SYSTEM OF HODGE BUNDLES AND GENERALIZED OPERS ON SMOOTH PROJECTIVE VARIETIES

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ABSTRACT. Let k be an algebraically closed field of any characteristic. Let X be a polarized irreducible smooth projective algebraic variety over k. We give criterion for semistability and stability of system of Hodge bundles on X. We define notion of generalized opers on X, and prove semistability of the Higgs bundle associated to generalized opers. We also show that existence of partial oper structure on a vector bundle E together with a connection ∇ over X implies semistability of the pair (E, ∇).

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $char(k) \geq 0$. Let X be a polarized irreducible smooth projective algebraic variety over k. Let E be a vector bundle on X together with a filtration

$$\mathcal{F}^\bullet(E) : \ \ E = \mathcal{F}^0(E) \supsetneq \mathcal{F}^1(E) \supsetneq \cdots \supsetneq \mathcal{F}^{n-1}(E) \supsetneq \mathcal{F}^n(E) = 0 \tag{1.1}$$

where $\mathcal{F}^i(E)$ are subbundles of E, for all i. Suppose that E admits a flat algebraic connection $\nabla : E \to E \otimes \Omega^1_X$ such that the filtration (1.1) is Griffiths transversal with respect to ∇; meaning that $\nabla(\mathcal{F}^i(E)) \subseteq \mathcal{F}^{i-1}(E) \otimes \Omega^1_X$, for all $i = 1, \ldots, n-1$. Then ∇ induces a Higgs field θ_∇ on the associated vector bundle $\text{gr}(\mathcal{F}^\bullet(E)) := \bigoplus_{i=0}^{n-1} \text{gr}^i(\mathcal{F}^\bullet(E))$, where $\text{gr}^i(\mathcal{F}^\bullet(E)) := \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$, for all $i = 0, 1, \ldots, n-1$.

In [Si1], it is shown that given a flat connection ∇ on E, there exists a Griffiths transversal filtration $\mathcal{F}^\bullet(E)$ such that the associated Higgs bundle $(\text{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable. However, given a Griffiths transversal filtration $\mathcal{F}^\bullet(E)$ of E with respect to ∇, it is not known, in general, whether the associated Higgs bundle $(\text{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable. When X is a compact Riemann surface, in [Bi, p. 156], the following possible solution to this problem is proposed:

Let E be a holomorphic vector bundle on X whose all indecomposable components has degree zero. Let $\mathcal{F}^\bullet(E)$ be a filtration of E by its subbundles on X. Then E admits a holomorphic connection ∇ such that $\mathcal{F}^\bullet(E)$ is Griffiths transversal with respect to ∇ if and only if $\text{gr}(\mathcal{F}^\bullet(E))$ admits a holomorphic Higgs field θ such that

- $(\text{gr}(\mathcal{F}^\bullet(E)), \theta)$ is semistable, and

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- $\theta(\mathrm{gr}^i(\mathcal{F}^\bullet(E))) \subseteq \mathrm{gr}^{i-1}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X$, for all i.

Note that, the Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ admits a structure of a system of Hodge bundles on X; meaning that, $\theta_\nabla(\mathrm{gr}^i(\mathcal{F}^\bullet(E))) \subseteq \mathrm{gr}^{i-1}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X$, for all i. Therefore, it is natural to ask, more generally, when does a Higgs bundle (not necessarily of degree zero) on X having a structure of a system of Hodge bundles is semistable and when it is stable. We give a criterion for this.

Fix an ample line bundle on X. We prove the following results :

Theorem 1.1. *Assume that $char(k) \geq 0$, and Ω^1_X is semistable with $\deg(\Omega^1_X) \geq 0$. Let (E, θ) be a Higgs bundle on X having a structure of a system of Hodge bundles: $E = \bigoplus_{i=0}^{n} E_i$ such that $\theta|_{E_i} : E_i \xrightarrow{\simeq} E_{i-1} \otimes \Omega^1_X$ is an isomorphism, for all $i = 1, \ldots, n$. Then (E, θ) is semistable if E_i is semistable, for all $i = 0, 1, \ldots, n$. The converse holds if $char(k) = 0$.*

Theorem 1.2. *Assume that $char(k) \geq 0$, and $\deg(\Omega^1_X) > 0$. The Higgs bundle (E, θ) in Theorem 1.1 is stable if E_i is stable, for all $i = 0, 1, \ldots, n$. Converse holds if $\dim_k(X) = 1$.*

We give some examples to show that the isomorphism conditions

$$\theta|_{E_i} : E_i \xrightarrow{\simeq} E_{i-1} \otimes \Omega^1_X\,,\ \forall\ i = 1, \ldots, n$$

and semistability of E_i, for all $i = 0, 1, \ldots, n$, in the Theorem 1.1 are crucial for semistability of (E, θ).

Finally, we give a criteria on the Griffiths transversal filtration for a flat connection, which we refer to as "*generalized oper*" so that the associated Higgs bundle becomes semistable. We define notion of semistability of connections, and prove the following :

Theorem 1.3. *Let X be a polarized smooth projective variety over k, and let Ω^1_X be semistable of non-negative degree. Let E be a vector bundle on X together with a connection (not necessarily flat) $\nabla : E \longrightarrow E \otimes \Omega^1_X$. Let*

$$\mathcal{F}^\bullet(E) : 0 = \mathcal{F}^n(E) \subsetneq \mathcal{F}^{n-1}(E) \subsetneq \cdots \subsetneq \mathcal{F}^1(E) \subsetneq \mathcal{F}^0(E) = E$$

be a ∇-Griffiths transversal filtration of E by its subbundles such that the induced $\mathcal{O}_X$–module homomorphism $\theta_\nabla : \mathrm{gr}(\mathcal{F}^\bullet(E)) \longrightarrow \mathrm{gr}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X$ is a Higgs filed on $\mathrm{gr}(\mathcal{F}^\bullet(E))$ (i.e., $\theta_\nabla \wedge \theta_\nabla = 0$ in $H^0(X, \mathcal{E}nd(\mathrm{gr}(\mathcal{F}^\bullet(E))) \otimes \Omega^2_X)$). If the Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable (respectively, stable), then the pair (E, ∇) is semistable (respectively, stable).

Theorem 1.3 is a sort of converse of [LSYZ, Theorem 2.2] by Lan-Sheng-Yang-Zuo. This also generalize [JP, Proposition 3.4.4] of Joshi-Pauly proved for the case of curve in positive characteristic.

