Module 1 : Solution of Linear System of Equations

Introduction :

Numerical methods provide various techniques to find approximate solution to difficult problems using simple operations.

Numerical methods involve sequential steps, hence they are easily adoptable to solve problems using computers.

1.1 : Solution of Linear System of Equations

Problems like

(i) analysis of electronic circuits consisting of invariant elements,

(ii) analysis of a network under sinusoidal steady-state conditions,

(iii) determination of the output of a chemical plant,

(iv) finding the cost of chemical reactions, etc

all these depend on the solution of simultaneous linear equations.

To solve the linear system of equations we have the following methods

(1) Direct Method

(2) Iterative Method.

Direct method of solution

LU Decomposition method

LU Decomposition method is based on the fact that " every square matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular".

i.e if $A = [a_{ij}]$ then $a_{11} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \dots\dots$$

such a factorization if exists, it is unique.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix form we have $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now let $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Consider $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

By matrix multiplication

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now compute the elements of L & U in the following order.

(i) From first row for U

$$u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$$

(ii) From first column for L

$$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = \frac{a_{21}}{u_{11}}, \quad l_{31}u_{11} = a_{31} \Rightarrow l_{31} = \frac{a_{31}}{u_{11}}$$

(iii) From second row for U

$$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - l_{21}u_{12}$$

$$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - l_{21}u_{13}$$

(iv) From second column for L

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{1}{u_{22}}\left[a_{32} - l_{31}u_{12}\right]$$

(v) From third row for U

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \Rightarrow u_{33} = a_{33} - l_{13}u_{13} - l_{32}u_{23}$$

Now substitute $A = LU$ in equation ①

$\therefore$ eqn ① becomes

$$LUX = B \quad \text{— ②}$$

Let $UX = V$ — ③

$\therefore$ eqn ② becomes

$$LV = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ l_{21}v_1 + v_2 \\ l_{31}v_1 + l_{32}v_2 + v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\therefore$ $v_1 = b_1$, $l_{21}v_1 + v_2 = b_2$, $l_{31}v_1 + l_{32}v_2 + v_3 = b_3$

solving these equations we get v_1, v_2, v_3 and substituting in ③ we get.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 \\ u_{22}x_2 + u_{23}x_3 \\ u_{33}x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\Rightarrow u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1$, $u_{22}x_2 + u_{23}x_3 = v_2$, $u_{33}x_3 = v_3$

solving these equations we get x_1, x_2, x_3 which is the solution of given system of equations.

Procedure in brief

Write the given system of equations in the matrix form $AX = B$ — ①

Let $A = LU$ — ②

Find the elements of L & U

using eqn ② in eqn ① we get

$$LUX = B \quad \text{— ③}$$

Now let $UX = V$

∴ eqn ③ becomes

$$LV = B$$

Find the elements of V and then

Consider $UX = V$ and find elements of

X,

∴ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution of given system of equations.

Solved Problems

1) Apply LU decomposition method to solve the equations
$3x+2y+7z=4$, $2x+3y+z=5$, $3x+4y+z=7$

Sol^n: Given
$$3x+2y+7z=4$$
$$2x+3y+z=5$$
$$3x+4y+z=7$$

In matrix form we have $AX=B$ where

$$A=\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad B=\begin{bmatrix}4\\5\\7\end{bmatrix}$$

Let $LU=A$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}$$

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}$$

Now from elements of

(i) first row for U : $\boxed{u_{11}=3}$, $\boxed{u_{12}=2}$, $\boxed{u_{13}=7}$

(ii) first column for L : $l_{21}u_{11}=2 \Rightarrow l_{21}=\frac{2}{u_{11}}=\frac{2}{3}$

$l_{31}u_{11}=3 \Rightarrow l_{31}=\frac{3}{u_{11}}=\frac{3}{3}=1$

$\therefore \boxed{l_{21}=\frac{2}{3}}$, $\boxed{l_{31}=1}$

(iii) second row of U : $l_{21}u_{12}+u_{22}=3$

$\frac{2}{3}\cdot 2+u_{22}=3 \Rightarrow \boxed{u_{22}=\frac{5}{3}}$

& $l_{21}u_{13}+u_{23}=1 \Rightarrow \frac{2}{3}\cdot 7+u_{23}=1 \Rightarrow \boxed{u_{23}=-\frac{11}{3}}$

