MA 232 - Linear Algebra

Homework 3 - Solutions

Problem 1 [15 pts]

Construct a matrix whose nullspace consists of all combinations of $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

pivot free

Special solution 1

set $x_3 = 1$, $x_4 = 0$

$$S_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Special solution 2

set $x_3 = 0$, $x_4 = 1$

$$S_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 2 [15 pts]

Construct a matrix whose column space contains $\begin{bmatrix}1\\1\\5\end{bmatrix}$ and $\begin{bmatrix}0\\3\\1\end{bmatrix}$ and whose nullspace contains $\begin{bmatrix}1\\1\\2\end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \quad \text{we must have} \quad A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0$$

Hence,
$$\left.\begin{aligned} 1 + 2a &= 0 \\ 1 + 3 + 2b &= 0 \\ 5 + 1 + 2c &= 0 \end{aligned}\right\} \quad \begin{aligned} a &= -\frac{1}{2} \\ b &= -2 \\ c &= -3 \end{aligned}$$

Thus,
$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

Problem 3 [10 pts]

Let $u_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $u_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $u_4 = \begin{bmatrix}2\\3\\4\end{bmatrix}$. Show that u_1, u_2, u_3 are independent but u_1, u_2, u_3, u_4 are dependent.

(i) u_1, u_2, u_3 are independent iff $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$. has three pivots. Indeed,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) A 3×4 matrix always has dependent columns. $\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}$ is 3×4, hence

u_1, u_2, u_3, u_4 are dependent.

Problem 4 [15 pts]

For which numbers c, d does the following matrix have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

The third column is a pivot unless $c=0$

Suppose $c \neq 0$, then d must be 0, otherwise the fourth column is a pivot as well, hence the rank would be 3.
Since, $d=0$, the fourth column is a pivot, thus again we have rank 3.

So, we need to have $c=0$.

Now, the fourth column is a pivot and in order to not have the fifth column as a pivot we must have $d=2$.

Problem 5 [15 pts]

Find a basis for each of the four fundamental subspaces (column, null, row, left null) associated with the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

special solutions

free

$$A \sim \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

$$S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Column space $C(A) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\rangle$

Nullspace $N(A) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\rangle$

Rowspace $C(A^T) = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \right\rangle$

Left nullspace $N(A^T)$: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim$

free

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots

Special solution

$$S_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Hence $N(A^T) = \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

Problem 6 [15 pts]

Suppose that S is spanned by $s_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$, $s_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$. Find two vectors that span the orthogonal complement $S^{\perp}$. (Hint: this is the same as solving $Ax = 0$ for some A)

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$. The left nullspace is orthogonal to the column space. Hence, we need to find $N(A^T)$

free

$$A^T = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

pivot

Special solutions

$$s_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad \text{Hence} \quad S^{\perp} = \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

Problem 7 [15 pts]

Suppose P is the subspace of $\mathbb{R}^4$ that consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that satisfy $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for the perpendicular complement $P^\perp$ of P.

P is the nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = A$

The orthogonal complement of $N(A)$ is the row space of A which is $C(A^T) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$