Computer Graphics (CSE2066)

Module 4

Plane curves and surfaces

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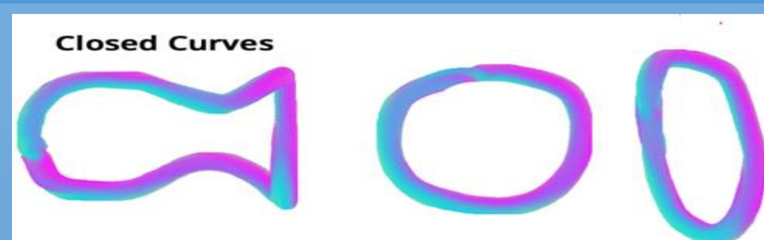
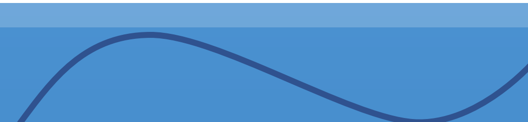
Curve

- A curve is a **smoothly flowing continuous line** that has bent.
- It does not have **any sharp turns.**
- The technique to identify the curve is that the **line bends and changes** its direction at least once for all. Various curve shapes other than the ones mentioned in the above image are **circles, ellipses, parabolas, and hyperbolas, even arcs, sectors, and segments**, they are all two-dimensional curved shapes.
- However, curves are three-dimensional shapes as well, such as **spheres, cylinders, and cones; we all have these three-dimensional curved shapes.**

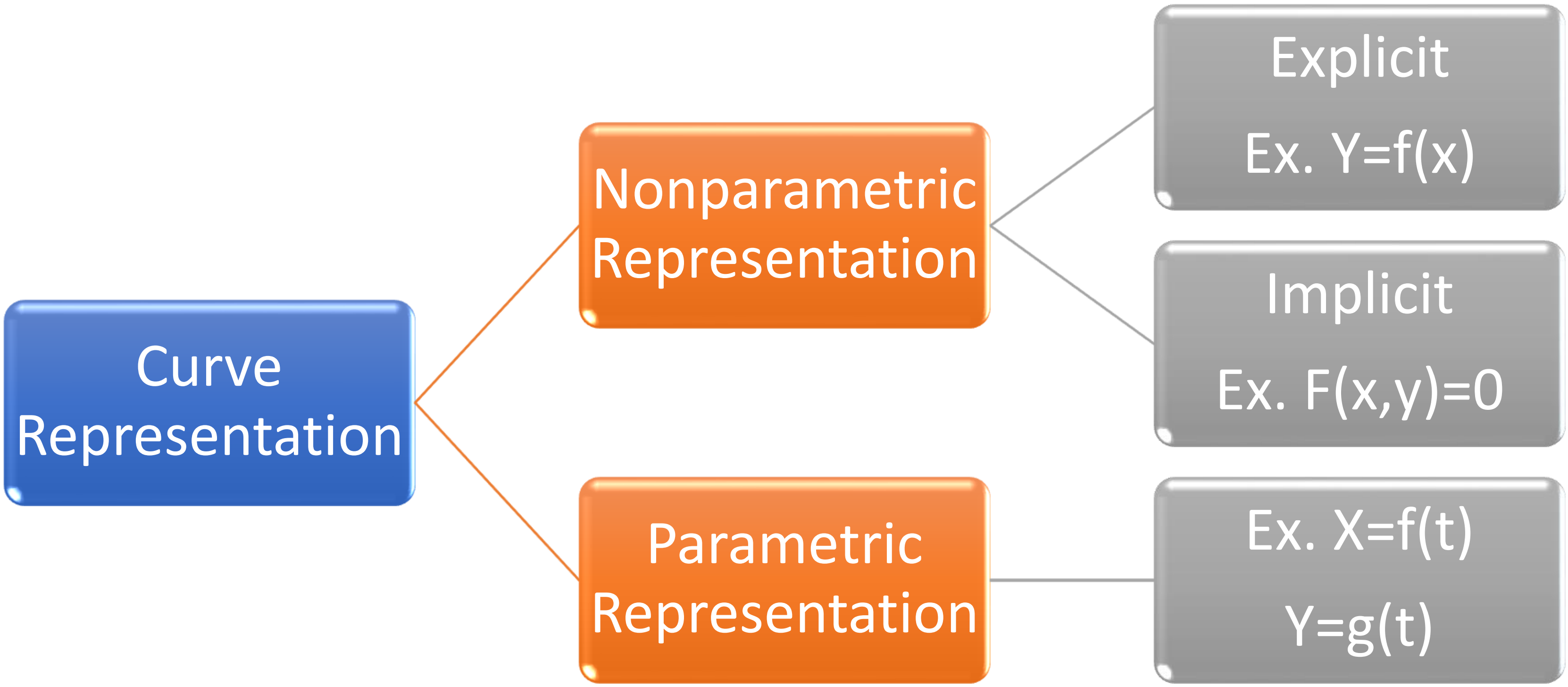
Different Types of Curves

Simple Curve	
Non-Simple Curve	
Open Curve	
Closed Curve	
Curved Line	

In Maths:

- Apart from the real-life examples, we can also observe the curve-shaped lines in Maths;
- for example, the graph of a quadratic polynomial including **parabola, ogive curve, arrows,** etc.
- So, this is how we understand curve Maths and the types of curves we find in our surroundings.

Curve representation



Non Parametric Representation

- The generic form in which any generic point (x, y, z) satisfies a **relationship** in implicit form in x, y, & z i.e. $f(x, y, z) = 0$.
- A single such constraints generally describe a surface while two constraints considered together can be thought of as a curve which is the intersection of two surface.
- This may expressed in an explicit form in the following manner:

$$x = g^1 (y, z); \qquad y = g^2 (x, z); \qquad z = g^3 (x, y)$$

- **Ex-** $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ (General equation of Pair of straight line)

Form of Non Parametric Representation

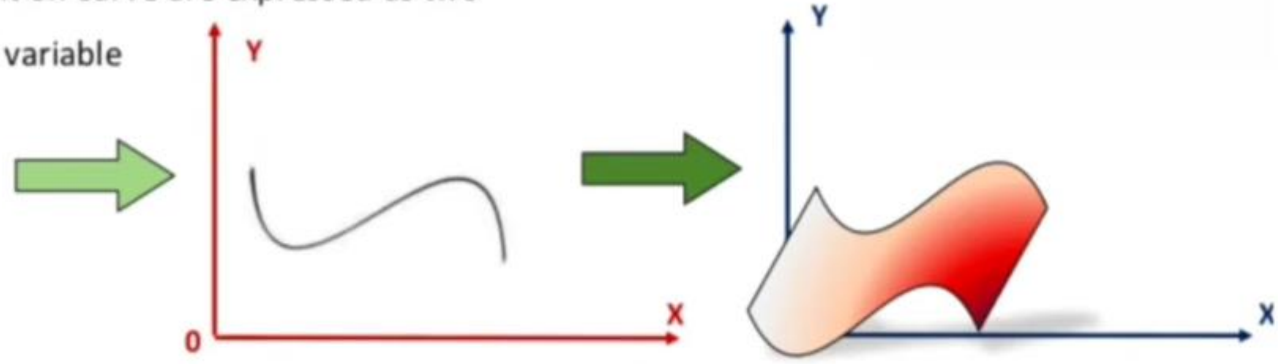
1) Explicit form (Clearly Expressed):-

- In this coordinates of y & z of a point on curve are expressed as two separate function of x independent variable

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ f(x) \\ g(x) \end{bmatrix}$$

$$P = [x \;\; y \;\; z]^T = [x \;\; f(x) \;\; g(x)]^T$$

- **Ex-** $y = mx + c$



1) Implicit form (Not Clearly Expressed):-

- In this, coordinates of x, y & z of a point on curve are related together by two function.

$$f(x, y) = 0$$

$$f(x, y, z) = 0$$

$$g(x, y, z) = 0$$

- **Ex-** $ax + by + c = 0$

$$\mathbf{P} = [x \quad y \quad z]^{\mathrm{T}} = [x \quad f(x) \quad g(x)]^{\mathrm{T}}$$

nonparametric explicit form

$$F(x, y, z) = 0$$
$$G(x, y, z) = 0$$

nonparametric implicit form

- Explicit functions

$$y = f(x),\ z = g(x)$$

- Implicit equations

$$f(x,y) = 0$$

- Parametric – Cubic Curve

$$x = x(t),\ y = y(t),\ z = z(t)$$

Disadvantages of Explicit form

- impossible to get multiple values for a single x
  - break curves like circles and ellipses into segments
- problem with curves with vertical tangents
  - infinite slope is difficult to represent

Disadvantages of Implicit form

- problem to join curve segments together
  - difficult to determine if their tangent directions agree at their joint point

