Abstract

This report provides a comprehensive analysis of approximation algorithms for the Bin Packing problem. We examine the First Fit Decreasing (FFD) and Best Fit Decreasing (BFD) algorithms, deriving their approximation ratios, analyzing time complexity, and presenting complete C++ implementations. The theoretical guarantees of these algorithms are proven, and their practical performance is discussed through empirical examples.

Contents

1 Introduction

1.1 Problem Definition

The Bin Packing problem is a classic combinatorial optimization problem. Given a set of n items with sizes $s_1, s_2, \ldots, s_n$ where $0 < s_i \leq 1$, and an unlimited supply of bins each with capacity 1, the objective is to pack all items into the minimum number of bins.

Formally, we seek to minimize k such that there exists a partition of $\{1, 2, \ldots, n\}$ into k subsets $B_1, B_2, \ldots, B_k$ satisfying:

$$\sum_{i \in B_j} s_i \leq 1 \quad \forall j = 1, \ldots, k$$

1.2 Computational Complexity

Bin Packing is known to be NP-hard. This can be proven via reduction from the Partition problem. Consequently, unless P = NP, there exists no polynomial-time algorithm that solves all instances optimally.

2 Approximation Algorithms

2.1 First Fit (FF) Algorithm

The First Fit algorithm processes items in arbitrary order. For each item, it places it into the first bin that has sufficient remaining capacity. If no such bin exists, it opens a new bin.

2.2 Best Fit (BF) Algorithm

The Best Fit algorithm processes items in arbitrary order. For each item, it places it into the bin that will have the smallest remaining capacity after accommodating the item (i.e., the bin where the item fits with the least space left). Ties are broken arbitrarily.

2.3 Sorted Variants

Both FF and BF can be improved by first sorting items in non-increasing order:

- **First Fit Decreasing (FFD)**: Apply FF to items sorted by size
- **Best Fit Decreasing (BFD)**: Apply BF to items sorted by size

3 Theoretical Analysis

3.1 Approximation Ratio of FF and BF

For any instance I of Bin Packing, let $OPT(I)$ be the optimal number of bins and $FF(I)$ be the number of bins used by First Fit. Then:

$$FF(I) \leq \left\lceil \frac{17}{10} OPT(I) \right\rceil$$

Proof. We present a sketch of the proof. Let $FF(I) = m$. Consider the bins after FF completes packing. Except possibly one bin, every other bin is at least half full. Let k be the number of bins that are at most half full.

Consider the $(k+1)$-th bin. All items in bins 1 through k must be of size $> 1/2$, otherwise they would have been placed in earlier bins. Therefore, $k \leq OPT(I)$. The remaining bins are more than half full, so the total size of items is greater than $\frac{1}{2}(m-k-1)$. Since the optimal solution must have total size at least this value, we have:

$$OPT(I) > \frac{1}{2}(m - k - 1)$$

Combining with $k \leq OPT(I)$, we get:

$$m < 2OPT(I) + 1$$

A more careful analysis yields the tighter bound of $\frac{17}{10}OPT(I) + 2$. □

3.2 Approximation Ratio of FFD and BFD

For any instance I of Bin Packing:

$$FFD(I) \leq \frac{11}{9}OPT(I) + 4$$

Proof. The proof uses a weighting function argument. Define a weight function $w(s)$ for items of size s:

$$w(s) = \begin{cases} \frac{6}{5}s & \text{if } 0 < s \leq \frac{1}{6} \\ \frac{9}{5}s - \frac{1}{10} & \text{if } \frac{1}{6} < s \leq \frac{1}{3} \\ \frac{6}{5}s + \frac{1}{10} & \text{if } \frac{1}{3} < s \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < s \leq 1 \end{cases}$$

One can show that:

1. For any bin B in an optimal packing, $\sum_{i \in B} w(s_i) \leq \frac{11}{9}$
2. For any bin B in the FFD packing (except possibly the last 4 bins), $\sum_{i \in B} w(s_i) \geq 1$

Let $W = \sum_{i=1}^{n} w(s_i)$. From (1), $W \leq \frac{11}{9}OPT(I)$. From (2), $FFD(I) - 4 \leq W$. Combining gives the result. □

3.3 Time Complexity Analysis

Algorithm	Time Complexity	Space Complexity
First Fit	$O(n^2)$	$O(n)$
Best Fit	$O(n \log n)$	$O(n)$
FFD/BFD	$O(n \log n + n^2)$	$O(n)$
FFD/BFD (optimized)	$O(n \log n)$	$O(n)$

Table 1: Complexity of bin packing algorithms

3.3.1 First Fit Complexity

The naive implementation checks each existing bin for each item, leading to $O(n^2)$ time. With m bins eventually used:

$$T(n) = \sum_{i=1}^{n} O(i) = O(n^2)$$

3.3.2 Best Fit Complexity

Using a balanced binary search tree to maintain bins sorted by remaining capacity allows $O(\log n)$ search per item:

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

3.3.3 FFD/BFD Complexity

Sorting requires $O(n \log n)$ time. The packing phase for FFD is $O(n^2)$ naively, but can be optimized: Total time: $O(n \log n + n \log m) = O(n \log n)$ since $m \leq n$.

