Interpolation formulae for unequal intervals:

Divided differences:

Let $f(x_0), f(x_1), \dots f(x_n)$ be the values of an unknown function $y = f(x)$ corresponding to the values of $x : x_0, x_1, \dots, x_n$ at unequal intervals.

The first order divided differences are defined as,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

$$\dots \quad f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

The second order divided differences are

$$f(x_0 x_1, x_2) = \frac{f(x_1 x_2) - f(x_0 x_1)}{x_2 - x_0}, \quad f(x_1 x_2 x_3) = \frac{f(x_2 x_3) - f(x_1 x_2)}{x_3 - x_1},$$

$$\dots \quad f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1} x_n) - f(x_{n-2} x_{n-1})}{x_n - x_{n-2}}.$$

Similarly, the higher order divided differences are defined.

Divided difference table:

x	$f(x)$	I D.D.	II D.D.	III D.D.
$x_0 = 2$	$f(x_0) = 4$	$\frac{56-4}{4-2} = 26 = f(x_0 x_1)$	$\frac{131-26}{9-2} = 15 = f(x_0 x_1 x_2)$	$\frac{23-15}{10-2} = 1 = f(x_0 x_1 x_2 x_3)$
$x_1 = 4$	$f(x_1) = 56$	$\frac{711-56}{9-4} = 131 = f(x_1 x_2)$	$\frac{269-131}{10-4} = 23 = f(x_1 x_2 x_3)$	
$x_2 = 9$	$f(x_2) = 711$	$\frac{980-711}{10-9} = 269 = f(x_2 x_3)$		
$x_3 = 10$	$f(x_3) = 980$			

We have the following two methods:

1. Newton's divided difference formula
2. Lagrange's Interpolation formula.

1. Newton's divided difference formula: is given by

$$y = f(x) = f(x_0) + (x - x_0) f(x_0 x_1) + (x - x_0)(x - x_1) \cdot f(x_0 x_1 x_2) + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) \cdot f(x_0 x_1 \cdots x_n)$$

Examples:-

1. Use Newton's divided difference formula to find $f(4)$ given the data

x :	0	2	3	6
$f(x)$:	-4	2	14	158

Solⁿ:-

x	$f(x)$	I DD	II DD	III DD
$x_0 = 0$	$f(x_0) = -4$	$\frac{2+4}{2-0} = 3 = f(x_0 x_1)$	$\frac{12-3}{3-0} = 3 = f(x_0 x_1 x_2)$	$\frac{9-3}{6-0} = 1 = f(x_0 x_1 x_2 x_3)$
$x_1 = 2$	$f(x_1) = 2$	$\frac{14-2}{3-2} = 12 = f(x_1 x_2)$	$\frac{48-12}{6-2} = 9 = f(x_1 x_2 x_3)$	
$x_2 = 3$	$f(x_2) = 14$	$\frac{158-14}{6-3} = 48 = f(x_2 x_3)$		
$x_3 = 6$	$f(x_3) = 158$			

By Newton's divided difference formula, we have

$$y = f(x) = f(x_0) + (x - x_0) f(x_0 x_1) + (x - x_0)(x - x_1) f(x_0 x_1 x_2) + (x - x_0)(x - x_1)(x - x_2) \cdot f(x_0 x_1 x_2 x_3)$$

$$= -4 + (x-0)(3) + (x-0)(x-2)(3) + (x-0)(x-2)(x-3)(1)$$

$$= -4 + 3x + 3x(x-2) + x(x-2)(x-3)$$

Take $x = 4$

$$f(4) = -4 + 12 + 3(4)(2) + 4(2)(1)$$

$$\therefore f(4) = -4 + 12 + 24 + 8 = 40$$

$$\therefore \boxed{f(4) = 40}$$ //.

