MA5202: Algebraic Geometry

Arjun Paul
Department of Mathematics and Statistics
Indian Institute of Science Education and Research Kolkata,
Mohanpur - 741 246, Nadia,
West Bengal, India.
Email: arjun.paul@iiserkol.ac.in.

Version: January 13, 2023 at 11:24pm (IST).

This note will be updated from time to time.
If there are any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\bigsqcup$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
A := ...	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Basic Theory of Schemes

1.1 Classical variety

Let k be a field and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ and coefficients from the field k. Given a subset $E \subseteq k[x_1, \ldots, x_n]$, let

$$\mathcal{Z}(E) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0, \ \forall\, f \in E\}$$

be the subset of all common zeros of the polynomials in E. We are interested to study geometry of $\mathcal{Z}(E)$. If $f \in E$ is a linear polynomial with zero constant term, i.e., $f(0, \ldots, 0) = 0$, the map $T_f : k^n \to k$ defined by

$$T_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n), \ \ \forall\, (a_1, \ldots, a_n) \in k^n,$$

is a k-linear map, and that $\mathcal{Z}(f) = \mathrm{Ker}(T_f)$ is a k-linear subspace of k^n. Therefore, if all the polynomials in E are linear with zero constant terms, then

$$\mathcal{Z}(E) = \bigcap_{f \in E} \mathrm{Ker}(T_f)$$

is a k-linear subspace of k^n, and in this case standard linear algebra machinery could be used to study the space $\mathcal{Z}(E)$. However, when $f \in A$ is not a linear polynomial, $\mathcal{Z}(f)$ is no longer a liner space, and hence we cannot use linear algebra machinery to study geometry of $\mathcal{Z}(f)$. In this situation, the techniques from commutative algebra comes into the picture.

Proposition 1.1.1. *Let E be a non-empty subset of the polynomial ring $k[x_1, \ldots, x_n]$. Then $\mathcal{Z}(E) = \mathcal{Z}(\langle\, E\, \rangle)$, where $\langle\, E\, \rangle$ is the ideal of $k[x_1, \ldots, x_n]$ generated by E.*

Proof. Since $E \subseteq \langle\, E\, \rangle$, it follows from the definition of $\mathcal{Z}(E)$ that $\mathcal{Z}(\langle\, E\, \rangle) \subseteq \mathcal{Z}(E)$. Conversely, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{Z}(E)$. Let $f \in \langle\, E\, \rangle$ be arbitrary. Then $f = \sum\limits_{j=1}^{m} \phi_j f_j$, for some $\phi_j \in k[x_1, \ldots, x_n]$ and $f_j \in E$, for all $j \in \{1, \ldots, m\}$. Since $f_j(\mathbf{a}) = 0$, for all j, we have $f(\mathbf{a}) = \sum\limits_{j=1} \phi_j(\mathbf{a}) f_j(\mathbf{a}) = 0$. Therefore, $\mathbf{a} \in \mathcal{Z}(\langle\, E\, \rangle)$. $\square$

We now introduce a class of commutative rings with identity for which every ideals are finitely generated. Such a ring is called Noetherian. We show that polynomial ring

$k[x_1, \ldots, x_n]$ and its quotient rings are Noetherian. One of the advantage to work with such rings is that all of its ideals being finitely generated, zero locus of a given family of possibly infinitely many polynomials is determined by a finite number of polynomials among them.

Let A be a commutative ring with identity.

Definition 1.1.2. An A-module M is said to be *finitely generated* if there exists finitely many elements $x_1, \ldots, x_n \in M$ such that given any $x \in M$ there exists $a_1, \ldots, a_n \in A$ such that $x = a_1 x_1 + \cdots + a_n x_n$.

Example 1.1.3. For each $n \in \mathbb{N}$, the A-module $A^{\oplus n}$ is finitely generated. Indeed, it is generated by $\{e_1, \ldots, e_n\}$, where $e_j \in A^{\oplus n}$ is the ordered n-tuple of elements of A whose j-th coordinate is $1 \in A$, and all other coordinates are $0 \in A$.

Example 1.1.4. Let $f : M \to N$ be a surjective A-module homomorphism. If M is a finitely generated A-module, so is N. Indeed, if M is generated by $\{x_1, \ldots, x_n\}$ as an A-module, then N is generated by $\{f(x_1), \ldots, f(x_n)\}$ as an A-module.

Corollary 1.1.5. *An A-module M is finitely generated if and only if there exists a surjective A-module homomorphism $f : A^{\oplus n} \to M$, for some $n \in \mathbb{N}$.*

Note that, an A-submodule of a finitely generated A-module need not be finitely generated. For example, take $A = k[X_1, X_2, \ldots]$ be the polynomial ring over a field k with countably infinitely many variables $\{X_n : n \in \mathbb{N}\}$. Clearly, A is a commutative ring with identity. Let $\mathfrak{a}$ be the ideal of A generated by its variables $(X_n : n \in \mathbb{N})$. Clearly $\mathfrak{a}$ is a non-zero proper ideal of A (and hence an A-module), which is clearly not finitely generated.

Definition 1.1.6. An A-module M is said to be *noetherian* if every A-submodule of M is finitely generated. We say that A is *noetherian* if it is noetherian as a module over itself.

In particular, a noetherian A-module is a finitely generated A-module. However, the converse may not be true (see above Example).

Proposition 1.1.7. *A is noetherian if and only if every ideal of A is finitely generated.*

Proof. Since any A-submodule of A is an ideal of A, the result follows. □

Lemma 1.1.8. *Let*

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

be a short exact sequence of A-modules. Then M is noetherian if and only if both M' and M'' are noetherian.

Proof. Suppose that M is noetherian. Let N' be an A-submodule of M'. Then N is isomorphic to the A-submodule $\phi(N')$ of M, and hence is finitely generated. Therefore, M' is noetherian. Let N'' be an A-submodule of M''. Then $N := \psi^{-1}(N'')$ is an A-submodule of M, and so is finitely generated. Since the A-module homomorphism $\psi|_N : N \to N''$ is surjective and N is finitely generated, N'' is finitely generated. Therefore, M'' is noetherian.