2. Griffiths Transversal Filtration

2.1. **Preliminaries.** Let k be an algebraically closed field of characteristic $char(k) \geq 0$. Let X be an irreducible smooth projective algebraic variety over k. Let $\mathcal{O}_X$ be the sheaf of regular functions on X. Let Ω^1_X be the cotangent bundle of X. Let E be a coherent sheaf of $\mathcal{O}_X$–modules on X. The rank of E is defined to be the dimension of the generic fiber of E. We denote it by $\mathrm{rk}(E)$. Since X is irreducible, this is well-defined. We say that E is a *vector bundle* on X if it is locally free and of finite rank on X. An $\mathcal{O}_X$–submodule F of a vector bundle E is said to be a *subbundle* of E if F is locally free and the quotient sheaf E/F is torsion free on X.

Let E be a vector bundle on X.

Definition 2.1. A *connection* on E is a k–linear sheaf homomorphism

$$\nabla : E \longrightarrow E \otimes \Omega^1_X\,, \tag{2.1}$$

satisfying the following Leibniz identity:

$$\nabla(f \cdot s) \;=\; s \otimes \mathrm{d}f \;+\; f \cdot \nabla(s)\,, \tag{2.2}$$

for every section $s \in E(U)$ and regular function $f \in \mathcal{O}_X(U)$, for any open subset $U \subset X$.

Let $\Omega^2_X := \bigwedge^2 \Omega^1_X$. Given a connection ∇ on E, we can extend it to a k–linear sheaf homomorphism (denoted by the same symbol)

$$\nabla : E \otimes \Omega^1_X \longrightarrow E \otimes \Omega^2_X$$

satisfying $\nabla(s \otimes \omega) = s \otimes d\omega - \nabla(s) \wedge \omega$, for all local sections $s \in E(U)$ and $\omega \in \Omega^1_X(U)$. This defines an element

$$\kappa(\nabla) := \nabla \circ \nabla \in H^0(X, \mathcal{E}nd(E) \otimes \Omega^2_X)\,,$$

called the *curvature* of ∇. A connection ∇ is said to be *flat* if $\kappa(\nabla) = 0$.

Definition 2.2. Let E be a vector bundle on X and let

$$\mathcal{F}^\bullet(E) \;:\; E = \mathcal{F}^0(E) \supsetneq \mathcal{F}^1(E) \supsetneq \cdots \supsetneq \mathcal{F}^{n-1}(E) \supsetneq \mathcal{F}^n(E) = 0\,, \tag{2.3}$$

be a filtration of E by its subbundles. The filtration $\mathcal{F}^\bullet(E)$ is said to be *Griffiths transversal* for a flat connection ∇ on E if it satisfies the following conditions:

$$\nabla(\mathcal{F}^i(E)) \subseteq \mathcal{F}^{i-1}(E) \otimes \Omega^1_X\,,\;\; \forall\, i = 1, \ldots, n-1\,. \tag{2.4}$$

2.2. **Associated Higgs Bundle.**

Definition 2.3. A *Higgs sheaf* on X is a pair (E, θ), where E is a coherent sheaf of $\mathcal{O}_X$–modules on X and $\theta : E \longrightarrow E \otimes \Omega^1_X$ is an $\mathcal{O}_X$–module homomorphism such that the following composite $\mathcal{O}_X$–module homomorphism vanishes identically:

$$\theta \wedge \theta : E \xrightarrow{\theta} E \otimes \Omega^1_X \xrightarrow{\theta \otimes \mathrm{Id}_{\Omega^1_X}} E \otimes \Omega^1_X \otimes \Omega^1_X \xrightarrow{\mathrm{Id}_E \otimes (-\wedge -)} E \otimes \Omega^2_X\,. \tag{2.5}$$

Consider a triple $(E, \mathcal{F}^\bullet(E), \nabla)$, where E is a vector bundle on X together with a flat connection ∇, and a filtration $\mathcal{F}^\bullet(E)$ on E, as in (2.3), which is Griffiths transversal for ∇ (see (2.4)). Then ∇ induces an $\mathcal{O}_X$–linear homomorphism

$$\theta^i_\nabla : \mathrm{gr}^i(\mathcal{F}^\bullet(E)) \longrightarrow \mathrm{gr}^{i-1}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X\,, \tag{2.6}$$

where $\mathrm{gr}^i(\mathcal{F}^\bullet(E)) = \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$, for all $0 \leq i \leq n-1$, and $\mathrm{gr}^{-1}(\mathcal{F}^\bullet(E)) := 0$; the $\mathcal{O}_X$–linearity of θ^i_∇ follows from the Leibniz identity (2.2). Thus we have an $\mathcal{O}_X$–linear homomorphism

$$\theta_\nabla : \mathrm{gr}(\mathcal{F}^\bullet(E)) \longrightarrow \mathrm{gr}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X\,, \tag{2.7}$$

where

$$\mathrm{gr}(\mathcal{F}^\bullet(E)) = \bigoplus_{i=0}^{n-1} \mathrm{gr}^i(\mathcal{F}^\bullet(E))\,. \tag{2.8}$$

Note that, the flatness of ∇ ensures that $\theta_\nabla \wedge \theta_\nabla = 0$. Therefore, $(\mathrm{gr}(\mathcal{E}^\bullet), \theta_\nabla)$ is a Higgs bundle over X. Note that the Higgs field θ_∇ satisfies $\theta^n_\nabla = 0$, and hence is nilpotent in the graded k–algebra $\bigoplus\limits_{i=0}^{n} H^0\left(X\,,\, \mathcal{E}nd(E) \otimes (\Omega^1_X)^{\otimes i}\right)$, where $\mathcal{E}nd(E)$ is the sheaf of $\mathcal{O}_X$–module endomorphisms of E.

A *polarization* on X is given by choice of an ample line bundle L on it. Fix an ample line bundle L on X. Let E be a non-zero coherent sheaf of $\mathcal{O}_X$–modules on X. Then the degree of E with respect to L is defined by

$$\deg(E) := c_1(\det(E)) \cdot [L]^{n-1}\,,$$

where $\det(E)$ is the *determinant* line bundle of E. If $\mathrm{rk}(E) > 0$, the ratio $\mu(E) := \deg(E)/\,\mathrm{rk}(E)$ is called the *slope* of E.

