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HOPF- bifurcation analysis of delayed computer virus model with holling type iii incidence function and treatment

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ABSTRACT

Computer viruses have become a threatening challenge to most network users. Due to enormous operational constraints, distortions, and alterations caused by malware spread, research on network security is deemed essential. These malicious codes are sufficiently equipped to distribute themselves over the whole system. Consequently, infected hosts can rapidly pollute adjoining nodes. In the past, many research efforts have derived analytical models for computer viruses under several scenarios of infectiousness. Therefore, we considered the Susceptible-Exposed-Infectious-Recovered-Vaccinated (SVEIR) model with Holling Type III incidence function and treatment. This is to address the ubiquitous application of the bilinear contagion rate. Here, the delay parameter was chosen and, through numerous analyses, we demonstrated the presence of a Hopf bifurcation as it crosses a critical value. Furthermore, we examined the attributes of the Hopf bifurcation by employing the centre manifold theorem and the normal form theory. Then, we performed numerical simulation experiments for different conditions in order to underpin the derived theoretical conclusions.

Introduction

In every organization that makes use of an information and communication technology infrastructure, a trustworthy defence structure is always desired to ensure the total protection of useful mobile and static data/information warehoused on their machines. To achieve this aim, it has become critical to investigate and be informed about the innate features of popular malicious codes, for instance, worms and viruses. Viruses are malevolent, spiteful software codes intended to mimic themselves while they are transferred to other neighbouring machines [1,2] through human intervention. Besides infecting neighbouring computers, their harmful activities include attacking storage media and applications, expunging data,

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distorting files, and altering the typical operations. With the rapid expansion and commercialization of networks, computer viruses have had a remarkable negative impact on businesses and daily life, which equals losses and damage worth millions of dollars. Recently, researchers started expressing fears about the emergence of swarm intelligence-based viruses, i.e., the swarm [3] and neural swarm [4] viruses, etc. The swarm viruses, which have no centralised communication point, make use of some of the characteristics of natural swarm systems, or swarm algorithms, to mimic the behaviours of birds and ants [5]. A neural swarm virus models the behaviour of swarm intelligence and incorporates a neural network to improve efficiency. As these types of complex malware are imminent, network managers have to come up with remedies if they are ever going to achieve a malware-free cyberspace for their organizations.

Recent developments have shown that information security is the most significant factor during the exchange of information, and computer viruses prominently pose a serious threat. Regardless of the substantial progress of anti-virus as the foremost and most common means of shielding against viruses, they are still a major concern on networks. Another viable option focuses on deciphering the prevalent nuances and intentions of computer viruses and developing network structures, behaviours, and strategies to prevent future attacks and occurrences. Although most malicious objects may possess varied methods of dissemination, they all have similar features, which include infectiousness and the ability to be invincible, destructible, unpredictable, and dormant for some time before full infectivity [6]. Generally speaking, in the light of epidemiology and public health, several similarities have been noted between diseases in biological networks and virtual malware in communication networks. In fact, the most prominent epidemiological models are enhanced descriptions of the conventional Kermack & McKendrick's [7] model, more universally recognised as the Susceptible/Infected/Recovered (SIR) model.

Considering the standardised collaborating node communication scenario, Mishra et al [8,9]. developed a model for virus dissemination on the internet. Their study examined the spread theory using epidemic thresholds, thus envisaging tendencies of viral growth in the network. An infected computer may exist in the exposed phase, where it's not contagious but may subsequently attain the full capability to infect other machines. Consequent upon this, delay is introduced to illustrate that although a certain computer is not infectious, it possesses some low infectivity [10,11]. However, we propose the Susceptible-Exposed-Infectious-Recovered-Vaccinated (SEIR-V) model but with the Holling Type III incidence function. Note that older models of computer viruses either use mass action or standard incidence types. The Holling Type II incidence function was used in the basic Susceptible-Exposed-Infected-Recovered (SEIR) model by Safi & Garba [12] for the human immunodeficiency virus. Therefore, its application to computer viruses here attests to the semblance between biological viral diseases and virtual malware. Besides this special kind of incidence function, our model herein extends the SEIR model [12] with the addition of the vaccination compartment, delay, and bifurcation analyses.

This paper is organised as follows: Section 2 contains the related works, while Section 3 is the description of the model with explanations and nomenclature. In Section 4, the analysis of delay is discussed, while Section 5 contains descriptions of the direction and stability of Hopf bifurcation. In Section 6, the obtained analytical findings are justified through numerical simulations, and Section 7 contains the concluding remarks and future directions.

Related works

Yang et al [13,14] proposed the Susceptible-Latent-Breaking out (SLB) and the Susceptible-Latent-Breaking out-Susceptible (SLBS) models, which confirmed that viruses possess a certain infectivity during the dormancy stage when the malware is inactive. However, they neither presented the length of inactivity nor addressed the impact of using a protective mechanism. Insights on computer virus propagation in networks have been elicited using the following models that consider the latency stage: e-SEIR [15], Susceptible-Exposed-Infected-Quarantine-Recovered-Susceptible [16], SLB [17,22], Susceptible-Exposed-Infected-Susceptible (SEIS) [18], Susceptible Internal computers-Infected Internal Computers-External Computers (SIE) [19], SEIR [20,21,23], and Susceptible-Exposed-Infected-Benign-Recovered (e–SEIAR) [24]. Mishra & Jha [16] suggested and investigated the SEIQR-S model. The studies in refs [15,20,21, and 23] conceived the SEIR models, taking into consideration the recovery of computers, which possessed an enduring vaccination time span. These models confirm the fact that a computer can transmit the infection instantly after it gets infected [22]. For the SEIR computer virus propagation model, most authors suggest that the improved computers have a perpetual immunisation time span without reinfection. Although, they addressed computers in the expose class, they did not consider bifurcation.

Time delays of various types have been incorporated into various virus models; these can be as a result of the latent period [25,26] using the SIR model, the transient immunity period [27,28] using the SIR and SEIR models, quarantine [29] using the SEIQR-S model, or two or more delays [30] using the SEIR model. As quantified in [26], the manifestation of Hopf bifurcation implies that the state transforms from equilibrium to a limit cycle, a scenario that is unanticipated since the intermittent performance is undesirable from the epidemiological perspective. With the SLBR (equalling the SEIR) model, the authors [31] investigated the effect of the anti-malicious codes. All these conceptions fulfil critical roles depending on the modeler's objective (s). It is possible for time delays to result in the loss of stability and can lead to Hopf bifurcation and intermittent (periodic) solutions. Despite the fact that refs [25–31] catered for different computer network phenomena, these models have some limitations. Considering the objectives of this study, these models are limited in the sense that they use the simple bilinear incidence function and do not possess a vaccination compartment. Additionally, while refs [26–31] addressed both delay and bifurcation, Muroya *et al.* [25] did not consider the latter.

Note that Zhao et al [32] worked on the SVEIR model alongside the Holling Type II incidence, but we apply the Holling Type III incidence function. While the former is density-dependent, the latter gives rise to a ratio-dependent functional response. Besides these modifications, the installed anti-malware also requires some time to eliminate viruses from the contagious nodes. Therefore, considering the Hopf bifurcation with time delay is a worthy research aim.

Mathematical Model

Most characterizations of computer infection propagation utilize the bilinear contamination type, and there is some rationale for its improvement. Firstly, malware infections can be immensely influenced by the physical arrangement of nodes, thus leading to some unique nonlinear conditions. Secondly, the choice of treatment function type plays a significant role in illustrating the dimensions of malware spread. Therefore, we propose the susceptible, exposed, infected, recovered, and vaccinated (S, E, I, R, V) model considering the Holling Type III incidence function.

$$S'(t) = \wedge - \delta_0 S - \frac{\alpha S^2 I}{S^2 + I^2} + \eta V - \mu S \tag{1}$$

$$E'(t) = \frac{\alpha S^2 I}{S^2 + I^2} - \delta_0 E - \delta_1 E \tag{2}$$

$$I'(t) = \delta_1 E - (\delta_0 + \delta_3) I - \delta_2 I(t-\tau) - \frac{\beta I^2(t-\tau)}{I^2(t-\tau) + a^2} \tag{3}$$

$$R'(t) = \delta_2 I(t-\tau) - \delta_0 R + \frac{\beta I^2(t-\tau)}{I^2(t-\tau) + a^2} \tag{4}$$

$$V'(t) = \mu S - (\delta_0 + \eta) V \tag{5}$$

Using the compartmental model, *S(t), E(t), I(t), R(t), and V(t)* represent vulnerable, exposed, infectious, recovered, and vaccinated computers at time t, respectively. Other model parameters are as follows: $\wedge$ = the rate at which new computers are added to the network, α = infectivity rate of computers prone to virus infection, η = the rate of transfer of computers from the vaccinated to the susceptible class, β = maximal treatment capacity of a network, δ_0 = rate of natural death for reasons other than virus attack, δ_1= rate of transfer of computers from the latent phase to the fully infectious class, δ_2 = rate of transfer of computers from infectious to recovered class, δ_3 = death rate of computers as a result of virus attack, a = half saturation constant for the infectious computers, μ = rate at which susceptible computers are vaccinated to gain transient immunity, and τ = time delay as a result of cleaning infective computer nodes using updated anti-malware.