Conversely, suppose that both M' and M'' are noetherian A-modules. Let N be an A-submodule of M. Since $N' := \phi^{-1}(N)$ and $N'' := \psi(N)$ are A-submodule of noetherian A-modules, they are finitely generated. Then we have an exact sequence of A-modules

$$0 \to N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \to 0.$$

Suppose that $\phi^{-1}(N)$ and $\psi(N)$ are generated as A-modules by $x_1, \ldots, x_m \in \phi^{-1}(N)$ and $y_1, \ldots, y_n \in \psi(N)$, respectively. Fix an element $z_j \in \psi^{-1}(y_j) \subseteq N$, for each $j \in \{1, \ldots, n\}$. Let $x \in N$ be given. Then $\psi(x) = b_1 y_1 + \cdots + b_n y_n$, for some $b_1, \ldots, b_n \in A$. Consider the element $w = x - (b_1 z_1 + \cdots + b_n z_n) \in N$. Since $\psi(w) = 0$, there exists $a_1, \ldots, a_m \in A$ such that $w = a_1 x_1 + \cdots + a_m x_m$. Then we have $x = a_1 x_1 + \cdots + a_m x_m + b_1 z_1 + \cdots + b_n z_n$. Therefore, N is generated as an A-module by $\{x_1, \ldots, x_m\} \cup \{z_1, \ldots, z_n\}$. Therefore, M is noetherian. □

Corollary 1.1.9. *If M and N are noetherian A-modules, so is $M \oplus N$.*

Proof. Follows from Lemma 1.1.8. □

Corollary 1.1.10. *Any finitely generated module over a noetherian ring is noetherian.*

Proof. Let A be a noetherian ring and let M be a finitely generated A-module. Then there exists a surjective A-module homomorphism

$$\varphi : A^{\oplus n} \to M,$$

for some $n \in \mathbb{N}$. Since $A^{\oplus n}$ is noetherian by Corollary 1.1.9, that M is noetherian by Lemma 1.1.8. □

Theorem 1.1.11 (Hilbert's basis theorem)**.** *If A is a noetherian ring, the polynomial ring $A[x_1, \ldots, x_n]$ is noetherian.*

Proof. Since the polynomial ring $A[x_1, \ldots, x_n]$ is isomorphic to the polynomial ring $B[x_n]$, where $B = A[x_1, \ldots, x_n]$, using induction it suffices to prove the result for $n = 1$. Consider the polynomial ring $A[x]$. Let $I \subset A[x]$ be an ideal of $A[x]$. Since the cases $I = 0$ and $I = A[x]$ are trivial, we assume that $I \neq 0$ and $I \neq A[x]$. Let

$$J = \{0\} \cup \{\text{set of all leading coefficients of non-zero polynomials in } I\}.$$

Clearly J is an ideal of A, and hence is finitely generated because A is noetherian. Let $c_1, \ldots, c_r \in A$ be non-zero elements of A that generates J as an ideal of A. Each c_j is a leading coefficient of a non-zero element, say f_j, of I. Let $J' = (f_1, \ldots, f_r)$ be the ideal of $A[x]$ generated by $f_1, \ldots, f_r$. Let $m := \max\limits_{1 \leq j \leq r} \deg(f_j)$, and let

$$M := I \cap \left(A + Ax + \cdots + Ax^{m-1}\right).$$

Then M is an A-submodule of $A[x]$. We claim that $I = M + J'$. Since both M and J' are subsets of I and I is an ideal, $M + J' \subseteq I$. To show the reverse inclusion, we need to show that every $f \in I$ is in $M + J'$. We show this by induction on $d = \deg(f)$. If $d \leq m - 1$, then $f \in M \subseteq M + J'$. Suppose that $d := \deg(f) \geq m$, and assume that for given any $g \in I$ with $\deg(g) < d$ we have $g \in M + J'$. Let c be the leading coefficient of f. Since $J = (c_1, \ldots, c_r)$ and $c \in J$, we have $c = \sum\limits_{j=1}^{r} a_j c_j$, for some $a_1, \ldots, a_r \in A$. Since $g := f - \sum\limits_{j=1}^{r} a_j x^{d-\deg(f_j)} f_j \in I$ with $\deg(g) \leq d-1$, by induction hypothesis $g \in M + J'$. Then $f = g + \sum\limits_{j=1}^{r} a_j x^{d-\deg(f_j)} f_j \in M + J'$, as required. Therefore, by induction $I = M + J'$. Since A is noetherian and $A + Ax + \cdots + Ax^{m-1}$ is a finitely generated A-module, that $A + Ax + \cdots + Ax^{m-1}$ is a noetherian A-module by Corollary 1.1.10. Since M is an A-submodule of $A + Ax + \cdots + Ax^{m-1}$, M is a finitely generated A-module, generated by, say $g_1, \ldots, g_n$. Then $I = M + J'$ is generated as an $A[x]$-module by $f_1, \ldots, f_r, g_1, \ldots, g_n$. This completes the proof. $\square$

By Hilbert basis theorem, every ideal of $A = k[x_1, \ldots, x_n]$ are finitely generated. Then every generating subset E of a finitely generated ideal $\mathfrak{a}$ of A contains a finite subset that generates the ideal $\mathfrak{a}$. Therefore, for every $E \subseteq A$, there exists finitely many elements $f_1, \ldots, f_n \in E$ such that $\mathcal{Z}(E) = \mathcal{Z}(f_1, \ldots, f_n) = \bigcap\limits_{j=1}^{n} \mathcal{Z}(f_j)$. Note that, given $E_1 \subseteq E_2 \subseteq A$, we have $\mathcal{Z}(E_2) \subseteq \mathcal{Z}(E_1)$.

Proposition 1.1.12. *The sets $\mathcal{Z}(\mathfrak{a})$, where $\mathfrak{a}$ runs over the set of all ideals of $k[x_1, \ldots, x_n]$ satisfy axioms for closed subsets for a topology on k^n, called the Zariski topology.*

Proof. The proposition follows from the following observations.

(i) $\mathcal{Z}(1) = \emptyset$ and $\mathcal{Z}(0) = k^n$;

(ii) Given any family of ideals $\{\mathfrak{a}_j : j \in I\}$ of $k[x_1, \ldots, x_n]$, we have

$$\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) = \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big);$$

(iii) given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $k[x_1, \ldots, x_n]$, we have

$$\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{Z}(\mathfrak{a}\mathfrak{b}).$$

The first point is obvious. To see the second, note that

$$\begin{aligned}
\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) &= \bigcap_{j \in I} \{x \in k^n : f(x) = 0, \ \forall\, f \in \mathfrak{a}_j\} \\
&= \{x \in k^n : f(x) = 0, \ \forall\, f \in \mathfrak{a}_j, \ \forall\, j \in I\} \\
&= \mathcal{Z}\big(\bigcup_{j \in I} \mathfrak{a}_j\big) \\
&= \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big).
\end{aligned}$$

To see the third point, note that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, which is a subset of both $\mathfrak{a}$ and $\mathfrak{b}$. Therefore, $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a}\mathfrak{b})$. Conversely, if $x \in \mathcal{Z}(\mathfrak{a}\mathfrak{b})$ and $x \notin \mathcal{Z}(\mathfrak{a})$, then there exists $f \in \mathfrak{a}$ such that $f(x) \neq 0$, and for any $g \in \mathfrak{b}$ that $f(x)g(x) = (fg)(x) = 0$, since $fg \in \mathfrak{ab}$ and $x \in \mathcal{Z}(\mathfrak{ab})$. Since k is an integral domain, we must have $g(x) = 0$, for all $g \in \mathfrak{b}$. Therefore, $x \in \mathcal{Z}(\mathfrak{b})$. This completes the proof. □

Definition 1.1.13. The set k^n together with the Zariski topology on it is called the *affine n-space over k* and is denoted by $\mathbb{A}^n(k)$. A closed subspace of $\mathbb{A}^n(k)$ is called an *algebraic set*.

