

FUNDAMENTAL GROUP SCHEMES OF n -FOLD SYMMETRIC PRODUCT OF A SMOOTH PROJECTIVE CURVE

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ABSTRACT. Let k be an algebraically closed field of characteristic $p > 0$. Let X be an irreducible smooth projective curve of genus g over k . Fix an integer $n \geq 2$, and let $S^n(X)$ be the n -fold symmetric product of X . In this article we find the S -fundamental group scheme and Nori's fundamental group scheme of $S^n(X)$.

1. INTRODUCTION

For a connected reduced complete scheme X defined over a perfect field k and having a k -rational point x , in [Nor76, Nor82], Nori introduced an affine k -group scheme $\pi^N(X, x)$ associated to the neutral Tannakian category of essentially finite vector bundles on X , known as *Nori's fundamental group scheme*. This group scheme carries more informations than the étale fundamental group scheme $\pi^{\text{ét}}(X, x)$ in positive characteristic, and is the same as $\pi^{\text{ét}}(X, x)$ when $k = \mathbb{C}$. For a connected smooth projective curve defined over an algebraically closed field k , in [BPS06], Biswas, Parameswaran and Subramanian defined and studied the S -fundamental group scheme $\pi^S(X, x)$ of X . This is further generalized and extensively studied for higher dimensional smooth projective varieties over algebraically closed fields by Langer in [Lan11, Lan12]. It is an interesting question to find $\pi^{\text{ét}}(X, x)$, $\pi^N(X, x)$ and $\pi^S(X, x)$ for well-known algebraic varieties.

Let X be a connected smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$. Fix an integer $n \geq 2$, and let S_n be the permutation group of n symbols. Then S_n acts on X^n by permutation of its factors, and the associated quotient $S^n(X) := X^n/S_n$ is a connected smooth projective variety over k . For any affine k -group scheme G we denote by G_{ab} its abelianization. In this article we prove the following results.

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Theorem 1 (Theorem 3.5.1). *For any closed point $x \in X(k)$, there is an isomorphism of affine k -group schemes*

$$\widetilde{\psi}_*^S : \pi^S(X, x)_{\text{ab}} \longrightarrow \pi^S(S^n(X), nx).$$

Theorem 2 (Theorem 3.5.3). *For any closed point $x \in X(k)$, there is an isomorphism of affine k -group schemes*

$$\widetilde{\psi}_*^N : \pi^N(X, x)_{\text{ab}} \longrightarrow \pi^N(S^n(X), nx).$$

As a consequence we also obtain the following result, which is already contained in [BH15], and proved using a different method. For any closed point $x \in X(k)$, there is an isomorphism of affine k -group schemes

$$\widetilde{\psi}_*^{\text{ét}} : \pi^{\text{ét}}(X, x)_{\text{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx).$$

Note that when $n > 2g - 2$, where g is the genus of X , these isomorphisms can be easily obtained from results in [Lan12, Section 7], since $S^n(X)$ is a projective bundle over $\text{Alb}(X)$. We prove the above results without any restriction on n . Our initial strategy was to use the same method as in [PS19] under the assumption that $\text{char}(k) > 3$. However, we observed that one can avoid using the characterization of numerically flat sheaves as strongly semistable reflexive sheaves with vanishing Chern classes; proved in [Lan11]. Instead, we first show that $\widetilde{\psi}_*^S$ is faithfully flat and then use [Lan12, Section 7] to conclude that it is an isomorphism.

2. FUNDAMENTAL GROUP SCHEMES

Let k be an algebraically closed field. Let X be a reduced proper k -scheme, which is connected in the sense that $H^0(X, \mathcal{O}_X) \cong k$.

2.1. S -fundamental group scheme. Let $\text{Coh}(X)$ be the category of coherent sheaf of \mathcal{O}_X -modules on X . This has a full subcategory $\text{Vect}(X)$, whose objects are locally free coherent sheaves (vector bundles) on X . A vector bundle E on X is said to be *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$. An object E of $\text{Coh}(X)$ is said to be *numerically flat* if E is locally free and both E and its dual E^\vee are nef. Let $\mathcal{E}^{\text{nf}}(X)$ be the full subcategory of $\text{Coh}(X)$, whose objects are numerically flat vector bundles on X . It is known that, $E \in \text{Ob}(\text{Coh}(X))$ is an object of $\mathcal{E}^{\text{nf}}(X)$ if and only if E is locally free and for any smooth projective curve C over k and any morphism $f : C \longrightarrow X$, its pullback f^*E on C is slope semistable and of degree 0 (see [Lan11, Remark 5.2]). Note that $\mathcal{E}^{\text{nf}}(X)$ is closed under finite direct sum and tensor products. Choosing a closed point $x \in X(k)$, one can define a fiber functor

