ERRATUM TO "SYSTEM OF HODGE BUNDLES AND GENERALIZED OPERS ON SMOOTH PROJECTIVE VARIETIES"

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ABSTRACT. In our recent paper "System of Hodge Bundles and Generalized Opers on Smooth Projective Varieties" [BPS], there is a miscalculation in the proof of Theorem 3.1 as pointed out by Ronnie Sebastian. In this erratum article, we give a corrected proof of this theorem.

1. CRITERION FOR SEMISTABILITY OF A SYSTEM OF HODGE BUNDLES

Let X be a polarized smooth projective variety of dimension $d \geq 1$ over an algebraically closed field k. We refer [BPS] for definitions related to Higgs bundles and system of Hodge bundles. Then we have the following.

Theorem 1.1. [BPS, Theorem 3.1] *Assume that* $\deg(\Omega^1_X) \geq 0$. *Let* (E, θ) *be a Higgs bundle on* X *which admits a structure of a system of Hodge bundles* $E = \bigoplus_{i=0}^n E_i$. *Suppose that,* $\theta|_{E_i} : E_i \longrightarrow E_{i-1} \otimes \Omega^1_X$ *is an isomorphism of* $\mathcal{O}_X$*–modules, for all* $i \in \{1, \ldots, n\}$. *If* E_i *is semistable, for all* $i \in \{1, \ldots, n\}$, *then* (E, θ) *is a semistable Higgs bundle.*

The mistake in the proof of [BPS, Theorem 3.1] is that the inequality (3.10) in that proof holds only for $d = 1$. However, for $d > 1$, the corrected inequality is $\mathrm{rk}(F_i) \leq d \cdot \mathrm{rk}(F_{i-1})$, for all $i = 1, \ldots, r$. But then our earlier method of using Chebyshev's inequality is not applicable there. This necessitates us to use the following inequality to give a corrected proof of [BPS, Theorem 3.1].

Lemma 1.2. *Let* ℓ *and* d *be positive integers. Then for a finite sequence of real numbers* $r_0, r_1, \ldots, r_\ell$ *with* $r_j \leq d \cdot r_{j-1}$, *for all* $j = 1, \ldots, \ell$, *we have*

$$\left(\sum_{i=0}^{\ell} d^i\right)\left(\sum_{j=0}^{\ell} j \cdot r_j\right) \leq \left(\sum_{i=0}^{\ell} i \cdot d^i\right)\left(\sum_{j=0}^{\ell} r_j\right).$$

Proof. This can be proved by induction. Let $P_\ell := \sum\limits_{i=0}^{\ell} d^i \sum\limits_{j=0}^{\ell} j r_j$ and $Q_\ell := \sum\limits_{i=0}^{\ell} i d^i \sum\limits_{j=0}^{\ell} r_j$. Clearly for $\ell = 1$, $P_1 \leq Q_1$. So assume that $\ell \geq 2$, and $P_{\ell-1} \leq Q_{\ell-1}$. Now we have

$$P_\ell = P_{\ell-1} + d^\ell \sum_{j=0}^{\ell-1} j r_j + \ell r_\ell \sum_{i=0}^{\ell-1} d^i + \ell d^\ell r_\ell,$$

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and

$$Q_\ell = Q_{\ell-1} + \ell d^\ell \sum_{j=0}^{\ell-1} r_j + r_\ell \sum_{i=0}^{\ell-1} i d^i + \ell d^\ell r_\ell \,.$$

Therefore, it is enough to show that

$$d^\ell \sum_{j=0}^{\ell-1} j r_j + \ell r_\ell \sum_{i=0}^{\ell-1} d^i \leq \ell d^\ell \sum_{j=0}^{\ell-1} r_j + r_\ell \sum_{i=0}^{\ell-1} i d^i \,. \tag{1.1}$$

Since $d^i r_\ell \leq d^\ell r_i$, for all i, we have

$$\begin{aligned} & \ell d^i r_\ell = i d^i r_\ell + (\ell - i) d^i r_\ell \leq i d^i r_\ell + (\ell - i) d^\ell r_i \\ \Rightarrow \quad & \ell d^i r_\ell + i d^\ell r_i \leq i d^i r_\ell + \ell d^\ell r_i \end{aligned} \tag{1.2}$$

Now summing up the inequality (1.2) from $i = 0$ to ℓ, we get (1.1). This completes the proof. □

Proof of Theorem 1.1. Since $E_i \cong E_0 \otimes (\Omega^1_X)^{\otimes i}$, for all $i \in \{0, 1, \dots, n\}$, we have,

$$\deg(E_i) = i \cdot d^{i-1} \cdot \deg(\Omega^1_X) \cdot \mathrm{rk}(E_0) + d^i \cdot \deg(E_0) \,, \tag{1.3}$$

and

$$\mathrm{rk}(E_i) = d^i \cdot \mathrm{rk}(E_0) \,, \ \ \forall\, i = 0, \dots, n \,. \tag{1.4}$$

