1. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Which of the following vectors is not an eigenvector of this matrix ?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Solution: Option D is the correct answer. For each of the vectors given in the options you can compute the product Ax. For example, consider $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4x$$

Hence, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. Similarly, you can show that $x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are also eigenvectors of this matrix with the corresponding eigen values being 1 and -1 respectively. You can then also check that $x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is not an eigenvector of this matrix.

2. Consider a square matrix $A \in \mathbb{R}^{3\times3}$ such that $A^T = A$. My friend told me that the following three vectors are the eigenvectors of this matrix A:

$$x = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is my friend telling the truth ?

A. Yes

B. No

C. Can't say without knowing all the elements of A

D. Yes, only if all the diagonal elements of A are 1

Solution: Note that A is a square symmetric matrix ($\because A \in \mathbb{R}^{3\times3}$ and $A^T = A$). We know that the eigenvectors of a square symmetric matrix are orthogonal. In other words, if x, y, z are the eigenvectors of A then $x^T y = x^T z = y^T z = 0$. You can easily verify that this is not the case. Hence, my friend is not telling the truth. **Option B** is the correct answer.

3. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

What can you say about the series $x, Ax, A^2xA^3x, ...$?

A. It will diverge (explode)

B. It will converge (vanish)

C. It will reach a steady state

D. Can't say without knowing all the elements of x

Solution: Referring to the solution for question 1, we know that the dominant eigenvalue of this matrix is 4. From slide 10 of Lecture 6 we know that if the dominant eigenvalue $\lambda_d > 1$ then the series will diverge (explode) irrespective of which x we start with. **Option A** is the correct answer.

4. Which of the following sets of vectors **does not** form a valid basis in $\mathbb{R}^3$

A. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}$

B. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\4\\5\end{bmatrix}$

C. $\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\end{bmatrix}, \begin{bmatrix}6\\5\\4\end{bmatrix}$

D. $\begin{bmatrix}3\\5\\7\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}4\\7\\4\end{bmatrix}$

Solution: Option C is the correct answer. A set of 3 vectors can form a basis in $\mathbb{R}^3$ if the vectors in the set are linearly independent. Now, consider the vectors $\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\-1\\2\end{bmatrix}, \begin{bmatrix}6\\5\\4\end{bmatrix}$. We observe that,

$$2\begin{bmatrix}1\\3\\1\end{bmatrix} + \begin{bmatrix}4\\-1\\2\end{bmatrix} - \begin{bmatrix}6\\5\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence, the vectors are linearly dependent and thus cannot form a basis in $\mathbb{R}^3$.

5. Consider the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Now consider the following optimization problem:

$$\min_x \; x^T A x$$
$$\text{s.t } \|x\| = 1$$

Which of the following vectors is a solution to the above minimization problem?

A. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

C. $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

D. None of the above

Solution: From the Theorem on Slide 26 of Lecture 6 we know that the solution to the above minimization problem is the eigenvector corresponding to the smallest eigenvalue of A. From the solution to question 1 we know that the eigenvector corresponding to the smallest eigenvalue of A is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (the corresponding eigenvalue is -1 and the other two eigenvalues are 1 and 4). Hence **Option C** is the correct answer.

6. Consider a row stochastic matrix $M \in \mathbb{R}^3$. The sum of the elements of each row of this matrix is 1. Is the vector $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of this matrix?

    A. Yes

    B. No

    C. Can't say without knowing the elements of A

    D. Yes, only if each row represents a uniform distribution

Solution: Any row stochastic matrix $M \in \mathbb{R}^3$ will have the following form:

$$= \begin{bmatrix} a & b & 1-(a+b) \\ m & n & 1-(m+n) \\ p & q & 1-(p+q) \end{bmatrix}$$

where the sum of the elements of each row is 1. If we multiply such a row stochastic matrix by the vector $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ we get

$$Mx = \begin{bmatrix} a & b & 1-(a+b) \\ m & n & 1-(m+n) \\ p & q & 1-(p+q) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of any row stochastic matrix $M \in \mathbb{R}^3$.

7. Consider a set of points $x_1, x_2, ..., x_m \in \mathbb{R}^2$ represented using the standard basis $x = \begin{bmatrix}1 & 0\end{bmatrix}$ and $y = \begin{bmatrix}0 & 1\end{bmatrix}$. Let $X \in \mathbb{R}^{m\times 2}$ be a matrix such that $x_1, x_2, ..., x_m$ are the rows of this matrix. Using PCA, we want to represent this data using a new basis. To do so, we find the eigenvectors of X^TX, which happen to be $u_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$ and $u_2 = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$. Now suppose, we want to represent one of the m points, say $x_i = \begin{bmatrix}2.1 & 2.4\end{bmatrix}$ using only u_1 (*i.e.*, we want to represent the data using fewer dimensions then what would be the squared error in reconstructing x_i using only u_1 ?

   A. 0.045
   B. 0.030
   C. 0.015
   D. 0

Solution:

Consider the point $x_i = \begin{bmatrix}2.1 & 2.4\end{bmatrix}$ represented using the standard basis $x = \begin{bmatrix}1 & 0\end{bmatrix}$ and $y = \begin{bmatrix}0 & 1\end{bmatrix}$. We want to represent it using only $u_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$. To do this we first need to find the projection of x_i on u_1 and u_2 as:

$$\alpha_1 = x_i^T u_1 = \frac{4.5}{\sqrt{2}}$$
$$\alpha_2 = x_i^T u_2 = \frac{0.3}{\sqrt{2}}$$

We can then see that,

$$x_i = \alpha_1 u_1 + \alpha_2 u_2 = \begin{bmatrix}2.1\\2.4\end{bmatrix}$$

This is the full error-free reconstruction of x_i using both u_1 and u_2. However, in the question we are asked to reconstruct x_i using only u_1. Hence, we get,

$$\hat{x}_i = \alpha_1 u_1 = \begin{bmatrix}2.25\\2.25\end{bmatrix}$$

We can now compute the squared error between x_i and $\hat{x}_i$ as,

$$error = (2.25 - 2.1)^2 + (2.25 - 2.4)^2 = 0.045$$

Hence, **Option A** is the correct answer