MA5202: Algebraic Geometry

Arjun Paul
Department of Mathematics and Statistics
Indian Institute of Science Education and Research Kolkata,
Mohanpur - 741 246, Nadia,
West Bengal, India.
Email: arjun.paul@iiserkol.ac.in.

Version: January 4, 2023 at 10:40am (IST).

This note will be updated from time to time.
If there are any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\bigsqcup$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
A := ...	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Basic Theory of Schemes

1.1 Classical variety

Let k be a field and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ and coefficients from the field k. Given a subset $E \subseteq k[x_1, \ldots, x_n]$, let

$$\mathcal{Z}(E) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0, \ \forall\, f \in E\}$$

be the subset of all common zeros of the polynomials in E. We are interested to study geometry of $\mathcal{Z}(E)$. If $f \in E$ is a linear polynomial with zero constant term, i.e., $f(0, \ldots, 0) = 0$, the map $T_f : k^n \to k$ defined by

$$T_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n), \ \ \forall\, (a_1, \ldots, a_n) \in k^n,$$

is a k-linear map, and that $\mathcal{Z}(f) = \mathrm{Ker}(T_f)$ is a k-linear subspace of k^n. Therefore, if all the polynomials in E are linear with zero constant terms, then

$$\mathcal{Z}(E) = \bigcap_{f \in E} \mathrm{Ker}(T_f)$$

is a k-linear subspace of k^n, and in this case standard linear algebra machinery could be used to study the space $\mathcal{Z}(E)$. However, when $f \in A$ is not a linear polynomial, $\mathcal{Z}(f)$ is no longer a liner space, and hence we cannot use linear algebra machinery to study geometry of $\mathcal{Z}(f)$. In this situation, the techniques from commutative algebra comes into the picture.

Proposition 1.1.1. *Let E be a non-empty subset of the polynomial ring $k[x_1, \ldots, x_n]$. Then $\mathcal{Z}(E) = \mathcal{Z}(\langle E \rangle)$, where $\langle E \rangle$ is the ideal of $k[x_1, \ldots, x_n]$ generated by E.*

Proof. Since $E \subseteq \langle E \rangle$, it follows from the definition of $\mathcal{Z}(E)$ that $\mathcal{Z}(\langle E \rangle) \subseteq \mathcal{Z}(E)$. Conversely, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{Z}(E)$. Let $f \in \langle E \rangle$ be arbitrary. Then $f = \sum\limits_{j=1}^{m} \phi_j f_j$, for some $\phi_j \in k[x_1, \ldots, x_n]$ and $f_j \in E$, for all $j \in \{1, \ldots, m\}$. Since $f_j(\mathbf{a}) = 0$, for all j, we have $f(\mathbf{a}) = \sum\limits_{j=1} \phi_j(\mathbf{a}) f_j(\mathbf{a}) = 0$. Therefore, $\mathbf{a} \in \mathcal{Z}(\langle E \rangle)$. $\square$

We now introduce a class of commutative rings with identity for which every ideals are finitely generated. Such a ring is called Noetherian. We show that polynomial ring

$k[x_1, \ldots, x_n]$ and its quotient rings are Noetherian. One of the advantage to work with such rings is that all of its ideals being finitely generated, zero locus of a given family of possibly infinitely many polynomials is determined by a finite number of polynomials among them.

Let A be a commutative ring with identity.

Definition 1.1.2. An A-module M is said to be *noetherian* if every A-submodule of M is finitely generated. We say that A is *noetherian* if it is noetherian as a module over itself.

Proposition 1.1.3. *A is noetherian if and only if every ideal of A is finitely generated.*

Proof. Since any A-submodule of A is an ideal of A, the result follows. □

Lemma 1.1.4. *Let*

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

be a short exact sequence of A-modules. Then M is noetherian if and only if both M' and M'' are noetherian.

Proof. Suppose that M is noetherian. Let N' be an A-submodule of M'. Then N is isomorphic to the A-submodule $\phi(N')$ of M, and hence is finitely generated. Therefore, M' is noetherian. Let N'' be an A-submodule of M''. Then $N := \psi^{-1}(N'')$ is an A-submodule of M, and so is finitely generated. Since the A-module homomorphism $\psi|_N : N \to N''$ is surjective and N is finitely generated, N'' is finitely generated. Therefore, M'' is noetherian.