Given a point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$, consider the evaluation map

$$ev_a : k[x_1, \ldots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a_1, \ldots, a_n), \ \ \forall\, f \in k[x_1, \ldots, x_n].$$

Note that ev_a a surjective ring homomorphism with kernel

$$\mathrm{Ker}(ev_a) = \mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n).$$

Therefore, $\{a\} = \mathcal{Z}(\mathfrak{m}_a)$ is a closed subset of $\mathbb{A}^n(k)$. As a result, any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.

Example 1.1.14. For $n = 1$, the polynomial ring $k[x]$ is a principal ideal domain. So every ideal of $k[x]$ is generated by a single polynomial. Since a polynomial in $k[x]$ has only finite number of roots in k, any closed subset of $\mathbb{A}^1(k)$ is either finite or $\mathbb{A}^1(k)$ itself.

Example 1.1.15. For $n = 2$, the situation is more complicated. Here is an obvious list of closed subsets of $\mathbb{A}^2(k)$.

- $\emptyset$ and $\mathbb{A}^2(k)$.
- any finite subset of $\mathbb{A}^2(k)$.
- $\mathcal{Z}(f)$, where $f \in k[x_1, x_2]$ is an irreducible polynomial.

In fact, we shall see later that the no-empty closed subsets of $\mathbb{A}^2(k)$ listed above are of the form $\mathcal{Z}(\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of $k[x_1, x_2]$. Moreover, any closed subsets of $\mathbb{A}^2(k)$ is a finite union of the closed subsets of the form listed above.

Connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz and its corollaries.

Theorem 1.1.16 (Hilbert's Nullstellensatz)**.** *Let K be a field that is not necessarily algebraically closed, and let A be a finitely generated K-algebra. Then A is Jacobson; i.e., every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is an intersection of all maximal ideals of A containing $\mathfrak{p}$.*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{p})} \mathfrak{m},$$

where $V_{\max}(\mathfrak{p})$ is the set of all maximal ideals of A containing $\mathfrak{p}$. Moreover, if $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite degree field extension of K.

Before proving Hilbert's Nullstellensatz, lets discuss some of its consequences.

Corollary 1.1.17. *Let k be an algebraically closed field, and let A be a finitely generated k-algebra.*

(i) Then $A/\mathfrak{m} = k$, for every maximal ideal $\mathfrak{m}$ of A.

(ii) Let $\mathfrak{m}$ be a maximal ideal of the polynomial ring $k[x_1, \ldots, x_n]$. Then there exists $(a_1, \ldots, a_n) \in \mathbb{A}^n(k)$ such that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

Proof. (i) Since the ring homomorphism $k \to A/\mathfrak{m}$ is a finite degree field extension of k by Hilbert's Nullstellensatz (Theorem 1.1.16), $A/\mathfrak{m}$ is an algebraic field extension of k. Since k is algebraically closed, we have $k \to A/\mathfrak{m}$ is an isomorphism of rings.

(ii) Let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is a finitely generated k-algebra and k is algebraically closed, by part (i) the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is isomorphic to the field k. Note that the quotient map

$$\varphi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/\mathfrak{m} = k$$

is a k-algebra homomorphism. For each $i \in \{1, \ldots, n\}$, let $a_i = \varphi(x_i) \in k$. Then $\varphi(x_j - a_j) = 0,\ \forall\, j \in \{1, \ldots, n\}$. Therefore, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is contained in the maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$. Since $(x_1 - a_1, \ldots, x_n - a_n)$ is also maximal ideal of $k[x_1, \ldots, x_n]$, it follows that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. $\square$

Corollary 1.1.18. *Let k be an algebraically closed field. Let $\mathfrak{a}$ be an ideal of $k[x_1, \ldots, x_n]$, and let $\mathcal{Z}(\mathfrak{a})$ be the algebraic subset of $\mathbb{A}^n(k)$ defined by $\mathfrak{a}$. Then there is a one-to-one correspondence between the points of $\mathcal{Z}(\mathfrak{a})$ and the set of all maximal ideals of $k[x_1, \ldots, x_n]/\mathfrak{a}$.*

Proof. Let A be the quotient ring $k[x_1, \ldots, x_n]/\mathfrak{a}$. By correspondence theorem, the maximal ideals of A are in one-to-one correspondence with the maximal ideals of $k[x_1, \ldots, x_n]$ containing $\mathfrak{a}$. Let $a = (a_1, \ldots, a_n) \in \mathcal{Z}(\mathfrak{a})$ be given. Since $\mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$ is the kernel of the surjective ring homomorphism

$$ev_a : k[x_1, \ldots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a),\ \ \forall\, f \in k[x_1, \ldots, x_n],$$

$\mathfrak{m}_a$ is a maximal ideal of $k[x_1, \ldots, x_n]$. Since $a \in \mathcal{Z}(\mathfrak{a})$, we have $ev_a(f) = f(a) = 0$, $\forall\, f \in \mathfrak{a}$. Therefore, $\mathfrak{a} \subseteq \mathrm{Ker}(ev_a) = \mathfrak{m}_a$. Let $\mathrm{MaxSpec}(A)$ be the set of all maximal ideals of $A = k[x_1, \ldots, x_n]/\mathfrak{a}$. Thus we get a map

$$\psi : \mathcal{Z}(\mathfrak{a}) \longrightarrow \mathrm{MaxSpec}(A)$$

defined by sending $a \in \mathcal{Z}(\mathfrak{a})$ to the maximal ideal $\overline{\mathfrak{m}_a}$ of A associated to $\mathfrak{m}_a$. Clearly ψ is injective by construction. To see ψ is surjective, note that a maximal ideal of A is given by a maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ containing $\mathfrak{a}$. Since k is algebraically closed, by Hilbert's Nullstellensatz (Corollary 1.1.17) we have $\mathfrak{m} = \mathfrak{m}_a$ for some $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$. Since $\mathfrak{a} \subseteq \mathfrak{m}_a$, given any $f \in \mathfrak{a}$, there exists $g_1, \ldots, g_n \in k[x_1, \ldots, x_n]$ such that

$f = \sum_{i=1}^{n} (x_i - a_i) g_i$. Then $f(a) = 0$. Therefore, $a \in \mathcal{Z}(\mathfrak{a})$, and hence ψ is surjective. This completes the proof. $\square$

We now give a proof of Theorem 1.1.16 (Hilbert's Nullstellensatz). The main ingredient is Noether's normalization lemma (for a proof, see e.g. Basic Commutative Algebra book by Balwant Singh).

