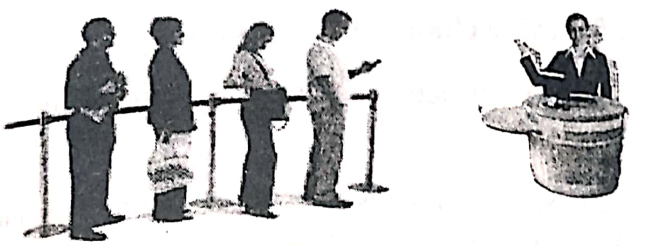
CHAPTER 4

QUEUEING THEORY

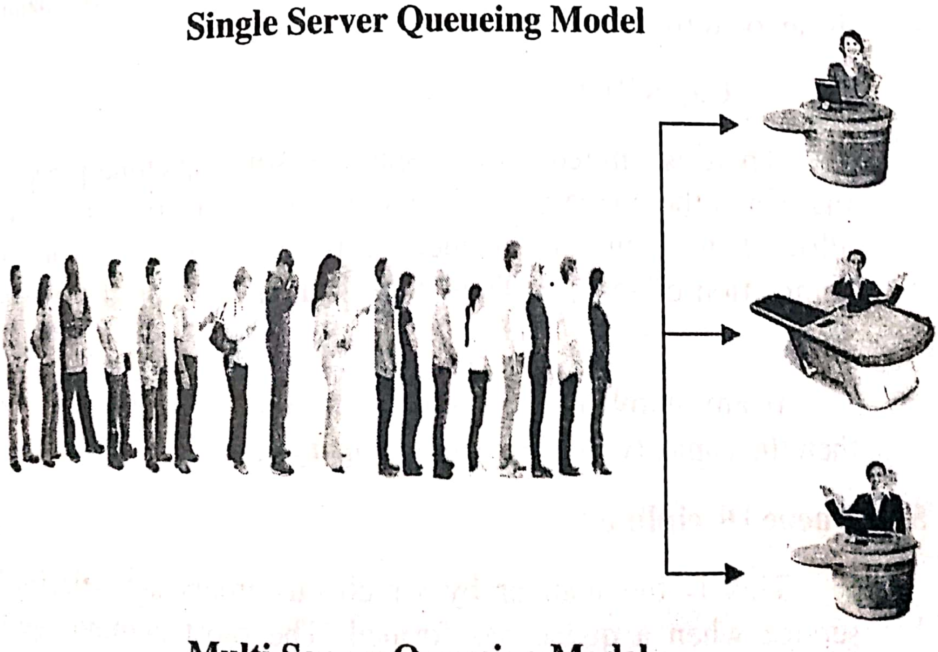
One can easily understand what is meant by queue from the examples.

Example	Members of Queue	Server(s)
Bank Counter	Account Holders	Counter Clerk
Toll Gate	Vehicles	Toll Collectors
Library	Students	Counter Clerk
Traffic Signal	Vehicles	Signal Point
Airport runways	Planes	Runways
Maintenance Shop	Breakdown Machines	Mechanics

We represent a simple queueing system as a diagram.



Single Server Queueing Model



Multi Server Queueing Model

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CHARACTERISTICS OF QUEUEING SYSTEM

The basic characteristics of a queueing system are

(1) **Arrival pattern of customers**

(2) **Service pattern of servers**

(3) **Number of service channels**

(4) **System capacity**

(5) **Queue discipline**

1. **Arrival Pattern :**

The arrival pattern is measured by the mean arrival rate or inter-arrival time. Here we assume that arrival process follows Poisson process. The arrival rate follows a Poisson distribution and hence the inter-arrival time follows an exponential distribution. Arrival rate is denoted by 'λ'.

2. **Service rate :**

Service time distribution is assumed to be exponential and mean service rate is usually denoted by 'μ'.

3. **Number of service channels (counters)**

To provide service we may have one counter or many counters.

In multi-server queues, there are many channels which provides the same service facilities. Number of service channels is denoted by 'c'.

4. **System Capacity :**

There is limited waiting space in some queuing process so that when the queue becomes large, further customers cannot be allowed to join the queue, until space is available after completion of service. Thus there is a finite limit of the system size.

If any number of customers are allowed to join the queue then the capacity of the system is ***infinite***.

5. **Queue Discipline :**

This is the manner by which customers are selected for service when a queue has formed. The most common queues disciplines are

a	FIFO (or) FCFS	First - In - First - Out (or) First - Come - First - Served
b	LIFO (or) LCFS	Last - In - First - Out (or) Last - Come - First - Served
c	SIRO	Selection - In - Random - Order
d	PIR	Priority in Section

KENDAL'S NOTATION FOR QUEUEING MODELS

Generally Queueing models may be completely specified in the following symbol form: $(a/b/c):(d/e)$, where

a = Probability law for the arrival (or inter arrival) time,

b = Probability law according to which the customers are being served.

c = Number of service stations

d = The maximum number allowed in the system (in service and waiting)

e = Queue Discipline

The above notation is called **Kendal's Notation.**

FOUR IMPORTANT MODELS :

1. $(M/M/1):(\infty/FIFO)$
2. $(M/M/c):(\infty/FIFO)$
3. $(M/M/1):(k/FIFO)$
4. $(M/M/c):(k/FIFO)$

NOTATIONS AND TERMINOLOGIES

	Notations	Terminology
1.	Queue size (or) Line length	No. of customers in the system.
2.	n	No. of customers in the system.
3.	S_n	The state in which there are 'n' customers in the system.

