L. Vandenberghe 4/29/15
EE133A

Midterm Exam

- You have time until 9:50AM.
- Only this booklet should be on your desk. Calculators are not allowed. Please turn off and put away your cellphones.

Name: ____________________

UID#: ____________________

Name of left neighbor: ____________________

Name of right neighbor: ____________________

Problem 1	/25
Problem 2	/25
Problem 3	/25
Problem 4	/25
Total	/100

Formulas

- Inner product, norm, angle.
  - Relation between inner product, norms, and angle:
$$a^T b = \|a\| \, \|b\| \cos \angle(a, b).$$
  - Average value of elements of an n-vector: $\mathrm{avg}(a) = (\mathbf{1}^T a)/n$.
  - Root-mean-square value of an n-vector: $\mathrm{rms}(a) = \|a\|/\sqrt{n}$.
  - Standard deviation of an n-vector:
$$\mathrm{std}(a) = \mathrm{rms}(\tilde{a}) \quad \text{where } \tilde{a} = a - \mathrm{avg}(a)\mathbf{1}.$$
  - Correlation coefficient of two n-vectors:
$$\rho = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \|\tilde{b}\|} \quad \text{where } \tilde{a} = a - \mathrm{avg}(a)\mathbf{1} \text{ and } \tilde{b} = b - \mathrm{avg}(b)\mathbf{1}.$$
- Complexity of basic matrix and vector operations (α is a scalar, x and y are n-vectors, A is an $m \times n$ matrix, B is an $n \times p$ matrix).
  - Inner product $x^T y$: $2n - 1$ flops ($\approx 2n$ flops for large n).
  - Vector addition $x + y$: n flops.
  - Scalar-vector multiplication αx: n flops.
  - Scalar-matrix multiplication αA: mn flops.
  - Matrix-vector multiplication Ax: $m(2n - 1)$ flops ($\approx 2mn$ flops for large n).
  - Matrix-matrix multiplication AB: $mp(2n - 1)$ flops ($\approx 2mpn$ flops for large n).
- Pseudo-inverses.
  - Pseudo-inverse of left invertible matrix A: $A^\dagger = (A^T A)^{-1} A^T$.
  - Pseudo-inverse of right invertible matrix A: $A^\dagger = A^T (A A^T)^{-1}$.
- Complexity of forward or back substitution with triangular $n \times n$ matrix: n^2 flops.
- Complexity of matrix factorizations.
  - QR factorization of $m \times n$ matrix: $2mn^2$ flops.
  - LU factorization of $n \times n$ matrix: $(2/3)n^3$ flops.

Problem 1. Let A be a tall $m \times n$ matrix with linearly independent columns. Define

$$P = A(A^T A)^{-1} A^T.$$

1. Show that the matrix $2P - I$ is orthogonal.
2. Use the Cauchy-Schwarz inequality to show that the inequalities

   $$-\|x\|\|y\| \leq x^T(2P - I)y \leq \|x\|\|y\|$$

   hold for all m-vectors x and y.
3. Take $x = y$ in part 2. Show that the right-hand inequality implies that $\|Px\| \leq \|x\|$ for all m-vectors x.

Answer for problem 1.

Answer for problem 1 (continued).

Problem 2. A lower triangular matrix A is *bidiagonal* if $A_{ij} = 0$ for $i > j+1$:

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & A_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & A_{n-1,n-2} & A_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_{n,n-1} & A_{nn} \end{bmatrix}.$$

Assume A is a nonsingular bidiagonal and lower triangular matrix of size $n \times n$.

1. What is the complexity of solving $Ax = b$?
2. What is the complexity of computing the inverse of A?

State the algorithm you use in each subproblem, and give the dominant term (exponent and coefficient) of the flop count. If you know several methods, consider the most efficient one.

Answer for problem 2.

Answer for problem 2 (continued).

Problem 3. Let B be an $m \times n$ matrix.

1. Prove that the matrix $I + B^T B$ is nonsingular. Since we do not impose any conditions on B, this also shows that the matrix $I + BB^T$ is nonsingular.

2. Show that the matrix
$$A = \begin{bmatrix} I & B^T \\ -B & I \end{bmatrix}$$
is nonsingular and that the following two expressions for its inverse are correct:
$$A^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -B^T \\ I \end{bmatrix} (I + BB^T)^{-1} \begin{bmatrix} B & I \end{bmatrix},$$
$$A^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I \\ B \end{bmatrix} (I + B^T B)^{-1} \begin{bmatrix} I & -B^T \end{bmatrix}.$$

3. Now assume B has orthonormal columns. Use the result in part 2 to formulate a simple method for solving $Ax = b$. What is the complexity of your method? If you know several methods, give the most efficient one.

Answer for problem 3.

Answer for problem 3 (continued).

Problem 4. We have defined the pseudo-inverse of a right invertible matrix B as the matrix

$$B^\dagger = B^T(BB^T)^{-1}.$$

Note that $B^\dagger B$ is a symmetric matrix. It can be shown that $B^\dagger$ is the only right inverse X of B with the property that XB is symmetric.

1. Assume A is a nonsingular $n \times n$ matrix and b is an n-vector. Show that the $n \times (n+1)$ matrix
$$B = \left[\begin{array}{cc} A & b \end{array}\right]$$
is right invertible and that
$$X = \left[\begin{array}{c} A^{-1} - A^{-1}by^T \\ y^T \end{array}\right]$$
is a right inverse of B, for any value of the n-vector y.

2. Show that XB is symmetric (hence, $X = B^\dagger$) if
$$y = \frac{1}{1 + \|A^{-1}b\|^2} A^{-T}A^{-1}b.$$

3. What is the complexity of computing the vector y in part 2 using an LU factorization of A? Give a flop count, including all cubic and quadratic terms. If you know several methods, consider the most efficient one.

Answer for problem 4.

Answer for problem 4 (continued).