Definition 2.4. A torsion free Higgs sheaf (E, θ) on X is said to be *semistable* (respectively, *stable*) if for any non-zero proper subsheaf $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ and $\theta(F) \subseteq F \otimes \Omega^1_X$, we have

$$\mu(F) \leq \mu(E) \;\; (\text{respectively, } \mu(F) < \mu(E))\,.$$

Remark 2.1. A torsion free coherent sheaf E on X can be considered as a Higgs sheaf (E, θ) with zero Higgs field $\theta = 0$ on E. Then the above notion of semistability and stability coincides with the corresponding notions for torsion free coherent sheaves.

Definition 2.5. Let (E_1, θ_1) and (E_2, θ_2) be two Higgs sheaves on X. A *Higgs homomorphism* from (E_1, θ_1) to (E_2, θ_2) is given by an $\mathcal{O}_X$–module homomorphism $\varphi : E_1 \longrightarrow E_2$ such that $\theta_2 \circ \varphi = (\varphi \times \mathrm{Id}_{\Omega^1_X}) \circ \theta_1$.

Lemma 2.1. *Let (E, θ) and (F, ϕ) be two Higgs bundles on X. Let $\Theta = \theta \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \phi$. If $(E \otimes F, \Theta)$ is semistable, then both (E, θ) and (F, ϕ) are semistable. Converse holds if the characteristic of k is zero.*

Proof. Suppose that $(E \otimes F, \Theta)$ is semistable. If (E, θ) were not semistable, then there is a maximal destabilizing Higgs subsheaf $(E_0, \theta|_{E_0})$ of (E, θ) with $\mu(E_0) > \mu(E)$. Since the functor $- \otimes F$ is left exact, $(E_0 \otimes F, \theta|_{E_0} \otimes \mathrm{Id}_F + \mathrm{Id}_{E_0} \otimes \phi)$ is a destabilizing subsheaf of $(E \otimes F, \Theta)$, with $\mu(E_0 \otimes F) = \mu(E_0) + \mu(F) > \mu(E) + \mu(F) = \mu(E \otimes F)$, contradicting Higgs semistability of $(E \otimes F, \Theta)$. Therefore, both (E, θ) and (F, ϕ) are semistable. For the converse part, see [Si2, Corollary 3.8, p. 38]. $\square$

3. SYSTEM OF HODGE BUNDLES AND SEMISTABILITY

Let X be an irreducible smooth projective algebraic variety over k together with a fixed ample line bundle on it.

Definition 3.1. A Higgs bundle (E, θ) is said to have a structure of a *system of Hodge bundles* if E has a direct sum decomposition $E = \bigoplus_{i=0}^{n} E_i$ by its subbundles E_i such that $\theta(E_i) \subseteq E_{i-1} \otimes \Omega^1_X$, for all $0 \leq i \leq n$, with $E_{-1} = 0$.

3.1. **Criterion for semistability of a system of Hodge bundles.** Now we give a criterion for semistability of a Higgs bundle having a structure of a system of Hodge bundles.

Theorem 3.1. *Assume that* $\deg(\Omega^1_X) \geq 0$. *Let* (E, θ) *be a Higgs bundle on* X *which admits a structure of a system of Hodge bundles* $E = \bigoplus_{i=0}^n E_i$. *Suppose that,* $\theta|_{E_i} : E_i \longrightarrow E_{i-1} \otimes \Omega^1_X$ *is an isomorphism of* $\mathcal{O}_X$*–modules, for all* $i \in \{1, \dots, n\}$. *If* E_i *is semistable, for all* $i \in \{1, \dots, n\}$, *then* (E, θ) *is a semistable Higgs bundle.*

To prove this theorem, we need the following useful inequalities:

Lemma 3.2 (Chebyshev's sum inequalities). *Let* $(a_i)_{i=1}^n$ *and* $(b_j)_{j=1}^n$ *be two finite sequence of real numbers.*

(i) If $a_1 \geq a_2 \geq \cdots \geq a_n$ *and* $b_1 \leq b_2 \leq \cdots \leq b_n$*, then we have*

$$n\left(\sum_{i=1}^n a_i b_i\right) \leq \left(\sum_{i=1}^n a_i\right)\left(\sum_{j=1}^n b_j\right). \tag{3.1}$$

(ii) If $a_1 \leq a_2 \leq \cdots \leq a_n$ *and* $b_1 \leq b_2 \leq \cdots \leq b_n$*, then we have*

$$\left(\sum_{j=1}^n b_j\right)\left(\sum_{i=1}^n a_i\right) \leq n\left(\sum_{i=1}^n a_i b_i\right). \tag{3.2}$$

Lemma 3.3. *Let* $d \geq 1$ *be an integer. Then for any integers* r *and* n*, with* $0 \leq r \leq n$*, we have*

$$\left(\sum_{i=0}^r i \cdot d^{i-1}\right)\left(\sum_{j=0}^n d^j\right) \leq \left(\sum_{i=0}^n i \cdot d^{i-1}\right)\left(\sum_{j=0}^r d^j\right). \tag{3.3}$$

Proof of Theorem 3.1. Since $E_i \cong E_0 \otimes (\Omega^1_X)^{\otimes i}$, for all $i \in \{0, 1, \dots, n\}$, we have,

$$\deg(E_i) = i \cdot d^{i-1} \cdot \deg(\Omega^1_X) \cdot \mathrm{rk}(E_0) + d^i \cdot \deg(E_0)\,, \tag{3.4}$$

and

$$\mathrm{rk}(E_i) = d^i \cdot \mathrm{rk}(E_0)\,, \ \ \forall\, i = 0, \dots, n\,. \tag{3.5}$$

Now for any integer $k \in \{0, 1, \cdots, n\}$, by (3.4) and (3.5) we have,

$$\mu\left(\bigoplus_{i=0}^{k} E_i\right) = \frac{\sum_{i=0}^{k} \deg(E_i)}{\sum_{i=0}^{k} \operatorname{rk}(E_i)} = \frac{\left(\deg(\Omega_X^1)\operatorname{rk}(E_0)\sum_{i=0}^{r} i \cdot d^{i-1} + \deg(E_0)\sum_{i=0}^{k} d^i\right)}{\operatorname{rk}(E_0)\sum_{i=0}^{k} d^i}$$
$$= \frac{\deg(\Omega_X^1) \cdot \sum_{i=0}^{r} i \cdot d^{i-1}}{\sum_{i=0}^{r} d^i} \tag{3.6}$$

It follows from (3.6) and Lemma 3.3 that

$$\mu\left(\bigoplus_{i=0}^{k} E_i\right) \leq \mu(E)\,, \quad \forall\; k = 0, \ldots, n\,. \tag{3.7}$$

Suppose on the contrary that (E, θ) is not semistable. Let F be the unique maximal semistable proper Higgs subsheaf of (E, θ) with

$$\mu(F) > \mu(E)\,. \tag{3.8}$$

It follows from [LSYZ, Lemma 2.4] that F admits a structure of system of Hodge bundle; in particular, $F \cong \bigoplus_{i=0}^{n} F_i$, with $F_i = F \cap E_i$, for all $i = 0, 1, \ldots, n$.

