Stochastic Calculus Lecture 3

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Introduction

In this lecture, we begin by introducing the role of random variables in finance and examining the payoff structure of a forward contract. We then develop the concept of arbitrage and the idea of replication in markets. Finally, we illustrate how the price of one asset can be replicated by another through the principle known as the law of one price.

1 Random Variables and Payoffs

Stock prices are **stochastic** in nature, meaning that their future values are not known in advance and can only be described probabilistically.

Example 1.1. *Suppose today ($t = 0$) the stock price is $S_0 = 100$. In one period, the stock may either go up to 120 with probability 0.6, or go down to 90 with probability 0.4.*

$$S_1 = \begin{cases} 120 & \text{with probability } 0.6, \\ 90 & \text{with probability } 0.4. \end{cases}$$

Here, S_1 is a random variable describing the uncertain stock price tomorrow. The expected value is

$$\mathbb{E}[S_1] = (0.6)(120) + (0.4)(90) = 108.$$

Thus, although the actual value of S_1 will be either 120 or 90, on average we expect the stock to be worth 108 tomorrow.

Definition 1.1. *A **forward contract** is an agreement made at time $t = 0$ to buy or sell an asset at a predetermined price F_T (called the forward price) at a future time T (the maturity date).*

- *The buyer of the forward contract agrees to purchase the asset at T for F_T.*
- *The seller of the forward contract agrees to deliver the asset at T for F_T.*

The payoff at maturity is given by

$$Payoff = S_T - F_T,$$

where S_T is the spot price of the asset at maturity and S_T is the realization *of the random variable at maturity T.*

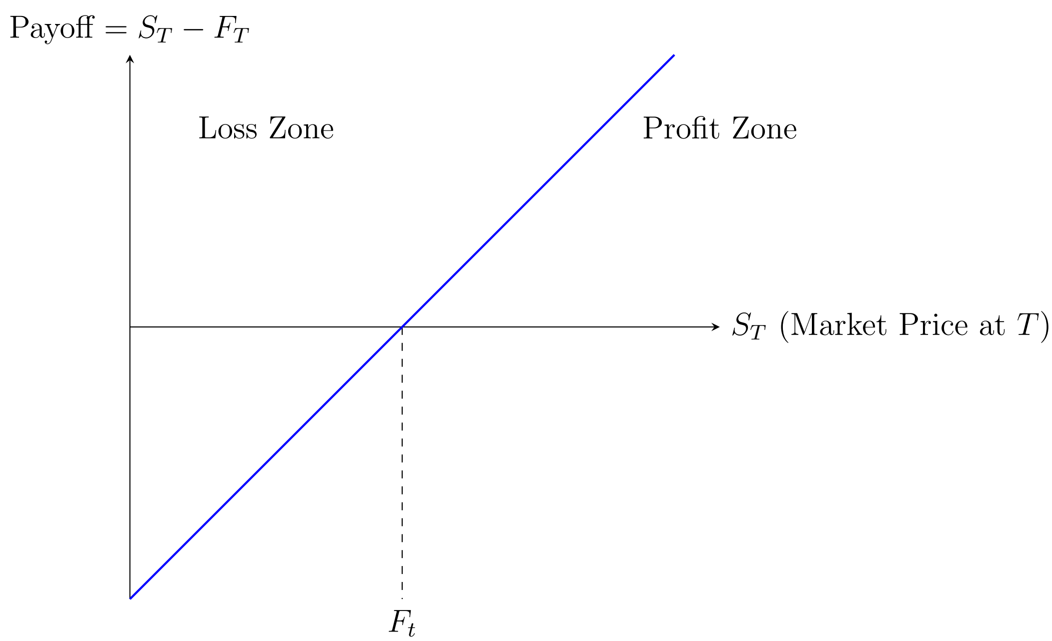


Figure 1: Payoff Diagram of a Long Forward Contract

Here $S_t(W)$ represents the realization of a random variable. This diagram tells us where exactly the buyers gets money in hand and where he looses.

