EE2703: Applied Programming Lab
Convolution of Signals using the Fast Fourier Transform Algorithm

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Contents

1 Introduction

In this program, convolution of signals with an FIR (Finite Impulse Response) filter is computed in time and frequency domains. To compute the linear convolution of signals with long support, the signal is broken into blocks of smaller length and then convolved with the filter (enabling real-time processing). This is known as block convolution; in this program this is implemented using the overlap-and-save method. Finally, the auto-correlation of the Zadoff-Chu sequence with a cyclically shifted version of itself is computed.

1.1 Overlap and Save

This method can be used for block convolution of a signal with FIR filters. For Infinite Impulse Response filters, the processing is done either in the FFT domain or by recursively implementing the associated Linear Differential Equation.
Consider the support of the input signal x to be [0, $N_x - 1$] and that of the filter h to be [0, P-1]. The signal x is divided into multiple blocks, each of length L (L > P, the final block may require zero-padding).
Consider the circular convolution of a sequence of length L with h. The first P samples of the output are corrupt due to time-aliasing while the samples [P : L-1] match the results of linear convolution. Each block of the input signal therefore contains P samples from the previous block and L-P new samples. The code for the algorithm is given in the next section.

1.2 Auto-Correlation

The cross-correlation of two complex valued signals with support [0, N-1] is defined as:

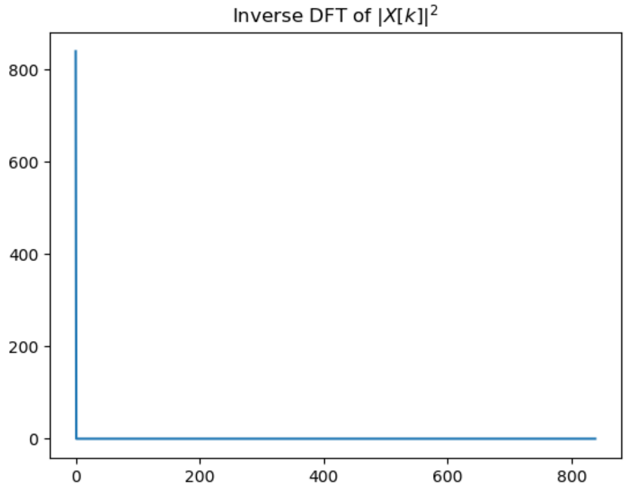
$$z[n] = \sum_{r=0}^{N-1} x[r]y^*[r+n]$$

The DFT of z[n] is:

$$Z[k] = X[-k]Y^*[-k]$$

Here, the sequence y is a cyclically shifted version of x. Let the shift be n_o, then:

$$\begin{aligned} Z[k] &= X[-k]X^*[-k]e^{j\frac{2\pi}{N}kn_o} && (1) \\ &= |X[-k]|^2 e^{j\frac{2\pi}{N}kn_o} && (2) \end{aligned}$$



The inverse DFT of $|X[-k]|^2$ of the Zadoff-Chu sequence has a non-zero value only at index 0

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2 Approach

2.1 Important blocks used in the program

1: Block Convolution- Overlap and Save

```
def overlap_save(x, h, L=32, P=16):
  x_d = np.concatenate((np.zeros(P), x, np.zeros(L)))
  if h.shape != P:
    h_2m = np.concatenate((h, np.zeros(P - len(h))))
  n_os = (len(x)+L)/(L-P)
  y = np.zeros(len(x)+len(h_2m)+L, dtype = complex)
  for i in range(int(n_os)):
   y_d = ifft(fft(x_d[i*(L-P):i*(L-P)+L])\
       *fft(np.concatenate((h_2m, np.zeros(L-P)))))
   y[i*(L-P):(i+1)*(L-P)] = y_d[P:L]
  y = y[0:len(x)+len(h)-1]
  return y
```

2: Filter Characteristics

```
def filter_char(b, a, show= True):
  # Pole-zero characteristics
```

```
  zero, pole, __ = sp.tf2zpk(b, a)
  if (pole.shape < zero.shape):
    pole = np.concatenate((pole, np.zeros(zero.shape[0] - pole.shape[0], \
        dtype = complex)))
  zp = [zero, pole]
  # Magnitude and Phase Characteristics
  w1, H = sp.freqz(b, a, 1024, whole= True)
  # Time domain sequence (= Coefficients for given filter since it is FIR)
  h = ifft(H)
  ii = np.where(abs(h) > 1e-4)
  h = np.real(h[ii]) # remove imaginary components- h is a real FIR filter
  # Group Delay of Filter
  w2, gd = sp.group_delay((b, a), whole= True)
  return zp, H, h, gd
```

3: Auto-correlation

```
def fft_corr(x, y, circ_shift= 0):
  y = np.roll(y, circ_shift)
  Y = fft(y)
  X = fft(x)
  Y[1:] = np.flip(Y[1:])
  X[1:] = np.flip(X[1:])
  z = ifft(X*np.conj(Y))
  Z = fftshift(fft(z))
  return z, Z
```