Since $\theta|_{E_i}$ is an isomorphism, we have

$$F_i \cong \theta(F_i) \subseteq F_{i-1} \otimes \Omega_X^1\,, \quad \forall\; i = 0, 1, \ldots, n\,. \tag{3.9}$$

Therefore, $F_i \neq 0$ implies $F_{i-1} \neq 0$, for all $1 \leq i \leq n$. Let $r \in \{0, \cdots, n\}$ be the largest integer such that $F_r \neq 0$. Then $F = \bigoplus_{i=0}^{r} F_i$. Now form (3.9), we have

$$0 < \operatorname{rk}(F_r) \leq \operatorname{rk}(F_{r-1}) \leq \cdots \leq \operatorname{rk}(F_0)\,. \tag{3.10}$$

Since $F_i \neq 0$ and E_i is semistable by assumption, using (3.5), we have

$$\deg(F_i) \leq \frac{\operatorname{rk}(F_i) \cdot \deg(E_i)/d^i}{\operatorname{rk}(E_0)}\,, \forall\, i = 0, 1, \ldots, r\,. \tag{3.11}$$

Therefore, using (3.10) and (3.4), applying Lemma 3.2 (i), from (3.11), we have

$$\deg(F) \leq \frac{\sum_{i=0}^{r} \operatorname{rk}(F_i)\deg(E_i)/d^i}{\operatorname{rk}(E_0)} \leq \frac{\left(\sum_{i=0}^{r} \operatorname{rk}(F_i)\right)\left(\sum_{j=0}^{r} \deg(E_j)/d^j\right)}{(r+1)\operatorname{rk}(E_0)} \tag{3.12}$$

Now from (3.4) and (3.12), applying Lemma 3.2 (ii), we have

$$\mu(F) \leq \frac{\left(\sum_{j=0}^{r} \deg(E_j)/d^j\right)}{(r+1)\operatorname{rk}(E_0)} \leq \frac{\sum_{i=0}^{r} \deg(E_i)}{\operatorname{rk}(E_0) \cdot \sum_{i=0}^{r} d^i} = \mu\left(\bigoplus_{i=0}^{r} E_i\right)\,. \tag{3.13}$$

Then from (3.13) and (3.7), we have

$$\mu(F) \leq \mu(E)\,,$$

which contradicts (3.8). Therefore, (E, θ) is semistable. $\square$

Remark 3.1. Note that, semistability of $E_1 \cong E_0 \otimes \Omega^1_X$ forces Ω^1_X to be semistable. It follows from the relation (3.4) and (3.5) that $E = \bigoplus_{i=0}^{n} E_i$, in Theorem 3.1, is semistable if and only if $\deg(\Omega^1_X) = 0$. Therefore, we get many examples of semistable Higgs bundles on X whose underlying vector bundle is not semistable.

Remark 3.2. If $char(k) > 0$, it is expected that, if Ω^1_X and all E_i are strongly semistable, then (E, θ) is strongly semistable; meaning that all the Frobenius pullbacks of (E, θ) are semistable.

We now give an example to show that a semistable Higgs bundle in Theorem 3.1 may not be stable, in general.

Example 3.1. Let X be an irreducible smooth complex projective algebraic curve of genus $g \geq 1$. Then $K_X := \Omega^1_X$ is a line bundle of degree $2g - 2$ on X. Let $Q = (\mathcal{O}_X(1))^{\otimes(g-1)}$ and set $E_1 = Q \oplus Q$. Then E_1 is a rank 2 strictly semistable vector bundle of degree $2g-2$ on X. Take $E_0 := E_1 \otimes K_X^{-1}$ and define $E = E_0 \oplus E_1$. Clearly, $\deg(E) = 0$. Fix $\alpha \in \mathrm{Aut}_X(E_1)$ and consider it as an $\mathcal{O}_X$–module isomorphism $\alpha : E_1 \xrightarrow{\simeq} E_0 \otimes K_X$. Define an $\mathcal{O}_X$–module homomorphism $\theta : E \longrightarrow E \otimes K_X$ by the matrix

$$\theta := \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}.$$

Then (E, θ) is a system of Hodge bundles on X satisfying all conditions in Theorem 3.1. So (E, θ) is a semistable Higgs bundle on X. Since E_1 is not stable, there is a line subbundle L_1 of E_1 with $\deg(L_1) = \mu(E_1) = \deg(E_1)/2$. Let $L_0 = \theta(L_1) \otimes K_X^{-1} \subset E_0$ and define $F := L_0 \oplus L_1$. Then $\theta(F) \subset F \otimes K_X$ and

$$\deg(F) = \deg(L_0) + \deg(L_1) = 2 \cdot \deg(L_1) - \deg(K_X) = \deg(E_1) - (2g - 2) = 0\,.$$

Therefore, (E, θ) is not stable.

Unless otherwise mentioned, from now on, we assume that $char(k) = 0$.

Lemma 3.4. *Let V be an unstable torsion free coherent sheaf of $\mathcal{O}_X$–modules on X. Let*

$$0 = V_m \subset V_{m-1} \subset \cdots \subset V_0 = V$$

be the Harder-Narasimhan filtration of V. Then for any semistable vector bundle W on X,

$$0 = V_m \otimes W \subset V_{m-1} \otimes W \subset \cdots \subset V_0 \otimes W = V \otimes W$$

is the Harder-Narasimhan filtration of $V \otimes W$.