By direct computation the system (1-5) has a unique positive equilibrium point $D_*(S^*, E^*, I^*, R^*, V^*)$in which

$$S^* = I^* \left[\sqrt{\frac{(\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3)\left((I^*)^2 + a^2\right) + \beta(I^*)(\delta_0+\delta_1)}{[\alpha\delta_1 - (\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3)]\left((I^*)^2 + a^2\right) - \beta(I^*)(\delta_0+\delta_1)}} \right]$$

$$E^* = \frac{\delta_0+\delta_2+\delta_3}{\delta_1} I^* + \frac{\beta (I^*)^2}{\delta_1\left((I^*)^2 + a^2\right)}$$

$$R^* = \frac{\delta_2}{\delta_0} I^* + \frac{\beta (I^*)^2}{\delta_0\left((I^*)^2 + a^2\right)}$$

$$V^* = \frac{\mu S^*}{\delta_0 + \eta}$$

I^* is the positive root of the equation (6)

$$C_0(I^*)^8 + C_1(I^*)^7 + C_2(I^*)^6 + C_3(I^*)^5 + C_4(I^*)^4 + C_5(I^*)^3 + C_6(I^*)^2 + C_7(I^*) + C_8 = 0 \tag{6}$$

where

$$C_0 = \Gamma_1^2\Gamma_6 - \Gamma_3^2\Gamma_4;\ C_1 = 2\Gamma_1\Gamma_2\Gamma_6 - 2\wedge\Gamma_1\Gamma_6 - \Gamma_1^2\Gamma_5$$
$$C_2 = \wedge^2\Gamma_6 + 3a^2\Gamma_1^2\Gamma_6 + \Gamma_2^2\Gamma_6 + 2\wedge\Gamma_1\Gamma_5 - 2\Gamma_1\Gamma_2\Gamma_5 - 2\wedge\Gamma_2\Gamma_6 - 3a^2\Gamma_3^2\Gamma_4$$
$$C_3 = 4a^2\Gamma_1\Gamma_2\Gamma_6 + 2\wedge\Gamma_2\Gamma_5 - \Gamma_3^2\Gamma_5 - \wedge^2\Gamma_5 - 2a^2\Gamma_1^2\Gamma_5 - \Gamma_2^2\Gamma_5 - 6a^2\wedge\Gamma_1\Gamma_6$$
$$C_4 = 3a^2\wedge^2\Gamma_6 + 3a^4\Gamma_1^2\Gamma_6 + a^2\Gamma_2^2\Gamma_6 + 4a^2\wedge\Gamma_1\Gamma_5 - 2a^2\Gamma_1\Gamma_2\Gamma_5 - 4a^2\wedge\Gamma_2\Gamma_6 - 3a^4\Gamma_3^2\Gamma_4$$
$$C_5 = 2a^4\Gamma_1\Gamma_2\Gamma_6 + 2a^2\wedge\Gamma_2\Gamma_5 - 2\wedge^2 a^2\Gamma_5 - a^4\Gamma_1^2\Gamma_5 - 6a^4\wedge\Gamma_1\Gamma_6 - a^2\Gamma_3^2\Gamma_5$$
$$C_6 = 3a^4\wedge^2\Gamma_6 + a^6\Gamma_1^2\Gamma_6 + 2a^4\wedge\Gamma_1\Gamma_5 - 2\wedge a^4\Gamma_2\Gamma_6 - a^6\Gamma_3^2\Gamma_4$$
$$C_7 = -\wedge^2 a^4\Gamma_5 - 2\wedge a^6\Gamma_1\Gamma_6$$
$$C_8 = \wedge^2 a^6\Gamma_6$$

with

$$\Gamma_1 = \frac{(\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3)}{\delta_1};\ \Gamma_2 = \frac{\beta(\delta_0+\delta_1)}{\delta_1};\ \Gamma_3 = \frac{\delta_0(\delta_0+\eta+\mu)}{\delta_0+\eta}$$
$$\Gamma_4 = (\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3);\ \Gamma_5 = (\delta_0+\delta_1);\ \Gamma_6 = (\alpha\delta_1 - (\delta_0+\delta_1)(\delta_0+\delta_2+\delta_3))$$

Delay Analysis

The linear system of (1)-(5) about endemic equilibrium point $D_*(S^*, E^*, I^*, R^*, V^*)$is given by

$$S'(t) = h_{11}S(t) + h_{13}I(t) + h_{15}V(t) \tag{7}$$

$$E'(t) = h_{21}S(t) + h_{22}E(t) + h_{23}I(t) \tag{8}$$

$$I'(t) = h_{32}E(t) + h_{33}I(t) + d_{33}I(t-\tau) \tag{9}$$

$$R'(t) = h_{44}R(t) + d_{44}I(t-\tau) \tag{10}$$

$$V'(t) = h_{51}S(t) + h_{55}V(t) \tag{11}$$

Where$h_{11} = -(\delta_0+\mu) - \frac{2\alpha I^{*3}S^*}{(S^{*2}+I^{*2})^2}$;$h_{13} = -\frac{\alpha S^{*2}(S^{*2}-I^{*2})}{(S^{*2}+I^{*2})^2}$; $h_{15} = \eta$;

$$h_{21} = \frac{2\alpha S^* I^{*3}}{\left(S^{*2}+I^{*2}\right)^2};\ h_{22} = -(\delta_0+\delta_1);\ h_{23} = \frac{\alpha S^{*2}\left(S^{*2}-I^{*2}\right)}{\left(S^{*2}+I^{*2}\right)^2};\ h_{32} = \delta_1;$$

$$h_{33} = -(\delta_0+\delta_3);\ d_{33} = -\delta_2 - \frac{2\beta a^2 I^*}{\left(a^2+I^{*2}\right)^2};\ h_{44} = -\delta_0;\ d_{44} = \delta_2 + \frac{2\beta a^2 I^*}{\left(a^2+I^{*2}\right)^2};$$

$$h_{51} = \mu;\ h_{55} = -(\delta_0+\eta);$$

Then the associated characteristic equation is

$$\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5 + \left(B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5\right)e^{-\lambda\tau} = 0 \tag{12}$$

where $A_1 = -(h_{11}+h_{22}+h_{33}+h_{44}+h_{55})$

$$A_2 = \begin{pmatrix} h_{11}h_{22}+h_{11}h_{33}+h_{11}h_{44}+h_{22}h_{33}+h_{22}h_{44}+h_{33}h_{44}-h_{23}h_{32}-h_{15}h_{51} \\ +h_{55}h_{11}+h_{55}h_{22}+h_{55}h_{33}+h_{55}h_{44} \end{pmatrix}$$

$$A_3 = h_{23}h_{32}(h_{11}+h_{44}+h_{55}) - h_{13}h_{21}h_{32} - h_{15}h_{51}(h_{22}+h_{33}+h_{44})$$

$$-h_{11}h_{22}(h_{33}+h_{44}) - h_{33}h_{44}(h_{11}+h_{22}) - h_{55}(h_{11}h_{22}+h_{33}h_{44}) + (h_{11}+h_{22})(h_{33}+h_{44})$$

$$A_4 = h_{55}[h_{11}h_{22}(h_{33}+h_{44}) + h_{33}h_{44}(h_{11}+h_{22}) + h_{11}h_{22}h_{33}h_{44} + h_{15}h_{23}h_{32}h_{51} + h_{13}h_{21}h_{32}(h_{44}+h_{55})$$

