Assignment 10

Solutions

There are total 20 points in this assignment. 18 points is considered 100%. If you go over 18 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

This assignment covers section 5.3, the end of 5.6, and a part of 6.1 in Bartle–Sherbert.

1. Continuous Functions

(1) [2pt] (Part of 5.3.5) Show that the polynomial $x^4 + 7x^3 - 9$ has at least two real roots.

$\triangleright$ Note that $f(-10) > 10000 - 8000 > 0$, $f(0) = -9 < 0$, and $f(10) > 10000 - 9 > 0$. By the Location of roots theorem (or by the Bolzano's Intermediate value theorem), there is a root of f on the interval $(-10, 0)$ and a root of f on the interval $(0, 10)$. Since these two intervals do not have any common points, the two roots are distinct.

(2) [2pt] (5.3.6) Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. (*Hint:* Consider $g(x) = f(x) - f(x + \frac{1}{2})$.)
Note. Therefore, there are, at any time, antipodal points on the earth's equator that have the same temperature.

$\triangleright$ Consider $g : [0, 1/2] \to \mathbb{R}$ given by $g(x) = f(x) - f(x + \frac{1}{2})$. Note that $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = f(1/2) - f(0) = -g(0)$. So $g(0)$ and $g(1/2)$ have opposite signs (or both zero), so by the Location of roots theorem (or by the Bolzano's Intermediate value theorem), there is a point $c \in [0, 1/2]$ such that $g(c) = 0$, i.e. $f(c) = f(c + 1/2)$.

(3) (a) [3pt] (5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Location of Roots Theorem.)

$\triangleright$ Suppose $f(w) \neq 0$. Then either $f(w) > 0$, or $f(w) < 0$.
Let $f(w) > 0$. Then by Problem 1 of HW9 (or by definition of continuity), there is a neighborhood $(w - \delta, w + \delta)$ of w where f is positive. But then $w - \delta/2$ is an upper bound of W, so w is not the least upper bound.
Let $f(w) < 0$. Then by the same reason, there is a neighborhood $(w - \delta, w + \delta)$ of w where f is negative. But then $w + \delta/2 \in W$, so w is not an upper bound of W.
Both options lead to a contradiction, so $f(w)$ must be 0.

(b) [1pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)

$\triangleright$ W can be empty, so its sup is not defined.

The next two problems are required for the introduction of rational power functions. In particular, in solving them you cannot use properties of rational powers (doing so would be a vicious circle), but rather only the definition of n-th root function (i.e., that for $x \geq 0$, $n \in \mathbb{N}$ we have $(x^{1/n})^n = (x^n)^{1/n} = x$), and the in-class statement that $(x^{1/n})^m = (x^m)^{1/n}$.

(4) (a) [1pt] Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $q \in \mathbb{N}$, and let $x \in \mathbb{R}$. Show that

$$(x^n)^{\frac{1}{nq}} = x^{\frac{1}{q}}.$$

$\triangleright$ Let $(x^n)^{\frac{1}{nq}} = y$. Then using the in-class statement that $(x^{1/n})^m = (x^m)^{1/n}$ we get

$$y^q = ((x^n)^{\frac{1}{nq}})^q = ((x^n)^q)^{\frac{1}{nq}} = (x^{nq})^{\frac{1}{nq}} = x,$$

which precisely means $y = x^{1/q}$, as required.

(b) [2pt] ($\sim$Def. 5.6.6) Let $m, p \in \mathbb{Z}$, $n, q \in \mathbb{N}$, and $x \in \mathbb{R}$, $x > 0$. Show that if $\frac{m}{n} = \frac{p}{q}$, then

$$(x^{1/n})^m = (x^{1/q})^p.$$

(*Hint:* Extract root nq from the equality $x^{mq} = x^{np}$. Use (4a).)
Comment. This problem explains that the function x^r given by $x^r = (x^{1/n})^m$ for $r = m/n \in \mathbb{Q}$ is well-defined.

