Introduction to Bridgeland Stability

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ABSTRACT. In this expository article, we discuss the concept of stability conditions in triangulated category and derived category as introduced by Tom Bridgeland in his famous 2007 paper [Bri07]. After reviewing some basic theories, we discuss some of its interesting applications in algebraic geometry.

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Caution! I have started writing this note for myself to learn derived category, triangulated category, Bridgeland stability conditions, and their applications in various areas of algebraic geometry. The present note is unorganized, incomplete, and may contains many critical bugs to be fixed. Stable version 1.0 will be released publicly in my home page maybe after a year. After few years of improvements, a polished version may be released in arXiv as expository article. Any suggestions to improve the exposition are welcome.

Rolling version: https://arjunpaul29.github.io/home/notes/stability.pdf.

0. Introduction

One of my preliminary motivation to start this series of discussions is to understand *stability condition* in triangulated category as introduced by Tom Bridgeland in his celebrated 2007 paper published in Annals of Mathematics. We also would like to learn some of its application in some other areas, like birational geometry, mirror symmetry etc.

Bridgeland's original motivation was to mathematically formulate the concept of Π-*stability* in theoretical physics as formulated by Douglas. In physics, Π-stability is something to relate a *super-symmetric non-linear sigma model* with a *(2, 2) Super Conformal Field Theory (SCFT)*. Let's have a quick tour into an interesting intersection of geometry and physics.

0.1. **Motivation from modern physics.** Let us start with a tailor of a largely speculating theory, known as *mirror symmetry*. A *super-symmetric non-linear sigma model* consists of a complex Calabi-Yau variety $X = (M, I)$ admitting a Ricci flat Kähler form ω and a "B-field". Let us explain the terminologies:

- M is the underlined real manifold of X and I is a complex structure on it,
- the variety X is *Calabi-Yau* means that the canonical line bundle K_X is trivial,
- the Kähler form ω is *Ricci flat* means that the curvature $F_{\det(\nabla_\omega)} = 0$, where $\det(\nabla_\omega)$ is the connection on $\det(TX)$ induced by the Chern connection ∇_ω on TX with respect to the Kähler form ω, and
- that "B-field" is something mysterious.

  In the context of SYZ mirror symmetry (an attempt to understand mathematically original version of mirror symmetry in physics), a B-field should be a class of a *unitary flat gerbe*, as suggested by Hitchin.

It is expected from physical ground that such a *super-symmetric non-linear sigma model* should give us a *(2, 2) Super Conformal Field Theory (SCFT)*. But wait! What is this (2, 2) SCFT? We don't know any precise mathematical formulation of it, except for few cases! Roughly, a (2, 2) SCFT is some physical theory that depends on both complex and symplectic structures of varieties, and using *topological twists* one may separate its parts:

- *A-side*: depend only on symplectic structure, and
- *B-side*: depend only on complex structure.

In his famous ICM talk in 1994, Maxim Kontsevich proposed that the mathematical objects obtained from these topological twists should be in the derived category of coherent sheaves on the B-side (algebraic side), and in the derived Fukaya category of Lagrangian submanifolds on the A-side (symplectic side). Physically, objects of these categories are considered to be boundary conditions, known as *branes*. In this sense, Fukaya category is the category of A-branes and the derived category of coherent sheaves is the category of B-branes.

Conjecture 0.1.1 (Kontsevich)**.** *If two super-symmetric non-linear sigma models (X, ω, B) and (X', ω', B'), as described above, defines mirror symmetric SCFTs, then there are equivalences of categories:*

$$D^b(X) \simeq D^b(\mathrm{Fukaya}(X', \omega')) \quad \textit{and} \quad D^b(\mathrm{Fukaya}(X, \omega)) \simeq D^b(X') .$$

Well, this is mathematically quite vague because we don't have precise mathematical formulation of SCFT!

But, from the mathematical ground, we may consider the above Conjecture as a definition of mirror symmetric super-symmetric non-linear sigma models; i.e., we may call (X, ω, B) and (X', ω', B') to be mirror partner to each other if there are equivalences of such derived categories.

Remark 0.1.2. I have not seen such a definition in literature, but I think, it will be a very non-trivial problem to find out examples of such mirror symmetric pairs of super-symmetric non-linear sigma models. There is a notion of mirror symmetric varieties in SYZ sense, which identifies X and X' as dual to each other in an appropriate sense; see e.g., works of Hitchin, Hausel-Thaddues, Donagi-Pantev etc. This notion is quite different from my proposed definition above.

But where is the *Bridgeland stability conditions* (waiting for it for $\sim$15 minutes!) in the above picture? We shall see from construction of $D^b(X)$ that the derived category $D^b(X)$ depends only on complex/algebraic structure of X, and so $D^b(X)$ keeps only half information of the SCFT. Douglas argued that for any Ricci flat Kähler metric ω on X, there is a subcategory of $D^b(X)$, whose objects are physical branes, and these subcategories changes as the Kähler class ω moves in the stringy Kähler moduli.

To get an intutive idea what this mathematically means, instead of looking at whole $D^b(X)$, consider the abelian category $\mathfrak{Coh}(X)$. Then a cohice of Kähler class (or polarization) singles out semistable and stable objects of $\mathfrak{Coh}(X)$, and as we change the polarization, the collection of stable/semistable objects changes. Thus, there might be some way to encode more informations of SCFT purely in terms of triangulated category $D^b(X)$ together with some extra structure on it. In a series of papers, Bridgeland set out to put these ideas on a mathematical setting and introduced the notion of *stability conditions* on a triangulated category. He has shown that the space of such stability conditions forms an (infinite) dimensional manifold, and this can be thought of an approximation of the stringy Kähler moduli space.

Mathematically, interesting point is that this new theory associates to a very algebraic object, like a triangulated category, a moduli space with meaningful geometric structure.

Roughly, a *stability condition* on a triangulated category $\mathcal{A}$ is given by a heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{A}$ and an additive group homomorphism $Z : K_0(\mathcal{H}) \to \mathbb{C}$, called the "central charge", satisfying Harder-Narasimhan property.

Well, enough introduction, and we shall see these in detail! Let's first set up some languages from category theory.

1. Category Theory

1.1. **Abelian category.**

Definition 1.1.1. A *category* $\mathscr{A}$ consists of the following data:

(i) a class of objects, denoted $\mathrm{Ob}(\mathscr{A})$,
(ii) for $X, Y \in \mathrm{Ob}(\mathscr{A})$, a class of morphisms from X into Y, denoted $\mathrm{Hom}_{\mathscr{A}}(X, Y)$,
(iii) for each $X, Y, Z \in \mathrm{Ob}(\mathscr{A})$, a *composition map*

$$\mathrm{Hom}_{\mathscr{A}}(X, Y) \times \mathrm{Hom}_{\mathscr{A}}(Y, Z) \to \mathrm{Hom}_{\mathscr{A}}(X, Z), \quad (f, g) \mapsto g \circ f\,,$$

which satisfies associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{A}}(Z, W)$, for all $X, Y, Z, W \in \mathrm{Ob}(\mathscr{A})$.

A category $\mathscr{A}$ is said to be *locally small* if $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is a set, for all $X, Y \in \mathrm{Ob}(\mathscr{A})$. A category $\mathscr{A}$ is said to be *small* if it is locally small and the class of objects $\mathrm{Ob}(\mathscr{A})$ is a set.

Example 1.1.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{A}$ such that $g \circ f = \mathrm{Id}_{A_1}$ and $f \circ g = \mathrm{Id}_{A_2}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,
(ii) for $X, Y \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, there is a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$,

which are compatible with the composition maps.

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(F(A_1), F(A_2))$$

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ is given by the following data: for each object $A \in \mathscr{A}$, a map $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_A"'] & \mathcal{F}(A') \arrow[d, "\varphi_{A'}"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A') \end{tikzcd} \tag{1.1.3}$$

Definition 1.1.4. A morphism $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathcal{F}un(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 1.1.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 1.1.6. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Hom}_{\mathscr{A}}(X, Y) \times \mathrm{Hom}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.1.7 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.

(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0,0)$ is the trivial group with one element.

(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $\mathrm{j}_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 1.1.8. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A, B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A,B) \times \mathrm{Hom}_{\mathscr{A}}(B,C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A,C)\,, \quad (f,g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 1.1.9. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1,A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 1.1.10. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
& C \arrow[dl, dashed, "\exists ! \widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined by reversing the arrows of the above diagram.

