Numerical ~~Integ~~ Integration

The problem of numerical Integration is to find an approximate value of the integral

$$I = \int_a^b f(x)\,dx$$

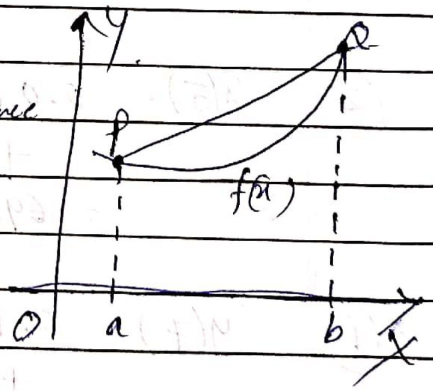
Methods :-

1. Trapizodial Rule
2. Simpson's one-third Rule
3. Simpson's three-Eighth Rule.

Trapizodial Rule

Let the curve $y = f(x)$, $a \le x \le b$ be approximated by the line joining points $P(a, f(a))$ and $Q(b, f(b))$ on the curve.

By Newton's forward difference formula, the linear polynomial approximation to $f(x)$, is given by



$$f(x) = f(x_0) + r\Delta f(x_0)$$

$$= f(x_0) + \frac{(x - x_0)}{h}\Delta f(x_0) \quad \text{--- (1)}$$

where $x_0 = a$ and $x_1 = b$ and $h = b - a$

Q.-> Evaluate $\int_0^6 3x^2\,dx$ by dividing the interval into six equal parts by applying Trapizodial Rule.

Soln :-> With $N=6$, steplength $h = \frac{b-a}{N}$.

when $a=0$ $b=6$.

Hence $h = \frac{6-0}{6} = 1$

and the nodal points are
$0, 1, 2, 3, 4, 5, 6$.

We have the following table of values

x	0	1	2	3	4	5	6
$f(x)$	0	3	12	27	48	75	108

$$\int_a^b f(x)\,dx = \int_0^6 f(x)\,dx = \frac{h}{2}\Big[f(x_0) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\big) + f(x_6)\Big]$$

$$= \frac{1}{2}\Big[0 + 2[3+12+27+48+75] + 108\Big]$$

$$= 219.$$

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Simpson 1/3 Rd Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\{\text{sum of functional values at odd nodal points}\}$$
$$+ 2\{\text{sum of functional values at even nodal points}\}$$
$$+ f(x_{2N}) \Big] \quad \text{—— (1)}$$

Q:7. Evaluate $\int_0^6 3x^2\,dx$. dividing the intervals $[0,6]$ into six equal parts.

Sol:→ $N = 6$. step length $= h = \frac{b-a}{N} = \frac{6-0}{6} = 1$

nodal points are

x:	0	1	2	3	4	5	6
$f(x)$:	0	3	12	27	48	75	108

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\big(f(x_1) + f(x_3) + f(x_5)\big) + 2\big(f(x_2) + f(x_4)\big) + f(x_6)\Big]$$

$$= \frac{1}{3}\Big[0 + 4(3 + 27 + 75) + 2(12 + 48) + 108\Big]$$

$$= \frac{1}{3}\Big[0 + \overset{420}{\cancel{100}} + 120 + 108\Big]$$

$$= \cancel{309.3}\ 216$$

$$\int_0^6 3x^2\,dx = 3\frac{x^3}{3} = \left[x^3\right]_0^6 = 6^3 - 0 = 216.$$

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Simpson 3/8th Rule :→

$$\int_a^b f(x)dx = \frac{3h}{8}\Big[f(x_0) + 3\{\text{sum of functional values not multiple of } 3\} + 2\{\text{multiple of } 3\} + f(x_n)\Big]$$

$$\int_a^b f(x)dx = \frac{3h}{8}\Big[f(x_0) + 3[f(x_1)+f(x_2)+f(x_4)+f(x_5)] + 2[f(x_3)] + f(x_6)\Big]$$

$$= \frac{3}{8}\Big[0 + 3(3+12+48+75) + 2(27) + 108\Big]$$

$$= \frac{3}{8}\Big[0 + 414 + 54 + 108\Big]$$

$$= 216.$$

Q:→ Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Trapizodial Simpson 1/3rd & 3/8th Rule.

Soln:→ Let $N=4$. h = Step length $= \frac{b-a}{N} = \frac{1-0}{4} = 0.25$.

