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Wijsman and Wijsman regularly triple ideal convergence sequences of sets

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ABSTRACT

This paper introduces the notions of I_{W_3}-convergence, $I^*_{W_3}$-convergence, I_{W_3}-Cauchy, $I^*_{W_3}$-Cauchy, regularly (I_{W_3}, I_W)-convergence and regularly $(I^*_{W_3}, I^*_W)$-convergence and investigates the relationship among them.

Introduction

The concept of convergence of sequences of real numbers has been extended to statistical convergence independently by Fast [6,20]. Later, it was defined for triple sequences by Sahiner et al. [18,19]. A triple sequence (x_{nmj}) is called convergent to L in Pringsheim's sense if for every $\varepsilon > 0$ there exists $p_0(\varepsilon) \in \mathbb{N}$ such that $|x_{nmj} - L| < \epsilon$, for $n, m, j \geq p_0$. A subset P of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have density $\rho(P)$ if the limit given by

$$\rho(P) = \lim_{q,w,e\to\infty} \frac{1}{qwe} \sum_{n\leq q} \sum_{m\leq w} \sum_{j\leq e} \chi_P(n, m, j) \text{ exists.}$$

On the other hand, the idea of I-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of the set of natural numbers. Sahiner and Tripathy [19] introduced the notion I-convergence for triple sequences in a metric space and studied their properties. This notion has been studied by many mathematicians in different fields of mathematics (see [5,21,22]),

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets, (see, [1–3]). Nuray and Rhoades [16] studied statistical convergence set sequences and gave some basic theorems. Then, Nuray et al. [17]studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigated the relationship between them. Kisi and Nuray [15] introduced a new convergence notion for sequences of sets which is called Wijsman I-convergence. Recently, Dundar and Pancaroglu [4] extended this notion to Wijsman regularly ideal convergence of double sequences of sets. In this paper, we extend those

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notions for triple sequences of sets and investigate some relationship among them. For more notions related to double or triple sequences, we refer the reader to Granados [7–13,23–29].

Throughout this paper, $I_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ denotes a strongly admissible ideal, and (X,ρ) denotes a separable metric space and Q, Q_{nmj} are any non-empty closed subsets of X. Besides, for any point $x \in X$ and any non-empty subset Q of X, we define the distance from x to Q by $d(x,Q) = \inf_{q\in Q} \rho(x,q)$. Furthermore, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. Next, we recall some definitions and notions which are useful for the development of this paper.

Definition 1.1. An ideal I on X is a collection of non-empty subsets of X which satisfies the following properties:

1. $\emptyset \in I$,
2. If $A, B \in I$, then $A \cup B \in I$,
3. If $A \subset B$ and $A \in I$, then $B \in I$.

Additionally, if $X \notin I$, then I is called non-trivial. A non-trivial ideal I on X is called admissible if $\{x\} \in I$ for each $x \in X$. On the other hand, if I is a non-trivial ideal in X, the class $F(I) = \{M \subset X : (\exists A \in I)(M = X - A)\}$ is a filter on X which is called the filter associated with I.

Definition 1.2. A non trivial ideal I_3 of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to be strongly admissible if $\{i\}\times\mathbb{N}\times\mathbb{N}$, $\mathbb{N}\times\mathbb{N}\times\{i\}$ and $\mathbb{N}\times\{i\}\times\mathbb{N}$ belong to I_3 for each $i \in \mathbb{N}$. It is clear that a strongly admissible ideal is an admissible ideal.

Definition 1.3. $I_3^0 = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j, k \geq m(A) \Rightarrow (i, j, k) \notin A)\}$. Then, I_3^0 is a non-trivial strongly admissible ideal and we can see that I_3 is a strongly admissible ideal if and only if $I_3^0 \subset I_3$.

Definition 1.4. An admissible ideal $I_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP3) if for every countable family of mutually disjoint sets $\{Q_1, Q_2, ...\}$ belong to I_3, there exits a countable family of sets $\{W_1, W_2, ...\}$ such that $Q_j \Delta W_j \in I_3^0$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $W = \bigcup_{j=1}^{\infty} W_j \in I_3$, therefore $W_j \in I_3$ for each $j \in \mathbb{N}$.

Wijsman I_3-convergence of triple sequences

In this section, we introduce ans study the notions of I_{W_3}-convergence, $I^*_{W_3}$-convergence, I_{W_3}-Cauchy, $I^*_{W_3}$-Cauchy and investigate relations among them.