Definition 1.1.11. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists ! \widetilde{g}"] \\
& C &
\end{tikzcd}$$

Definition 1.1.12. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 1.1.13. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.
(ii) If a cokernel of f exists, then it is an epimorphism.
(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.
(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). □

Exercise 1.1.14. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 1.1.15. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$ (see Lemma 1.1.13 (iii)). This completes the proof.

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\
& & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & &
\end{tikzcd} \tag{1.1.16}$$

□

Definition 1.1.17. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 1.1.12).

Example 1.1.18.
(1) For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.
(2) Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules on X. Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{QCoh}(X)$ (reps., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of $\mathcal{O}_X$-modules on X, are also abelian. However, the full subcategory $\mathcal{V}ect(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of $\mathcal{O}_X$-modules on X, is not abelian, because kernel of a morphism in $\mathcal{V}ect(X)$ may not be in $\mathcal{V}ect(X)$.

1.2. **Triangulated category.** Let $\mathscr{A}$ be an additive category. A *shift functor* is an additive functor

(1.2.1) $$T : \mathscr{A} \longrightarrow \mathscr{A}\,,$$

which is an equivalence of categories. A *triangle* in $(\mathscr{A}, T)$ is given by a diagram

(1.2.2) $$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,,$$

with objects and arrows in $\mathscr{A}$. A *morphism of triangles* in $(\mathscr{A}, T)$ is given by a commutative diagram

(1.2.3) $$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

where $f[1] := T(f) \in \mathrm{Hom}_{\mathscr{A}}(A[1], A'[1])$. If, in addition, f, g, h are isomorphisms in $\mathscr{A}$, we say that (1.2.3) is an isomorphism of triangles. We denote by $A[n]$ the object $T^n(A) \in \mathscr{A}$, and denote by $f[n]$ the morphism $T^n(f) \in \mathrm{Hom}_{\mathscr{A}}(A[n], B[n])$, for $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$.

Definition 1.2.4. A *triangulated category* is an additive category $\mathscr{A}$ together with an additive equivalence (*shift functor*)

(1.2.5) $$T : \mathscr{A} \longrightarrow \mathscr{A}\,,$$

and a set of *distinguished triangles*

(1.2.6) $$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,,$$

satisfying the following axioms (TR1) – (TR4) below.

(TR1) (i) Any triangle of the form

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is a distinguished triangle.
(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]\,.$$

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is a distinguished triangle.

(TR3) Any commutative diagram of distinguished triangles with vertical arrows f and g

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dashed, "\exists\, h"'] & A[1] \arrow[d, "f[1]"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} \tag{1.2.7}$$

can be completed to a commutative diagram (not necessarily in a unique way).

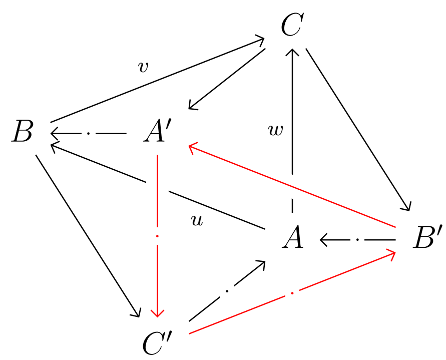
(TR4) (Octahedral axiom) Given any three distinguished triangles

$$A \xrightarrow{u} B \longrightarrow C' \longrightarrow A[1]$$
$$B \xrightarrow{v} C \longrightarrow A' \longrightarrow B[1]$$
$$A \xrightarrow{w} C \longrightarrow B' \longrightarrow A[1]$$

there is a distinguished triangle $C' \longrightarrow B' \longrightarrow A' \longrightarrow C'[1]$ such that the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\mathrm{Id}"] & B \arrow[r] \arrow[d, "v"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, "\mathrm{Id}"] \\ A \arrow[r, "w"] \arrow[d, "u"] & C \arrow[r] \arrow[d, "\mathrm{Id}"] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "u[1]"] \\ B \arrow[r, "v"] \arrow[d] & C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\mathrm{Id}"] & B[1] \arrow[d] \\ C' \arrow[r] & B' \arrow[r] & A' \arrow[r] & C'[1] \end{tikzcd}$$

This axiom is called *"octahedron axiom"* because of its original formulation: given composable morphisms $A \xrightarrow{u} B \xrightarrow{v} C$, with $w := v \circ u$, we have the following octahedron diagram.



where any triangle of the form
$$\begin{tikzcd}[column sep=small] & Z \arrow[dl, dashed] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$
are distinguished triangles, and the arrow $Z \dashrightarrow X$ stands for $Z \longrightarrow X[1]$.

Remark 1.2.8. A triangulated category need not be abelian, in general. In triangulated category, distinguished triangles play the roles of exact sequences in abelian categories. Examples of triangulated categories, we will be interested in, are derived categories of abelian categories.

1.3. **Semi-orthogonal decomposition.** Let $\mathscr{D}$ be a k-linear triangulated category. An object $E \in \mathscr{D}$ is said to be *exceptional* if

$$\operatorname{Hom}_{\mathscr{D}}(E, E[\ell]) = \begin{cases} k & \text{if } \ \ell = 0\,, \\ 0 & \text{if } \ \ell \neq 0\,. \end{cases} \tag{1.3.1}$$

An *exceptional sequence* in $\mathscr{D}$ is a sequence of exceptional objects $E_1, E_2, \ldots, E_n$ of $\mathscr{D}$ such that $\operatorname{Hom}_{\mathscr{D}}(E_i, E_j[\ell]) = 0$, for all $i > j$ and all ℓ. In other words, if

$$\operatorname{Hom}_{\mathscr{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if } \ i = j \text{ and } \ell = 0\,, \\ 0 & \text{if } \ i > j, \text{ or if } \ell \neq 0 \text{ and } i = j\,. \end{cases} \tag{1.3.2}$$

An exceptional sequence $\{E_i\}_{i=1}^n$ is said to be *full* if, as a triangulated category, $\mathscr{D}$ is generated by $\{E_i\}_{i=1}^n$; i.e., if $\mathscr{D}'$ is a triangulated full subcategory of $\mathscr{D}$ containing E_i, for all $i = 1, \ldots, n$, then the inclusion morphism $\mathscr{D}' \hookrightarrow \mathscr{D}$ is an equivalence of categories.

1.4. **t-structure and heart.** Let $(\mathscr{A}, T)$ be a triangulated category. Let $\mathscr{B}$ be a subcategory of $\mathscr{A}$. For an integer n, we denote by $\mathscr{B}[n]$ the full subcategory of $\mathscr{A}$, whose objects are of the form $X[n]$, with $X \in \mathscr{B}$. In other words, $\mathscr{B}[n] = T^n(\mathscr{B}) \subset \mathscr{A}$.

Definition 1.4.1. Let $\mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 0}$ be two full subcategories of $\mathscr{A}$. For an integer n, let $\mathscr{A}^{\leq n} := \mathscr{A}^{\leq 0}[-n]$ and $\mathscr{A}^{\geq n} := \mathscr{A}^{\geq 0}[-n]$. A *$t$-structure* on $\mathscr{A}$ is given by a pair $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ of full subcategories of $\mathscr{A}$ satisfying the following axioms.

(t1) $\mathscr{A}^{\leq -1} \subset \mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 1} \subset \mathscr{A}^{\geq 0}$.

(t2) For any $X \in \mathscr{A}^{\leq 0}$ and $Y \in \mathscr{A}^{\geq 1}$, we have $\mathrm{Hom}_{\mathscr{A}}(X, Y) = 0$.
(t3) For any $X \in \mathscr{A}$, there is a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]\,,$$

with $X_0 \in \mathscr{A}^{\leq 0}$ and $X_1 \in \mathscr{A}^{\geq 1}$.

In this case, the full subcategory $\mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ of $\mathscr{A}$ is called the *heart* (or, *core*) of the t-structure $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$.

