Enhanced Proof of the Traveling Salesman Problem

February 8, 2024

1 Introduction

The Traveling Salesman Problem (TSP) is a classic problem in combinatorial optimization. It involves finding the shortest possible route that visits each city exactly once and returns to the origin city. The TSP is known for its NP-hardness, making it a central problem in theoretical computer science and operations research.

2 Problem Statement

Consider n cities represented on a 2-dimensional coordinate plane, within a unit square $[0, 1]^2$. The goal is to find an ordering of these cities that minimizes the total distance traveled.

3 Lipschitz Continuity and Space-Filling Curves

We utilize a space-filling curve $f : [0, 1] \to [0, 1]^2$ with Lipschitz continuity of order $\frac{1}{2}$. This implies that for any two points $t_1, t_2 \in [0, 1]$, the distance between $f(t_1)$ and $f(t_2)$ is at most $2\sqrt{5}$ times the square root of the distance between t_1 and t_2. Formally, $\|f(t_1) - f(t_2)\| \leq 2\sqrt{5}|t_1 - t_2|^{1/2}$.

4 Optimization Approach

Given the cities' coordinates $x_1, \ldots, x_n$, we define a permutation map Π, where $k \in (1, \ldots, n)$ and π_k denotes an ordering. The objective is to minimize the distance:

$$\min_{\pi_k} dist(\pi_k) = \min_{\pi_k} \sum_{x_i : \pi_k} \|x_i - x_{i-1}\|$$

Let t_k be the preimage of x_k under f. By the property of Lipschitz continuity, the sum of distances between successive cities in the permutation can be

bounded:
$$\sum_{j=1}^{n} \|x_{\pi_j} - x_{\pi_{j-1}}\| \leq 2\sqrt{5} \sum_{j=1}^{n} |t_j - t_{j-1}|$$

This provides a constant upper bound for the sum of the squares of the distances:

$$\begin{aligned}\sum_{j=1}^{n} \|x_{\pi_j} - x_{\pi_{j-1}}\|^2 \leq 20 \sum_{j=1}^{n} |t_j - t_{j-1}|^2 \\ &\leq 40 \text{ (since in the worst case, } t_1 = 0 \text{ and } t_n = 1)\end{aligned}$$

5 Visualizations

The following figure illustrates the application of the TSP optimization approach to a randomly generated dataset of points within the unit square. The visualization helps in understanding the effectiveness of the approach in reducing the total distance traveled.

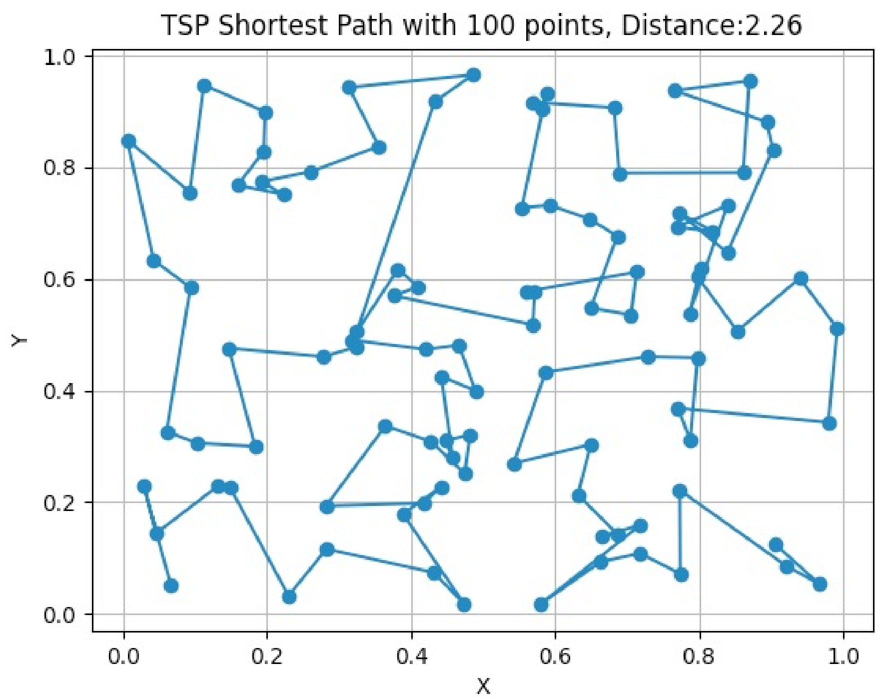


Figure 1: Visualization of the TSP solution for 100 points.

