Homework 10

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April 1, 2022 (Revised May 10, 2022)

Problem 1

Theorem. *If a finite deterministic two-player game with perfect information allows draws, then one player has a winning strategy or both players have drawing strategies.*

Proof. Let us name two players, Player A and Player B. We can represent any such game as a directional tree, with each node representing a position in the game and each branch representing a player choice. The parity of the distance from any given node and the root node determines which player's turn it is. For example, in a game where A goes first, every node an even distance from the root represents a position in which A must make a decision, and odd distances represent positions for B's decisions. At the endpoint for every branch, it follows that the endpoint can be a win for Player A, a loss for Player A, or a draw. Therefore we have two cases for the positions before the endpoint.

<u>Case 1:</u> The position before the endpoint of the tree is odd.
Since the position right before the endpoint is odd, it follows that it will be Player A's turn. Player A aims to make a winning move. If Player A can not do this, they will aim to make a drawing move. And lastly, if Player A can not do this, they will make a move that results in a loss.

<u>Case 2:</u> The position before the endpoint of the tree is even.
Since the position right before the endpoint is even, it follows that it will be Player B's turn. Player B aims to make a winning move. If Player B can not do this, they will aim to make a drawing move. And lastly, if Player B can not do this, they will make a move that results in a loss.

We can then catergorize each endpoint as W (win), D (draw), and L (loss) for Player A. Through the use of induction, we can repeat the process until we reach the starting position of the directional tree. It is event that Player A will pick the best option for them; however, the only options that are possible include W, D, or L (which is a W for Player B) for Player A. The same can be said for Player B, since player B's only options include W, D or L (which is a W for Player A). Thus the theorem holds true. □

Problem 2.1

Theorem. *The position* $(n, 1, 1, ..., 1)$*,* $n \geq 2$ *is a winning position.*

Proof. Suppose that for two players A and B, the position of the game is at $(n, 1, 1, ..., 1)$, $n \geq 2$. We have the following 2 cases.

Case 1: The number of 1-token heaps is even.
If we have an even number of 1-token heaps, the best strategy for player A is to remove $n-1$ tokens from n-heap. This would result in an odd number of 1-token heaps with player B's turn. We can see that player A and player B can now only remove one 1-heap. Once we get to the last 1-heap, it will be player B's turn to take the last token and result in a win for player A.

Case 2: The number of 1-token heaps is odd.
If we have an odd number of 1-token heaps, the best strategy for player A is to remove n tokens from n-heap. This would result in an odd number of 1-token heaps with player B's turn. We can see that player A and player B can now only remove one 1-heap. Once we get to the last 1-heap, it will be player B's turn to take the last token and result in a win for player A.

Since we have established a winning strategy in both cases, we can state that the position $(n, 1, 1, ..., 1)$, $n \geq 2$ is a winning position. □

Problem 2.2

Theorem. *A winning strategy can be created for a game of Nim where the player who takes the last token loses. Take for granted that $(n, 1, 1, ..., 1)$, $n \geq 2$ is unbalanced.*

Proof. We know that $(n, 1, 1, ..., 1)$, $n \geq 2$ is unbalanced, so we aim to balance the board when it is unbalanced. If we leave the game in a balanced state, our opponent will never have the game in the form of $(n, 1, 1, ..., 1)$, $n \geq 2$, since that form is unbalanced. Balancing the position will always yield the opposing player to unbalance the position, and then for us to balance it again in this cycle. Once we reach the position $(n, 1, 1, ..., 1)$, $n \geq 2$, we know that from Problem 2.1 that it is a winning position, therefore we can win the game by following the strategy described in Problem 2.1. □