Introduction

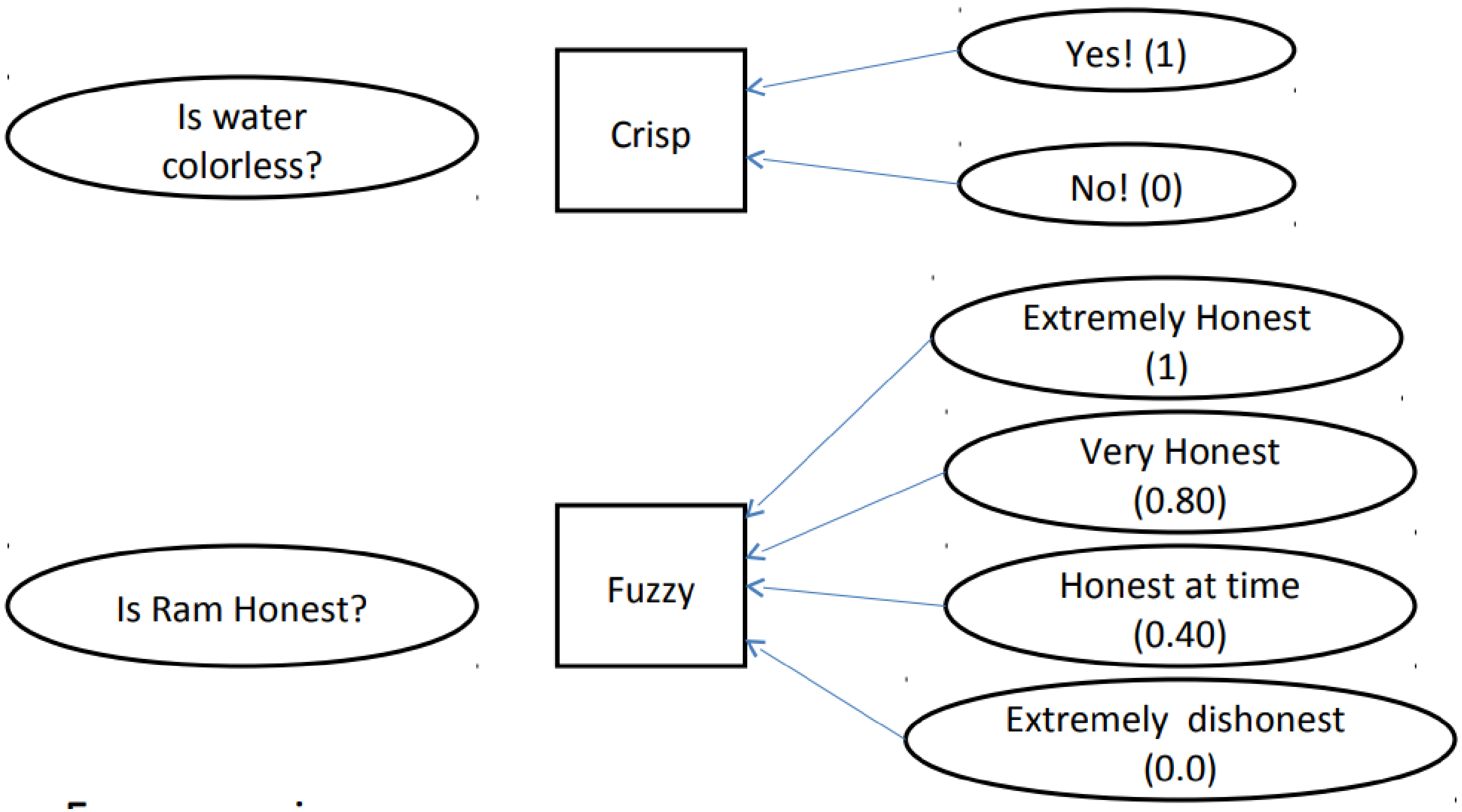
Fuzzy Set Theory

- Fuzzy set theory is a generalization of the classical set theory so that the classical set theory is a special case of the fuzzy set theory.
- It takes into consideration the natural vagueness that we the human beings deal with in our practical, daily, life.
- The word “fuzzy” means “vaguness (ambiguity)”.
- Fuzziness occurs when the boundary of a piece of information is not clear-cut.
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Classical set theory allows the membership of the elements in the set in binary terms.
- Fuzzy set theory permits membership function valued in the interval [0,1].

Fuzzy Set Theory

- Example: Words like young, tall, good or high are fuzzy.
- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Example



Is water colorless?
Crisp
Yes! (1)
No! (0)
Is Ram Honest?
Fuzzy
Extremely Honest (1)
Very Honest (0.80)
Honest at time (0.40)
Extremely dishonest (0.0)

Fuzzy Set vs Crisp Set

BASIS FOR COMPARISON	FUZZY SET	CRISP SET
Basic	Prescribed by vague or ambiguous properties.	Defined by precise and certain characteristics.
Property	Elements are allowed to be partially included in the set.	Element is either the member of a set or not.
Applications	Used in fuzzy controllers	Digital design
Logic	Infinite-valued	bi-valued

Example

#	Family member	Age	Gender
1	Grand-pa	72	Male
2	Grand-ma	63	Female
3	Dad	41	Male
4	Mom	38	Female
5	Daughter	15	Female
6	Son	13	Male
7	Aunty	52	Female

Table 1

Example

- The crisp set of the family members U = {*Grand-pa*, *Grand-ma*, *Dad*, *Mom*, *Sister*, *Brother*, *Aunt*} may be treated as the *reference set*, or the *universe of discourse*.
- Now, consider the sets M and F of the male family members and female family members respectively.
- These are crisp sets because for any arbitrary element x of U, it is possible to decide precisely whether x is member of the set, or not.
- There is no intermediate status regarding the membership of x to the respective set.
- However, deciding the membership of an arbitrary $x \in U$ to the set of *senior* persons of the family is not as straightforward as in case of M or F.
- The status of membership of an element x with respect to a given set S is expressed with the help of a *membership function m*.
- A set, crisp or fuzzy, may be defined in terms of membership function.

Membership of a Set

- Description of a set in terms of its membership function is presented:
- *Membership Function* of a Crisp Set :
  Given an element *x* and a set *S*, the membership of *x* with respect to *S*, denoted as *m S* (*x*), is defined as *mS* (*x) is defined as* –

$$\mu_S(x) = 1, \quad \text{if } x \in S$$
$$= 0, \quad \text{if } x \notin S$$

Membership of a Set

- Let us consider the set *M* of male family members and set *F* of female family members with reference to the family presented in Table 1.
- We see that $\mu_M(Dad) = 1$, and $\mu_M(Mom) = 0$.
- Similarly, $\mu_F(Dad) = 0$, and $\mu_F(Mom) = 1$.
- Membership values of the other family members in *M* and *F* can be ascertained in similar manner.

