Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam – 603 110

(An Autonomous Institution, Affiliated to Anna University, Chennai)

Department of Mathematics

Continuous Assessment Test – II

Answer Key

Part – A (6×2 = 12 Marks)

1. $(1+a)P(X>k)=a$

$$P(X>k)=\frac{a}{1+a}$$

$$\int_k^\infty f(x)\,dx=\frac{a}{1+a}$$ -----------------(1m)

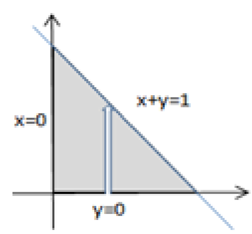
$$\lambda\int_k^\infty e^{-\lambda x}\,dx=\frac{a}{1+a}= \quad k=\frac{1}{\lambda}\log\left(\frac{1+a}{a}\right).$$ -----------------(1m)

2. $f(x)=1/\pi$ --------------------------(1m)

$f(y)=f(x)\left|\frac{dx}{dy}\right|=\frac{1/\pi}{1+y^2}, -\infty<y<\infty$ ------------------(1m)

3.

$$P[X+Y\le 1]=\int_0^1\int_0^{1-y}\frac{1}{4}dxdy=\frac{1}{4}\int_0^1(1-y)dy=\frac{1}{8}.$$



-----------------------(2m)

4.

Given X is uniformly distributed in (-1,1),pdf of X is $f(x)=\frac{1}{b-a}=\frac{1}{2}, \quad -1\le x\le 1$

$E(X)=\frac{1}{2}\int_{-1}^{1}x\,dx=0$ and $E(XY)=E(X^3)=0$ ———————— (1m)

$\therefore \operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=0 \quad \Rightarrow r(X,Y)=0$

Hence X and Y are uncorrelated. ———————— (1m)

5.

If $x_1, x_2, \ldots\ldots\ldots, x_n$ are n independent identically distributed RVs with men μ and S.D σ and if $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, then the variate $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ has a distribution that approaches the standard normal distribution as $n \to \infty$ provided the m.g.f of x_i exist.

------------------------(2m)

6.

Given $f(x,y) = 6e^{-2x-3y}$, $x \geq 0, y \geq 0,$

The Marginal p.d.f of X:

$$f(x) = \int_0^\infty 6e^{-2x-3y}dy = 2e^{-2x},\ x \geq 0$$ ———————— (1m)

Conditional density of Y given X:

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{6e^{-2x-3y}}{2e^{-2x}} = 3e^{-3y}, y \geq 0.$$ ———————— (1m)

Part – B (3×6 = 18 Marks)

7.

We know that $$\sum_{j=1}^{3}\sum_{i=0}^{2} P(x_i, y_j) = 1 \Rightarrow 72k = 1 \; \boxed{k = \frac{1}{72}}$$

------------------------(1m)

MARGINAL DISTRIBUTION OF *X*:

$$P(X=0) = \sum_{j=1}^{3} P(X=0, Y=j) = 3k + 6k + 9k = 18k = \frac{18}{72}$$

$$P(X=1) = \sum_{j=1}^{3} P(X=1, Y=j) = 5k + 8k + 11k = 24k = \frac{24}{72}$$

$$P(X=2) = \sum_{j=1}^{3} P(X=2, Y=j) = 7k + 10k + 13k = 30k = \frac{30}{72}$$

MARGINAL DISTRIBUTION OF *Y*:

$$P(Y=1) = \sum_{i=0}^{2} P(X=i, Y=1) = 3k + 5k + 7k = 15k = \frac{15}{72}$$

$$P(Y=2) = \sum_{i=0}^{2} P(X=i, Y=2) = 6k + 8k + 10k = 24k = \frac{24}{72}$$

$$P(Y=3) = \sum_{i=0}^{2} P(X=i, Y=3) = 9k + 11k + 13k = 33k = \frac{33}{72}$$

------------------------(2m)

