Winter 2021
BUS 37904
Professor Jean-Pierre Dubé
Professor Günter J. Hitsch

Advanced Quantitative Marketing: Bayesian Learning

Computational preliminaries

Numerical integration

See the example in `Integration.r`. The goal is to approximate the integral

$$I = \int f(\boldsymbol{x})\phi(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})d\boldsymbol{x}.$$

The vector $\boldsymbol{x}$ has D components and $\phi(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the pdf of multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Specify the number of quadrature nodes M in each dimension. The function `GaussHermite` in `Gauss-Hermite-Quadrature.R` returns a list with a $D \times M^D$ matrix of quadrature nodes, $\boldsymbol{X}$, and a vector of quadrature weights $\boldsymbol{\omega}$. To approximate the integral I first calculate $y_i = f(\boldsymbol{X}_i)$ for each row i of $\boldsymbol{X}$. The approximation is then given by

$$I^{\text{approx}} = \pi^{-\frac{D}{2}}\boldsymbol{\omega}^T\boldsymbol{y}.$$

Expectation of the value function

Consider the following computational step. For some vector $\boldsymbol{x}$ and distribution $p(\boldsymbol{x}'|\boldsymbol{x}, j)$ we need to calculate the expectation

$$\int v(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x}, j)d\boldsymbol{x}'.$$

The value function v is represented as a Chebyshev polynomial in computer memory,

$$v(\boldsymbol{x}) = T(\boldsymbol{x})^T\boldsymbol{\theta} = \sum_k \theta_k T_k(\boldsymbol{x}).$$

The integral is calculated based on a weighted sum,

$$\int v(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x}, j)d\boldsymbol{x}' = \sum_q \omega_q v(\boldsymbol{x}'_q).$$

Combining the numerical integration method and the approximation method we see that

$$\begin{aligned}\int v(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x},j)d\boldsymbol{x}' &= \sum_q \omega_q \left(\sum_k \theta_k T_k(\boldsymbol{x}_q)\right) \\ &= \sum_k \theta_k \left(\sum_q \omega_q T_k(\boldsymbol{x}_q)\right) \\ &= \sum_k \theta_k \left(\int T_k(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x},j)d\boldsymbol{x}'\right) \\ &= \sum_k \theta_k E_k(\boldsymbol{x},j),\end{aligned}$$

where $E_k(\boldsymbol{x})$ is defined as

$$E_k(\boldsymbol{x},j) = \int T_k(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x},j)d\boldsymbol{x}'.$$

The significance of this representation of the value function will become clear below when we discuss how to iteratively solve for the optimal value function v.

Updating the value function

Solving for the optimal value function v requires us to solve for a sequence of $\boldsymbol{\theta}^{(n)}$ Chebyshev coefficients. $\boldsymbol{\theta}^{(n)}$ represents our current guess of the value function $v^{(n)}$, and the corresponding choice-specific value functions

$$\begin{aligned}v_j^{(n)}(\boldsymbol{x}) &= u_j(\boldsymbol{x}) + \beta \int v^{(n)}(\boldsymbol{x}')p(\boldsymbol{x}'|\boldsymbol{x},j)d\boldsymbol{x}' \\ &= u_j(\boldsymbol{x}) + \beta \sum_k \theta_k^{(n)} E_k(\boldsymbol{x},j).\end{aligned}$$

Based on the choice-specific value functions we can update the value function:

$$v^{(n+1)}(\boldsymbol{x}) \leftarrow \int \max_j \{v_j^{(n)}(\boldsymbol{x}) + \epsilon_j\} g(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon}.$$

Let $\boldsymbol{X}$ be a matrix where each row represents a Chebyshev node vector. D is the dimension of the state space. In each dimension we choose a Chebyshev approximation of degree N with M nodes. Then $\boldsymbol{X}$ has M^D rows, and the corresponding polynomial matrix $\boldsymbol{T}$ has $(N+1)^D$ columns. In order to update the value function we first need to compute

$$y^{(n)}(\boldsymbol{x}_i) = \int \max_j \{v_j^{(n)}(\boldsymbol{x}_i) + \epsilon_j\} g(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon}$$

for each row i in $\boldsymbol{X}$. We obtain the updated Chebyshev coefficient vector $\boldsymbol{\theta}^{(n+1)}$ based on the regression

$$y^{(n)}(\boldsymbol{X}) = \boldsymbol{T}\boldsymbol{\theta}.$$

The updated value function $v^{(n+1)}$ is then given by $v^{(n+1)}(\boldsymbol{x}) = \sum_k \theta_k^{(n+1)} T_k(\boldsymbol{x})$.

