Homework - Basic numerical methods and Risk measurement

FE-620 - Fall 2022

December 11, 2022

Problem 5.1
Apply the control variate method to the two American options on futures in Problem 3.5, using an European option as control variate, in order to obtained an improved estimate for the American option prices.

Use the same option and market parameters as in Problem 3.5.

Problem 5.2
Explain why the Monte Carlo simulation approach cannot be used easily for American-style derivatives.

Hint. Recall that when pricing American options, we have to compare the *continuation value* with the *exercise value* at any intermediate time. Which do you expect to be more difficult to compute?

Problem 5.3
Build a Monte Carlo simulation of two normally distributed random variables $\varepsilon_1, \varepsilon_2$ with mean zero and unit variance, with correlation $\rho = +0.7$. Using 1000 MC samples, compute the expectation

$$M = \mathbb{E}[max(\varepsilon_1, \varepsilon_2)] \tag{1}$$

Hint. You can construct the random variables from two independent N(0,1) random variables, as explained in class. The simulation can be performed in Excel.

Problem 5.4
Consider a portfolio which can have a loss over 1 year of 1m with probability 98%, or a loss of 10m with probability 2%. Compute the value-at-risk and expected shortfall for this portfolio at confidence level 97.5%.

Problem 5.5

Use the historical simulation method described in class (Ch. 22.2 in Hull, page 496) to compute the daily 95% Confidence Level Value-at-Risk of a portfolio consisting of 1000 shares of TSLA stock on 17-Oct-2022. Denote $PnL_t = V_{t+1} - V_t$ the portfolio PnL (Price-and-Loss) for day t.

At the close of the day on 17-Oct-2022 we do not know yet the PnL for the next day, and we would like to estimate its probability distribution, in order to compute the VaR.

Use the past 1 year of data (17-Oct-2021 to 17-Oct-2022) as inputs to the computation.

You can organize the computation using Excel, by modifying the spreadsheet used in class.

Solution 5.1

For the control variate approach, we have to compute also the European option prices on the tree, and also their exact values using the Black-Scholes formula. These values are shown in the upper two rows in Figure 1. Note that the European call has the same price as the American call, which is expected since the stock does not pay a dividend.

The control variate results for the American options are obtained using the formula from class, which follows from the assumption that the errors introduced by using the tree method are the same for the european and american options)

$$Am_{CV} = Am_{tree} + (Eur_{BS} - Eur_{tree}) \tag{2}$$

Applying this formula gives the results shown in the last column (lowest rows) in Figure 1: the improved American call price is 11.010 and the improved American put price is 8.750.

Solution 5.2 When pricing derivatives with American-style exercise we have to know at each exercise date (in the future) the continuation value $p_{\text{cont}}(t_{ex}|S)$. This has to be compared with the immediate exercise value. For example for an American put option this is $p_{\text{ex}}(t_{ex}|S) = \max(K-S, 0)$. Then the true value at this time is the maximum over the exercise and continuation

$$p(t_{ex}|S) = \max(p_{\text{cont}}(t_{ex}|S), p_{\text{ex}}(t_{ex}|S)) \tag{3}$$

The difficulty lies in computing the continuation value by Monte Carlo methods, since we would have to generate MC paths starting at the same

K=100	Tree	Black-Scholes
European Call	11.819	11.0104
European Put	9.35	8.54142
	Tree	**Control Variate**
American Call	11.819	11.0104
American Put	9.559	8.75042

Figure 1: The control variate improved prices of the American options for Problem 5.1.

value S from a time in the future. In other words we would have to perform as many MC simulations as the number of decision nodes. This would be very time consuming.

A possible approach is to use regression methods to *approximate* the continuation values from a small number of MC paths.This is the idea of the Longstaff-Schwarz approach for pricing American options in MC simulations, also known as Least Squares MC. This is covered in Chapter 27.8 of Hull.

Solution 5.3 The expectation $M = \mathbb{E}[\max(\varepsilon_1, \varepsilon_2)]$ can be evaluated by taking n MC samples $z_i = \max(\varepsilon_1, \varepsilon_2)$ and averaging them.

The result depends on the sample, of course. For example with one $n = 1000$ sample I get

$$\hat{M} = \frac{1}{n} \sum_{i=1}^{n} z_i = 0.269 \tag{4}$$

and the sample payoff standard deviation is $stdev(z_i) = 0.978$.

The 95% confidence region for the true expectation is obtained using the equation on page 472 in Hull

$$\left[0.269 - 1.96 \frac{0.978}{\sqrt{1000}}, 0.269 + 1.96 \frac{0.978}{\sqrt{1000}}\right] = [0.208, 0.330] \tag{5}$$

A simple estimate is often quoted with error $sd/\sqrt{n}$, which gives 0.269 ± 0.031.

Running the computation in R with $n = 100k$ samples gives 0.313 ± 0.003.

Solution 5.4

The Value-at-Risk at confidence level X is determined as the smallest value q_X which can be chosen such that the probability of a loss larger than q_X is less than or equal to $1 - X$.

For the case given, $X = 0.975$, so we have to find the smallest $q_{97.5}$ such that the probability of a loss larger than this value is less than 2.5%. This is clearly $1m$, thus we have

$$VaR(97.5\%) = 1m \tag{6}$$

The expected shortfall is computed as

$$(7)\quad ES_{97.5} = \frac{1}{2.5\%}(2.0\% \cdot 10m + 0.5\% \cdot 1m) = 80\% \cdot 10m + 20\% \cdot 1m = 8.2m$$

Solution 5.5 Implementation in Excel gives

$$VaR_{97.5} = 1000 \cdot \$15.27 = \$15,270 \tag{8}$$

The computation can be done for one share of TSLA, and the result is multiplied with 1000 to obtain the VaR of a portfolio of 1000 shares. The scenario losses ranked in decreasing order are shown below. 95% VaR corresponds to the rank $12.6 = 0.05 \cdot 252$, which is roughly the average of the ranks 12 and 13 losses.

Loss	Rank
26.7258712	1
26.3007302	2
25.3440837	3
20.2226662	4
19.9015134	5
18.8959428	6
18.7341789	7
18.2688178	8
18.1064275	9
16.5511664	10
15.5785748	11
15.3499322	**12**
15.1910072	**13**
14.9378464	14
14.9188261	15

Figure 2: The scenario losses, ranked in decreasing order.