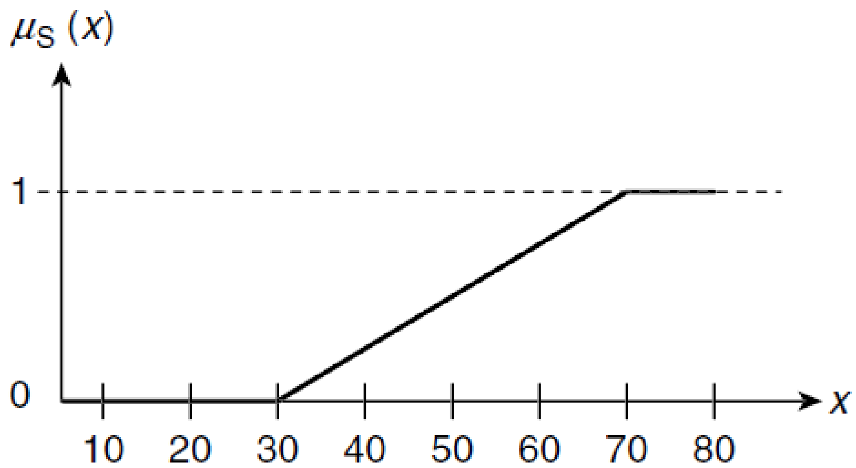
Membership of a Set

- Now, consider *A* to be the set of *senior* persons in the family.
- *Seniority* is a familiar and frequently used attribute to a person.
- But is there any clear and unambiguous way to decide whether a person
- should be categorized as senior or not?
- Let us see with reference to *U*, the universe of discourse.
- We may agree without any hesitation that Grand-pa, being 72 years old, is a senior person, so that *Grand-pa* $\in$ *A.*
- On the other hand the brother and the sister are both too young to be categorized as senior persons. Therefore, we may readily accept that $Daughter \notin A$ and $Son \notin A$. What about *Mom*, or *Dad*?
- They are not as aged as Grand-pa but neither as young as the daughter or the son. Moreover, Grand-ma is almost a senior person, being at 63 years, but she might be categorized as a middle-aged person as well.

Membership of a Set

- The point is, the concept of a senior person is not as clearly defined as the gender of the person.
- In fact, there is a whole range of gray area between total inclusion and total exclusion, over which the degree of membership of a person in the set of senior persons varies.
- This intuitive notion of partial membership of an element in a set can be formalized if one allows the membership function to assume any real value between 0 and 1, including both.
- This means that an element may belong to a set to any extent within the range [0, 1].
- Hence it is now possible for an element x to be 0.5 member, or $1/\sqrt{2}$ member of a set S so that we may say $m\,S(x) = 0.5$, or $m\,S(x) = 1/\sqrt{2}$.

Membership of a Set



Membership function for the fuzzy set of senior family members

Membership Profile

- **Membership profiles/functions** Quite often it is convenient to express the membership of various elements with respect to a fuzzy set with the help of a function, referred to as the **membership function**.
- For example the fuzzy set *A* of senior persons on the universe of discourse *U* described in Table 1.
- For any $x \in U$, we may determine the membership value of *x* in *A* with the help of the following membership function.
- $$\mu_A(x) = \begin{cases} 0, & \text{if} \quad x < 30 \\ \dfrac{x-30}{40}, & \text{if} \quad 30 \le x < 70 \\ 1, & \text{if} \quad x \ge 70 \end{cases} \text{---------------------(i)}$$

- Here *x* represents the age of the concerned person.

Example

- Formula (i) may be applied to find to what extent a person is a member of the fuzzy set *A* of senior persons.
- For example, as Grand-pa is more than 70 years old, he is a full member of *A*, μ_A (*Grand-pa*) = 1.0.
- However for Grand-ma we have μ_A (*Grand-ma*) = (63 – 30) / 40 = 33 / 40 = 0.825.
- Hence Grand-ma is a senior person to a large extent, but not as fully as Grand-pa.
- Table 2 shows the membership values of all the family members.

Example

#	Family member	Age	$\mu_A(x)$
1	Grand-pa	72	1.0
2	Grand-ma	63	0.825
3	Dad	41	0.275
4	Mom	38	0.200
5	Daughter	15	0.0
6	Son	13	0.0
7	Aunty	52	0.55

Table 2

Fuzzy Set

- A fuzzy set F on a given universe of discourse U is defined as a collectionof ordered pairs $(x, \mu F(x))$ where x U, and for all $x \in U$, $0.0 \leq \mu F(x) \leq 1.0$.
- $F = \{(x, \mu F(x)) \mid x \in U, 0.0 \leq \mu F(x) \leq 1.0\}$
- Classical sets, often referred to as *crisp* sets to distinguish them from fuzzy sets, are special cases of fuzzy sets where the membership values are restricted to either 0, or 1. Each pair $(x, \mu F(x))$ of the fuzzy set F is known as a **singleton**.
- Fuzzy sets are frequently expressed as the union of all singletons where a singleton is denoted as $\mu F(x) / x$. Using this notation
- $$F = \sum_{x \in U} \mu_F(x)/x \qquad \text{----------------------------(i)}$$
- Here the summation sign Σ is to be interpreted as union over all singletons, and not arithmetic sum. Formula (ii) is appropriate for discrete sets. For continuous sets, the summation notation is replaced by the integral sign $\int$, as shown below.

Fuzzy Set Example 1

Let a, b, c, d, and e be five students who scored 55, 35, 60, 85 and 75 out of 100 respectively in Mathematics. The students constitute the universe of discourse $U = \{a, b, c, d, e\}$ and a fuzzy set M of the students who are *good in Mathematics* is defined on U with the help of the following membership function.

$$\mu_M(x) = \begin{cases} 0, & \text{if} \quad x < 40 \\ \dfrac{x-40}{40}, & \text{if} \quad 40 \le x < 80 \\ 1, & \text{if} \quad x \ge 80 \end{cases} \tag{2.4}$$

The membership function is graphically shown in Fig. 2.8. Computing the membership value of each student with the help of the Formula 2.4 we get $M = \{(a, 0.375), (c, 0.5), (d, 1.0), (e, 0.875)\}$, or equivalently

$$M = \frac{0.375}{a} + \frac{0.5}{c} + \frac{1.0}{d} + \frac{0.875}{e}$$

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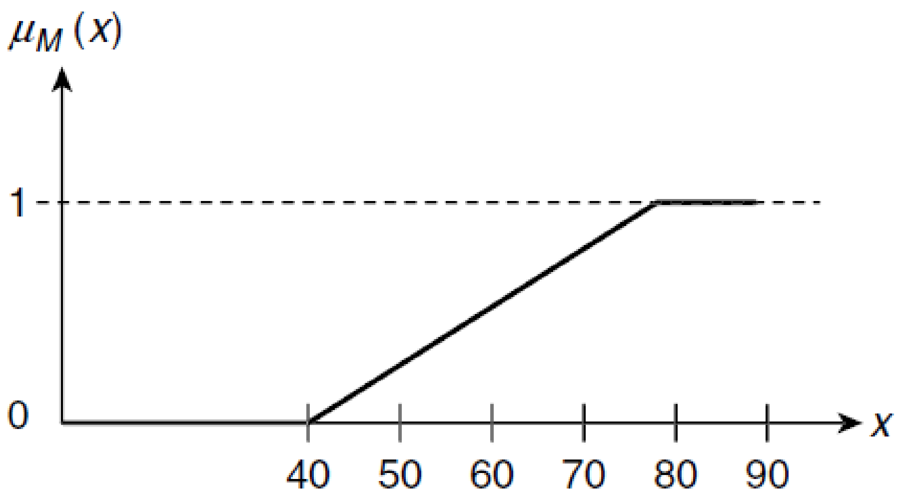
Fuzzy Set Example



