Homework 4

NE 795-001: Fall 2023

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October 29th, 2023

Problem 1 - UQ with gPCE

The Generalized Polynomial Chaos Expansion (gPCE) is a method used to represent uncertain parameters in a stochastic system. It allows us to express the uncertain parameters as a series of orthogonal polynomials with respect to the probability distribution of the underlying random variables. Let's denote ω as the random variable with a known probability distribution, and $\alpha(\omega)$ and $\gamma(\omega)$ as the uncertain parameters in the Lotka-Volterra equations.

The Lotka-Volterra equations with stochastic parameters are:

$$\frac{dx}{dt} = \alpha(\omega)x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma(\omega)y$$

To apply the gPCE to these equations, we first need to expand the uncertain parameters $\alpha(\omega)$ and $\gamma(\omega)$ into a polynomial chaos series:

$$\alpha(\omega) = \sum_{i=0}^{P} a_i \Psi_i(\omega)$$
$$\gamma(\omega) = \sum_{i=0}^{P} g_i \Psi_i(\omega)$$

where a_i and g_i are the deterministic coefficients of the expansion, $\Psi_i(\omega)$ are the orthogonal polynomials with respect to the probability distribution of ω, and P is the order of the expansion.

Next, we also expand the state variables $x(t, \omega)$ and $y(t, \omega)$ into gPCE series:

$$x(t, \omega) = \sum_{i=0}^{P} x_i(t) \Psi_i(\omega)$$
$$y(t, \omega) = \sum_{i=0}^{P} y_i(t) \Psi_i(\omega)$$

Substituting these expansions into the Lotka-Volterra equations, we get:

$$\frac{d}{dt}\left(\sum_{i=0}^{P} x_i(t)\Psi_i(\omega)\right) = \left(\sum_{i=0}^{P} a_i\Psi_i(\omega)\right)\left(\sum_{j=0}^{P} x_j(t)\Psi_j(\omega)\right) - \beta\left(\sum_{k=0}^{P} x_k(t)\Psi_k(\omega)\right)\left(\sum_{l=0}^{P} y_l(t)\Psi_l(\omega)\right)$$

$$\frac{d}{dt}\left(\sum_{i=0}^{P} y_i(t)\Psi_i(\omega)\right) = \delta\left(\sum_{k=0}^{P} x_k(t)\Psi_k(\omega)\right)\left(\sum_{l=0}^{P} y_l(t)\Psi_l(\omega)\right) - \left(\sum_{i=0}^{P} g_i\Psi_i(\omega)\right)\left(\sum_{j=0}^{P} y_j(t)\Psi_j(\omega)\right)$$

Now, we apply the Galerkin projection, which involves multiplying both sides of the equations by $\Psi_m(\omega)$ and integrating over the probability space of ω:

$$\int \Psi_m(\omega)\frac{d}{dt}\left(\sum_{i=0}^{P} x_i(t)\Psi_i(\omega)\right) d\mathbb{P}(\omega) = \int \Psi_m(\omega)\left[\left(\sum_{i=0}^{P} a_i\Psi_i(\omega)\right)\left(\sum_{j=0}^{P} x_j(t)\Psi_j(\omega)\right) - \beta\left(\sum_{k=0}^{P} x_k(t)\Psi_k(\omega)\right)\left(\sum_{l=0}^{P} y_l(t)\Psi_l(\omega)\right)\right] d\mathbb{P}(\omega)$$

$$\int \Psi_m(\omega)\frac{d}{dt}\left(\sum_{i=0}^{P} y_i(t)\Psi_i(\omega)\right) d\mathbb{P}(\omega) = \int \Psi_m(\omega)\left[\delta\left(\sum_{k=0}^{P} x_k(t)\Psi_k(\omega)\right)\left(\sum_{l=0}^{P} y_l(t)\Psi_l(\omega)\right) - \left(\sum_{i=0}^{P} g_i\Psi_i(\omega)\right)\left(\sum_{j=0}^{P} y_j(t)\Psi_j(\omega)\right)\right] d\mathbb{P}(\omega)$$

Since the polynomials $\Psi_i(\omega)$ are orthogonal with respect to the probability distribution of ω, the integrals of the products of different polynomials are zero, and the integral of the square of a polynomial is not zero. This property simplifies the above equations significantly.

To simplify the equations obtained from the Galerkin projection, we will use the orthogonality property of the polynomial basis $\Psi_i(\omega)$. The orthogonality condition states that:

$$\int \Psi_i(\omega)\Psi_j(\omega)d\mathbb{P}(\omega) = \begin{cases} 0 & \text{if } i \neq j \\ \langle \Psi_i^2 \rangle & \text{if } i = j \end{cases}$$

where $\langle \Psi_i^2 \rangle$ is the expected value of $\Psi_i^2(\omega)$, which is a constant.