$$T_x : \mathcal{E}^{\text{nf}}(X) \longrightarrow \text{Vect}_k$$

by sending an object E of $\mathcal{C}^{\text{nf}}(X)$ to its fiber E_x at x . The quadruple $(\mathcal{C}^{\text{nf}}(X), \otimes, \mathcal{O}_X, T_x)$ is a neutral Tannakian category (see [Lan11, Proposition 5.5]), and the affine k -group scheme $\pi^S(X, x)$ Tannaka dual to this is known as the *S-fundamental group scheme* of X with base point x .

Let X be a connected smooth projective variety of dimension d over k . Fix an ample divisor H on X . Let $\text{Vect}_0^s(X)$ be the full subcategory of $\text{Coh}(X)$, whose objects are reflexive coherent sheaves E on X , that are strongly H -semistable and $\text{ch}_1(E) \cdot H^{d-1} = \text{ch}_2(E) \cdot H^{d-2} = 0$, where $\text{ch}_i(E)$ is the i -th Chern character of E . It is shown in [Lan11, Proposition 5.1] that the objects of the category $\text{Vect}_0^s(X)$ are in fact locally free coherent sheaves on X and all of their Chern classes vanishes. It follows from [Lan11, Proposition 4.5] that the category $\text{Vect}_0^s(X)$ does not depend on choice of H . For X smooth, the categories $\mathcal{C}^{\text{nf}}(X)$ and $\text{Vect}_0^s(X)$ are the same (see [Lan11, Proposition 5.1], [Lan12, Theorem 2.2]). We will not use this characterization here, however, this was crucial in [PS19].

It is clear from the definition of the categories $\text{Vect}_0^s(X)$ and $\text{EF}(X)$ that $\pi^S(X, x)$ carries more informations than $\pi^N(X, x)$. In fact, there are natural faithfully flat homomorphisms of affine k -group schemes $\pi^S(X, x) \longrightarrow \pi^N(X, x) \longrightarrow \pi^{\text{ét}}(X, x)$, (see [Lan11, Lemma 6.2]).

2.2. Nori's fundamental group scheme.

Definition 2.2.1. *A vector bundle E on X is said to be finite if there are two distinct non-zero polynomials f and g with positive integer coefficients such that $f(E) \cong g(E)$.*

A vector bundle E on X is said to be essentially finite if there are finitely many finite vector bundles E_1, \dots, E_n and two numerically flat vector bundles V_1 and V_2 with $V_2 \subseteq V_1 \subseteq \bigoplus_{i=1}^n E_i$ such that $E \cong V_1/V_2$.

Let $\text{EF}(X)$ be the full subcategory of $\text{Vect}(X)$ whose objects are essentially finite vector bundles on X . Then $\text{EF}(X)$ is an abelian rigid tensor category. Let Vect_k be the category of k -vector spaces. Fixing a closed point $x \in X(k)$, we have a fiber functor

$$T_x : \text{EF}(X) \longrightarrow \text{Vect}_k$$

defined by sending a vector bundle $E \in \text{Ob}(\text{EF}(X))$ to its fiber E_x at x . This makes the quadruple $(\text{EF}(X), \otimes, \mathcal{O}_X, T_x)$ a neutral Tannakian category. The affine k -group scheme $\pi^N(X, x)$ Tannaka dual to this category is called *Nori's fundamental group scheme* of X with base point x .

3. FUNDAMENTAL GROUP SCHEMES OF $S^n(X)$

3.1. Symmetric product of curve. Let k be an algebraically closed field of characteristic $p > 0$. Let X be an irreducible smooth projective curve over

k . Fix an integer $n \geq 2$, and let us denote by S_n the permutation group of n symbols. There is a natural action of S_n on the n -fold product X^n , and the associated quotient X^n/S_n , denoted by $S^n(X)$, is a smooth projective variety of dimension n over k . Note that any closed point $q \in S^n(X)$ can be uniquely written as $\sum_{i=1}^r n_i x_i$, where x_1, \dots, x_r are distinct closed points of X and n_1, \dots, n_r are integers with

$$n_1 \geq \dots \geq n_r \geq 1.$$

We call $\langle n_1, \dots, n_r \rangle$ the *type* of q . The quotient morphism

$$(3.1.1) \quad \psi : X^n \longrightarrow S^n(X)$$

is a faithfully flat finite morphism of k -schemes.

