Industrial Organization II: Problem Set 1

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Question 1

1)

Define $\mu = \ln\left[\sum_{i=1}^n \exp\left(\frac{\mu_i}{\sigma}\right)\right]$. Then

$$\begin{aligned}
\Pr\{Y \leq x\} &= \Pr\{\bigcap_{i=1}^n (X_i \leq x)\} \\
&= \prod_{i=1}^n \Pr\{X_i \leq x; \mu_i, \sigma\} && \text{by indep.} \\
&= \exp\left\{-\sum_{i=1}^n \exp\left(-\frac{x-\mu_i}{\sigma}\right)\right\} \\
&= \exp\left\{-\exp\left(\frac{-x}{\sigma}\right)\sum_{i=1}^n \exp\left(\frac{\mu_i}{\sigma}\right)\right\} \\
&= \exp\left\{-\exp\left(-\frac{x-\mu\sigma}{\sigma}\right)\right\} && \text{by def'n of } \mu \\
&\sim T1EV(\mu\sigma, \sigma)
\end{aligned}$$

2)

(I skip some steps in this derivation because it is too much typing!)

$$
\begin{aligned}
\Pr\{X - Y \le z\} &= \Pr\{X \le Y + z\} \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{Y+z} f_X(x)dx \right) f_Y(y)dy \\
&= \int_{-\infty}^{\infty} f_Y(y) F_X(y+z)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma} \exp\left\{ -\frac{y-\mu_Y}{\sigma} \right\} \exp\left\{ -\exp\left(-\frac{y-\mu_Y}{\sigma} \right) \left(1 + \exp\left(-\frac{z+\mu_Y-\mu_X}{\sigma} \right) \right) \right\} dy \\
&\qquad \text{define } a \equiv 1 + \exp\left(-\frac{z+\mu_Y-\mu_X}{\sigma} \right) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma} \exp\left\{ -\frac{y-\mu_Y}{\sigma} \right\} \exp\left\{ -\exp\left(-\frac{y-\mu_Y}{\sigma} \right) a \right\} dy \\
&= \frac{1}{a} \int_{-\infty}^{\infty} a \frac{1}{\sigma} \exp\left\{ -\frac{y-\mu_Y}{\sigma} \right\} \exp\left\{ -a \exp\left(-\frac{y-\mu_Y}{\sigma} \right) \right\} dy
\end{aligned}
$$

We now make a change of variables[1]: Define $u \equiv a \exp\left(-\frac{y-\mu_Y}{\sigma}\right)$. Then $du = \frac{-a}{\sigma} \exp\left(-\frac{y-\mu_Y}{\sigma}\right)$, and as $y \to -\infty$, $u \to \infty$, and $y \to \infty$, $u \to 0$. So, we may write the above as

$$
\begin{aligned}
\Pr\{X - Y \le z\} &= \frac{1}{a} \int_{\infty}^{0} -\exp\left\{ -a \exp\left(-\frac{y-\mu_Y}{\sigma} \right) \right\} \frac{a}{\sigma} \exp\left\{ -\frac{y-\mu_Y}{\sigma} \right\} dy \\
&= \frac{1}{a} \int_{\infty}^{0} -\exp(-u)du \\
&= \frac{1}{a} \int_{0}^{\infty} \exp(-u)du \\
&= \frac{1}{a} \\
&= \frac{\exp\left(-\frac{z-(\mu_X-\mu_Y)}{\sigma} \right)}{1 + \exp\left(-\frac{z-(\mu_X-\mu_Y)}{\sigma} \right)}
\end{aligned}
$$

which we recognize as the cdf of a Logistic$(\mu_X - \mu_Y, \sigma)$ random variable.