Conversely, suppose that both M' and M'' are noetherian A-modules. Let N be an A-submodule of M. Since $N' := \phi^{-1}(N)$ and $N'' := \psi(N)$ are A-submodule of noetherian A-modules, they are finitely generated. Then we have an exact sequence of A-modules

$$0 \to N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \to 0.$$

Suppose that $\phi^{-1}(N)$ and $\psi(N)$ are generated as A-modules by $x_1, \ldots, x_m \in \phi^{-1}(N)$ and $y_1, \ldots, y_n \in \psi(N)$, respectively. Fix an element $z_j \in \psi^{-1}(y_j) \subseteq N$, for each $j \in \{1, \ldots, n\}$. Let $x \in N$ be given. Then $\psi(x) = b_1 y_1 + \cdots + b_n y_n$, for some $b_1, \ldots, b_n \in A$. Consider the element $w = x - (b_1 z_1 + \cdots + b_n z_n) \in N$. Since $\phi(w) = 0$, there exists $a_1, \ldots, a_m \in A$ such that $w = a_1 x_1 + \cdots + a_m x_m$. Then we have $x = a_1 x_1 + \cdots + a_m x_m + b_1 z_1 + \cdots + b_n z_n$. Therefore, N is generated as an A-module by $\{x_1, \ldots, x_m\} \cup \{z_1, \ldots, z_n\}$. Therefore, M is noetherian. □

Corollary 1.1.5. *If M and N are noetherian A-modules, so is $M \oplus N$.*

Proof. Follows from Lemma 1.1.4. □

Corollary 1.1.6. *Any finitely generated module over a noetherian ring is noetherian.*

Proof. Let A be a noetherian ring and let M be a finitely generated A-module. Then there exists a surjective A-module homomorphism

$$\varphi : A^{\oplus n} \to M,$$

for some $n \in \mathbb{N}$. Since $A^{\oplus n}$ is noetherian by Corollary 1.1.5, that M is noetherian by Lemma 1.1.4. $\square$

Theorem 1.1.7 (Hilbert's basis theorem)**.** *If A is a noetherian ring, the polynomial ring $A[x_1, \ldots, x_n]$ is noetherian.*

Proof. Since the polynomial ring $A[x_1, \ldots, x_n]$ is isomorphic to the polynomial ring $B[x_n]$, where $B = A[x_1, \ldots, x_n]$, using induction it suffices to prove the result for $n = 1$. Consider the polynomial ring $A[x]$. Let $I \subset A[x]$ be an ideal of $A[x]$. Since the cases $I = 0$ and $I = A[x]$ are trivial, we assume that $I \neq 0$ and $I \neq A[x]$. Let

$$J = \{0\} \cup \{\text{set of all leading coefficients of non-zero polynomials in } I\}.$$

Clearly J is an ideal of A, and hence is finitely generated because A is noetherian. Let $c_1, \ldots, c_r \in A$ be non-zero elements of A that generates J as an ideal of A. Each c_j is a leading coefficient of a non-zero element, say f_j, of I. Let $J' = (f_1, \ldots, f_r)$ be the ideal of $A[x]$ generated by $f_1, \ldots, f_r$. Let $m := \max\limits_{1 \leq j \leq r} \deg(f_j)$, and let

$$M := I \cap \left(A + Ax + \cdots + Ax^{m-1}\right).$$

Then M is an A-submodule of $A[x]$. We claim that $I = M + J'$. Since both M and J' are subsets of I and I is an ideal, $M + J' \subseteq I$. To show the reverse inclusion, we need to show that every $f \in I$ is in $M + J'$. We show this by induction on $d = \deg(f)$. If $d \leq m - 1$, then $f \in M \subseteq M + J'$. Suppose that $d := \deg(f) \geq m$, and assume that for given any $g \in I$ with $\deg(g) < d$ we have $g \in M + J'$. Let c be the leading coefficient of f. Since $J = (c_1, \ldots, c_r)$ and $c \in J$, we have $c = \sum\limits_{j=1}^{r} a_j c_j$, for some $a_1, \ldots, a_r \in A$. Since $g := f - \sum\limits_{j=1}^{r} a_j x^{d - \deg(f_j)} f_j \in I$ with $\deg(g) \leq d - 1$, by induction hypothesis $g \in M + J'$. Then $f = g + \sum\limits_{j=1}^{r} a_j x^{d - \deg(f_j)} f_j \in M + J'$, as required. Therefore, by induction $I = M + J'$. Since A is noetherian and $A + Ax + \cdots + Ax^{m-1}$ is a finitely generated A-module, that $A + Ax + \cdots + Ax^{m-1}$ is a noetherian A-module by Corollary 1.1.6. Since M is an A-submodule of $A + Ax + \cdots + Ax^{m-1}$, M is a finitely generated A-module, generated by, say $g_1, \ldots, g_n$. Then $I = M + J'$ is generated as an $A[x]$-module by $f_1, \ldots, f_r, g_1, \ldots, g_n$. This completes the proof. $\square$

By Hilbert basis theorem, every ideal of $A = k[x_1, \ldots, x_n]$ are finitely generated. Then every generating subset E of a finitely generated ideal $\mathfrak{a}$ of A contains a finite subset that generates the ideal $\mathfrak{a}$. Therefore, for every $E \subseteq A$, there exists finitely many elements $f_1, \ldots, f_n \in E$ such that $\mathcal{Z}(E) = \mathcal{Z}(f_1, \ldots, f_n) = \bigcap\limits_{j=1}^{n} \mathcal{Z}(f_j)$. Note that, given $E_1 \subseteq E_2 \subseteq A$, we have $\mathcal{Z}(E_2) \subseteq \mathcal{Z}(E_1)$.

