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Midterm solutions

Problem 1. In the homework you derived the following factorization of a circulant Toeplitz matrix $T(a)$ with the n-vector a as its first column:

$$T(a) = \begin{bmatrix} a_1 & a_n & a_{n-1} & \cdots & a_3 & a_2 \\ a_2 & a_1 & a_n & \cdots & a_4 & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_5 & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_n \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix} = \frac{1}{n} W^H \,\mathbf{diag}(Wa) W.$$

Here W is the $n \times n$ discrete Fourier Transform matrix and $\mathbf{diag}(Wa)$ is the diagonal matrix with the vector Wa (the discrete Fourier transform of a) on its diagonal.

1. Suppose $T(a)$ is nonsingular. Show that its inverse $T(a)^{-1}$ is a circulant Toeplitz matrix. Give a fast method for computing the vector b that satisfies $T(b) = T(a)^{-1}$.

2. Let a and b be two n-vectors. Show that the product $T(a)T(b)$ is a circulant Toeplitz matrix. Give a fast method for computing the vector c that satisfies $T(c) = T(a)T(b)$.

Solution.

1. The inverse is given by
$$T(a)^{-1} = \frac{1}{n} W^H \,\mathbf{diag}(Wa)^{-1} W.$$
because $W^{-1} = (1/n)W^H$. This can be written as
$$T(a)^{-1} = \frac{1}{n} W^H \,\mathbf{diag}(Wb) W.$$
if we define $b = (1/n)W^H \,\mathbf{diag}(Wa)^{-1}\mathbf{1}$.

   The vector b can be computed in order $n \log n$ flops by calculating the DFT Wa of a, inverting it componentwise to get $\mathbf{diag}(Wa)^{-1}\mathbf{1}$, and then taking the inverse DFT of the result.

2. Using $W^H W = WW^H = nI$ to simplify the product gives
$$\begin{aligned} T(a)T(b) &= \left(\frac{1}{n} W^H \,\mathbf{diag}(Wa) W\right)\left(\frac{1}{n} W^H \,\mathbf{diag}(Wb) W\right) \\ &= \frac{1}{n} W^H \,\mathbf{diag}(Wa)\,\mathbf{diag}(Wb) W \\ &= \frac{1}{n} W^H \,\mathbf{diag}\left((Wa) \circ (Wb)\right) W. \end{aligned}$$

This can be written as

$$T(a)T(b) = \frac{1}{n} W^H \,\mathbf{diag}(Wc) W$$

with $c = (1/n)W^H((Wa) \circ (Wb))$. The vector c can be computed in order $n \log n$ flops by taking the DFTs of a and b, multiplying them componentwise, and taking the inverse DFT.

Problem 2. The *Kronecker product* of two $n \times n$ matrices A and B is the $n^2 \times n^2$ matrix

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \otimes \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ -5 & 6 & 0 & 0 \\ 6 & 8 & -3 & -4 \\ -10 & 12 & 5 & -6 \end{bmatrix}.$$

Suppose A and B are orthogonal. Is $A \otimes B$ orthogonal? Explain your answer.

Solution. If A and B are orthogonal, then $A \otimes B$ is orthogonal. To show this, we need to prove that $(A \otimes B)^T(A \otimes B) = I$.

First, we note that the transpose of the Kronecker product is the Kronecker product of the transposes: $(A \otimes B)^T = A^T \otimes B^T$. Therefore

$$\begin{aligned}
&(A \otimes B)^T(A \otimes B) \\
&\quad = (A^T \otimes B^T)(A \otimes B) \\
&\quad = \begin{bmatrix} A_{11}B^T & A_{21}B^T & \cdots & A_{n1}B^T \\ A_{12}B^T & A_{22}B^T & \cdots & A_{n2}B^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}B^T & A_{2n}B^T & \cdots & A_{nn}B^T \end{bmatrix} \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix} \\
&\quad = \begin{bmatrix} (A_{11}^2 + \cdots + A_{n1}^2)I & (A_{11}A_{12} + \cdots + A_{n1}A_{n2})I & \cdots & (A_{11}A_{1n} + \cdots + A_{n1}A_{nn})I \\ (A_{12}A_{11} + \cdots + A_{n2}A_{n1})I & (A_{12}^2 + \cdots + A_{n2}^2)I & \cdots & (A_{12}A_{1n} + \cdots + A_{n2}A_{nn})I \\ \vdots & \vdots & \ddots & \vdots \\ (A_{1n}A_{11} + \cdots + A_{nn}A_{n1})I & (A_{1n}A_{12} + \cdots + A_{nn}A_{n2})I & \cdots & (A_{1n}^2 + \cdots + A_{nn}^2)I \end{bmatrix} \\
&\quad = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}
\end{aligned}$$

because $A^TA = I$ and $B^TB = I$.