4: Input Format

```
python3 EE18B104_Assign10.py --help
usage: EE18B104_Assign10.py [-h] [--fn_choice FN_CHOICE]

optional arguments:
  -h, --help            show this help message and exit
  --fn_choice FN_CHOICE
                        Choose function:
                        1: FIR Filter Characteristics
                        2: Linear and Circular Convolution with FIR filter
                        3: Autocorrelation of Zadoff-Chu sequence
```

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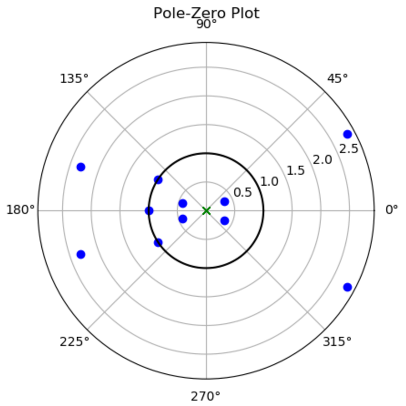
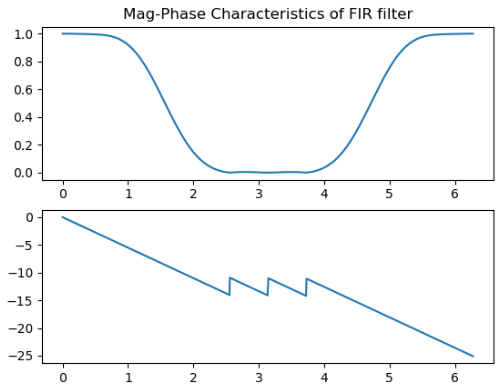
3 Analysis

3.1 Filter Characteristics

The given signal is a Linear Phase FIR filter (Type 2) as is clear from the pole-zero plot. The discontinuities in the phase plot of the filter occur at the zeroes (zeroes on the Unit Circle in z-plane). The filter is low pass.

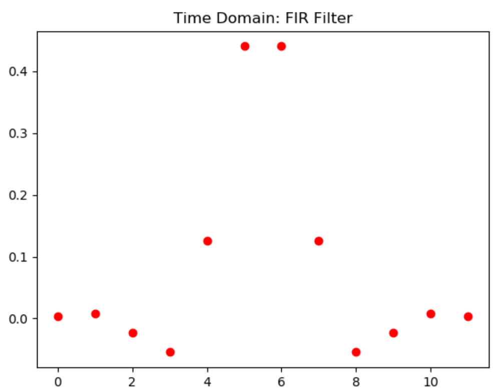


Pole-Zero Plot of the filter

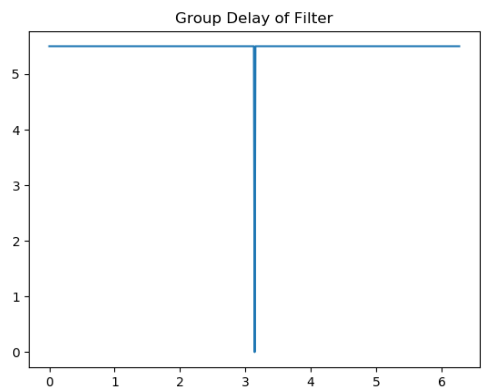


FFT of filter

Since the filter is linear phase, the group delay is constant (except at zeroes) and as the filter is Type 2, the time-domain sequence is symmetric and has even length.



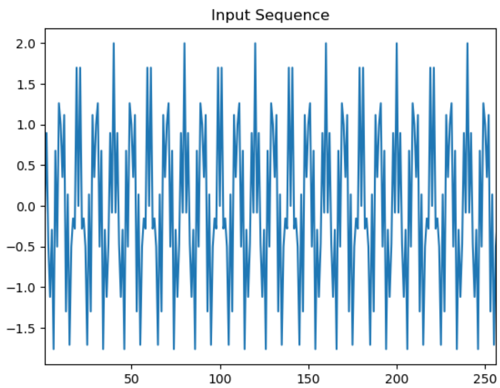
Time-Domain Sequence



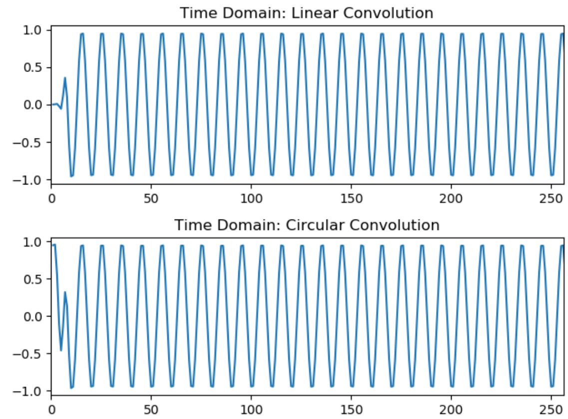
Group Delay- the singularity at π is replaced by 0

3.2 Convolution

The sequence $cos(0.2\pi n) + cos(0.85\pi n)$ evaluated at 1024 points in [1, 1024] is convolved with the filter.

