Proof of the Radon-Nikodym Theorem

1 Introduction

The Radon-Nikodym theorem is a cornerstone of measure theory, providing a fundamental relationship between absolutely continuous measures.

2 Theorem Statement

Theorem 1 (Radon-Nikodym)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space, and let μ and ν be σ-finite measures on this space. If $\nu \ll \mu$, then there exists a μ-measurable function $f : \Omega \to [0, \infty)$ such that for all $A \in \mathcal{F}$,*

$$\nu(A) = \int_A f d\mu.$$

The function f is called the Radon-Nikodym derivative of ν with respect to μ and is denoted by $\frac{d\nu}{d\mu}$.

3 Proof

3.1 Uniqueness

If f and g are two Radon-Nikodym derivatives of ν with respect to μ, then for every measurable set A,

$$\int_A f d\mu = \nu(A) = \int_A g d\mu.$$

By the property of integrals, $f = g$ μ-almost everywhere.

3.2 Existence

The existence part is more involved and requires several steps. We outline the key components of the proof here:

3.2.1 Reduction to Finite Measures

We first reduce the problem to the case where μ and ν are finite measures by considering a countable partition of Ω into sets of finite measure under both μ and ν.

3.2.2 Verifying the Existence of Conditional Expectation

This section helps verify the existence of conditional expectation, which is crucial in the construction of the Radon-Nikodym derivative. Assume that $X \geq 0$ and $\mu = P$:

$$v(A) = \int_A X dP$$

By the Dominated Convergence Theorem, v is a measure and the integral definition implies that $v << \mu$. For any $A \in \mathcal{F}$ and $dv/d\mu \in \mathcal{F}$:

$$\int_A X dP = v(A) = \int_A \frac{dv}{d\mu} dP$$

Considering $A = \Omega$ and $dv/d\mu \geq 0$ being integrable shows that $dv/d\mu$ is a version of $E[X \mid \mathcal{F}]$. For $X = X^+ - X^-$, and defining $Y_1 = E[X^+|\mathcal{F}]$ and $Y_2 = E[X^-|\mathcal{F}]$, a version of $Y_2 - Y_1$ demonstrates conditional expectation:

$$\begin{aligned}
\int_A X dP &= \int_A X^+ dP - \int_A X^- dP \\
&= \int_A (Y_1 - Y_2) dP
\end{aligned}$$

3.2.3 Construction of the Radon-Nikodym Derivative

We define a class of functions C and identify a "largest" function z in this class, which serves as the Radon-Nikodym derivative. This involves showing that the supremum of integrals of functions in C is achieved and that z satisfies the integral property for ν.

3.2.4 Verification of Measure λ

We define a new measure $\lambda(A) = \nu(A) - \int_A z d\mu$ and show that λ is the zero measure. This involves constructing a sequence of "good" sets and demonstrating that λ behaves as expected over these sets.

3.3 Conclusion

By showing both the uniqueness and existence of the Radon-Nikodym derivative, we complete the proof of the Radon-Nikodym theorem.

4 Discussion on Conditional Expectation and Measure Theory

The concept of conditional expectation is central to understanding the Radon-Nikodym theorem. In the context of measure theory, conditional expectation reflects our "best guess" about the value of a random variable given the available information, represented by the σ-field $\mathcal{F}$. This allows for a more nuanced understanding of probability, especially when conditioning on more complex events or collections of events, as represented by various σ-fields.

Furthermore, the ability to work under multiple measures is critical in fields such as finance, where one must account for a wide range of possible events, including those with low probability but high impact. The Radon-Nikodym derivative plays a key role in this context by facilitating the transition between different probability measures, ensuring that events with zero probability under one measure have zero probability under another, and allowing for the comparison of expected values across measures.

5 Conclusion

The Radon-Nikodym theorem plays a crucial role in measure theory and its applications, providing a powerful tool for analyzing absolutely continuous measures and understanding the relationship between different measures through the Radon-Nikodym derivative.

6 Application to Finance

Suppose we had a call option C on some underlying stock S, and the stock has only two possibilities: going up or down at time T. We can denote these as our two possible outcomes: $\Omega = (\omega_u, \omega_d)$.

By the fundamental theorem of asset pricing, there must be some risk-neutral probability measure that makes this call act like a martingale — the discounted future value of the asset must be equal to the price of the asset currently. Or else, there would be arbitrage to exploit. So there must be some measure $\mathbb{Q}$ on the set of events (here just the power set of Ω) such that

$$C_0 = e^{-rT} E_{\mathbb{Q}}[C_T]$$

, where r is the risk-free rate.

What if there's a discrepancy between your model's predictions of the probability of the stock going up or down, and the actual price? There could be many reasons, but a common one is the idea of a risk premium. Investors typically want insurance against bad events, and as such, often overweight the chance of "black swan" events for their own portfolio's security. We can try to measure this.

Let $\mathbb{P}$ be the measure of the physical, real-world probabilities of the stock going up and down. Let $\mathbb{P}(\omega_u) = p_u$ and $\mathbb{P}(\omega_d) = p_d = 1 - p_u$. Similarly, for our risk-neutral measure $\mathbb{Q}$, let $\mathbb{Q}(\omega_u) = q_u$ and $\mathbb{Q}(\omega_d) = q_d = 1 - p_u$.

Then, note that

$$\mathbb{Q}(\omega_u) = \frac{q_u}{p_u} \mathbb{P}(\omega_u)$$

and similarly,

$$\mathbb{Q}(\omega_d) = \frac{1 - q_u}{1 - p_u} \mathbb{P}(\omega_d)$$

.

This is the Radon-Nikodym derivative between our two measures, or $\frac{d\mathbb{Q}}{d\mathbb{P}}$. And more generally, we can convert our earlier risk-neutral pricing equation:

$$C_0 = e^{-rT} E_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} C_T \right]$$

With this, we can use a Radon-Nikodym derivative to measure the distance between an implied distribution of risk versus the one priced into the market, and potentially factor that into pricing calculations in the future.