(iv) second column for L

$l_{31}u_{12} + l_{32}u_{22} = 4$

$1(2) + l_{32} \cdot \frac{5}{3} = 4 \Rightarrow$

$\frac{5}{3} l_{32} = 4 - 2 = 2 \Rightarrow \boxed{l_{32} = \frac{6}{5}}$

(v) third row for U

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$1(7) + \frac{6}{5}\left(-\frac{11}{3}\right) + u_{33} = 1 \Rightarrow \boxed{u_{33} = -\frac{8}{5}}$

Now substituting $A = LU$ in $AX = B$ we get

$LUX = B$.

and substituting $UX = V$ we get.

$LV = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} * \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$\therefore$ we get $\begin{bmatrix} V_1 \\ \frac{2}{3}V_1 + V_2 \\ V_1 + \frac{6}{5}V_2 + V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

$\therefore \boxed{V_1 = 4}$, $\frac{2}{3}V_1 + V_2 = 5 \Rightarrow \boxed{V_2 = \frac{7}{3}}$, $V_1 + \frac{6}{5}V_2 + V_3 = 7$

$\Rightarrow \boxed{V_3 = \frac{1}{5}}$

$\therefore UX = V$ becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$\Rightarrow 3x + 2y + 7z = 4$

$5/3\, y - \frac{11}{3} z = 7/3$

$-\frac{8}{5} z = \frac{1}{5} \Rightarrow \boxed{z = -\frac{1}{8}}$

By back substitution we get

$\boxed{y = \frac{9}{8}}$ & $\boxed{x = \frac{7}{8}}$

2) Solve $2x+3y+z=9$, $x+2y+3z=6$, $3x+y+2z=8$ using (8)
LU decomposition method

Sol^n: Given
$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2z=8$$

In matrix form we have $AX=B$ where

$$A=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}, \quad B=\begin{bmatrix}9\\6\\8\end{bmatrix}$$

Let $LU=A$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

Now from

(i) first row for U : $\boxed{u_{11}=2}$ $\boxed{u_{12}=3}$ $\boxed{u_{13}=1}$

(ii) first column for L : $l_{21}u_{11}=1 \Rightarrow l_{21}=\frac{1}{u_{11}}=\frac{1}{2}$

$l_{31}u_{11}=3 \Rightarrow l_{31}=\frac{3}{u_{11}}=\frac{3}{2}$

$\therefore \boxed{l_{21}=\frac{1}{2}}, \boxed{l_{31}=\frac{3}{2}}$

(iii) second row for U : $l_{21}u_{12}+u_{22}=2$

$\frac{1}{2}(3)+u_{22}=2 \Rightarrow \boxed{u_{22}=\frac{1}{2}}$

& $l_{21}u_{13}+u_{23}=3$

$\frac{1}{2}(1)+u_{23}=3 \Rightarrow \boxed{u_{23}=\frac{5}{2}}$

(iv) second column for L : $l_{31}u_{12}+l_{32}u_{22}=1$

$\frac{3}{2}(3)+l_{32}\left(\frac{1}{2}\right)=1 \Rightarrow \boxed{l_{32}=-7}$

(v) third row for U:

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$
$$\frac{3}{2}(1) + (-7)\left(\frac{5}{2}\right) + u_{33} = 2 \Rightarrow \boxed{u_{33} = 18}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Now substituting $A = LU$ in $AX = B$ we get

$$LUX = B$$

and substituting $UX = V$ we get

$$LV = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\Rightarrow \boxed{V_1 = 9}$ $\quad \frac{1}{2}V_1 + V_2 = 6 \Rightarrow \boxed{V_2 = \frac{3}{2}}$, $\quad \frac{3}{2}V_1 - 7V_2 + V_3 = 8$

$\Rightarrow \boxed{V_3 = 5}$

$\therefore$ $UX = V$ becomes

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad 2x + 3y + z = 9$$
$$\frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$$
$$18z = 5$$
$$\Rightarrow \quad z = \frac{5}{18}$$