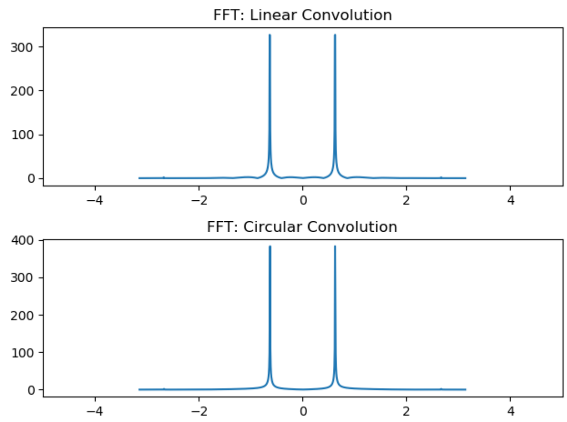


Input Sequence

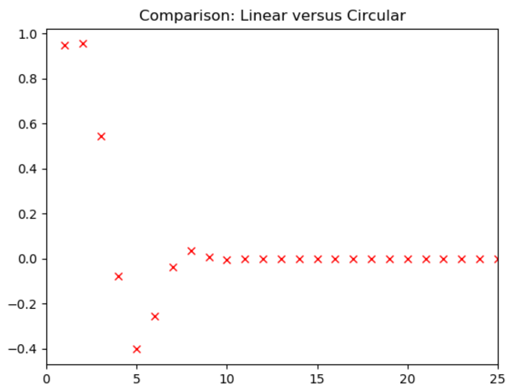


Convolved Outputs- Linear and Frequency Domain

As expected, the convolved signal is predominantly low-pass corresponding to $\omega = 0.2\pi$. The linear and circular convolution outputs differ in their initial values, due to time-aliasing in circular convolution.



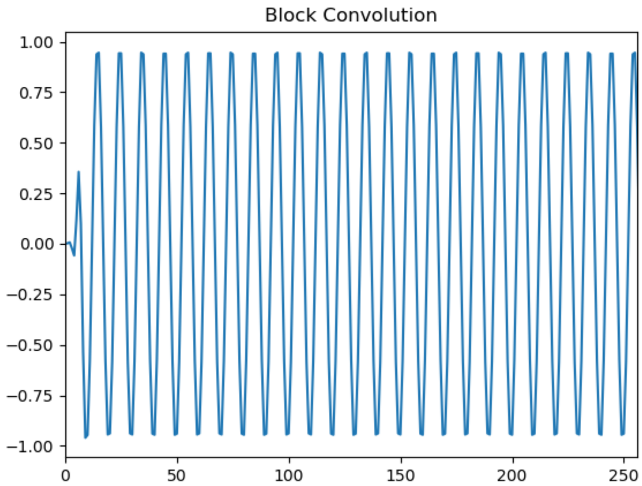
FFTs of Convolved Outputs



The linear and circular outputs have a significant difference only for the first 12 indices, i.e.- the length of the filter

3.2.1 Block Convolution

The Block Convolution output exactly matches the linear convolution output. The sum of absolute errors between y_block and y_lin is 1.2e-13, which is due to floating point errors.



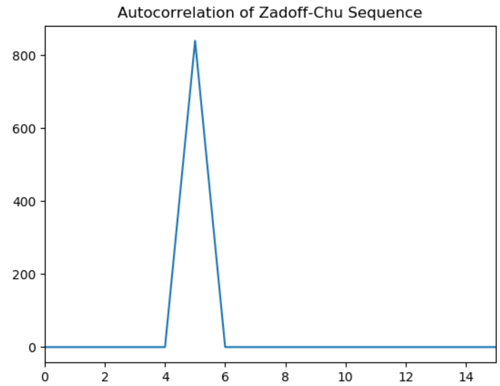
Linear Convolution by Overlap and Save

3.3 Auto-correlation of the Zadoff-Chu sequence

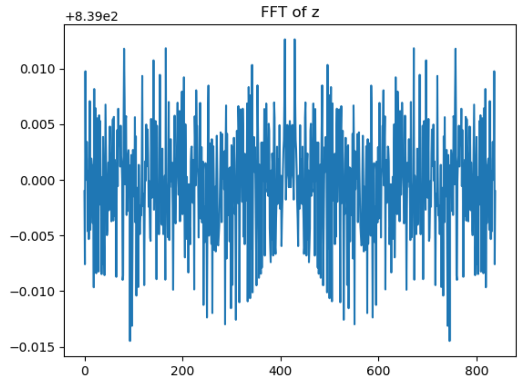
The Zadoff-Chu sequence is a complex-valued sequence with the following properties:

1. The sequence is periodic and has constant amplitude
2. The auto-correlation of a Zadoff-Chu sequence gives a sequence with a non-zero value only at 0
3. The cross-correlation of two Zadoff-Chu sequences with different parameters is a constant.

Here, we examine the 2^{nd} property by computing the auto-correlation of a Zadoff-Chu sequence with a cyclically shifted version of itself.



Auto-correlation for a delay of 5



Magnitude of FFT of the Auto-correlated sequence- the magnitude of fft does not vary with the delay (from (2))

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