3.2. A group theoretic lemma. A proof of the following easy Lemma can be found in [PS19].

Lemma 3.2.1. *Let G and H be two group schemes over k . For an integer $n \geq 2$, we denote by G^n the group scheme $G \times \dots \times G$. Then S_n acts on G^n by permuting the factors. Let f_0 be the following composite group homomorphism*

$$f_0 : G^n \xrightarrow{\alpha^n} (G_{\text{ab}})^n \xrightarrow{m} G_{\text{ab}},$$

where $\alpha : G \rightarrow G_{\text{ab}} := G/[G, G]$ denotes the abelianization homomorphism and m denotes the multiplication homomorphism. Then a homomorphism of k -group schemes $f : G^n \rightarrow H$ is S_n -invariant if and only if there is a homomorphism $\tilde{f} : G_{\text{ab}} \rightarrow H$ of affine k -group schemes such that $\tilde{f} \circ f_0 = f$. In other words, f is S_n -invariant iff there is \tilde{f} which makes the following diagram commutes.

$$\begin{array}{ccc} G^n & \xrightarrow{f} & H \\ & \searrow f_0 & \nearrow \tilde{f} \\ & G_{\text{ab}} & \end{array}$$

3.3. Construction of homomorphism. The functor which sends $E \in \mathcal{C}^{\text{nf}}(S^n(X))$ to $\psi^* E \in \mathcal{C}^{\text{nf}}(X^n)$ defines a morphism of neutral Tannakian categories (for any closed point $p \in X^n(k)$)

$$(3.3.1) \quad \mathcal{F} : (\mathcal{C}^{\text{nf}}(S^n(X)), \otimes, \mathcal{O}_{S^n(X)}, T_{\psi(p)}) \rightarrow (\mathcal{C}^{\text{nf}}(X^n), \otimes, \mathcal{O}_{X^n}, T_p).$$

Thus, we get a homomorphism

$$\psi_*^S : \pi^S(X^n, p) \longrightarrow \pi^S(S^n(X), \psi(p)).$$

It follows from [Lan12, Theorem 4.1, p. 842] that, for any closed point $x \in X(k)$, the natural homomorphism of affine k -group schemes

$$\pi^S(X^n, (x, \dots, x)) \xrightarrow{\simeq} \pi^S(X, x)^n.$$

is an isomorphism. By abuse of notation, denote by ψ_*^S the composite of the inverse of this isomorphism and ψ_*^S . So now

$$(3.3.2) \quad \psi_*^S : \pi^S(X, x)^n \rightarrow \pi^S(S^n(X), nx),$$

where $nx = \psi(x, \dots, x)$.

The natural S_n -action on X^n gives rise to automorphisms σ_* of the affine k -group scheme $\pi^S(X^n, (x, \dots, x)) \cong \pi^S(X, x)^n$, for all $\sigma \in S_n$. Now one can check that $\psi_*^S \circ \sigma_* = \psi_*^S$, where ψ_*^S is the homomorphism defined in (3.3.2) with $p = (x, \dots, x)$. Consider the natural homomorphism of affine k -group schemes

$$\phi : \pi^S(X, x)^n \longrightarrow \pi^S(X, x)_{\text{ab}}$$

defined as the following composite homomorphism

$$\pi^S(X, x)^n \longrightarrow (\pi^S(X, x)_{\text{ab}})^n \xrightarrow{m} \pi^S(X, x)_{\text{ab}},$$

where the first homomorphism is given by abelianization at each factor, and the second homomorphism is the multiplication. Then it follows from Lemma 3.2.1 that the homomorphism ψ_*^S in (3.3.2) factors through a homomorphism

$$(3.3.3) \quad \widetilde{\psi}_*^S : \pi^S(X, x)_{\text{ab}} \longrightarrow \pi^S(S^n(X), nx).$$

We record the above discussion in the following proposition.