By back substitution we get $y = \frac{29}{18}$, $x = \frac{35}{18}$

$\therefore$ $x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$ is the solution of given system of equations

3) solve $10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$ (10)
using LU decomposition method

Soln: Given $10x+y+z=12$
$2x+10y+z=13$
$2x+2y+10z=14$

In matrix form we have $AX=B$ where

$$A=\begin{bmatrix}10&1&1\\2&10&1\\2&2&10\end{bmatrix}\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad B=\begin{bmatrix}12\\13\\14\end{bmatrix}$$

Let $LU=A$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}10&1&1\\2&10&1\\2&2&10\end{bmatrix}$$

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}10&1&1\\2&10&1\\2&2&10\end{bmatrix}$$

Now from

(i) first row for U : $\boxed{u_{11}=10}$, $\boxed{u_{12}=1}$, $\boxed{u_{13}=1}$

(ii) first column for L : $l_{21}u_{11}=2 \Rightarrow l_{21}(10)=2 \Rightarrow \boxed{l_{21}=\frac{1}{5}}$

$l_{31}u_{11}=2 \Rightarrow l_{31}(10)=2 \Rightarrow \boxed{l_{31}=\frac{1}{5}}$

(iii) second row for U : $l_{21}u_{12}+u_{22}=10$
$\frac{1}{5}(1)+u_{22}=10 \Rightarrow \boxed{u_{22}=\frac{49}{5}}$

& $l_{21}u_{13}+u_{23}=1$
$\frac{1}{5}(1)+u_{23}=1 \Rightarrow \boxed{u_{23}=\frac{4}{5}}$

(iv) second column for L : $l_{31}u_{12}+l_{32}u_{22}=2$
$\frac{1}{5}(1)+l_{32}\cdot\left(\frac{49}{5}\right)=2 \Rightarrow \boxed{l_{32}=\frac{9}{49}}$

(v) third row for U : $l_{31}u_{13}+l_{32}u_{23}+u_{33}=10$
$\frac{1}{5}(1)+\frac{9}{49}\cdot\frac{4}{5}+u_{33}=10 \Rightarrow \boxed{u_{33}=\frac{473}{49}}$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{1}{5} & \frac{9}{49} & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 10 & 1 & 1 \\ 0 & \frac{49}{5} & \frac{4}{5} \\ 0 & 0 & \frac{473}{49} \end{bmatrix}$$

Now substituting $A = LU$ in $AX = B$ we get

$$LUX = B$$

and substituting $UX = V$ we get

$$LV = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{1}{5} & \frac{9}{49} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix}$$

$\Rightarrow \boxed{V_1 = 12}$, $\frac{1}{5}V_1 + V_2 = 13 \Rightarrow \boxed{V_2 = \frac{53}{5}}$, $\frac{1}{5}V_1 + \frac{9}{49}V_2 + V_3 = 14$

$\Rightarrow \boxed{V_3 = \frac{473}{49}}$

$\therefore$ $UX = V$ becomes

$$\begin{bmatrix} 10 & 1 & 1 \\ 0 & \frac{49}{5} & \frac{4}{5} \\ 0 & 0 & \frac{473}{49} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 53/5 \\ \frac{473}{49} \end{bmatrix}$$

$$\Rightarrow 10x + y + z = 12$$

$$\frac{49}{5}y + \frac{4}{5}z = \frac{53}{5}$$

$$\frac{473}{49}z = \frac{473}{49} \Rightarrow \boxed{z = 1}$$

By back substitution we get $\boxed{y = 1}$, $\boxed{x = 1}$

$\therefore$ $x = 1$, $y = 1$, $z = 1$ is the solution of given system of equations.

Tutorial Problems

Solve the following system of equations by LU decomposition method.

(1) $x + 2y + 3z = 14$,
$2x + 3y + 4z = 20$
$3x + 4y + z = 14$.

Ans : $x = 1,\ y = 2,\ z = 3$

(2) $3x + 2y - 3z = 6$
$2x + 2y + 5z = -3$
$x + y - z = 2$.