2. For the given data find $f(8)$ using Newton's divided difference formula

x :	1	2	4	7	12
$f(x)$:	576	168	-30	48	378

Sol^n:-

~~x $f(x)$~~

x	$f(x)$	I DD	II DD	III DD	IV DD
$x_0 = 1$	$f(x_0) = 576$	$\frac{168-576}{2-1} = -408$ $f(x_0x_1)$	$\frac{99+408}{4-1} = 169$ $f(x_0x_1x_2)$	$\frac{-14.6-169}{7-1} = 30.6$ $f(x_0x_1x_2x_3)$	$\frac{1.96-30.6}{12-1} = -2.60 = f(x_0x_1x_2x_3x_4)$
$x_1 = 2$	$f(x_1) = 168$	$\frac{-30-168}{4-2} = 99$ $f(x_1x_2)$	$\frac{26-99}{7-2} = -14.6$ $f(x_1x_2x_3)$	$\frac{5+14.6}{12-2} = 1.96$ $f(x_1x_2x_3x_4)$	
$x_2 = 4$	$f(x_2) = -30$	$\frac{48+30}{7-4} = 26$ $f(x_2x_3)$	$\frac{66-26}{12-4} = 5$ $f(x_2x_3x_4)$		
$x_3 = 7$	$f(x_3) = 48$	$\frac{378-48}{12-7} = 66$ $f(x_3x_4)$			
$x_4 = 12$	$f(x_4) = 378$				

By Newton's divided difference formula, we have

$$y = f(x) = f(x_0) + (x-x_0) f(x_0x_1) + (x-x_0)(x-x_1) f(x_0x_1x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0x_1x_2x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)$$
$$\times f(x_0x_1x_2x_3x_4)$$

$$= 576 + (x-1)(-408) + (x-1)(x-2)(169) + (x-1)(x-2)(x-4)(30.6)$$
$$+ (x-1)(x-2)(x-4)(x-7)(-2.60)$$

Take $x = 8$;

$$f(8) = 576 + (7)(-408) + (7)(6)(169) + (7)(6)(4)(30.6)$$
$$+ (7)(6)(4)(1)(-2.60)$$
$$= 576 - 2856 + 7098 + 5140.8 - 436.8$$
$$= 12814.8 - 3292.8$$

$\therefore f(8) = 9522$

2. Lagrange's Interpolation formula

If $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, ..., $y_n = f(x_n)$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of $x_0, x_1, x_2, \ldots, x_n$ at unequal intervals then,

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \cdot y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \cdot y_1$$
$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} \cdot y_2 + \ldots$$
$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} \cdot y_n$$

Examples:-

1. Apply Lagrange's interpolation formula to find y at $x = 10$ given

x	5	6	9	11
y	12	13	14	16

Solⁿ:- Let $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$] at $x = 10$
$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$]

$$\therefore y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

$$\therefore y = f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{(\cancel{4})(1)(-1)}{(-1)(-\cancel{4})(-\cancel{6})}(\cancel{12})^{2} + \frac{(\cancel{5})(1)(-1)}{(\cancel{1})(-3)(-\cancel{5})}(13)$$

$$+ \frac{(5)(\cancel{4})(-1)}{(\cancel{4})(3)(-\cancel{7})}(\cancel{14})^{7} + \frac{(\cancel{5})(4)(1)}{(\cancel{6}_{3})(\cancel{5})(\cancel{2})}(\cancel{16})^{\cancel{8}\,4}$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= 2 + \frac{35+16-13}{3}$$

$$= 2 + \frac{38}{3}$$

$$\therefore\ y = f(10) = \frac{44}{3} \simeq 14.66$$

2. The observed values of a function are respectively 168, 120, 72, 63 at the four positions 3, 7, 9, 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable.

Solⁿ:- Let $x_0 = 3,\ x_1 = 7,\ x_2 = 9,\ x_3 = 10$ and $y_0 = 168,\ y_1 = 120,\ y_2 = 72,\ y_3 = 63$ at $x = 6$.

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$= \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168) + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}(120)$$

$$+ \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}(-168) + \frac{(3)(-3)(-4)}{(4)(-2)(-3)}(120)$$

$$+ \frac{(3)(-1)(-4)}{(6)(2)(-1)}(72) + \frac{(3)(-1)(-3)}{(7)(3)(1)}(63)$$

$$= 12 + 180 - 72 + 27$$

$$\therefore y = f(6) = 147$$

Practice Problems:

1. Newton's divided difference formula

1. Find the cubic polynomial which passes through the points (2, 4), (4, 56), (9, 711), (10, 980) using Newton's divided difference formula & hence find $f(5)$.