The above two equalities (1.3) and (1.4) gives

$$\mu(E_i) = \frac{\deg(E_i)}{\mathrm{rk}(E_i)} = \frac{i}{d} \cdot \deg(\Omega^1_X) + \mu(E_0) \,, \ \ \forall\, i = 0, \dots, n \,. \tag{1.5}$$

Now for any integer $k \in \{0, 1, \cdots, n\}$, by (1.3) and (1.4) we have,

$$\begin{aligned} \mu\left(\bigoplus_{i=0}^{k} E_i\right) = \frac{\sum\limits_{i=0}^{k} \deg(E_i)}{\sum\limits_{i=0}^{k} \mathrm{rk}(E_i)} &= \frac{\left(\deg(\Omega^1_X)\,\mathrm{rk}(E_0) \sum\limits_{i=0}^{k} i \cdot d^{i-1} + \deg(E_0) \sum\limits_{i=0}^{k} d^i\right)}{\mathrm{rk}(E_0) \sum\limits_{i=0}^{k} d^i} \\ &= \frac{\deg(\Omega^1_X) \cdot \sum\limits_{i=0}^{k} i \cdot d^{i-1}}{\sum\limits_{i=0}^{k} d^i} + \mu(E_0). \end{aligned} \tag{1.6}$$

It follows from (1.6) and [BPS, Lemma 3.3] that

$$\mu\left(\bigoplus_{i=0}^{k} E_i\right) \leq \mu(E) \,, \ \ \forall\ k = 0, \dots, n \,. \tag{1.7}$$

Suppose on the contrary that (E, θ) is not semistable. Let F be the unique maximal semistable proper Higgs subsheaf of (E, θ) with

$$\mu(F) > \mu(E) \,. \tag{1.8}$$

It follows from [LSYZ, Lemma 2.4] that F admits a structure of system of Hodge bundle; in particular, $F \cong \bigoplus\limits_{i=0}^{n} F_i$, with $F_i = F \cap E_i$, for all $i = 0, 1, \dots, n$.

Since $\theta|_{E_i}$ is an isomorphism, we have

$$F_i \cong \theta(F_i) \subseteq F_{i-1} \otimes \Omega^1_X \,, \ \ \forall\ i = 0, 1, \dots, n \,. \tag{1.9}$$

Therefore, $F_i \neq 0$ implies $F_{i-1} \neq 0$, for all $1 \leq i \leq n$. Let $r \in \{0, \cdots, n\}$ be the largest integer such that $F_r \neq 0$. Then $F = \bigoplus_{i=0}^{r} F_i$. Now from (1.9), we have

$$\mathrm{rk}(F_i) \leq d \cdot \mathrm{rk}(F_{i-1}), \quad \forall\ i = 1, \ldots, r. \tag{1.10}$$

Since $F_i \neq 0$ and E_i is semistable by assumption, using (1.5), for each $i = 0, 1, \ldots, r$, we have

$$\deg(F_i) \leq \mathrm{rk}(F_i) \cdot \mu(E_i) = \mathrm{rk}(F_i) \left(\frac{i}{d} \cdot \deg(\Omega_X^1) + \mu(E_0) \right). \tag{1.11}$$

Then using (1.11), we have

$$\begin{aligned} \mu(F) = \frac{\sum\limits_{i=0}^{r} \deg(F_i)}{\mathrm{rk}(F)} \quad &\leq \quad \frac{1}{\mathrm{rk}(F)} \sum_{i=0}^{r} \mathrm{rk}(F_i) \left(\frac{i}{d} \cdot \deg(\Omega_X^1) + \mu(E_0) \right) \\ &= \quad \mu(E_0) + \frac{\deg(\Omega_X^1)}{d \cdot \mathrm{rk}(F)} \sum_{i=1}^{r} i \cdot \mathrm{rk}(F_i). \end{aligned} \tag{1.12}$$

Since $\deg(\Omega_X^1) \geq 0$, using (1.6) and (1.10), it follows from (1.12) and Lemma 1.2 that

$$\mu(F) \leq \mu\left(\bigoplus_{i=0}^{r} E_i \right) \leq \mu(E).$$

$\square$

Remark 1.1. In the proof of [BPS, Theorem 3.8], we have referred the same calculation as in proof of [BPS, Theorem 3.1], which is not correct because of the same mistake (Chebyshev's inequality is not applicable there). However, this can easily be fixed by using above Lemma 1.2 as in the proof of Theorem 1.1.

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References

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