Ans : $x = 1,\ y = 0,\ z = -1$

Iterative Method of solution

The repeated execution of the same process where at each step the result of the preceding step is used. This is known as iteration process and this process is repeated till the result is obtained to a desired degree of accuracy.

We have the following iterative methods

1) Jacobi's iteration method
2) Gauss - seidel iteration method.

These two methods are applicable to system of equations in which the numerically large coefficients are along the principal diagonal of the coefficient matrix, associated with the system of equations.

Gauss-seidel Iterative Method

The modification of Jacobi's method is the Gauss-seidel iterative method. In this method the convergence is twice as fast as in Jacobi's method.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The system of equations can be written as

$$x = \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right]$$
$$y = \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right]$$
$$z = \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right]$$

Start with the initial approximation x_0, y_0, z_0

1^{st} iteration : $x_1 = \frac{1}{a_1}\left[d_1 - b_1 y_0 - c_1 z_0\right]$

$$y_1 = \frac{1}{b_2}\left[d_2 - a_2 x_1 - c_2 z_0\right]$$
$$z_1 = \frac{1}{c_3}\left[d_3 - a_3 x_1 - b_3 y_1\right]$$

i.e as soon as a new approximation ~~is~~ for an unknown is found, it is immediately used in next step

2^{nd} iteration : $x_2 = \frac{1}{a_1}\left[d_1 - b_1 y_1 - c_1 z_1\right]$

$$y_2 = \frac{1}{b_2}\left[d_2 - a_2 x_2 - c_2 z_1\right]$$
$$z_2 = \frac{1}{c_3}\left[d_3 - a_3 x_2 - b_3 y_2\right]$$

Repeat the process till two consecutive values of x, y, z are same.

Solved Problems

1) Apply Gauss-Seidel iteration method to solve the equations $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$

Sol: Given equations are diagonally dominant. They can be written in the form

$$x = \frac{1}{20}[17-y+2z]$$

$$y = \frac{1}{20}[-18-3x+z]$$

$$z = \frac{1}{20}[25-2x+3y]$$

Start with the initial approximation $x_0=y_0=z_0=0$

1st iteration:

$$x_1 = \frac{1}{20}[17-y_0+2z_0] = \frac{1}{20}[17-0+2(0)] = \frac{17}{20} = 0.8500$$

$$y_1 = \frac{1}{20}[-18-3x_1+z_0] = \frac{1}{20}[-18-3(0.8500)+0] = -1.0275$$

$$z_1 = \frac{1}{20}[25-2x_1+3y_1] = \frac{1}{20}[25-2(0.8500)+3(-1.0275)] = 1.0109$$

2nd iteration:

$$x_2 = \frac{1}{20}[17-y_1+2z_1] = \frac{1}{20}[17-(-1.0275)+2(1.0109)] = 1.0025$$

$$y_2 = \frac{1}{20}[-18-3x_2+z_1] = \frac{1}{20}[-18-3(1.0025)+1.0109] = -0.9998$$

$$z_2 = \frac{1}{20}[25-2x_2+3y_2] = \frac{1}{20}[25-2(1.0025)+3(-0.9998)] = 0.9998$$

3rd iteration:

$$x_3 = \frac{1}{20}[17-y_2+2z_2] = \frac{1}{20}[17-(-0.9998)+2(0.9998)] = 1.0000$$

$$y_3 = \frac{1}{20}[-18-3x_3+z_2] = \frac{1}{20}[-18-3(1.0000)+0.9998] = -1.0000$$

$$z_3 = \frac{1}{20}[25-2x_3+3y_3] = \frac{1}{20}[25-2(1.0000)+3(-1.0000)] = 1.0000$$

4th iteration :

$$x_4 = \frac{1}{20}[17 - y_3 + 2z_3] = \frac{1}{20}[17-(-1.0000)+2(1.0000)] = 1.0000$$

$$y_4 = \frac{1}{20}[-18-3x_4+z_3] = \frac{1}{20}[-18-3(1.0000)+1.0000] = -1.0000$$

$$z_4 = \frac{1}{20}[25-2x_4+3y_4] = \frac{1}{20}[25-2(1.0000)+3(-1.0000)] = 1.0000$$

