Assignment 11

There are total 18 points in this assignment. 16 points is considered 100%. If you go over 16 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 6.1, 6.2 in Bartle–Sherbert.

In this assignment, assume that we defined $\sin x$, $\cos x$, and x^α for $\alpha \in \mathbb{R}$, and their derivatives are as they should be: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(x^\alpha)' = \alpha x^{\alpha-1}$.

1. Basic properties of the derivative

(1) [2pt] ($\sim$6.1.4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for x rational, $f(x) = 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$. (Hint: Use the limit of ratio definition of derivative.)

(2) [3pt] *(This is problem 7 of HW10. It was not properly stated there. If you did that problem by saying "f, g are not even defined at 0", which is technically correct, do this problem now. If you already did it assuming $f, g = 0$ at 0, I'll take your solution from HW10.)*

Using the "limit of ratio" definition of the derivative, establish whether the following functions are differentiable at 0. In the case of positive answer, find the derivative at 0.

(a) $f(x) = \begin{cases} x\sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$

(b) $g(x) = \begin{cases} x^2\sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$

(3) [2pt] (6.1.10) Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $h(0) = 0$. Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative h' is not bounded on the interval $[-1, 1]$.

(4) [2pt] ($\sim$6.1.14) Given that the function $h(x) = x^3+2016x+1$, $x \in \mathbb{R}$, has an inverse h^{-1} on $\mathbb{R}$, find the value of $(h^{-1})'(y)$ at the points y corresponding to $x = 0, 1, -1$.

(5) [2pt] (6.1.16) Given that the restriction of the tangent function tan to $I = (-\pi/2, \pi/2)$ is strictly increasing and $\tan(I) = \mathbb{R}$, let $\arctan : \mathbb{R} \to \mathbb{R}$ be the function inverse to the restriction of tan to I. Show that arctan is differentiable on $\mathbb{R}$ and $(\arctan y)' = (1 + y^2)^{-1}$ for $y \in \mathbb{R}$.

2. Mean Value Theorem

(6) [2pt] (6.2.6) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (Hint: Apply the Mean Value theorem to sine on the interval $[x, y]$.)

(7) [3pt] (Example 6.2.10(c)) (Bernoulli's inequality) Let $\alpha \in \mathbb{R}$, $\alpha > 1$. Prove that

$$(1 + x)^\alpha \geq 1 + \alpha x \text{ for all } x > -1.$$

(*Hint:* Apply the Mean Value theorem to $(1 + x)^\alpha$ on $[0, x]$.)

(8) [2pt] (6.2.17) Let f, g be differentiable on $\mathbb{R}$ and suppose that $f(0) = g(0)$, and $f'(x) \leq g'(x)$ for all $x \geq 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$. (*Hint:* Use the Mean Value Theorem.)