Proposition 3.3.4. *The map*

$$\psi_*^S : \pi^S(X^n, (x, \dots, x)) \longrightarrow \pi^S(S^n(X), \psi(x, \dots, x))$$

factors to give a homomorphism $\widetilde{\psi}_^S : \pi^S(X, x)_{\text{ab}} \longrightarrow \pi^S(S^n(X), nx)$.*

A vector bundle E on X^n is said to be S_n -invariant if $\sigma^*E \cong E$, for all $\sigma \in S_n$.

Proposition 3.3.5. *Let E be a vector bundle in $\mathcal{C}^{\text{nf}}(X^n)$ associated to a representation $\rho : \pi^S(X^n, (x, \dots, x)) \cong \pi^S(X, x)^n \rightarrow \text{GL}(V)$. If ρ factors through $\pi^S(X, x)_{\text{ab}}$, as in Lemma 3.2.1, then E is S_n -invariant.*

Proof. From the hypothesis it follows that $\rho \circ \sigma_* = \rho$. The proposition follows. \square

3.4. Faithfully flat. In this subsection we use [DMOS82, Proposition 2.21] to show that the homomorphism $\widetilde{\psi}_*^S$ in (3.3.3) is faithfully flat. We begin by recalling this result for the convenience of the reader. Let $\theta : G \longrightarrow G'$ be a homomorphism of affine group schemes over k , and let

$$(3.4.1) \quad \widetilde{\theta} : \text{Rep}_k(G') \longrightarrow \text{Rep}_k(G)$$

be the functor given by sending $\rho' : G' \rightarrow \text{GL}(V)$ to $\rho' \circ \theta : G \rightarrow \text{GL}(V)$. An object $\rho : G \rightarrow \text{GL}(V)$ in $\text{Rep}_k(G)$ is said to be a *subquotient* of an object $\eta : G \rightarrow \text{GL}(W)$ in $\text{Rep}_k(G)$ if there are two G -submodules $V_1 \subset V_2$ of W such that $V \cong V_2/V_1$ as G -modules.

Proposition 3.4.2 (Proposition 2.21, [DMOS82]). *Let $\theta : G \longrightarrow G'$ be a homomorphism of affine algebraic groups over k . Then*

- (a) *θ is faithfully flat if and only if the functor $\tilde{\theta}$ in (3.4.1) is fully faithful and given any subobject $W \subset \tilde{\theta}(V')$, with $V' \in \text{Rep}_k(G')$, there is a subobject $W' \subset V'$ in $\text{Rep}_k(G')$ such that $\tilde{\theta}(W') \cong W$ in $\text{Rep}_k(G)$.*
- (b) *f is a closed immersion if and only if every object of $\text{Rep}_k(G)$ is isomorphic to a subquotient of an object of the form $\tilde{\theta}(V')$, for some $V' \in \text{Rep}_k(G')$.*

Proposition 3.4.3. *The homomorphism*

$$\widetilde{\psi}_*^S : \pi^S(X, x)_{\text{ab}} \longrightarrow \pi^S(S^n(X), nx)$$

defined in (3.3.3) is faithfully flat.

Proof. We will apply [DMOS82, Proposition 2.21 (a)]. Let E_1 be an object in the category $\mathcal{C}^{\text{nf}}(S^n(X))$. Clearly ψ^*E_1 has the same rank as that of E_1 . If $\mathcal{E}_2 \subset \mathcal{E}_1 := \psi^*E_1$ is a subbundle corresponding to a representation of $\pi^S(X, x)_{\text{ab}}$, we need to show that there is a subbundle $E_2 \subset E_1$ such that $\psi^*E_2 = \mathcal{E}_2$. We will prove this by induction on rank of E_1 . If $\text{rank}(E_1) = 1$, there is nothing to prove. Assume that $\text{rank}(E_1) \geq 2$.

The vector bundles \mathcal{E}_i corresponds to a representation

$$\pi^S(X^n, (x, \dots, x)) \xrightarrow{f_0} \pi^S(X, x)_{\text{ab}} \xrightarrow{\rho_i} \text{GL}(V_i), \quad \forall i = 1, 2.$$

It follows from Proposition 3.3.5 that \mathcal{E}_2 is a S_n -invariant numerically flat vector bundle on X^n . Since $\pi^S(X, x)_{\text{ab}}$ is an abelian k -group scheme, it follows from [Wat79, Theorem 9.4, p. 70], that we can find a surjective $\pi^S(X, x)_{\text{ab}}$ -module homomorphism $V_1 \rightarrow L_1$, where L_1 is one dimensional and V_2 is a $\pi^S(X, x)_{\text{ab}}$ -submodule of the kernel of this homomorphism. Let \mathcal{L} be the line bundle on X^n corresponding to the representation L_1 . Then it is clear that \mathcal{L} is S_n -invariant (see Proposition 3.3.5) and there is an S_n -equivariant exact sequence of vector bundles

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{L} \longrightarrow 0$$

on X^n such that $\mathcal{E}_2 \subset \mathcal{K}$.