PROBABILITY DISTRIBUTION OF $(X+Y)$ **:**

$$P(X+Y=1)=P(X=0,\,Y=1)=3k=\frac{3}{72}$$

$$P(X+Y=2)=P(X=0,\,Y=2)+P(X=1,\,Y=1)=6k+5k=\frac{11}{72}$$

$$P(X+Y=3)=P(X=0,\,Y=3)+P(X=1,\,Y=2)$$

$$+P(X=2,\,Y=1)=9k+8k+7k=24k=\frac{24}{72}$$

$$P(X+Y=4)=P(X=1,\,Y=3)$$

$$+P(X=2,\,Y=2)=11k+10k=21k=\frac{21}{72}$$

$$P(X+Y=5)=P(X=2,\,Y=3)=13k=\frac{13}{72}$$

-----------------------(3m)

8. $E(U)=55, E(V)=5, COV(U,V)=0, E(X^2)=29, E(Y^2)=109, E(XY)=50, r(U,V)=0.$

-----------------------(6m)

9.

Let X denote the life time of the 60 bulbs.

Then $\mu = E(X) = 1200$ hrs. and $Var(X)=(S.D)^2=\sigma^2=(250)^2$hrs.

Let $\overline{X}$ denote the average life time of 60 bulbs.

By Central Limit Theorem, $\overline{X}$ follows $N\left(\mu,\frac{\sigma^2}{n}\right)$. ———————— (2)

Let $Z=\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ be the standard normal variable

$$P[\overline{X}>1250]=P[Z>1.55]$$

$$=0.5-P[0<Z<1.55]$$

$$=0.5-0.4394=0.0606$$ ———————— (4)

Part – C (2×10 = 20 Marks)

10. $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}$ $---------(6m)$

$$E(X) = \frac{K}{\lambda} -----------(2m)$$

$$E(X^2) = \frac{K(K+1)}{\lambda^2} ------------- (2m)$$

$$V(X) = \frac{K}{\lambda^2} ----------- (2m)$$

11.

$$P(X > 1/Y < \frac{1}{2}) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)} = \frac{5/24}{1/4} = \frac{5}{6} \quad \text{——————— (4m)}$$

$$P(Y < \frac{1}{2}/X > 1) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P(X > 1)} = \frac{5/24}{19/24} = \frac{5}{19} \quad \text{——————— (4m)}$$

$$P(X < Y) = \underset{\substack{R_4 \\ (x<y)}}{\int}\int \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{53}{480} \quad \text{——————— (4m)}$$

$$P(X + Y \le 1) = \underset{\substack{R_5 \\ (x+y\le 1)}}{\int}\int \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{13}{480} \quad \text{——————— (4m)}$$

12.

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2 - x - y)\,dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1 = \left[2 - x - \frac{1}{2}\right] = \frac{3}{2} - x,\ 0 \le x \le 1 \text{--------(1m)}$$

The marginal pdf of Y is

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1 (2-x-y)\,dx=\left[2x-\frac{x^2}{2}-yx\right]_0^1=\left[2-\frac{1}{2}-yx\right]=\frac{3}{2}-y,\ 0\le y\le 1 \text{-------(1m)}$$

$$E(X)=\int_{-\infty}^{\infty} xf_X(x)\,dx=\int_0^1 x.\left(\frac{3}{2}-x\right)dx=\int_0^1\left(\frac{3}{2}x-x^2\right)dx=\left[\frac{3}{2}.\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\frac{3}{4}-\frac{1}{3}=\frac{5}{12} \text{-------(1m)}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_0^1 x^2.\left(\frac{3}{2}-x\right)dx=\int_0^1\left(\frac{3}{2}x^2-x^3\right)dx=\left[\frac{3}{2}.\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{2}-\frac{1}{4}=\frac{1}{4} \text{-------(1m)}$$