$$-h_{15}h_{51}(h_{22}h_{33}+h_{22}h_{44}+h_{33}h_{44}) - h_{23}h_{32}(h_{11}h_{44}+h_{11}h_{55})$$

$$A_5 = h_{44}(h_{22}h_{33}-h_{23}h_{32})(h_{15}h_{51}-h_{11}h_{55}) - h_{13}h_{21}h_{32}h_{44}h_{55}$$

$$B_1 = -d_{33};\ B_2 = -d_{33}(h_{11}+h_{22}+h_{44}+h_{55});$$

$$B_3 = h_{15}h_{51}d_{33} - d_{33}(h_{11}h_{22}+h_{41}h_{55}) + (h_{11}+h_{22})(h_{44}+h_{55})$$

$$B_4 = h_{11}h_{22}d_{44}(h_{14}+h_{55}) + h_{44}h_{55}d_{33}(h_{11}+h_{22}) - h_{15}h_{51}d_{44}(h_{22}+h_{44})$$

$$B_5 = h_{22}h_{55}d_{33}(h_{15}h_{51}-h_{11}h_{55})$$

Put $\tau = 0$ in (12), we get

$$\lambda^5 + (A_1 + B_1)\lambda^4 + (A_2 + B_2)\lambda^3 + (A_3 + B_3)\lambda^2 + (A_4 + B_4)\lambda + A_5 = 0 \tag{13}$$

From (13), $A_1 + B_1 = s\delta_0 + (\delta_1 + \delta_2 + \delta_3 + \eta + \mu) + \frac{2\alpha I^{*3} S^*}{(S^{*2} + I^{*2})^2} + \frac{2\beta a^2 I^*}{(a^2 + I^{*2})^2} > 0$

By using Routh-Hurwitz criteria, sufficient conditions for all roots of equation (13) to be negative real part are given in the following form.

$$M_2 = \begin{vmatrix} A_1 + B_1 & 1 \\ A_3 + B_3 & A_2 + B_2 \end{vmatrix} \tag{14}$$

$$M_3 = \begin{vmatrix} A_1 + B_1 & 1 & 0 \\ A_3 + B_3 & A_2 + B_2 & A_1 + B_1 \\ 0 & A_4 + B_4 & A_3 + B_3 \end{vmatrix} \tag{15}$$

$$M_4 = \begin{vmatrix} A_1 + B_1 & 1 & 0 & 0 \\ A_3 + B_3 & A_2 + B_2 & A_1 + B_1 & 1 \\ A_5 + B_5 & A_4 + B_4 & A_3 + B_3 & A_2 + B_2 \\ 0 & 0 & A_5 + B_5 & A_4 + B_4 \end{vmatrix} \tag{16}$$

$$M_5 = \begin{vmatrix} A_1 + B_1 & 1 & 0 & 0 & 0 \\ A_3 + B_3 & A_2 + B_2 & A_1 + B_1 & 1 & 0 \\ A_5 + B_5 & A_4 + B_4 & A_3 + B_3 & A_2 + B_2 & A_1 + B_1 \\ 0 & 0 & A_5 + B_5 & A_4 + B_4 & A_3 + B_3 \\ 0 & 0 & 0 & 0 & A_5 + B_5 \end{vmatrix} > 0 \tag{17}$$

This, if conditions (14)-(17) hold, $E_*$is locally asymptotically stable in the absence of delay
For $\tau > 0$, Put $\lambda = i\omega$ in equation (12) we have

$$(i\omega^5 + A_1\omega^4 - iA_2\omega^3 - A_3\omega^2 + iA_4\omega + A_5) + (B_1\omega^4 - iB_2\omega^3 - B_3\omega^2 + iB_4\omega + A_5)(\cos\omega\tau - i\sin\omega\tau) = 0 \tag{18}$$

Equating real and imaginary parts we have

$$(B_1\omega^4 - B_3\omega^2 + B_5)\cos\omega\tau + (B_4\omega - B_2\omega^3)\sin\omega\tau = \left(A_3\omega^2 - A_1\omega^4 - A_5\right) \tag{19}$$

$$(B_4\omega - B_2\omega^3)\cos\omega\tau - (B_1\omega^4 - B_3\omega^2 + B_5)\sin\omega\tau = \left(A_2\omega^3 - \omega^5 - A_4\omega\right) \tag{20}$$

Squaring and Adding (19) and (20) we get

$$\omega^{10} + P_1\omega^8 + P_2\omega^6 + P_3\omega^4 + P_4\omega^2 + P_5 = 0 \tag{21}$$

Where

$$P_1 = A_1^2 - B_1^2 - 2A_2;\ P_2 = A_2^2 - 2A_3A_1 - B_2^2 + 2B_1B_3 + 2A_4;\ P_3 = A_3^2 - 2A_4A_1 - B_3^2 - 2B_4B_2 + 2A_5A_1;$$
$$P_4 = A_4^2 - 2A_5A_3 - B_4^2 + 2B_5B_3;\ P_5 = A_5^2 - B_5^2;$$

Now By Assuming $\omega^2 = u$ then the equation (21) becomes

$$u^5 + P_1u^4 + P_2u^3 + P_3u^2 + P_4u + P_5 = 0 \tag{22}$$

Define the function

$$f(u) = u^5 + P_1u^4 + P_2u^3 + P_3u^2 + P_4u + P_5 = 0 \tag{23}$$

Clearly $\lim_{u\to\infty} f(u) = \infty$. Thus if $P_5 < 0$, then equation (23) has at least one positive root.
Solving from (19) and (20), we get

$$\cos\omega\tau = \frac{Q_1\omega^8 + Q_2\omega^6 + Q_3\omega^4 + Q_4\omega^2 + Q_5}{Q_6\omega^8 + Q_7\omega^6 + Q_8\omega^4 + Q_9\omega^2 + Q_{10}}$$

where

$$Q_1 = B_2 - A_1B_1;\ Q_2 = A_3B_1 + A_1B_3 + A_2B_2 - B_4;\ Q_3 = A_4B_2 + A_2B_4 - A_3B_3 - A_5B_1 - A_1B_5;\ Q_4 = A_5B_3 + A_3B_5 - A_4B_4;$$
$$Q_5 = -A_5B_5;\ Q_6 = B_1^2;\ Q_7 = B_2^2 - 2B_1B_3;\ Q_8 = B_3^2 + 2B_4B_2 + 2B_1B_5;\ Q_9 = B_4^2 - 2B_3B_5;\ Q_{10} = B_5^2$$

So, corresponding to$\lambda = i\omega_0$, there exists

$$\tau_{0n} = \frac{1}{\omega_0}\cos^{-1}\left[\frac{Q_1\omega^8 + Q_2\omega^6 + Q_3\omega^4 + Q_4\omega^2 + Q_5}{Q_6\omega^8 + Q_7\omega^6 + Q_8\omega^4 + Q_9\omega^2 + Q_{10}}\right] + \frac{2n\pi}{\omega_0};\ \text{Where}\, n = 0, 1, 2, \ldots \tag{24}$$

Differentiate (12) with respect toτ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{5\lambda^4 + 4A_1\lambda^3 + 3A_2\lambda^2 + 2A_3\lambda + A_4}{\lambda e^{-\lambda\tau}\left(B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5\right)} - \frac{\tau}{\lambda} + \frac{4B_1\lambda^3 + 3B_2\lambda^2 + 2B_3\lambda + B_4}{\lambda e^{-\lambda\tau}\left(B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5\right)}$$

$$\frac{5\lambda^4 + 4A_1\lambda^3 + 3A_2\lambda^2 + 2A_3\lambda + A_4}{-\lambda\left(\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5\right)} + \frac{4B_1\lambda^3 + 3B_2\lambda^2 + 2B_3\lambda + B_4}{-\lambda\left(\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5\right)} - \frac{\tau}{\lambda}$$

$$\frac{5\lambda^5 + 4A_1\lambda^4 + 3A_2\lambda^3 + 2A_3\lambda^2 + A_4\lambda}{-\lambda^2\left(\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5\right)} + \frac{4B_1\lambda^4 + 3B_2\lambda^3 + 2B_3\lambda^2 + B_4\lambda}{-\lambda^2\left(\lambda^5 + A_1\lambda^4 + A_2\lambda^3 + A_3\lambda^2 + A_4\lambda + A_5\right)} - \frac{\lambda\tau}{\lambda^2}$$