Definition 2.1. A triple sequence of sets $\{Q_{nmj}\}$ is Wijsman convergent to Q if for each $x \in X$,
$\lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$

Definition 2.2. A triple sequence of sets $\{Q_{nmj}\}$ is I_{W_3}-convergent to Q if for every $\varepsilon > 0$ and each $x \in X$,
$\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x, Q_{nmj}) - d(x, Q)| \geq \varepsilon\} \in I_3$
Is is denotes by $I_{W_3} - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(X, Q)$.

Definition 2.3. A triple sequence of sets $\{Q_{nmj}\}$ is $I^*_{W_3}$-convergent to Q if there exits a set $M_3 \in F(I_3)$, this means $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M_3 = H \in I_3$ such that for each $x \in X$,
$\lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$ and $(n, m, j) \in M_3$
It is denotes by $I^*_{W_3} - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$.

Theorem 2.4. *If a triple sequence of sets $\{Q_{nmj}\}$ is $I^*_{W_3}$-convergent, then it is I_{W_3}-convergent.*

Proof. Since $I^*_{W_3} - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$, there exists a set $M_3 \in F(I_3)$ (i.e. $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M_3 = H \in I_3$) such that for each $x \in X$
$\lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$, where $(n, m, j) \in M_3$.

Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that for each $x \in X$, $|d(x, Q_{nmj}) - d(x, Q)| < \epsilon$ for all $(n, m, j) \in M_3$ and $n, m, j \geq n_0$. Then, for each $\varepsilon > 0$ and $x \in X$, we have that

$Y(\varepsilon, x) = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x, Q_{nmj}) - d(x, Q)| \geq \varepsilon\} \subset H \cup [M_3 \cap ((\{1, 2, ..., (n_0 - 1)\}\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\{1, 2, ..., (n_0 - 1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1, 2, ..., (n_0 - 1)\}))]$.

Since $H \cup [M_3 \cap ((\{1, 2, ..., (n_0 - 1)\}\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\{1, 2, ..., (n_0 - 1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1, 2, ..., (n_0 - 1)\}))] \in I_3$. We have $Y(\varepsilon, x) \in I_3$. Therefore, $I_{W_3} - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$. □

Theorem 2.5. *If the ideal I_3 has the property (AP3), then I_{W_3}-convergent implies $I^*_{W_3}$-convergence of triple sequences of sets.*

Proof. Suppose that I_3 satisfies property (AP3). Now, let $I_{W_3} - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$. Then,

$Y(\varepsilon, x) = Y_\varepsilon = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x, Q_{nmj}) - d(x, Q)| \geq \varepsilon\} \in I_3$ for each $\epsilon > 0$ and for each $x \in X$. Take
$Y_1 = Y(1, x) = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x, Q_{nmj}) - d(x, Q)| \geq 1\}$ and

$Y_n = Y(n,x) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{n} \le |d(x,Q_{nmj}) - d(x,Q)| < \frac{1}{n-1}\}$ for $n \ge 2$ and $n \in \mathbb{N}$. It is clear that $Y_i \cap Y_k = \epsilon$ for $i \ne k$ and $Y_i \in I_3$ for each $i \in \mathbb{N}$. By the property (AP3) there exits a sequence of sets $\{V_n\}_{n\in\mathbb{N}}$ such that $Y_k \Delta V_k$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ for each k and $V = \bigcup_{k=1}^{\infty} V_k \in I_3$. Now, we shall prove that for $M_3 = \mathbb{N}\times\mathbb{N}\times\mathbb{N} - V$ we have

$\lim_{n,m,j\to\infty} d(x,Q_{nmj}) = d(x,Q)$, where $(n,m,j) \in M_3$.

Let $\gamma > 0$ be given. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \gamma$. Then,

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x,Q_{nmj}) - d(x,Q)| \ge \gamma\} \subset \bigcup_{n=1}^{k} Y_n.$$

Since $Y_k \Delta V_k$, $k = 1,2,\ldots$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$(\bigcup_{n=1}^{k} Y_n) \cap \{(n,m,j) : n \ge n_0 \wedge m \ge n_0 \wedge j \ge n_0\} = (\bigcup_{n=1}^{k} V_n) \cap \{(n,m,j) : n \ge n_0 \wedge m \ge n_0 \wedge j \ge n_0\}$

If $n,m,j > n_0$ and $(n,m,j) \notin V$, then $(n,m,j) \notin \bigcup_{n=1}^{k} V_n$ and $(n,m,j) \notin \bigcup_{n=1}^{k} T_n$. This implies that $|d(x,Q_{nmj}) - d(x,Q)| < \frac{1}{n} < \gamma$.