Definition 1.1.19. Let $\phi : A \to B$ be a ring homomorphism. An element $\beta \in B$ is said to be *integral over* A if there exists a non-constant monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_n \in A[x]$ such that $\beta^n + \phi(a_1)\beta^{n-1} + \cdots + \phi(a_n) = 0$. If every element of B is integral over A, then B is called *integral* over A.

Lemma 1.1.20. *Let $\phi : A \to B$ be a ring homomorphism. Then B is a finitely generated A-algebra that is integral over A if and only if B is a finitely generated A-module.*

Definition 1.1.21. Let K be a field and let A be a finitely generated K-algebra. A finite subset $\{a_1, \ldots, a_n\}$ of A is said to be *algebraically dependent over* K if there exists a non-zero polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$. If $\{a_1, \ldots, a_n\} \subset A$ is said to be *algebraically independent over* K if it is not algebraically dependent over K.

Theorem 1.1.22 (Noether's Normalization lemma)**.** *Let K be a field (not necessarily algebraically closed), and let $A \neq 0$ be a finitely generated K-algebra. Then there exists an integer $n \geq 0$ and $t_1, \ldots, t_n \in A$ such that the K-algebra homomorphism*

$$K[x_1, \ldots, x_n] \longrightarrow A, \quad x_j \mapsto t_j, \ \forall\, j,$$

is injective, and A is a finitely generated $K[x_1, \ldots, x_n]$-algebra that is integral over $K[x_1, \ldots, x_n]$.

To deduce Hilbert's Nullstellensatz from Noether's normalization lemma, we need the following two lemmas.

Lemma 1.1.23. *Let A and B be integral domains and let $A \to B$ be an injective ring homomorphism. If B is integral over A, then A is a field if and only if B is a field.*

Proof. Suppose that A is a field. Let $b \in B \setminus \{0\}$. Since b is integral over A, $A[b]$ is a finite dimensional A-vector space. Since B is an integral domain, the multiplication by b map

$$\mu_b : A[b] \to A[b], \quad f(b) \mapsto bf(b)$$

is injective. Clearly μ_b is A-linear, and hence is bijective. Then $bf(b) = 1$, for some $f(b) \in A[b]$, and hence b is a unit in $A[b]$. Thus B is a field.

Conversely, suppose that B is a field. Let $a \in A \setminus \{0\}$. Let $b = a^{-1}$ in B. Since B is integral over A, there exists $a_1, \ldots, a_n \in A$ such that $b^n + a_1 b^{n-1} + \cdots + a_n = 0$. Multiplying both sides by $a^{n-1} \neq 0$ and using the fact that $ab = 1$, we see that

$$b = -(a_1 + a_2 b + \cdots + a_n a^{-1}) \in A.$$

Therefore, A is a field. $\square$

Lemma 1.1.24. *Let K and L be a fields such that L is a finitely generated K-algebra. Then L is a finite degree field extension of K.*

Proof. Since L is a finitely generated K-algebra, by Noether's normalization lemma (Theorem 1.1.22) there exists an injective K-algebra homomorphism $\varphi : K[x_1, \ldots, x_n] \to L$ that makes L integral over $K[x_1, \ldots, x_n]$, for some integer $n \geq 0$. Since L is a field, by Lemma 1.1.23 we conclude that $n = 0$. Then L is a finitely generated K-algebra that is integral over K, and hence L is a finitely generated K-vector space. Therefore, L is a finite degree field extension of K. □

Proof of Hilbert's Nullstellensatz (Theorem 1.1.16). Let A be a finitely generated K-algebra. To see the second part, let $\mathfrak{m}$ be a maximal ideal of A. Since the composite map

$$K \longrightarrow A \xrightarrow{\pi} A/\mathfrak{m}$$

is a non-zero K-algebra homomorphism, $A/\mathfrak{m}$ is a field extension of K. Since $A/\mathfrak{m}$ is a finitely generated K-algebra, it follows from Lemma 1.1.24 that $A/\mathfrak{m}$ is a finite degree field extension of K.

For the first part, we begin with the following remark. If L is a finite degree field extension of K and $\varphi : A \to L$ is a K-algebra homomorphism, the image of φ is an integral domain that is finitely generated K-vector space. Then $\varphi(A)$ is a field by Lemma 1.1.24. Therefore, $\mathrm{Ker}(\varphi)$ is a maximal ideal of A.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Replacing A by $A/\mathfrak{p}$, if required, it suffices to show that if A is an integral domain that is a finitely generated K-algebra then intersection of all maximal ideals of A is the zero ideal. For this we show that, given any $\alpha \in A \setminus \{0\}$ there is a maximal ideal $\mathfrak{m}$ of A such that $\alpha \notin \mathfrak{m}$. Note that, for $\alpha \neq 0$ in A, the ring $A[\alpha^{-1}] \subseteq Q(A)$ is a non-zero finitely generated K-algebra. Let $\mathfrak{n}$ be a maximal ideal of $A[\alpha^{-1}]$. Then $L := A[\alpha^{-1}]/\mathfrak{n}$ is a finite degree field extension of K by the second assertion. Then the kernel of the composite map

$$\varphi : A \to A[\alpha^{-1}] \to L$$

is a maximal ideal, say $\mathfrak{m}$, of A by above remark. Clearly $\alpha \notin \mathfrak{m}$. □

Let A be a commutative ring with identity. Given an ideal $\mathfrak{a}$ of A, the subset

$$\mathrm{rad}(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a}, \text{ for some } n \in \mathbb{N}\}$$

forms an ideal of A, called the *radical of* $\mathfrak{a}$ *in* A. Note that, $\mathfrak{a} \subseteq \mathrm{rad}(\mathfrak{a})$. If $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$, then we call $\mathfrak{a}$ a *radical ideal* of A. For example, prime ideals are radical ideals.