2. Find $f(2)$ using Newton's divided difference formula

x ;	0	1	4	8	10
$f(x)$:	-5	-14	-125	-21	355

2. Lagrange's Interpolation formula

1. The following table gives the normal weights of babies during the first eight months of life

Age (in months)	0	2	5	8
Weight (in months)	6	10	12	16

Estimate the weight of the baby at the age of seven months using Lagrange's interpolation formula.

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2. Use Lagrange's interpolation method, find the value of $f(x)$ at $x=5$ given the data

x :	1	3	4	6
$f(x)$:	3	9	30	132

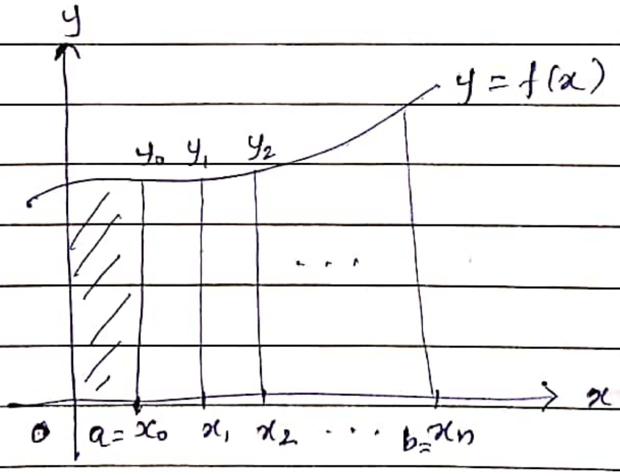
Numerical Integration :- The problem of numerical integration is to find an approximate value of the integral

$$I = \int_a^b f(x)\,dx.$$

Methods :

1. Trapezoidal Rule
2. Simpson's $1/3^{rd}$ Rule
3. Simpson's $3/8^{th}$ Rule

Let us divide the interval $[a,b]$ into 'n' number of equal subintervals so that length of each subinterval is $h = \frac{b-a}{n}$



1. **Trapezoidal Rule of Integration** :

This rule is used to find the area under the curve when we consider only one strip at a time.

Formula :- $$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

which is called Trapezoidal Rule.

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Ex 1. Evaluate $\int_0^6 3x^2\,dx$ by dividing the interval into six equal parts by applying Trapezoidal rule.

Solⁿ:- Let ~~n=6~~, $n = 6$, & $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

(1) We have the following table

x :	0	1	2	3	4	5	6
$f(x)$:	$y_0=0$	$y_1=3$	$y_2=12$	$y_3=27$	$y_4=48$	$y_5=75$	$y_6=108$

$$\therefore \int_a^b f(x)\,dx = \int_0^6 f(x)\,dx$$

$$= \int_0^6 3x^2\,dx$$

$$= \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{1}{2}\left[(0+108) + 2(3+12+27+48+75)\right]$$

$$= \frac{1}{2}\left[108 + 330\right]$$

$$= \frac{438}{2}$$

$$= 219.$$

(2) Simpson's $1/3^{rd}$ rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots) + 2(y_2+y_4+\cdots)\right]$$

$$\therefore \int_a^b 3x^2\,dx = \frac{1}{3}\left[(0+108) + 4(3+27+75) + 2(12+48)\right]$$

$$= \frac{1}{3}\left[108 + 420 + 120\right] = 216$$

3. Simpson's $3/8^{th}$ rule:

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 2(y_3+y_6+\cdots) + 3(y_1+y_2+y_4+\cdots)\right]$$

$$\therefore \int_0^6 3x^2 dx = \frac{3}{8}\left[(0+108) + 2(27) + 3(3+12+48+75)\right]$$

$$= \frac{3}{8}\left[108 + 54 + 3(\overset{(414)}{138})\right]$$

$$= 216$$

2. Simpson's $\frac{1}{3}^{rd}$ Rule:

To find area under a curve we take two strips at a time.