Example 1.4.2 (Standard t-structure on $D^b(X)$)**.** Consider the full subcategories

$$\begin{aligned}\mathcal{A}^{\leq 0} &:= \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i > 0\} \text{ and} \\ \mathcal{A}^{\geq 0} &:= \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i < 0\}\end{aligned}$$

of $D^b(X)$. The axiom (T1) is easy to see. To check axiom (T2), we need some notations. For an integer $n \in \mathbb{Z}$, let $D^{\leq n}(X)$ (resp., $D^{\geq n}$) be the full subcategory of $D^b(X)$, whose objects are $E^\bullet \in D^b(X)$ satisfying $E^i = 0$, for all $i \leq n$ (resp., for all $i \geq n$). Consider the *truncation functors* $\tau^{\leq n} : D^b(X) \longrightarrow D^{\leq n}(X)$ and $\tau^{\geq n} : D^b(X) \longrightarrow D^{\geq n}(X)$ defined by

$$\begin{aligned}\tau^{\leq n}(E^\bullet) &:= (\cdots \to E^{n-2} \to E^{n-1} \to \mathrm{Ker}(d_E^n : E^n \to E^{n+1}) \to 0 \to \cdots), \text{ and} \\ \tau^{\geq n}(E^\bullet) &:= (\cdots \to 0 \to \mathrm{Ker}(d^n : E^n \to E^{n+1}) \to E^n \to E^{n+1} \to \cdots)\,.\end{aligned}$$

Now take $E^\bullet \in \mathcal{A}^{\leq 0}$ and $F^\bullet \in \mathcal{A}^{\geq 1}$. If $f \in \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet)$, then f factors as

$$\begin{tikzcd} E^\bullet \arrow[rrr, "f"] \arrow[dr, "\tau^{\leq 0}"', "\cong"] & & & F^\bullet \\ & \tau^{\leq 0}(E^\bullet) \arrow[r, "\tau^{\leq 0}(f)"] & \tau^{\leq 0}(F^\bullet) \arrow[ur] & \end{tikzcd}\,.$$

Since $\tau^{\leq 0}(F^\bullet) = 0$ in $D^b(X)$, we conclude that $f = 0$. Axiom (T3) follows from the exact triangle

$$\tau^{\leq 0}(E^\bullet) \longrightarrow E^\bullet \longrightarrow \tau^{\geq 1}(E^\bullet) \longrightarrow \tau^{\leq 0}(E^\bullet)[1],\ \ \forall\, E^\bullet \in D^b(X)\,.$$

Thus $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a t-structure on $D^b(X)$, and the associated heart $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ is isomorphic to $\mathfrak{Coh}(X)$.

The above mentioned t-structure on $D^b(X)$ is not interesting, and somehow useless. We shall be interested in some non-trivial t-structures on $D^b(X)$ giving more interesting and useful hearts different from $\mathfrak{Coh}(X)$.

Lemma 1.4.3. *The heart of a t-structure on $\mathcal{A}$ is an abelian category.*

Remark 1.4.4. If $D^b(\mathcal{A}) \cong D^b(X)$ for some abelian subcategory $\mathcal{A}$ of $D^b(X)$, then $\mathcal{A}$ is a heart of a t-structure on $D^b(X)$. However, the converse is not true, in general.

1.5. **Derived category of an abelian category.** Let $\mathscr{A}$ be an abelian category. A *complex* in $\mathscr{A}$ is given by

$$A^\bullet \quad : \quad \cdots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \cdots , \tag{1.5.1}$$

where A^i are objects of $\mathscr{A}$ and d_A^i are morphisms in $\mathscr{A}$ such that $d_A^i \circ d_A^{i-1} = 0$, for all $i \in \mathbb{Z}$. A complex $A^\bullet$ in $\mathscr{A}$ is said to be *bounded above* (resp., *bounded below*) if there is an integer i_0 such that $A^i = 0$, for all $i \geq i_0$ (resp., if there is an integer j_0 such that $A^j = 0$, for all $j \leq j_0$). If $A^\bullet$ is both bounded above and bounded below, we say that $A^\bullet$ is *bounded*.

A morphism $f^\bullet : A^\bullet \to B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ of objects and morphisms from $\mathscr{A}$ is given by a collection of morphisms $\{f^i : A^i \to B^i\}_{i \in \mathbb{Z}}$ in $\mathscr{A}$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & A^i \arrow[r, "d_B^i"] & A^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots
\end{tikzcd} \tag{1.5.2}$$

Let $\mathcal{K}om(\mathscr{A})$ be the category, whose objects are complexes of objects and morphisms from $\mathscr{A}$, and morphisms are given by morphism of complexes, as defined in (1.5.2). Denote by $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ the full subcategories of $\mathcal{K}om(\mathscr{A})$, whose objects are bounded above complexes, resp., bounded below complexes, resp., bounded complexes. Then we have the following.

Proposition 1.5.3. *For any abelian category $\mathscr{A}$, the categories $\mathcal{K}om(\mathscr{A})$, $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ are abelian.*

Definition 1.5.4. For any complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ and $k \in \mathbb{Z}$, we define its k^{th}-shift to be the complex $A[k]^\bullet \in \mathcal{K}om(\mathscr{A})$ satisfying

(i) $A[k]^i := A^{k+i}$, for all $i \in \mathbb{Z}$, and
(ii) $d^i_{A[k]^\bullet} := (-1)^k d^{i+k}_{A^\bullet} : A[k]^i \longrightarrow A[k]^{i+1}$, for all $i \in \mathbb{Z}$.

Proposition 1.5.5. *For any integer k, the k^{th}-shift functor*

$$\mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{A})\,, \quad A^\bullet \longmapsto A[k]^\bullet$$

is an equivalence of categories.

Proof. Clearly, the $(-k)^{\text{th}}$-shift functor $A^\bullet \longmapsto A[-k]^\bullet$ defines the inverse functor of the k^{th}-shift functor. $\square$

Remark 1.5.6. We shall see later that the category $\mathcal{K}om(\mathscr{A})$ together with the shift functor do not form a triangulated category, in general. However, we shall construct the derived category $D^b(\mathscr{A})$ from $\mathcal{K}om(\mathscr{A})$, which will turn out to be a triangulated category.

Given a complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, we define its i^{th} cohomology sheaf

$$\mathcal{H}^i(A^\bullet) := \frac{\mathrm{Ker}(d^i_{A^\bullet})}{\mathrm{Im}(d^{i-1}_{A^\bullet})} \in \mathscr{A}\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{1.5.7}$$

A complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ is said to be *acyclic* if $\mathcal{H}^i(A^\bullet) = 0$, for all $i \in \mathbb{Z}$. Any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$ of complexes gives rise to natural homomorphisms

$$\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet)\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{1.5.8}$$

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories. Let

$$\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B} \tag{1.5.9}$$

be an additive functor. Then $\mathcal{F}$ induces a functor, also denoted by the same symbol,

$$\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{B}) \tag{1.5.10}$$

defined by sending $A^\bullet \in \mathcal{K}om(\mathscr{A})$ to the complex $\mathcal{F}(A^\bullet)$, defined by

(i) $\mathcal{F}(A^\bullet)^i := \mathcal{F}(A^i)$, for all $i \in \mathbb{Z}$, and
(ii) $d^i_{\mathcal{F}(A^\bullet)} : \mathcal{F}(A^i) \xrightarrow{F(d^i_{A^\bullet})} \mathcal{F}(A^{i+1})$, for all $i \in \mathbb{Z}$,

and for any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$, we have a natural morphism of complexes

$$\mathcal{F}(f^\bullet) : \mathcal{F}(A^\bullet) \longrightarrow \mathcal{F}(B^\bullet)$$

defined by $\mathcal{F}(f^\bullet)^i := \mathcal{F}(f^i) : \mathcal{F}(A^i) \longrightarrow \mathcal{F}(B^i)$, for all $i \in \mathbb{Z}$.

Definition 1.5.11. An additive functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *exact* if it takes exact sequence to exact sequence.

Remark 1.5.12. Note that $\mathcal{F}$ is exact if and only if for any acyclic complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, its image $\mathcal{F}(A^\bullet) \in \mathcal{K}om(\mathscr{B})$ is acyclic.

Since $\mathcal{K}om(\mathscr{A})$ is abelian for $\mathscr{A}$ abelian, we can talk about short exact sequences in $\mathcal{K}om(\mathscr{A})$. Then by standard techniques from homological algebra, any short exact sequence

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0 \tag{1.5.13}$$

gives rise to a long exact sequence of cohomologies (which are objects of $\mathscr{A}$)

$$\cdots \longrightarrow \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \longrightarrow \mathcal{H}^i(C^\bullet) \longrightarrow \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \cdots\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{1.5.14}$$

Definition 1.5.15. A morphism of complexes $f^\bullet : A^\bullet \longrightarrow B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is called *quasi-isomorphism* if the induced morphism

$$\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \tag{1.5.16}$$

is an isomorphism, for all $i \in \mathbb{Z}$.

Example 1.5.17. Let X be a smooth projective k-variety and let E be a coherent sheaf on X. Then we can find a finite resolution

$$0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0 \to E \to 0\,.$$

of E with E^i projective (locally free) $\mathcal{O}_X$-modules. (We can use this to study many properties of E in terms of locally free coherent sheaves.) This gives rise to a morphism of complexes

$$f^\bullet : (0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0) \longrightarrow (\cdots \to 0 \to E \to 0 \to \cdots),$$

which is a quasi-isomorphism.