Bayesian learning model

Setup

Consider a Bayesian learning model in the spirit of Erdem and Keane (1996) or Crawford and Shum (2005). A consumer, indexed by i, chooses among J experience goods, $j = 0$ denotes the no-purchase option. For now we focus on the choice behavior of one consumer and suppress the index i to simplify the notation. The realized quality of product j is

$$\xi_{jt} = \vartheta_j + \nu_{jt}, \qquad \nu_{jt} \sim N(0, \sigma_\nu^2).$$

ϑ_j is the match value based on the product attributes and the consumer's idiosyncratic preferences over these attributes. Consumers have normal priors on the match values for each product, $\pi_{jt} = N(\mu_{jt}, \sigma_{jt}^2)$. We assume that the priors are independent and that the noise terms ν_{jt} are i.i.d.

The realized utility (net of the latent utility draw ϵ_{jt}) conditional on the purchase of product j is

$$\mathfrak{u}_{jt} = \gamma - \exp(-\rho\xi_{jt}) - \alpha p_{jt}.$$

p_{jt} is the price of product j. The state vector is $\boldsymbol{x}_t = (\boldsymbol{p}_t, \boldsymbol{\pi}_t) = (p_{1t}, \ldots, p_{Jt}, \mu_{1t}, \sigma_{1t}^2, \ldots, \mu_{Jt}, \sigma_{Jt}^2)$. The expected utility conditional on the purchase of j and given the consumer's prior belief about the match value of product j is

$$u_j(\boldsymbol{x}_t) = \mathbb{E}\left(\gamma - \exp(-\rho\xi_{jt}) - \alpha p_{jt} | \boldsymbol{x}_t\right).$$

$\rho > 0$ measures risk aversion. Assume that the latent utility draws are Type I extreme value, i.i.d., and centered at 0. Normalize the utility from the outside option such that $u_0(\boldsymbol{x}_t) = 0$. Note the role of the parameter γ in the utility function: Without γ, i.e. if $\gamma = 0$, the expected utility is always negative. This imposes restrictions on the market shares. For example in the case of one good, $J = 1$, the outside share would always be larger than 1/2 because $u_j(\boldsymbol{x}_t) < 0$.

Consumers are forward-looking and discount the future using a discount factor $\beta > 0$. Optimal choices are characterized by the choice-specific value functions $v_j(\boldsymbol{x})$, such that consumers choose product j if and only if $v_j(\boldsymbol{x}_t) + \epsilon_{jt} \geq u_k(\boldsymbol{x}_t) + \epsilon_{kt}$ for all $k \neq j$.

The match value for consumer i from product j is drawn from a normal distribution,

$$\vartheta_{ij} \sim N(\bar{\vartheta}_j, \tau_j^2). \tag{1}$$

We assume that consumers have rational expectations such that $\pi_{ij0} = N(\bar{\vartheta}_j, \tau^2)$. After a purchase (= consumption) the consumer observes the signal ξ_{ijt} and updates the prior in a rational, Bayesian fashion.

The price vectors $\boldsymbol{p}_t$ are drawn from a discrete distribution with support $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_K\}$. We assume that prices are i.i.d. across periods, and we define and $\omega_k = \Pr\{\boldsymbol{p}_t = \boldsymbol{p}_k\}$. This assumption simplifies the model solution and is not essential to the learning story.

Solution strategy

With some abuse of notation (because v is in general a function of $\boldsymbol{x}_t$), define the expected value function

$$v(\boldsymbol{\pi}_t) = \sum_{k=1}^{K} \omega_k \left(\int \max_j \{v_j(\boldsymbol{p}_k, \boldsymbol{\pi}_t) + \epsilon_j\} g(\boldsymbol{\epsilon}) d\boldsymbol{\epsilon} \right).$$

Make sure you understand what $v(\boldsymbol{\pi}_t)$ captures: the value of the decision problem, conditional on the prior belief $\boldsymbol{\pi}_t$, before prices and the latent utility draws are realized.

