Assignment 10

There are total 20 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%). Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 5.3, the end of 5.6, and a part of 6.1 in Bartle–Sherbert.

1. Continuous Functions

(1) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.

(2) [2pt] (5.3.6) Let f be continuous on the interval $[0,1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.)
Note. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.

(3) (a) [3pt] (5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Location of Roots Theorem.)

(b) [1pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)

The next two problems are required for the introduction of rational power functions. In particular, in solving them you cannot use properties of rational powers (doing so would be a vicious circle), but rather only the definition of n-th root function (i.e., that for $x \geq 0$, $n \in \mathbb{N}$ we have $(x^{1/n})^n = (x^n)^{1/n} = x$), and the in-class statement that $(x^{1/n})^m = (x^m)^{1/n}$.

(4) (a) [1pt] Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $q \in \mathbb{N}$, and let $x \in \mathbb{R}$. Show that

$$(x^n)^{\frac{1}{nq}} = x^{\frac{1}{q}}.$$

(*Hint:* Denote LHS by y, show that $y^q = x$ using Problem (**??**).)

(b) [2pt] (~Def. 5.6.6) Let $m, p \in \mathbb{Z}$, $n, q \in \mathbb{N}$, and $x \in \mathbb{R}$, $x > 0$. Show that if $\frac{m}{n} = \frac{p}{q}$, then

$$(x^{1/n})^m = (x^{1/q})^p.$$

(*Hint:* Extract root nq from the equality $x^{mq} = x^{np}$. Use (4a).)
Comment. This problem explains that the function x^r given by $x^r = (x^{1/n})^m$ for $r = m/n \in \mathbb{Q}$ is well-defined.

(5) Prove that if $x > 0$ and $r, s \in \mathbb{Q}$, then

$$x^r x^s = x^{r+s} \text{ and } (x^r)^s = x^{rs}.$$

(*Hint:* Raise the equalities to the power equal to a common denominator of the involved fractions.)

— see next page —

2. The Derivative

Recall that we say that a function $f : I \to \mathbb{R}$ is differentiable at a point c of the interval I if there exists limit

$$lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

In such case the value $f'(c)$ of the limit is called the derivative of f at c. Since we will have couple more definitions of the derivative later, we will refer to this one as the "limit of ratio" definition.

(6) [4pt] (Part of 6.1.1) Use the "limit of ratio" definition to find derivative of each of the following functions:
  (a) $f(x) = x^2$, $x \in \mathbb{R}$.
  (b) $f(x) = x^3$, $x \in \mathbb{R}$.
  (c) $f(x) = 1/\sqrt{x}$, $x > 0$.
  (d) ($\sim$6.1.2) Show that $f(x) = x^{1/7}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

(7) [3pt] Using the "limit of ratio" definition of the derivative, establish whether the following functions are differentiable at 0. In the case of positive answer, find the derivative at 0.
  (a) $f(x) = x \sin(1/x)$.
  (b) $g(x) = x^2 \sin(1/x)$.