The main idea for definition of derived category is: *quasi-isomorphism of complexes should become isomorphism in the derived category*. Therefore, the derived category $D(\mathscr{A})$ is the localization of $\mathcal{K}om(\mathscr{A})$ by quasi-isomorphisms. This can be done by passing to the appropriate homotopy category.

Theorem 1.5.18. *Let $\mathscr{A}$ be an abelian category, and $\mathcal{K}om(\mathscr{A})$ the category of complexes in $\mathscr{A}$. Then there is a category $D(\mathscr{A})$, known as the derived category of $\mathscr{A}$, together with a functor*

$$Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A}) \tag{1.5.19}$$

such that:

(i) If $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a quasi-isomorphism, then $Q(f^\bullet)$ is an isomorphism in $D(\mathscr{A})$,
(ii) if a functor $\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathscr{D}$ satisfies property (i), there is a unique functor $\widetilde{\mathcal{F}} : D(\mathscr{A}) \longrightarrow \mathscr{D}$ such that $\widetilde{\mathcal{F}} \circ Q \cong \mathcal{F}$.

$$\begin{tikzcd} \mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "\mathcal{F}"'] & & D(\mathscr{A}) \arrow[dl, dashed, "\exists\,!\,\widetilde{\mathcal{F}}"] \\ & \mathscr{D} & \end{tikzcd} \tag{1.5.20}$$

Now we go ahead for construction of the derived category $D(\mathscr{A})$ of $\mathscr{A}$. Since we want any quasi-isomorphism $C^\bullet \to A^\bullet$ of complexes in $\mathcal{K}om(\mathscr{A})$ to become isomorphism in the derived category $D(\mathscr{A})$, any morphism of complexes $C^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ should give rise to a morphism $A^\bullet \to B^\bullet$ in $D^b(\mathscr{A})$. This leads to the definition of morphisms in $D^b(\mathscr{A})$ as diagrams of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet\,, \end{tikzcd}$$

where "qis" stands for "quasi-isomorphism" of complexes. To make this more precise, we need to define when two such roofs should considered to be equal, and how to define their compositions. Then natural context for both problems is to consider the homotopy category $K(\mathscr{A})$ of complexes in $\mathcal{K}om(\mathscr{A})$, which is an intermediate step for going from $\mathcal{K}om(\mathscr{A})$ to $\mathscr{D}(\mathscr{A})$.

$$\begin{tikzcd}
\mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, bend right] & & D(\mathscr{A}) \\
& K(\mathscr{A}) \arrow[ur, bend right, "Q'"'] &
\end{tikzcd}$$

Definition 1.5.21. Two morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ are said to be *homotopically equivalent*, written as $f^\bullet \sim g^\bullet$, if there is a morphism of complexes $h^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

$$\begin{tikzcd}
A^\bullet \arrow[d, shift right, "g^\bullet"'] \arrow[d, shift left, "f^\bullet"] & \cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, shift right, "g^{i-1}"'] \arrow[d, shift left, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] \arrow[dl, "h^i"'] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, shift right, "g^{i+1}"'] \arrow[d, shift left, "f^{i+1}"] \arrow[dl, "h^{i+1}"'] & \cdots \\
B^\bullet & \cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots
\end{tikzcd}$$

Let $K(\mathscr{A})$ be the category, whose objects are the same as objects of $\mathcal{K}om(\mathscr{A})$ and morphisms are given by $\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, for all $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$.

Following proposition is an easy consequence of the above definition.

Proposition 1.5.22. *(i) Homotopy equivalence of morphisms $A^\bullet \to B^\bullet$ of complexes is an equivalence relation.*

(ii) Homotopically trivial morphisms of complexes form an 'ideal' in the morphisms of $\mathcal{K}om(\mathscr{A})$.

(iii) If $f^\bullet$ and $g^\bullet$ are two homotopically equivalent morphisms of complexes in $\mathcal{K}om(\mathscr{A})$, then the induced morphisms $\mathcal{H}^i(f^\bullet)$ and $\mathcal{H}^i(g^\bullet)$ from $\mathcal{H}^i(A^\bullet)$ to $\mathcal{H}^i(B^\bullet)$ coincides.

(iv) Let $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to A^\bullet$ be two morphisms of complexes. If $f^\bullet \circ g^\bullet \sim \mathrm{Id}_{B^\bullet}$ and $g^\bullet \circ f^\bullet \sim \mathrm{Id}_{A^\bullet}$, then $f^\bullet$ and $g^\bullet$ are quasi-isomorphisms, and $\mathcal{H}^i(f^\bullet)^{-1} = \mathcal{H}^i(g^\bullet)$, for all $i \in \mathbb{Z}$.

Now we complete the construction of derived category $D(\mathscr{A})$. Take $\mathrm{Ob}(D(\mathscr{A})) := \mathrm{Ob}(\mathcal{K}om(\mathscr{A}))$. As discussed before, a morphism $f : A^\bullet \to B^\bullet$ in $D(\mathscr{A})$ is given by equivalence class of roofs of the form

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

where $C^\bullet \xrightarrow{qis} A^\bullet$ is a quasi-isomorphism of complexes in $\mathcal{K}om(\mathscr{A})$, and two such roofs

$$\begin{tikzcd}
& C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & A^\bullet & & B^\bullet
\end{tikzcd}$$

are considered to be equivalent if they are dominated by a third one of the same type

$$
\begin{tikzcd}
 & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] \arrow[ddll, dashed, bend right, "qis"'] & & \\
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "qis"] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}
\tag{1.5.23}
$$

such that the above diagram commutes in the homotopy category $K(\mathscr{A})$. In particular, the compositions $C_0^\bullet \to C_1^\bullet \to A^\bullet$ and $C_0^\bullet \to C_2^\bullet \to A^\bullet$ are homotopically equivalent. To define composition of morphisms in $D(\mathscr{A})$, consider two roofs

$$
\begin{tikzcd}
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
B^\bullet & & C^\bullet
\end{tikzcd}
$$

representing two morphisms in $D(\mathscr{A})$. It is natural to guess that, one should be able to define their composition to be a morphism represented by 'the' following roof

$$
\begin{tikzcd}
 & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet ,
\end{tikzcd}
\tag{1.5.24}
$$

which commutes in the homotopy category $K(\mathscr{A})$. Now we need to ensure that such a diagram exists uniquely, up to *equivalence of roofs* as defined in (1.5.23) (c.f., Proposition 1.5.29). For this, we need the concept of "mapping cone".

Definition 1.5.25. The *mapping cone* of a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a complex $C(f^\bullet)$ defined as follow:

$$C(f^\bullet)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f^\bullet)} = \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix}, \ \forall\, i \in \mathbb{Z}\,.$$

Note that $C(f^\bullet)$ is a complex. Moreover, there are natural morphisms of complexes

$$\tau : B^\bullet \to C(f^\bullet) \quad \text{and} \quad \pi : C(f^\bullet) \to A^\bullet[1] \tag{1.5.26}$$

given by natural injection $B^i \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively, for all i. Then we have the following.

(i) The composition $B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$ is trivial. In fact,

$$0 \longrightarrow B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1] \longrightarrow 0$$

is a short exact sequence in $\mathcal{K}om(\mathscr{A})$, and gives us a long exact sequence of cohomologies

$$\cdots \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C(f^\bullet)) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots .$$

(ii) The composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet)$ is homotopic to the trivial morphism. Indeed, take $h^\bullet : A^\bullet \longrightarrow C(f^\bullet)$ to be morphism of complexes defined by the natural injective morphism $h^i : A^i \to C(f^\bullet)^{i-1} = A^i \oplus B^{i-1}$, for all i. Then we have

$$h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{C(f^\bullet)} \circ h^i = \tau^i \circ f^i, \quad \forall\, i \in \mathbb{Z}\,.$$

Remark 1.5.27. It follows from the above construction that any commutative diagram of complexes

$$\begin{tikzcd}
A_1^\bullet \arrow[r, "f_1^\bullet"] \arrow[d, "\phi"] & B_1^\bullet \arrow[r, "\tau_1"] \arrow[d, "\psi"] & C(f_1^\bullet) \arrow[r, "\pi_1"] \arrow[d, dashed, "\phi[1]\oplus\psi", "\exists"'] & A_1^\bullet \arrow[d, "\phi[1]"] \\
A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r, "\tau_2"] & C(f_2) \arrow[r, "\pi_2"] & A_2^\bullet
\end{tikzcd}$$

can be completed by a dashed arrow as above (c.f., axiom (TR3) in Definition 1.2.4).