Every S_n -invariant line bundle on X^n is the pullback of a line bundle from $S^n(X)$ (see [Fog77, Proposition 3.6], also [PS19, Proposition 5.1.1]). Therefore, it follows that $L := (\psi_*\mathcal{L})^{S_n}$ is a line bundle on all of $S^n(X)$. We now show that L is numerically flat on $S^n(X)$. Given a morphism $C \longrightarrow S^n(X)$ from a smooth projective curve C into $S^n(X)$, we can find a curve

\tilde{C} and a morphism $\tilde{C} \rightarrow C$ making the following diagram commutative.

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & X^n \\ \downarrow & & \downarrow \psi \\ C & \longrightarrow & S^n(X) \end{array}$$

Since \mathcal{L} is numerically flat on X^n and $\mathcal{L} \cong \psi^*L$, it follows that L is numerically flat.

We claim that

$$(3.4.4) \quad 0 \rightarrow (\psi_*\mathcal{K})^{S_n} \rightarrow (\psi_*\mathcal{E}_1)^{S_n} \cong E_1 \xrightarrow{q} (\psi_*\mathcal{L})^{S_n} = L \rightarrow 0$$

is exact. The sequence (3.4.4) can fail to be exact only on the right. Since both E_1 and L are numerically flat and L is a line bundle, q is surjective since it is nonzero. This proves the exactness of (3.4.4). It follows that $K := (\psi_*\mathcal{K})^{S_n}$ is locally free and numerically flat on $S^n(X)$. It is clear that $\psi^*K = \mathcal{K}$ on X^n . Since $\mathcal{E}_2 \subset \mathcal{K}$ the assertion that there is $E_2 \subset E_1$ such that $\mathcal{E}_2 = \psi^*E_2$ on X^n follows by induction on rank.

To complete the proof of the Proposition, we need to show that if E_1 and E_2 are numerically flat vector bundles on $S^n(X)$ then the natural map

$$\mathrm{Hom}_{S^n(X)}(E_1, E_2) \rightarrow \mathrm{Hom}_{X^n}(\psi^*E_1, \psi^*E_2)$$

is bijective. It is clear that this natural map is injective (faithful). Therefore, it suffices to show that given any numerically flat vector bundle E on $S^n(X)$, any nonzero homomorphism $\phi : \mathcal{O}_{X^n} \rightarrow \psi^*E$ comes from a nonzero homomorphism $\tilde{\phi} : \mathcal{O}_{S^n(X)} \rightarrow E$. Since the homomorphism $\pi^S(X^n, x) \rightarrow \pi^S(X, x)_{\mathrm{ab}}$ is faithfully flat, and ψ^*E corresponds to a representation of $\pi^S(X, x)_{\mathrm{ab}}$, it follows that ϕ is a morphism between two representations of $\pi^S(X, x)_{\mathrm{ab}}$. This shows that ϕ is S_n -equivariant on X^n . Now from the preceding discussion it follows that ϕ arises from a morphism $\mathcal{O}_{S^n(X)} \rightarrow E$. \square

3.5. Proofs of Theorems. Let X be a connected smooth projective variety over k and $f : \mathbb{P}(E) \rightarrow X$ be a projective bundle over X . It is easy to see, using $\pi^S(\mathbb{P}^n, s) = \{1\}$ and [Har77, Corollary 12.9, Chapter III], that for a numerically flat sheaf F on $\mathbb{P}(E)$, the sheaf f_*F is locally free and the natural map $f^*f_*F \rightarrow F$ is an isomorphism. From this it easily follows that the homomorphism of S -fundamental group schemes

$$f_*^S : \pi^S(\mathbb{P}(E), y) \rightarrow \pi^S(X, f(y))$$

is an isomorphism, for all $y \in \mathbb{P}(E)$.