$\triangleright$ Observe that $\frac{m}{n} = \frac{p}{q}$ implies $mq = np$. As the Hint suggest, extract root nq from the equality $x^{mq} = x^{np}$. Using Problem (4a) to split nq-th root into a composition of n-th and q-th roots, we get

$$(x^{mq})^{1/nq} = \left(((x^m)^q)^{\frac{1}{q}}\right)^{\frac{1}{n}} = (x^m)^{\frac{1}{n}}.$$

By the in-class statement, the latter is equal to $(x^{1/n})^m$.
Similarly, we get

$$(x^{np})^{1/nq} = (x^{1/q})^p.$$

Since $x^{mq} = x^{np}$, we get $(x^{1/n})^m = (x^{1/q})^p$.

(5) Prove that if $x > 0$ and $r, s \in \mathbb{Q}$, then

$$x^r x^s = x^{r+s} \text{ and } (x^r)^s = x^{rs}.$$

(*Hint:* Raise the equalities to the power equal to a common denominator of the involved fractions.)

$\triangleright$ Let $r = m/n$ and $s = p/q$. Using the above statements, we obtain

$$x^r x^s = (x^{1/n})^m (x^{1/q})^p = (x^{1/nq})^{mq} (x^{1/nq})^{np} = (x^{1/nq})^{mq+np} = x^{r+s}.$$

The same can be seen by raising $x^r x^s$ and x^{r+s} to the power nq.
For the second equality, we have

$$(x^r)^s = ((x^{1/n})^m)^{1/q})^p = ((x^{1/n})^{1/q})^m)^p = (x^{1/nq})^{mp} = x^{rs}.$$

The same can be seen by raising $(x^r)^s$ and x^{rs} to the power nq.

2. The Derivative

(6) [4pt] (Part of 6.1.1) Use the "limit of ratio" definition to find derivative of each of the following functions:

(a) $f(x) = x^2$, $x \in \mathbb{R}$.

$\triangleright$ Compute derivative at the point c:

$$\frac{x^2 - c^2}{x - c} = \frac{(x-c)(x+c)}{x-c} = x + c \to 2c \text{ as } x \to c.$$

(b) $f(x) = x^3$, $x \in \mathbb{R}$.

$\triangleright$ Compute derivative at the point c:

$$\frac{x^3 - c^3}{x - c} = \frac{(x-c)(x^2 + xc + c^2)}{x-c} = x^2 + xc + c^2 \to 3c^2 \text{ as } x \to c.$$

(c) $f(x) = 1/\sqrt{x}$, $x > 0$.

$\triangleright$

$$\frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{c}}}{x-c} = \frac{\frac{\sqrt{c}-\sqrt{x}}{\sqrt{xc}}}{x-c} = \frac{\frac{(\sqrt{c}-\sqrt{x})(\sqrt{c}+\sqrt{x})}{\sqrt{xc}(\sqrt{c}+\sqrt{x})}}{x-c} = \frac{x-c}{\sqrt{xc}(\sqrt{c}+\sqrt{x})(x-c)} \to \frac{1}{2c\sqrt{c}}$$

as $x \to c$.

(d) ($\sim$6.1.2) Show that $f(x) = x^{1/7}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

$\triangleright$

$$\lim_{x\to 0} \frac{x^{1/7} - 0}{x - 0} = \lim_{x\to 0} \frac{1}{x^{6/7}} = +\infty.$$

(7) [3pt] Using the "limit of ratio" definition of the derivative, establish whether the following functions are differentiable at 0. In the case of positive answer, find the derivative at 0.

(a) $f(x) = x\sin(1/x)$.

$\triangleright$ We have $\lim\limits_{x\to 0} \frac{x\sin(1/x)-0}{x-0} = \lim\limits_{x\to 0} \sin(1/x)$. This limit does not exist, which can be seen by taking $x_n = 1/(\pi n + \frac{\pi}{2})$. Thefore, f is not differentiable at 0.

(b) $g(x) = x^2\sin(1/x)$.

$\triangleright$ $g'(0) = \lim\limits_{x\to 0} \frac{x^2\sin(1/x)-0}{x-0} = \lim\limits_{x\to 0} x\sin(1/x)$. The latter function is squeezed between $|x|$ and $-|x|$, so the limit and, therefore, the value of the derivative at 0 is 0.