Parametric Representation of curve

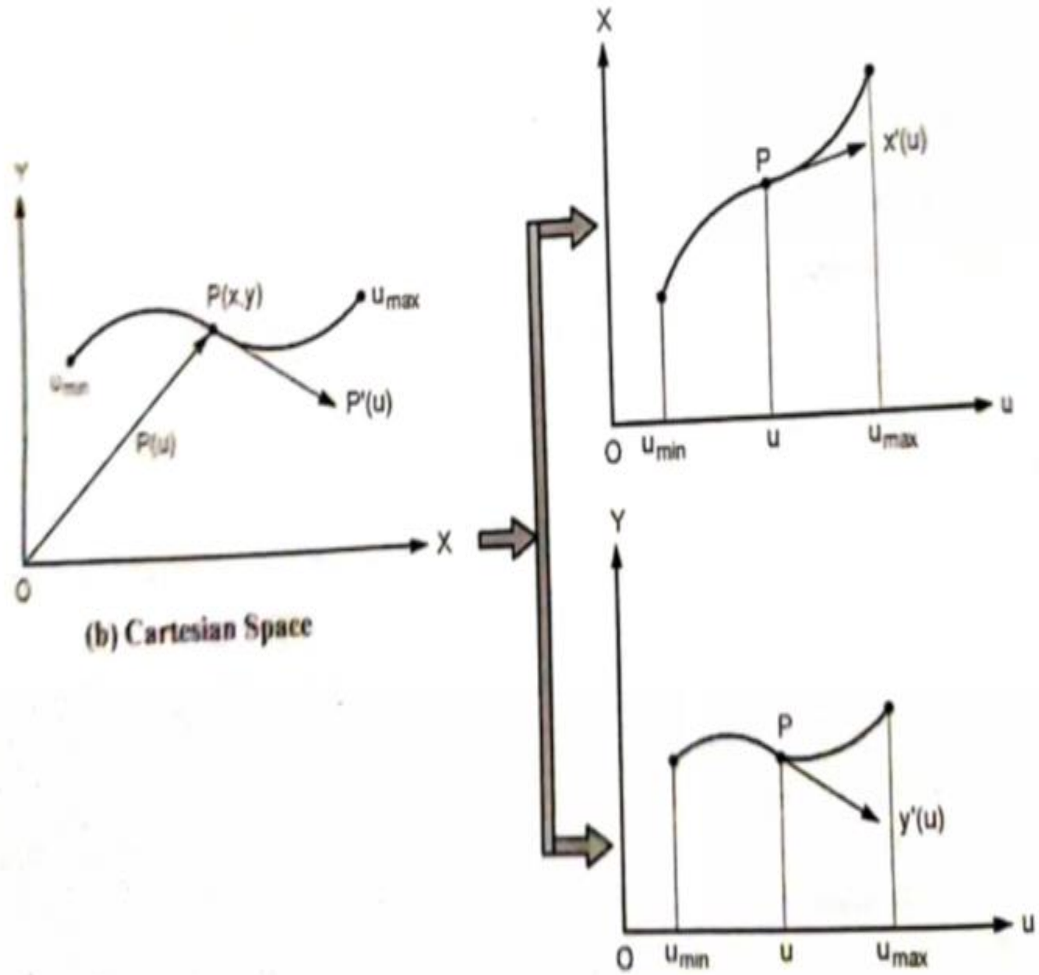
- The curve is not represented the relationship bet^n x, y & z.
- It's the coordinates of x, y & z are expressed as functions of this independent parameters '**u or θ**'
- This parameters acts as a local coordinates for a points on curve.

$P(u) = [x \;\; y] \qquad u_{min} \leq u \leq u_{max}$

$= [x(u) \;\; y(u)]$

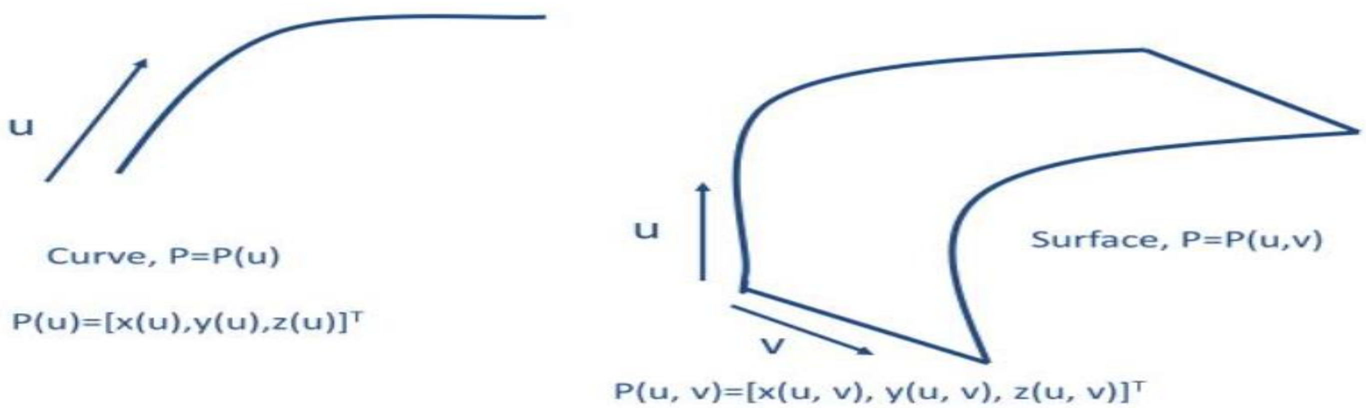
or $x = r \cos \theta$

$y = r \sin \theta$



(b) Cartesian Space

Parametric Representation



In parametric form,

- Each point on a **curve is expressed** as a function of a **parameter u.**
- The parametric equation for a three-dimensional curve in space takes the following vector form:

$$\mathbf{P}(u) = [x \quad y \quad z]^{\mathrm{T}} = [x(u) \quad y(u) \quad z(u)]^{\mathrm{T}}, \quad u_{\min} \le u \le u_{\max}$$

The tangent vectors at point P is given by

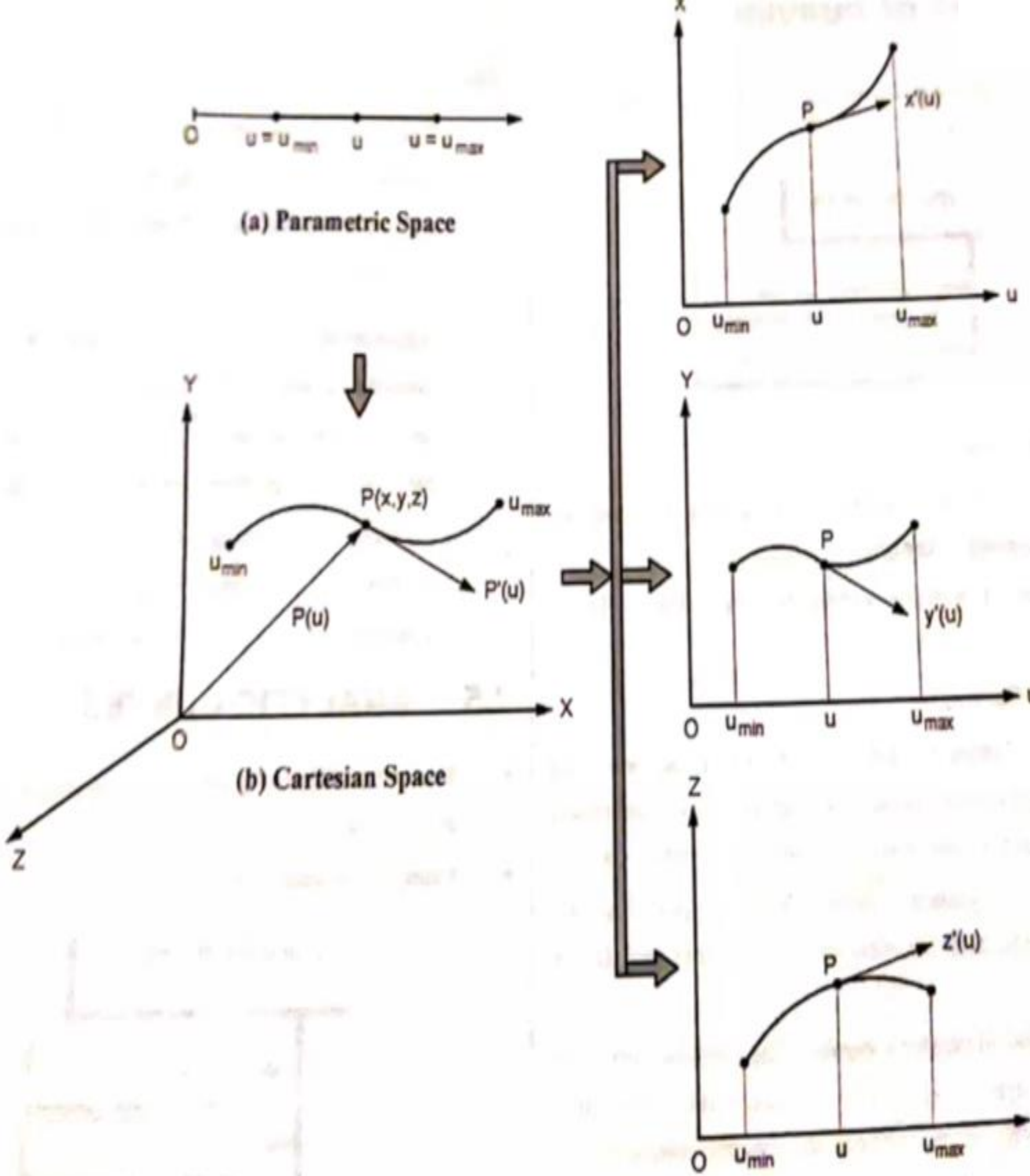
$$P'(u) = \frac{dP}{du}(u) \quad \ldots u_{min} \le u \le u_{max}$$