Exercise 1.1.25. Let A be a commutative ring with identity. Given an ideal $\mathfrak{a}$ of A, let $V(\mathfrak{a})$ be the set of all prime ideals of A containing $\mathfrak{a}$. Show that $\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

Exercise 1.1.26. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients for a field k. Let $\mathfrak{a}$ be an ideal of A. Use Hilbert's Nullstellensatz to show that

$$\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{a})} \mathfrak{m},$$

where $V_{\max}(\mathfrak{a})$ is the set of all maximal ideals of A containing $\mathfrak{a}$.

Exercise 1.1.27. Given an ideal $\mathfrak{a}$ of the polynomial ring $k[x_1, \ldots, x_n]$, show that $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathrm{rad}(\mathfrak{a}))$.

Definition 1.1.28. Given an algebraic subset Z of $\mathbb{A}^n(k)$, the subset

$$\mathcal{I}(Z) := \{f \in k[x_1, \ldots, x_n] : f(a) = 0, \forall\, a \in Z\}$$

is an ideal of $k[x_1, \ldots, x_n]$, called the *ideal of polynomials vanishing on Z*.

Let Z be an algebraic subset of $\mathbb{A}^n(k)$. Since for given any $f \in k[x_1, \ldots, x_n]$ and $a \in \mathbb{A}^n(k)$, we have $f(a) = 0$ if and only if $f \in \mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$, it follows that $\mathcal{I}(Z) = \bigcap\limits_{a \in Z} \mathfrak{m}_a$.

Proposition 1.1.29. *(i) Let $\mathfrak{a}$ be an ideal of $k[x_1, \ldots, x_n]$. Then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.*

(ii) Let $Z \subseteq \mathbb{A}^n(k)$ be a subset (not necessarily Zariski closed), and let $\overline{Z}$ be its closure in $\mathbb{A}^n(k)$. Then $\mathcal{Z}(\mathcal{I}(Z)) = \overline{Z}$.

Proof. (i) Since k is algebraically closed and $k[x_1, \ldots, x_n]$ is a finitely generated k-algebra, it follows from Hilbert's Nullstellensatz (Corollary 1.1.17) that every prime ideal $\mathfrak{p}$ of $A := k[x_1, \ldots, x_n]$ is the intersection of all maximal ideals of A containing $\mathfrak{p}$. Since $\mathrm{rad}(\mathfrak{a}) = \bigcap\limits_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$, it follows from Corollary 1.1.17 that $\mathrm{rad}(\mathfrak{a}) = \bigcap\limits_{\mathfrak{m} \in V_{\max}(\mathfrak{a})} \mathfrak{m}$, where $V_{\max}(\mathfrak{a})$ is the set of all maximal ideals of A containing $\mathfrak{a}$.

Let $a \in Z$ be given. Then for any $f \in \mathcal{I}(Z)$ we have $f(a) = 0$. and so $a \in \mathcal{Z}(\mathcal{I}(Z))$. Thus $Z \subseteq \mathcal{Z}(\mathcal{I}(Z))$. Since $\mathcal{Z}(\mathcal{I}(Z))$ is closed in $\mathbb{A}^n(k)$, we have $\overline{Z} \subseteq \mathcal{Z}(\mathcal{I}(Z))$. To show the converse, it suffices to show that any closed subset of $\mathbb{A}^n(k)$ containing Z contains $\mathcal{Z}(\mathcal{I}(Z))$. Let W be any closed subset of $\mathbb{A}^n(k)$ containing Z. Then $W = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. Then $\mathrm{rad}(\mathfrak{a}) = \mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subseteq \mathcal{I}(Z)$. Then $\mathcal{Z}(\mathcal{I}(Z)) \subseteq \mathcal{Z}(\mathrm{rad}(\mathfrak{a})) = \mathcal{Z}(\mathfrak{a}) = W$. Therefore, $\overline{Z} = \mathcal{Z}(\mathcal{I}(Z))$. $\square$

Definition 1.1.30. Given an affine algebraic subset $X \subseteq \mathbb{A}^n(k)$ the quotient ring

$$k[X] := k[x_1, \ldots, x_n]/\mathcal{I}(X)$$

is called the *affine coordinate ring* of X.

Exercise 1.1.31. Let A be a commutative ring with identity. Let $\mathfrak{a}$ be an ideal of A. Show that, $\mathfrak{a} = \mathrm{rad}(\mathfrak{a})$ if and only if $\mathrm{Nil}(A/I) = 0$.

Lemma 1.1.32. *Let $\mathfrak{a}$ be an ideal of a commutative ring A with identity, and let $A/\mathfrak{a}$ be the associated quotient ring. Then there is a one-to-one correspondence between the set of all radical ideals of A containing $\mathfrak{a}$ and the set of all radical ideals of $A/\mathfrak{a}$.*

Proof. Let $\pi : A \to A/\mathfrak{a}$ be the natural surjective ring homomorphism. Given an ideal I of A containing $\mathfrak{a}$, its image $I/\mathfrak{a}$ is an ideal of $A/\mathfrak{a}$. This gives a one-to-one correspondence between the set of all ideals of A containing $\mathfrak{a}$ and the set of all ideals of $A/\mathfrak{a}$. By third isomorphism theorem we have an isomorphism of quotient rings $(A/\mathfrak{a})/(I/\mathfrak{a}) \cong A/I$. Then $\mathrm{Nil}\left((A/\mathfrak{a})/(I/\mathfrak{a})\right) \cong \mathrm{Nil}(A/I)$. Therefore, I is a radical ideal of A if and only if $I/\mathfrak{a}$ is a radical ideal of $A/\mathfrak{a}$. Hence the result follows. $\square$

Corollary 1.1.33. *Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic subset with affine coordinate ring $k[X]$. Equip X with the subspace topology induced from the Zariski topology on $\mathbb{A}^n(k)$. Then there is a one-to-one correspondence between the set of all closed subsets of X and the set of all radical ideals of $k[X]$.*

Proof. Let Z be a closed subset of X. Since X is closed in $\mathbb{A}^n(k)$, so is Z. Then $Z = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. Since $\mathcal{Z}(\operatorname{rad}(\mathfrak{a})) = \mathcal{Z}(\mathfrak{a})$, we may assume that $\mathfrak{a} = \operatorname{rad}(\mathfrak{a})$. Since $Z \subseteq X$ if and only if $\mathcal{I}(X) \subseteq \mathcal{I}(Z) = \operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$, using Lemma 1.1.32 we have an one-to-one correspondence between the set of all closed subsets of X and the set of all radical ideals of $k[X]$. $\square$

Definition 1.1.34. Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be two affine algebraic subsets. A *morphism* from X into Y is a map $\varphi : X \to Y$ such that there exists polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ such that $\varphi(a) = (f_1(a), \ldots, f_n(a))$, $\forall\, a \in X$. The set of all morphisms from X into Y is denoted by $\operatorname{Hom}(X, Y)$.

Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic subsets. Let $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ be two ordered n-tuples of polynomials in $k[x_1, \ldots, x_m]$ defining morphisms φ and ψ from X into Y. Then $\varphi = \psi$ if and only if for all $j \in \{1, \ldots, n\}$ we have

$$\begin{aligned} &(f_j - g_j)(a) = 0, \ \ \forall\, a \in X \\ \Leftrightarrow\ & f_j - g_j \in \mathcal{I}(X) \\ \Leftrightarrow\ & \overline{f_j} = \overline{g_j} \text{ in } k[X]. \end{aligned}$$

Proposition 1.1.35. *Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be two affine algebraic subsets. Then there is a one-to-one correspondence between* $\operatorname{Hom}(X, Y)$ *and the set of all k-algebra homomorphisms from $k[Y]$ into $k[X]$.*

Proof. Let $\varphi = (f_1, \ldots, f_n) \in \operatorname{Hom}(X, Y)$ be given. Define a k-algebra homomorphism

$$\widetilde{\varphi} : k[y_1, \ldots, y_n] \to k[X] = k[x_1, \ldots, x_m]/\mathcal{I}(X)$$

by sending y_j to the image of f_j in the coordinate ring $k[X]$ under the quotient map $k[x_1, \ldots, x_m] \to k[X]$, for all $j = 1, \ldots, n$. Note that $g \in \operatorname{Ker}(\widetilde{\varphi})$ if and only if $g(f_1, \ldots, f_n) \in \mathcal{I}(X)$ if and only if $g(\varphi(a)) = g(f_1(a), \ldots, f_n(a)) = 0$, $\forall\, a \in X$ if and only if $g \in \mathcal{I}(\varphi(X))$. In particular, $\operatorname{Ker}(\widetilde{\varphi}) = \mathcal{I}(Y)$ if and only if φ is surjective. Since $\varphi(X) \subseteq Y$, we have $\mathcal{I}(Y) \subseteq \mathcal{I}(\varphi(X))$. Therefore, $\mathcal{I}(Y) \subseteq \operatorname{Ker}(\widetilde{\varphi})$. Then there is a unique k-algebra homomorphism

$$\varphi^{\#} : k[Y] \to k[X]$$

such that the following diagram commutes.

$$\begin{tikzcd} k[y_1, \ldots, y_n] \arrow[r, "\widetilde{\varphi}"] \arrow[d] & k[X] \\ k[Y] \arrow[ur, "\varphi^{\#}"'] & \end{tikzcd}$$

Conversely, given any k-algebra homomorphism

$$f : k[y_1, \dots, y_n]/\mathcal{I}(Y) \to k[x_1, \dots, x_m]/\mathcal{I}(X),$$

fix a polynomial $f_j \in k[x_1, \dots, x_m]$ whose image in the coordinate ring $k[X]$ is $f(\overline{y_j})$, where $\overline{y_j}$ is the image of y_j in $k[Y]$, for all $j = 1, \dots, n$. Then f defines a morphism of affine algebraic sets

$$\varphi : X \to Y, \;\; a \mapsto (f_1(a), \dots, f_n(a))$$

such that $\varphi^{\#} = f$. This completes the proof. □

Proposition 1.1.36. *Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic subsets, and let $\varphi : X \to Y$ be a morphism of affine algebraic sets. Then φ is continuous for the subspace Zariski topologies on X and Y.*

Proof. It suffices to assume that $Y = \mathbb{A}^n(k)$ and show that $\varphi^{-1}(Z)$ is closed in X, for every closed subset Z of $\mathbb{A}^n(k)$. Let $f_1, \dots, f_n \in k[x_1, \dots, x_m]$ be such that $\varphi = (f_1, \dots, f_n)$. Let Z be a closed subset of $\mathbb{A}^n(k)$. Then $Z = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[y_1, \dots, y_n]$. Let

$$\varphi^{\#} : k[y_1, \dots, y_m] \to k[X], \;\; g \mapsto g(f_1, \dots, f_n),$$

be the k-algebra homomorphism induced by φ. Let $\mathfrak{b} = \langle \varphi^{\#}(\mathfrak{a}) \rangle$ be the ideal of $k[X]$ generated by the image of $\mathfrak{a}$ under $\varphi^{\#}$. Then it is easy to check (verify!) that

$$\varphi^{-1}(\mathcal{Z}(\mathfrak{a})) = \{a \in X : (f_1(a), \dots, f_n(a)) \in \mathcal{Z}(\mathfrak{a})\} = \mathcal{Z}(\mathfrak{b}).$$

This completes the proof. □

Let X be an affine algebraic subset of $\mathbb{A}^n(k)$. Given an element $f \in k[x_1, \dots, x_n]$, let

$$D(f) = X \setminus \mathcal{Z}(f) = \{a \in X : f(a) \neq 0\}.$$

Then $D(f)$ is an open subset of X, called a *principal open subset* of X.

Proposition 1.1.37. *Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic subset. The collection of all principal open subsets of X forms a basis for the subspace topology on X induced from $\mathbb{A}^n(k)$.*

Proof. Note that, $D(f) \cap D(g) = D(fg)$, for all $f, g \in k[x_1, \dots, x_n]$. Let $\mathfrak{a} = \mathcal{I}(X)$. Since X is an affine algebraic subset of $\mathbb{A}^n(k)$, $\mathfrak{a}$ is a radical ideal of $k[x_1, \dots, x_n]$ and $X = \mathcal{Z}(\mathfrak{a})$. Let U be an open subset of X. Since X is closed in $\mathbb{A}^n(k)$, so is $Z := X \setminus U$. Then it follows from Corollary 1.1.33 that $Z = \mathcal{Z}(\mathfrak{b})$, for some radical ideal $\mathfrak{b}$ of $k[x_1, \dots, x_n]$ containing $\mathfrak{a}$. Since $k[x_1, \dots, x_n]$ is a noetherian ring, $\mathfrak{b} = (f_1, \dots, f_r)$, for some $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. Then $Z = \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(f_1, \dots, f_r) = \bigcap\limits_{j=1}^{r} \mathcal{Z}(f_j)$, and hence $U = X \setminus Z = \bigcup\limits_{j=1}^{r} D(f_j)$. This completes the proof. □