Theorem 3.5.1. *The homomorphism of affine k -group schemes*

$$\widetilde{\psi}_*^S : \pi^S(X, x)_{\mathrm{ab}} \rightarrow \pi^S(S^n(X), nx)$$

is an isomorphism, for all $x \in X(k)$.

Proof. Let $\text{Alb}(X)$ be the Albanese variety of X . Let g be the genus of the curve X . Fix a closed point $x \in X(k)$. If $n \geq 2g - 1$, then the morphism $\eta : S^n(X) \rightarrow \text{Alb}(X)$ given by

$$\sum_{i=1}^n x_i \mapsto \sum_{i=1}^n x_i - nx,$$

makes $S^n(X)$ into a projective bundle over $\text{Alb}(X)$. It follows that the induced homomorphism of affine k -group schemes is an isomorphism,

$$\eta_* : \pi^S(S^n(X), nx) \xrightarrow{\sim} \pi^S(\text{Alb}(X), 0).$$

From [Lan12, Section 7] it follows that the Albanese morphism $\text{alb}_X : X \rightarrow \text{Alb}(X)$ given by $t \mapsto t - x$ induces maps

$$\text{alb}_{X*} : \pi^S(X, x) \xrightarrow{a_0} \pi^S(X, x)_{\text{ab}} \xrightarrow{\widetilde{\text{alb}_{X*}}} \pi^S(\text{Alb}(X), 0),$$

where $\widetilde{\text{alb}_{X*}}$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a} & X^n \\ & \searrow \psi \circ a & \downarrow \psi \\ & & S^n(X) \end{array} \xrightarrow{\eta} \text{Alb}(X),$$

where $a(t) = (t, x, x, \dots, x)$. At the level of S -fundamental group schemes we get the commutative diagram

$$\begin{array}{ccccc} \pi^S(X, x) & \xrightarrow{a_*} & \pi^S(X, x)^n & \longrightarrow & \pi^S(X, x)_{\text{ab}}^n \\ & \searrow a_0 & \downarrow \psi_0 & \swarrow m & \\ & & \pi^S(X, x)_{\text{ab}} & \xrightarrow{\widetilde{\psi}_*^S} & \pi^S(S^n(X), nx) \end{array} \xrightarrow{\eta_*} \pi^S(\text{Alb}(X), 0),$$

We have $\eta \circ \psi \circ a = \text{alb}_X$. It is easy to check that $(\psi \circ a)_* = \widetilde{\psi}_*^S \circ a_0$. Since a_0 is faithfully flat and

$$\widetilde{\psi}_*^S \circ a_0 = \eta_*^{-1} \circ \widetilde{\text{alb}_{X*}} \circ a_0,$$

it follows that $\widetilde{\psi}_*^S = \eta_*^{-1} \circ \widetilde{\text{alb}_{X*}}$ and so the theorem is true is $n \geq 2g - 1$.

Assume that $n < 2g - 1$. Consider the maps

$$X \xrightarrow{a} X^n \xrightarrow{\psi} S^n(X) \xrightarrow{c} S^{2g-1}(X) \xrightarrow{\eta} \text{Alb}(X),$$

where $a(t) = (t, x, x, \dots, x)$ and $c(\sum_{i=1}^n x_i) = \sum_{i=1}^n x_i + (2g - 1 - n)x$. It is clear that the composite morphism is alb_X . As above, one easily checks that

$$\eta_* \circ c_* \circ \widetilde{\psi}_*^S = \widetilde{\text{alb}_{X*}}.$$

Thus, we get homomorphisms of affine k -group schemes

$$\pi^S(X, x)_{\text{ab}} \xrightarrow{\widetilde{\psi}_*^S} \pi^S(S^n(X), nx) \xrightarrow{\eta_* \circ c_*} \pi^S(\text{Alb}(X), 0),$$

such that the composite homomorphism is an isomorphism. This forces that the first map is a closed immersion. Since we know from Proposition 3.4.3 that the first map is faithfully flat, the theorem follows. \square

Remark 3.5.2. That $\widetilde{\psi}_*^S$ is a closed immersion could have been proved using the same method in [PS19] under the assumption that $\text{char}(k) > 3$.

