Springs

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1 Energy, Force Vector, Stiffness Matrix

We define the spring energy, V, between two nodes to be

$$V = \frac{1}{2}K(l - l_0)^2, \tag{1}$$

where K is the stiffness constant, l is the current spring length, and l_0 is the spring length at rest. Denoting the two nodes by $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$, the vector between them is $\Delta\boldsymbol{x}$, and the length is $l = \|\Delta\boldsymbol{x}\|$.

The force acting on the first node is the negative gradient of the energy.

$$\boldsymbol{f}_0 = \frac{\partial V}{\partial \boldsymbol{x}_0} = \frac{\partial V}{\partial l}\frac{\partial l}{\partial \Delta\boldsymbol{x}}\frac{\partial \Delta\boldsymbol{x}}{\partial \boldsymbol{x}_0}. \tag{2}$$

Using the identity

$$\frac{\partial\|a\|}{\partial a} = \frac{a}{\|a\|}, \tag{3}$$

the force is

$$\boldsymbol{f}_0 = K(l - l_0)\frac{\Delta\boldsymbol{x}}{l}. \tag{4}$$

The force acting on the second node is simply the negation.

$$\boldsymbol{f}_1 = \frac{\partial V}{\partial \boldsymbol{x}_1} = -\boldsymbol{f}_0. \tag{5}$$

The stiffness matrix is the Hessian of the energy.

$$\mathsf{K}_{00} = \frac{\partial \boldsymbol{f}_0}{\partial \boldsymbol{x}_0} = K\left(\frac{\partial}{\partial \boldsymbol{x}_0}\{l - l_0\}\left(\frac{\Delta \boldsymbol{x}}{l}\right) + (l - l_0)\frac{\partial}{\partial \boldsymbol{x}_0}\left\{\frac{\Delta \boldsymbol{x}}{l}\right\}\right). \tag{6}$$

Using the identity

$$\frac{\partial}{\partial a}\left\{\frac{a}{\|a\|}\right\} = \frac{a^T a I - a a^T}{\|a\|^3}, \tag{7}$$

the stiffness matrix is

$$\mathsf{K}_{00} = \frac{K}{l^2}\left(\Delta \boldsymbol{x} \Delta \boldsymbol{x}^T + \frac{l - l_0}{l}\left(\Delta \boldsymbol{x}^T \Delta \boldsymbol{x} I - \Delta \boldsymbol{x} \Delta \boldsymbol{x}^T\right)\right) \tag{8}$$

The stiffness matrix for the whole element is

$$\mathsf{K} = \begin{pmatrix} \mathsf{K}_{00} & \mathsf{K}_{01} \\ \mathsf{K}_{10} & \mathsf{K}_{11} \end{pmatrix}, \tag{9}$$

where $\mathsf{K}_{01} = \mathsf{K}_{10} = -\mathsf{K}_{00}$, and where $\mathsf{K}_{11} = \mathsf{K}_{00}$.

2 Time Stepper

Conceptually, there are three types of things in the scene

- DoF: describes the configuration of the object.
- Elements: has geometry and mass
- Force emitters: produces forces

In this simple example, the nodes are the DoFs, and the springs are both the Elements and Force emitters.

2.1 Simulation Step

• Loop through all the *DoFs*, and collect the current positions and velocities into global vectors.

• Loop through all the *elements*, and distribute the element masses to the DoFs that are attached to the elements. In our example, we assign half the mass of the spring to each of the two incident nodes. Since this is constant, we could put this outside of the simulation loop, but in general, the mass matrix changes at every time step.

- Loop through all the *force emitters*, and compute the forces and the force gradient (stiffness matrix in this case).

- We then take a velocity step. We approximate the acceleration with finite differencing, so that the equations of motion is

$$\mathsf{M}\frac{\mathsf{v}' - \mathsf{v}}{h} = \mathsf{f}', \tag{10}$$

where we denote the quantities at the next time with a prime. M, v, and f are the global mass matrix, velocity vector, and force vector, respectively. Rearranging and expanding f', we get

$$\begin{align} \mathsf{M}\mathsf{v}' &= \mathsf{M}\mathsf{v} + h\mathsf{f}' \tag{11} \\ &= \mathsf{M}\mathsf{v} + h\left(\mathsf{f} + \frac{\partial \mathsf{f}}{\partial \mathsf{x}}(\mathsf{x}' - \mathsf{x})\right) \tag{12} \\ &= \mathsf{M}\mathsf{v} + h\left(\mathsf{f} + \frac{\partial \mathsf{f}}{\partial \mathsf{x}}h\mathsf{v}'\right). \tag{13} \end{align}$$

Since the force gradient is the stiffness matrix in our simple example with springs, we have

$$\left(\mathsf{M} - h^2 K\right)\mathsf{v}' = \mathsf{M}\mathsf{v} + h\mathsf{f}, \tag{14}$$

which we solve for the new velocities, v'. The sparsity of the LHS matrix depends on the connectivity of the springs.

- The position step is trivial: $\mathsf{x}' = \mathsf{x} + h\mathsf{v}'$.

- Finally, copy back the new positions and velocities back to the DoFs.