3rd and 4th iteration values are same

$\therefore$ $x=1$, $y=-1$, $z=1$ is the required solution

2) Solve $27x+6y-z=85$, $x+y+54z=110$, $6x+15y+2z=72$ using Gauss seidel iteration method correct to three decimal places

Sol^n: Given equations are not diagonally dominant

$\therefore$ Rearranging we get

$$27x+6y-z=85$$

$$6x+15y+2z=72$$

$$x+y+54z=110$$

Now writing these equations in the form

$$x = \frac{1}{27}[85-6y+z]$$

$$y = \frac{1}{15}[72-6x-2z]$$

$$z = \frac{1}{54}[110-x-y]$$

start with initial approximation $x_0=y_0=z_0=0$

1st iteration:

$$x_1 = \frac{1}{27}[85-6y_0+z_0] = \frac{1}{27}[85-6(0)+0] = 3.148$$

$$y_1 = \frac{1}{15}[72-6x_1-2z_0] = \frac{1}{15}[72-6(3.148)-2(0)]$$

$$= 3.541$$

$$z_1 = \frac{1}{54}[110 - x_1 - y_1] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$$

2nd iteration:

$$x_2 = \frac{1}{27}[85 - 6y_1 + z_1] = \frac{1}{27}[85 - 6(3.541) + 1.913] = 2.432$$

$$y_2 = \frac{1}{15}[72 - 6x_2 - 2z_1] = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572$$

$$z_2 = \frac{1}{54}[110 - x_2 - y_2] = \frac{1}{54}[110 - 2.432 - 3.572] = 1.926$$

3rd iteration:

$$x_3 = \frac{1}{27}[85 - 6y_2 + z_2] = \frac{1}{27}[85 - 6(3.572) + 1.926] = 2.426$$

$$y_3 = \frac{1}{15}[72 - 6x_3 - 2z_2] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z_3 = \frac{1}{54}[110 - x_3 - y_3] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$$

4th iteration:

$$x_4 = \frac{1}{27}[85 - 6y_3 + z_3] = \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.425$$

$$y_4 = \frac{1}{15}[72 - 6x_4 - 2z_3] = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573$$

$$z_4 = \frac{1}{54}[110 - x_4 - y_4] = \frac{1}{54}[110 - 2.425 - 3.573] = 1.926$$

3rd and 4th iteration values are same

$\therefore$ $x = 2.425$, $y = 3.573$ & $z = 1.926$ is the solution

3) solve the following equations using Gauss-Seidel (18) iteration method correct to three decimal places. carry out 5 iterations,

~~Sum~~: $2x+y+6z=9$, $8x+3y+2z=13$, $x+5y+z=7$

Soln: Given equations are not diagonally dominant

$\therefore$ rearranging the equations

$$8x+3y+2z=13$$
$$x+5y+z=7$$
$$2x+y+6z=9$$

Now writing the equations in the form

$$x=\frac{1}{8}[13-3y-2z]$$
$$y=\frac{1}{5}[7-x-z]$$
$$z=\frac{1}{6}[9-2x-y]$$

start with initial approximation $x_0=y_0=z_0=0$

1st iteration:

$$x_1=\frac{1}{8}[13-3y_0-2z_0]=\frac{1}{8}[13-3(0)-2(0)]=\frac{13}{8}=1.625$$
$$y_1=\frac{1}{5}[7-x_1-z_0]=\frac{1}{5}[7-1.625-0]=1.075$$
$$z_1=\frac{1}{6}[9-2x_1-y_1]=\frac{1}{6}[9-2(1.625)-1.075]=0.779$$

2nd iteration

$$x_2=\frac{1}{8}[13-3y_1-2z_1]=\frac{1}{8}[13-3(1.075)-2(0.779)]=1.027$$
$$y_2=\frac{1}{5}[7-x_2-z_1]=\frac{1}{5}[7-1.027-0.779]=1.039$$
$$z_2=\frac{1}{6}[9-2x_2-y_2]=\frac{1}{6}[9-2(1.027)-1.039]=0.985$$

