1. Model Answer

Question. Use Plancherel's identity to find a relation between the solution and the initial data in terms of mean square norm called L^2 norm of the Cauchy problem:

$$u_t + u_{xxxx} + 2u = 0,\ -\infty < x < \infty,\ t > 0$$

with initial data $u(x,0) = u_0(x), \ \ -\infty < x < \infty$; and also find $\lim\limits_{t\to\infty} \int_{-\infty}^{\infty} |u(x,t|^2\, dx$.

Model Answer: We have given

$$u_t + u_{xxxx} + 2u = 0,\ -\infty < x < \infty,\ t > 0, \tag{1.1}$$

with initial data $u(x,0) = u_0(x)$, $-\infty < x < \infty$. Taking Fourier transform with respect to x, from equation (1.1), we get

$$\begin{aligned} &\widehat{u}_t(w,t) + (iw)^4\widehat{u} + 2\widehat{u} = 0 \\ \Longrightarrow\ &\widehat{u}_t(w,t) + (w^4+2)\widehat{u} = 0 \\ \Longrightarrow\ &\widehat{u}(w,t) = \widehat{u}_0(w)e^{-t(w^4+2)} \qquad (1.2) \end{aligned}$$

Using Plancherel's identity, we get

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t)|^2 dx &= \int_{-\infty}^{\infty} |\widehat{u}(w,t)|^2 dw \\ &= e^{-4t}\int_{-\infty}^{\infty} |e^{-2tw^4}|\cdot|\widehat{u}_0(w)|^2 dw, \text{ using (1.2).} \\ &\le e^{-4t}\int_{-\infty}^{\infty} |\widehat{u}_0(w)|^2 dw, \ \text{ since } |e^{-2tw^4}| \le 1, \forall\, t > 0. \qquad (1.3) \end{aligned}$$

Therefore, the relation between the solution and the initial data is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \le e^{-4t}\int_{-\infty}^{\infty} |\widehat{u_0}(w)|^2 dw, \forall\, t > 0.$$

Taking limit as $t \to \infty$, we get

$$\begin{aligned} &0 \le \lim_{t\to\infty}\int_{-\infty}^{\infty} |u(x,t)|^2 dx \le \lim_{t\to\infty} e^{-4t}\int_{-\infty}^{\infty} |\widehat{u_0}(w)| dw = 0. \\ \Longrightarrow\ &\lim_{t\to\infty}\int_{-\infty}^{\infty} |u(x,t)|^2 dx = 0. \end{aligned}$$

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