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4.	$P_n(t)$	Transient state probability that exactly 'n' customers are in the system at time 't'.
5.	P_n	Steady state probability of having 'n' customers in the system.
6.	λ	Mean arrival rate (number of arrivals per unit time)
7.	λ_n	Mean arrival rate when there are 'n' customers in the system (expected number of arrivals per unit time when there are 'n' customers in the system).
8.	μ	Mean service rate (number of customers being served in unit time)
9.	μ_n	Mean service rate when there are 'n' customers in the system (expected number of customers served per unit time when there are 'n' customers in the system).
10.	L_s	Expected number of customers in the system. (or) Expected queue size (waiting persons + person who is served)
11.	L_q	Expected (average) number of customers in the queue (only waiting persons).
12.	W_s	Expected (average) waiting time of a customer in the system
13.	W_q	Expected (average) waiting time of a customer in the queue.

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MODEL – I

SINGLE SERVER POISSON QUEUE : (M/M/1): (∞/ FIFO)

We assume

(a) the average arrival rate is constant, $\lambda_n = \lambda$ for all n.

(b) the average service rate is constant, $\mu_n = \mu$ for all n.

(c) the average arrival rate is less than the average service rate; $\lambda < \mu$ which assures that an infinite queue will not form.

Put $\lambda_n = \lambda$, $\mu_n = \mu$ in equation (7),

$$P_n = \frac{\lambda^n}{\mu^n} P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad \text{... (A)}$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

$$= \frac{1}{1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \dots}$$

$$= \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{-1}}$$

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SUMMARY

1. First find $W_s = \frac{1}{\mu - \lambda}$, then $W_q = W_s - \frac{1}{\mu}$,

   $L_s = \lambda W_s$ and $L_q = \lambda W_q$ can be found.

2. $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

3. Average length of non-empty queue $L_w = \mu . W_s$

4. $P(N \geq K) = \left(\frac{\lambda}{\mu}\right)^K$

   = Probability that the length of the queue system $\geq$ K.

5. P (Channel busy) $= \frac{\lambda}{\mu}$ and P (idle system) $= P_0 = 1 - \frac{\lambda}{\mu}$

6. P (Waiting time in the system $> t$) $= e^{-(\mu - \lambda)t}$

7. Probability density function of waiting time in the system is given by $f(w) = (\mu - \lambda)\, e^{-(\mu - \lambda)w}$, $w \geq 0$.

8. Probability density function of waiting time in the queue is given by $g(w) = \frac{\lambda}{\mu}(\mu - \lambda)\, e^{-(\mu - \lambda)w}$

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WORKED EXAMPLES

Example 6.1 : In the railway marshalling yard goods trains arrive at a rate of 30 trains per day. Assume that the inter arrival time follows exponential distribution and the service time distribution is also exponential with an average of 36 minutes. Calculate the following

(i) the mean queue size

(ii) the probability that the queue size exceeds 10.

If the input of trains increases to an average of 33 per day, what will be the change in the above quantities ? **[M.U 1990, A.U-2006]**

Solution : Given mean arrival rate $\lambda = 30$ per day

$$= \frac{30}{24 \times 60} \text{ per minute}$$

$$\Rightarrow \lambda = \frac{1}{48} \text{ trains/minute}$$

Service time is exponential with mean equal to 36 minutes.

$$\therefore \mu = \frac{1}{36} \text{ per minute.} \qquad [\because \text{mean} = \frac{1}{\mu}]$$

Then $$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{36} - \frac{1}{48}} = \frac{1}{\frac{4-3}{144}} = 144$$

$$W_q = W_s - \frac{1}{\mu} = 144 - 36 = 108$$

(i) Average queue size $L_q = \lambda W_q$

$$= \frac{1}{48} \times 108 = 2.25 \text{ trains.}$$

(ii) $$P(N \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10}$$

$$= \left(\frac{36}{48}\right)^{10} = (0.75)^{10} = 0.056$$

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If the input increases to 33 trains per day, then

$$\lambda = \frac{33}{24 \times 60} = \frac{11}{480} \text{ trains/minute}$$

and $$\mu = \frac{1}{36} \text{ trains/minute}$$

$$\therefore \quad W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{36} - \frac{11}{480}}$$

$$= \frac{1440}{40 - 33} = \frac{1440}{7}$$

$$W_q = W_s - \frac{1}{\mu}$$

$$= \frac{1440}{7} - 36$$

$$= \frac{1440 - 252}{7} = \frac{1188}{7}$$

$$L_q = \lambda W_q = \frac{11}{480} \times \frac{1188}{7}$$

$$= 3.89 \text{ trains.}$$

$\therefore$ Change in the size of the queue $= 3.89 - 2.25$

$= 1.64$ trains

$$\text{Now } P(N \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10}$$

$$= \left(\frac{11}{480} \times 36\right)^{10}$$

$$= \left(\frac{33}{40}\right)^{10} = 0.146$$

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$\therefore$ Change in probability $= 0.146 - 0.056$

$= 0.09$

Example 6.2 : Customers arrive at a one-man barber-shop according to a poisson process with mean interval arrival time of 20 minutes. Customers spend an average of 15 minutes in the barber's chair. If an hour is used as unit of time, then

(i) What is the probability that a customer need not wait for a hair cut ?

(ii) What is the expected number of customers in the barber shop and in the queue ?

(iii) How much time can a customer expect to spend in the barber shop ?

(iv) Find the average time that the customer spend in the queue.

(v) What is the probability that there will be more than 3 customers in the system ?

Solution : Model (M|M|1) : (∞|FIFO)

If the arrival is poisson with rate λ then the inter arrival is exponential with mean $\frac{1}{\lambda}$

Given $\frac{1}{\lambda} = 20 \Rightarrow \lambda = \frac{1}{20}$ per minute

$\frac{1}{\mu} = 15 \Rightarrow \mu = \frac{1}{15}$ per minute

(i) Probability that the queue is empty or the system is idle is

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{15}{20} = \frac{5}{20} = 0.25$$