Therefore, $\lim_{n,m,j\to\infty} d(x,Q_{nmj}) = d(x,Q)$, where $(n,m,j) \in M_3$. □

Definition 2.6. A triple sequence of sets $\{Q_{nmj}\}$ is I_{W_3}-Cauchy if for every $\varepsilon > 0$ and each $x \in X$, there exits $p = p(\varepsilon)$, $q = q(\varepsilon)$ and $r = r(\varepsilon)$ in $\mathbb{N}$ such that

$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x,Q_{nmj}) - d(x,Q_{pqr})| \ge \varepsilon\} \in I_3$

Theorem 2.7. *If a triple sequence of sets $\{Q_{nmj}\}$ is I_{W_3}-convergent, then it is I_{W_3}-Cauchy.*

Proof. Let $I_{W_3} - \lim Q_{nmj} = Q$. Then, for each $\varepsilon > 0$ and for each $x \in X$, we have

$V(x,\varepsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x,Q_{nmj}) - d(x,Q)| \ge \varepsilon\} \in I_3$

Since I_3 is a strongly admissible ideal, there exists a $p,q,r \in \mathbb{N}$ such that $(p,q,r) \notin V(x,\varepsilon)$. Now, let

$U(x,\varepsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x,Q_{nmj}) - d(x,Q_{pqr})| \ge 2\varepsilon\}$ taking into account the inequality

$|d(x,Q_{nmj}) - d(x,Q_{pqr})| \le |d(x,Q_{nmj}) - d(x,Q)| + |d(x,Q_{pqr} - d(x,Q)|$,

We can see that if $(n,m,j) \in U(x,\epsilon)$ then $|d(x,Q_{nmj}) - d(x,Q)| + |d(x,Q_{pqr} - d(x,Q)| \ge 2\varepsilon$. On the other hand, since $(n,m,j) \notin V(x,\varepsilon)$, we have $|d(x,Q_{pqr}) - d(x,Q)| < \varepsilon$.

By the mentioned above, we can imply that $|d(x,Q_{nmj}) - d(x,Q)| \ge \epsilon$, hence $(n,m,j) \in V(x,\varepsilon)$. Therefore, $U(x,\varepsilon) \subset V(x,\varepsilon) \in I_3$ for each $\varepsilon > 0$ and for each $x \in X$. This shows that $U(x,\varepsilon) \in I_3$ and then $\{Q_{nmj}\}$ is Wijsman I_3-Cauchy triple sequence. □

Definition 2.8. A triple sequence of sets $\{Q_{nmj}\}$ is $I^*_{W_3}$-Cauchy if there exits a set $M_3 \in F(I_3)$, this means that $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M_3 = H \in I_3$ such that for each $x \in X$ and $(n,m,j),(p,q,r) \in M_3$,

$\lim_{n,m,j,p,q,r\to\infty} |d(x,Q_{nmj}) - d(x,Q_{pqr})| = 0$

Theorem 2.9. *If a triple sequence of sets $\{Q_{nmj}\}$ is $I^*_{W_3}$-Cauchy, then it is I_{W_3}-Cauchy.*

Proof. Let $\{Q_{nmj}\}$ is Wijsman $I^*_{W_3}$-Cauchy triple sequence, then by the definition, there exist a set $M_3 \in F(I_3)$ (i.e., $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M_3 = H \in I_3$) such that for each $\varepsilon > 0$ and for each $x \in X$,

$|d(x,Q_{nmj}) - d(x,Q_{pqr})| < \varepsilon$ for all $(n,m,j),(p,q,r) \in M_3$, $n,m,j,p,q,r > N = N(x,\varepsilon)$ and $N \in \mathbb{N}$. Then, for each $\varepsilon > 0$ and $x \in X$, we have

$V(\varepsilon,x) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |d(x,Q_{nmj}) - d(x,Q_{pqr})| \ge \varepsilon\} \subset H \cup (M_3 \cap ((\{1,2,\ldots,(N-1)\}\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(N-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1,2,\ldots,(N-1)\})))$