$$P'(u) = [x'(u) \;\; y'(u)]$$

The parameters acts as a local coordinates for a points on curve

$$P(u) = [x \;\; y \;\; z] \qquad u_{min} \le u \le u_{max}$$

$$= [x(u) \;\; y(u) \;\; z(u)]$$



Advantages

- Parameter space is represented by coordinates of a point on the curve as position vector
- Bounded by parameter values $u_{min} \leq u \leq u_{max}$
- Parametric form becomes useful for CG operations like clipping, trimming, segmentation etc
- Computation easy as can be solved using vectors and matrices Eg circle.

Other advantages include :

- Parametric equation provides more degree of freedom for controlling the shape of the curve and surfaces than non parametric form

Eg cubic curve in both forms

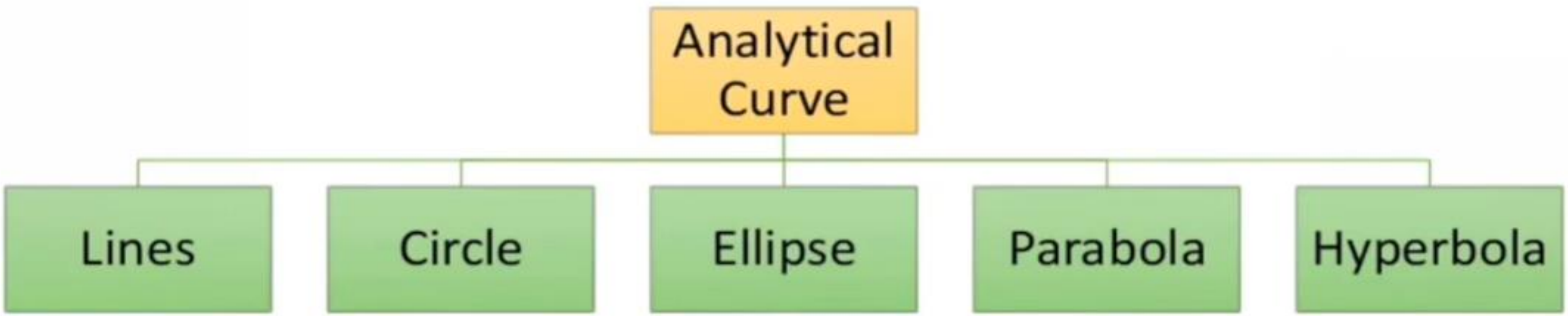
- Parametric form readily handles infinite slopes
- Uses polynomials instead of roots
- Transformations can be directly applied

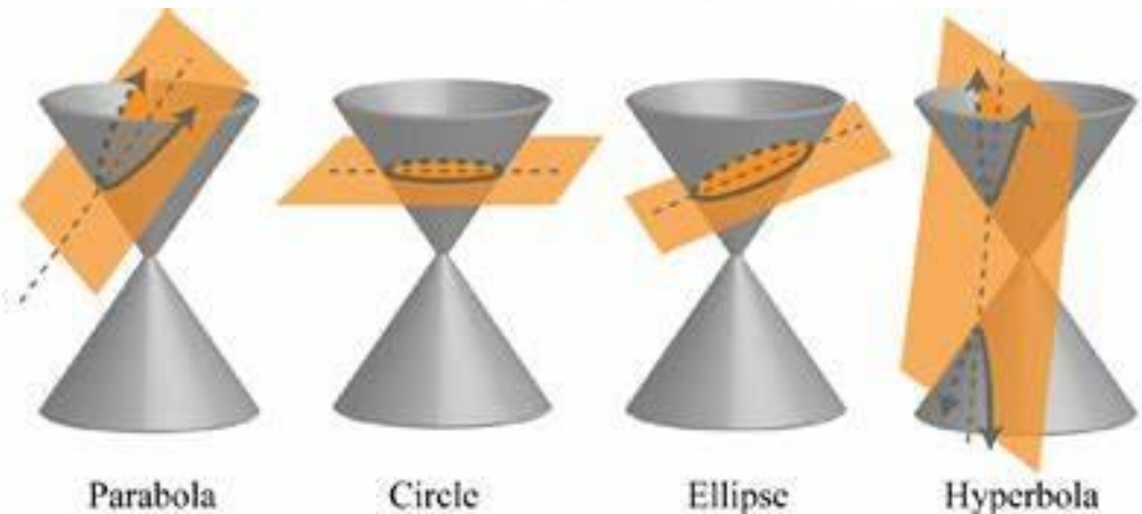
There are two categories of curves that can be represented parametrically.

- Analytic curves
- Synthetic curves

Analytical Curve

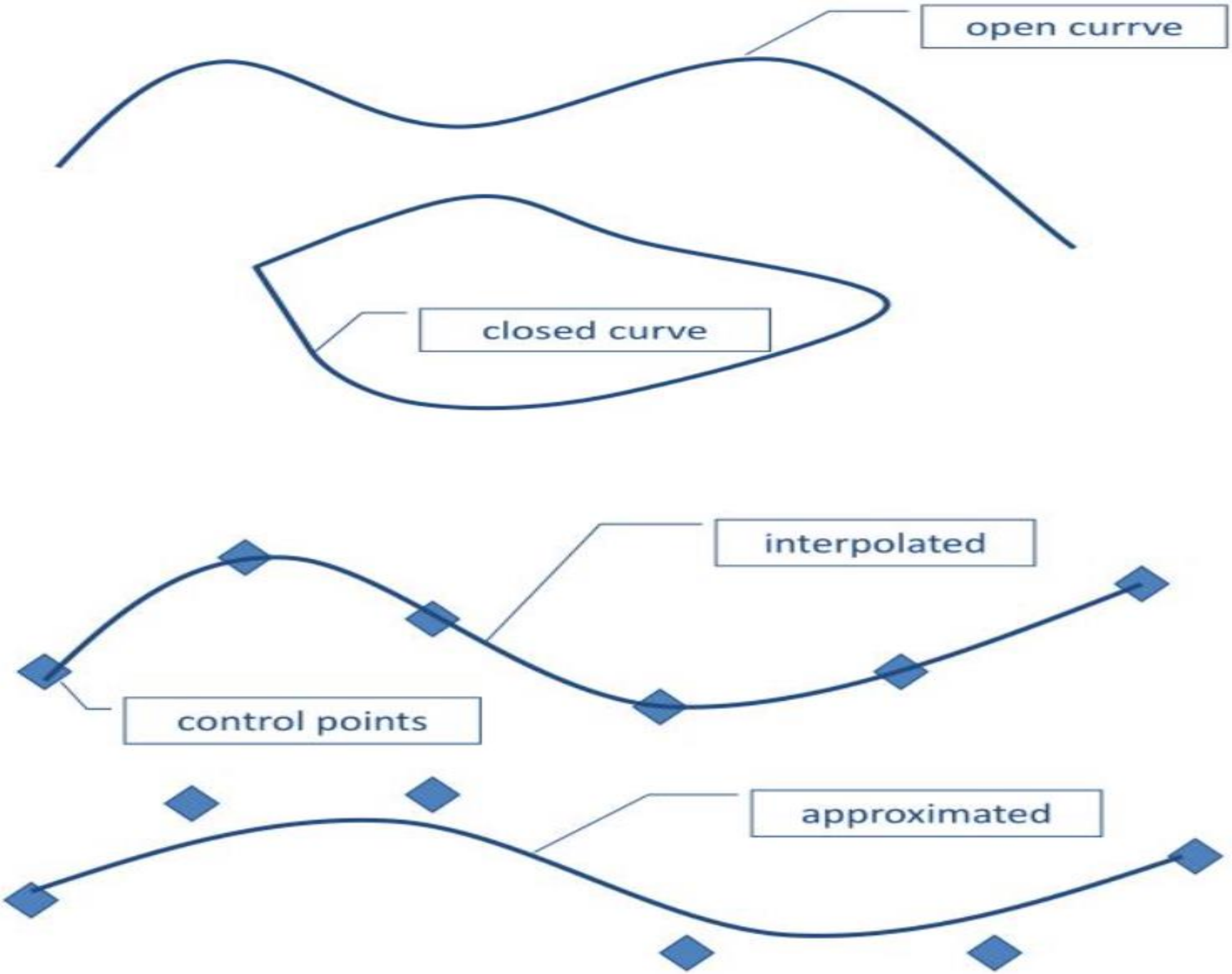
- The curve which are defined by the analytical equations are known as Analytical Curves.
- It shows simple mathematical equations.
- They have fixed form & cannot be modified to achieve a shape that violets mathematical equation.
- Its basically known <u>form curve</u>





