POLLACZEK – KHINCHINE FORMULA (PK–formula)

Let us assume that the arrivals follow a Poisson process with rate of arrival λ. We also assume that service times are independently and identically distributed random variables with an arbitrary (or general) probability distribution. Let $b(t)$ be the p.d.f of service time T between 2 departures.

Let $N(t)$ be the number of customers in the system at time $t \geq 0$. Let t_n be the time instant at which the n^{th} customer completes service and departs. Let $X_n = N(t_n)$, $n = 1, 2, 3, \ldots$. Then X_n represents the number of customers in the system when the n^{th} customer departs. Also, the sequence of random variables $\{X_n : n = 1, 2, 3, \ldots\}$ is a Markov chain. Hence, we have,

$$X_{n+1} = \begin{cases} X_n - 1 + A, \text{ if } X_n > 0 \text{ i.e. } X_n \geq 1 \\ A \text{ if } X_n = 0 \end{cases}$$

where A is the number of customers arriving during the service time "T" of the $(n+1)^{th}$ customer.

We know that, if $U(X_n)$ denotes the unit step function, then we can write,

$$U(X_n) = \begin{cases} 1, & \text{if } X_n > 0 \text{ or } X_n \geq 1 \\ 0, & \text{if } X_n = 0 \end{cases}$$

$\therefore$ X_{n+1} can be written as

$$X_{n+1} = X_n - U(X_n) + A \qquad \ldots (1)$$

Suppose the system is in steady state, then the probability of the number of customers in the system is independent of time and hence is a constant.

That is, $E(X_{n+1}) = E(X_n)$ (the average size of the system at departure points).

Taking expectation on both sides of (1), we get

$$E(X_{n+1}) = E[X_n - U(X_n) + A]$$

$$\Rightarrow \quad E(X_{n+1}) = E(X_n) - E[U(X_n)] + E(A) \qquad \ldots (2)$$

Since $E(X_{n+1}) = E(X_n)$, we get

$$E(X_n) = E(X_n) - E[U(X_n)] + E(A)$$

$$\Rightarrow \quad E[U(X_n)] = E(A) \qquad \ldots (3)$$

Squaring equation (1), we have

$(a - b + c)^2$

$$X_{n+1}^2 = [X_n - U(X_n) + A]^2$$

$a^2 + b^2 + c^2 - 2ab$

$$= X_n^2 + U^2(X_n) + A^2 - 2X_n U(X_n)$$

$+2ca - 2cb$

$$+ 2AX_n - 2AU(X_n) \qquad \ldots (4)$$

$$[\because (a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac]$$

But

$$U^2(X_n) = \begin{cases} 1 & \text{if } X_n^2 > 0 \\ 0 & \text{if } X_n^2 = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } X_n > 0 \\ 0 & \text{if } X_n = 0 \end{cases}$$

[$\because$ X_n denotes the number of customers and hence X_n cannot be – ve]

$$= U(X_n)$$

Also, $X_n U(X_n) = X_n$ $\qquad [\because U(X_n) = 1 \text{ or } 0]$

Hence (4) becomes,

$$X_{n+1}^2 = X_n^2 + U(X_n) + A^2 - 2X_n + 2AX_n - 2AU(X_n)$$

$$2X_n - 2AX_n = X_n^2 - X_{n+1}^2 + U(X_n) + A^2 - 2AU(X_n)$$

$$2X_n(1-A) = X_n^2 - X_{n+1}^2 + U(X_n) + A^2 - 2AU(X_n)$$

Taking expectation on both sides, we get

$$2[E(X_n) - E(AX_n)] = E(X_n^2) - E(X_{n+1}^2) + E[U(X_n)] + E(A^2) - 2E[AU(X_n)]$$

$$2[E(X_n) - E(A)E(X_n)] = E(X_n^2) - E(X_{n+1}^2) + E[U(X_n)] + E(A^2) - 2E(A)E[U(X_n)]$$

[$\because$ A and X_n are independent]

$$\Rightarrow 2E(X_n)[1-E(A)] = E(A^2) - \cancel{E(A^2)} + E(A) + \cancel{E(A^2)} - 2E(A)E(A)$$

$$\Rightarrow 2E(X_n)[1-E(A)] = E(A^2) + E(A) - 2[E(A)]^2$$

$$\Rightarrow \quad E(X_n) = \frac{E[A^2] + E(A) - 2[E(A)]^2}{2[1-E(A)]} \quad \ldots (5)$$

Since the arrivals during "T" is a Poisson process with rate λ,

$$E(A/T) = \lambda T$$

$$E(A^2/T) = \lambda^2 T^2 + \lambda T \quad \ldots (6)$$

(Refer to the Poisson process in Unit 3).

Also, $E(A) = E[E(A/T)]$

$$= E(\lambda T) = \lambda E(T) \quad \text{(From (6))} \quad \ldots (7)$$

Similarly, $E(A^2) = E[E(A^2/T)]$

$$= E(\lambda^2 T^2 + \lambda T)$$

$$= \lambda^2 E(T^2) + \lambda E(T) \quad \ldots (8)$$

(from (6))

$\therefore$ (5) becomes,

$$E(X_n) = \frac{\lambda^2 E(T^2) + \lambda E(T) + \lambda E(T) - 2[\lambda E(T)]^2}{2[1-\lambda E(T)]}$$

$$\Rightarrow \quad L_s = \frac{\lambda^2 E(T^2) + 2\lambda E(T) - 2\lambda^2 [E(T)]^2}{2(1 - \lambda E(T))}$$

$$= \frac{2\lambda E(T)[1 - \lambda E(T)] + \lambda^2 E(T^2)}{2(1 - \lambda E(T))}$$

$$= \frac{2\lambda E(T)[1 - \lambda E(T)]}{2(1 - \lambda E(T))} + \frac{\lambda^2 E(T^2)}{2(1 - \lambda E(T))}$$

We know that

$$\mathrm{Var}(T) = E(T^2) - [E(T)]^2$$

$$\Rightarrow E(T^2) = \mathrm{Var}(T) + [E(T)]^2$$

$$\therefore \quad \mathbf{L_s = \lambda E(T) + \frac{\lambda^2 [Var(T) + (E(T))^2]}{2(1 - \lambda E(T))}}, \quad \text{where } \lambda E(T) < 1$$

This is called Pollaczek-khinchine formula (PK – formula)