3^{rd} iteration

$$x_3 = \frac{1}{8}\left[13 - 3y_2 - 2z_2\right] = \frac{1}{8}\left[13 - 3(1.039) - 2(0.985)\right] = 0.989$$

$$y_3 = \frac{1}{5}\left[7 - x_3 - z_2\right] = \frac{1}{5}\left[7 - 0.989 - 0.985\right] = 1.005$$

$$z_3 = \frac{1}{6}\left[9 - 2x_3 - y_3\right] = \frac{1}{6}\left[9 - 2(0.989) - 1.005\right] = 1.003$$

4^{th} iteration

$$x_4 = \frac{1}{8}\left[13 - 3y_3 - 2z_3\right] = \frac{1}{8}\left[13 - 3(1.005) - 2(1.003)\right] = 0.997$$

$$y_4 = \frac{1}{5}\left[7 - x_4 - z_3\right] = \frac{1}{5}\left[7 - 1.003 - 1.003\right] = 1.000$$

$$z_4 = \frac{1}{6}\left[9 - 2x_4 - y_4\right] = \frac{1}{6}\left[9 - 2(1.003) - 0.999\right] = 1.001$$

5^{th} iteration

$$x_5 = \frac{1}{8}\left[13 - 3y_4 - 2z_4\right] = 1.000$$

$$y_5 = \frac{1}{5}\left[7 - x_5 - z_4\right] = 1.000$$

$$z_5 = \frac{1}{6}\left[9 - 2x_5 - y_5\right] = 1.000$$

6^{th} iteration

$x_6 = 1.000$

$y_6 = 1.000$

$z_6 = 1.000$

From the 6^{th} iteration, the solution of given equations is $x = 1,\ y = 1,\ z = 1$

Tutorial Problems

Solve the following system of equations using Gauss Seidel iterative method to three decimal places.

1) $10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14.$

Ans: $x = 1,\ y = 1,\ z = 1$

2) $5x + 2y + z = 12,\ x + 4y + 2z = 15,\ x + 2y + 5z = 20,$ with initial approximation $(1, 0, 3)$.

Ans: $x = 1,\ y = 2,\ z = 3$

Solution of Algebraic and Transcendental Equations:

A problem of great importance in applied mathematics and engineering is to find the roots of an equation of the form $f(x)=0$. The function $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-1}x+a_n$ — ① where n is a positive integer and $a_0, a_1, a_2 \dots a_n$ are constants with $a_0 \neq 0$ is known as a polynomial of degree n. The values of x making $f(x)$ zero are known as zeroes or roots of the polynomial $f(x)$ and every polynomial of degree 'n' has 'n' roots.

The equation of the form $f(x)=0$ is called Algebraic or Transcendental according as $f(x)$ is purely a polynomial in x or contains some other functions such as logarithmic, exponential and trignometric functions etc.

EX: 1. $x^4+2x^3-3x^2+x+5=0$ is an algebraic equation

2. $2x^2+\log(x-1)+e^x+\sin x=0$ is a transcendental equation.

NOTE: A transcendental equation may have a finite or an infinite number of real roots and many have no real roots at all.

Numerical methods of finding approximate roots of the given equation is a repetative type of process known as iteration process. In each step the result of the previous step is used and the process is carried out till we get the result of the desired degree of accuracy. All the numerical methods are only approximate techniques for the solution of any problem and computers play an important role in various numerical methods for obtaining the result to the highest degree of accuracy.

We study the following methods of finding solution of Algebraic and transcendental Equations:

1. Bisection method
2. Method of false position or Regula-falsi method
3. New-ton Raphson method

Newton - Raphson Method:

In this method we locate an approximate root of the given equation and improve its accuracy by an iterative process.

Assuming that x_0 is an approximate value of a real root of the equation $f(x)=0$, let x_1 be the exact root and $x_1 = x_0 + h$ where h is a small correction.

We have $f(x_1)=0$ i.e., $f(x_0+h)=0$, using Tayler's expansion, we write $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$

Since h is a small quantity, $h^2, h^3 \ldots$ being still smaller can be neglected. Thus we have

$$f(x_0) + h f'(x_0) = 0 \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

Using this in $x_1 = x_0 + h$, we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$ (provided $f'(x_0) \neq 0$) which is the first approximation.