Proof. For each $i \in \{0, 1, \cdots, m\}$, consider the exact sequence of coherent sheaves :

$$0 \longrightarrow V_{i+1} \longrightarrow V_i \longrightarrow V_i/V_{i+1} \longrightarrow 0\,. \tag{3.14}$$

Since W is locally free, tensoring (3.14) with W, we get

$$(V_i \otimes W)/(V_{i+1} \otimes W) \cong (V_i/V_{i+1}) \otimes W\,, \ \ \forall\ i = 0, 1, \ldots, m-1\,.$$

Then the result follows from the fact that $(V_i/V_{i+1}) \otimes W$ is semistable (see e.g., [HL]) and $\mu((V_i/V_{i+1}) \otimes W) = \mu((V_i/V_{i+1})) + \mu(W)$, for all $i = 0, 1, \ldots, m-1$. ☐

Lemma 3.5. *Let E and F be two isomorphic unstable torsion free coherent sheaf of $\mathcal{O}_X$–modules on X. Let $G \subset E$ be the maximal destabilizing subsheaf of E. Then for any two $\mathcal{O}_X$–module isomorphisms $f_1, f_2 : E \longrightarrow F$, we have $f_1(G) = f_2(G)$, and this is the maximal destabilizing subsheaf of F.*

Proof. This follows from the fact that the maximal destabilizing subsheaf is invariant under all $\mathcal{O}_X$–module automorphisms of the coherent sheaf. ☐

Theorem 3.6. *Assume that Ω^1_X is semistable with $\deg(\Omega^1_X) \geq 0$. Let (E, θ) be a Higgs bundle on X admitting a structure of a system of Hodge bundles given by $E = \bigoplus_{i=0}^{n} E_i$ with $\theta|_{E_i} : E_i \longrightarrow E_{i-1} \otimes \Omega^1_X$ isomorphisms, for all $i = 1, \ldots, n$. Then (E, θ) is semistable if and only if E_0 is semistable.*

Proof. Since $E_p \cong E_0 \otimes (\Omega^1_X)^{\otimes p}$, for all $p = 0, 1, \ldots, n$, and Ω^1_X is semistable, for any $p \in \{0, 1, \cdots, n\}$ we have, E_p is semistable if and only if E_0 is semistable. Therefore, if E_0 is semistable, then (E, θ) is semistable by Theorem 3.1. We now show the converse part.

Let (E, θ) be semistable. Tensoring E with a sufficiently large degree line bundle, if required, we may assume that $\deg(E_p) > 0$, for all $p = 0, 1, \ldots, n$. Suppose that, E_0 is not semistable. Let $F_p \subset E_p$ be the maximal destabilizing subsheaf of E_p, for all $p = 0, 1, \ldots, n$. Since $\theta|_{E_p} : E_p \to E_{p-1} \otimes \Omega^1_X$ is an isomorphism, it follows from Lemma 3.4 and Lemma 3.5 that $\theta(F_p) = F_{p-1} \otimes \Omega^1_X$, for all $p = 0, 1, \ldots, n$. Therefore, we have

$$\mathrm{rk}(E_p) = d^p \cdot \mathrm{rk}(E_0) \ \ \text{and} \ \ \mathrm{rk}(F_p) = d^p \cdot \mathrm{rk}(F_0)\,, \ \forall\ p = 0, 1, \ldots, n\,, \tag{3.15}$$

where $d = \mathrm{rk}(\Omega^1_X) = \dim(X)$. Clearly $F = \bigoplus_{p=0}^{n} F_p$ is a Higgs subsheaf of (E, θ). Now from (3.15) we have,

$$\deg(F) = \sum_{p=0}^{n} \deg(F_p) > \frac{\mathrm{rk}(F_0)}{\mathrm{rk}(E_0)} \sum_{p=0}^{n} \deg(E_p) = \frac{\mathrm{rk}(F_0)}{\mathrm{rk}(E_0)} \deg(E)\,.$$

Therefore, $\mu(F) > \mu(E)$, which contradicts the fact that (E, θ) is semistable. ☐

3.2. **Criterion for stability of a system of Hodge bundles.** Let $char(k) \geq 0$.

Definition 3.2. A Higgs bundle (E, θ) is said to be *simple* if any non-zero Higgs endomorphism of (E, θ) is an isomorphism.

Proposition 3.7. *Assume that* $\deg(\Omega^1_X) > 0$. *Let* (E, θ) *be a Higgs bundle on* X *having a structure of a system of Hodge bundles:* $E = \bigoplus_{p=0}^{n} E_p$, *with* $\theta|_{E_p} : E_p \to E_{p-1} \otimes \Omega^1_X$ *isomorphism, for all* $p = 1, \ldots, n$. *If* E_p *is stable, for all* $p = 0, 1, \ldots, n$, *then* (E, θ) *is simple.*

Proof. Since $\theta(E_p) \subseteq E_{p-1} \otimes \Omega^1_X$, for all $p = 0, 1, \ldots, n$, the matrix of θ is strictly block-upper triangular and of the form :

$$\theta = \begin{pmatrix} 0 & \theta_{01} & 0 & \cdots & 0 \\ 0 & 0 & \theta_{12} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\theta_{ij} \in \mathrm{Iso}_{\mathcal{O}_X}(E_j, E_i \otimes \Omega^1_X)$, for all i, j. Let $\varphi : E \longrightarrow E$ be a non-zero $\mathcal{O}_X$–module homomorphism with

$$\theta \circ \varphi = (\varphi \otimes \mathrm{Id}_{\Omega^1_X}) \circ \theta\,. \tag{3.16}$$

For any $i, j \in \{0, 1, \cdots, n\}$, let φ_{ij} be the composite homomorphism

$$\varphi_{ij} : E_j \hookrightarrow E \xrightarrow{\varphi} E \xrightarrow{\pi_i} E_i\,,$$

where π_i is the projection of E onto the i-th factor. It follows from (3.16) that the matrix of φ is block-upper triangular :

$$\varphi = \begin{pmatrix} \varphi_{00} & \varphi_{01} & \cdots & \varphi_{0n} \\ 0 & \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_{nn} \end{pmatrix}$$

Since $\mu(E_i) = \mu(E_0) + i \cdot \mu(\Omega^1_X)$, and E_i are stable by assumption, $\varphi_{ii} = \lambda_i \,\mathrm{Id}_{E_i}$, for some $\lambda_i \in k$, for all $i = 0, 1, \ldots, n$, and $\varphi_{ij} = 0$, for all $i < j$. Therefore, φ is the diagonal matrix $\varphi = \mathrm{diag}(\varphi_{00}, \cdots, \varphi_{nn})$. It follows from the relation (3.16) that $\lambda_0 = \lambda_1 = \cdots = \lambda_n$. Since $\varphi \neq 0$, we must have $\lambda_i \neq 0$, and hence φ is an isomorphism. $\square$

Definition 3.3. A semistable Higgs bundle is said to be *polystable* if it is a direct sum of stable Higgs bundles.