. *Membership function for students good in Mathematics.*

Fuzzy Set Example 2

Let us consider the infinite set of all real numbers between 0 and 1, both inclusive, to be the universe of discourse, or, the reference set, $U = [\ 0, 1\ ]$. We define a fuzzy set $C0.5$ as the set of all real numbers in U that are *close* to 0.5 in the following way

$$C0.5 = \{x \in [\ 0, 1\ ] \mid x \text{ is } \textit{close} \text{ to } 0.5\}$$

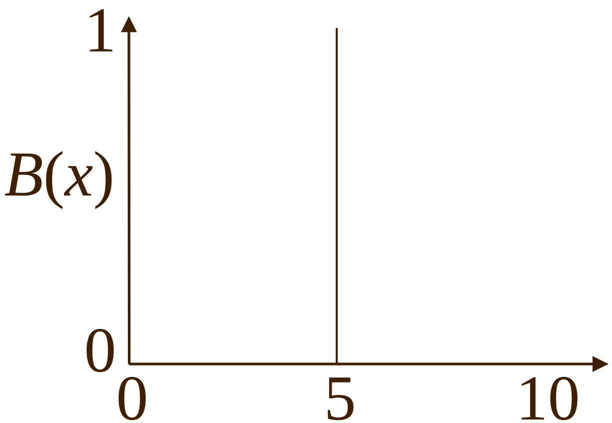
The highest membership value would be attained by the point $x = 0.5$ because that is the number closest to 0.5, and the membership is understandably 1. On the other hand since both 0 and 1 are furthest points from 0.5 (within the interval $[\ 0, 1\ ]$) they should have zero membership to $C0.5$. Membership values should increase progressively as we approach 0.5 from both ends of the interval $[\ 0, 1\ ]$. The membership function for $C0.5$ may be defined in the following manner.

$$\mu_{C0.5}(x) = 1 - |\ 2x - 1\ |, \qquad \forall x \in [\ 0, 1\ ] \tag{2.5}$$

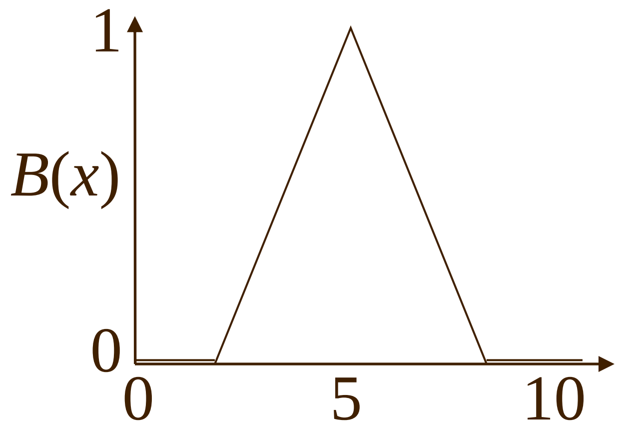
Since $C0.5$ is a fuzzy set in a continuous domain it can be expressed with the help of the notation

$$C0.5 = \int_{x\in[0,1]} \frac{\mu_{c0.5}(x)}{x} = \int_{x\in[0,1]} \frac{1-|2x-1|}{x} \tag{2.6}$$

