\cos^2 t+\sin^2 t}{1}\,dt = 2\pi \neq 0$$

not exact

(b) Assume $u>0$

we look for f, s.t.

$$f_u = \frac{-v}{u^2+v^2}, \qquad f_v = \frac{u}{u^2+v^2}$$

Assuming $u>0$

$$\int f_v\,dv = u\int \frac{1}{u^2+v^2}\,dv$$

$$= \frac{u}{u^2}\int \frac{1}{1+\left(\frac{v}{u}\right)^2}\,dv$$

$$= \frac{1}{u}\,u\arctan\left(\frac{v}{u}\right) + g(u)$$

$$= \arctan\left(\frac{v}{u}\right) + g(u)$$

$$\Rightarrow f_u = -u^{-2}v\frac{1}{\dots\left(\frac{v}{u}\right)^2} + g'(u)$$

$$\Rightarrow \quad f_u = -u^{-2}v \frac{1}{1+\left(\frac{v}{u}\right)^2} + g'(u)$$

$$= -\frac{u^{-2}v}{u^{-2}} \frac{1}{u^2+v^2} + g'(u)$$

$$= \frac{-v}{u^2+v^2} + g'(u) \Rightarrow g'(u) = K$$

$$\Rightarrow \quad f = \arctan\left(\frac{v}{u}\right) + K \quad \text{is s.t.}$$

$$df = \psi \quad \text{if} \quad u > 0$$

Q3

Let $f = f(x,y) - z \Rightarrow \nabla f = \langle f_x, f_y, -1\rangle$

normal

$\Rightarrow \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{1}{\sqrt{f_x^2+f_y^2+1}}\langle f_x, f_y, -1\rangle = u$

However $S_0(u_1) = f_{xx}(0)\, u_1 + f_{yx}(0)\, u_2$

$S_0(u_2) = f_{xy}(0)\, u_1 + f_{yy}(0) u_2$

(a) $f = z = xy \Rightarrow f_x = y \Rightarrow \begin{cases} f_{xx} = 0 \\ f_{xy} = 1 \end{cases}$

$f_y = x \Rightarrow \begin{cases} f_{yx} = 1 \\ f_{yy} = 0 \end{cases}$

$\Rightarrow S_0(u_1) = u_2 \qquad S_0(u_2) = u_1$

$\Rightarrow S_0(au_1 + bu_2) = a\, S_0(u_1) + b\, S_0(u_2)$
$= bu_1 + au_2$

$S_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow$ rank 2

(c) $z = f = (x+y)^2$

$f_x = 2(x+y) \Rightarrow \begin{cases} f_{xx} = 2 \\ f_{xy} = 2 \end{cases}$

$f_y = 2(x+y) \Rightarrow \begin{cases} f_{yy} = 2 \\ f_{yx} = 2 \end{cases}$

$\Rightarrow S_0(u_1) = 2u_1 + 2u_2$

$S_0(u_2) = 2u_1 + 2u_2$

$$\Rightarrow \quad S_0 (a u_1 + b u_2) = 4 a u_1 + 4 b u_2$$

$$S_0 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \leftarrow \text{rank } 1$$