Synthetic Curve

- The curve which is defined by the set of data points are known as **synthetic curves.**
- Its represent by the polynomial.
- It is free form curve or Freedom curve
- Its needed when a curve is represented by a collection of data point. **(Control Points).**
- **Ex-** Cubic Spline, B-Spline, Beta Spline, nu spline & Bezier Curve.


open currve
closed curve
interpolated
control points
approximated

Applications

Ship hulls

Airplane wings

Car Bodies

Bottles

Propeller Blade

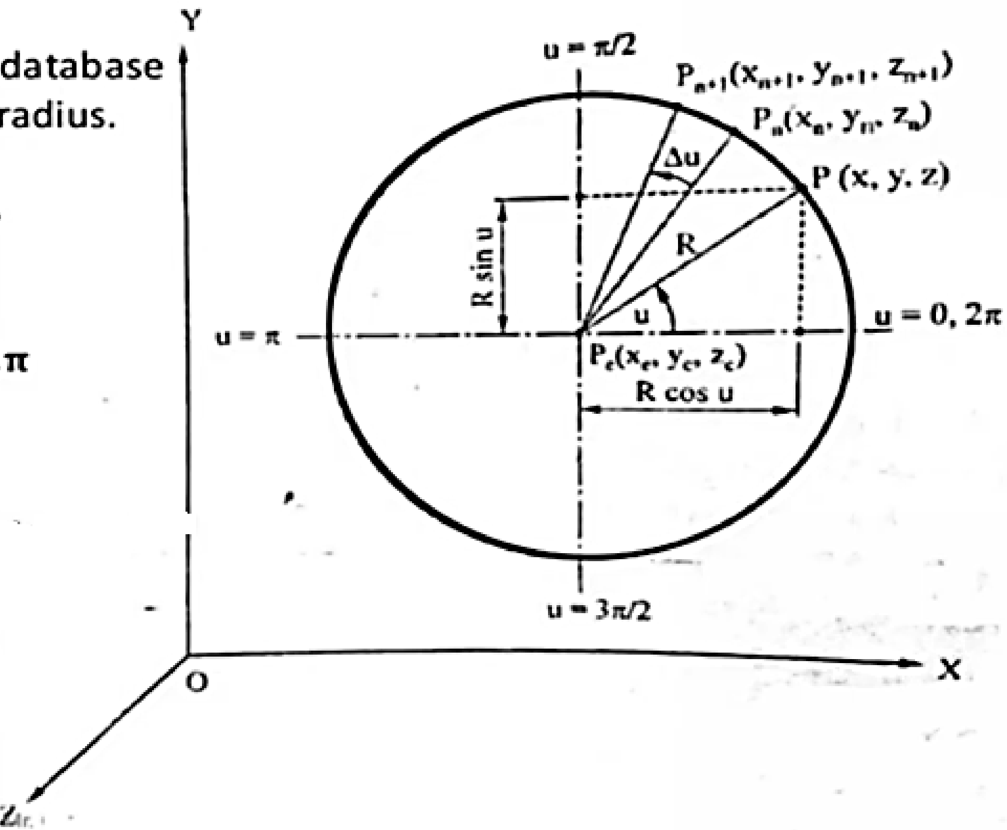
Shoe Insoles

Parametric Representation of Circle

Circle

- Circle is represented in the CAD/CAM database by storing the value of its centre & its radius.
- It can be represented by the equation,

$$X = X_C + R\cos u$$
$$Y = Y_C + R\sin u \qquad 0 \le u \le 2\pi$$
$$Z = Z_C$$

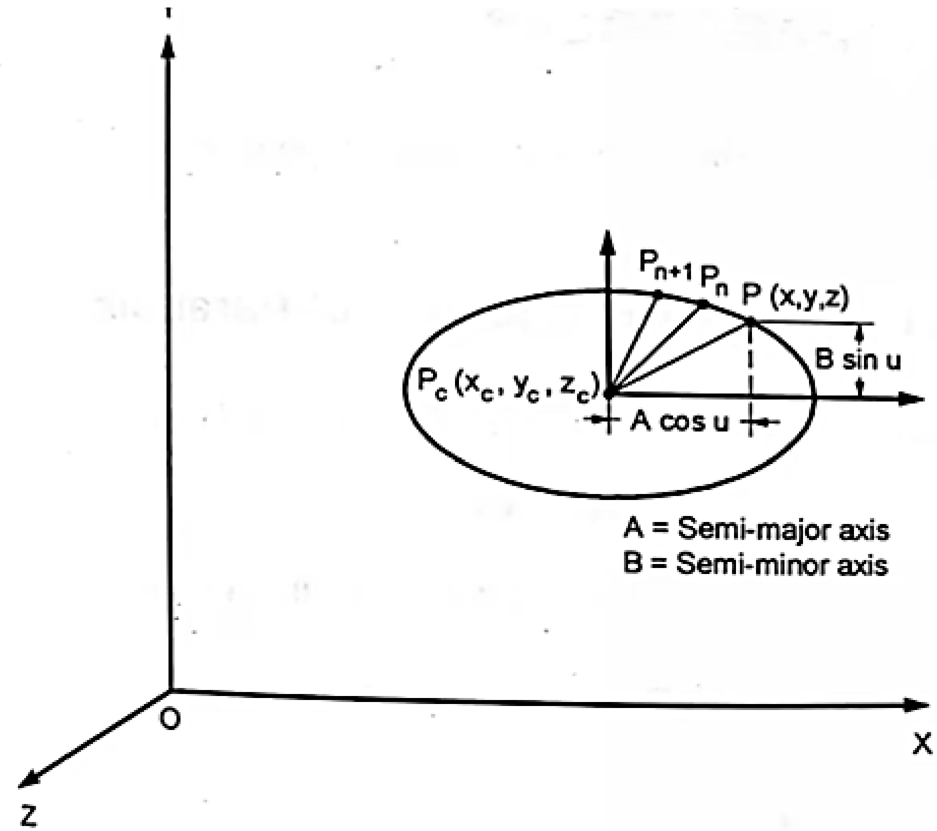


Parametric Representation of Ellipse

Circle is represented in computer graphics b
Storing the value of its center and its radius

- It can be represented by the equation,

$X = X_C + A \cos u$
$Y = Y_C + B \sin u$ $\quad 0 \le u \le 2\pi$
$Z = Z_C$

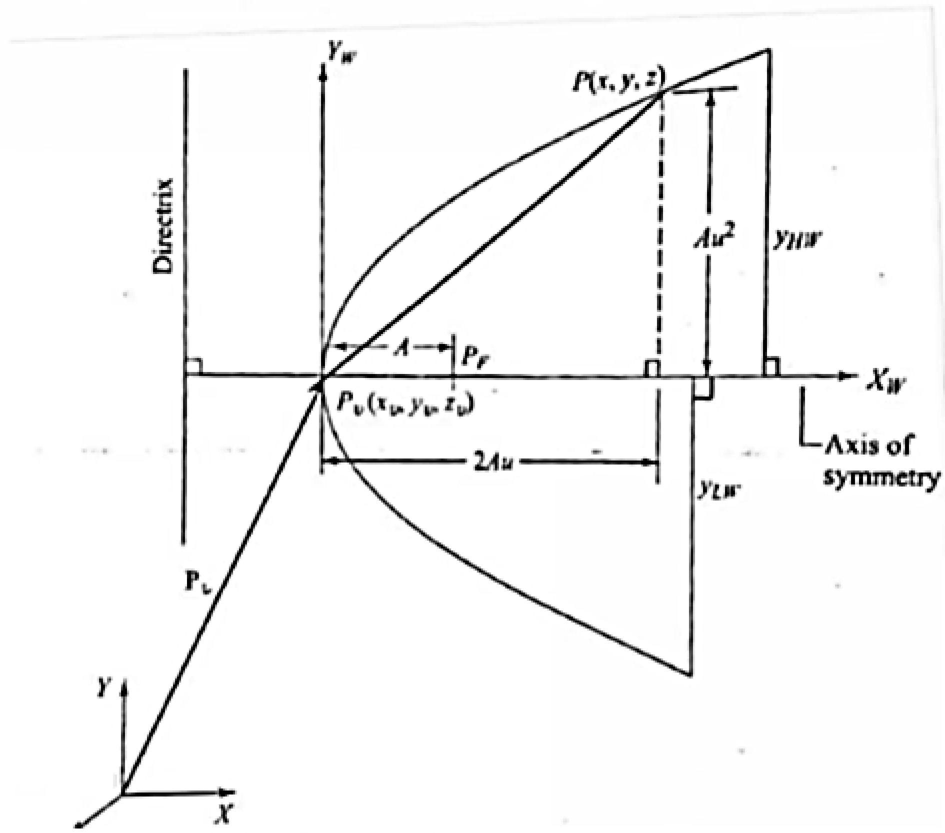


Parametric Representation of Parabola

Its defined mathematically as a curve generated by a point that moves such that its distance from a fixed point (the focus P_F) is always equal to its distance to a fixed line (Directrix)