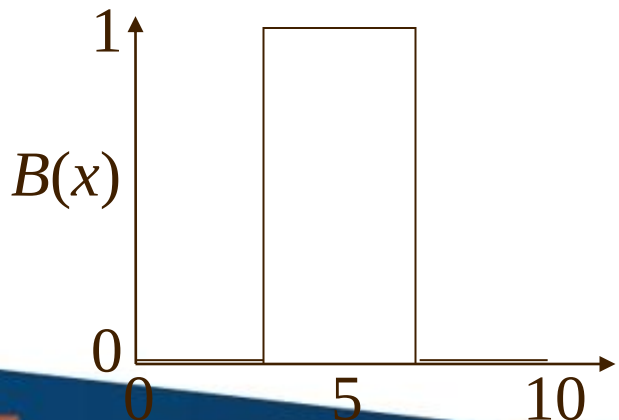
Types of Uncertainty



Certainty

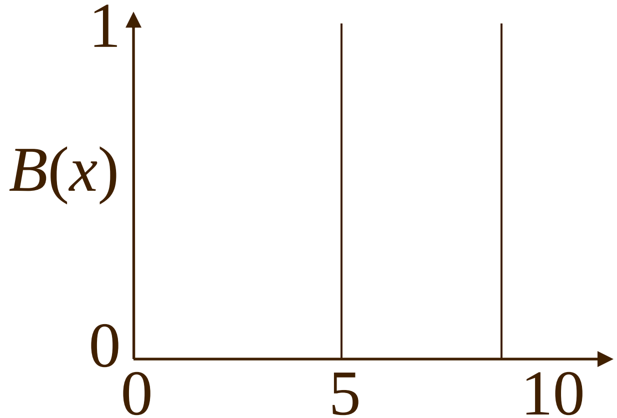


Vagueness

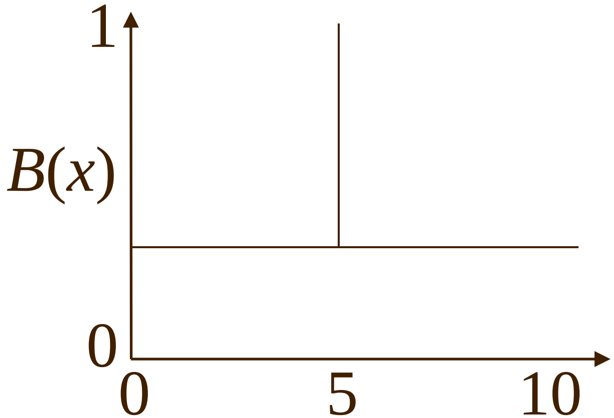


Imprecision

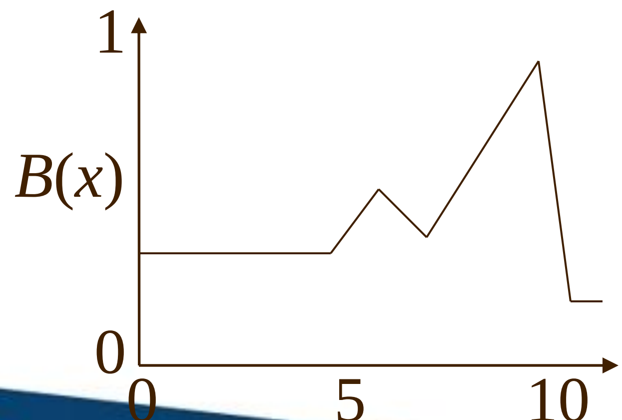
Types of Uncertainty



Ambiguity



Unreliability



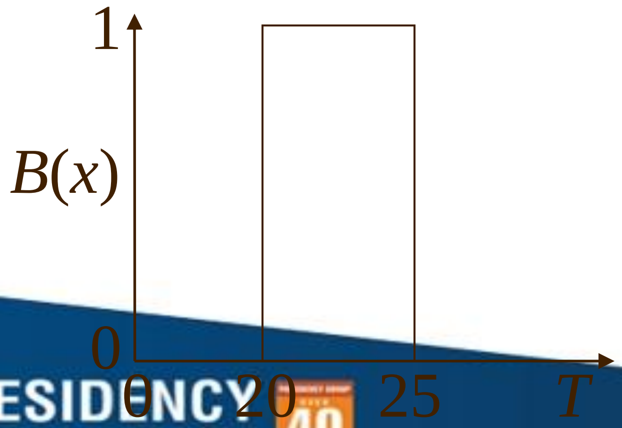
Combined

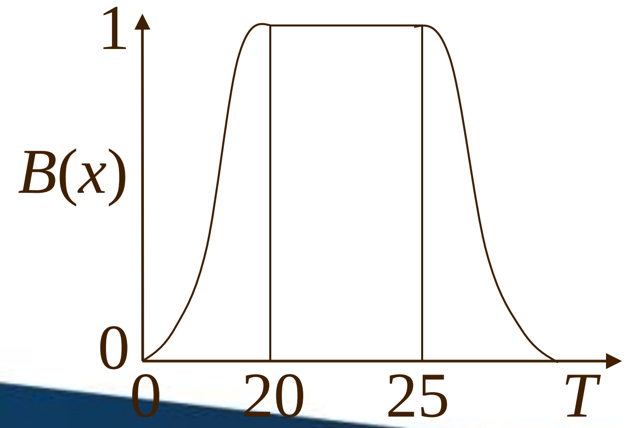
What is Fuzzy Logic

- It is a generalization of classical logic which manifests in crisp quantities.
- It represents concepts with unclear boundaries.
- Crisp logic deals with crisp sets having sharp boundaries. The inherent logic is Boolean in nature (i.e. either TRUE or FALSE).
- Fuzzy logic deals with fuzzy sets having indistinct boundaries. The inherent logic is multivalued in nature.

What is Fuzzy Logic

- Consider the temperature on a sunny day.
- It can either be represented in terms of temperature values e.g. 20^0-25^0C.
- It can also be represented as a “warm” day with temperature, $T \in [20^0\ 25^0]$.





Definitions

- Fuzziness measures the degree to which an event occurs, not whether it occurs.

- It exists when the law of non-contradiction [$A \cap \tilde{A} = \Phi$] (or the law of excluded middle [$A \cup \tilde{A} = U$]) is violated.

- Fuzziness primarily describes partial truth or imprecision.

Fuzzy Sets

- A fuzzy set is a set of ordered pairs.

$$\underline{A} = \{(x, \mu_{\underline{A}}(x)) \mid x \in X, \mu_{\underline{A}}(x) \in [0,1]\}$$

Here X is the universe of discourse. It is given by

$$X = \{x_1, x_2, x_{3,}, \ldots, x_n\}$$

According to Zadeh's notation:

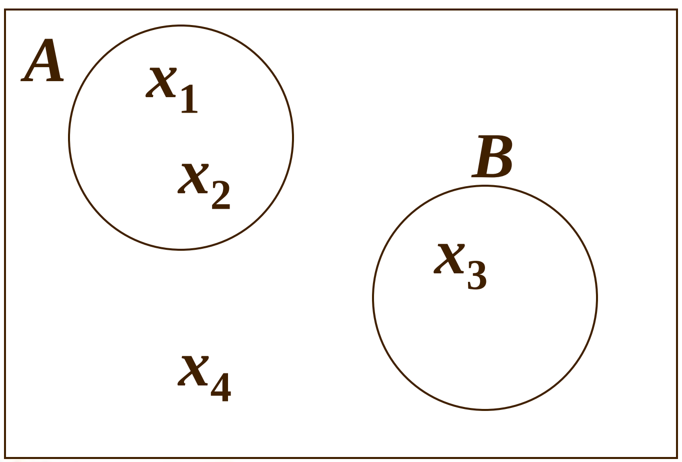
$$\underline{A} = \{(\frac{\mu_{\underline{A}}(x_i)}{x_i}) \mid x_i \in X\}$$

$$or \quad \underline{A} = \frac{\mu_{\underline{A}}(x_1)}{x_1} + \frac{\mu_{\underline{A}}(x_2)}{x_2} + \frac{\mu_{\underline{A}}(x_3)}{x_3} + \ldots\ldots + \frac{\mu_{\underline{A}}(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_{\underline{A}}(x_i)}{x_i}$$

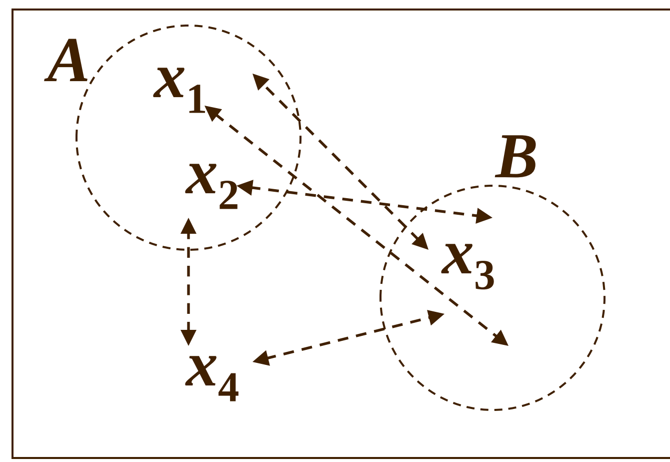
Fuzzy Sets

- If X is continuous, $\underline{A}(x) = \int_X \frac{\mu_{\underline{A}}(x)}{x}$

$\sum$ $\int$ represent the union of membership degrees.