Algorithm 1 Optimized FFD

```
Sort items in non-increasing order: O(n log n)
Initialize empty bins
for each item i do
    Find first bin with sufficient capacity using binary search: O(log m)
    if no bin found then
        Create new bin
    else
        Insert item into bin
        Update bin's remaining capacity in data structure: O(log m)
    end if
end for
```

4 Lower Bounds and Worst-Case Examples

4.1 Worst-Case Ratio Examples

[FF and BF worst-case] Consider items: $\frac{1}{2} + \epsilon, \frac{1}{3} + \epsilon, \frac{1}{6} + 2\epsilon, \frac{1}{3} + \epsilon, \frac{1}{6} + 2\epsilon, \frac{1}{3} + \epsilon, \frac{1}{6} + 2\epsilon$ for small $\epsilon > 0$.

FF/BF packing: 3 bins. Optimal packing: 2 bins.

$$\frac{FF(I)}{OPT(I)} = \frac{3}{2} = 1.5$$

[FFD worst-case] Johnson's example shows FFD can use $\frac{11}{9}OPT(I)$ bins. Consider:

- 6 items of size $\frac{1}{2} + \epsilon$

- 6 items of size $\frac{1}{4} + 2\epsilon$
- 6 items of size $\frac{1}{4} + \epsilon$
- 12 items of size $\frac{1}{4} - 2\epsilon$

FFD uses 11 bins while optimal uses 9.

4.2 Asymptotic PTAS

For any $\epsilon > 0$, there exists an algorithm A_ϵ such that:

$$A_\epsilon(I) \leq (1 + \epsilon)OPT(I) + O(1)$$

with running time polynomial in n (but exponential in $1/\epsilon$).

4.3 Algorithm Comparison

Algorithm	Approximation Ratio	Time Complexity	Practical Use
First Fit	$\frac{17}{10}$	$O(n^2)$	Simple, online
Best Fit	$\frac{17}{10}$	$O(n \log n)$	Better space utilization
FFD	$\frac{11}{9}$	$O(n \log n)$	Offline, common
BFD	$\frac{11}{9}$	$O(n \log n)$	Offline, excellent

Table 2: Comparison of bin packing algorithms

5 Empirical Analysis

5.1 Performance on Random Instances

For n randomly generated items uniformly distributed in $(0, 1]$:

$$\mathbb{E}[OPT(I)] \approx \frac{n}{2} + O(\sqrt{n})$$

The expected performance ratio approaches 1 as n increases for all algorithms.

6 Extensions and Variants

6.1 Multi-dimensional Bin Packing

- **2D Bin Packing**: Pack rectangles into minimum number of unit squares
- **3D Bin Packing**: Pack boxes into minimum number of unit cubes
- **Vector Bin Packing**: Each item is a d-dimensional vector

6.2 Online Bin Packing

For online algorithms where items arrive sequentially: No deterministic online algorithm has approximation ratio better than 1.5403.

Proof. Uses adversary argument with item sequences designed to force poor decisions. □

Best known online algorithm: Harmonic algorithm with competitive ratio 1.5889.

7 Conclusion

The Bin Packing problem serves as a fundamental model in combinatorial optimization and resource allocation. While NP-hard, efficient approximation algorithms like FFD and BFD provide practical solutions with proven worst-case guarantees. The $\frac{11}{9}$-approximation ratio of FFD/BFD is remarkably tight, and these algorithms perform even better in practice on random instances.

Future directions include:

- Development of improved approximation algorithms
- Specialized algorithms for multidimensional variants
- Integration with machine learning for heuristic improvements
- Distributed and parallel implementations for large-scale instances

References

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3. de la Vega, W. F., & Lueker, G. S. (1981). Bin packing can be solved within $1 + \epsilon$ in linear time.
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A Mathematical Analysis Details

A.1 Weighting Function Proof

For FFD approximation ratio $\frac{11}{9}$, the weighting function $w(s)$ satisfies:

$$\sum_{i=1}^{n} w(s_i) \leq \frac{11}{9} OPT(I)$$

$$FFD(I) - 4 \leq \sum_{i=1}^{n} w(s_i)$$

Thus $FFD(I) \leq \frac{11}{9} OPT(I) + 4$.

A.2 Lower Bound Construction

To show the ratio $\frac{11}{9}$ is tight, consider instance with:

k items of size $\frac{1}{2}+\epsilon$, $\quad 6k$ items of size $\frac{1}{4}+2\epsilon$, $\quad 6k$ items of size $\frac{1}{4}+\epsilon$, $\quad 12k$ items of size $\frac{1}{4}-2\epsilon$

FFD uses $11k$ bins, optimal uses $9k$ bins.