Formula: $\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots)+2(y_2+y_4+\cdots)\right]$

$= \frac{h}{3}\left[(1^{st}\text{ term} + \text{last term}) + 4(\text{odd terms}) + 2(\text{Even terms})\right]$

3. Simpson's $\frac{3}{8}^{th}$ Rule:

To find area under a curve we take three strips at a time.

Formula: $\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+\cdots)+3(y_1+y_2+y_4+\cdots)\right]$

$= \frac{3h}{8}\left[(1^{st}\text{ term} + \text{last term}) + 2(\text{multiple of } 3) + 3(\text{remaining terms})\right].$

2). Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using
a) Trapezoidal rule
b) Simpson's $\frac{1}{3}^{rd}$ rule
c) Simpson's $\frac{3}{8}^{th}$ rule

Solⁿ:- Here,
$n=4$ & $h = \text{step length} = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1
y	$y_0=1$	$y_1=0.9411$	$y_2=0.8$	$y_3=0.64$	$y_4=0.5$

a. Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{2}\left[(1+0.5)+2(0.9411+0.8+0.64)\right]$$

$$= 0.125\left[1.5+4.7622\right]$$

$$= 0.7827$$

b. Simpson's $\frac{1}{3}^{rd}$ rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+\cdots)+2(y_2+y_4+\cdots)\right]$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{3}\left[(1+0.5)+4(y_1+y_3)+2(y_2+y_4)\right]$$

$$= (0.0833)\left[1.5+4(0.9411+0.64)+2(0.8)\right]$$

$$= (0.0833)\left[1.5+4(1.5811)+2(1.6)\right]$$

$$= (0.0833)\left[1.5+6.3244+1.6\right]$$

$$\cancel{= (0.0833)[}$$

$$= 0.7850$$

c. Simpson's $\frac{3}{8}^{th}$ rule

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}\left[(y_0+y_4)+2y_3+3(y_1+y_2)\right]$$

$$= \frac{3(0.25)}{8}\left[(1+0.5)+2(0.64)+3(0.9411+0.8)\right]$$

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$= (0.0937)\left[1.5 + 1.28 + 5.2233\right]$

$= \cancel{0.7499} = 0.7503.$

Q:→ find $\int_0^1 \frac{1}{1+x^2}\,dx$ by Trapezodial, Simpson 1/3rd Rule, Simpson 3/8th Rule, where interval is divided into 6 equal parts.

Soln:→ $n=6, \quad h = \frac{1-0}{6} = 1/6.$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1	0.97297	0.9	0.8	0.6923	0.59016	0.5

Trapizodial Rule :→ ($n=6$)

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[\left(f(x_0)+f(x_6)\right) + 2\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\right)\right]$$

$$= \frac{1}{12}\left[(1+0.5) + 2(0.97297 + 0.9 + 0.8 + 0.69230 + 59016)\right]$$

$$= \frac{9.41086}{12} = 0.78424.$$

Simpson 1/3rd Rule :→ ($n=6$).

$$I = \frac{h}{3}\left[\left(f(x_0)+f(x_6)\right) + 4\left(f(x_1)+f(x_3)+f(x_5)\right) + 2\left(f(x_2)+f(x_4)\right)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 4(0.97297 + 0.8 + 0.59016) + 2(0.9 + 0.6923)\right]$$

$=$

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$$= \frac{14.13712}{18} = 0.78539$$

Simpson's 3/8 th Rule : $(n=6)$.

$$I = \frac{3h}{8}\Big[\big(f(x_0)+f(x_6)\big) + 3\big(f(x_1)+f(x_2) + f(x_4)+f(x_5)\big) + 2\big(f(x_3)\big)\Big]$$

$$= \frac{3}{8}\cdot\frac{1}{6}\Big[(1+0.5) + 3(0.97297 + 0.9 + 0.692230 + 0.59016) + 2(0.8)\Big]$$

$$= \frac{12.56575}{16} = 0.7853593$$

Practice Problems :->

1. Compute $\int_0^{0.6} e^{-x^2}\,dx$ using
   1. Trapezodial Rule
   2. Simpson's $1/3$rd Rule
   3. Simpson's 3/8th Rule