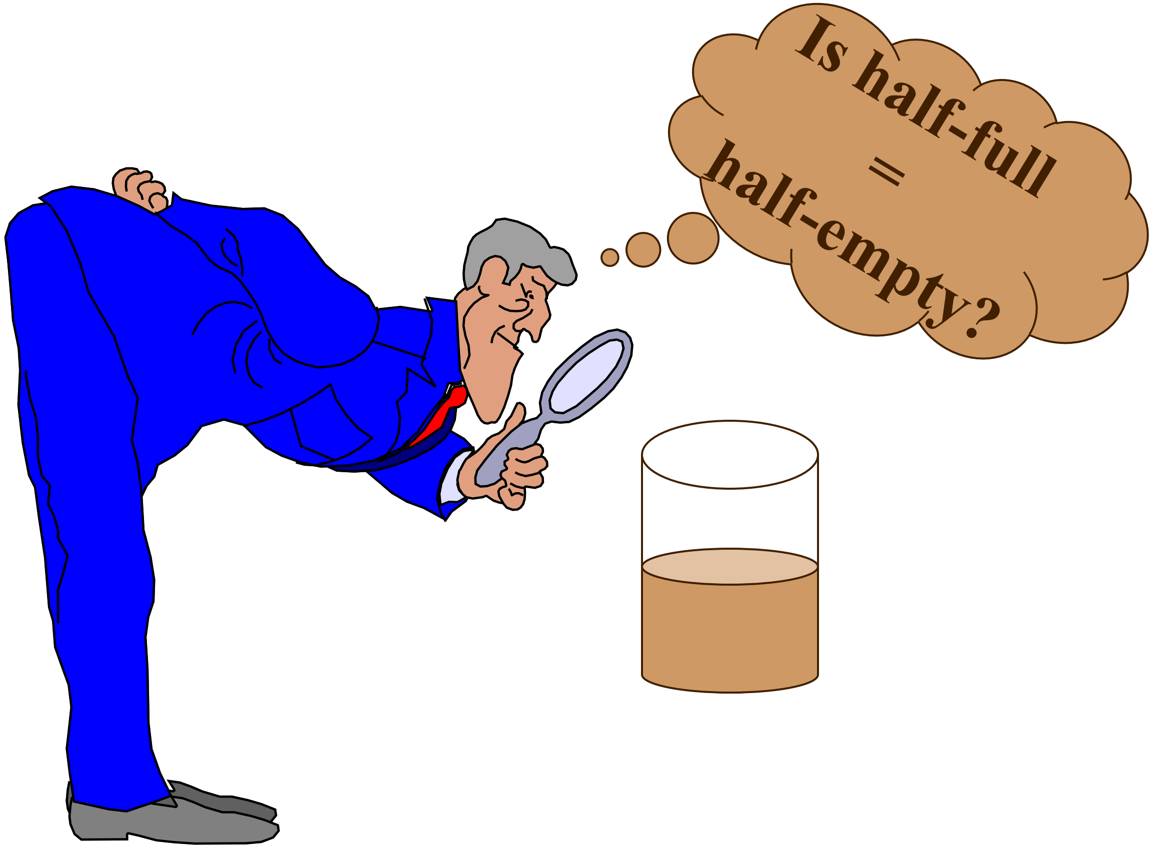


Crisp Set



Fuzzy Set

Fuzzy Sets - Classic Example



Fuzzy Sets - Classic Example

μ = 0.5 **p = 0.5**

Fuzzy Logic and probability are not the same thing.

Fuzzyness vs Probability

The membership of a fuzzy set relate to *vagueness* and not to probability which is a measure of *uncertainty*.

Sr.No.	**Fuzzy Logic**	**Probability**
1.	Here, everything is a matter of degree.	It is specific within the range between 0 and 1.
2.	It is based upon natural language processing.	Not used for high approximations.
3.	The Best suitable for approximation use cases.	It captures partial knowledge.
4.	Readily integrable with programming.	Deals with likelihood.
5.	Used by Quantitative Analysts too for the improvisation of their algorithms.	Not capable of capturing any type of uncertainties.

Fuzzyness vs Probability

- Differences are as follows:

Sr. No.	Fuzzy Logic	Probability
1	Trying to understand the concept of vagueness.	The main association is with events and to Check whether the events will occur or not.
2	This captures the meaning of partial truth.	This captures partial knowledge.
3	The degree of membership is in a set.	The probability event is in a set.

Fuzzyness vs Probability : Example

Example 1: There is a chance of 95% rain this evening.
Example 2: It is going to rain very heavily in the evening today.
So here in the above examples, we can see that example 1 says that there is a very well chance of raining in the evening since it is strictly mentioned as 95% is the probability of the same. Whereas if we look through the second example, there is fuzziness in it, and we get to know that it is going to rain heavily, but it is not very clear how much heavy it is going to be. Here the 'heavy' term can be subjective depending upon the person interpreting it.

Features of Fuzzy Set

- Example: Words like young, tall, good or high are fuzzy.
- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Fuzzy Set Theory

- Fuzzy sets are often characterized with certain features, *e.g.*, *normality*, *height*, *support*, *core*, *cardinality* etc.
- These features reveal the nature and structure of a fuzzy set.
- *Normality*: A fuzzy set *F* is said to be normal if there exists an element *x* that
- completely belongs to *F*, $\mu_F(x) = 1$. A fuzzy set which is not normal is said to be sub-normal.
- *Height*: The height of a fuzzy set is defined as the maximal membership value
- attained by its elements.

$$height\,(F) = \underset{x \in U}{Max}\, \mu_F(x)$$

- *U* is the universe of discourse, or, the reference set, for *F*, and *height* (*F*) is the height of the fuzzy set *F*. Obviously, *F* is a normal fuzzy set if *height* (*F*) = 1.

Fuzzy Set Theory

- *Support*: Th e support of a fuzzy set *F*, denoted by *supp* (*F*), is the set of all elements of the reference set *U* with non-zero membership to *F*

$$Supp\ (F) = \{x \mid x \in U, \text{and } \mu_F(x) > 0\}$$

- *Core*: The *core* of a fuzzy set *F* is the set of all elements of the reference set *U* with complete membership to *F*.

$$Core\ (F) = \{x \mid x \in U, \text{and } \mu_F(x) = 1\}$$

- Both *supp* (*F*) and *core* (*F*) are crisp sets. It is easy to see that *core* (*F*) ⊆ *supp* (*F*).
- *Cardinality*: The sum of all membership values of the members of a fuzzy set *F* is said to be the cardinality of *F*.

$$|F| = \sum_{x \in U} \mu_F(x)$$

Fuzzy Set Theory : Example

Let us consider the fuzzy set M defined on the reference set $U = \{a, b, c, d, e\}$ as described in Example 2.8.

$$M = \frac{0.375}{a} + \frac{0.5}{c} + \frac{1.0}{d} + \frac{0.875}{e}$$

The fuzzy set M is normal, because we have $\mu_M(d) = 1.0$. Its height is 1.0. Moreover, as we see, *supp* $(M) = \{a, c, d, e\}$, *core* $(M) = \{d\}$, and the cardinality of M is $|M| = 0.375 + 0.5 + 1.0 + 0.875 = 2.750$.

Operations on Fuzzy Set

- Most of the usual set theoretic operations e.g., union, intersection, complementation etc. are readily extended to fuzzy sets.

Operation/Relation	Description
Union ($P \cup Q$)	$\mu_{P \cup Q}(x) = max\ \{\mu_P(x), \mu_Q(x)\}, \forall x \in U$
Intersection ($P \cap Q$)	$\mu_{P \cap Q}(x) = min\ \{\mu_P(x), \mu_Q(x)\}, \forall x \in U$
Complementation (P')	$\mu_{P'}(x) = 1 - \mu_P(x), \forall x \in U$
Equality ($P = Q$)	Two fuzzy sets P and Q are equal if and only if $\forall x \in U, \mu_P(x) = \mu_Q(x)$
Inclusion ($P \subseteq Q$)	P is included in Q, i.e., P is a subset of Q, written as $P \subseteq Q$, if and only if $\forall x \in U, \mu_P(x) \leq \mu_Q(x)$
Product ($P \cdot Q$)	$\mu_{P.Q}(x) = \mu_P(x) \times \mu_Q(x)\}, \forall x \in U$
Difference ($P - Q$)	$P - Q = P \cap Q'$
Disjunctive sum ($P \oplus Q$)	$P \oplus Q = (P \cap Q') \cup (P' \cap Q)$

Example

- Example of family set

#	Operation		Resultant fuzzy set
1.	*A* ∪ *B*	*senior* OR *active*	{(*Grand-pa*, 1.0), (*Grand-ma*, 0.825), (*Dad*, 1.0), (*Mom*, 1.0), (*Daughter*, 0.333), (*Son*, 0.2), (*Aunty*, 0.933)}
2.	*A* ∩ *B*	*senior* AND *Active*	{(*Grand-pa*, 0.267), (*Grand-ma*, 0.567), (*Dad*, 0.275), (*Mom*, 0.2), (*Aunty*, 0.55)}
3.	*A′*, *B′*	NOT *senior*, NOT *active*	A′ = {(*Grand-ma*, 0.175), (*Dad*, 0.725), (*Mom*, 0.8), (*Sister*, 1.0), (*Brother*, 1.0), (*Aunty*, 0.45)} B′ = {(*Grand-pa*, 0.733), (*Grand-ma*, 0.433), (*Sister*, 0.667), (*Brother*, 0.8), (*Aunty*, 0.067)}