Put $\lambda = i\omega_0$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{5(i\omega_0)^5 + 4A_1(i\omega_0)^4 + 3A_2(i\omega_0)^3 + 2A_3(i\omega_0)^2 + A_4(i\omega_0)}{\omega_0{}^2\left((i\omega_0)^5 + A_1(i\omega_0)^4 + A_2(i\omega_0)^3 + A_3(i\omega_0)^2 + A_4(i\omega_0) + A_5\right)}$$
$$+ \frac{4B_1(i\omega_0)^4 + 3B_2(i\omega_0)^3 + 2B_3(i\omega_0)^2 + B_4(i\omega_0)}{\omega_0{}^2\left((i\omega_0)^5 + A_1(i\omega_0)^4 + A_2(i\omega_0)^3 + A_3(i\omega_0)^2 + A_4(i\omega_0) + A_5\right)} + \frac{i\tau}{\omega_0}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{5i\omega_0^5 + 4A_1\omega_0^4 - 3A_2i\omega_0^3 - 2A_3\omega_0^2 + A_4(i\omega_0)}{\omega_0{}^2\left(i\omega_0^5 + A_1\omega_0^4 - A_2i\omega_0^3 - A_3\omega_0^2 + A_4(i\omega_0) + A_5\right)}$$
$$+ \frac{4B_1\omega_0{}^4 - 3B_2i\omega_0{}^3 - 2B_3\omega_0{}^2 + B_4(i\omega_0)}{\omega_0{}^2\left(i\omega_0{}^5 + A_1\omega_0{}^4 - A_2i\omega_0{}^3 - A_3\omega_0{}^2 + A_4(i\omega_0) + A_5\right)} + \frac{i\tau}{\omega_0}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{5i\omega_0^5 + 4A_1\omega_0^4 - i3A_2\omega_0^3 - 2A_3\omega_0^2 + A_4(i\omega_0)}{\omega_0{}^2\left((A_1\omega_0^4 - A_3\omega_0^2 + A_5) + i(\omega_0^5 - A_2\omega_0^3 + A_4\omega_0)\right)}$$
$$+ \frac{4B_1\omega_0{}^4 - 3B_2i\omega_0{}^3 - 2B_3\omega_0{}^2 + B_4(i\omega_0)}{\omega_0{}^2\left((A_1\omega_0{}^4 - A_3\omega_0{}^2 + A_5) + i(\omega_0{}^5 - A_2\omega_0{}^3 + A_4\omega_0)\right)} + \frac{i\tau}{\omega_0}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}$$

$$\frac{5\omega_0^8 + \omega_0^6(4A_1^2 - 3A_2) + \omega_0^4(3A_2^2 + 6A_4 - 6A_1A_3) + \omega_0^2(2A_3^2 + 4A_1A_5 - 4A_2A_4) - (A_4^2 - 2A_3A_5)}{\omega_0^{10} + \omega_0^8(A_1^2 - 2A_2) + \omega_0^6(A_2^2 + 2A_4 - 2A_1A_3) + \omega_0^4(A_3^2 + 2A_1A_5 - 2A_2A_4) + \omega_0^2(A_4^2 - 2A_3A_5 + A_5^2)}$$
$$+ \frac{\omega_0^6(4A_1B_1 - 3B_2) - \omega_0^4(B_4 + 3A_2B_2 - 4A_3B_1 - 2A_1B_3) + \omega_0^2(4A_5B_1 + 2A_3B_3 + 3B_2A_4 + A_2B_4) + A_4B_4 - 2A_5B_3}{\omega_0^{10} + \omega_0^8(A_1^2 - 2A_2) + \omega_0^6(A_2^2 2A_4 - 2A_1A_3) + \omega_0^4(A_3^2 + 2A_1A_5 - 2A_2A_4) + \omega_0^2(A_4^2 - 2A_3A_5 + A_5^2)}$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1} > 0$$

Therefore the transversability conditions hold and hence Hopf bifurcation occurs at $\tau = \tau_0$

Theorem 2. If D_* exists with the condition (14)-(17) and $u = \omega^2$ be a positive root of (22) then there exists $\tau = \tau_0$ such that

(i) D_* is locally asymptotically stable for $0 \le \tau < \tau_0$,
(ii) D_* is unstable for $\tau > \tau_0$,
(iii) The system (1)-(5) undergoes Hopf bifurcation around D_* at$\tau = \tau_0$, where $\tau_{0n} = \frac{1}{\omega_0}\cos^{-1}[\frac{Q_1\omega^8+Q_2\omega^6+Q_3\omega^4+Q_4\omega^2+Q_5}{Q_6\omega^8+Q_7\omega^6+Q_8\omega^4+Q_9\omega^2+Q_{10}}] + \frac{2n\pi}{\omega_0}$; where $n = 0, 1, 2, \ldots$

Direction and Hopf-Bifurcation Analysis

Theorem 3

(i) The sign of Γdetermines the direction of the hopf bifurcation if $\Gamma > 0$then the Hopf bifurcation is supercritical otherwise it is subcritical.

$$\Gamma = -\frac{\mathrm{Re}\{\Delta(0)\}}{\mathrm{Re}\{\lambda^1(\tau_0)\}}$$

(ii) The sign of Ydetermines the stability of the bifurcated periodic solutions if $Y < 0$then the bifurcated periodic solutions are stable otherwise it is unstable.

$$Y = 2\mathrm{Re}\{\Delta(0)\}$$

(iii) The sign of Ω determines the period of the bifurcated periodic solutions if $\Omega > 0$, then the period of the bifurcated solutions increases otherwise it is decreases.

$$\Omega = -\frac{\mathrm{Im}\{\Delta(0)\} + \Gamma \mathrm{Im}\left\{\lambda^1(\tau_0)\right\}}{\tau_0\omega_0}$$

and $\Delta(0) = \frac{i}{2\tau_0\omega_0}[z_{11}z_{20} - 2|z_{11}|^2 - \frac{|z_{02}|^2}{3}] + \frac{z_{21}}{2}$

Proof: Let$t = \frac{t}{\tau}$, $u_1(t) = S(t) - S^*$, $u_2(t) = E(t) - E^*$, $u_3(t) = I(t) - I^*$, $u_4(t) = R(t) - R^* u_5(t) = V(t) - V^*$ and $\tau = \tau_0 + \Omega$, where $\Omega \in R$.Then the Hopf bifurcation occurs at $\Omega = 0$.Thus the system (1)-(5) becomes

$$\dot{u}(t) = L_\Omega u_t + G(\Omega, u_t) \tag{25}$$

Where$u_t = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))^T = (S, E, I, R, V)^T \in R^5$,$u_t(\phi) = u(t+\phi) \in C$=$C([-1, 0], R^5)$, and $L_\Omega : C \to R^5$and $G(\Omega, u_t) \to R^5$are given by

$$L_\Omega\Theta = (\tau_0 + \Omega)(H_{\max}\Theta(0) + J_{\max}\Theta(-1)),\ G(\Omega, \Theta) = (G_1, G_2, G_3, G_4, 0),\ \text{with}$$

$$H_{\max} = \begin{bmatrix} h_{11} & 0 & h_{13} & 0 & h_{15} \\ h_{21} & h_{22} & h_{23} & 0 & 0 \\ 0 & h_{32} & h_{33} & 0 & 0 \\ 0 & 0 & 0 & h_{44} & 0 \\ h_{51} & 0 & 0 & 0 & h_{55} \end{bmatrix}, J_{\max} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{33} & 0 & 0 \\ 0 & 0 & h_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1 = h_{16}\Theta_1^2(0) + h_{17}\Theta_3^2(0) + h_{18}\Theta_1(0)\Theta_3(0) + h_{19}\Theta_1^2(0)\Theta_3(0) + h_{110}\Theta_1(0)\Theta_3^2(0) + h_{111}\Theta_1^3(0) \\ +h_{112}\Theta_3^3(0) + \ldots,$$