3)

For this derivation, first define $u_{-j} = \max_{k \ne j}\{\mu_k + \epsilon_k\}$ and $\mu_{-j} = \ln\left(\sum_{k \ne j} \exp(\mu_k)\right)$. By lemma 2.2.2 in the textbook, $u_{-j} \sim T1EV(\mu_{-j})$. So:

$$
\begin{aligned}
\Pr\{u_j > u_k, \forall k \ne j\} &= \Pr\{\mu_j + \epsilon_j > \max_{k \ne j}\{\mu_k + \epsilon_k\}\} \\
&= \Pr\{u_j \ge u_{-j}\} \\
&= \Pr\{u_{-j} - u_j \le 0\} \\
&= \frac{\exp(\mu_j)}{\exp(\mu_{-j}) + \exp(\mu_j)} && \text{by lemma 2.2.3 in textbook} \\
&= \frac{\exp(\mu_j)}{\sum_k \exp(\mu_k)}
\end{aligned}
$$

[1]Thanks to Conroy and Feng for help with this part.

4)

i)

We have that the latent utilities are independently distributed according to $u_{ij} \sim T1EV(\alpha(y_i - p_j), 1)$. Therefore, from theorem 2.2.1 in the textbook, we have

$$s_{ij} = \frac{\exp(\alpha(y_i - p_j))}{\sum_{k \in \mathcal{J}} \exp(\alpha(y_i - p_k))} = \frac{\exp(-\alpha p_j)}{\sum_{k \in \mathcal{J}} \exp(-\alpha p_k)}$$

and

$$\frac{\partial s_j(i)}{\partial y_i} = 0$$

i.e., the demand elasticity with respect to income is 0. This is because income enters utility linearly, so y_i just cancels out in the probability.

ii)

Maket share of product j:

$$\begin{aligned} s_j &= \int_{\mathcal{Y}} s_{ij} dF(y_i) \\ &= \int_{\mathcal{Y}} \frac{\exp(-\alpha p_j)}{\sum_{k \in \mathcal{J}} \exp(-\alpha p_k)} dF(y_i) \\ &= \frac{\exp(-\alpha p_j)}{\sum_{k \in \mathcal{J}} \exp(-\alpha p_k)} \\ &= s_{ij} \end{aligned}$$

The own-price elasticity is

$$\frac{\partial s_j}{\partial p_j} \frac{p_j}{s_j} = -\alpha s_j (1 - s_j) \frac{p_j}{s_j} = -\alpha (1 - s_j) p_j$$

and the cross-price elasticity is

$$\frac{\partial s_j}{\partial p_k} \frac{p_k}{s_j} = \alpha s_j s_k \frac{p_k}{s_j} = \alpha s_k p_k$$

The own and cross-price elasticities depend only on prices and market shares. This is probably not reasonable. Own-price elasticity should realistically be affected by the utility that can be gained from other products (are there close substitutes or complements?). And as discussed in class through the BMW-Mercedes-Kia example, this expression for cross-price elasticity places strong (and unrealistic) restrictions on substitution patterns.

iii)

Assume $\beta_i = \beta$

Remark that in this case, heterogeneity between consumers is due to y_i only. However, y_i does not affect how consumers make their choices, so all consumers will choose the same good. One good will have a market share of 1 and the rest 0.

The probability that consumer i chooses product j:

$$
\begin{aligned}
s_{ij} &= \Pr\{\alpha(y_i - p_j) + \beta x_j > \max_{k \neq j} \alpha(y_i - p_k) + \beta x_k\} \\
&= \prod_{k \neq j} \Pr\{\beta(x_j - x_k) > \alpha(p_j - p_k)\} \\
&= \prod_{k \neq j} \mathbb{1}\left[\beta(x_j - x_k) > \alpha(p_j - p_k)\right]
\end{aligned}
$$

The market share of product j:

$$
s_j = s_{ij} = \prod_{k \neq j} \mathbb{1}\left[\beta(x_j - x_k) > \alpha(p_j - p_k)\right]
$$

because, as explained above, all consumers make the same choice.