- The parametric equation

$$\left.\begin{aligned} X &= X_v + A u^2 \\ Y &= Y_v + 2Au^2 \\ Z &= Z_v \end{aligned}\right\} \quad 0 \leq u \leq \infty$$

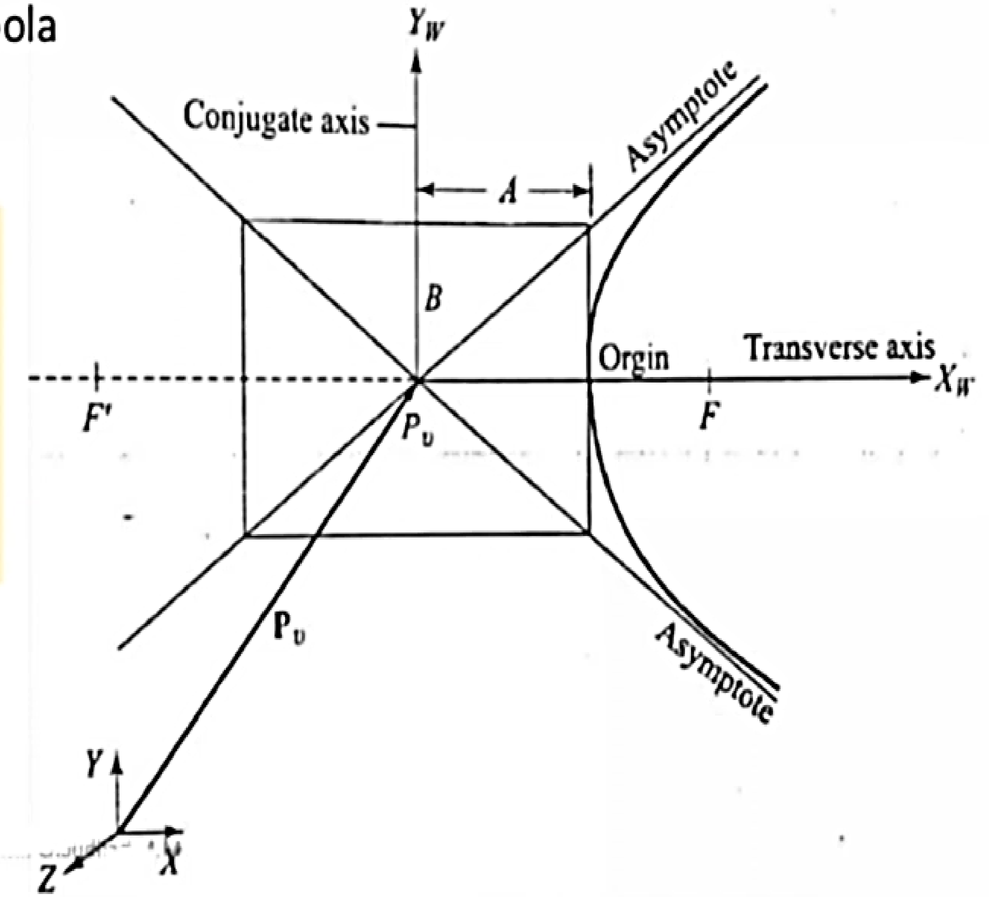


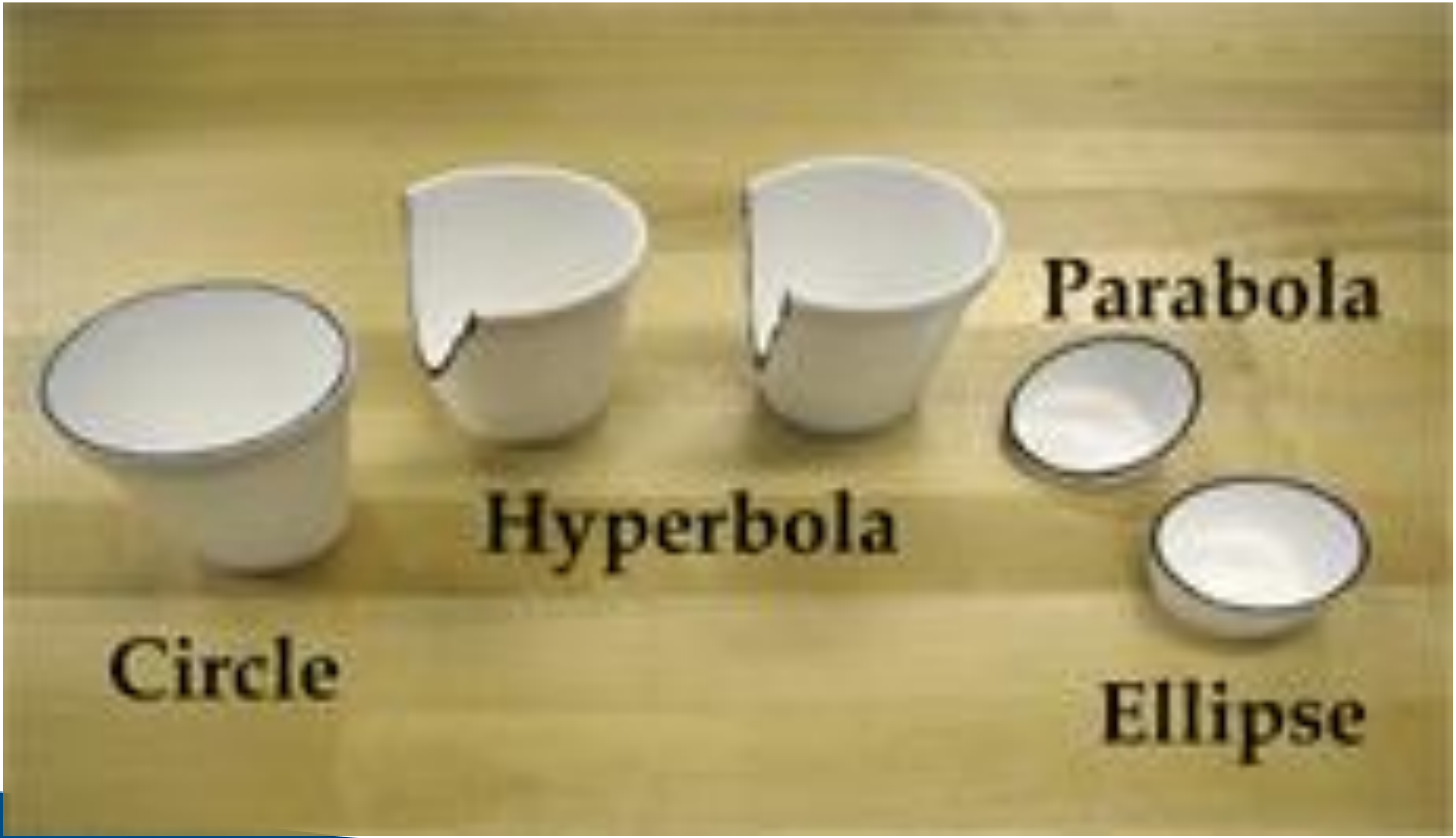
Parametric Representation of Hyperbola

Its defined mathematically as a curve generated by a point that moving such that at any position he difference of its distance from the fixed positions (foci) F & F′ is a constant & equal to the transverse axis of the hyperbola

