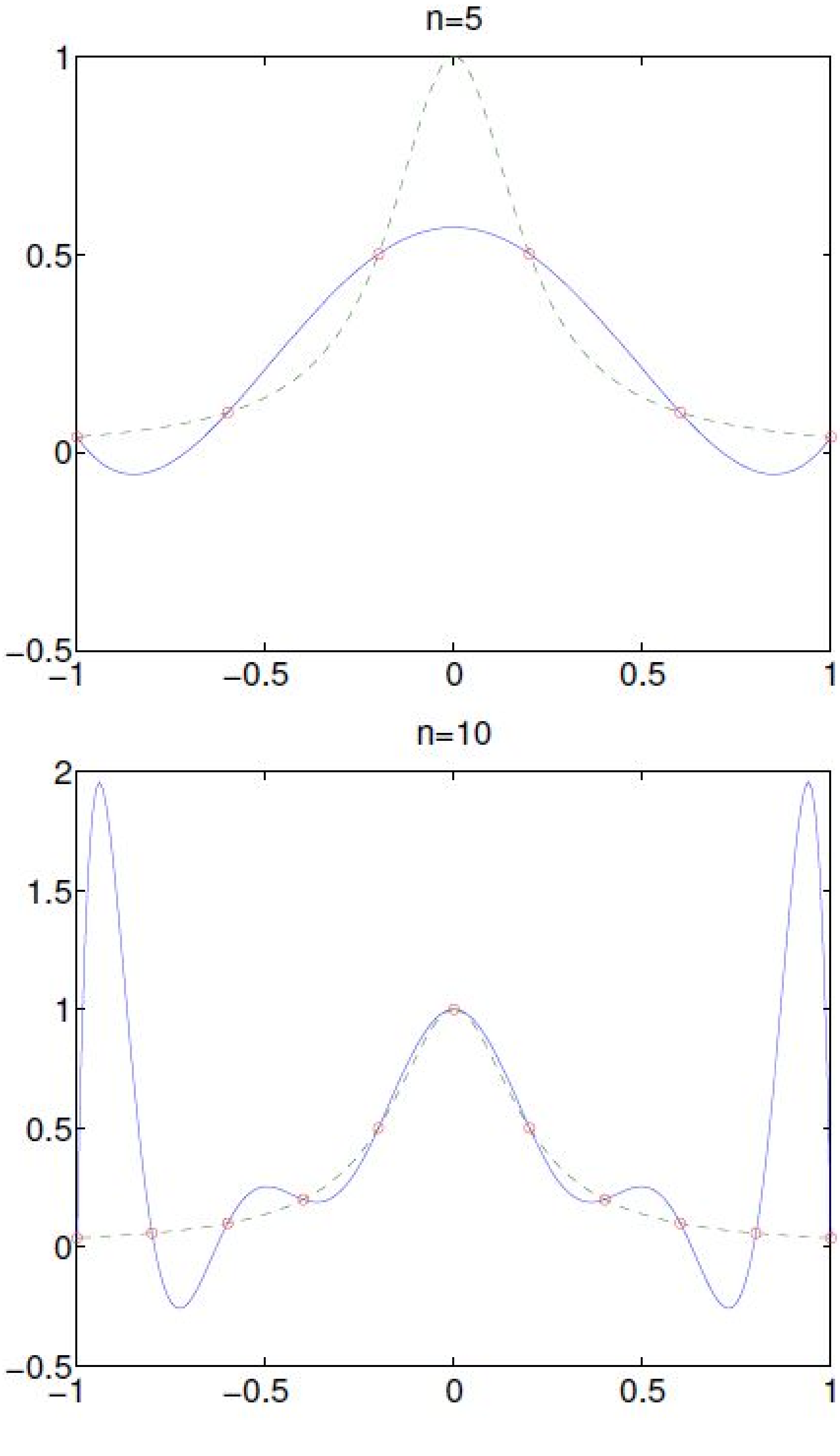
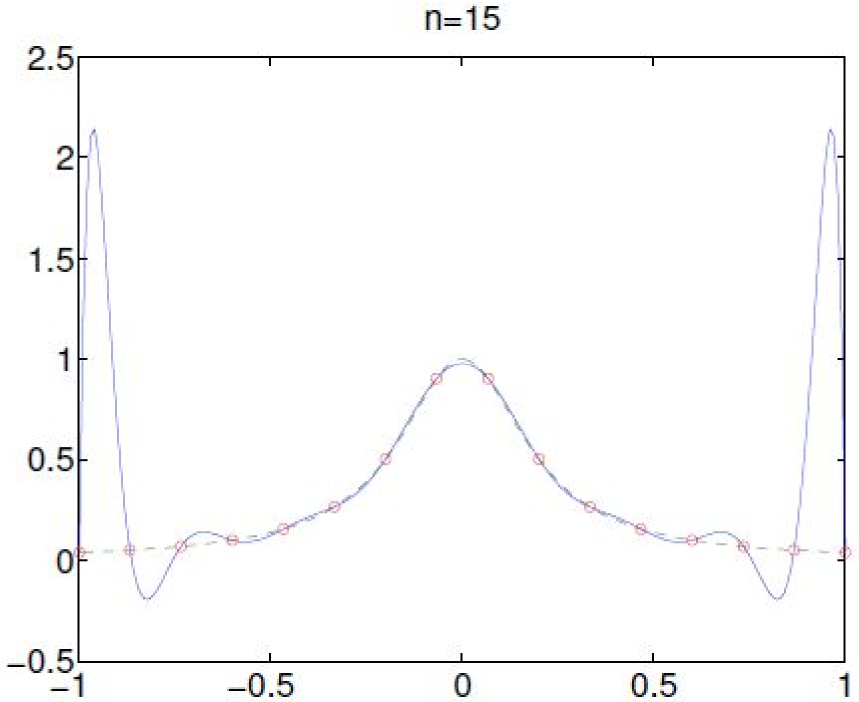
**3.1**






```
n = 16;

t = linspace(-1, 1, n)';     % column vector with n+1 points in [-1,1]
y = 1 ./ (1 + 25*t.^2);      % column vector with values of f(ti)
x = fliplr(vander(t)) \ y;   % coefficients of the polynomial

%
% plot f(t) and the interpolating polynomial
%

% generate 200 points in  [-1,1]
t2 = linspace(-1, 1, 200)';

% evaluate f at the points t2
yf = 1 ./ (1 + 25*t2.^2);

% evaluate the interpolating polynomial
ypol = x(1)*ones(200,1);
for i=1:n-1
   ypol = ypol + x(i+1)*t2.^i;
end;

% plot function and interpolating polynomial
plot(t2, ypol, '-', t2, yf, '--', t, y, 'o');
```