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8	0.64	0.5

Trapizodial Rule :→

$$I = \frac{h}{2}\Big[f(x_0) + 2(f(x_1)+f(x_2)+f(x_3)) + f(x_4)\Big]$$

$$= \frac{0.25}{2}\Big[1 + 2(0.9411 + 0.8 + 0.64)

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$= 0.782775$

Simpson 1/3 Rd Rule

$$I = \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2\left(f(x_2)\right) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[1 + 4(0.9411 + 0.64) + 2(0.8) + 0.5\right]$$

$= 0.7854.$

Simpson 3/8th Rule

$$I = \frac{3h}{8}\left[f(x_0) + 3\left(f(x_1) + f(x_2)\right) + 2\left(f(x_3)\right) + f(x_4)\right]$$

$$= \frac{3 \times 0.25}{8}\left[1 + 3(0.9411 + 0.8) + 2(0.64) + 0.5\right]$$

$= 0.75030$

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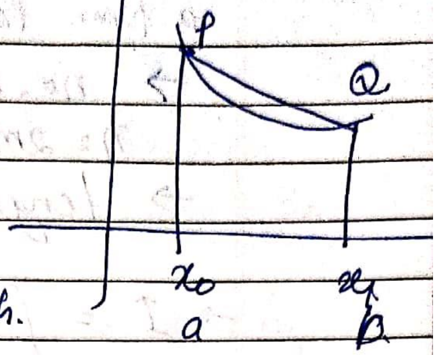
area of trapezoid

$$= \frac{1}{2} h (b_1 + b_2)$$

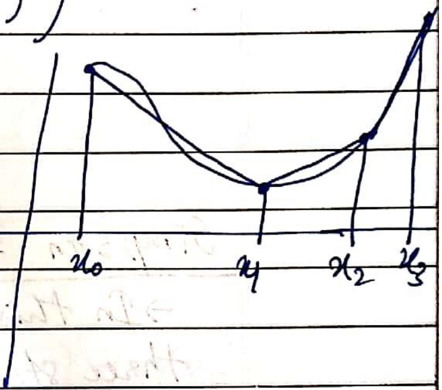


$b_1, b_2 \to$ base length.

$h \to$ height

$$= \frac{1}{2}(x_1 - x_0)\left(f(x_0) + f(x_1)\right)$$

$$= \frac{1}{2} h \left(f(x_0) + f(x_1)\right)$$



$$\frac{1}{2} h \left(f(x_0) + f(x_1)\right) + \frac{1}{2} h \left(f(x_1) + f(x_2)\right) + \frac{1}{2} h \left(f(x_2) + f(x_3)\right)$$

$$\Rightarrow \frac{1}{2} h \left\{ f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right\}$$

Thm :→ $f(x)$ is continuous over $[a,b]$. Let n be a positive integer (no. of subintervals) and $\Delta x = h = \frac{b-a}{n}$.

$P = \{x_0, x_1, \ldots x_n\}$

$$\int_a^b f(x)\,dx = I_n = \frac{h}{2}\left\{ f(x_0) + 2\left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right) + f(x_n) \right\}$$

Simpson 1/3 Rule

In simpson 1/3rd Rule, we are taking two strips at a time to get a quadratic polynomial.

→ no. of subintervals should be even.
$n = 2m$.

→ length of subinterval $= h = \frac{b-a}{2m}$.

$$\rightarrow I = \int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_{2m}) + 4(y_1 + y_3 + y_5 + \cdots + y_{2m-1}) + 2(y_2 + y_4 + \cdots + y_{2m-2})\Big]$$

$$= \frac{h}{3}\Big[(\text{1st term} + \text{last term}) + 4(\text{odd terms}) + 2(\text{even terms})\Big]$$

Simpson 3/8th Rule :→

→ In this method, we are taking three strips at a time.

→ no. of subintervals must be multiple of 3 ⇒ $n = 3m$.

→ length of subintervals $= \frac{b-a}{3m}$.

$$\rightarrow I = \int_a^b f(x)\,dx = \frac{3h}{8}\Big[(y_0 + y_{3m}) + 3(y_1 + y_2 + \cdots + y_{3m-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{3m-3})$$

$$= \frac{3h}{8}\Big[(\text{1st} + \text{last term}) + 3(\text{remaining}) + 2(\text{multiple of 3})\Big]$$

Q.7 $\int_0^6 \frac{1}{1+x^2}\,dx$ (0–6) into 6 parts.

(I) 1.4108 (II) 1.3662, (III) 1.3571.