Problem 3. Let A be an $m \times n$ matrix with linearly independent columns. The Householder algorithm for the QR factorization of A computes an orthogonal $m \times m$ matrix Q such that

$$Q^T A = \left[\begin{array}{c} R \\ 0 \end{array} \right]$$

where R is upper triangular with nonzero diagonal elements. The matrix Q is computed as a product $Q = Q_1 Q_2 \cdots Q_{n-1}$ of orthogonal matrices. In this problem we discuss the first step, the calculation of Q_1. This matrix has the property that

$$Q_1^T A = \left[\begin{array}{cccc} R_{11} & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{array} \right].$$

The '$\times$' symbols denote elements that may or may not be zero.

Let $a = (A_{11}, A_{21}, \ldots, A_{m1})$ be the first column of A. Define an m-vector

$$v = \frac{1}{\sqrt{1 + |A_{11}|/\|a\|}} \left(\frac{1}{\|a\|} a + s e_1 \right)$$

where $s = 1$ if $A_{11} \geq 0$ and $s = -1$ if $A_{11} < 0$. The vector e_1 is the first unit vector $(1, 0, \ldots, 0)$.

1. Show that v has norm $\sqrt{2}$.
2. Define $Q_1 = I - vv^T$. Show that Q_1 is orthogonal.
3. Show that $Q_1^T a = R_{11} e_1$, where $R_{11} = -s\|a\|$.
4. Give the complexity (dominant term in the flop count for large m, n) of computing the matrix-matrix product
$$Q_1^T A = (I - vv^T) A.$$

Solution.

1. The square of $\|v\|$ is

$$\begin{aligned} v^T v &= \frac{1}{1 + |A_{11}|/\|a\|} \left(\frac{1}{\|a\|} a + s e_1 \right)^T \left(\frac{1}{\|a\|} a + s e_1 \right) \\ &= \frac{1}{1 + |A_{11}|/\|a\|} \left(\frac{a^T a}{\|a\|^2} + 2s \frac{A_{11}}{\|a\|} + s^2 \right) \\ &= \frac{2 + 2|A_{11}|/\|a\|}{1 + |A_{11}|/\|a\|} \\ &= 2. \end{aligned}$$

2. We verify that $Q_1^T Q_1 = I$. Since $v^T v = 2$,

$$Q_1^T Q_1 = (I - vv^T)(I - vv^T) = I - vv^T - vvv^T + (v^T v) vv^T = I.$$

3. First note that

$$v^T a = \frac{a^T a/\|a\| + s e_1^T a}{\sqrt{1 + |A_{11}|/\|a\|}} = \frac{\|a\| + |A_{11}|}{\sqrt{1 + |A_{11}|/\|a\|}} = \|a\| \sqrt{1 + |A_{11}|/\|a\|}.$$

   Therefore

$$Q_1^T a = a - (v^T a) v = a - \|a\| (\frac{1}{\|a\|} a + s e_1) = -s \|a\| e_1$$

4. We compute $Q_1^T A$ as

$$Q_1^T a = A - v(v^T A).$$

   This takes $2mn$ flops for the product $y = v^T A$, another mn for the outer product vy^T, and mn for the addition to A. The total is $4mn$.

Problem 4. Let A be an $n \times m$ matrix and B an $m \times n$ matrix. We compare the complexity of two methods for solving

$$(I + AB)x = b.$$

We assume the matrix $I + AB$ is nonsingular.

1. In the first method we compute the matrix $C = I + AB$ and then solve $Cx = b$ using the standard method (LU factorization). Give the complexity of this method.

2. Suppose the matrix $I + BA$ is nonsingular. Show that

$$(I + AB)^{-1} = I - A(I + BA)^{-1} B.$$

3. This suggests a second method for solving the equation: compute the solution via the formula

$$x = \left(I - A(I + BA)^{-1} B\right) b.$$

   Describe an efficient method for evaluating this formula and give the complexity. Which of the two methods is faster when $m \ll n$? Explain your answer.

Solution.

1. Computing C costs $2n^2 m$ flops. Solving the equation costs $(2/3)n^3$ flops.

2. We check that $(I + AB)(I - A(I + BA)^{-1}B) = I$:

$$\begin{aligned}
(I + AB)(I - A(I + BA)^{-1}B &= I + AB - A(I + BA)^{-1}B - ABA(I + BA)^{-1}B \\
&= I + AB - A(I + BA)(I + BA)^{-1}B \\
&= I + AB - AB \\
&= I.
\end{aligned}$$

3. The complexity is $2nm^2 + (2/3)m^3$ flops plus lower order terms.

- Compute $v = Bb$ ($2mn$ flops).
- Compute $C = I + BA$ ($2nm^2$ flops).
- Solve $Cu = v$ using the LU factorization of C ($(2/3)m^3$ flops).
- Compute $x = b - Au$ ($2mn$ flops).

Method 2 is faster when $m \ll n$.