Assume $\beta_i \sim Unif[0, \overline{\beta}]$, iid
The probability that consumer i chooses product j:

$$
\begin{aligned}
s_{ij} &= \Pr\{\alpha(y_i - p_j) + \beta_i x_j > \max_{k \neq j} \alpha(y_i - p_k) + \beta_i x_k\} \\
&= \prod_{k \neq j} \Pr\{\beta_i(x_j - x_k) > \alpha(p_j - p_k)\} \\
&= \prod_{k \neq j} \left(1 - \Pr\left\{\beta_i \leq \alpha \frac{p_j - p_k}{x_j - x_k}\right\}\right) \\
&= \prod_{k \neq j} \left(1 - \frac{\alpha}{\overline{\beta}} \frac{p_j - p_k}{x_j - x_k}\right)
\end{aligned}
$$

Notice that once again, s_{ij} does not depend on i. This is because the unobserved heterogeneity (β_i) is iid. So the market share of product j is

$$
s_j = s_{ij} = \prod_{k \neq j} \left(1 - \frac{\alpha}{\overline{\beta}} \frac{p_j - p_k}{x_j - x_k}\right)
$$

The own-price elasticity is:

$$
\begin{aligned}
\frac{\partial s_j}{\partial p_j} &= \frac{\partial}{\partial p_j} \prod_{k \neq j} \left(1 - \frac{\alpha}{\overline{\beta}} \frac{p_j - p_k}{x_j - x_k}\right) \\
&= \prod_{l \neq j} \left[-\frac{\alpha}{\overline{\beta}} \frac{1}{x_j - x_l} \prod_{k \neq j, l} \left(1 - \frac{\alpha}{\overline{\beta}} \frac{p_j - p_k}{x_j - x_k}\right)\right] \\
&= \prod_{l \neq j} \left[\frac{-\frac{\alpha}{\overline{\beta}} \frac{1}{x_j - x_l}}{1 - \frac{\alpha}{\overline{\beta}} \frac{p_j - p_l}{x_j - x_l}} s_j\right]
\end{aligned}
$$

and the cross-price elasticity is

$$
\begin{aligned}
\frac{\partial s_j}{\partial p_l} &= \frac{\partial}{\partial p_l} \prod_{k \neq j} \left(1 - \frac{\alpha}{\beta} \frac{p_j - p_k}{x_j - x_k} \right) \\
&= \frac{\frac{\alpha}{\beta} \frac{1}{x_j - x_l}}{1 - \frac{\alpha}{\beta} \frac{p_j - p_l}{x_j - x_l}} s_j
\end{aligned}
$$

Here the cross-price elasticities depend on how close the products are in characteristics space $(x_j - x_l)$. This is more reasonable than the result in (ii). Consider the BMW, Mercedes-Benz, and Kia example again. Suppose $i =$BMW, $j =$Mercedes-Benz, and $k =$Kia. Write the cross-price elasticities:

$$
\begin{aligned}
\frac{\partial s_i}{\partial p_j} \frac{p_j}{s_i} &= \frac{\frac{\alpha}{\beta} \frac{p_j}{x_i - x_j}}{1 - \frac{\alpha}{\beta} \frac{p_i - p_j}{x_i - x_j}} \\
\frac{\partial s_j}{\partial p_k} \frac{p_k}{s_j} &= \frac{\frac{\alpha}{\beta} \frac{p_k}{x_j - x_k}}{1 - \frac{\alpha}{\beta} \frac{p_j - p_k}{x_j - x_k}}
\end{aligned}
$$

Because of our assumption that $\frac{p_i - p_j}{x_i - x_j} \geq \frac{p_j - p_k}{x_j - x_k}$ whenever $x_i \geq x_j \geq x_k$, and because $\frac{p_j}{x_i - x_j} \geq \frac{p_k}{x_j - x_k}$ very likely holds (since BMW and Mercedes are closer in characteristics space than Mercedes-Benz and Kia, and the price of Mercedes-Benz is likely higher than Kia), we have that

$$
\frac{\partial s_i}{\partial p_j} \frac{p_j}{s_i} \geq \frac{\partial s_j}{\partial p_k} \frac{p_k}{s_j}
$$

i.e., demand for BMWs is more sensitive to changes in the price of Mercedes-Benz than demand for Mercedes-Benz is to changes in the price of Kia. We do not get this result in (ii), where cross-price elasticities depended only on market shares and prices.

iv)

Now we have $u_{ij} = \alpha(y_i - p_j) + \beta_i x_j + \nu_{ij}$, where $\beta_i \sim^{iid} F(.)$ and $\nu_{ij} \sim^{iid} T1EV(0,1)$.