Since $H \cup (M_3 \cap ((\{1,2,\ldots,(N-1)\}\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(N-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1,2,\ldots,(N-1)\}))) \in I_3$, so we have $V(\varepsilon,x) \in I_3$. Therefore, $\{Q_{nmj}\}$ is I_{W_3}-Cauchy triple sequence. □

Theorem 2.10. *A triple sequence of sets $\{Q_{nmj}\}$ is $I^*_{W_3}$-convergent, then it is I_{W_3}-Cauchy.*

Proof. Let $I^*_{W_3} - \lim_{n,m,j\to\infty} d(x,Q_{nmj}) = d(x,Q)$, so there exists a set $M_3 \in F(I_3)$ (i.e., $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M_3 = H \in I_3$) such that for each $x \in X$

$\lim -n,m,j \to \infty d(x,Q_{nmj}) = d(x,Q)$, where $(n,m,j) \in M_3$

Let $\varepsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that for each $x \in X$, $|d(x,Q_{nmj}) - d(x,Q)| < \frac{\varepsilon}{2}$, for all $(n,m,j) \in M_3$ and $n,m,j \ge n_0$. Then for each $\varepsilon > 0$ and $x \in X$, we have

$|d(x,Q_{nmj}) - d(Q_{pqr})\| < |d(x,Q_{nmj}) - d(x,Q)| + d(x,Q_{pqr}) - d(x,Q)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$.

Therefore, for each $x \in X$ and $(n,m,j),(p,q,r) \in M_3$, we have

$\lim_{n,m,j,p,q,r\to\infty} |d(x,Q_{nmj}) - d(x,Q_{pqr})| = 0$

This implies that $\{Q_{nmj}\}$ is $I^*_{W_3}$-Cauchy triple sequence and by the Theorem 2.9, $\{Q_{nmj}\}$ is I_{W_3}-Cauchy triple sequence. □

The following diagrams show the results that we obtained in this section.

Diagram I

$I^*_{W_3}$-convergence $\rightarrow$ I_{W_3}-convergence $\rightarrow$ I_{W_3}-Cauchy
$\uparrow$
$I^*_{W_3}$-Cauchy
If I_3 has $AP3$ property,
Diagram II
$I^*_{W_3}$-convergence $\leftrightarrow$ I_{W_3}-convergence $\rightarrow$ I_{W_3}-Cauchy
$\uparrow$
$I^*_{W_3}$-Cauchy

Wijsman regularly I_3-convergence of triple sequences

In this section, we introduce and study the notions of regularly (I_{W_3}, I_W)-convergence and regularly $(I^*_{W_3}, I^*_W)$-convergence and show relations among them.

Theorem 3.1. *Let $\{Q_{nmj}\}$ be a triple sequence of sets. Then, for each $x \in X$,* $\lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$ *implies* $I_3 - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$.

Proof. Let $\lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$. Then, for every $\epsilon > 0$ and each $x \in X$ there exists $n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ such that $|d(x, Q_{nmj}) - d(x, Q)| < \varepsilon$, for all $n, m, j > n_0$. Hence, for each $x \in X$ we have $P(\varepsilon)) = \{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |d(x, Q_{nmj}) - d(x, Q)| \geq \varepsilon\} \subset ((\mathbb{N} \times \mathbb{N} \times \{1, 2, ..., n_0\}) \cup (\mathbb{N} \times \{1, 2, ..., n_0\} \times \mathbb{N}) \cup (\{1, 2, ..., n_0\} \times \mathbb{N} \times \mathbb{N})) \in I_3$ and then, we have that $P(\epsilon) \in I_3$. □

Definition 3.2. A triple sequence $\{Q_{nmj}\}$ is said to be Wijsman regularly convergent ($R(W_3, W)$-convergent) if it is convergent in Pringsheim's sense and fore each $x \in X$ the limits $\lim_{m\to\infty} d(x, Q_{nmj})(n \in \mathbb{N})$, $\lim_{n\to\infty} d(x, Q_{nmj})(j \in \mathbb{N})$ and $\lim_{j\to\infty} d(x, Q_{nmj})(m \in \mathbb{N})$ exist.