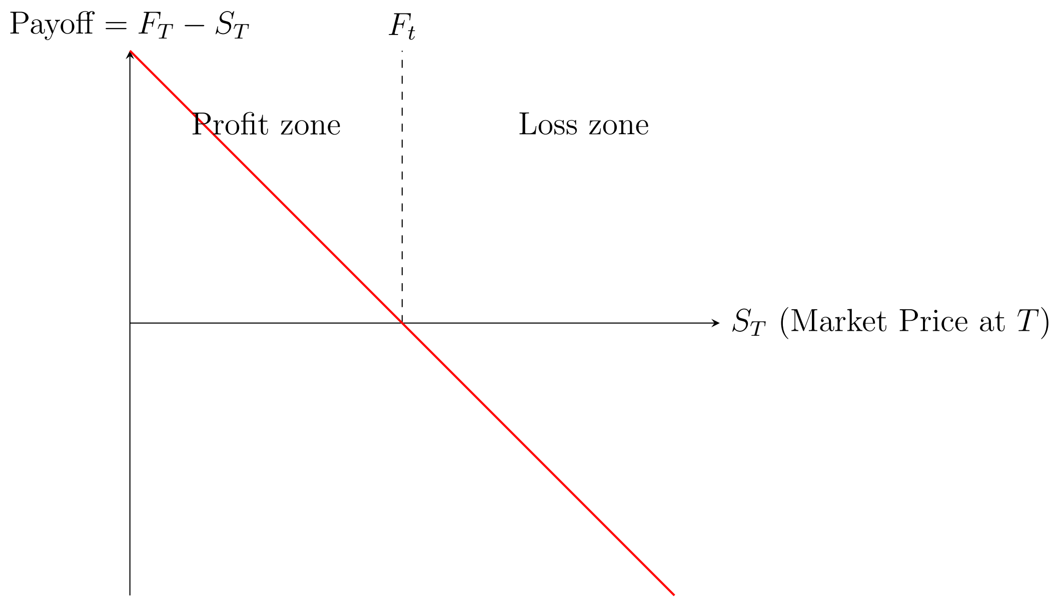


Figure 2: Payoff Diagram of a Short Forward Contract

In a short forward position, the seller (short) benefits when the market price S_T falls below the agreed forward price F_T, since they can sell the asset at the higher contract price. Conversely, if $S_T > F_T$, the seller incurs a loss.

Timeline of a Forward Contract

- At $t = 0$, the forward price $F_t(T)$ is fixed.
- At maturity T, the spot price S_T is realized.
- At time t there is no money being exchanged.

2 Arbitrage

Definition 2.1. *An **arbitrage** is a strategy that yields riskless profit by exploiting inefficiencies in asset pricing.*

Example 2.1. *Consider two assets that are guaranteed to give the same payoff at maturity. Suppose asset A is priced at $100 while asset B, which has the identical payoff, is priced at $102.*

An investor can buy asset A and simultaneously sell asset B. At maturity, the payoffs from both assets are the same, so the investor has no risk. However, at the initial time the investor collects a net gain of $2.

*This is an **arbitrage opportunity**, and it violates the **Law of One Price**, which requires that two assets with identical future payoffs must trade for the same price today.*

3 Replication in Markets

Suppose we know the payoff of an asset $\vec{P}_a$ but not its price. The goal is to construct another portfolio that replicates this payoff:

$$\vec{P}_a = \vec{P}_b$$

where $\vec{P}_b$ may consist of multiple assets. The main objective is to replicate $\vec{P}_a$.

3.1 Strategy Representation

Let M be the matrix whose columns are the assets from the market, it can *Reliance*, $Tata$, *Apple* etc.

$$M\vec{X} = \vec{P}_a \in \mathrm{C}(M)$$

- $\vec{X}$ is the **strategy vector**. It specifies how many units of each asset from the market are to be held in order to form a portfolio. Positive entries in $\vec{X}$ represent long positions (buying the asset), while negative entries represent short positions (selling the asset).

