Module 2: Numerical Interpolation, Differentiation & Integration

Numerical Interpolation:

1) Interpolation with equal intervals — XOR

(a) Newton's forward interpolation formula

(b) Newton's backward interpolation formula

Interpolation:-

Suppose we have the values of $y=f(x)$ for a set of values of x as follows:

x:	x_0	x_1	x_2	...	x_n
y:	y_0	y_1	y_2	...	y_n.

Then, the process of finding the value of y corresponding to the value of x between x_0 & x_n is called interpolation.

Also, the process of computing the value of the function y outside the given range of x is called extrapolation.

e.g:

x :	20	25	30	35	40	45
$y=f(x)$:	354	332	291	260	231	204

(1) Finding y when $x=23$ → Interpolation.

(2) $f(28)$ → Interpolation.

(3) Finding y when $x=18$ → Extrapolation.

(4) $f(46)$ → Extrapolation.

How to create a difference table :-

e.g.

Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$2 = \Delta y_0$			
1	3		$6 = \Delta^2 y_0$		
		8		$6 = \Delta^3 y_0$	
2	11		12		$0 = \Delta^4 y_0$
		20		6	
3	31		18		0
		38		6	
4	69		24		$0 = \nabla^4 y_n$
		62		$6 = \nabla^3 y_n$	
5	131		$30 = \nabla^2 y_n$		
		$92 = \nabla y_n$			
$x_n = 6$	$y_n = 223$				
		∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$

a) Newton's forward interpolation formula:

The value of $y = f(x)$ at $x = x_0 + rh$ is given by

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots,$$

where r is any real no., $r = \frac{x - x_0}{h}$,

$x_0 \to$ first value of x & h is the step length.

also, $\Delta y, \Delta^2 y, \Delta^3 y, \dots$ are forward differences.

b) Newton's backward interpolation formula:

The value of $y = f(x)$ at $x = x_n + rh$ is given by

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

where 'r' is any real number, $r = \frac{x - x_n}{h}$,

$x_n \to$ last value of x & $h \to$ step length

also, $\nabla y, \nabla^2 y, \nabla^3 y, \dots$ are backward differences.

Note:

1. Newton's forward interpolation is used to interpolate the value of y when x is near to x_0 (first value).

2. Newton's backward interpolation is used to interpolate the value of y when x is near the end value (x_n) of the set of tabular values.

Examples:-

1. Find $y(1.4)$, given the data

x	1	2	3	4	5
y	10	26	58	112	194

Solⁿ:- Here, the given interval is equal.
Also, 1.4 value is close to $x_0 = 1$.
So, we apply Newton's forward interpolation formula.

The forward difference table:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	10				
		$\boxed{16 = \Delta y_0}$			
2	26		$\boxed{16 = \Delta^2 y_0}$		
		32		$\boxed{6 = \Delta^3 y_0}$	
3	58		22		$\boxed{0 = \Delta^4 y_0}$
		54		6	
4	112		28		
		$82 = \nabla y_n$			
5	194				

Newton's forward interpolation formula is given by

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Here, $x_0 = 1$, $x = 1.4$, $h = 1$

$$\therefore r = \frac{x - x_0}{h} = \frac{1.4 - 1}{1} = 0.4$$

$$\therefore y_r = 10 + (0.4)(16) + \frac{(0.4)(0.4-1)}{2}(16) + \frac{(0.4)(0.4-1)(0.4-2)}{6} \times 6$$

$$= 10 + 6.4 - 1.920 + 0.384$$

$$= 14.846$$

$$\therefore y(1.4) \simeq 14.85.$$

2. The population of a town is given by the table

Year	1971	1981	1991	2001	2011	2021
Population in lakhs	12	15	20	27	39	52

Estimate the increase in population during the period from 1975 to 2015.

Solⁿ: ~~Case 1 :- For the year 1975,~~ we use ~~Newton's~~ forward interpolation formula.

Difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_0 = 1971$	$y_0 = 12$					
		$\boxed{3 = \Delta y_0}$				
1981	15		$\boxed{2 = \Delta^2 y_0}$			
		5		$\boxed{0 = \Delta^3 y_0}$		
1991	20		2		$\boxed{3 = \Delta^4 y_0}$	
		7		3		$\boxed{-10 = \Delta^5 y_0}$ $\boxed{= \nabla^5 y_n}$
2001	27		5		$\boxed{-7 = \nabla^4 y_n}$	
		12		$\boxed{-4 = \nabla^3 y_n}$		
2011	39		$\boxed{1 = \nabla^2 y_n}$			
		$\boxed{13 = \nabla y_n}$				
$x_n = 2021$	$y_n = 52$					
		∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	

Case 1 : For the year 1975, we use Newton's forward interpolation formula : ie,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

Here, $r = \frac{x - x_0}{h} = \frac{1975 - 1971}{10} = \frac{4}{10} = 0.4$

$$\therefore y_r = 12 + (0.4)(3) + \frac{(0.4)(0.4-1)}{2}(2) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(0)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(3) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{120}(-10)$$

$$= 12 + 1.2 - 0.24 + 0 - 0.12 - 0.30$$

$$= 13.2 - 0.66$$

$\therefore y(1975) = 12.54$

Case 2 : For the year 2015, we use Newton's backward interpolation formula,

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n$$

$$+ \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \frac{r(r+1)(r+2)(r+3)(r+4)}{5!}\nabla^5 y_n$$

Here, $r = \frac{x - x_n}{h} = \frac{2015 - 2021}{10} = \frac{-6}{10} = -0.6$

$$\therefore y_r = 52 + (-0.6)(13) + \frac{(-0.6)(-0.6+1)}{2}(1) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-4)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-7) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{120}(-10)$$

$= 52 - 7.8 + 0.12 + 0.22 + 0.24 + 0.23$

$\therefore$ ~~$y(2015) = 44.01$~~

$\therefore y(2015) = 44.77$

Practice Problems:

1) In the table given below, the values of y are consecutive terms of a series of which 14.5 is the 5^{th} term. Find the first & tenth term of the series

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

2) The population of a town is given by the table

Year	1951	1961	1971	1981	1991
Population in thousands	19.96	39.65	58.81	77.21	94.61

Calculate the increase in population from the year 1995 to 1985 using appropriate interpolation formula.

Interpolation formulae for unequal intervals:

Divided differences:

Let $f(x_0), f(x_1), \dots f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of x: $x_0, x_1, \dots, x_n$ at unequal intervals.

The first order divided differences are defined as,

$$f(x_0 x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

$$\dots \quad f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

The second order divided differences are

$$f(x_0 x_1 x_2) = \frac{f(x_1 x_2) - f(x_0 x_1)}{x_2 - x_0}, \quad f(x_1 x_2 x_3) = \frac{f(x_2 x_3) - f(x_1 x_2)}{x_3 - x_1},$$

$$\dots \quad f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1} x_n) - f(x_{n-2} x_{n-1})}{x_n - x_{n-2}}.$$

Similarly, the higher order divided differerences are defined.

Divided difference table:

x	$f(x)$	I D.D.	II D.D.	III D.D.
$x_0 = 2$	$f(x_0) = 4$			
		$\frac{56-4}{4-2} = 26 = f(x_0 x_1)$		
$x_1 = 4$	$f(x_1) = 56$		$\frac{131-26}{9-2} = 15 = f(x_0 x_1 x_2)$	
		$\frac{711-56}{9-4} = 131 = f(x_1, x_2)$		$\frac{23-15}{10-2} = 1 = f(x_0 x_1 x_2 x_3)$
$x_2 = 9$	$f(x_2) = 711$		$\frac{269-131}{10-4} = 23 = f(x_1 x_2 x_3)$	
		$\frac{980-711}{10-9} = 269 = f(x_2 x_3)$		
$x_3 = 10$	$f(x_3) = 980$			

We have the following two methods:

1. Newton's divided difference formula
2. Lagrange's Interpolation formula.

1. Newton's divided difference formula: is given by

$$y = f(x) = f(x_0) + (x-x_0) f(x_0 x_1) + (x-x_0)(x-x_1) \cdot f(x_0 x_1 x_2) + \cdots + (x-x_0)(x-x_1) \cdots (x-x_{n-1}) \cdot f(x_0 x_1 \cdots x_n)$$

Examples:-

1.