We can see that if $\{Q_{nmj}\}$ is Wijsman regularly convergent to Q, the limits

$$\lim_{n\to\infty}\lim_{m\to\infty}\lim_{j\to\infty} d(x, Q_{nmj} = d(x, Q),$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{j\to\infty} d(x, Q_{nmj}) = d(x, Q),$$

$$\lim_{m\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} d(x, Q_{nmj}) = d(x, Q),$$

$$\lim_{j\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} d(x, Q_{nmj}) = d(x, Q),$$

$$\lim_{n\to\infty}\lim_{j\to\infty}\lim_{m\to\infty} d(x, Q_{nmj}) = d(x, Q)$$

$$\lim_{j\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} d(x, Q_{nmj}) = d(x, Q)$$

exist. In this case we write $R(W_3, W) - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$ or $Q_{nmj} \xrightarrow{R(W_3,W)} Q$

Definition 3.3. A triple sequence $\{Q_{nmj}\}$ is said to be regularly (I_{W_3}, I_W)-convergent ($R(I_{W_3}, I_W)$-convergent) if it is I_{W_3}-convergent in Pringsheim's sense and for every $\varepsilon > 0$ and each $x \in X$, the following hold:
$\{j \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, K_m)| \geq \varepsilon\} \in I$, for some $K_m \in X$ and each $m \in \mathbb{N}$,
$\{m \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, L_n)| \geq \varepsilon\} \in I$, for some $L_n \in X$ and each $n \in \mathbb{N}$,
$\{n \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, O_j)| \geq \varepsilon\} \in I$, for some $O_j \in X$ and each $j \in \mathbb{N}$.
If $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent to Q, then we write

$$R(I_{W_3}, I_W) - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q) \text{ or } Q_{nmj} \xrightarrow{R(I_{W_3},I_W)} Q.$$

Theorem 3.4. *If a triple sequence $\{Q_{nmj}\}$ is $R(W_3, W)$-convergent, then $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent.*

Proof. Let $\{Q_{nmj}\}$ is $R(W_3, W)$-convergent to Q. Then, $\{Q_{nmj}\}$ is convergent to Q in Pringsheim's sense and for each $x \in X$ the limits $\lim_{m\to\infty} d(x, Q_{nmj})(n \in \mathbb{N})$, $\lim_{n\to\infty} d(x, Q_{nmj})(j \in \mathbb{N})$ and $\lim_{j\to\infty} d(x, Q_{nmj})(m \in \mathbb{N})$ exist. By the Theorem 3.1, we get that $\{Q_{nmj}\}$ is I_{W_3}-convergent. Besides, for every $\epsilon > 0$ and each $x \in X$ there exist $j = j - 0(\varepsilon, x)$, $n = n_0(\varepsilon, x)$ and $m = m_0(\varepsilon, x)$ such that $|d(x, Q_{nmj}) - d(x, Q)| < \varepsilon$ for each fixed $n \in \mathbb{N}$ and $j > j_0$, $|d(x, Q_{nmj}) - d(x, P)| < \varepsilon$ for each fixed $m \in \mathbb{N}$ and every $n > n_0$, and $|d(x, Q_{nmj}) - d(x, O)| < \epsilon$ for each fixed $j \in \mathbb{N}$ and every $m > m_0$. Now, since I is an admissible ideal, for every $\epsilon > 0$ and each $x \in X$ we have $\{n \in \mathbb{N} : d(x, Q_{nmj}) - d(x, Q) \geq \varepsilon\} \subset \{1, ..., n_0\} \in I$, $\{m \in \mathbb{N} : d(x, Q_{nmj}) - d(x, Q) \geq \varepsilon\} \subset \{1, ..., m_0\} \in I$ and $\{j \in \mathbb{N} : d(x, Q_{nmj}) - d(x, Q) \geq \varepsilon\} \subset \{1, ..., j_0\} \in I$. Therefore, $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent. □

Definition 3.5. A triple sequence $\{Q_{nmj}\}$ is said to be regularly $(I^*_{W_3}, I^*_W)$-convergent ($R(I^*_{W_3}, I^*_W)$-convergent) if there exist the sets $M \in F(I_3)$, $M_1 \in F(I)$, $M_2 \in F(I)$ and $M_3 \in F(I)$ (i.e. $\mathbb{N}\times\mathbb{N}\times\mathbb{N} - M \in I_3$, $\mathbb{N} - M_1 \in I$, $\mathbb{N} - M_2 \in I$ and $\mathbb{N} - M_3 \in I$) such that the limits $\lim_{n,m,j\to\infty} d(x, Q_{nmj})$ where $(n, m, j) \in M$, $\lim_{n\to\infty} d(x, Q_{nmj})$ where $n \in M_1$, $\lim_{m\to\infty} d(x, Q_{nmj})$ where $m \in M_2$ and $\lim_{j\to\infty} d(x, Q_{nmj})$ where $j \in M_3$ exist for each fixed $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $j \in \mathbb{N}$, respectively.