- The parametric equation

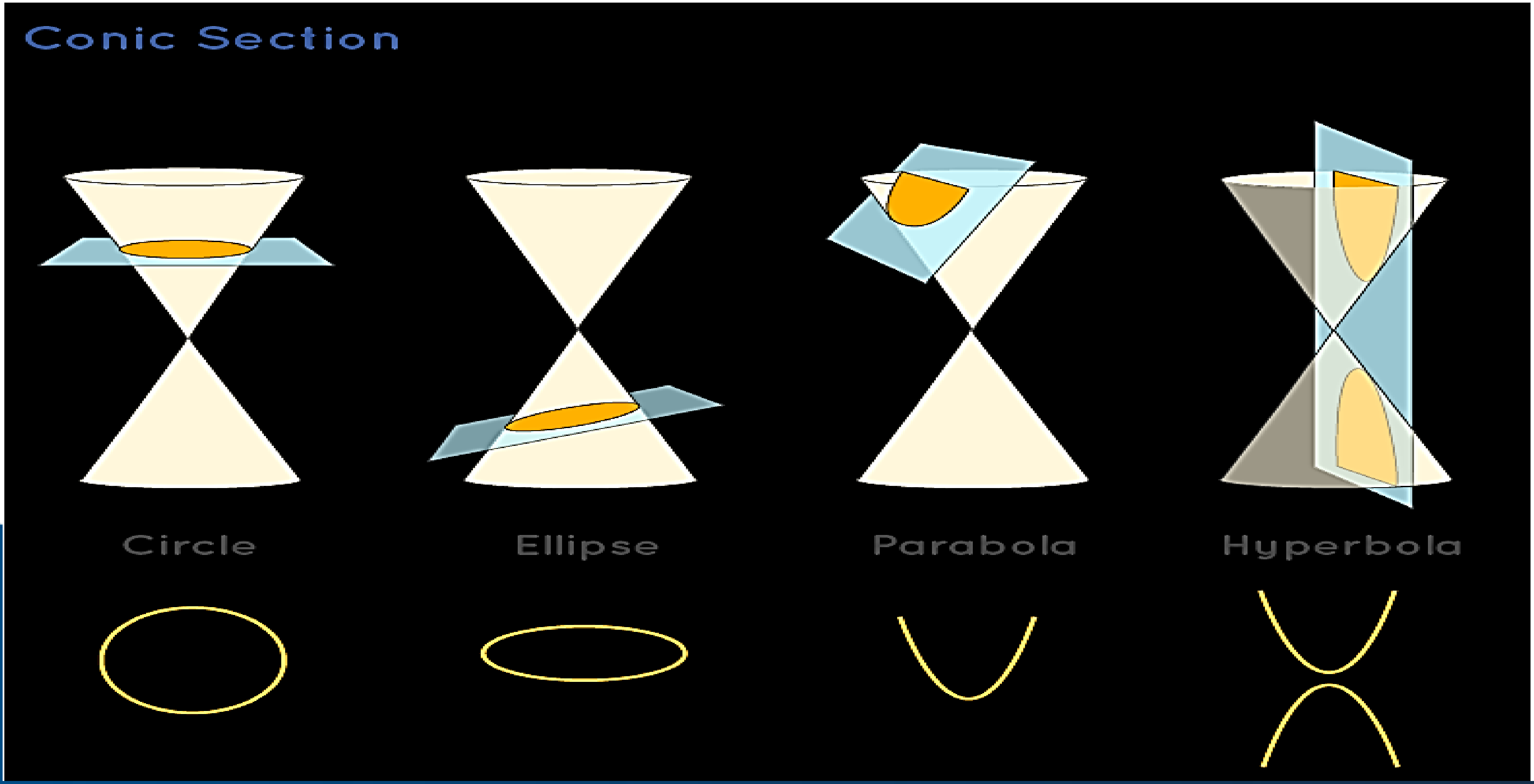
$$\left.\begin{aligned} X &= X_v + A\cosh(u) \\ Y &= Y_v + B\sinh(u) \\ Z &= Z_v \end{aligned}\right\} \quad 0 \le u \le 2\pi$$




Parabola
Hyperbola
Circle
Ellipse

Conic Sections

- Conic sections or sections of a cone are the curves obtained by the intersection of a plane and cone.
- here are three major sections of a cone or **conic sections**: parabola, hyperbola, and ellipse(the circle is a special kind of ellipse).
- A cone with two identical nappes is used to produce the conic sections**.**



The General Conic Equation

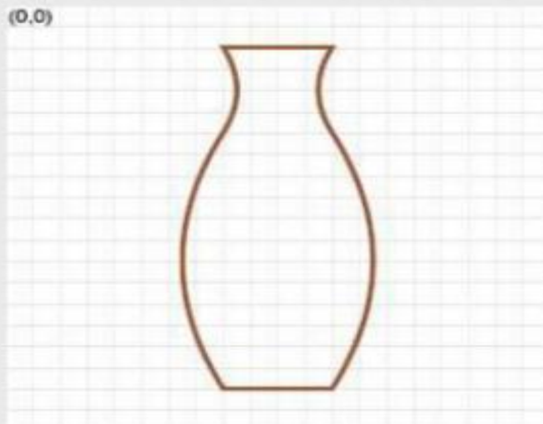
Conic Sections Equations

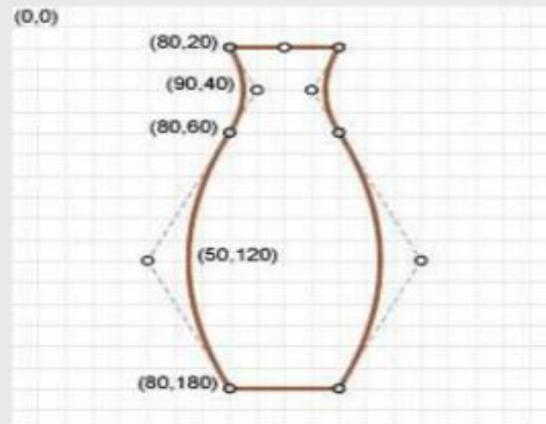
Conic section Name	Equation when the centre is at the Origin, i.e. (0, 0)	Equation when centre is (h, k)
Circle	$x^2 + y^2 = r^2$; r is the radius	$(x - h)^2 + (y - k)^2 = r^2$; r is the radius
Ellipse	$(x^2/a^2) + (y^2/b^2) = 1$	$(x - h)^2/a^2 + (y - k)^2/b^2 = 1$
Hyperbola	$(x^2/a^2) - (y^2/b^2) = 1$	$(x - h)^2/a^2 - (y - k)^2/b^2 = 1$
Parabola	$y^2 = 4ax$, where a is the distance from the origin to the focus	

Basics of Surfaces
Curve:

How to represent curves

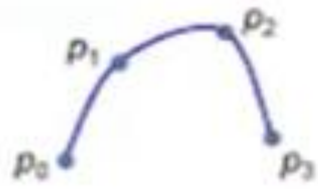
- Specify every point along a curve?
  - Used sometimes as “freehand drawing mode” in 2D applications
  - Hard to get precise results
  - Too much data, too hard to work with generally
- Specify a curve using a small number of “control points”
  - Known as a *spline curve* or just *spline*





Interpolation and Approximation Spline:

- **Interpolation:** When polynomial sections are fitted so that the curve passes through each control point.



- interpolation curves are commonly used to digitize drawings or to specify animation paths
- **Approximation:** when the polynomials are fitted to the general control-point path without necessarily passing through all control points.

Polynomial Functions

- Linear:
  (1st order)

$$f(t) = at + b$$

- Quadratic:
  (2nd order)

$$f(t) = at^2 + bt + c$$

- Cubic:
  (3rd order)

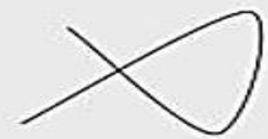
$$f(t) = at^3 + bt^2 + ct + d$$

Point-valued Polynomials (Curves)

- Linear: $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$
  (1st order)

- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$
  (2nd order)

- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$
  (3rd order)



- Each is 3 polynomials "in parallel":

$$x_x(t) = a_x t + b_x \qquad x_y(t) = a_y t + b_y \qquad x_z(t) = a_z t + b_z$$

- We usually define the curve for $0 \le t \le 1$

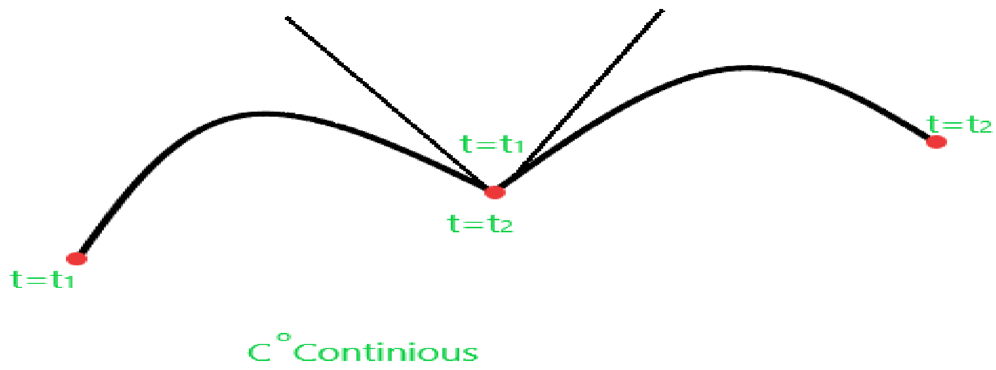
Parametric & Geometric Continuity of Curves

Parametric Continuity of Curves

There are three kinds of Parametric continuities that exist:

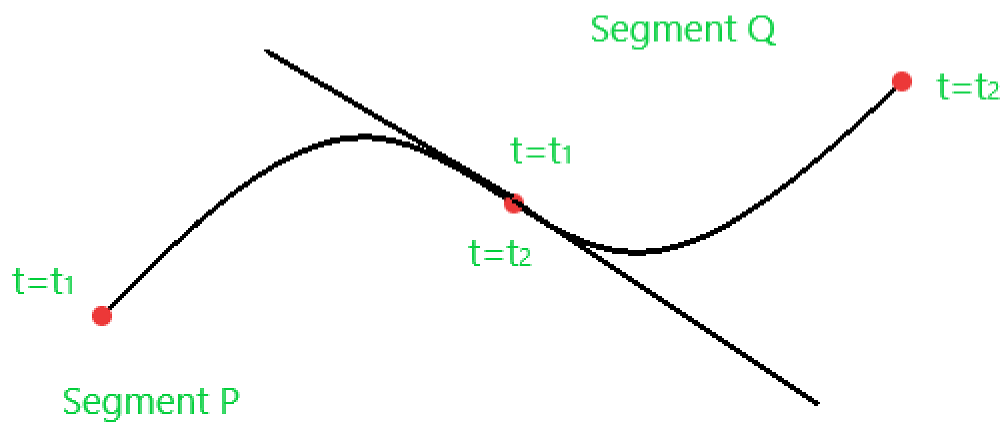
(a) **Zero-order parametric continuity(C^0) :** if both segments of the curve intersect at one endpoint.