The probability that i chooses j is

$$
s_{ij} = \frac{\exp(-\alpha p_j + \beta_i x_j)}{\sum_k \exp(-\alpha p_k + \beta_i x_k)}
$$

and the market share of product j is

$$
\begin{aligned}
s_j &= \int s_{ij} dF(\beta_i) \\
&= \int \frac{\exp(-\alpha p_j + \beta_i x_j)}{\sum_k \exp(-\alpha p_k + \beta_i x_k)} dF(\beta_i)
\end{aligned}
$$

The own price elasticity is

$$
\begin{aligned}
\frac{\partial s_j}{\partial p_j} \frac{p_j}{s_j} &= \frac{p_j}{s_j} \int \frac{\partial}{\partial p_j} \frac{\exp(-\alpha p_j + \beta_i x_j)}{\sum_k \exp(-\alpha p_k + \beta_i x_k)} dF(\beta_i) \\
&= \frac{p_j}{s_j} \int -\alpha s_{ij}(1 - s_{ij}) dF(\beta_i)
\end{aligned}
$$

and the cross-price elasticity is

$$\begin{aligned}\frac{\partial s_j}{\partial p_k}\frac{p_k}{s_j} &= \frac{p_k}{s_j}\int \frac{\partial}{\partial p_k}\frac{\exp(-\alpha p_j+\beta_i x_j)}{\sum_k \exp(-\alpha p_k+\beta_i x_k)}dF(\beta_i)\\ &= \frac{p_k}{s_j}\int \alpha s_{ij}s_{ik}dF(\beta_i)\end{aligned}$$

The price elasticities are now determined in part by the distribution of β_i in the population. Since the price elasticities do not depend solely on market shares and prices, we no longer have independence of irrelevant alternatives (IIA) at the market level, unlike in (ii).

v)

Rewrite W as a function of the market share of the outside good s_0:

We have that $u_{ij} \sim T1EV(\alpha(y_i - p_j), 1)$, independently distributed. By lemma 2.2.2 in the textbook,

$$\mathbb{E}\left[\max_j u_{ij}\right] = \ln\left(\sum_j \exp(\alpha(y_i - p_j))\right) + \gamma$$

We also have that

$$s_0 = \frac{1}{\sum_j \exp(\alpha(y_i - p_j))} \Rightarrow \sum_j \exp(\alpha(y_i - p_j)) = \frac{1}{s_0}$$

So,

$$W \equiv E\max_j u_{ij} = \ln\left(\frac{1}{s_0}\right) + \gamma$$

What happens to W when a new product $J+1$ is introduced in the market?:

Define $W^+ = \mathbb{E}\left[\max\{\max_{j\in\mathcal{J}} u_{ij}, u_{i,J+1}\}\right]$. By lemma 2.2.2 in the textbook, and using our result that $\sum_j^J \exp(\alpha(y_i - p_j)) = \frac{1}{s_0}$ we have that

$$\begin{aligned}\mathbb{E}\left[\max\{\max_{j\in\mathcal{J}} u_{ij}, u_{i,J+1}\}\right] &= \ln\left(\sum_j^{J+1}\exp(\alpha(y_i - p_j))\right) + \gamma\\ &= \ln\left(\sum_j^{J}\exp(\alpha(y_i - p_j)) + \exp(\alpha(y_i - p_{J+1}))\right) + \gamma\\ &= \ln\left(\frac{1}{s_0} + \exp(\alpha(y_i - p_{J+1}))\right) + \gamma\end{aligned}$$

Then $W^+ - W = \ln(1 + s_{i0}\exp(\alpha(y_i - p_{J+1})) \geq 0$. W increases when a new good is introduced to the market. This is what we'd expect, because no consumer is made worse off by having an additional option to choose from.

Question 2

(i)

β reflects the effect of product characteristics on utility that is common to all individuals. Γ reflects the effect of product characteristics on utility that varies by demographics.