Example 3.1. *Suppose there are two assets: a stock and a bond. If* $\vec{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$*, this means the strategy consists of taking a long position in* 2 *units of the stock and a short position in* 1 *unit of the bond.*

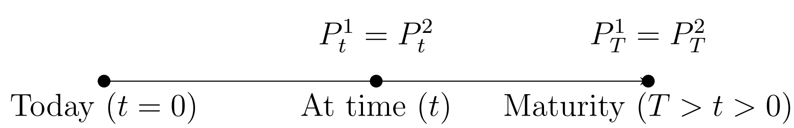
- $\mathrm{C}(M)$ denotes the **column space** of M. It is the set of all possible payoff vectors that can be generated by portfolios of the available assets. In other words, if a target payoff lies in $\mathrm{C}(M)$, then it can be replicated exactly by choosing an appropriate strategy $\vec{X}$.

  Example 3.2. *Let*

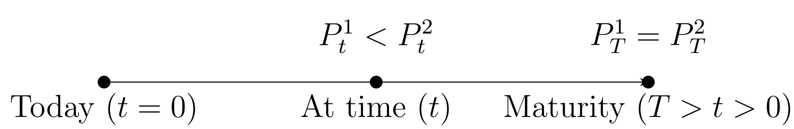
  $$M = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

  The first column represents the payoff of asset 1, the second column that of asset 2. Then $C(M)$ is the set of all linear combinations $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Any payoff vector in this set can be replicated using a portfolio of assets 1 and 2.

4 Arbitrage Strategy: Law of One Price



Remark 4.1. *Here the both the assets at time t is equal, showing there is no arbitrage opportunity.*



Remark 4.2. *Here the asset 1 at time t is less than asset 2 at time t. Here exists an arbitrage opportunity.*

The **law of one price** states that if two assets give the same payoff at maturity, their prices must be equal at all earlier times (in an arbitrage-free market). The example of the arbitrage opportunity is given up.

4.1 Case 1: Arbitrage Opportunity

$$P_T^1 = P_T^2 \quad \text{but} \quad P_t^1 < P_t^2$$

At time t, asset 1 is cheaper than asset 2, although both have the same value at T. Therefore:

- Go **long** in asset 1: $+P_t^1$
- Go **short** in asset 2: $-P_t^2$

The total value at t is

$$P_t^1 - P_t^2 < 0,$$

which means money flows into your pocket immediately. The major rule of finance is **Buy Cheap and Sell Expensive**. Here as the price of asset 1 is cheap we are buying it at time t and shorting the asset 2 which is more expensive than asset 1. So we are doing two trades(buy and sell).

4.2 Strategy Vector

$$\vec{X} = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, \qquad \vec{P}_t = \begin{bmatrix} P_t^1 \\ P_t^2 \end{bmatrix}, \qquad \vec{P}_T = \begin{bmatrix} P_T^1 \\ P_T^2 \end{bmatrix}$$

- At t, long asset 1 = + P_t^1
- At t, short asset 2 = - P_t^2

The payoff can be written as:

$$\vec{X}^T \vec{P}_t = P_t^1 - P_t^2 < 0, \qquad \vec{X}^T \vec{P}_T = P_T^1 - P_T^2$$

$$Proof for: \vec{X}^T \vec{P}_t = P_t^1 - P_t^2 < 0$$

$$\vec{X}^T \vec{P}_t = \begin{bmatrix} +1 - 1 \end{bmatrix} * \begin{bmatrix} P_t^1 \\ P_t^2 \end{bmatrix}$$

$$\vec{X}^T \vec{P}_t = P_t^1 - P_t^2$$

Remark 4.3. *Every asset can, in principle, be replicated.*