## 3.3

**Solution.** We first derive expressions for $f(t_1)$, $f(t_2)$, $f'(t_1)$, $f'(t_2)$:

$$f(t_1) = c_1, \qquad f(t_2) = c_1 + c_2 h + c_3 h^2, \qquad f'(t_1) = c_2, \qquad f'(t_2) = c_2 + 2c_3 h + c_4 h^2$$

where $h = t_2 - t_1$. In matrix notation, the four interpolation conditions are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & h & h^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2h & h^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ s_1 \\ s_2 \end{bmatrix}.$$

If we exchange the second and third rows, we can solve this by forward substitution. The solution is

$$\begin{aligned} c_1 &= y_1, \\ c_2 &= s_1, \\ c_3 &= \frac{y_2 - c_1 - hc_2}{h^2} \\ &= \frac{(y_2 - y_1)/h - s_1}{h}, \\ c_4 &= \frac{s_2 - c_2 - 2hc_3}{h^2} \\ &= \frac{s_2 - s_1 - 2((y_2 - y_1)/h - s_1)}{h^2} \\ &= \frac{s_2 + s_1 - 2(y_2 - y_1)/h}{h^2}. \end{aligned}$$

## 3.5

**Solution.** We can write $f(u_1, u_2)$ as

$$f(u_1, u_2) = u_1^2 p_{11} + 2u_1 u_2 p_{12} + u_2^2 p_{22} + u_1 q_1 + u_2 q_2 + r.$$

For given $u_1$ and $u_2$, this is a linear function of the unknowns $p_{11}$, $p_{12}$, $p_{22}$, $q_1$, $q_2$, $r$. For example, $f(0,1) = 6$ means

$$p_{22} + q_2 + r = 6.$$

We therefore obtain the following set of equations:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & 1 \\ 1 & 4 & 4 & 1 & 2 & 1 \\ 4 & 4 & 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \\ q_1 \\ q_2 \\ r \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \\ 7 \\ 2 \\ 6 \end{bmatrix}.$$

```
>> A = [0  0  1  0  1  1;
        1  0  0  1  0  1;
        1  2  1  1  1  1;
        1  2  1 -1 -1  1;
        1  4  4  1  2  1;
        4  4  1  2  1  1]
>> b = [6; 6; 3; 7; 2; 6 ];
>> x = A\b

x =

    3.0000
   -2.0000
    1.0000
   -2.0000
    0.0000
    5.0000
```

We can check the answer as follows.

```
>> P = [x(1), x(2); x(2), x(3)];
>> q = [x(4); x(5)];
>> r = x(6);
>> u = [0;1];  u'*P*u + q'*u + r
ans =
     6
>> u = [1;0];  u'*P*u + q'*u + r
ans =
    6.0000
>> u = [1;1];  u'*P*u + q'*u + r
ans =
    3.0000
>> u = [-1;-1]; u'*P*u + q'*u + r
ans =
    7.0000
>> u = [1;2];  u'*P*u + q'*u + r
ans =
     2
>> u = [2;1];  u'*P*u + q'*u + r
ans =
     6
```

## 3.6

**Solution.** Squaring the norms gives four nonlinear equations

$$\begin{aligned} \|x\|^2 - 2a^Tx + \|a\|^2 &= r_a^2 \\ \|x\|^2 - 2b^Tx + \|b\|^2 &= r_b^2 \\ \|x\|^2 - 2c^Tx + \|c\|^2 &= r_c^2 \\ \|x\|^2 - 2d^Tx + \|d\|^2 &= r_d^2. \end{aligned}$$