Proposition 1.5.28. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone. Let $\tau : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$ be the natural morphisms as in (1.5.26). Then there is a morphism of complexes $g : A^\bullet[1] \longrightarrow C(\tau)$ which is an isomorphism in $K(\mathscr{A})$ such that the following diagram commutes in the homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \\
 & & C(\tau) \arrow[ur, "\pi_\tau"'] &
\end{tikzcd}$$

Proof. Define $g : A^\bullet[1] \longrightarrow C(\tau)$ by setting

$$g^i : A^\bullet[1]^i = A^{i+1} \longrightarrow C(\tau)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$$

to be the morphism $g^i = (-f^{i+1}, \mathrm{Id}_{A^{i+1}}, 0)$, for all i. Clearly, g is a morphism of complexes, and its inverse (in $K(\mathscr{A})$) is given by the morphism of complexes $g^{-1} : C(\tau) \to A^\bullet[1]$ defined by projection onto the middle factor. Clearly, $\pi_\tau \circ g = -f$ in $\mathcal{K}om(\mathscr{A})$. However, the diagram

$$\begin{tikzcd}
C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[d, "g"] \\
 & C(\tau)
\end{tikzcd}$$

does not commute in $\mathcal{K}om(\mathscr{A})$. We show that, $\pi \circ g \sim \tau_\tau$. For this, note that $g^{-1} \circ \tau_\tau = \pi$ in $\mathcal{K}om(\mathscr{A})$. Since $g \circ g^{-1} \sim \mathrm{Id}_{C(\tau)}$, we have $\tau_\tau \sim g \circ \pi$. This completes the proof. $\square$

Now we use the above proposition to complete the proof of existence and uniqueness of composition of morphisms in $D(\mathscr{A})$.

Proposition 1.5.29. *Let $f : A^\bullet \to B^\bullet$ and $g : C^\bullet \to B^\bullet$ be morphism of complexes with f a quasi-isomorphism. Then there is a complex $C_0^\bullet$ together with a quasi-isomorphism $C_0^\bullet \to C^\bullet$ and a morphism $C_0^\bullet \to A^\bullet$ such that the following diagram commutes in the homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd}
C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet \arrow[d, "g"] \\
A^\bullet \arrow[r, "f", "qis"'] & B^\bullet
\end{tikzcd}$$

Proof. Note that, there is a natural morphism of complexes $\phi^i : C(\tau \circ g) \to C(\tau)$ given by the natural projection

$$\phi^i : C(\tau \circ g)^i = C^{i+1} \oplus C(f)^i = C^{i+1} \oplus A^{i+1} \oplus B^i \xrightarrow{pr_2} A^{i+1} = A^\bullet[1]^i$$

onto the middle factor, for each i. By Proposition 1.5.28, there is a quasi-isomorphism of complexes $\psi : C(\tau) \xrightarrow{qis} A^\bullet[1]$. This gives us a morphism of complexes $\Phi[-1] : C(\tau \circ g)[-1] \longrightarrow A^\bullet$, where $\Phi = \psi \circ \phi$, such that the following diagram is commutative.

$$\begin{tikzcd}
C_0^\bullet := C(\tau \circ g)[-1] \arrow[r] \arrow[d, dashed, "{\Phi[-1]}"] & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, "g"] & C(f) \arrow[r, "\tau_{\tau\circ g}"] \arrow[d, equal] & C(\tau \circ g) \arrow[d, "\phi"] \arrow[dr, "\Phi:=\psi\circ\phi"] & \\
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & C(\tau) \arrow[r, "\psi", "qis"'] & A^\bullet[1]
\end{tikzcd}$$

Define $C_0^\bullet := C(\tau \circ g)[-1]$. Since f is a quasi-isomorphism, it follows from the commutativity of the above diagram that $C_0^\bullet \to C^\bullet$ is a quasi-isomorphism. □

Remark 1.5.30. (i) Existence and uniqueness of composition of morphisms in $D(\mathscr{A})$ follows from the above Proposition 1.5.29 (c.f., diagram (1.5.24)). This completes the construction of the derived category $D(\mathscr{A})$.

(ii) The category $D(\mathscr{A})$ is additive.

Definition 1.5.31. A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $K(\mathscr{A})$ (resp., in $D(\mathscr{A})$) is said to be a *distinguished triangle* if it is isomorphic in $K(\mathscr{A})$ (resp., $D(\mathscr{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]\,,$$

where f is a morphism of complexes with mapping cone $C(f)$, and τ and π are natural morphisms as defined in (1.5.26).

Proposition 1.5.32. *The categories $D(\mathscr{A})$ and $K(\mathscr{A})$ together with the shift functor are triangulated. Moreover, the natural functor $Q : K(\mathscr{A}) \longrightarrow D(\mathscr{A})$ is an exact functor of triangulated categories.*

Proof. Let $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$. Since $\mathcal{K}om(\mathscr{A})$ is an abelian category, it follows from Proposition 1.5.22 that the quotient

$$\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim,$$

is an abelian group. Thus $K(\mathscr{A})$ is an additive category. To see $D(\mathscr{A})$ is an additive category, let $f, g \in \mathrm{Hom}_{D(\mathscr{A})}(A^\bullet, B^\bullet)$ be two morphisms in $D(\mathscr{A})$ represented by the following equivalence classes of roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\phi_1"', "qis"] \arrow[dr, "\psi_1"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_1^\bullet \arrow[dl, "\phi_2"', "qis"] \arrow[dr, "\psi_2"] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

respectively. Let $C^\bullet$ be the complex defined by

$$C^i = C_1^i \oplus C_2^i, \quad \text{and} \quad d^i_{C^\bullet} = \begin{pmatrix} d^i_{C_1} & 0 \\ 0 & d^i_{C_2} \end{pmatrix}, \quad \forall\, i \in \mathbb{Z}\,.$$

Then we define $f + g$ in $D(\mathscr{A})$ to be the morphism represented by the roof

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\phi_1\oplus\phi_2"', "qis"] \arrow[dr, "\psi_1\oplus\psi_2"] & \\ A^\bullet & & B^\bullet\,. \end{tikzcd}$$

One need to check that, $f + g$ is well-defined in $D(\mathscr{A})$, and gives $\mathrm{Hom}_{D(\mathscr{A})}(A^\bullet, B^\bullet)$ a structure of an abelian group.

Now triangulated structure on both $K(\mathscr{A})$ and $D(\mathscr{A})$ are given by shift functor $A^\bullet \mapsto A^\bullet[1]$. Verification of axioms (TR1) – (TR4) requires crucial use of mapping cone. □

Corollary 1.5.33. *(a) The functor $Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A})$ identifies set underlying set of objects of both categories (Apply property (ii) to the identity functor $\mathcal{K}om(\mathscr{A}) \to \mathcal{K}om(\mathscr{A})$).*

(b) For any complex $A^\bullet \in D(\mathscr{A})$, its cohomology objects $\mathcal{H}^i(A^\bullet)$ are well-defined objects in $\mathscr{A}$. (This is because, quasi-isomorphisms of $\mathcal{K}om(\mathscr{A})$ turns into isomorphisms in $D(\mathscr{A})$.)

(c) Considering $A \in \mathscr{A}$ as a complex $A^\bullet \in D(\mathscr{A})$ concentrated at degree zero only, gives an equivalence between $\mathscr{A}$ and the full subcategory of objects of $D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$.

Remark 1.5.34. Contrary to the category $\mathcal{K}om(\mathscr{A})$ of complexes in $\mathscr{A}$, the derived category $D(\mathscr{A})$ is not abelian, in general. However, $D(\mathscr{A})$ is always triangulated. $D^b(\mathscr{A})$ is abelian if and only if $\mathscr{A}$ is semisimple
(see https://math.stackexchange.com/questions/189769).

Example 1.5.35. Let $\mathscr{A} = \mathcal{V}ect_{fd}(k)$ be the category of all finite dimensional k-vector spaces with k-linear homomorphisms as morphisms between its objects. Then $D(\mathscr{A})$ is equivalent to $\prod_{i\in\mathbb{Z}} \mathcal{A}$. Note that, any complex of k-vector spaces $A^\bullet \in D(\mathcal{V}ect_{fd}(k))$

is isomorphic to its cohomology complex $\bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$ with trivial differentials. More generally, this holds for any *semisimple* abelian category $\mathscr{A}$ (i.e., when $\mathscr{A}$ is abelian and any short exact sequence in $\mathscr{A}$ splits).