$$G_2 = h_{24}\Theta_1^2(0) + h_{25}\Theta_3^2(0) + h_{26}\Theta_1(0)\Theta_3(0) + h_{27}\Theta_1^2(0)\Theta_3(0) + h_{28}\Theta_1(0)\Theta_3^2(0) \\ h_{29}\Theta_1^3(0) + h_{210}\Theta_3^3(0) + \ldots,$$

$$G_3 = h_{34}\Theta_3^2(-1) + h_{35}\Theta_3^3(-1) + \ldots,$$

$$G_4 = h_{45}\Theta_3^2(-1) + h_{46}\Theta_3^3(-1) + \ldots,$$

$$h_{16} = -\frac{\alpha I^3\left(I^2 - 3S^2\right)}{\left(S^2 + I^2\right)^3};\ h_{17} = \frac{\alpha S^2 I\left(S^2 + I^2\right) + 2\alpha S^2 I\left(S^2 - I^2\right)}{\left(S^2 + I^2\right)^3};\ h_{18} = \frac{\alpha S I^2\left(S^2 + I^2\right) - 4\alpha S^3 I^2}{\left(S^2 + I^2\right)^3};$$

$$h_{19} = \frac{-\alpha S^2 I^2\left(9S^2 - 14I^2\right) - \alpha I^6}{\left(S^2+I^2\right)^4};\ h_{110} = \frac{-12\alpha S^5 I + 5\alpha S^3 I^3 + 6\alpha S^4 I^3 - \alpha S I^5}{\left(S^2+I^2\right)^4};\ h_{111} = \frac{\alpha S I^3\left(I^2 - S^2\right)}{\left(S^2+I^2\right)^4};\ h_{112} = \frac{\alpha S^4 - 5\alpha S^4 I^2}{\left(S^2+I^2\right)^4};$$

$$h_{24} = \frac{\alpha S I^2\left(I - S^2\right)}{\left(S^2+I^2\right)^3};\ h_{25} = \frac{\alpha S^2 I^3 - 3\alpha S^4 I}{\left(S^2+I^2\right)^3};\ h_{26} = \frac{3\alpha S^3 I^2 - \alpha S I^4}{\left(S^2+I^2\right)^3};$$

$$h_{27} = \frac{-9\alpha S^4 I^2 + 14\alpha S^2 I^4 - \alpha I^6}{\left(S^2 + I^2\right)^4};\ h_{28} = \frac{-8\alpha S^3 I^3 + 3\alpha S^5 I + \alpha S I^5}{\left(S^2 + I^2\right)^4};$$

$$h_{29} = \frac{-5\alpha S^2 I^3 + 3\alpha S^4 I^2 - 3\alpha S^2 I^4 + \alpha I^5}{\left(S^2+I^2\right)^2};\ h_{210} = \frac{3\alpha S^4 I - \alpha S^2 I^3 + \alpha S^2 I^4 - \alpha S^6}{\left(S^2+I^2\right)^4};\ h_{34} = \frac{\beta a^2\left(3I^2 - a^2\right)}{\left(I^2+a^2\right)^3};$$

$$h_{35} = \frac{\beta a^2\left(2a^2 I - I^3\right)}{\left(I^2+a^2\right)^4};\ h_{45} = \frac{\beta a^2\left(a^2 - 3I^2\right)}{\left(I^2+a^2\right)^3};\ h_{46} = \frac{\beta a^2\left(I^3 - a^2 I\right)}{\left(I^2+a^2\right)^4};$$

By the Riesz representation theorem, there exist a matrix $\Xi(\phi,\Omega)$such that

$$L_\Omega\Theta=\int_{-1}^{0} d\Xi(\phi,\Omega)\Theta(\phi)$$

We choose $\Xi(\phi,\Omega)=(\tau_0+\Omega)(M_{\max}\delta(\phi)+N_{\max}\delta(\phi+1))$,where $\delta(\phi)$is the Dirac delta function. For$\Theta\varepsilon C([-1,0],R^5)$, define

$$A(\Omega)\Theta=\begin{cases}\frac{d\Theta(\phi)}{d\phi}, -1\le\phi<0,\\ \int_{-1}^{0} d\Xi(\phi,\Omega)\Theta(\phi),\ \phi=0\end{cases} \text{and}\, R(\Omega)\Theta=\begin{cases}0 \quad , \ -1\le\phi<0,\\ G(\Omega,\Theta), \quad \phi=0.\end{cases}$$

Then system (1)-(5) is equivalent to$\dot{w}(t)=A(\Omega)w_t+R(\Omega)w_t$.
For $\upsilon\varepsilon C^1([0,1],(R^5)^*)$, the adjoint operator $A^*$of $A(0)$ can be defined as

$$A^*(\upsilon)=\begin{cases}-\frac{d\upsilon(s)}{ds}, \quad 0<s\le1\\ \int_{-1}^{0} d\Xi^T(s,0)\upsilon(s),\ s=0\end{cases}$$

For $\Theta\varepsilon C([-1,0],R^5)$and $\upsilon\varepsilon C^1([0,1],(R^5)^*)$, define

$$\langle\upsilon(s),\Theta(\phi)\rangle=\bar{\upsilon}(0)\Theta(0)-\int_{\phi=-1}^{0}\int_{\xi=0}^{\phi}\bar{\upsilon}(\xi-\phi)d\Xi(\phi)\Theta(\xi)d\xi$$

Define the vector$\aleph(\phi)=(1,\aleph_2,\aleph_3,\aleph_4,\aleph_5)^Te^{i\omega_0\tau_0\phi}$,$\phi\in[-1,0]$is the eigenvector of A(0) corresponding to $i\omega_0\tau_0$and $\aleph_*(\phi)=(1,\aleph_{2_*},\aleph_{3_*},\aleph_{4_*},\aleph_{5_*})e^{i\omega_0\tau_0 s}$,$s\varepsilon(0,1)$, is the eigenvector of $A^*(0)$corresponding to $-i\omega_0\tau_0$.Then, we have

$$\aleph_2=\frac{a_{21}+a_{23}\aleph_3}{i\omega_0-a_{22}},\aleph_3=\frac{i\omega_0-a_{11}}{a_{13}}-\frac{a_{15}a_{51}}{a_{13}(i\omega_0-a_{55})},\aleph_4=\frac{b_{43}e^{-i\tau_0\omega_0}\aleph_3}{i\omega_0-a_{44}},\aleph_5=\frac{a_{51}}{i\omega_0-a_{55}}$$

$$\aleph_{2_*}=\frac{a_{15}a_{51}}{a_{21}(i\omega_0+a_{55})}-\frac{i\omega_0+a_{11}}{a_{21}},\aleph_{3_*}=-\frac{(i\omega_0+a_{22})\aleph_2}{a_{32}},$$

$$\aleph_{4_*}=-\frac{\left(i\omega_0+a_{33}+b_{33}e^{i\tau_0\omega_0}\right)\aleph_{3_*}-a_{23}\aleph_{2_*}+a_{13}}{b_{43}e^{i\tau_0\omega_0}},\aleph_{5_*}=-\frac{a_{15}}{i\omega_0+a_{55}}$$

Furthermore, we have

$$\bar{V}=\left[1+\aleph_2\bar{\aleph}_2+\aleph_3\bar{\aleph}_3+\aleph_4\bar{\aleph}_4+\aleph_5\bar{\aleph}_5+\tau_0e^{-i\tau_0\omega_0}\aleph_3\left(b_{33}\bar{\aleph}_3+b_{43}\bar{\aleph}_4\right)\right]^{-1},$$