$P(t_2) = Q(t1)$



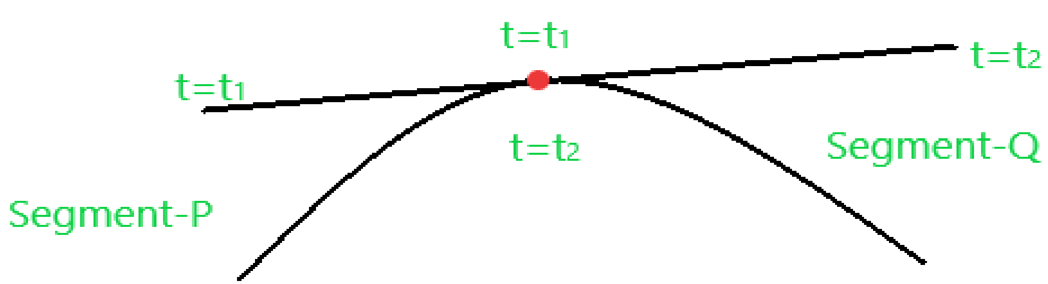
(b) First-order parametric continuity(C^1): kinds of curves have the same tangent line at the intersection point.

$P'(t_2) = Q'(t1)$



(c) Second-order parametric continuity(C^2) : A curve is said to be second-order parametric continuous if it is C^o and C^1 Continuous and the second-order derivative of the segment P at $t=t_1$ is equal to the second-order derivative of segment Q at $t=t_2$.

$P''(t_2) = Q''(t1)$



Geometric Continuity : It is an alternate method for joining two curve segments, where it requires the parametric derivation of both segments which are proportional to each other rather than equal to each other.

(a) Zero-order parametric continuity(C^0) : P(t2) = Q(t1)

(b) First-order geometric continuity(G^1) : P'(t2) = k * Q'(t1)   for all x, y, z and k > 0.

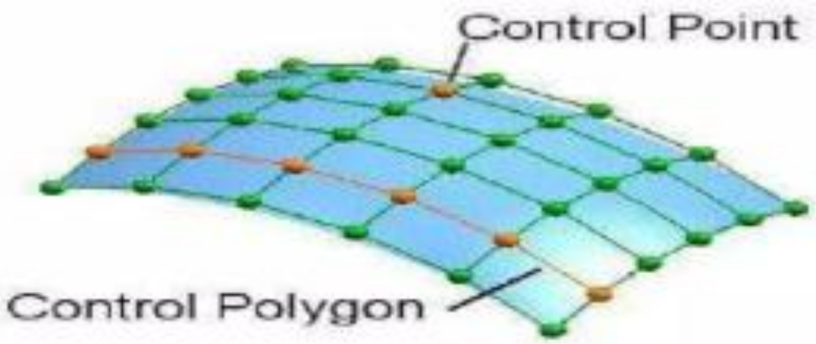
(c) Second-order geometric continuity(G^2) : P''(t2) = k * Q''(t1)   for all x, y, z and k > 0.

Surfaces

Surface :

Objects are represented as a collection of surfaces. Most common representation for surfaces:

- Polygon mesh
- Parametric surfaces
- Quadric surfaces



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PARAMETRIC CUBIC CURVE

Polylines and polygons:

- Large amounts of data to achieve good accuracy.
- Interactive manipulation of the data is tedious.

Higher-order curves:

- More compact (use less storage).
- Easier to manipulate interactively.

Possible representations of curves:

- explicit, implicit, and parametric

TYPES

There are Three Types of Parametric Cubic Curves.

Hermite Curves:

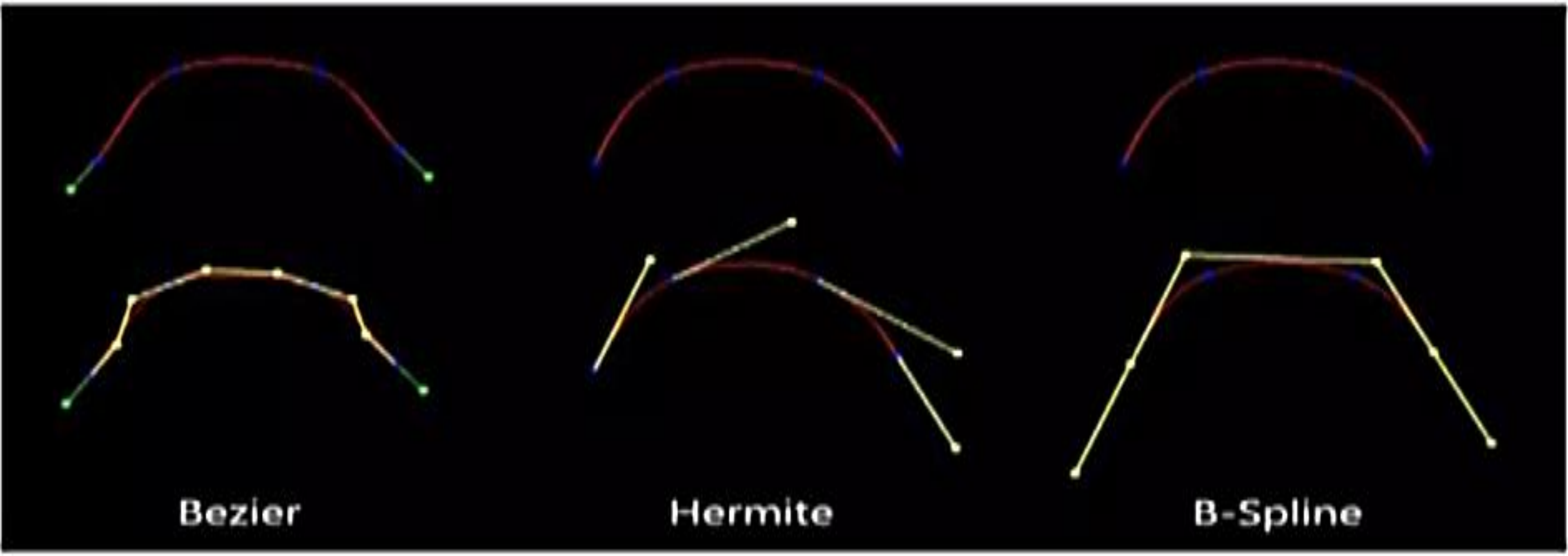
Defined by two **endpoints** and two endpoint **tangent vectors (used 1st order)**

Bézier Curves:

Defined by two **endpoints** and two **control points** which control the endpoint' **tangent vectors**

Splines:

Defined by four **control points**



WHY CUBIC POLYNOMIAL SUITABLE FOR CURVE REPRESENTATION

The degree of the polynomial defining the curve segment is one less that the number of defining polygon point. Therefore, for 4 control points, the degree of the polynomial is 3, i.e. cubic polynomial. A Bezier curve generally follows the shape of the defining polygon.

Degree= no. of control points - 1

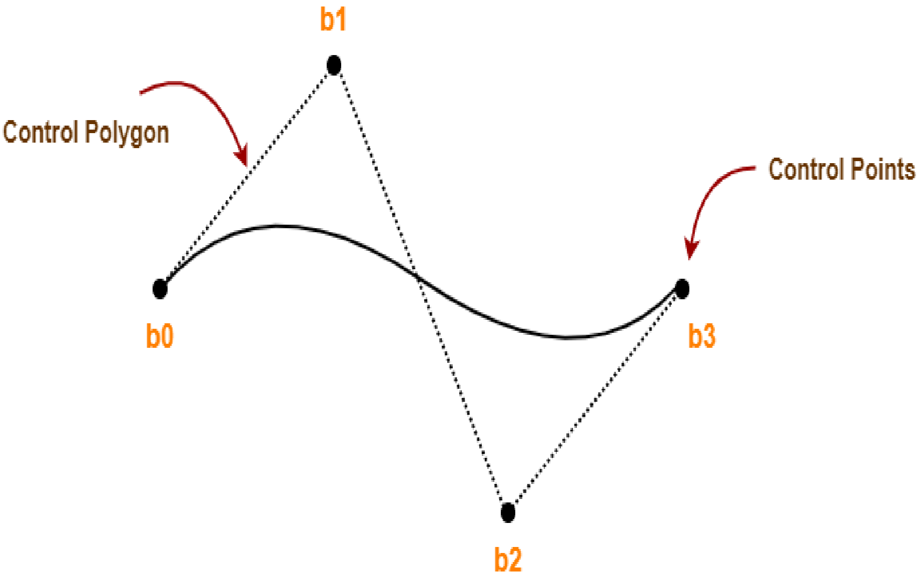
- Cubic: (3rd order) $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$

Bezier Curve

Definition:

- Bezier Curve is parametric curve defined by a set of control points.
- Two points are ends of the curve.
- Other points determine the shape of the curve

Bezier Curve Example-



Bezier Curve Example

Here,

This bezier curve is defined by a set of control points b_0, b_1, b_2 and b_3.

