Module 3

Numerical differentiation

order → no. of times differentiated.
degree → Power of the highest derivative term.

Ex: → $\frac{d^2y}{dx^2} + y = x$

$\Rightarrow$ order = 2, degree = 1.

Ex: → $\frac{dy}{dx} + y = x$ $\Rightarrow$ order = 1
degree = 1

Ex: → $\left(\frac{dy}{dx}\right)^2 + y = x$

$\Rightarrow$ order = 1 degree = 2

First order Initial value Problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

order = 1 degree = 1. ~~is called~~.

→ Higher order differential equations. Consist of two or more points (conditions) which is known as Boundary Value problems.

Modified Euler method :-

$$\frac{dy}{dx} = f(x,y), \; y(x_0) = y_0$$

we need to find y at $x_1 = x_0 + h$.

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first approximation by Euler's Method is given by

$$y_1 = y_0 + h(f(x_0, y_0))$$

first modified value of y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

The second modified Euler value of y_1

$$y_1^{(2)} = y_0 + h/2[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

The Third modified value of y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

and so on.

The generalized formula is.

$$\cancel{y_1^{(n+1)} = y_0 + \frac{h}{2}[f(}$$

$$y_1^{(n)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n-1)})]$$

which gives approximate value of y_1 at $x_1 = x_0 + h$ ~~upto~~ correct upto 3 decimal places.

Similarily, if we continue the same process.

$$y_2^{(n)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_1, y_2^{(n-1)})]$$

which gives approximate values of y_2 at $x_2 = x_0 + 2h$. upto some decimal places

(3) Using modified Euler method, find y at $x=0.2$ given $\frac{dy}{dx} = 3x + \frac{y}{2}$ with $y(0)=1$ & $h=0.1$

$\Rightarrow$ $f(x,y) = 3x + \frac{y}{2}$ $\quad x_0 = 0 \quad y_0 = 1 \quad$ find $x=0.2$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.2\left(3x_0 + \frac{y_0}{2}\right)$$

$$= 1 + 0.2\left(2\times 0 + 1/2\right)$$

$$= 1 + 0.2\,(1/2)$$

$$= 1.1$$

1st modification value

(i) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$

$$y_1^{(1)} = 1 + \frac{0.2}{2}\left[3x_0 + \frac{y_0}{2} + 3x_1 + \frac{y_1}{2}\right]$$

$$= 1 + 0.1\left[0 + \frac{1}{2} + 3\times 0.2 + \frac{1.3}{2}\right]$$

$$= 1 + 0.1\left[\frac{1}{2} + 0.6 + 0.65\right]$$

$$= 1 + 0.1\,[2.05]$$

$$= \cancel{2.05} \; 1.175$$

(ii) $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$= 1 + \frac{0.2}{2}\left[1/2 + 3x_1 + \frac{y_1^{(1)}}{2}\right]$$

$$= 1 + 0.1\left[0.5 + 0.6 + \frac{2.2}{2}\right]$$

$= 1 + 0.1[2.2]$

$= 1.22$

(iii) $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + 0.1[1/2 + 0.6 + 0.61]$

$= 1.171$

(iv) $y_1^{(4)} = 1 + 0.1[1/2 + 0.6 + 0.5850]$

$= 1.1685$

(v) $y_1^{(5)} = 1 + 0.1[0.5 + 0.6 + 0.5842]$

$= 1.1684$

Modified Euler's Method

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$. We need to find y at $x_1 = x_0 + h$. We first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the first approximation and is given by $y_1 = y_0 + hf(x_0, y_0)$.

Now by modified Euler's method, the first modified value of y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)].$$

The second modified value of y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right].$$

The third modified value of y_1 is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] \text{ and so on.}$$

1. Using Modified Euler's method, find an approximate value of y when $x = 0.3$ given that $\frac{dy}{dx} = x + y$, y(0) = 1. (carry out computations correct to 5 decimal places)

Solution: We need to find $y(0.3)$ by taking $h = 0.3$.

Given $\boldsymbol{x_0 = 0}$, $\boldsymbol{y_0 = 1}$, $\boldsymbol{f(x, y) = x + y}$. $x_1 = x_0 + h = 0 + 0.3 \Rightarrow \boldsymbol{x_1 = 0.3}$.