$$E(Y)=\int_{-\infty}^{\infty} yf_Y(y)\,dy=\int_0^1 y.\left(\frac{3}{2}-y\right)dy=\int_0^1\left(\frac{3}{2}y-y^2\right)dx=\left[\frac{3}{2}.\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=\frac{3}{4}-\frac{1}{3}=\frac{5}{12} \text{-------(1m)}$$

$$E(Y^2)=\int_{-\infty}^{\infty} y^2 f_Y(y)\,dy=\int_0^1 y^2.\left(\frac{3}{2}-y\right)dy=\int_0^1\left(\frac{3}{2}y^2-y^3\right)dy=\left[\frac{3}{2}.\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1=\frac{1}{2}-\frac{1}{4}=\frac{1}{4} \text{-------(1m)}$$

$$\therefore Var(X)=E(X^2)-(E(X))^2=\frac{1}{4}-\left(\frac{5}{12}\right)^2=\frac{1}{4}-\frac{25}{144}=\frac{11}{144}\Rightarrow \sigma_X=\frac{\sqrt{11}}{12} \text{-------(1m)}$$

$$\therefore Var(Y)=E(Y^2)-(E(Y))^2=\frac{1}{4}-\left(\frac{5}{12}\right)^2=\frac{1}{4}-\frac{25}{144}=\frac{11}{144}\Rightarrow \sigma_X=\frac{\sqrt{11}}{12} \text{-------(1m)}$$

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy=\int_0^1\int_0^1 xy(2-x-y)dx\,dy=\int_0^1 y\left(\int_0^1(2x-x^2-xy)dx\right)dy$$

$$=\int_0^1 y\left[2\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^2}{2}y\right]_0^1 dy=\int_0^1 y\left(1-\frac{1}{3}-\frac{y}{2}\right)dy=\int_0^1\left(\frac{2}{3}y-\frac{y^2}{2}\right)dy \qquad \text{-------(2m)}$$

$$=\left[\frac{2}{3}\frac{y^2}{2}-\frac{y^3}{6}\right]_0^1=\left[\frac{1}{3}-\frac{1}{6}\right]=\frac{1}{6}$$

Cov(X , Y) = $\frac{1}{6}-\frac{5}{12}.\frac{5}{12}=-\frac{1}{144}$

$$\therefore r_{XY}=\frac{-\frac{1}{144}}{\frac{\sqrt{11}}{12}.\frac{\sqrt{11}}{12}}=-\frac{1}{11}=-0.0909 \text{-------(2m)}$$

13.

The joint pdf $f_{XY}(x,y)=f_X(x).f_Y(y)=e^{-x}.e^{-y}=e^{-(x+y)},\ x\ge 0,\ y\ge 0$ -------(1m)

The transformation functions are $u = \dfrac{x}{x+y}$ and $v = x + y$

Solving for x and y, we get $u = \dfrac{x}{v} \Rightarrow x = uv$

y = v - x $\Rightarrow$ y = v – vu

The Jacobian of the transformation is J =
$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v$$ -------(3m)

$\therefore x \geq 0$ and $y \geq 0$ we $have$ $uv \geq 0$ and $v(1-u) \geq 0$

$\Rightarrow v \geq 0$ and $u \geq 0$ and $u \leq 1$

$$f_{UV}(u,v) = \begin{cases} v e^{-v}, \ v \geq 0, 0 \leq u \leq 1 \\ 0 \ , \ elsewhere, \end{cases}$$ -------(3m)

The pdf of U is the marginal density function of U,

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)dv = \int_0^{\infty} v e^{-v} \, dv = \left[v.\frac{e^{-v}}{-1} - 1.\frac{e^{-v}}{(-1)^2} \right]_0^{\infty} = 0 + 1 = 1 \ , \ 0 \leq u \leq 1$$

The pdf of U is the marginal density function of U,

$$f_V(v) = \int_{-\infty}^{\infty} f(u,v)du = \int_0^1 v.e^{-v} du = v e^{-v} \ , v \geq 0$$

$f_U(u).\, f_V(v) = 1.v e^{-v} = f_{UV}(u,v)$ -------(6m)

****