If $R(I^*_{W_3}, I^*_W)$-convergent to Q, then for each $x \in X$ the limits

$$\lim_{n\to\infty}\lim_{m\to\infty}\lim_{j\to\infty} d(x, Q_{nmj}),$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{j\to\infty} d(x, Q_{nmj}),$$
$$\lim_{m\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} d(x, Q_{nmj}),$$
$$\lim_{j\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} d(x, Q_{nmj}),$$
$$\lim_{n\to\infty}\lim_{j\to\infty}\lim_{m\to\infty} d(x, Q_{nmj})$$
$$\lim_{j\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} d(x, Q_{nmj})$$

exist and are equal to Q. In this case we write $R(I^*_{W_3}, I_W) - \lim_{n,m,j\to\infty} d(x, Q_{nmj}) = d(x, Q)$ or $Q_{nmj} \xrightarrow{R(I^*_{W_3}, I^*_W)} Q$.

Theorem 3.6. *If a triple sequence $\{Q_{nmj}\}$ is $R(I^*_{W_3}, I^*_W)$-convergent, then $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent.*

Proof. Consider that $\{Q_{nmj}\}$ is $R(I^*_{W_3}, I^*_W)$-convergent. Then, it is $I^*_{W_3}$-convergent and then, by the Theorem 2.4, it is I_{W_3}-convergent. Furthermore, there exist the sets $M_1, M_2, M_3 \in F(I)$ such that for every $\epsilon > 0$ and each $x \in X$, $(\forall\varepsilon > 0)(\exists j_0 \in \mathbb{N})(\forall j \geq j_0)(j \in M_1)$ $|d(x, Q_{nmj}) - d(x, K_m)| < \varepsilon$, for some $K_m \in X$ and each $m \in \mathbb{N}$, $(\forall\epsilon > 0)(\exists m_0 \in \mathbb{N})(\forall m \geq m_0)(m \in M_2)$ $|d(x, Q_{nmj}) - d(x, L_n)| < \epsilon$, for some $L_n \in X$ and each $n \in \mathbb{N}$, $(\forall\epsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(n \in M_2)$ $|d(x, Q_{nmj}) - d(x, O_j)| < \varepsilon$, for some $O_j \in X$ and each $j \in \mathbb{N}$. Therefore, for every $\varepsilon > 0$ and each $x \in X$ we have $P(\epsilon) = \{n \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, O_j)| \geq \varepsilon\} \subset W_1 \cup \{1, 2, ..., n_0 - 1\}$ $(j \in \mathbb{N})$, $R(\varepsilon) = \{m \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, L_n)| \geq \epsilon\} \subset W_2 \cup \{1, 2, ..., m_0 - 1\}$ $(n \in \mathbb{N})$ and $U(\varepsilon) = \{j \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, K_m)| \geq \epsilon\} \subset W_3 \cup \{1, 2, ..., j_0 - 1\}$ $(m \in \mathbb{N})$, for $W_1, W_2, W_3 \in I$. Since I is an admissible ideal, we have $W_1 \cup \{1, 2, ..., n_0 - 1\} \in I$, $W_2 \cup \{1, 2, ..., m_0 - 1\} \in I$ and $W_3 \cup \{1, 2, ..., j_0 - 1\} \in I$, in consequence $P(\varepsilon) \in I$, $R(\varepsilon) \in I$ and $U(\varepsilon) \in I$. Therefore, this proves that $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent. □

Theorem 3.7. *Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal with the property and $I_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP3). If a triple sequence $\{Q_{nmj}\}$ is $R(I_{W_3}, I_W)$-convergent, then $\{Q_{nmj}\}$ is $R(I^*_{W_3}, I^*_W)$-convergent.*

Proof. Let $\{Q_{nmj}\}$ be $R(I_{W_3}, I_W)$-convergent. Then, $\{Q_{nmj}\}$ is I_{W_3}-convergent and then, by Theorem 2.5, $\{Q_{nmj}\}$ is $I^*_{W_3}$-convergent. Besides, for every $\varepsilon > 0$ and each $x \in X$ we have

$P(\epsilon) = \{j \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, K_m)| \geq \varepsilon\} \in I$, for some $K_m \in X$ and each $m \in \mathbb{N}$,