Definition 1.5.36. Let $\mathcal{K}om^*(\mathscr{A})$, with $* = +, -$, or b, be the full subcategory of $\mathcal{K}om(\mathscr{A})$, whose objects are complexes $A^\bullet \in \mathcal{K}om(\mathscr{A})$ with $A^i = 0$ for all $i \ll 0$, $i \gg 0$, or $|i| \ll 0$, respectively.

Note that, $\mathcal{K}om^*(\mathscr{A})$ is again abelian, for $* \in \{+, -, b\}$. So following similar construction (i.e., by dividing out first by homotopy equivalence, and then by localizing with respect to quasi-isomorphisms), we can construct a category, denoted by $D^*(\mathscr{A})$. There is a natural forgetful functor

$$\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A}),$$

which just forgets boundedness condition.

Proposition 1.5.37. *The natural forgetful functor $\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A})$, where $* = +, -$, or b, gives an equivalence of $D^*(\mathscr{A})$ with the full triangulated subcategories of all complexes $A^\bullet \in D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for $i \ll 0$, for $i \gg 0$, or for $|i| \ll 0$, respectively.*

Definition 1.5.38. Let $\mathscr{A}$ be an abelian category. A full subcategory $\mathscr{B} \subset \mathscr{A}$ of $\mathscr{A}$ is called *thick* if for any short exact sequence (in $\mathscr{A}$)

$$0 \to A \to B \to C \to 0$$

with $A, C \in \mathscr{B}$, we have $B \in \mathscr{B}$.

Proposition 1.5.39. *Let $\mathscr{A} \subset \mathscr{B}$ be a thick abelian subcategory of an abelian category $\mathscr{B}$. Suppose that any object $A \in \mathscr{A}$ is embedded in an object $I \in \mathscr{A}$, which is injective as an object of $\mathscr{B}$. Then the natural forgetful functor $D(\mathscr{A}) \longrightarrow D(\mathscr{B})$ induces an equivalence*

$$D^*(\mathscr{A}) \longrightarrow D^*_{\mathscr{A}}(\mathscr{B}), \quad \textit{where } * = + \textit{ or } b,$$

of $D^*(\mathscr{A})$ and the full triangulated subcategory $D^*_{\mathscr{A}}(\mathscr{B}) \subset D^*(\mathscr{B})$ of complexes with cohomologies in $\mathscr{A}$.

1.6. **Derived functors.**

1.7. **Grothendieck group.**

Definition 1.7.1. Let $\mathscr{A}$ be a small abelian category. The *Grothendieck group* of $\mathscr{A}$, denoted by $K_0(\mathscr{A})$, is the quotient of the free abelian group generated by the set of all isomorphism classes of objects of $\mathscr{A}$ by its normal subgroup generated by all elements $[B] - [A] - [C]$, where $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathscr{A}$.

Let $\mathscr{A}$ be an abelian category. For an object $A \in \mathscr{A}$, we denote by $A^\bullet \in \mathcal{K}om(\mathscr{A})$ the complex defined by

$$A^i = \begin{cases} A, & \text{if} \quad i = 0, \\ 0, & \text{if} \quad i \neq 0\,. \end{cases}$$

We also denote by $A^\bullet \in D^b(\mathscr{A})$, the object in $D^b(\mathscr{A})$ represented by the complex $A^\bullet$.

Definition 1.7.2. Let $\mathscr{T}$ be a small triangulated category. For example, $\mathscr{T}$ can be the bounded derived category $D^b(\mathscr{A})$ of a small abelian category $\mathscr{A}$. Let $F(\mathscr{T})$ be the free abelian group generated by the set of all objects of $\mathscr{T}$. Let $\mathcal{R}(\mathscr{T})$ be the normal subgroup of $F(\mathscr{T})$ generated by the elements $[A]+[C]-[B] \in F(\mathscr{A})$, whenever there is a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathscr{T}$. The quotient abelian group

$$K_0(\mathscr{T}) := F(\mathscr{T})/\mathcal{R}(\mathscr{T}) \tag{1.7.3}$$

is called the *Grothendieck group* of $\mathscr{T}$.

Remark 1.7.4. It follows from the axiom (TR3) in Definition 1.2.4 that $A[1] = 0$ in $K_0(\mathscr{T})$.

The following proposition establishes relation between $K_0(\mathscr{A})$ and $K_0(D^b(\mathscr{A}))$.

Proposition 1.7.5. *Let $\mathscr{A}$ be a small abelian category, and let $D^b(\mathscr{A})$ be the bounded derived category of $\mathscr{A}$. Then the natural functor $\iota : \mathscr{A} \longrightarrow D^b(\mathscr{A})$ defined by sending an object $A \in \mathscr{A}$ to $A^\bullet \in D^b(\mathscr{A})$ induces an isomorphism of their Grothendieck groups $K_0(\mathscr{A}) \xrightarrow{\simeq} K_0(D^b(\mathscr{A}))$.*

1.8. **Serre functor.** Let k be a field. Let $\mathcal{A}$ be a k-linear additive category.

Definition 1.8.1. A *Serre functor* on $\mathcal{A}$ is a k-linear equivalence of categories

$$S : \mathcal{A} \longrightarrow \mathcal{A}$$

such that for any two objects $A, B \in \mathcal{A}$, there is a natural k-linear isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(B, S(A))^*,$$

which is functorial in both A and B. We write the induced k-bilinear pairing as

$$\mathrm{Hom}(B, S(A)) \times \mathrm{Hom}(A, B) \longrightarrow k\,, \quad (f, g) \longmapsto \langle f|g\rangle\,.$$

Proposition 1.8.2. *Let $\mathcal{A}$ be a k-linear additive category together with a Serre functor $S : \mathcal{A} \to \mathcal{A}$. Then for any $A, B \in \mathcal{A}$, the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] \arrow[d, "S_{A,B}"'] & \mathrm{Hom}(B,S(A))^* \\
\mathrm{Hom}(S(A),S(B)) \arrow[r, "\eta_{S(A),S(B)}"] \arrow[ur, dashed, "\exists\, \eta^*_{B,S(A)}"] & \mathrm{Hom}(S(B),S^2(A))^* \arrow[u, "S^*_{B,S(A)}"']
\end{tikzcd} \tag{1.8.3}$$

Proof. By abuse of notation, we denote by $\eta^*_{B,S(A)}$ the composite k-linear homomorphism

$$\eta^*_{B,S(A)} : \mathrm{Hom}(S(A), S(B)) \hookrightarrow \mathrm{Hom}(S(A), S(B))^{**} \xrightarrow{\eta^*_{B,S(A)}} \mathrm{Hom}(B, S(A))^*.$$

Therefore, it suffices to show that both upper and lower triangles in (1.8.3) commutes. Note that, commutativity of upper triangle is equivalent to

$$\langle f|g\rangle = \langle S_{A,B}(g)|f\rangle, \quad \forall\ f \in \mathrm{Hom}(B, S(A)),\ g \in \mathrm{Hom}(A, B)\,.$$

Applying functoriality of η in the second variable, we have the following commutative diagram.

$$\begin{tikzcd} \mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] & \mathrm{Hom}(B,S(A))^* \\ \mathrm{Hom}(B,B) \arrow[u, "-\circ g"] \arrow[r, "\eta_{B,B}"] & \mathrm{Hom}(B,S(B))^* \arrow[u, "(S(g)\circ -)^*"'] \end{tikzcd} \tag{1.8.4}$$

Applying commutativity of (1.8.4) to $\mathrm{Id}_B \in \mathrm{Hom}(B,B)$ we have $\langle f|g\rangle = \langle S(g)\circ f|\,\mathrm{Id}\rangle$. Applying functoriality of η in the first variable, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(B,B) \arrow[r, "\eta_{B,B}"] \arrow[d, "f\circ -"'] & \mathrm{Hom}(B,S(B))^* \arrow[d, "(-\circ f)^*"] \\ \mathrm{Hom}(B,S(A)) \arrow[r, "\eta_{B,S(A)}"] & \mathrm{Hom}(S(A),S(B))^* , \end{tikzcd} \tag{1.8.5}$$

which gives $\langle (S(g)\circ f)|\,\mathrm{Id}_B\rangle = \langle S(g)|f\rangle$. This completes the proof. □

Remark 1.8.6. In order to avoid trouble with identifying $\mathrm{Hom}(A,B)$ with its double dual $\mathrm{Hom}(A,B)^{**}$, we always assume that a k-linear additive category $\mathcal{A}$ has finite dimensional Home's (i.e., $\dim_k \mathrm{Hom}_{\mathcal{A}}(A,B) < \infty$, for all $A, B \in \mathrm{Ob}(\mathcal{A})$).