Points b_0 and b_3 are ends of the curve.

Points b_1 and b_2 determine the shape of the curve

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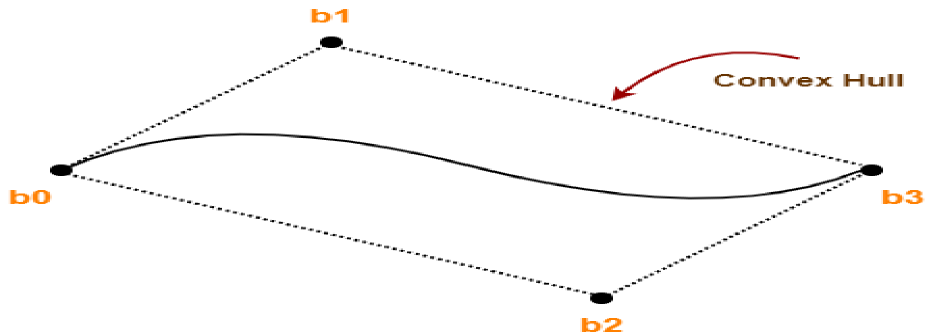
Bezier Curve Properties

Property-01:

Bezier curve is always contained within a polygon called as convex hull of its control points.



Bezier Curve With Convex Hull

Property-02:

Bezier curve generally follows the shape of its defining polygon.

The first and last points of the curve are coincident with the first and last points of the defining polygon.

Property-03:

The degree of the polynomial defining the curve segment is one less than the total number of control points.

Degree = Number of Control Points – 1

Property-04:

The order of the polynomial defining the curve segment is equal to the total number of control points.

Property-05:

No straight line intersects a Bezier curve more times than it intersects its control polygon.

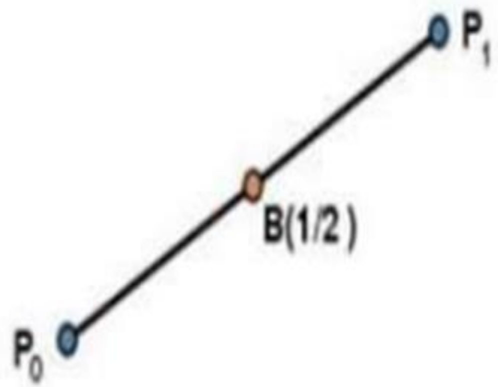
Bezier Curve Equation-

A bezier curve is parametrically represented by-

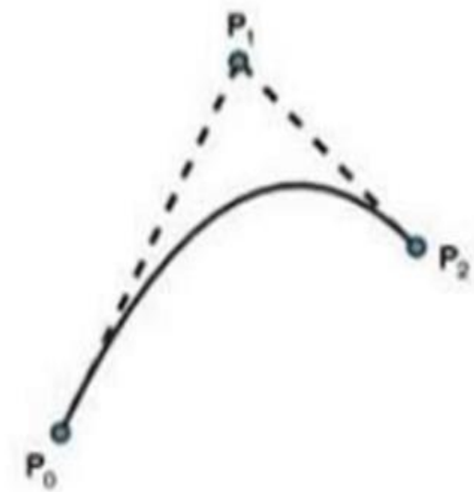
$$\boxed{P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t)}$$

Bezier Curve Equation

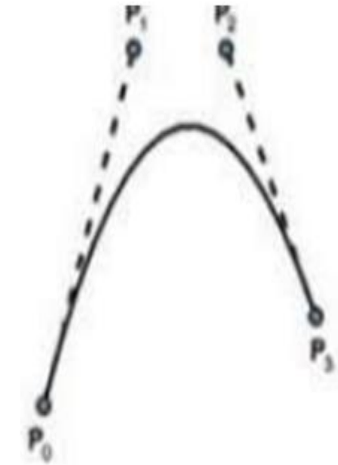
Bezier Curve Types



simple Bézier curve

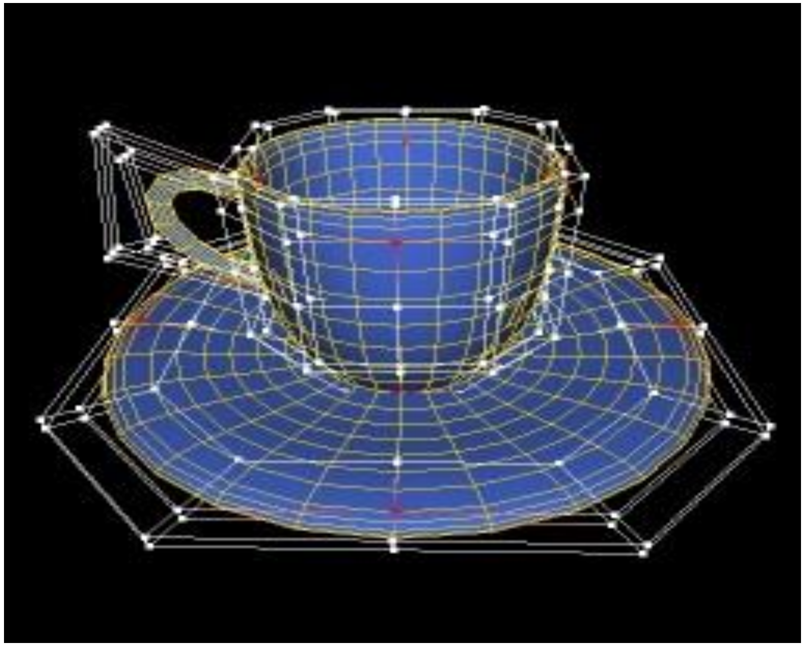


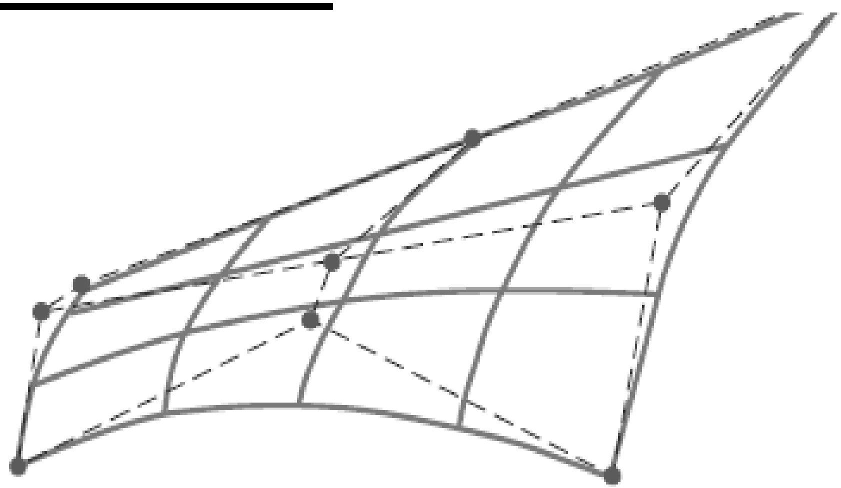
Quadratic Bézier curve



Cubic Bézier curve

Bezier Surfaces





Wire-frame Bézier surfaces constructed with 9 control points arranged in a 3 × 3 mesh

Bezier Surfaces

- Two sets of orthogonal Bézier curves can be used to design an object surface.
- The parametric vector function for the Bézier surface is formed as the tensor product of Bézier blending functions:

$$\mathbf{P}(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} \mathbf{p}_{j,k} \, \mathrm{BEZ}_{j,m}(v) \, \mathrm{BEZ}_{k,n}(u)$$

with $\mathbf{p}_{j,k}$ specifying the location of the $(m + 1)$ by $(n + 1)$ control points

Quadric Surfaces

What Are Quadric Surfaces?

Quadric surfaces are surfaces that are defined by different types of second-order equations with three variables: x, y, and z. These surfaces are defined by the general form shown below.

$$Ax^2 + By^2 + Cz^2 + J = 0$$

$$Ax^2 + By^2 + Iz = 0$$