$Y(\epsilon) = \{m \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, L_n)| \geq \varepsilon\} \in I$, for some $L_n \in X$ and each $n \in \mathbb{N}$,

$W(\epsilon) = \{n \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, O_j)| \geq \varepsilon\} \in I$, for some $O_j \in X$ and each $j \in \mathbb{N}$. Now, for each $x \in X$ take

$P_1 = \{j \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, K_m)| \geq 1\}$ and $P_i = \{j \in \mathbb{N} : \frac{1}{i} \leq |d(x, Q_{nmj}) - d(x, K_m)| < \frac{1}{i-1}\}$ for $i \geq 2$, for some $K_m \in X$and for each $m \in \mathbb{N}$. We can see that $Q_k \cap Q_t = \emptyset$ for $k \neq t$ and $Q_k \in I$ for each $k \in \mathbb{N}$. By the property (AP), there is a countable family of sets $\{C_1, C_2, ...\}$ in I such that $Q_t \Delta C_t$ is a finite set for each $t \in \mathbb{N}$ and $C = \bigcup_{t=1}^{\infty} C_t \in I$. We will show that for some K_m and each $m \in \mathbb{N}$, $\lim_{j\to\infty} d(x, Q_{nmj}) = d(x, K_m)$, where $j \in M$, for some $M = \mathbb{N} - C \in F(I)$ and each $x \in X$. Now, let $\mu > 0$ be given. Take $i \in \mathbb{N}$ such that $1/i > \mu$. Then, for each $x \in X$ we have

$\{j \in \mathbb{N} : |d(x, Q_{nmj}) - d(x, K_m)| \geq \mu\} \subset \bigcup_{k=1}^{i} Q_k$, for some $K_m \in X$ and each $m \in \mathbb{N}$. Since $A_k \Delta C_k$ is a finite set for $k \in \{1, 2, ..., i\}$, there exists $j_0 \in \mathbb{N}$ such that

$$(\bigcup_{k=1}^{i} C_k) \cap \{j : j \geq j_0\}$$
$$= (\bigcup_{k=1}^{i} Q_k) \cap \{j : j \geq j_0\}.$$

If $j \geq j_0$ and $j \neq C$ then

$j \neq \bigcup_{k=1}^{i} C_k$ and so $j \neq \bigcup_{k=1}^{i} Q_k$.

Therefore, for every $\mu > 0$ and each $x \in X$ we have $|d(x, Q_{nmj}) - d(x, K_m)| < \frac{1}{i} < \mu$, for some $K_m \in \mathbb{N}$ and each $m \in \mathbb{N}$. This implies that $\lim_{j\to\infty} d(x, Q_{nmj}) = d(x, K_m)$, for some $j \in M$. Hence, for each $x \in X$ we have $I_W^* - \lim_{j\to\infty} d(x, Q_{nmj}) = d(x, K_m)$, for some $K_m \in X$ and each $m \in \mathbb{N}$. Similarly, for the set $Y(\varepsilon)$ and $W(\varepsilon)$. Hence, $\{Q_{nmj}\}$ is $R(I_{W_3}^*, I_W^*)$-convergent. □

The following diagrams show the results that we obtained in this section.
Diagram III
$R(W_3, W)$-convergence $\to R(I_{W_3}, I_W)$-convergence
$\uparrow$
$R(I_{W_3}^*, I_W^*)$-convergence
If I_3 is a strongly admissible ideal with $AP3$ property,
Diagram IV
$R(W_3, W)$-convergence $\to R(I_{W_3}, I_W)$-convergence
$\updownarrow$
$R(I_{W_3}^*, I_W^*)$-convergence

Conclusion

In this paper, the notions of Wijsman I_3-Convergence for triple sequences and Wijsman Regularly I_3-Convergence for triple sequences (see Diagrams I, II, III and IV) have been defined and studied. Besides, some interesting results and relations among these concepts were proved. For future work, definitions of $R(I_{W_3}, I_W)$-Cauchy and $R(I_{W_3}^*, I_W^*)$-Cauchy are suggested and then study some relations with the notions studied in this paper. On the other hand, these notions can be also studied for the lacunary sequence.

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The authors declare that they have no conflict of interest.

CRediT authorship contribution statement

Carlos Granados: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Bright O. Osu:** Data curation, Formal analysis, Writing – original draft, Writing – review & editing.

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