The second approximation is obtained by replacing x_0 by x_1 in the RHS of the above expression.

i.e., $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ and so on.

In general we can write,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is called Newton-Raphson iterative formula.

Problems

1. Use Newton-Raphson method to find a real root of the equation $x^3-2x-5=0$ correct to 3 decimal places.

Solution: Let $f(x)=x^3-2x-5$

$f(1)=-6$ $\quad f(2)=-1$ (-ve) $\quad f(3)=16$ (+ve).

Here $f(2)$ is negative and $f(3)$ is positive. Hence the root lies between 2 and 3. The root will be in the neighbourhood of 2 and let the approximate root $x_0=2$. [Since -1 is close to zero compared to 16]

The first approximation is given by $x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}$

$$\text{i.e., } x_1 = 2-\frac{f(2)}{f'(2)}$$

Here $f(x)=x^3-2x-5 \Rightarrow f(2)=-1$

$f'(x)=3x^2-2 \Rightarrow f'(2)=10$

$$\therefore x_1 = 2-\frac{(-1)}{10} = 2.1$$

The second approximation is given by $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$

$$\text{i.e., } x_2 = 2.1-\frac{f(2.1)}{f'(2.1)}$$

Here $f(2.1)=(2.1)^3-2(2.1)-5 = 0.061$

$f'(2.1)=3(2.1)^2-2 = 11.23$

$$\therefore x_2 = 2.1-\frac{(0.061)}{11.23} = 2.0946$$

The third approximation is given by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

i.e., $x_3 = 2.0946 - \frac{f(2.0946)}{f'(2.0946)}$

Here $f(2.0946) = (2.0946)^3 - 2(2.0946) - 5 = 0.00054155$

$f'(2.0946) = 3(2.0946)^2 - 2 = 11.1620$.

$\therefore x_3 = 2.0946 - \frac{0.00054155}{11.1620} = 2.0946$.

Since x_2 and x_3 are same, stop the process.
Hence the required approximate root correct to 3 decimal places is $\underline{2.095}$

2. Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.

Sol^n^ Let $f(x) = x^4 - x - 10$

$f(0) = -10$, $f(1) = -10$ (-ve) $f(2) = 4$ (+ve).

Here $f(1)$ is negative and $f(2)$ is positive. Hence the root lies between 1 and 2. The root will be in the neighbourhood of 2 and let the approximate root $x_0 = 2$. [Since 4 is close to zero compared to -10]

The first approximate is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

i.e., $x_1 = 2 - \frac{f(2)}{f'(2)}$

Here $f(x) = x^4 - x - 10 \Rightarrow f(2) = 4$

$f'(x) = 4x^3 - 1 \Rightarrow f'(2) = 31$

$\therefore\ x_1 = 2 - \frac{4}{31} = 1.871$

The second approximation is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

i.e., $x_2 = 1.871 - \frac{f(1.871)}{f'(1.871)}$

$f(1.871) = (1.871)^4 - (1.871) - 10 = 0.3835$

$f'(1.871) = 4(1.871)^3 - 1 = 25.199$

$\therefore\ x_2 = 1.871 - \frac{0.3835}{25.199} = 1.856$

The third approximation is given by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

i.e. $x_3 = 1.856 - \frac{f(1.856)}{f'(1.856)}$

$f(1.856) = (1.856)^4 - (1.856) - 10 = 0.010$

$f'(1.856) = 4(1.856)^3 - 1 = 24.574$

$\therefore\ x_3 = 1.856 - \frac{0.010}{24.574} = 1.856$

Since x_2 and x_3 are same, stop the process.
Hence the required root correct to 3 decimal places is
$= 1.856$.

3. Find the real root of the Equation, $3x = \cos x + 1$ using Newton-Raphson method correct to 4 decimal places.

Solution: Let $f(x) = 3x - \cos x - 1$ [Take x in radians]

$f(0) = -2$ (-ve) $f(1) = 1.46$ (+ve)

Here $f(0)$ is negative and $f(1)$ is positive. Hence the root lies between 0 and 1.