Properties of Fuzzy Set Operations

#	Law	Description
1	Associativity	(a) $(P \cup Q) \cup R = P \cup (Q \cup R)$ (b) $(P \cap Q) \cap R = P \cap (Q \cap R)$
2	Commutativity	(a) $P \cup Q = Q \cup P$ (b) $P \cap Q = Q \cap P$
3	Distributivity	(a) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$ (b) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$
4	Idempotency	(a) $P \cup P = P$ (b) $P \cap P = P$
5	De Morgan's law	(a) $(P \cup Q)' = P' \cap Q'$ (b) $(P \cap Q)' = P' \cup Q'$
6	Boundary Conditions	(a) $P \cup \Phi = P$, $P \cup U = U$ (b) $P \cap \Phi = \Phi$, $P \cap U = P$
7	Involution	$(P')' = P$
8	Transitivity	If $P \subseteq Q$ and $Q \subseteq R$ then $P \subseteq R$

Binary Fuzzy Relations

- Binary Fuzzy relations map elements of one universe, *X*, to those of another universe, *Y*, through the Cartesian product of the two universes
- This is also referred to as fuzzy sets defined on universal sets, which are Cartesian product
- A fuzzy relation is based on the concept that *everything* is related to *some* extent or unrelated

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Fuzzy Relations

- A fuzzy relation is a fuzzy set defined on the Cartesian product of classical sets $\{X_1, X_2, \ldots, X_n\}$ where n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degree of membership $\mu_{\underset{\sim}{R}}(x_1, x_2, \ldots, x_n)$ within the relation i.e.

$$\underset{\sim}{R}(X_1, X_2, \ldots X_n) = \int_{X_1, X_2, \ldots X_n} \frac{\mu_{\underset{\sim}{R}}(x_1, x_2, \ldots x_n)}{(x_1, x_2, \ldots x_n)}$$

Fuzzy Relations

- A fuzzy relation between two sets $\underline{X}$ & $\underline{Y}$ is called binary fuzzy relation & is denoted by:

$$\underset{\sim}{R}\left(\underline{X},\underline{Y}\right)$$

- A binary relation $\underset{\sim}{R}\left(\underline{X},\underline{Y}\right)$ is referred to as ***bipartite*** graph when $\underline{X} \neq \underline{Y}$
- The binary relation defined on a single set X is called ***directed*** graph or ***digraph***. This occurs when $\underline{X} = \underline{Y}$ & is denoted by $\underset{\sim}{R}\left(\underline{X},\underline{X}\right)$or $\underset{\sim}{R}\left(\underline{X}^{2}\right)$

Fuzzy Relations

Let $\underline{X} = \{x_1, x_2, \ldots, x_n\}$, $\underline{Y} = \{y_1, y_2, \ldots, y_m\}$. The fuzzy relation $\underset{\sim}{R}(\underline{X}, \underline{Y})$ can be expressed by $n \times m$ matrix called ***Fuzzy Matrix*** denoted as:

$$\underset{\sim}{R}(\underline{X}, \underline{Y}) = \begin{bmatrix} \mu_{\underline{R}}(x_1, y_1) & \mu_{\underline{R}}(x_1, y_2) & \ldots & \mu_{\underline{R}}(x_1, y_m) \\ \mu_{\underline{R}}(x_2, y_1) & \mu_{\underline{R}}(x_2, y_2) & \ldots & \mu_{\underline{R}}(x_2, y_m) \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{\underline{R}}(x_n, y_1) & \mu_{\underline{R}}(x_n, y_2) & \ldots & \mu_{\underline{R}}(x_n, y_m) \end{bmatrix}$$

Fuzzy Relations

A fuzzy relation is mapping from Cartesian space ($\underline{X}$, $\underline{Y}$) to the interval [0, 1] where the mapping strength is expressed by the membership function of the relation for ordered pairs from the two universes $\mu_{\underset{\sim}{R}}(x, y)$

A fuzzy graph is the graphical representation of a binary fuzzy relation

Domain & Range

The domain of the fuzzy relation $\underset{\sim}{R}(\underline{X}, \underline{Y})$ is the fuzzy set, $\text{dom}\, \underset{\sim}{R}(\underline{X}, \underline{Y})$ having the membership function as:

$$\mu_{\text{dom}\, \underset{\sim}{R}}(x) = \max_{y \in Y} \mu_{\underset{\sim}{R}}(x, y), \forall x \in X$$

The range of the fuzzy relation $\underset{\sim}{R}(\underline{X}, \underline{Y})$ is the fuzzy set, $\text{Range}\, \underset{\sim}{R}(\underline{X}, \underline{Y})$ having the membership function as:

$$\mu_{\text{Range}\, \underset{\sim}{R}}(y) = \max_{x \in X} \mu_{\underset{\sim}{R}}(x, y), \forall y \in Y$$

Example

Consider a universe $\underline{X} = \{x_1, x_2, x_3, x_4\}$ & a binary fuzzy relation $\underline{R}$ as:

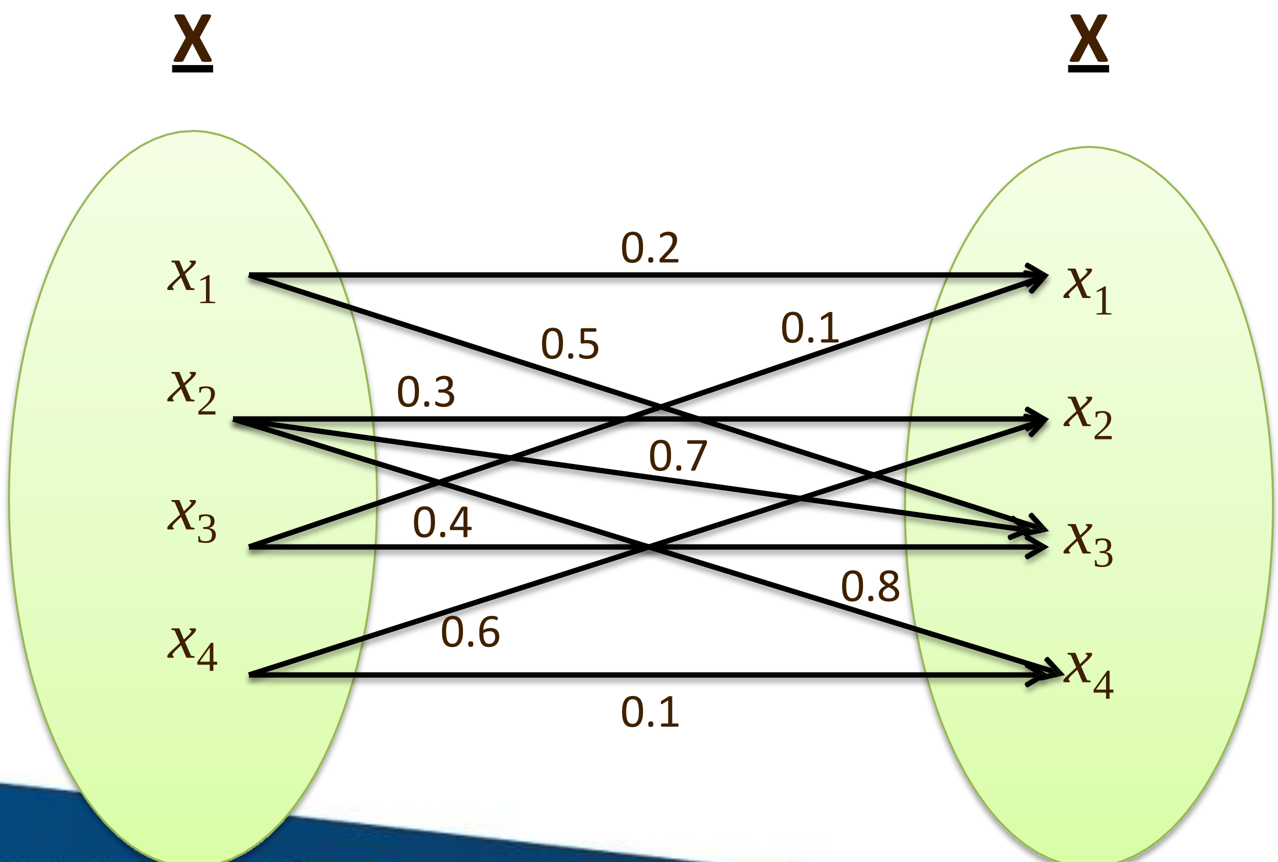
$$\underset{\sim}{R}(\underline{X}, \underline{X}) = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.1 \end{bmatrix} \end{array}$$