Subtracting the first equation from the three others gives

$$\begin{aligned} 2(a-b)^Tx &= r_b^2 - r_a^2 + \|a\|^2 - \|b\|^2 \\ 2(a-c)^Tx &= r_c^2 - r_a^2 + \|a\|^2 - \|c\|^2 \\ 2(a-d)^Tx &= r_d^2 - r_a^2 + \|a\|^2 - \|d\|^2. \end{aligned}$$

In matrix form,

$$\begin{bmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 \\ a_1-c_1 & a_2-c_2 & a_3-c_3 \\ a_1-d_1 & a_2-d_2 & a_3-d_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} r_b^2 - r_a^2 + \|a\|^2 - \|b\|^2 \\ r_c^2 - r_a^2 + \|a\|^2 - \|c\|^2 \\ r_d^2 - r_a^2 + \|a\|^2 - \|d\|^2 \end{bmatrix}.$$

The matrix is nonsingular because its rows are linearly independent.

The solution for the parameters in the problem is $x = (0.6, 0.4, -0.5)$.

## 4.1 (a,b,c,f)

(a) the two columns of $A$ are linearly independent. Suppose $Ax = 0$:

$$-x_1 + 2x_2 = 0, \qquad 3x_1 - 6x_2 = 0, \qquad 2x_1 - x_2 = 0.$$

The first and second equations are satisfied if $x_1 = 2x_2$. The third equation is satisfied if $2x_1 = x_2$. This is only possible if $x_1$ and $x_2$ are both zero. Therefore $Ax = 0$ implies $x = 0$.

(b) This is a wide matrix, so the columns are not linearly independent.

(c) The columns are not linearly independent: $Ax = 0$ for $x = (0, 1, 0)$ because the second column of $A$ is zero.

(f) This matrix has linearly independent columns. We show that $Ax = 0$ only if $x = 0$. Suppose $Ax = (I - ab^T)x = x - a(b^Tx) = 0$. Hence $x = a(b^Tx)$. Taking norms of both sides of the equality gives $\|x\| = \|a\||b^Tx|$. From the Cauchy-Schwarz inequality,

$$\|x\| = \|a\||b^Tx| \leq \|a\|\|b\|\|x\|.$$

If $x \neq 0$, we can divide by $\|x\|$ and obtain $1 \leq \|a\|\|b\|$. This contradicts the fact $\|a\|\|b\| < 1$. We conclude that $Ax = 0$ is only possible if $x = 0$.

## 4.2

**Solution.**

(a) True.

$$ABx = 0 \quad\Longrightarrow\quad Bx = 0 \quad\Longrightarrow\quad x = 0.$$

The first step follows because $A$ has linearly independent columns. The second step because $B$ has linearly independent columns.

(b) True.

$$(AB)^T(AB) = B^T(A^TA)B = B^TB = I.$$

(c) True. Left invertibility is the same as the property of having linearly independent columns. $YX$ is a left inverse because

$$(YX)(AB) = Y(XA)B = YB = I.$$

(d) False. A counterexample is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The pseudo-inverses are

$$A^\dagger = A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad B^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad C^\dagger = \frac{1}{5}\begin{bmatrix} 1 & 2 \end{bmatrix},$$

but

$$B^\dagger A^\dagger = \begin{bmatrix} 1/2 & 1/4 \end{bmatrix} \neq C^\dagger.$$

## 4.4

**Solution.** We verify that

$$(A + uv^T)(A^{-1} - \frac{1}{1 + v^TA^{-1}u}A^{-1}uv^TA^{-1}) = I.$$

This will show that the expression for the inverse is correct. By showing that the matrix is invertible, we also show that it is nonsingular.

$$\begin{aligned}
&(A + uv^T)(A^{-1} - \frac{1}{1 + v^TA^{-1}u}A^{-1}uv^TA^{-1}) \\
&= I + uv^TA^{-1} - \frac{1}{1 + v^TA^{-1}u}uv^TA^{-1} - \frac{1}{1 + v^TA^{-1}u}uv^TA^{-1}uv^TA^{-1} \\
&= I + uv^TA^{-1} - \frac{1}{1 + v^TA^{-1}u}uv^TA^{-1} - \frac{v^TA^{-1}u}{1 + v^TA^{-1}u}uv^TA^{-1} \\
&= I + (1 - \frac{1}{1 + v^TA^{-1}u} - \frac{v^TA^{-1}u}{1 + v^TA^{-1}u})uv^TA^{-1} \\
&= I.
\end{aligned}$$

The key observation to go from line 2 to line 3 is that $v^TA^{-1}u$ is a *scalar*. Therefore, we have

$$u(v^TA^{-1}u) = (v^TA^{-1}u)u$$

(*i.e.*, the matrix-matrix product of an $n \times 1$ matrix $u$ with the $1 \times 1$ matrix $v^TA^{-1}u$ is equal to the scalar multiplication of $u$ with the scalar $v^TA^{-1}u$).