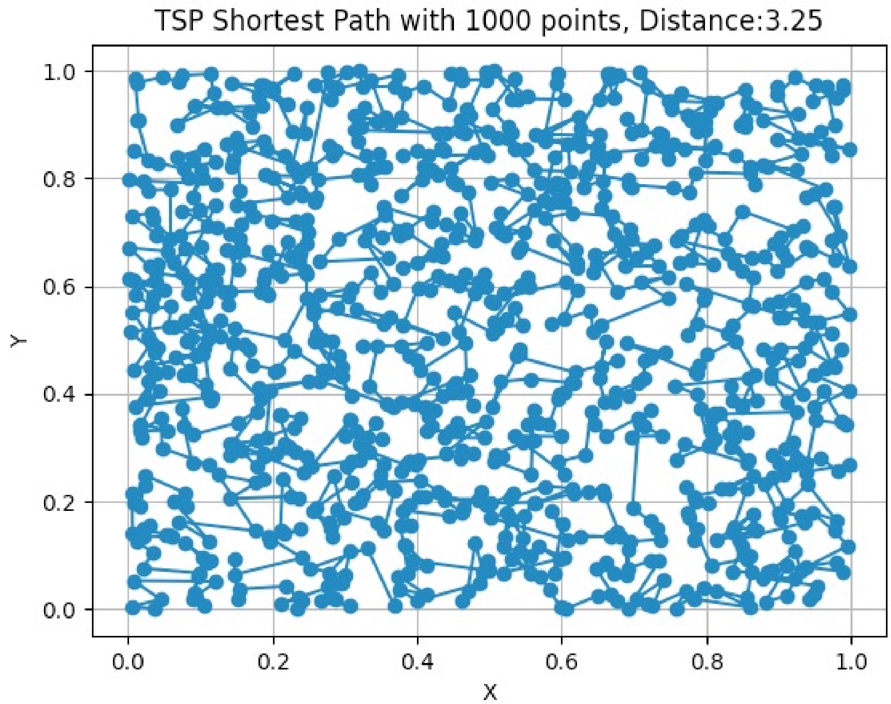


Figure 2: Visualization of the TSP solution for 1000 points.

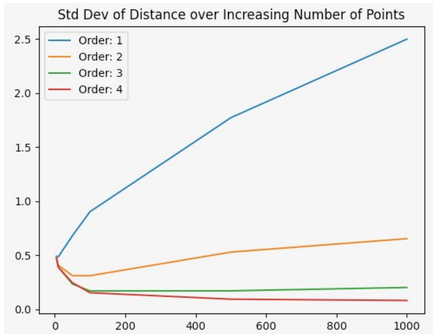
6 Simulation and Statistical Analysis

To further explore the effectiveness of our approach to the Traveling Salesman Problem, we conducted a series of simulations. Each simulation involved generating a random set of points within the unit square and applying our optimization algorithm to find the shortest path that visits each point exactly once.

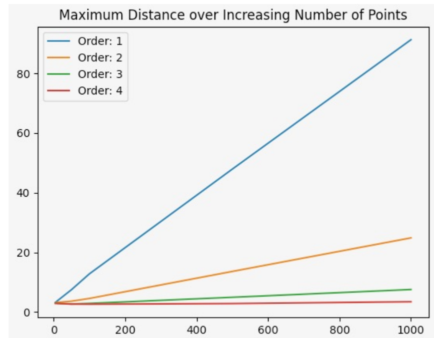
For each simulation, we recorded the total distance traveled in the optimal path. This process was repeated multiple times to gather a robust dataset of total distances for varying numbers of points. From this dataset, we calculated the following key statistics to assess the performance and reliability of our approach:

7 Complexity and Implications

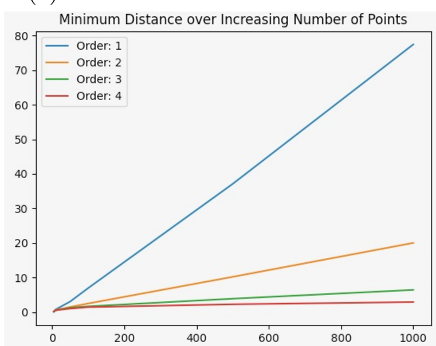
The TSP's complexity lies in the exponential number of possible tours. However, the Lipschitz continuity and the properties of space-filling curves provide a framework for approximating solutions, offering insights into heuristic and approximation algorithms for the TSP.



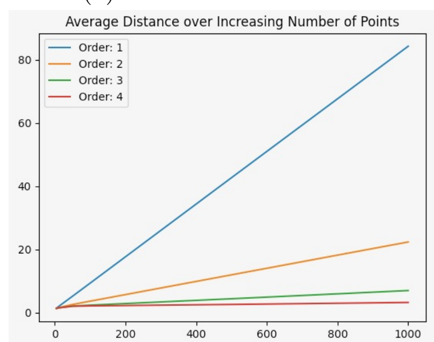
(a) Standard Deviation of Distances



(b) Maximum of Distances



(c) Minimum of Distances



(d) Average of Distances

Figure 3: A 2x2 grid showcasing the TSP Statistics for 100 and 1000 points.

8 Conclusion

This proof leverages space-filling curves and Lipschitz continuity to provide a bounded approximation for the Traveling Salesman Problem. It highlights the interplay between geometric properties and optimization strategies, offering a unique perspective on this classical problem.