Math189R SU17
Homework 5
Monday, June 5, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\|\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k}\lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d}\lambda_j$ into $\sum_{j=1}^{k}\lambda_j$ and $\sum_{j=k+1}^{d}\lambda_j$.

a) $$\left\| x_i - \sum_{j=1}^{k} z_{ij} v_j \right\|^2 = \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)^T \left(x_i - \sum_{j=1}^{k} z_{ij} v_j\right)$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} z_{ij} v_j^T x_i + \sum_{j=1}^{k} v_j^T z_{ij}^T z_{ij} v_j$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} v_j^T x_i z_{ij} + \sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$

$$= x_i^T x_i - 2\sum_{j=1}^{k} v_j^T x_i x_i^T v_j + \sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$

$$= x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j .$$

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b) So,

$$\frac{1}{n}\sum_{i=1}^{n}\left(x_i^T x_i - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} v_j^T \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i^T\right) v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{K} v_j^T \operatorname{cov}(x_i)\, v_j$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{k} \lambda_j .$$

c) From our previous result and

$$\sum_{j=1}^{d}\lambda_j = \sum_{j=1}^{k}\lambda_j + \sum_{j=k+1}^{d}\lambda_j ,$$

we can substitute to say that

$$J_k = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{d}\lambda_j + \sum_{j=k+1}^{d}\lambda_j .$$

Furthermore, since $J_d = 0$, it must be that

$$J_d = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T - \sum_{j=1}^{d}\lambda_j = 0$$

$$\sum_{j=1}^{d}\lambda_j = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T .$$

Substituting yields

$$J_k = \sum_{j=1}^{d}\lambda_j - \sum_{j=1}^{d}\lambda_j + \sum_{j=k+1}^{d}\lambda_j$$

$$= \sum_{j=k+1}^{d}\lambda_j .$$

2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

Show that the optimization problem

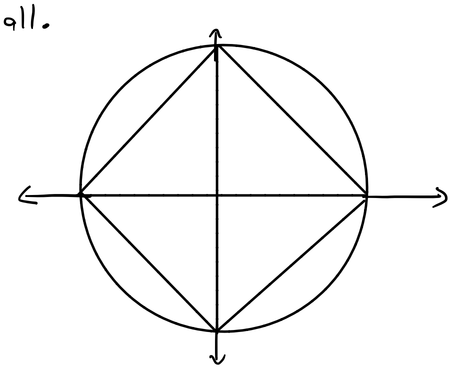
$$\begin{aligned} \text{minimize: } & f(\mathbf{x}) \\ \text{subj. to: } & \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda \|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.

The norm-ball B_k is the 4-sided polyhedron inside A_k, the circular Euclidean norm-ball.



Consulted solutions.

So minimizing $f(x)$ such that $\|x\|_p \leq k$ is equivalent to

$$\inf_x \sup_{\lambda \geq 0} \mathcal{L}(x, \lambda) = \inf_x \sup_{\lambda \geq 0} f(x) + \lambda(\|x\|_p - k)$$

by definition of the Lagrangian. Then, this is equivalent to

$$\sup_{\lambda \geq 0} \inf_x f(x) + \lambda \|x\|_p - \lambda k,$$

but since λk does not have an x term, it does not affect our minimization so we can just minimize $f(x) + \lambda \|x\|_p$.

ℓ_1 regularization yields sparser solutions than ℓ_2 because it is infinitely more likely to land on a vertex with the ℓ_1 ball than an ℓ_2 due to its rotational symmetry. ℓ_1 will encourage more weights to 0, which aids our objective function. ■

Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

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