Which leads to $\langle\aleph_*,\aleph\rangle=1$and $\langle\aleph_*,\bar{\aleph}\rangle=0$.Next based on the algorithms in [23,25–27] and the similar computation process as that in [23,25–27], we obtain

$$h_{20}=2\tau_0\bar{V}\left[a_{16}+a_{17}\aleph_3^2+a_{18}\aleph_3+\bar{\aleph}_{2_*}\left(a_{24}+a_{25}\aleph_3^2+a_{26}\aleph_3\right)+\left(a_{34}\bar{\aleph}_{3_*}+a_{45}\bar{\aleph}_{4_*}\right)\aleph_3^2e^{-2i\tau_0\omega_0}\right],$$

$$h_{11}=\tau_0\bar{V}\begin{bmatrix}2a_{16}+2a_{17}\aleph_3\bar{\aleph}_3+2a_{18}\mathrm{Re}\{\aleph_3\}+\bar{\aleph}_{2_*}\left(2a_{24}+2a_{25}\aleph_3\bar{\aleph}_3+2a_{16}\mathrm{Re}\{\aleph_3\}\right)\\+2\left(a_{34}\bar{\aleph}_{3_*}+a_{45}\bar{\aleph}_{4_*}\right)\aleph_3\bar{\aleph}_3\end{bmatrix},$$

$$h_{02}=2\tau_0\bar{V}\left[a_{16}+a_{17}\bar{\aleph}_3^2+a_{18}\bar{\aleph}_3+\bar{\aleph}_{2_*}\left(a_{24}+a_{25}\bar{\aleph}_3^2+a_{26}\bar{\aleph}_3\right)+\left(a_{34}\bar{\aleph}_{3_*}+a_{45}\bar{\aleph}_{4_*}\right)\bar{\aleph}_3^2e^{2i\tau_0\omega_0}\right],$$

$$h_{21}=2\tau_0\bar{V}\begin{bmatrix}a_{16}\left(2w_{11}^{(1)}(0)+w_{20}^{(1)}(0)\right)+h_{17}\left(2w_{11}^{(3)}(0)\aleph_3+w_{20}^{(3)}(0)\bar{\aleph}_3\right)\\+h_{18}\begin{pmatrix}w_{11}^{(1)}(0)\aleph_3+\\ \frac{1}{2}w_{20}^{(1)}(0)\bar{\aleph}_3+w_{11}^{(3)}(0)+\frac{1}{2}w_{20}^{(3)}(0)\end{pmatrix}\\+h_{19}\left(\bar{\aleph}_3+2\aleph_3\right)+h_{110}\left(\aleph_3^2+2\aleph_3\bar{\aleph}_3\right)+3h_{111}+3h_{112}\aleph_3^2\bar{\aleph}_3\\+\bar{\aleph}_{2_*}\begin{pmatrix}+h_{25}\left(2w_{11}^{(3)}(0)\aleph_3+w_{20}^{(3)}(0)\bar{\aleph}_3\right)\\+h_{26}\left(w_{11}^{(1)}(0)\aleph_3+\frac{1}{2}w_{20}^{(1)}(0)\bar{\aleph}_3+w_{11}^{(3)}(0)+\frac{1}{2}w_{20}^{(3)}(0)\right)\\+h_{27}\left(2\aleph_3+\bar{\aleph}_3\right)+\\h_{28}\left(\aleph_3^2+2\aleph_3\bar{\aleph}_3\right)+3h_{29}+3h_{210}\aleph_3^2\bar{\aleph}_3\end{pmatrix}\\+\bar{\aleph}_{3_*}\left(h_{34}\left(2w_{11}^{(3)}(-1)\aleph_3e^{-i\tau_0\omega_0}+w_{20}^{(3)}(-1)\bar{\aleph}_3e^{i\tau_0\omega_0}\right)+3h_{35}\aleph_3^2\bar{\aleph}_3e^{-i\tau_0\omega_0}\right)\\+\bar{\aleph}_{4_*}\left(h_{45}\left(2w_{11}^{(3)}(-1)\aleph_3e^{-i\tau_0\omega_0}+w_{20}^{(3)}(-1)\bar{\aleph}_3e^{i\tau_0\omega_0}\right)+3h_{46}\aleph_3^2\bar{\aleph}_3e^{-i\tau_{0\omega_0}}\right)\end{bmatrix}$$

Where

$$w_{20}(\phi) = \frac{ih_{20}\aleph(0)}{\tau_0 w_0} e^{i\tau_0\omega_0\phi} + \frac{i\bar{h}_{02}\bar{\aleph}(0)}{3\tau_0\omega_0} e^{-i\tau_0 w_0\phi} + Q_1 e^{2i\tau_0\omega_0\phi}, \quad w_{11}(\phi) = \frac{ih_{11}\aleph(0)}{\tau_0\omega_0} e^{i\tau_0\omega_0\phi} + \frac{i\bar{h}_{11}\bar{\aleph}(0)}{\tau_0\omega_0} e^{-i\tau_0 w_0\phi} + Q_2,$$

Also where Q_1 and Q_2can be obtained by the following two equations

$$Q_1 = 2\begin{bmatrix} 2i\omega_0 - a_{11} & 0 & -a_{13} & 0 & -a_{15} \\ -a_{21} & 2i\omega_0 - a_{22} & -a_{23} & 0 & 0 \\ 0 & -a_{32} & 2i\omega_0 - a_{33} - b_{33}e^{-2i\tau_0\omega_0} & 0 & 0 \\ 0 & 0 & -b_{43}e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_{44} & 0 \\ -a_{51} & 0 & 0 & 0 & 2i\omega_0 - a_{55} \end{bmatrix} X \begin{bmatrix} Q_1^{(1)} \\ Q_1^{(2)} \\ Q_1^{(3)} \\ Q_1^{(4)} \\ 0 \end{bmatrix}$$

$$Q_2 = -\begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} + b_{33} & 0 & 0 \\ 0 & 0 & b_{43} & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{bmatrix} \begin{bmatrix} Q_2^{(1)} \\ Q_2^{(2)} \\ Q_2^{(3)} \\ Q_2^{(4)} \\ 0 \end{bmatrix}$$

with

$$Q_1^{(1)} = h_{16} + h_{17}\aleph_3^2 + h_{18}\aleph_3, Q_1^{(2)} = h_{24} + h_{25}\aleph_3^2 + h_{26}\aleph_3, Q_1^{(3)} = h_{34}\aleph_3^2 e^{-2i\tau_0\omega_0}, Q_1^{(4)} = h_{45}\aleph_3^2 e^{-2i\tau_0\omega_0}$$

$$Q_2^{(1)} = 2h_{16} + 2h_{17}\aleph_3\bar{\aleph}_3 + 2h_{18}\mathrm{Re}\{\aleph_3\}, Q_2^{(2)} = 2h_{24} + 2h_{25}\aleph_3\bar{\aleph}_3 + 2h_{26}\mathrm{Re}\{\aleph_3\}, Q_2^{(3)} = 2h_{34}\aleph_3\bar{\aleph}_3,$$

$$Q_2^{(4)} = 2h_{45}\aleph_3\bar{\aleph}_{3.}$$

Then one can obtain the expressions of $h_{20}, h_{11}, h_{02} and\, h_{21}$,the equation (25) can be obtained. Thus we have theorem (3) and the proof is completed.

Numerical simulations

In this section, we present numerical simulation using Matlab software to validate our analytical findings. The numerical simulation was performed with the following parameters: $\wedge = 2$, $\alpha = 0.27$, $\beta = 0.003$, $\eta = 0.2$, $\mu = 0.003$, $\delta_0 = 0.025$, $\delta_2 = 0.045$, $\delta_1 = 0.2$, $\delta_3 = 0.003$.

In the absence of delay, the endemic equilibrium point D_*(13.3157, 7.3829, 19.9159, 37.8929, and 0.1774) is locally asymptotically stable and the corresponding time series is shown in Fig. 1(a). In the presence of delay, for the value of $\tau = 50.16 < 66.16$, the endemic equilibrium point is D_*(13.3157, 7.3829, 19.9159, 37.8929, 0.1774) is locally asymptotically stable and the dynamical behavior of time series is shown in Fig. 1(b). Furthermore, we increased the delay value; the system (1-5) undergoes a Hopf-Bifurcation at the endemic equilibrium point D_*(13.3157, 7.3829, 19.9159, 37.8929, and 0.1774) and a family of bifurcating periodic solutions Fig. 1.(c) shows the bifurcation from D_*(13.3157, 7.3829, 19.9159, 37.8929, and 0.1774) using the corresponding time series.

Fig. 2 shows the corresponding phase portraits of S-I-E for delay values of 50.16 and 66.16, respectively Fig. 3. shows the corresponding phase portraits of I-R-V for delay values of 50.16 and 66.16, respectively Fig. 4. shows the corresponding phase portraits of E-I-V for delay values of 50.16 and 66.16, respectively Fig. 5. shows the corresponding phase portraits of S-R-E for delay values of 50.16 and 66.16, respectively Fig. 6. shows the corresponding phase portraits of S-I-R for delay values of 50.16 and 66.16, respectively. Note that, with the exception of Fig. 1, in other Figs., the dynamical behaviours of $\tau = 50.16$ appear on the left (L) hand side, while those of $\tau = 66.16$ are shown on the right (R) hand side. By placing Figs. for different delay values side by side, one can clearly observe that those on the right display more oscillations than those on the left.