Lemma 1.8.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear additive categories with finite dimensional Hom's. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively, then any k-linear equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ commutes with Serre functors (i.e., there is an isomorphism of functors $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$).*

Let X be a smooth projective variety of dimension $n \geq 1$ defined over a field k. Note that, for any locally free coherent sheaf E on X, the functor

$$- \otimes E : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(X)\,, \quad F \longmapsto F \otimes E$$

is exact. Let ω_X be the dualizing sheaf on X. Let $D^*(X) = D^*(\mathfrak{Coh}(X))$, where $* \in \{\emptyset, b, -, +\}$. Consider the composite functor

$$S_X : D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X), \tag{1.8.8}$$

where $[n] : D^*(X) \to D^*(X)$ is the n-th shift functor given by sending a complex $E^\bullet$ to $E^\bullet[n]$. Since both $\omega_X \otimes -$ and $[n]$ are exact functors, the functor $S_X := \omega_X \otimes (-)[n]$ is exact.

Theorem 1.8.9 (Serre duality)**.** *Let X be a smooth projective variety over a field k. Then the functor $S_X : D^b(X) \longrightarrow D^b(X)$ as defined in* (1.8.8) *is a Serre functor in the sense of Definition 1.8.1.*

Proof. Let $E^\bullet, F^\bullet \in D^b(X)$ be any two bounded complexes of coherent sheaves on X. We need to define a isomorphism of k-vector spaces

$$\eta_{E^\bullet,F^\bullet} : \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \longrightarrow \mathrm{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet)) \tag{1.8.10}$$

which is functorial in both $E^\bullet$ and $F^\bullet$. For this, we note that, $\mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i]) = \mathrm{Ext}^i(E^\bullet, F^\bullet)$, for all i $\square$

Proposition 1.8.11. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for any $A^\bullet, B^\bullet \in \mathcal{K}om(\mathcal{A})$, we have natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{K}om(\mathcal{A})}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i])\,, \quad \forall\, i\,.$$

1.9. **Chern characters for complexes.** Let X be a smooth proper $\mathbb{C}$-variety. For each integer j, with $0 \leq j \leq \dim_{\mathbb{C}}(X)$, let $\mathrm{CH}^j(X)$ be the Chow group of j-cocycles on X. For any bounded complex of coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{E}^\bullet \in \mathcal{K}om^b(\mathfrak{Coh}(X))$, there is a bounded complex $E^\bullet$ of locally free coherent sheaves of $\mathcal{O}_X$-modules quasi-isomorphic to $\mathcal{E}^\bullet$. Then we define the *j-th Chern character* of $\mathcal{E}^\bullet$ by

$$\mathrm{ch}_j(\mathcal{E}^\bullet) := \sum_i (-1)^i \,\mathrm{ch}_j(E^i) \in \mathrm{CH}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \tag{1.9.1}$$

Proposition 1.9.2. *The map $\mathcal{E}^\bullet \longmapsto \mathrm{ch}_j(\mathcal{E}^\bullet)$ defined in (1.9.1) induces a map*

$$\mathrm{ch}_j : K_0(D^b(X)) \longrightarrow \mathrm{CH}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

Remark 1.9.3. Since X is smooth, for any $\mathcal{E}^\bullet \in \mathcal{K}om(\mathfrak{Mod}(X))$, there is a bounded complex $E^\bullet$ of locally free coherent sheaves of $\mathcal{O}_X$-modules quasi-isomorphic to $\mathcal{E}^\bullet$. Then the above definition of Chern character and Chern class map extend to define maps from $D(\mathfrak{Mod}(X))$.

Remark 1.9.4. To define Chern class and Chern character of objects of a general triangulated category $\mathscr{A}$, we need to give compatible symmetric monoidal structure on $\mathscr{A}$ (such a category is called a tensor triangulated category, c.f. Section §1.10). On a tensor triangulated category, one can define notion of algebraic cycle, Chow group etc., which recovers the classical algebraic cycles and Chow group when we take $\mathscr{A}$ modelled on a scheme X [Kle16, Theorem 3.2.6]. In fact, a tensor triangulated category associated with a scheme completely recovers the scheme [Bal10].

1.10. **Tensor Triangulated Category.**

2. Bridgeland Stability

The definition of stability condition generalized in different ways from curve to higher dimensional varieties. We follow [MS17, BBHR09, Huy06].

2.1. **Stability condition in an abelian category.**

Definition 2.1.1. Let $\mathscr{C}$ be an additive category. A *subobject* of an object E in $\mathscr{C}$ is a monomorphism $\iota : F \to E$ in $\mathscr{C}$.

Remark 2.1.2. Note that, "being a subobject" is a transitive relation on $\mathrm{Ob}(\mathscr{A})$. (This may fails to hold if we consider equivalence class of monomorphisms instead of just monomorphism).

Let $\mathscr{A}$ be an abelian category. Denote by $K_0(\mathscr{A})$ the Grothendieck group of $\mathscr{A}$ (c.f., Definition 1.7.1). For any complex number z, we denote by $\mathrm{Im}(z)$ (resp., $\mathrm{Re}(z)$) the *imaginary part* (resp., the *real part*) of z.

Definition 2.1.3. A *stability function* on $\mathscr{A}$ is an additive group homomorphism

$$Z : K_0(\mathscr{A}) \longrightarrow \mathbb{C}$$

such that for any non-zero object $E \in \mathscr{A}$, we have $\mathrm{Im}(Z(E)) \geq 0$, and if $\mathrm{Im}(Z(E)) = 0$, then $\mathrm{Re}(Z(E)) < 0$.

Note. Here "$\mathrm{Im}(Z(a)) \geq 0,\ \forall\ a \in \mathscr{A}$" does not imply that "$\mathrm{Im}(Z(a)) \geq 0, \forall\ a \in K_0(\mathscr{A})$".

Given a stability function $Z : K_0(\mathscr{A}) \to \mathbb{C}$ as in Definition 2.1.3, we may think of

$$\deg_Z(E) := -\mathrm{Re}(Z(E)) \quad \text{and} \quad \mathrm{rk}_Z(E) := \mathrm{Im}(Z(E))$$

to be the *generalized degree* and the *generalized rank* of E with respect to Z, respectively. Define the *generalized slope* of $E \in \mathscr{A} \setminus \{0\}$ with respect to Z by

$$\mu_Z(E) := \begin{cases} \dfrac{\deg_Z(E)}{\mathrm{rk}_Z(E)}, & \text{if } \mathrm{rk}_Z(E) \neq 0, \text{ and} \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1.4}$$

Example 2.1.5. Let X be a smooth projective curve defined over an algebraically closed field k. Let $\mathfrak{Coh}(X)$ be the category of coherent sheaves of $\mathcal{O}_X$-modules on X. Let $Z : K_0(\mathscr{A}) \longrightarrow \mathbb{C}$ be the additive group homomorphism defined by sending a non-zero object $E \in \mathscr{A}$ to

$$Z(E) := -\deg(E) + \sqrt{-1} \cdot \mathrm{rk}(E) \in \mathbb{C}\,.$$

Clearly, Z is a stability function on $\mathscr{A}$. Note that, $\mu_Z(E)$ coincides with the usual slope $\mu(E) := \deg(E)/\mathrm{rk}(E)$ of E, and hence in this case, Z-(semi)stability coincides with the usual slope (semi)stability of coherent sheaves on X.

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k. Fix an ample class $\omega \in \mathrm{Amp}(X) \subseteq N^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and a divisor class $B \in N^1(X)$.

Definition 2.1.6 (K. Matsuki, R. Wentworth)**.** The *B-twisted Chern character* of $E \in \mathfrak{Coh}(X)$ is defined by

$$\mathrm{ch}^B(E) := \mathrm{ch}(E) \cdot e^{-B} = \sum_{i \geq 0} \mathrm{ch}_i(E) \cdot \sum_{j \geq 0} \frac{(-1)^j}{j!} B^j\,.$$

Thus, for $i \geq 0$, the *B-twisted i-th Chern character of E*, denoted $\mathrm{ch}_i^B(E)$, are given by

$$\begin{aligned} \mathrm{ch}_0^B(E) &= \mathrm{ch}_0(E) = \mathrm{rk}(E), \\ \mathrm{ch}_1^B(E) &= \mathrm{ch}_1(E) - \mathrm{ch}_0(E) \cdot B, \\ \mathrm{ch}_2^B(E) &= \mathrm{ch}_2(E) - \mathrm{ch}_1(E) \cdot B + \frac{1}{2}\mathrm{rk}(E) \cdot B^2\,, \end{aligned}$$

and so on. Note that, taking $B = 0$, we get back the usual Chern characters.