Every semistable Higgs sheaf (E, θ) contains a unique maximal polystable Higgs subsheaf, called the *socle* of (E, θ). The socle of (E, θ) is invariant under all Higgs automorphisms of E.

Remark 3.3. Note that, a simple polystable Higgs bundle is necessarily stable.

Theorem 3.8. *Assume that* $char(k) \geq 0$, *and* $\deg(\Omega^1_X) > 0$. *Let* (E, θ) *be a Higgs bundle on* X *having a structure of a system of Hodge bundles:* $E = \bigoplus_{p=0}^{n} E_p$, *with* $\theta|_{E_p} : E_p \to E_{p-1} \otimes \Omega^1_X$ *isomorphism, for all* $p = 1, \ldots, n$. *If* E_p *is stable, for each* $p = 0, 1, \ldots, n$, *then* (E, θ) *is stable.*

Proof. Clearly (E,θ) is semistable by Theorem 3.1. Suppose that (E,θ) is not stable. Then its socle $(F,\theta_F)\subset(E,\theta)$ is the unique non-zero proper maximal polystable Higgs subsheaf with $\mu(F)=\mu(E)$. Clearly, (F,θ_F) is invariant under the $\mathbb{G}_m$–action on (E,θ). Therefore, (F,θ_F) admits a structure of a system of Hodge bundles, say $F=\bigoplus_{i=0}^{n} F_i$, with $\theta_F(F_i)\subseteq F_{i-1}\otimes\Omega^1_X$, for all $i=0,1,\ldots,n$. It follows form the proof of [LSYZ, Lemma 2.4] that, $F_i=F\cap E_i$, for all $i=0,1,\ldots,n$. Since $\theta|_{E_p}$ is an isomorphism, we have $F_p\cong\theta(F_p)\subseteq F_{p-1}\otimes\Omega^1_X$, for all $p=1,\ldots,n$. Let $r\in\{0,1,\cdots,n\}$ be the largest integer such that $F_r\neq 0$. Then $F=\bigoplus_{p=0}^{r} F_p$. Since F is a proper subsheaf of E, there is **at least one** $p\in\{0,1,\cdots,r\}$ such that $F_p\neq E_p$. Since all E_p are stable, we have,

$$\deg(F_p)\leq\frac{\mathrm{rk}(F_p)\cdot\deg(E_p)/d^p}{\mathrm{rk}(E_0)}\,,\quad\forall\ p=0,1,\ldots,r\,,\tag{3.17}$$

and the **inequality** (3.17) **is strict for at least one** $p\in\{0,1,\cdots,r\}$. Then from (3.17), following the inequality computations as in proof of Theorem 3.1, we have

$$\mu(F)<\mu\left(\bigoplus_{p=0}^{r}E_p\right)\leq\mu(E)\,,$$

which contradicts the fact that $\mu(F)=\mu(E)$. Therefore, (E,θ) is polystable. Then by Proposition 3.7, (E,θ) is stable. $\square$

Theorem 3.9. *Assume that $char(k)=0$. Let X be an irreducible smooth projective algebraic curve of genus $g\geq 2$. Let (E,θ) be a Higgs bundle on X admitting a structure of a system of Hodge bundles : $E=\bigoplus_{p=0}^{n}E_p$ with $\theta|_{E_p}:E_p\longrightarrow E_{p-1}\otimes\Omega^1_X$ an isomorphism, for all $p=1,\ldots,n$. Then (E,θ) is stable if and only if E_p is stable, for all $p=0,1,\ldots,n$.*

Proof. Suppose that, (E,θ) is stable. It follows from Theorem 3.6 that E_p is semistable, for all $p=0,1,\ldots,n$. Since $K_X:=\Omega^1_X$ is a line bundle on X, for any $p\in\{0,1,\cdots,n\}$ we see that, E_p stable if and only if E_0 is stable. Suppose that E_0 is not stable. Then there is a non-zero proper stable subsheaf $G_0\subset E_0$ with $\mu(G_0)=\mu(E_0)$. Since θ^p is an isomorphism of E_p onto $E_0\otimes K_X^{\otimes p}$, for all p, there is a subsheaf $G_p\subset E_p$ such that $\theta^p:G_p\longrightarrow G_0\otimes K_X^{\otimes p}$ is isomorphism, for all $p=0,1,\ldots,n$. Then $G=\bigoplus_{p=0}^{n}G_p$ is a Higgs subsheaf of (E,θ). Now a similar computation as in the proof of Theorem 3.6 shows that $\mu(G)=\mu(E)$, which contradicts the fact that (E,θ) is stable. $\square$

Remark 3.4. It is expected that for $\dim_k(X)\geq 2$ with Ω^1_X is stable and $\deg(\Omega^1_X)>0$, if (E,θ) in Theorem 3.9 is stable then all E_p are polystable.

Remark 3.5. Note that, in the proofs of all Theorems in this Section, we have used only semistability (or stability) of Ω^1_X and the condition $\deg(\Omega^1_X)\geq 0$ (>0). Therefore, with appropriate notion of semistability and stability of pairs (E,θ) with $\theta\in H^0(X,\mathcal{E}nd(E)\otimes V)$, all Theorems in this Section 3 hold if we replace Ω^1_X with any semistable (or stable) vector bundle V on X of degree ≥ 0 (or, >0).

3.3. **Examples of unstable system of Hodge bundles.** We now give two examples to show that the isomorphism conditions in Theorem 3.1 are crucial.

Example 3.2. Let X be a smooth complex projective curve of genus $g \geq 2$. Let L_0 be a line bundle of degree $d > 2g-2$ on X. Let E_1 be a non-trivial extension of $\mathcal{O}_X$ and $L_0 \otimes K_X$. So we have a short exact sequence of $\mathcal{O}_X$–modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E_1 \xrightarrow{\theta_1} L_0 \otimes K_X \longrightarrow 0\,.$$

Then E_1 is semistable, but not necessarily stable. Let $E = E_0 \bigoplus E_1$, where $E_0 = L_0$. Then $\deg(E) = \deg(E_0) + \deg(E_1) = 2d + 2g - 2$, and hence $\mu(E) = 2(d+g-1)/3$. Since $d > 2g-2$, we have $\mu(E) < d = \mu(L_0)$. Define a Higgs field $\theta \in H^0(X, \mathcal{E}nd(E) \otimes K_X)$ by

$$\theta = \begin{pmatrix} 0 & \theta_1 \\ 0 & 0 \end{pmatrix}\,.$$

Then (E, θ) is a Higgs bundle having a structure of a system of Hodge bundles on X. Note that, θ_1 **is surjective, but not isomorphism**. Since L_0 is a θ–invariant, (E, θ) is not semistable.