Let us take the first approximation x_0 of the root the average of 0 and 1, namely 0.5 i.e, $x_0 = 0.5$

The first approximation is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_1)}$

i.e., $x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$f(x) = 3x - \cos x - 1 \Rightarrow f(0.5) = 3(0.5) - \cos(0.5) - 1$

$= -0.3775$.

$f'(x) = 3 + \sin x \Rightarrow f'(0.5) = 3 + \sin(0.5) = 3.4794$.

$\therefore\ x_1 = 0.5 - \frac{(-0.3775)}{3.4794} = 0.6085$

The Second approximation is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

i.e $x_2 = 0.6085 - \frac{f(0.6085)}{f'(0.6085)}$.

$f(0.6085) = 3(0.6085) - \cos(0.6085) - 1 = 0.00493$

$f'(0.6085) = 3 + \sin(0.6085) = 3.5716$

$\therefore\ x_2 = 0.6085 - \frac{(0.00493)}{3.5716} = 0.6071$

The third approximation is given by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

i.e., $x_3 = 0.6071 - \frac{f(0.6071)}{f'(0.6071)}$

$f(0.6071) = 3(0.6071) - \cos(0.6071) - 1 = -0.000005884$

$f'(0.6071) = 3 + \sin(0.6071) = 3.5704$

$\therefore\ x_3 = 0.6071 - \frac{(-0.000005884)}{3.5704} = 0.6071$.

Since x_2 and x_3 are same, stop the process.
Hence the required root correct to 3 decimal places is 0.6071.

4. Find an approximate root of the equation $e^x \sin x = 1$ using the Newton Raphson method.

Solution: Let $f(x) = e^x \sin x - 1$ (x is in radians)

$f(0) = -1$ (-ve) $f(1) = 1.28735$ (+ve).

Since $f(0)$ is negative and $f(1)$ is positive. Then the root lies between 0 and 1. Taking first approximation as the average of 0 and 1, i.e., $x_0 = 0.5$.

The first approximate is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

i.e., $x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$f(x) = e^x \sin x - 1 \Rightarrow f(0.5) = e^{(0.5)} \sin(0.5) - 1 = -0.2096$.

$f'(x) = e^x (\sin x + \cos x) \Rightarrow f'(0.5) = e^{(0.5)} (\sin(0.5) + \cos(0.5))$

$= 2.2373$.

$\therefore x_1 = 0.5 - \frac{(-0.2096)}{2.2373} = 0.59368$.

The second approximation is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

i.e., $x_2 = 0.59368 - \frac{f(0.59368)}{f'(0.59368)}$

$f(0.59368) = e^{(0.59368)} \sin(0.59368) - 1 = 0.012898$

$f'(0.59368) = e^{(0.59368)} (\cos(0.59368) + \sin(0.59368)) = 2.51371$

$\therefore\ x_2 = 0.59368 - \dfrac{0.012898}{2.51371} = 0.58854$

The third approximation is given by $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

i.e., $x_3 = 0.58854 - \dfrac{f(0.58854)}{f'(0.58854)}$

$f(0.58854) = e^{(0.58854)} \sin(0.58854) - 1 = 0.000018127$

$f'(0.58854) = e^{(0.58854)} (\sin 0.58854 + \cos 0.58854) = 2.4983.$

$\therefore\ x_3 = 0.58854 - \dfrac{0.000018127}{2.4983} = 0.58853.$

Since x_2 and x_3 are same, stop the process. Hence the required root correct to 3 decimal places is $\underline{0.5885}$.

Tutorial problems:

1. Show that a root of the Equation $x^3+5x-11=0$ be between 1 and 2. Find the real root by Newton-Raphson method, carry 3 iterations only.
   Ans: The root lies b/n 1 and 2 and $x_0 = 1$.
   $x_1 = 1.625$, $x_2 = 1.5155$, $x_3 = 1.5106$.
2. Using Newton-Raphson method find the real root of $x \log_{10} x = 1.2$ correct to 3 decimal places.
   Ans: The root lies b/n 2 and 3 and $x_0 = 2$
   $x_1 = 2.8132$, $x_2 = 2.7412$, $x_3 = 2.74064$, $x_4 = 2.7406$.