Let X be a reduced proper k -scheme with $H^0(X, \mathcal{O}_X) = k$. Let E be an essentially finite vector bundle on X . Then there is a finite k -group scheme G , a principal G -bundle $p : P \rightarrow X$ and a finite dimensional k -linear representation $\rho : G \rightarrow \text{GL}(W)$ such that $E \cong P \times^\rho W$, the vector bundle associated to the representation ρ . It follows from the proof of [Nor76, Proposition 3.8] that there is a finite vector bundle \mathcal{V} on X such that E is a subbundle of \mathcal{V} .

As before, let X be a connected smooth projective curve over k and $S^n(X)$ the n -fold symmetric product of X . It is clear that ψ^* takes a finite vector bundle to a finite vector bundle. Thus, $\psi^*E \subset \psi^*\mathcal{V}$, which shows that \mathcal{F} takes essentially finite vector bundles to essentially finite vector bundles. Note that there is a commutative diagram of homomorphisms of affine k -group schemes

$$\begin{array}{ccc} \pi^S(X, x)_{\text{ab}} & \xrightarrow{\sim} & \pi^S(S^n(X), nx) \\ \downarrow & & \downarrow \\ \pi^N(X, x)_{\text{ab}} & \xrightarrow{\widetilde{\psi}_*^N} & \pi^N(S^n(X), nx) \end{array}$$

where the vertical arrows are faithfully flat by [Lan11, Lemma 6.2]. It follows that $\widetilde{\psi}_*^N$ is faithfully flat.

Now let \mathcal{E} be an essentially finite S_n -invariant vector bundle on X^n . It is easy to find a finite and S_n -invariant bundle \mathcal{V} on X^n and an S_n -equivariant inclusion $\mathcal{E} \subset \mathcal{V}$. Define $E = (\psi_*\mathcal{E})^{S_n}$ and $V := (\psi_*\mathcal{V})^{S_n}$. It is clear that V is a finite vector bundle and $E \subset V$. So E is essentially finite and $\mathcal{F}(E) = \mathcal{E}$. This shows that \widetilde{f}^N is a closed immersion. Thus, we have the following.

Theorem 3.5.3. *There is a natural isomorphism of affine k -group schemes*

$$\widetilde{\psi}_*^N : \pi^N(X, x)_{\text{ab}} \longrightarrow \pi^N(S^n(X), nx).$$

3.6. Étale Fundamental Group Scheme of $S^n(X)$. In this subsection we sketch how to deduce from Theorem 3.5.3 the same assertion for étale fundamental group schemes. This result is a special case of [BH15, Theorem 1.2]. Note that there is a commutative diagram

$$\begin{array}{ccccc} \pi^N(X, x) & \twoheadrightarrow & \pi^N(X, x)_{\text{ab}} & \xrightarrow{\sim} & \pi^N(S^n(X), nx) \\ \downarrow & & \downarrow & & \downarrow d \\ \pi^{\text{ét}}(X, x) & \twoheadrightarrow & \pi^{\text{ét}}(X, x)_{\text{ab}} & \longrightarrow & \pi^{\text{ét}}(S^n(X), nx). \end{array}$$

From this it follows that $\pi^{\text{ét}}(X, x)_{\text{ab}} \rightarrow \pi^{\text{ét}}(S^n(X), nx)$ is faithfully flat. Consider a homomorphism $\pi^{\text{ét}}(X, x)_{\text{ab}} \rightarrow \text{GL}(V)$. It follows using [Nor76, Proposition 3.10] that this homomorphism factors through a finite and reduced k -group scheme G . Now consider the diagram

$$\begin{array}{ccccc} \pi^N(X, x)_{\text{ab}} & \xrightarrow{\sim} & \pi^N(S^n(X), nx) & \xrightarrow{d} & \pi^{\text{ét}}(S^n(X), nx) \\ \downarrow & & \downarrow & \swarrow \text{---} & \\ \pi^{\text{ét}}(X, x)_{\text{ab}} & \longrightarrow & G & \longrightarrow & \text{GL}(V). \end{array}$$

The right vertical arrow is the unique map which makes the square commute. It factors through d since G is finite and reduced. Now it follows from [DMOS82, Proposition 2.21 (b)] that $\pi^{\text{ét}}(X, x)_{\text{ab}} \rightarrow \pi^{\text{ét}}(S^n(X), nx)$ is a closed immersion. This proves the following.

Theorem 3.6.1. *For any closed point $x \in X(k)$, there is an isomorphism of affine k -group schemes*

$$\widetilde{\psi}_*^{\text{ét}} : \pi^{\text{ét}}(X, x)_{\text{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx).$$

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