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 1 + 0.3f(0, 1) \Rightarrow y_1 = 1 + 0.3(1) \Rightarrow \boldsymbol{y_1 = 1.3}$$

From modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \Rightarrow y_1^{(1)} = 1 + \frac{0.3}{2}[f(0, 1) + f(0.3, 1.3)]$$

$$y_1^{(1)} = 1 + \frac{0.3}{2}[1 + 1.6] \Rightarrow \boldsymbol{y_1^{(1)} = 1.39000}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] \Rightarrow y_1^{(2)} = 1 + \frac{0.3}{2}[f(0, 1) + f(0.3, 1.39)]$$

$$y_1^{(2)} = 1 + \frac{0.3}{2}[1 + 1.69] => \boldsymbol{y_1^{(2)}} = \mathbf{1.40350}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] => y_1^{(3)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.4035)]$$

$$y_1^{(3)} = 1 + \frac{0.3}{2}[1 + 1.7035] => \boldsymbol{y_1^{(3)}} = \mathbf{1.40553}$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] => y_1^{(4)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40553)]$$

$$y_1^{(4)} = 1 + \frac{0.3}{2}[1 + 1.70553] => \boldsymbol{y_1^{(4)}} = \mathbf{1.40583}$$

$$y_1^{(5)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(4)}\right)\right] => y_1^{(5)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40583)]$$

$$y_1^{(5)} = 1 + \frac{0.3}{2}[1 + 1.70583] => \boldsymbol{y_1^{(2)}} = \mathbf{1.40587}$$

$$y_1^{(6)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(5)}\right)\right] => y_1^{(6)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40587)]$$

$$y_1^{(6)} = 1 + \frac{0.3}{2}[1 + 1.70587] => \boldsymbol{y_1^{(6)}} = \mathbf{1.40588}$$

$$y_1^{(7)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(6)}\right)\right] => y_1^{(7)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40588)]$$

$$y_1^{(7)} = 1 + \frac{0.3}{2}[1 + 1.70588] => \boldsymbol{y_1^{(7)}} = \mathbf{1.40588}$$

$$\therefore\ \ y(x_0 + h) = y(0 + 0.3) = \boldsymbol{y(0.3)} = \mathbf{1.40588}$$

2. Using Modified Euler's method, find y(0.2) and y(0.4) given $y' = y + e^x$, $y(0) = 0$. (carry out computations correct to 4 decimal places)

Solution:

I Stage: We need to find $y(0.2)$ by taking $h = 0.2$.

Given $\boldsymbol{x_0} = \mathbf{0}$, $\boldsymbol{y_0} = \mathbf{0}$, $\boldsymbol{f(x, y) = y + e^x}$. $x_1 = x_0 + h = 0 + 0.2 => \boldsymbol{x_1} = \mathbf{0.2}$.

From Euler's formula, $\quad y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 0 + 0.2f(0,0) => y_1 = 0 + 0.2(1) => \boldsymbol{y_1} = \mathbf{0.2}$$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \Rightarrow y_1^{(1)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2)]$$

$$y_1^{(1)} = 0 + (0.1)[1 + 1.4214] \Rightarrow \boldsymbol{y_1^{(1)} = 0.2421}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] \Rightarrow y_1^{(2)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2421)]$$

$$y_1^{(2)} = 0 + (0.1)[1 + 1.4635] \Rightarrow \boldsymbol{y_1^{(2)} = 0.2464}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] \Rightarrow y_1^{(3)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2464)]$$

$$y_1^{(3)} = 0 + (0.1)[1 + 1.4678] \Rightarrow \boldsymbol{y_1^{(3)} = 0.2468}$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] \Rightarrow y_1^{(4)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2468)]$$

$$y_1^{(4)} = 0 + (0.1)[1 + 1.4682] \Rightarrow \boldsymbol{y_1^{(4)} = 0.2468}$$

$$\therefore\ y(x_0 + h) = y(0 + 0.2) = \boldsymbol{y(0.2) = 0.2468}$$

II Stage: We need to find $y(0.4)$ using $y(0.2) = 0.2468$ as the initial condition and taking $h = 0.2$. Now $\boldsymbol{x_0 = 0.2}$, $\boldsymbol{y_0 = 0.2468}$, $\boldsymbol{f(x, y) = y + e^x}$.

$$x_1 = x_0 + h = 0.2 + 0.2 \Rightarrow \boldsymbol{x_1 = 0.4}.$$

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 0.2468 + 0.2f(0.2, 0.2468) \Rightarrow y_1 = 0.2468 + 0.2(1.4682) \Rightarrow \boldsymbol{y_1 = 0.5404}$$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$\Rightarrow y_1^{(1)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.5404)]$$

$$\Rightarrow y_1^{(1)} = 0.2468 + (0.1)[1.4682 + 2.0322] \Rightarrow \boldsymbol{y_1^{(1)} = 0.5968}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$\Rightarrow y_1^{(2)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.5968)]$$

$$\Rightarrow y_1^{(2)} = 0.2468 + (0.1)[1.4682 + 2.0886] \Rightarrow \boldsymbol{y_1^{(2)} = 0.6025}$$