Lecture 2 — Notes (simple terms + examples)

1) What is a Vector Space? (intuition first)

A vector space is a set of “things” (vectors) where you can:
• Add two vectors and still stay in the set.
• Scale a vector by any real number and still stay in the set.
Think of usual vectors in R^n (like (x,y,z)), but they can also be matrices, polynomials, etc., as long as the rules below hold.

Rules that must hold (over R):
• Addition behaves nicely: commutative, associative; there’s a zero vector; every vector has an additive inverse.
• Scaling (multiplying by a real number) behaves nicely:
• Distributive over vector addition: $\lambda(x+y)=\lambda x+\lambda y$
• Distributive over scalar addition: $(\lambda+\mu)x=\lambda x+\mu x$
• Associative with scalars: $\lambda(\mu x)=(\lambda\mu)x$
• Identity: $1\cdot x = x$

Examples:
• R^n with usual addition and scaling.
• $R^{(m\times n)}$ (all m×n real matrices) with elementwise addition and scalar multiplication.

2) Subspaces (how to test quickly)

A subspace U of a vector space V is a subset that is itself a vector space under the same operations.

Quick 2-step test (over R):
• U is nonempty and contains 0 (usually implied by step 2).
• Closed under addition and scaling:
If $u,v \in U$ then $u+v \in U$; if $\lambda \in R$, then $\lambda u \in U$.

Good examples:
• y-axis in R^2: $\{(0,y): y \in R\}$ → closed under + and scaling ⇒ subspace.
• Nullspace of a matrix A: $\{x: A x = 0\}$ ⇒ always a subspace.

Not a subspace:
• The vertical line x=1: scaling (1,2) by 2 gives (2,4) (not on x=1). Fails closure.

3) Linear Combination & Linear Independence (why it matters)

• A linear combination of $x_1,\ldots,x_k$ is $\lambda_1 x_1 + \cdots + \lambda_k x_k$.
• Vectors $x_1,\ldots,x_k$ are linearly independent if the only way to make the zero vector is with all coefficients zero:
$\lambda_1 x_1 + \cdots + \lambda_k x_k = 0 \Rightarrow \lambda_1 = \cdots = \lambda_k = 0$.
If there is any non-trivial solution, they’re dependent.

Mini-examples:
- In R^2, (1,2) and (2,4) are dependent because (2,4)=2(1,2).
- In R^2, (1,2) and (2,3) are independent (no scalar makes one a multiple of the other).

Practical test (matrices):
- Put the vectors as columns of a matrix and perform Gaussian elimination.
- If every column becomes a pivot column (has a leading 1 in RREF), they're independent.
- If some column is non-pivot, it can be written from earlier pivot columns → dependence.

Quick determinant test (square case):
- For n vectors in R^n: Put them as columns of A ∈ R^(n×n).
- If det(A) ≠ 0 ⇒ columns are independent. If det(A) = 0 ⇒ dependent.

4) Span, Basis, and Dimension

- Span[x1,…,xk]: all linear combinations of those vectors.
- A basis of a space V is a linearly independent set that spans V.
- Dimension dim(V): number of vectors in any basis of V (well-defined).

Examples:
- In R^3, the standard basis is e1=(1,0,0), e2=(0,1,0), e3=(0,0,1). dim(R^3)=3.
- If U=span{(1,0,0),(1,1,0)} ⊂ R^3, then dim(U)=2. A basis is {(1,0,0),(1,1,0)} (they're independent).

How to find a basis from a spanning set:
- Put spanning vectors as columns of a matrix A.
- Row-reduce to RREF.
- Columns of the original A that correspond to pivot columns in RREF form a basis of the span.

5) Determinant (only what you need + Sarrus' Rule)

Determinant tells you volume scaling and invertibility:
- det(A) ≠ 0 ⇔ A invertible ⇔ columns are independent ⇔ unique solution to A x = b.

Row operations effect:
- Swap two rows → multiply det by −1.
- Multiply a row by λ → det scales by λ.
- Add multiple of one row to another → det unchanged.

(5A) Sarrus' Rule (only for 3×3):
For A = [[a, b, c], [d, e, f], [g, h, i]],
det(A) = (a*e*i + b*f*g + c*d*h) − (c*e*g + a*f*h + b*d*i).

Worked example (Sarrus):
A = [[1, 2, 3], [0, 1, 4], [5, 6, 0]]
Down-right sum: 1*1*0 + 2*4*5 + 3*0*6 = 0 + 40 + 0 = 40
Up-right sum: 3*1*5 + 1*4*6 + 2*0*0 = 15 + 24 + 0 = 39
det(A) = 40 − 39 = 1 (non-zero ⇒ columns are independent).

(5B) 2×2 determinant & inverse (recall):
For A = [[a, b], [c, d]]:
• det(A) = a*d − b*c
• If det(A) ≠ 0 then A^{-1} = (1/(a*d − b*c)) * [[d, −b], [−c, a]]

6) Nullspace (kernel) — example

The nullspace N(A) = { x : A x = 0 } is always a subspace.

Example:
A = [[1, 2, 3], [2, 4, 6]], x = [x1, x2, x3]^T
Equations:
x1 + 2x2 + 3x3 = 0
2x1 + 4x2 + 6x3 = 0 (same constraint)
Let x2 = s, x3 = t. Then x1 = −2s − 3t.
N(A) = { [−2s−3t, s, t]^T : s,t ∈ R } = span{ [−2,1,0]^T, [−3,0,1]^T } (2D).