Proposition 1.1.38. *Let X be an affine algebraic subset of $\mathbb{A}^n(k)$. Then there is a k-algebra isomorphism $\Phi : \mathrm{Hom}(X, \mathbb{A}^1(k)) \to k[X]$.*

Proof. Note that each element $\varphi \in \operatorname{Hom}(X, \mathbb{A}^1(k))$ is given by a polynomial $f \in k[x_1, \ldots, x_n]$. Two polynomials $f, g \in k[x_1, \ldots, x_n]$ give rise to the same morphism $\varphi : X \to \mathbb{A}^1(k)$ if and only if $f - g \in \mathcal{I}(X)$. Therefore, we have a well-defined injective map

$$\Phi : \operatorname{Hom}(X, \mathbb{A}^1(k)) \to k[X] = k[x_1, \ldots, x_n]/\mathcal{I}(X)$$

defined by sending $\varphi = f$ to $\overline{f} \in k[X]$. This map is clearly surjective. Given $f, g \in \operatorname{Hom}(X, \mathbb{A}^1(k))$, we can define their addition and multiplication and k-multiplication in obvious way to get a natural k-algebra structure on $\operatorname{Hom}(X, \mathbb{A}^1(k))$. It follows that Φ is a k-algebra isomorphism. $\square$

1.2 Schemes

Let A be a commutative ring with identity. Let $\operatorname{Spec}(A)$ be the set of all prime ideals of A. Given a subset $E \subseteq A$, let

$$V(E) := \{\mathfrak{p} \in \operatorname{Spec}(A) : E \subseteq \mathfrak{p}\}.$$

Note that, $V(E) = V(\langle E \rangle)$, where $\langle E \rangle$ is the ideal of A generated by E. Moreover, if $E_1 \subseteq E_2 \subseteq A$, then $V(E_2) \subseteq V(E_1)$.

Proposition 1.2.1. *With the above notations the following holds.*

(i) $V(0) = \operatorname{Spec}(A)$ *and* $V(1) = \emptyset$;

(ii) Given a collection $\{\mathfrak{a}_j : j \in I\}$ *of ideals of* A, *we have* $\bigcap\limits_{j \in I} V(\mathfrak{a}_j) = V(\sum\limits_{j \in I} \mathfrak{a}_j)$; *and*

(iii) Given any two ideals $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* A, *we have* $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

Consequently, the collection $\tau_c := \{V(\mathfrak{a}) : \mathfrak{a}$ *is an ideal of* $A\}$, *satisfies axioms for closed subsets of a topology on* $\operatorname{Spec}(A)$, *called the Zariski topology on* $\operatorname{Spec}(A)$. *The set* $\operatorname{Spec}(A)$ *together with the Zariski topology on it is called an affine scheme.*

Proof. (i) is obvious. To see (ii), note that, $\mathfrak{a}_j \subseteq \sum\limits_{i \in I} \mathfrak{a}_i$ gives $V(\sum\limits_{i \in I} \mathfrak{a}_i) \subseteq V(\mathfrak{a}_j)$, $\forall\, j \in I$, and hence $V(\sum\limits_{i \in I} \mathfrak{a}_i) \subseteq \bigcap\limits_{j \in I} V(\mathfrak{a}_j)$. Conversely, if $\mathfrak{p} \in \bigcap\limits_{j \in I} V(\mathfrak{a}_j)$, then $\mathfrak{a}_j \subseteq \mathfrak{p}$, $\forall\, j \in I$, and so $\sum\limits_{j \in I} \mathfrak{a}_j \subseteq \mathfrak{p}$. Therefore, $\mathfrak{p} \in V(\sum\limits_{i \in I} \mathfrak{a}_i)$. This proves (ii). To see (iii), note that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$ gives $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$. Conversely, suppose that $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$. If $\mathfrak{p} \notin V(\mathfrak{a})$, then there exists $f \in \mathfrak{a}$ such that $f \notin \mathfrak{p}$. Let $g \in \mathfrak{b}$ be arbitrary. Then $fg \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal of A and $f \notin \mathfrak{p}$, we have $g \in \mathfrak{p}$. Therefore, $\mathfrak{b} \subseteq \mathfrak{p}$, and hence $\mathfrak{p} \in V(\mathfrak{b})$. This proves (iii). $\square$

Exercise 1.2.2. Show that $V(\mathfrak{a}) = V(\operatorname{rad}(\mathfrak{a}))$, for any ideal $\mathfrak{a}$ of A.

Example 1.2.3. Let R be a commutative ring with identity, and let $R[x_1, \ldots, x_n]$ be the polynomial ring in n-variables $x_1, \ldots, x_n$ over the ring R. The set $\operatorname{Spec}(R[x_1, \ldots, x_n])$ together with the Zariski topology on it is called the *affine n-space over R*, and is denoted by $\mathbb{A}^n_R$. When the base ring R is a field k, then the set of all closed points of $\mathbb{A}^n_k$ is precisely the affine n-space $\mathbb{A}^n(k)$ defined in the last section.

Exercise 1.2.4. Given a point $\mathfrak{p} \in \mathrm{Spec}(A)$, show that $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}}$ if and only if $\mathfrak{p}$ is a maximal ideal of A.

Given a closed subset Z of an affine scheme $X = \mathrm{Spec}(A)$, consider the subset

$$\mathcal{I}(Z) := \{f \in A : f \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z\}.$$

Clearly $0 \in \mathcal{I}(Z)$. Let $f, g \in \mathcal{I}(Z)$ and $h \in A$ be arbitrary. Then $f, g \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z$ implies that $f + gh \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z$. Therefore, $f + gh \in \mathcal{I}(Z)$. Therefore, $\mathcal{I}(Z)$ is an ideal of A. By definition of $\mathcal{I}(Z)$, we have

$$\mathcal{I}(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Proposition 1.2.5. *(i) Let $\mathfrak{a}$ be an ideal of A. Then $\mathcal{I}(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.*

(ii) Given any subset Z of $X = \mathrm{Spec}(A)$, we have $V(\mathcal{I}(Z)) = \overline{Z}$.