Let's simplify the first equation after applying the Galerkin projection:

$$\int \Psi_m(\omega)\frac{d}{dt}\left(\sum_{i=0}^{P} x_i(t)\Psi_i(\omega)\right) d\mathbb{P}(\omega) = \int \Psi_m(\omega)\left[\left(\sum_{i=0}^{P} a_i\Psi_i(\omega)\right)\left(\sum_{j=0}^{P} x_j(t)\Psi_j(\omega)\right) - \beta\left(\sum_{k=0}^{P} x_k(t)\Psi_k(\omega)\right)\left(\sum_{l=0}^{P} y_l(t)\Psi_l(\omega)\right)\right] d\mathbb{P}(\omega)$$

We can take the time derivative outside of the integral and the summation because it does not depend on ω:

$$\sum_{i=0}^{P}\frac{dx_i(t)}{dt}\int \Psi_m(\omega)\Psi_i(\omega)d\mathbb{P}(\omega) = \sum_{i=0}^{P}\sum_{j=0}^{P} a_i x_j(t)\int \Psi_m(\omega)\Psi_i(\omega)\Psi_j(\omega)d\mathbb{P}(\omega) - \beta\sum_{k=0}^{P}\sum_{l=0}^{P} x_k(t)y_l(t)\int \Psi_m(\omega)\Psi_k(\omega)\Psi_l(\omega)d\mathbb{P}(\omega)$$

Using the orthogonality condition, the integrals simplify to:

$$\frac{dx_m(t)}{dt}\langle \Psi_m^2 \rangle = \sum_{i=0}^{P} a_i x_i(t)\langle \Psi_m\Psi_i \rangle - \beta\sum_{k=0}^{P} x_k(t)y_k(t)\langle \Psi_m^2 \rangle$$

Since $\langle \Psi_m \Psi_i \rangle$ is zero for $i \neq m$ and $\langle \Psi_m^2 \rangle$ for $i = m$, we can simplify further:

$$\frac{dx_m(t)}{dt} \langle \Psi_m^2 \rangle = a_m x_m(t) \langle \Psi_m^2 \rangle - \beta x_m(t) y_m(t) \langle \Psi_m^2 \rangle$$

Dividing both sides by $\langle \Psi_m^2 \rangle$:

$$\frac{dx_m(t)}{dt} = a_m x_m(t) - \beta x_m(t) y_m(t)$$

We can perform a similar simplification for the second equation:

$$\frac{dy_m(t)}{dt} = \delta x_m(t) y_m(t) - g_m y_m(t)$$

These are the simplified deterministic ODEs for the coefficients $x_m(t)$ and $y_m(t)$ in the gPCE expansion of the state variables $x(t, \omega)$ and $y(t, \omega)$. The resulting system of ODEs can be solved to find the evolution of the moments of $x(t, \omega)$ and $y(t, \omega)$.

After simplifying the equations using the orthogonality property of the polynomial basis $\Psi_i(\omega)$, we obtain a system of deterministic ordinary differential equations (ODEs) for the coefficients $x_m(t)$ and $y_m(t)$ in the gPCE expansion of the state variables $x(t, \omega)$ and $y(t, \omega)$. The final form of these ODEs is:

For the prey population $x(t, \omega)$:

$$\frac{dx_m(t)}{dt} = a_m x_m(t) - \beta \sum_{k=0}^{P} \sum_{l=0}^{P} C_{mkl} x_k(t) y_l(t)$$

For the predator population $y(t, \omega)$:

$$\frac{dy_m(t)}{dt} = \delta \sum_{k=0}^{P} \sum_{l=0}^{P} C_{mkl} x_k(t) y_l(t) - g_m y_m(t)$$

where C_{mkl} are the triple product integrals of the polynomial basis:

$$C_{mkl} = \int \Psi_m(\omega) \Psi_k(\omega) \Psi_l(\omega) d\mathbb{P}(\omega)$$

These triple product integrals are typically nonzero only for certain combinations of indices m, k, and l, depending on the specific polynomial basis used and the probability distribution of ω. The coefficients a_m and g_m are the gPCE coefficients for the uncertain parameters $\alpha(\omega)$ and $\gamma(\omega)$, respectively.

The system of ODEs can be solved numerically to obtain the time evolution of the coefficients $x_m(t)$ and $y_m(t)$, which in turn provide the statistical moments of the prey and predator populations. This approach allows us to analyze the impact of uncertainty in the parameters $\alpha(\omega)$ and $\gamma(\omega)$ on the dynamics of the Lotka-Volterra system.