(ii)

First, define

$$s_{ij} := \Pr\{i \text{ chooses } j\} = \frac{\exp(\delta_j + d_i'\Gamma X_j)}{\sum_{k\in\mathcal{J}} \exp(\delta_k + d_i'\Gamma X_k)}$$

The log-likelihood of the data is then

$$\log L(y; \delta, \Gamma) = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \mathbf{1}\{i \text{ chooses } j\} \left(\delta_j + d_i'\Gamma X_j - \log(\sum_{k\in\mathcal{J}} \exp(\delta_k + d_i'\Gamma X_k)) \right)$$

(iii)

The FOC of the log-likelihood with respect to δ_j is

$$\begin{aligned} 0 &= \sum_{i\in\mathcal{I}} \mathbf{1}\{i \text{ chooses } j\} - \sum_{i\in\mathcal{I}} s_{ij} \sum_{k\in\mathcal{J}} \mathbf{1}\{i \text{ chooses } k\} \\ &= \sum_{i\in\mathcal{I}} \mathbf{1}\{i \text{ chooses } j\} - \sum_{i\in\mathcal{I}} s_{ij} \\ &= \sum_{i\in\mathcal{I}} (\mathbf{1}\{i \text{ chooses } j\} - s_{ij}) \end{aligned}$$

Interpretation: We can rewrite the above FOC as $\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} s_{ij} = \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} \mathbf{1}\{i \text{ chooses } j\}$. The LHS is the average probability that an individual chooses good j. The RHS is the proportion of individuals who choose product j. The LHS is a function of the δ_j's and the RHS is what we observe. We want to pick δ_j's such that these two quantities are equal.

The FOC of the log-likelihood with respect to Γ is

$$0 = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \mathbf{1}\{i \text{ chooses } j\} d_i X_j' (1 - s_{ij})$$

Interpretation: We can rewrite the FOC as
$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \mathbf{1}\{i \text{ chooses } j\} d_i X_j' = \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \mathbf{1}\{i \text{ chooses } j\} d_i X_j' s_{ij}$. On the LHS, we are summing all the $d_i X_j'$ for which individual i chose product j. On the RHS, we are summing all the $d_i X_j'$ for which individual i chose product j, scaled by the probability that individual i chooses j. I must admit I am not 100% sure how to interpret the FOC for Γ...

(iv)

See attached Jupyter Notebook for MLE code (done in Python). Table 1 contains the MLE results.

	delta		Gamma
δ_1	1.03	Γ_{11}	0.71
δ_2	0.17	Γ_{12}	0.10
δ_3	0.42	Γ_{13}	0.62
δ_4	0.41	Γ_{21}	0.51
δ_5	2.18	Γ_{22}	0.22
δ_6	1.62	Γ_{23}	0.77
δ_7	-0.23		.
δ_8	1.44		.
δ_9	1.10		.
δ_{10}	1.19		.
δ_{11}	1.31		.
δ_{12}	1.80		.
δ_{13}	0.96		.
δ_{14}	-0.93		.
δ_{15}	0.58		.
δ_{16}	1.40		.
δ_{17}	0.73		.
δ_{18}	1.09		.
δ_{19}	0.64		.
δ_{20}	0.46		.
δ_{21}	1.89		.
δ_{22}	1.41		.
δ_{23}	1.35		.
δ_{24}	2.95		.
δ_{25}	0.89		.
δ_{26}	1.45		.
δ_{27}	-1.89		.
δ_{28}	0.42		.
δ_{29}	3.21		.
δ_{30}	1.67		.

Table 1: Maximum Likelihood Estimates of δ and Γ coefficients

(v)

We defined $\delta_j = x_j'\beta + \xi_j$, where x_j are observed and ξ_j are unobserved product characteristics. One moment condition (albeit an unrealistic one) that we could assume is $\mathbb{E}[x_j \xi_j] = 0$. If this moment condition holds, we could regress δ_j on x_j using OLS to consistently estimate β.

(vi)

The table below shows the β coefficients estimated from OLS.

	(1)
x.1	-0.07
	(0.11)
x.2	0.82***
	(0.11)
x.3	0.18**
	(0.07)
N	30
R^2	0.88

Table 2: OLS estimates of β coefficients