Proposition 1.1.8. *The sets $\mathcal{Z}(\mathfrak{a})$, where $\mathfrak{a}$ runs over the set of all ideals of $k[x_1, \ldots, x_n]$ satisfy axioms for closed subsets for a topology on k^n, called the Zariski topology.*

Proof. The proposition follows from the following observations.

(i) $\mathcal{Z}(1) = \emptyset$ and $\mathcal{Z}(0) = k^n$;

(ii) Given any family of ideals $\{\mathfrak{a}_j : j \in I\}$ of $k[x_1, \ldots, x_n]$, we have

$$\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) = \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big);$$

(iii) given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $k[x_1, \ldots, x_n]$, we have

$$\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{Z}(\mathfrak{a}\mathfrak{b}).$$

The first point is obvious. To see the second, note that

$$\begin{aligned}\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) &= \bigcap_{j \in I} \{x \in k^n : f(x) = 0,\ \forall\, f \in \mathfrak{a}_j\} \\ &= \{x \in k^n : f(x) = 0,\ \forall\, f \in \mathfrak{a}_j,\ \forall\, j \in I\} \\ &= \mathcal{Z}\big(\bigcup_{j \in I} \mathfrak{a}_j\big) \\ &= \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big).\end{aligned}$$

To see the third point, note that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, which is a subset of both $\mathfrak{a}$ and $\mathfrak{b}$. Therefore, $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a}\mathfrak{b})$. Conversely, if $x \in \mathcal{Z}(\mathfrak{a}\mathfrak{b})$ and $x \notin \mathcal{Z}(\mathfrak{a})$, then there exists $f \in \mathfrak{a}$ such that $f(x) \neq 0$, and for any $g \in \mathfrak{b}$ that $f(x)g(x) = (fg)(x) = 0$, since $fg \in \mathfrak{a}\mathfrak{b}$ and $x \in \mathcal{Z}(\mathfrak{a}\mathfrak{b})$. Since k is an integral domain, we must have $g(x) = 0$, for all $g \in \mathfrak{b}$. Therefore, $x \in \mathcal{Z}(\mathfrak{b})$. This completes the proof. □

Definition 1.1.9. The set k^n together with the Zariski topology on it is called the *affine n-space over k* and is denoted by $\mathbb{A}^n(k)$. A closed subspace of $\mathbb{A}^n(k)$ is called an *algebraic set*.

Given a point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$, consider the evaluation map

$$ev_a : k[x_1, \ldots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a_1, \ldots, a_n),\ \ \forall\, f \in k[x_1, \ldots, x_n].$$

Note that ev_a a surjective ring homomorphism with kernel

$$\mathrm{Ker}(ev_a) = \mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n).$$

Therefore, $\{a\} = \mathcal{Z}(\mathfrak{m}_a)$ is a closed subset of $\mathbb{A}^n(k)$. As a result, any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.

Example 1.1.10. For $n = 1$, the polynomial ring $k[x]$ is a principal ideal domain. So every ideal of $k[x]$ is generated by a single polynomial. Since a polynomial in $k[x]$ has only finite number of roots in k, any closed subset of $\mathbb{A}^1(k)$ is either finite or $\mathbb{A}^1(k)$ itself.

Example 1.1.11. For $n = 2$, the situation is more complicated. Here is an obvious list of closed subsets of $\mathbb{A}^2(k)$.

- $\varnothing$ and $\mathbb{A}^2(k)$.
- any finite subset of $\mathbb{A}^2(k)$.
- $\mathcal{Z}(f)$, where $f \in k[x_1, x_2]$ is an irreducible polynomial.

In fact, we shall see later that the no-empty closed subsets of $\mathbb{A}^2(k)$ listed above are of the form $\mathcal{Z}(\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of $k[x_1, x_2]$. Moreover, any closed subsets of $\mathbb{A}^2(k)$ is a finite union of the closed subsets of the form listed above.

Connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz and its corollaries.

Theorem 1.1.12 (Hilbert's Nullstellensatz)**.** *Let k be a field that is not necessarily algebraically closed, and let A be a finitely generated k-algebra. Then A is Jacobson; i.e., every prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ is an intersection of all maximal ideals of A containing $\mathfrak{p}$.*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{p})} \mathfrak{m},$$

where $V_{\max}(\mathfrak{p})$ is the set of all maximal ideals of A containing $\mathfrak{p}$.

Proof. □