Example 3.3. Let X be a smooth projective curve of genus $g \geq 2$ over k. Let L_0 be a line bundle on X of positive degree. Let $L_1 = L_0^\vee$ and $E = L_0 \bigoplus L_1$. Since $\deg(\mathcal{H}om(L_1, L_0 \otimes K_X)) = 2 \cdot \deg(L_0) + (2g-2)$, choosing L_0 with $\deg(L_0)$ sufficiently large, we can find a non-zero $\mathcal{O}_X$–module homomorphism $\theta_1 : L_1 \longrightarrow L_0 \otimes K_X$. Note that, θ_1 **is injective**, because both L_1 and $L_0 \otimes K_X$ are line bundles, but θ_1 **is not an isomorphism**. Define an $\mathcal{O}_X$–module homomorphism $\theta : E \longrightarrow E \otimes K_X$ by

$$\theta := \begin{pmatrix} 0 & \theta_1 \\ 0 & 0 \end{pmatrix}\,.$$

Then (E, θ) is a Higgs bundle having a structure of a system of Hodge bundles on X. Since L_0 is a θ–invariant subbundle of positive degree, (E, θ) is not semistable.

We now shows that, if all E_p are not semistable (E, θ) may fail to be semistable.

Example 3.4. Let X be a smooth complex projective curve of genus $g \geq 2$. Fix a square root $K^{1/2}$ of the cotangent bundle K_X on X. Let L_0 be a positive degree line bundle on X. Consider the line bundles $L_1 = \left(L_0^\vee \otimes K_X^{-1/2}\right)^{\otimes 2}$ and $L_2 = L_0 \otimes K_X$ on X. Let $E = E_0 \bigoplus E_1$, where $E_0 = L_0$ and $E_1 = L_1 \bigoplus L_2$. Clearly, $\deg(E) = 0$. Consider the $\mathcal{O}_X$–module homomorphism $\theta : E \longrightarrow E \otimes K_X$ defined by

$$\theta = \begin{pmatrix} 0 & \theta_1 \\ 0 & 0 \end{pmatrix}\,,$$

where $\theta_1 : E_1 = L_1 \bigoplus (L_0 \otimes K_X) \longrightarrow E_0 \otimes K_X$ is the projection homomorphism onto the second factor. Then (E, θ) is a Higgs bundle of degree 0 having a structure of a system of Hodge bundles on X. Note that, E_1 **is not semistable**, and $\theta|_{E_1} : E_1 \to E_0 \otimes K_X$ is surjective, but not isomorphism. Since E_0 is a θ–invariant subbundle of positive degree, (E, θ) is not semistable.

4. Generalized Oper

4.1. **Semistability of generalized oper.** It is not know if the Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$, defined in Section 2.2 associated to a Griffiths transversal filtration $\mathcal{F}^\bullet(E)$ with respect to a flat connection ∇ on E, is semistable or not. In this section, we give a criterion for semistability of $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$.

Definition 4.1. An *oper* is a triple $(E, \mathcal{F}^\bullet(E), \nabla)$ consists of a vector bundle E on X together with a flat connection ∇ and a Griffiths transversal filtration $\mathcal{F}^\bullet(E)$ (see (2.3)) of E by its subbundles (see Definition 2.2), such that the quotients $\mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$ are line bundles on X and the $\mathcal{O}_X$–linear homomorphisms θ^i_∇ in (2.6) induced by ∇ are isomorphisms, for all $i = 1, \ldots, n-1$.

Let us first recall the following well-known result.

Proposition 4.1. [Si1, p. 186] *Let X be a connected smooth complex projective curve of genus $g \geq 1$. Let $(E, \mathcal{F}^\bullet(E), \nabla)$ be an oper on X. Then the associated Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ on X is semistable.*

We give a natural generalization of the above result for higher ranks and higher dimensional algebraic varieties.

Definition 4.2. A *generalized oper* is a triple $(E, \mathcal{F}^\bullet(E), \nabla)$ consists of a vector bundle E on X together with a flat connection ∇ and a Griffiths transversal filtration $\mathcal{F}^\bullet(E)$ (see (2.3)) of E by its subbundles (see Definition 2.2), such that the quotients $\mathrm{gr}^i(\mathcal{F}^\bullet(E)) := \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$ are semistable vector bundles on X and the $\mathcal{O}_X$–linear homomorphisms θ^i_∇ in (2.6) induced by ∇ are isomorphisms, for all $i = 1, \ldots, n-1$.

Remark 4.1. Note that, if $(E, \mathcal{F}^\bullet(E), \nabla)$ is a generalized oper on X and $\deg(\Omega^1_X) > 0$, then $\mathcal{F}^\bullet(E)$ is the Harder-Narasimhan filtration of E.

Theorem 4.2. *Let $(E, \mathcal{F}^\bullet(E), \nabla)$ be a generalized oper on X. If $\deg(\Omega^1_X) \geq 0$, then the associated Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ on X is semistable, where $\mathrm{gr}(\mathcal{F}^\bullet(E)) = \bigoplus_{i=0}^{n-1} \mathrm{gr}^i(\mathcal{F}^\bullet(E))$.*

Proof. Since $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ has a structure of a system of Hodge bundles, all $\mathrm{gr}^i(\mathcal{F}^\bullet(E))$ are semistable and all θ^i_∇ are isomorphisms, by Theorem 3.1, $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable. □

Remark 4.2.
1. If X is a smooth complex projective curve of genus $g \geq 1$, then we get Proposition 4.1 as a corollary to the Theorem 4.2.
2. With appropriate notion of logarithmic Higgs semistability, Theorem 4.2 holds for logarithmic connections singular over an effective divisor, using similar techniques.

4.2. **Semistability of Connections.** As before, let X be a smooth polarized projective variety over and algebraically closed filed k of characteristic $p \geq 0$, and the cotangent bundle Ω^1_X is semistable and of non-negative degree.

Let E be a pure coherent sheaf of $\mathcal{O}_X$–modules on X.

