Homework 4

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Problem 1

Theorem. *For all $n \in \mathbb{N}$, $F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}$.*

Proof. **Proposition :** Let P(n) be the statement $F_1^2 + F_2^2 + ... + F_n^2 = F_n F_{n+1}$.

Base Case : Show that P(1) is true.

$$\begin{aligned} F_1^2 &= F_n F_{n+1} \\ 1^2 &= 1 \cdot 1. \\ 1 &= 1. \end{aligned}$$

Inductive Hypothesis : Assume that the statement holds when $n = k$.
Let P(k) be the statement $F_1^2 + F_2^2 + ... + F_k^2 = F_k F_{k+1}$.

Inductive Step : Show that for any $k \geq 1$, if P(k) holds, then P($k+1$) also holds.

$$\begin{aligned} F_1^2 + F_2^2 + ... + F_k^2 + F_{k+1}^2 &= F_{k+1} F_{k+2} \\ F_1^2 + F_2^2 + ... + F_k^2 + F_{k+1}^2 &= F_{k+1}(F_k + F_{k+1}) \\ F_1^2 + F_2^2 + ... + F_k^2 + F_{k+1}^2 &= (F_k F_{k+1}) + (F_{k+1} F_{k+1}) \\ F_1^2 + F_2^2 + ... + F_k^2 + F_{k+1}^2 &= (F_k F_{k+1}) + (F_{k+1}^2). \end{aligned}$$

Thus, the statement P(n) holds for every $n \in \mathbb{N}$. □

Theorem. *Let S_n, where $n \in \mathbb{N}$, be the set of all n-digit binary strings that have no consecutive 1s. S_n contains exactly F_{n+2} elements.*

Proof. We can look at the set of binary strings at each level n where we categorize them by whether they begin by 0 or 1. If S_1 begins with 0, then we have 0, and if S_1 begins with 1, then we have 1. If S_2 begins with 0, then we have 00 and 01, and if S_2 begins with 1, then we have 10. We see that the statement holds true for S_1 and S_2, which are the base cases. Looking at S_3, if S_3 begins with 0, we have 000, 001, 010, and if S_3 begins with 1, we have 100, 101. For S_4, if S_4 begins with 0, we have 0000, 0001, 0010, 0100, 0101 and if S_4 begins with 1, we have 1000, 1001, 1010. We notice that from S_3 and onwards that the next level of strings beginning with a 1 can be made by adding a 1 in front of the strings beginning with a 0 on the previous level. This is due to the fact that since the strings on the previous level begin with only 0s, if we add a 1 in front, then it is impossible for the string to have consecutive $1s$. If we add a 0 to every string (both beginning with $0s$ and beginning with $1s$) in the n level (where $n \geq 3$), it holds true that it is impossible for the string to have consecutive $1s$, thus leading to the fact that for strings beginning with 0s in $n+1$ level will share the same number of elements as every string (both beginning with $0s$ and beginning with $1s$) in the n level. Considering the recursive pattern, we notice that the level of strings beginning with 1 have a magnitude that follows the fibonacci sequence F_n and the level of strings beginning with 0 have a magnitude that follows the fibonacci sequence F_{n+1}. Therefore for any level n of S_n, it follows that the magnitude of the set will always be $|F_n| + |F_{n+1}|$, which leads to $|S_n| = F_{n+2}$. □

Problem 2

Theorem. *No matter how straight lines on a piece of paper are drawn, it is always possible to color each region in two colors such that no two adjacent regions have the same color.*

Proof. **Proposition :** Let P(n) be the statement that the regions formed by any set of n distinct lines can be colored either red or blue in such a way that no two adjacent regions share the same color.

Base Case : Show that P(0) is true.

When $n = 0$, there are 0 distinct lines so there exists only one region, and thus no adjacent regions exist. Therefore we can color this region with either red or blue and still satisfy our hypothesis.

Inductive Hypothesis : Assume that the statement holds when $n = k$.
Let P(k) be the statement that the regions formed by any set of k distinct lines can be colored either red or blue in such a way that no two adjacent regions share the same color

Inductive Step : Show that for any $k \geq 0$, if P(k) holds, then P($k+1$) also holds.

When you add a new line $k+1$, you split existing regions the line passes through into two adjacent regions, now of the same color. From here, half of the adjacent regions must swap color to maintain color alternation, and this then affects other regions adjacent to these, further away from the line. Once the colors of all regions on one side are inverted, all adjacent regions on either side are still painted in different colors, and now newly created adjacent regions (which share the same side, which is a part of the new line) are opposite colors as well, because one of them was inverted. Thus, this statement P(n) holds for every $n \in \mathbb{N}$. □