Proof. (i) Note that, $\mathfrak{p} \in V(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$. Therefore, we have

$$\mathcal{I}(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p} = \mathrm{rad}(\mathfrak{a}).$$

(ii) Since $\mathcal{I}(Z) = \bigcap\limits_{\mathfrak{q} \in Z} \mathfrak{q}$, we have $\mathcal{I}(Z) \subseteq \mathfrak{p},\ \forall\, \mathfrak{p} \in Z$. Therefore, $\mathfrak{p} \in V(\mathcal{I}(Z)),\ \forall\, \mathfrak{p} \in Z$, and hence $Z \subseteq V(\mathcal{I}(Z))$. Since $V(\mathcal{I}(Z))$ is closed in $X = \mathrm{Spec}(A)$, we have $\overline{Z} \subseteq V(\mathcal{I}(Z))$. To show the reverse inclusion, it suffices to show that $V(\mathcal{I}(Z)) \subseteq V$, for any closed subset V of X containing Z. Let $\mathfrak{a}$ be an ideal of A be such that $Z \subseteq V(\mathfrak{a})$. Then for any $\mathfrak{p} \in Z$ we have $\mathfrak{a} \subseteq \mathfrak{p}$. Therefore, $\mathfrak{a} \subseteq \bigcap\limits_{\mathfrak{p} \in Z} \mathfrak{p} = \mathcal{I}(Z)$, and hence $V(\mathcal{I}(Z)) \subseteq V(\mathfrak{a})$, as required. Thus, $V(\mathcal{I}(Z)) = \overline{Z}$. □

Definition 1.2.6. Let X be a non-empty topological space. Then X is said to be *reducible* if there exists two non-empty proper closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$. If X is not reducible, then we call it *irreducible*.

Example 1.2.7. The affine line $\mathbb{A}^1(k)$ over a field k is irreducible if and only if k is infinite.

Proposition 1.2.8. *Let A be a non-zero commutative ring with identity, and let Z be a closed subset of* $\mathrm{Spec}(A)$. *Then Z is irreducible if and only if $\mathcal{I}(Z)$ is a prime ideal of A. In particular,* $\mathrm{Spec}(A)$ *is irreducible if and only if the nil radical* $\mathrm{Nil}(A)$ *of A is prime.*

Proof. Let $Z = V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of A. Then $\mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$. Suppose that Z is irreducible. Let $f, g \in A$ be such that $fg \in \mathrm{rad}(\mathfrak{a})$ and $f \notin \mathrm{rad}(\mathfrak{a})$. Then

$$Z = V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a})) \subseteq V(fg) = V(f) \cup V(g).$$

Since Z is irreducible, either $Z \subseteq V(f)$ or $Z \subseteq V(g)$. If $Z \subseteq V(f)$, then $f \in \mathrm{rad}(f) = \mathcal{I}(V(f)) \subseteq \mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$. But $f \notin \mathrm{rad}(\mathfrak{a})$. Then we must have $Z \subseteq V(g)$, which gives $g \in \mathrm{rad}(\mathfrak{a})$. Therefore, $\mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$ is a prime ideal of A. Conversely, suppose that $\mathcal{I}(Z)$ is a prime ideal of A. Suppose that $Z \subseteq Y_1 \cup Y_2$, for some non-empty proper closed subsets Y_1 and Y_2 of $\mathrm{Spec}(A)$. Let $Y_1 = V(\mathfrak{a})$ and $Y_2 = V(\mathfrak{b})$, for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A. Since $Y_1 \cup Y_2 = V(\mathfrak{ab})$, we have $\mathfrak{ab} \subseteq \mathcal{I}(Z)$. Since $\mathcal{I}(Z)$ is prime, either $\mathfrak{a} \subseteq \mathcal{I}(Z)$ or $\mathfrak{b} \subseteq \mathcal{I}(Z)$. Since $Z = \overline{Z}$, the above two conditions gives either $Z \subseteq V(\mathfrak{a}) = Y_1$ or $Z \subseteq V(\mathfrak{b}) = Y_2$. Therefore, Z is irreducible.

Since $\mathrm{Spec}(A) = V(0)$, it follows that $\mathrm{Spec}(A)$ is irreducible if and only if $\mathrm{rad}(0) = \mathrm{Nil}(A)$ is a prime ideal of A. Note that, this happens if and only if A has a unique minimal prime ideal. $\square$

1.3 Appendix: Category Theory

Joke: Category theory is like Ramayana and Mahabharata — there are lots of arrows!

— *Nitin Nitsure*

Definition 1.3.1. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)
$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \ \ (f, g) \mapsto g \circ f,$$
such that the following conditions hold.

   (a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

   (b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

A category $\mathscr{A}$ is said to be *locally small* if $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ is a set, for all $X, Y \in \mathrm{ob}(\mathscr{A})$. A category $\mathscr{A}$ is said to be *small* if it is locally small and the class of objects $\mathrm{ob}(\mathscr{A})$ is a set.

Example 1.3.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{A}$ such that $g \circ f = \mathrm{Id}_{A_1}$ and $f \circ g = \mathrm{Id}_{A_2}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,

(ii) for $X, Y \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, there is $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, which are compatible with the composition maps.

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$$

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ is given by the following data: for each object $A \in \mathscr{A}$, a map $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes.

(1.3.3)
$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_A"'] & \mathcal{F}(A') \arrow[d, "\varphi_{A'}"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A')
\end{tikzcd}$$

Definition 1.3.4. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Mor}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathrm{Func}(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 1.3.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 1.3.6. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Mor}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Notation. For any pre-additive category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ the abelian group $\mathrm{Mor}_{\mathscr{A}}(X, Y)$, for all $X, Y \in \mathrm{ob}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.3.7 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A,B) \times \mathrm{Hom}_{\mathscr{A}}(B,C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A,C)$, written as $(f,g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.

(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0,0)$ is the trivial group with one element.

(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $\mathrm{j}_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 1.3.8. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A,B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A,B) \times \mathrm{Hom}_{\mathscr{A}}(B,C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A,C)\,, \;\; (f,g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 1.3.9. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1,A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 1.3.10. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
& C \arrow[dl, dashed, "\exists ! \widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ is defined by reversing the arrows of the above diagram.

Definition 1.3.11. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists ! \widetilde{g}"] \\ & C & \end{tikzcd}$$

Definition 1.3.12. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 1.3.13. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.

(ii) If a cokernel of f exists, then it is an epimorphism.

(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.

(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). □

Exercise 1.3.14. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 1.3.15. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$ (see Lemma 1.3.13 (*iii*)). This completes the proof.

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\ & & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & & \end{tikzcd} \tag{1.3.16}$$

□

Definition 1.3.17. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 1.3.12).

Example 1.3.18.

1. For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.

2. Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules on X. Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{QCoh}(X)$ (reps., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of $\mathcal{O}_X$-modules on X, are also abelian. However, the full subcategory $\mathcal{V}ect(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of $\mathcal{O}_X$-modules on X, is not abelian, because kernel of a morphism in $\mathcal{V}ect(X)$ may not be in $\mathcal{V}ect(X)$.