Example

Domain = {0.5, 0.8, 0.4, 0.6} (Take max on rows)
&

Range = {0.2, 0.6, 0.7, 0.8}
(Take max on columns)

Sagittal Diagram



Operations on Fuzzy Relations

Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space $X \times Y$. Then the following operations apply for the membership values for various set operations:

1. *Union:* $\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$

2. *Intersection:* $\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min[\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y)]$

3. *Complement:* $\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$

Operations on Fuzzy Relations

4. Containment: $\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$

- All properties like commutativity, associativity, distributivity, involution, idempotency, De'Morgan's laws also hold for fuzzy relation as they do for crisp relations.
- However, the law of excluded middle and law of contradiction does not hold good for fuzzy relations (as for fuzzy sets)

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq \underset{\sim}{E} \text{ and } \underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \underset{\sim}{O}$$

Fuzzy Cartesian Product

- Fuzzy relations are in general fuzzy sets
- We can define Cartesian product as a relation between two or more fuzzy sets
- Let $\underset{\sim}{A}$ & $\underset{\sim}{B}$ be two fuzzy sets defined on the universes $\underset{\sim}{X}$ & $\underset{\sim}{Y}$, then the Cartesian product between $\underset{\sim}{A}$ & $\underset{\sim}{B}$ will result in fuzzy relation $\underset{\sim}{R}$ which is contained in full Cartesian product space

Fuzzy Cartesian Product

i. e. $$\underline{A} \times \underline{B} = \underset{\sim}{R} \subset \underline{X} \times \underline{Y}$$

Where, the fuzzy relation $\underset{\sim}{R}$ has membership functi on

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underline{A} \times \underline{B}}(x, y) = \min[\mu_{\underline{A}}(x), \mu_{\underline{B}}(y)]$$

The Cartesian product defined by $\underline{A} \times \underline{B} = \underset{\sim}{R}$ is implemented in the same fashion as the cross product of two vectors

Again, the Cartesian product is ***not the same*** operation as the arithmetic product.

Fuzzy Relations

- Fuzzy relations also map elements of one universe, say X, to those of another universe, say Y, through the Cartesian product of the two universes. However, the "strength" of the relation between ordered pairs of the two universes is not measured with the characteristic function, but rather with a membership function expressing various "degrees" of strength of the relation on the unit interval [0,1] $\underset{\sim}{R}$
- Hence, a fuzzy relation is a mapping from the Cartesian space $X \times Y$ to the interval [0,1], where the strength of the mapping is expressed by the membership function of the $\mu_{\underset{\sim}{R}}(x, y)$. for ordered pairs from the two universes, or

Cardinality of Fuzzy Relations

- Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

Operations on Fuzzy Relations

Union $\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$

Intersection $\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$

Complement $\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$

Containment $\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$

Properties of Fuzzy Relations

- commutativity, associativity, distributivity, involution, idempotency and De Morgan's principles all hold for fuzzy relations.
- the null relation, **O**, and the complete relation, **E**, are analogous to the null set and the whole set in set-theoretic form.
- excluded middle axioms doesn't hold.

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq \mathbf{E}$$

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \mathbf{O}$$

Fuzzy Cartesian Product and composition

Let $\underset{\sim}{A}$ be a fuzzy set on universe X and $\underset{\sim}{B}$ be a

fuzzy set on universe Y; then the Cartesian product between fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will result in a fuzzy relation $\underset{\sim}{R}$, which is contained within the full Cartesian product space, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y \qquad (3.15)$$

where the fuzzy relation $\underset{\sim}{R}$ has membership function

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)$$

Example 3.5. Suppose we have two fuzzy sets, $\underset{\sim}{A}$ defined on a universe of three discrete temperatures, $X = \{x_1, x_2, x_3\}$, and $\underset{\sim}{B}$ defined on a universe of two discrete pressures, $Y = \{y_1, y_2\}$, and we want to find the fuzzy Cartesian product between them. Fuzzy set $\underset{\sim}{A}$ could represent the "ambient" temperature and fuzzy set $\underset{\sim}{B}$ the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature–pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \quad \text{and} \quad \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$$

Note that $\underset{\sim}{A}$ can be represented as a column vector of size 3×1 and $\underset{\sim}{B}$ can be represented by a row vector of 1×2. Then the fuzzy Cartesian product, using Eq. (3.16), results in a fuzzy relation $\underset{\sim}{R}$ (of size 3×2) representing "efficient" conditions, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy composition

- fuzzy max–min composition

$$\underset{\sim}{\mathrm{T}} = \underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}}$$

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \wedge \mu_{\underset{\sim}{\mathrm{S}}}(y, z))$$

- fuzzy max–product composition

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \bullet \mu_{\underset{\sim}{\mathrm{S}}}(y, z))$$

It should be pointed out that neither crisp nor fuzzy compositions are commutative in general; that is,

$$\underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}} \neq \underset{\sim}{\mathrm{S}} \circ \underset{\sim}{\mathrm{R}} \tag{3.18}$$

Example

$$\mu_{S \circ R}(x, y) = \max_v \min\left(\mu_R(x, v), \mu_S(v, y)\right)$$

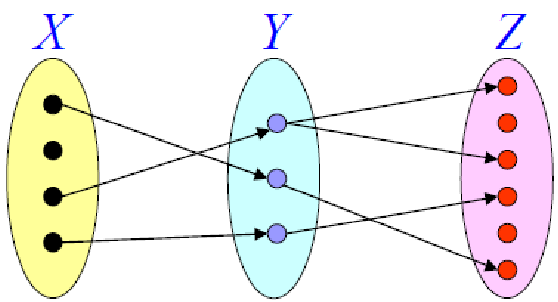
R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

	0.1	0.2	0.0	1.0
min	0.9	0.2	0.8	0.4
max	0.1	0.2	0.0	0.4

$R \circ S$	α	β	γ
1	$\boxed{0.4}$	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Max-Min Composition



R: fuzzy relation defined on X and Y.