Convince yourself that the choice-specific value functions are easy to calculate if $v(\boldsymbol{\pi})$ is known, and that correspondingly $v(\boldsymbol{\pi})$ can be calculated based on the iteration

$$v^{(n+1)}(\boldsymbol{\pi}) \leftarrow \sum_{k=1}^{K} \omega_k \left(\int \max_j \{ v_j^{(n)}(\boldsymbol{p}, \boldsymbol{\pi}) + \epsilon_j \} \mathrm{g}(\boldsymbol{\epsilon}) \mathrm{d}\boldsymbol{\epsilon} \right).$$

Structure of your report

1. Write down and explain all the model pieces, including everything that was left out above, such as an explanation of how consumers update their prior beliefs. Explain why it is possible to characterize the solution of the dynamic discrete choice problem based on $v(\boldsymbol{\pi})$ instead of $v(\boldsymbol{x}) = v(\boldsymbol{p}, \boldsymbol{\pi})$. Also explain the advantage of this characterization, and why the simplification would not be possible if prices were not i.i.d. but dependent across time.

2. Numerically solve for the choice-specific value functions and the corresponding conditional choice probabilities (CCP's), given the discount factor β, the preference parameters $\vartheta_{ij}, \ldots, \vartheta_{iJ}, \gamma, \rho, \alpha$, the learning parameters $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_J, \tau^2, \sigma_\nu^2$, (not a typo: assume that the prior variances and signal variances are the same across products), and the product price vectors$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_K\}$ and probabilities ω_k.

   We suggest that you solve the model for $J = 2$ products. Even better, write your code for an arbitrary number of products J.

3. Analyze and simulate the model. First, calibrate a simple price process that resembles real world data (to do so you could take a look at the price data shown at the beginning of the storable goods demand lecture). Two or three discrete price vectors are sufficient. Choose a discount factor $\beta = 0.998$. Calibrate the preference parameters such that you obtain choice probabilities under complete information somewhere in the range of 0.1 to 0.25 for each product in the case of $J = 2$.

   (a) Plot the value functions and CCP's for each product j on (μ_j, σ_j^2) given a specific price vector and holding the beliefs for the other products constant. Discuss the results.

   (b) Simulate choice sequences $(a_0, \ldots, a_T)^{(n)}$ for $t = 0, \ldots, T$. Condition on an initial prior and the true (but initially unknown to the consumer) match values ϑ_{ij}, and draw prices $\boldsymbol{p}_t$, latent utility terms $\boldsymbol{\epsilon}_t$, and signal noise terms $\boldsymbol{\nu}_t$. Predict the consumer's choices based on the choice-specific value functions and update the prior accordingly. Average over N choice sequences (choose a large N) to predict the choice probabilities $\phi_{jt} = N^{-1} \sum_{n=1}^{N} a_{jt}^{(n)}$.

   (c) Show and explain how the choice probabilities evolve depending on the prior mean, the difference between the prior mean and the true match value ϑ_{ij}, and the prior variance, i.e. initial uncertainty.

   (d) Do you find evidence for experimentation? For example, examine how the frequency of product switching evolves over time. Define a product switch as an instance where at time t a consumer purchases product $j \neq 0$ and at the most recent prior shopping trip at time $t' < t$ when the consumer purchased some product $k \neq 0$ the chosen product k was different from j. Calculate the ratio of the frequency of switches at time t relative to the inside share, i.e. the sum of choice frequencies for all products $j \neq 0$. This ratio measures the switching probability conditional on making a purchase in the category. Show how this switching frequency evolves over time.

   (e) Compare the simulations for different discount factors, $\beta = 0.998, 0.995, 0.99, 0$, and for different levels of the risk aversion parameter ρ.

References

Crawford, G. S. and M. Shum (2005): "Uncertainty and Learning in Pharmaceutical Demand," *Econometrica*, 73, 1137–1173.

Erdem, T. and M. P. Keane (1996): "Decision-Making Under Uncertainty: Capturing Dynamic Brand Choice Processes in Turbulent Consumer Goods Markets," *Marketing Science*, 15, 1–20.