Define an additive group homomorphism

$$Z_{\omega,B} : K_0(\mathfrak{Coh}(X)) \longrightarrow \mathbb{C}$$

by sending a non-zero object E of $\mathfrak{Coh}(X)$ to the complex number

$$Z_{\omega,B}(E) := -\omega^{n-1} \cdot \mathrm{ch}_1^B(E) + \sqrt{-1} \cdot \omega^n \cdot \mathrm{ch}_0^B(E). \tag{2.1.7}$$

If T is a torsion coherent sheaf on X supported in dimension $\leq n-2$, then $\mathrm{rk}(T) = 0$, and the line bundle $\det(E)$ admits a nowhere vanishing global section (c.f., [Kob87, Proposition 5.6.14]). Then $\det(T) \cong \mathcal{O}_X$, and hence $\mathrm{ch}_1(T) = 0$. Therefore, $Z_{\omega,B}(T) = 0$. This shows that, $Z_{\omega,B}$ is not a stability function.

Remark 2.1.8. Let $\mathfrak{Coh}_{\leq n-2}(X)$ be the full subcategory of coherent sheaves on X whose supports have dimension $\leq n-2$, and let $\mathscr{A}$ be the localized category

$$\mathfrak{Coh}_{n,n-2}(X) = \mathfrak{Coh}(X)/\,\mathfrak{Coh}_{\leq n-2}(X)\,.$$

Then the function

$$Z_{\omega,B} : K_0(\mathscr{A}) \longrightarrow \mathbb{C}$$

as defined in (2.1.7) above, is a stability function.

Definition 2.1.9. A *stability condition* is a pair $(\mathscr{A}, Z)$, where $\mathscr{A}$ is an abelian category and $Z : K_0(\mathscr{A}) \longrightarrow \mathbb{C}$ is a stability function such that for any non-zero object E of $\mathscr{A}$, there is a filtration of E by its subobjects

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E\,, \tag{2.1.10}$$

such that all E_i/E_{i-1} are Z-semistable, and their Z-slopes satisfies

$$\mu_Z(E_1) > \mu_Z(E_2/E_1) > \cdots > \mu_Z(E_\ell/E_{\ell-1}).$$

Such a filtration (2.1.10) is known as *Harder-Narasimhan filtration* of E.

Proposition 2.1.11. *Let $\mathscr{A}$ be an abelian category. Given a stability function $Z : K_0(\mathscr{A}) \to \mathbb{C}$, Harder-Narasimhan filtration of an object $E \in \mathscr{A}$, if it exists, is unique up to isomorphism in $\mathscr{A}$.*

Existence of Harder-Narasimhan filtration requires some additional assumption on the category $\mathscr{A}$ and the stability function Z. For this we need some definitions.

Definition 2.1.12. An additive category $\mathscr{A}$ is said to be *noetherian* if for any object $E \in \mathscr{A}$, and any any ascending chain of subobjects

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots\cdots \subseteq E,$$

of E, there is an integer $i_0 \geq 0$ such that $E_i = E_{i+1}$, for all $i \geq i_0$.

Lemma 2.1.13. *Let $\mathscr{A}$ be a noetherian abelian category. Let $Z : K_0(\mathscr{A}) \to \mathbb{C}$ be a stability function. If the image of $\mathrm{Im}Z$ is discrete in $\mathbb{R}$, then for any object $E \in \mathscr{A}$, there is a number $D_E \in \mathbb{R}$ such that for any subobject F of E in $\mathscr{A}$, we have $D(F) \leq D_E$.*

Proof. Suppose that $R(E) = 0$. Then $D(E) > 0$. Given a subobject $F \subset E$, from the exact sequence $0 \to F \to E \to E/F \to 0$, we have $R(E) = R(F) + R(E/F)$ and $D(E) = D(F) + D(E/F)$, since both $D := -\operatorname{Re} Z$ and $R := \operatorname{Im} Z$ are homomorphisms of additive groups. Since Z is a stability function, we have $R(F), R(E/F) \geq 0$. Then $R(F) = 0$, and hence $D(F) > 0$. Thus, $0 < D(F) \leq D(E)$ (note that, $D(E/F) > 0$ if $E/F = 0$).

Assume that $R(E) > 0$. Suppose on the contrary that there is an infinite sequence of subobjects $\{F_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} D(F_n) = +\infty$. If for some $n \in \mathbb{N}$, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ implies $D(E/F_n) \geq 0$, and so $D(F_n) \leq D(E)$. So, without loss of generality, we may assume that $R(F_n) < R(E)$, for all $n \in \mathbb{N}$.

□

Theorem 2.1.14. *Let $\mathscr{A}$ be a noetherian abelian category. Let $Z : K_0(\mathscr{A}) \longrightarrow \mathbb{C}$ be a stability function. Assume that $R := \operatorname{Im} Z : K_0(\mathscr{A}) \to \mathbb{R}$ has discrete image in $\mathbb{R}$. Then any non-zero object $E \in \mathscr{A}$ admits a unique Harder-Narasimhan filtration.*

2.2. **Stability conditions in a triangulated category.** The notion of stability conditions in triangulated category is introduced by Tom Bridgeland in his famous paper [Bri07] inspired by the works of Douglas [Dou01b, Dou02, Dou01a].

Recall that, a *triangulated category* is an additive category $\mathscr{A}$ together with an additive equivalence (*shift functor*)

$$T : \mathscr{A} \longrightarrow \mathscr{A}\,, \tag{2.2.1}$$

and a set of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,, \tag{2.2.2}$$

satisfying the following axioms (TR1) – (TR4) (c.f. Definition 1.2.4). We shall always assume that $\mathscr{A}$ is essentially small (i.e., $\mathscr{A}$ is equivalent to a small category).

Notation 2.2.3. *For any additive subcategory $\mathscr{B}$ of a triangulated category $\mathscr{A}$, we denote by $\mathscr{B}[1]$ the full subcategory of $\mathscr{A}$, whose objects are of the form $A[1] \in \mathscr{A}$, with $A \in \mathscr{B}$.*

Definition 2.2.4. A *stability condition* on a triangulated category $(\mathscr{A}, T)$ is a pair $(Z, \mathcal{P})$, where $Z : K_0(\mathscr{A}) \to \mathbb{C}$ is an additive group homomorphism (called the *central charge*) and full subcategories $\mathcal{P}(\phi)$ of $\mathscr{A}$, for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(i) if $E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \cdot \exp(2\pi i \phi)$, for some $m(E) \in \mathbb{R}_{>0}$.
(ii) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] := T(\mathcal{P}(\phi))$, for all $\phi \in \mathbb{R}$.
(iii) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $\operatorname{Hom}(A_1, A_2) = 0$.
(iv) for each object $E \in \mathscr{A}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_\ell,$$

(depending on E) and a sequence of distinguished triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{\ell-1} \arrow[rr] & & E_\ell = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & \cdots & & A_\ell \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$, for all $j \in \{1, \dots, \ell\}$.

Let $\mathscr{A}$ be an abelian category and $D^b(\mathscr{A})$ be the bounded derived category of $\mathscr{A}$.

Proposition 2.2.5. *Let $\mathscr{H}$ be a full additive subcategory of $D^b(\mathscr{A})$. Then $\mathscr{H}$ is a heart of a bounded t-structure on $D^b(\mathscr{A})$ if and only if $\mathscr{H}$ satisfies following properties.*

(1) for integers $i > j$ and objects $A, B \in \mathscr{H}$, we have $\mathrm{Hom}(A[i], B[j]) = \mathrm{Hom}(A, B[j-i]) = 0$, and
(2) for $E \in D^b(\mathscr{A})$, there are integers $k_1 > \cdots > k_m$, collection of objects $E_i \in D^b(\mathscr{A})$ and $A_i \in \mathscr{H}$, for each $i = 1, \dots, m$, and a sequence of distinguished triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E_m = E \arrow[d] \\
& A_1[k_1] \arrow[ul, dashed] & A_2[k_2] \arrow[ul, dashed] & & A_{m-1}[k_{m-1}] \arrow[ul, dashed] & A_m[k_m] \arrow[ul, dashed]
\end{tikzcd}$$

Definition 2.2.6. A *Bridgeland stability condition* on a triangulated category $\mathscr{A}$ is a pair $\sigma = (\mathcal{P}, Z)$, where

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