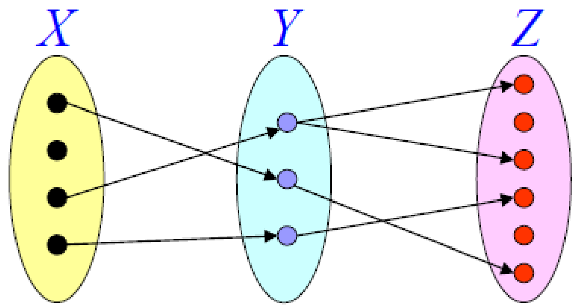
S: fuzzy relation defined on Y and Z.

$\underbrace{R \circ S}$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R \circ S}(x, z) = \max_y \min\left(\mu_R(x, y), \mu_S(y, z)\right)$$

Max-Product Composition



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$\underbrace{R \circ S}$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R\circ S}(x,y) = \max_v \left(\mu_R(x,v)\mu_S(v,y) \right)$$

Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested.

Example 3.6. Let us extend the information contained in the Sagittal diagram shown in Fig. 3.4 to include fuzzy relationships for X × Y (denoted by the fuzzy relation $\underset{\sim}{R}$) and Y × Z (denoted by the fuzzy relation $\underset{\sim}{S}$). In this case we change the elements of the universes to,

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Then the resulting relation, $\underset{\sim}{T}$, which relates elements of universe X to elements of universe Z, i.e., defined on Cartesian space X × Z, can be found by max–min composition, Eq. (3.17a), to be, for example,

$$\mu_{\underset{\sim}{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

and the rest,

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

and by max–product composition, Eq. (3.17b), to be, for example,

$$\mu_{\underset{\sim}{T}}(x_2, z_2) = \max[(0.8 \cdot 0.6), (0.4 \cdot 0.7)] = 0.48$$

and the rest,

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

Example 3.7. A certain type of virus attacks cells of the human body. The infected cells can be visualized using a special microscope. The microscope generates digital images that medical doctors can analyze and identify the infected cells. The virus causes the infected cells to have a black spot, within a darker grey region (Fig. 3.6).

A digital image process can be applied to the image. This processing generates two variables: the first variable, P, is related to black spot quantity (black pixels), and the second variable, S, is related to the shape of the black spot, i.e., if they are circular or elliptic. In these images it is often difficult to actually count the number of black pixels, or to identify a perfect circular cluster of pixels; hence, both these variables must be estimated in a linguistic way.

Suppose that we have two fuzzy sets, $\underset{\sim}{P}$ which represents the number of black pixels (e.g., none with black pixels, C_1, a few with black pixels, C_2, and a lot of black pixels, C_3), and $\underset{\sim}{S}$ which represents the shape of the black pixel clusters, e.g., S_1 is an ellipse and S_2 is a circle. So we have

$$\underset{\sim}{P} = \left\{ \frac{0.1}{C_1} + \frac{0.5}{C_2} + \frac{1.0}{C_3} \right\} \quad \text{and} \quad \underset{\sim}{S} = \left\{ \frac{0.3}{S_1} + \frac{0.8}{S_2} \right\}$$

and we want to find the relationship between quantity of black pixels in the virus and the shape of the black pixel clusters. Using a Cartesian product between $\underset{\sim}{P}$ and $\underset{\sim}{S}$ gives

$$\underset{\sim}{R} = \underset{\sim}{P} \times \underset{\sim}{S} = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \end{bmatrix} \end{array}$$

Now, suppose another microscope image is taken and the number of black pixels is slightly different; let the new black pixel quantity be represented by a fuzzy set, $\underset{\sim}{P}'$:

$$\underset{\sim}{P}' = \left\{ \frac{0.4}{C_1} + \frac{0.7}{C_2} + \frac{1.0}{C_3} \right\}$$

Using max–min composition with the relation $\underset{\sim}{R}$ will yield a new value for the fuzzy set of pixel cluster shapes that are associated with the new black pixel quantity:

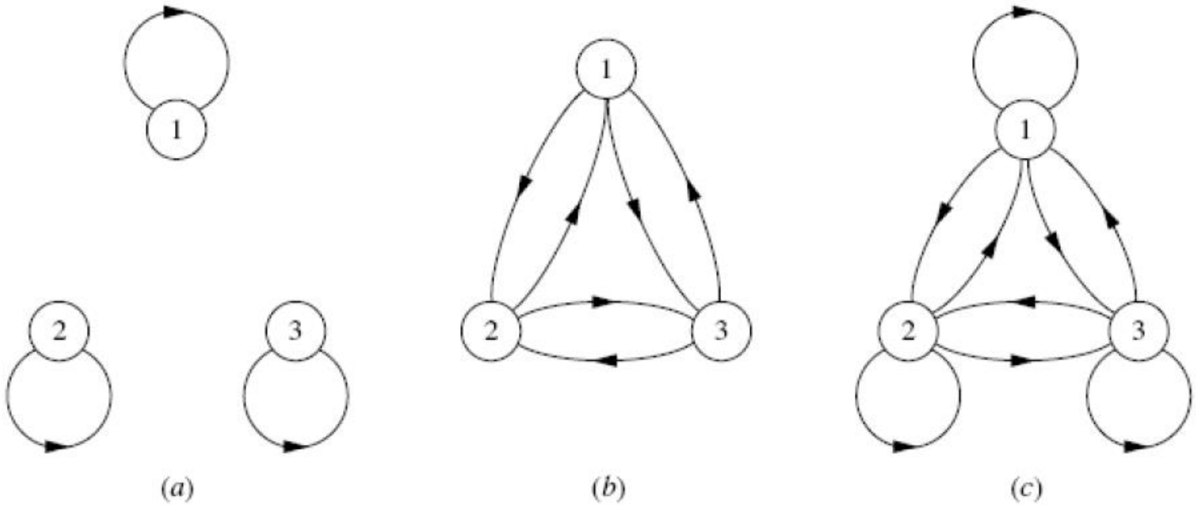
$$\underset{\sim}{S}' = \underset{\sim}{P}' \circ \underset{\sim}{R} = \begin{bmatrix} 0.4 & 0.7 & 1.0 \end{bmatrix} \circ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.8 \end{bmatrix}$$

Tolerance and Equivalence Relations

Relations can be used in graph theory

- When a relation is reflexive every vertex in the graph originates a single loop.
- If a relation is symmetric, then in the graph for every edge pointing (the arrows on the edge lines in Fig. 3.8*b*) from vertex *i* to vertex *j* (*i*,*j* = 1, 2, 3), there is an edge pointing in the opposite direction, i.e., from vertex *j* to vertex *i*.
- When a relation is transitive, then for every pair of edges in the graph, one pointing from vertex *i* to vertex *j*
  and the other from vertex *j* to vertex *k* (*i*,*j*,*k* = 1, 2, 3), there is an edge pointing from vertex *i* directly to vertex *k*,
  (e.g., an arrow from vertex 1 to vertex 2, an arrow from vertex 2 to vertex 3, and an arrow from vertex 1 to vertex 3).

Relations



(a) reflexivity, (b) symmetry, (c) transitivity

Fuzzy Tolerance & Equivalence

- Refer the following page:
- https://codecrucks.com/properties-of-relation-reflexivity-symmetricity-and-more/

Thank You!