Numerical Solutions of Partial Differential Equations

Partial differential equations arise in the study of many branches of applied mathematics, e.g., in fluid dynamics, heat transfer, boundary layer flow and electro magnetic theorey. only few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. of all the numerical methods available for the solution of partial differential equation, the finite difference method is commonly used.

Classification of PDEs of Second order

A general second order linear PDE in two independent values x, y is of the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

where A, B, C, D, E and F are in general ~~equ~~ functions of x and y

~~These~~ is equation ~~s~~ is said to be

1) Parabolic if $B^2-4AC=0$
2) Elliptic if $B^2-4AC<0$,
3) Hyperbolic if $B^2-4AC>0$

Ex:- ① one dimensional wave equation

$$c^2 u_{xx} - u_{tt} = 0$$

Here $A=c^2$, $B=0$, $C=-1$

$$B^2-4AC = 4c^2 > 0$$

$\therefore$ it is Hyperbolic equation

② one dimensional heat equation

$$c^2 u_{xx} - u_t = 0$$

Sol:- Here $A=c^2$, $B=0$, $C=0$

$$B^2-4AC = 0$$

$\therefore$ It is Parabolic equation

③ Two dimensional Laplace's equation

$$u_{xx} + u_{yy} = 0$$

Sol. Here $A=1$, $B=0$, $C=1$

$$B^2-4AC = -4 < 0$$

$\therefore$ It is Elliptic equation

Finite difference approximation to Partial Derivatives

Let $u = u(x,y)$ be a function of two independent variables. The finite difference approximation for the first order partial derivatives: u_x, u_y and second order Partial derivatives: u_{xx}, u_{yy} are as follows

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} \quad \text{(Forward)} \longrightarrow (1)$$

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x,y) - u(x-h,y)}{h} \quad \text{(Backward)} \longrightarrow (2)$$

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x-h,y)}{2h} \quad \text{(central)} \longrightarrow (3)$$

$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k} \quad \text{(Forward)} \longrightarrow (4)$$

$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y) - u(x,y-k)}{k} \longrightarrow (5)$$

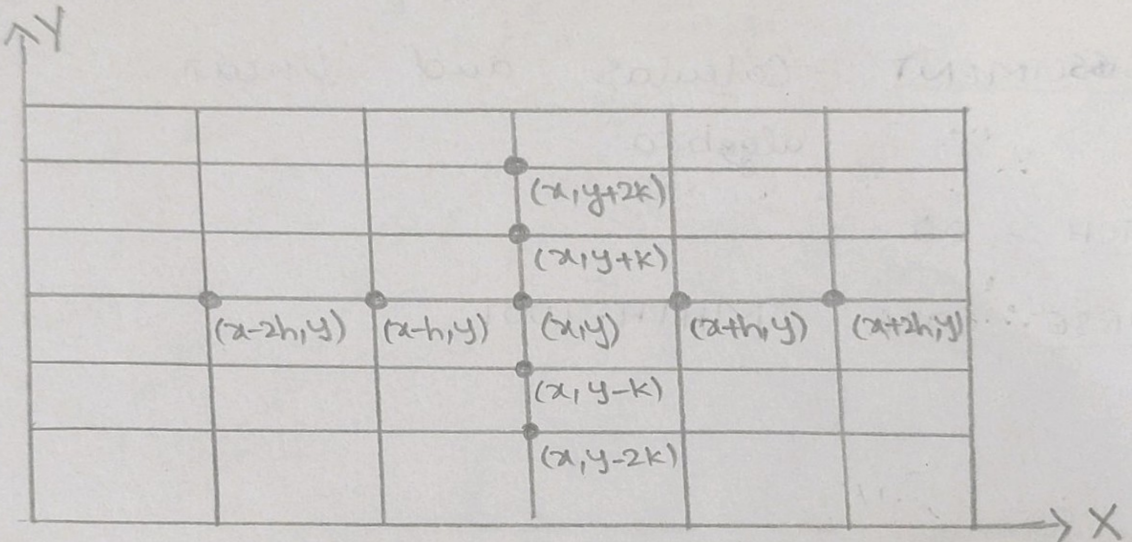
$$u_y = \frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y-k)}{2k} \longrightarrow (6)$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \lim_{h\to 0} \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right] \longrightarrow (7)$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \lim_{k\to 0} \frac{1}{k^2}\left[u(x,y+k) - 2u(x,y) + u(x,y-k)\right] \longrightarrow (8)$$

Numerical Solution of PDE

Consider a rectangular region R in the x-y plane. Let us divide this region into a network of rectangles of sides h and K. In other words, we draw lines $x = ih$, $y = jK$ where $i, j = 1, 2, 3 \ldots$ being parallel to the Y-axis and X-axis respectively as shown in the figure below. The points of intersection of these lines are called mesh points (or) grid points (or) pivotal points.



We write $u(x,y) = u(ih, jk)$ and finite difference approximation for the partial derivatives given by ① to ⑧ are put in the following modified form

$$\frac{\partial u}{\partial x} = \frac{1}{h}\left[u_{i+1,j} - u_{i,j}\right] + O(h) \longrightarrow (a)$$

$$\frac{\partial u}{\partial x} = \frac{1}{h}\left[u_{i,j} - u_{i-1,j}\right] + O(h) \longrightarrow (b)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2h}\left[u_{i+1,j} - u_{i-1,j}\right] + O(h^2) \longrightarrow (c)$$

$$\frac{\partial u}{\partial y} = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] + O(k) \longrightarrow (d)$$

$$\frac{\partial u}{\partial y} = \frac{1}{k}\left[u_{i,j} - u_{i,j-1}\right] + O(k) \longrightarrow (e)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2k}\left[u_{i,j+1} - u_{i,j-1}\right] + O(k^2) \longrightarrow (f)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] + O(h^2) \longrightarrow (g)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] + O(k^2) \longrightarrow (h)$$

The substitution of these finite difference approximations into the PDE converts the PDE into a finite difference approximations.

Crank - Nicolson Method

In numerical methods, the crank-Nicolson method is a finite difference method used for numerically solving precisely the heat equation, and diffusion equation and similar partial differential equations. This method helps to increase the convergence speed with respect to the finite variable of the numerical approximations.

The Crank-Nicolson one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given below

Sol:- Given one dimensional heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where } c^2 = \frac{k}{s\rho} \rightarrow (1)$$

K.i. k is the coefficient of conductivity of the material, ρ is the density and Cp is specifice heat.

In equation (1), if we replace $\frac{\partial^2 u}{\partial x^2}$ by the average of its finite difference approximations on the j^{th} and $(j+1)^{th}$ time levels, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2h^2}(u_{i-1,j} - 2u_{i,j} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j+1} + u_{i+1,j+1})$$

Hence Eq (1) is approximated by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{c^2}{2h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$$

which simplified to

$$-\lambda u_{i-1,j+1} + (2+2\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + (2-2\lambda)u_{i,j} + \lambda u_{i+1,j} \rightarrow (2)$$

we have 3 unknowns on the left side of Eq (2) and on the right side all are known quantities.

This is called crank-Nicolson formula for the one dimensional heat equation.