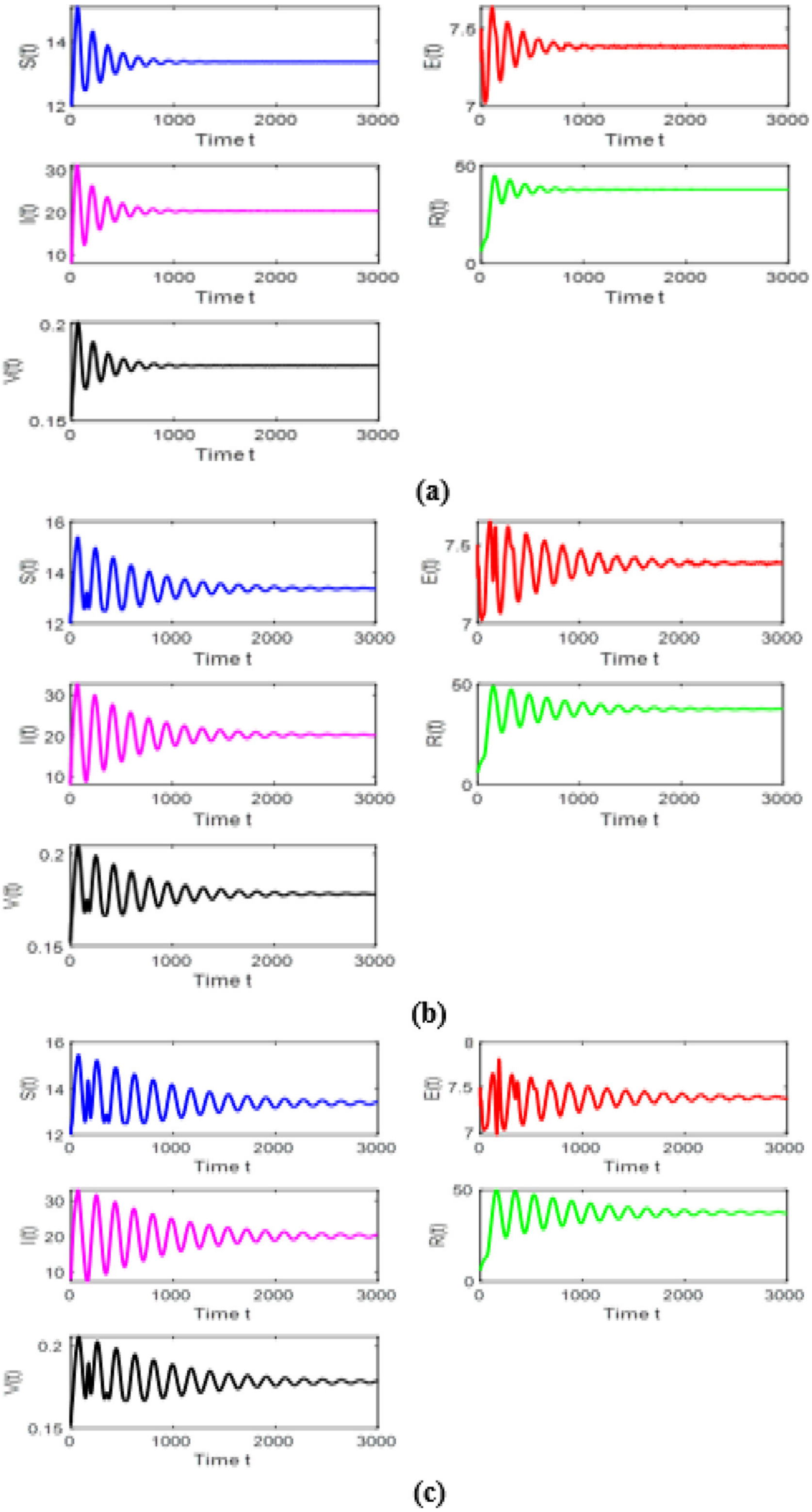

Fig. 1. Time series evaluation of S, E, I, R and V computers with attributes mentioned in section 5 for absence of delay (a), and at τ = 50.16 (b), 66.16 (c).

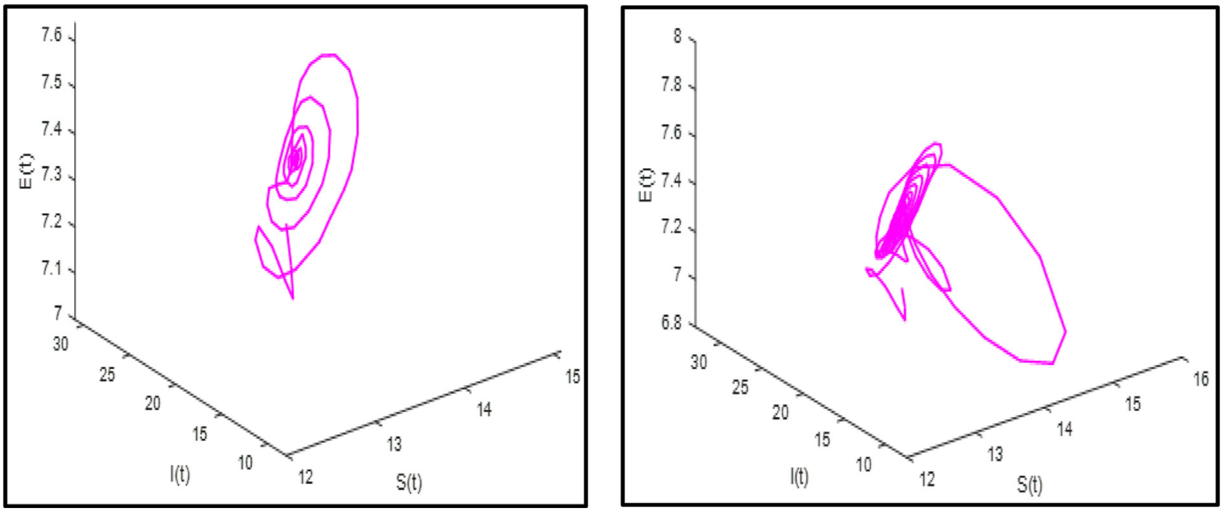

Fig. 2. Dynamical behaviour of S-I-E in system (1-5) with $\tau = 50.16$ (L) and 66.16 (R).

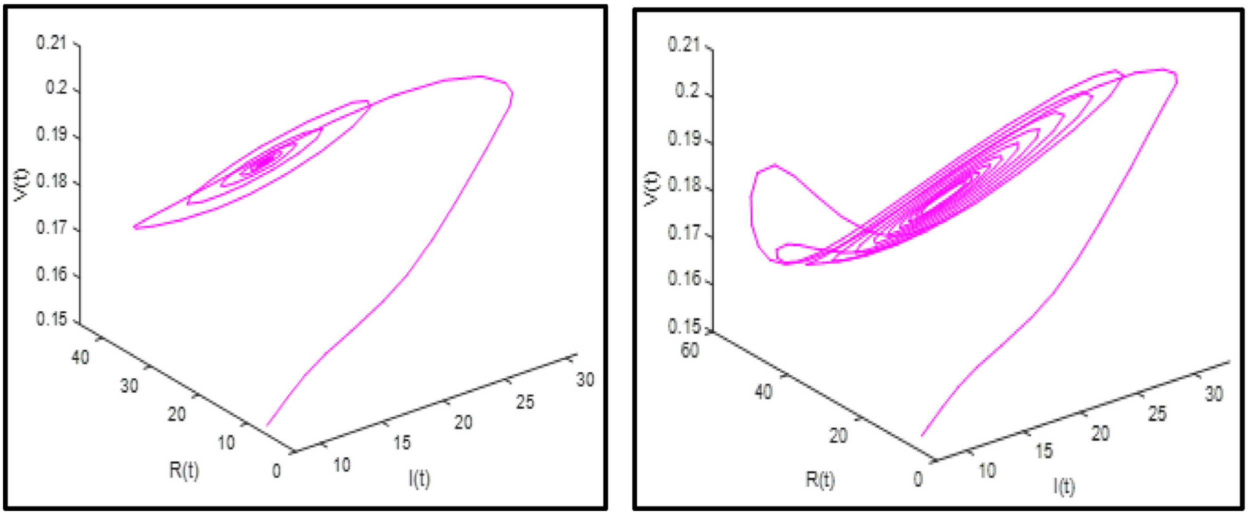

Fig. 3. Dynamical behaviour of I-R-V in system (1-5) with $\tau = 50.16$ (L) and 66.16 (R).