Definition 4.3. Let E be a vector bundle on X together with a connection $\nabla : E \to E \otimes \Omega^1_X$. Then the pair (E, ∇) is said to be *semistable* (respectively, *stable*) if for any non-zero proper $\mathcal{O}_X$–submodule $F \subset E$ with torsion free quotient sheaf E/F on X such that $\nabla(F) \subseteq F \otimes \Omega^1_X$, we have $\mu(F) \leq \mu(E)$ (respectively, $\mu(F) < \mu(E)$).

Definition 4.4. A *partial oper* is a triple $(E, \mathcal{F}^\bullet(E), \nabla)$ consisting of a vector bundle E on X together with a flat connection ∇ and a filtration $\mathcal{F}^\bullet(E)$ of E by its subbundles on X which is Griffiths transversal with respect to ∇ such that the induced Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable on X.

Given a vector bundle E on X together with a semistable flat connection ∇ on E, Lan-Sheng-Yang-Zuo proved that there is a filtration $\mathcal{F}^\bullet(E)$ of E such that the triple $(E, \mathcal{F}^\bullet(E), \nabla)$ is a partial oper on X (see [LSYZ, Theorem 2.2]). We now prove some sort of converse of the above result.

Theorem 4.3. *Let E be a vector bundle on a smooth projective variety X over an algebraically closed field k of positive characteristic. Let $\nabla : E \longrightarrow E \otimes \Omega^1_X$ be a connection (not necessarily flat) on E. Let*

$$\mathcal{F}^\bullet(E) : 0 = \mathcal{F}^n(E) \subsetneq \mathcal{F}^{n-1}(E) \subsetneq \cdots \subsetneq \mathcal{F}^1(E) \subsetneq \mathcal{F}^0(E) = E$$

be a ∇-Griffiths transversal filtration of E by its subbundles such that the induced $\mathcal{O}_X$–module homomorphism $\theta_\nabla : \mathrm{gr}(\mathcal{F}^\bullet(E)) \longrightarrow \mathrm{gr}(\mathcal{F}^\bullet(E)) \otimes \Omega^1_X$ is a Higgs filed on $\mathrm{gr}(\mathcal{F}^\bullet(E))$ (i.e., $\theta_\nabla \bigwedge \theta_\nabla = 0$ in $H^0(X, \mathcal{E}nd(\mathrm{gr}(\mathcal{F}^\bullet(E))) \otimes \Omega^2_X)$), and the Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable. Then the pair (E, ∇) is semistable.

Proof. If (E, ∇) were not semistable, then there is a non-zero proper $\mathcal{O}_X$–submodule $F \subset E$ such that $\nabla(F) \subseteq F \otimes \Omega^1_X$ and

$$\mu(F) > \mu(E)\,. \tag{4.1}$$

The filtration $\mathcal{F}^\bullet(E)$ induces a filtration on F

$$\mathcal{F}^\bullet(F) \;\;:\;\; 0 = \mathcal{F}^n(F) \subseteq \mathcal{F}^{n-1}(F) \subseteq \cdots \subseteq \mathcal{F}^1(F) \subseteq \mathcal{F}^0(F) = F\,,$$

where $\mathcal{F}^i(F) = \mathcal{F}^i(E) \cap F$, for all i. Since $\nabla(F) \subseteq F \otimes \Omega^1_X$, the injective homomorphisms $\iota_i : \mathcal{F}^i(F)/\mathcal{F}^{i+1}(F) \hookrightarrow \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$, induced by the inclusions $\mathcal{F}^i(F) \subseteq \mathcal{F}^i(E)$, fits into the following commutative diagram of $\mathcal{O}_X$–module homomorphisms

$$\begin{tikzcd}
\mathcal{F}^i(F)/\mathcal{F}^{i+1}(F) \arrow[r, hook, "\iota_i"] \arrow[d, "\theta'_i"'] & \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E) \arrow[d, "\theta_i"] \\
(\mathcal{F}^{i-1}(F)/\mathcal{F}^i(F)) \otimes \Omega^1_X \arrow[r, hook, "\iota_{i+1}"] & (\mathcal{F}^{i-1}(E)/\mathcal{F}^i(E)) \otimes \Omega^1_X
\end{tikzcd}$$

where θ_i' is the restriction of θ_i, for all i. So $(\mathrm{gr}(\mathcal{F}^\bullet(F)), \theta'_\nabla)$ is a non-zero Higgs subsheaf of the semistable Higgs bundle $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$. Now a simple degree rank computation shows that

$$\mu(F) = \mu(\mathrm{gr}(\mathcal{F}^\bullet(F))) \leq \mu(\mathrm{gr}(\mathcal{F}^\bullet(E))) = \mu(E)\,,$$

which contradicts the inequality (4.1). Therefore, (E, ∇) is semistable. $\square$

Corollary 4.4. *Let E be a vector bundle on X together with a flat connection $\nabla : E \to E \otimes \Omega^1_X$. Suppose that E admits a filtration by its subbundles*

$$\mathcal{F}^\bullet(E) \quad : \quad 0 = \mathcal{F}^n(E) \subsetneq \mathcal{F}^{n-1}(E) \subsetneq \cdots \subsetneq \mathcal{F}^1(E) \subsetneq \mathcal{F}^0(E) = E$$

which is Griffiths transversal with respect to ∇, and the ∇–induced $\mathcal{O}_X$–module homomorphisms $\theta_i : \mathcal{F}^i(E)/\mathcal{F}^{i+1}(E) \longrightarrow (\mathcal{F}^{i-1}(E)/\mathcal{F}^i(E)) \otimes \Omega^1_X$ are isomorphisms, for all $i = 1, \ldots, n-1$. Then the pair (E, ∇) is semistable if $\mathcal{F}^i(E)/\mathcal{F}^{i+1}(E)$ is semistable, for all $i = 0, 1, \ldots, n-1$.

Proof. If $p = char(k)$ is zero, then for any flat connection ∇ on E, the pair (E, ∇) is automatically semistable. This follows from the fact that any non-zero coherent sheaf F on X admitting a flat connection has zero first Chern class, and hence has zero slope with respect to any polarization on X. This is not the case if $p > 0$.

If $char(k) = p > 0$, since $(\mathrm{gr}(\mathcal{F}^\bullet(E)), \theta_\nabla)$ is semistable by Theorem 3.1, we are done by using Theorem 4.3. $\square$

Remark 4.3. Corollary 4.4 was proved over smooth projective curve in positive characteristics by Joshi-Pauly (see [JP, Proposition 3.4.4]).

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