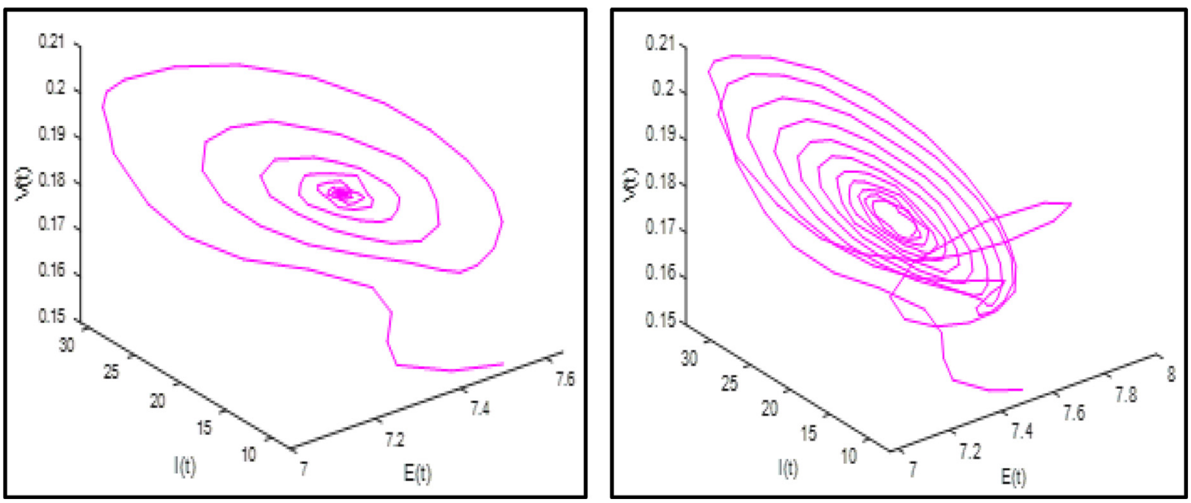

Fig. 4. Dynamical behaviour of E-I-V in system (1-5) with $\tau = 50.16$ (L) and 66.16 (R).

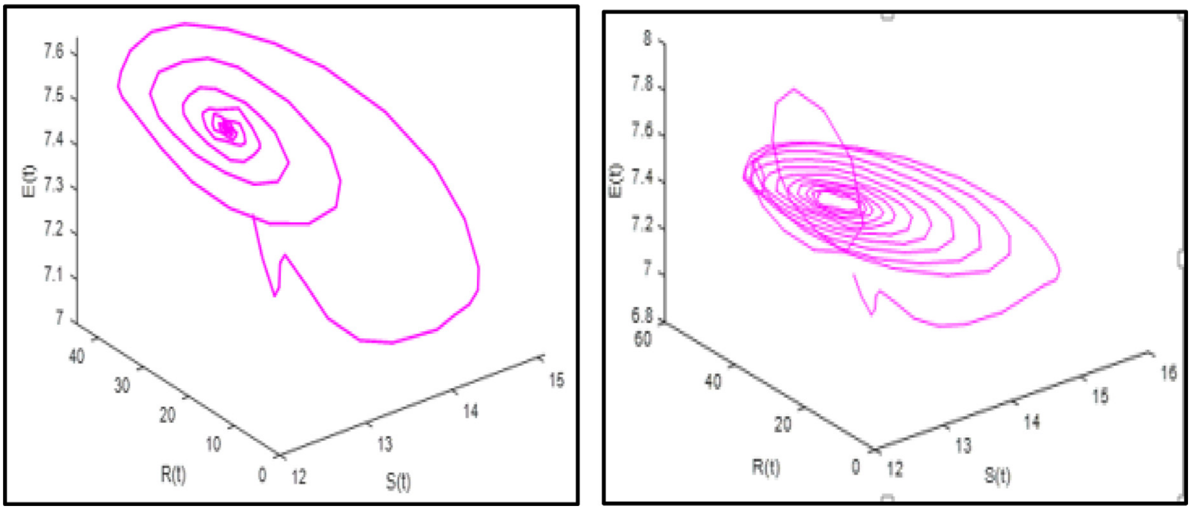

Fig. 5. Dynamical behaviour of S-R-E with τ = 50.16 (L) and 66.16 (L).

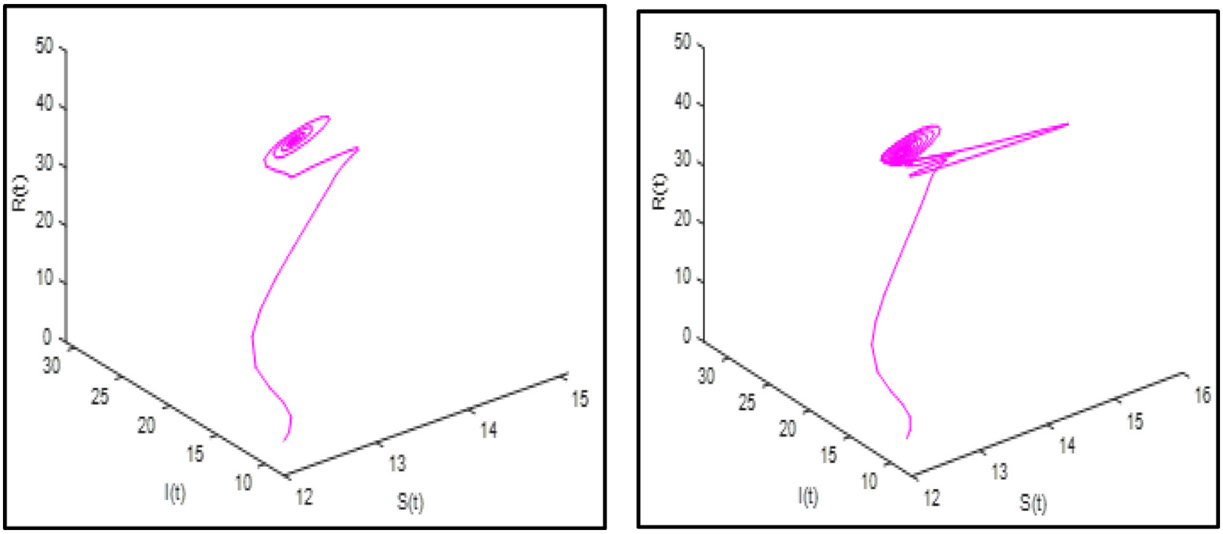

Fig. 6. Dynamical behaviour of S-I-R with τ = 50.16 (L) and 66.16 (L).

Conclusion

In this study, we suggested a delayed SVEIR model with the nonlinear Holling Type III incidence function as well as the integration of time delay caused by the cleaning of infected computers using antivirus software. The primary interest of the model considered in this paper is the effects due to the delay, and the key findings are expressed in terms of viral equilibrium stability and Hopf bifurcation. Additionally, we demonstrated that when the delay value is less than the critical value, the virus propagation can be managed. Considering the ratio-dependent functional response nature of this incidence type, our work above showed nice oscillations, unlike the output in ref [29]. This is clearly evident if one considers the susceptible and infected populations. As we envisaged, our outputs are finer compared to the density dependent approach. Furthermore, when the value of time delay goes past the critical value, a Hopf bifurcation takes place, indicating that computers in the compartments can exist in an oscillatory fashion that encourages the total elimination of the viral infection. As a result, bifurcation management techniques should be used to avoid the occurrence of the Hopf bifurcation; this will be the focus of our work in the near future. Also, the effects of the novel treatment function will be applied to some computer worm models that consider both horizontal and vertical transmission approaches to disease modelling.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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