FUNDAMENTAL GROUP-SCHEME OF HILBERT SCHEME OF n-POINTS OF A SMOOTH PROJECTIVE SURFACE

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Abstract. Let k be an algebraically closed field of characteristic $p > 3$. Let X be an irreducible smooth projective surface over k. Fix an integer $n \geq 1$, and let $\mathcal{H}ilb^n_X$ be the Hilbert scheme parametrizing effective 0-cycles of length n on X. The aim of the present article is to find the S-fundamental group scheme and the Nori's fundamental group scheme of the Hilbert scheme $\mathcal{H}ilb^n_X$.

1. Introduction

Let X be a connected, reduced and complete scheme over a perfect field k, and let $x \in X$ be a k-rational point. In [Nor76], Nori introduced a k-group scheme $\pi^N(X, x)$ associated to essentially finite vector bundles on X. In [Nor82], Nori extends the definition of $\pi^N(X, x)$ to connected and reduced k-schemes. In [BPS06], Biswas, Parameswaran and Subramanian defined the notion of *S-fundamental group scheme* $\pi^S(X, x)$ for X a smooth projective curve over any algebraically closed field k. This is generalized to higher dimensional connected smooth projective k-schemes and studied extensively by Langer in [Lan11, Lan12]. In general, $\pi^S(X, x)$ carries more information than $\pi^N(X, x)$ and $\pi^{\text{ét}}(X, x)$. There are natural faithfully flat morphisms of affine k-group schemes $\pi^S(X, x) \to \pi^N(X, x) \to \pi^{\text{ét}}(X, x)$. The reader is referred to the introductions in [Nor82] and [Lan11] for more details. Precise definitions of the above obejcts are given in the next section. It is an interesting problem to determine $\pi^{\text{ét}}(X, x)$, $\pi^N(X, x)$ and $\pi^S(X, x)$ for well-known varieties.

Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{H}ilb^n_X$ be the Hilbert scheme of n points on an irreducible smooth projective surface X over k. It is known that $\mathcal{H}ilb^n_X$ is an irreducible smooth projective variety of dimension $2n$ over k. The geometry of the scheme $\mathcal{H}ilb^n_X$ is extensively studied in the literature, see [Fog73, FGI+05, Iar72] and the references therein. It follows from [BH15, Theorem 1.1, Theorem 1.2] that the étale fundamental group $\pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$ is isomorphic to the abelianization of $\pi^{\text{ét}}(X, x)$, for any $x \in X(k)$. Therefore, it is natural to ask if a similar result holds for $\pi^N(\mathcal{H}ilb^n_X, n[x])$ and $\pi^S(\mathcal{H}ilb^n_X, n[x])$. In this paper we answer this question affirmatively when the base field k has characteristic $p > 3$. The following two theorems are the main results in this article.

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Theorem (Theorem 5.2.10). *Let* $\mathrm{char}(k) > 3$. *Then there is an isomorphism*

$$\widetilde{f} : \pi^S(X, x)_{\mathrm{ab}} \xrightarrow{\sim} \pi^S(\mathcal{H}ilb^n_X, n[x]).$$

Theorem (Theorem 5.3.1). *Let* $\mathrm{char}(k) > 3$. *Then there is an isomorphism*

$$\widetilde{f}^N : \pi^N(X, x)_{\mathrm{ab}} \xrightarrow{\sim} \pi^N(\mathcal{H}ilb^n_X, n[x]).$$

We can easily deduce from the above the same assertion about $\pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$. This is sketched in subsection 5.4. This assertion about $\pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$ is a corollary of the main result in [BH15], which is proved using a different method.

We briefly describe the organization of this paper. In §2 we recall the main definitions and results on fundamental group schemes that we need from [Nor82] and [Lan11]. In §3 we recall and prove results that we need about the Hilbert scheme and the Hilbert-Chow map. The main input in this paper is the construction in §4, which we briefly explain here. Let $\varphi : \mathcal{H}ilb^n_X \to S^n(X)$ denote the Hilbert-Chow morphism and let $\psi : X^n \to S^n(X)$ denote the the natural quotient map under the action of S_n on X^n. Given a numerically flat locally free sheaf E on $\mathcal{H}ilb^n_X$, a natural coherent sheaf we can associate on X^n is $\psi^*\varphi_*E$. However, it is not clear if this coherent sheaf is locally free. To remedy this, we associate to E a locally free sheaf on a large open subset of X^n and take its unique reflexive extension. Then we use the criterion [Lan12, Theorem 2.2] (this criterion is proved in [Lan11] but stated more precisely in loc. cit.) to check that this reflexive sheaf is locally free. From this construction we are able to define a morphism $\pi^S(X, x)_{\mathrm{ab}} \to \pi^S(\mathcal{H}ilb^n_X, n[x])$. In §5 we use the criterion in [DMOS82, Proposition 2.21] to show that this homomorphism is an isomorphism.

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2. Fundamental Group Schemes

In the rest of this article, unless mentioned otherwise, k will denote an algebraically closed field of characteristic $p > 3$.

2.1. **Nori's fundamental group scheme.** Let X be a connected, proper and reduced k-scheme. We denote by $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X. Consider the full subcategory whose objects are locally free coherent sheaves (vector bundles) of $\mathcal{O}_X$-modules of finite ranks on X, and denote it $\mathrm{Vect}(X)$. A vector bundle E on X is said to be *finite* if there are distinct non-zero polynomials $f, g \in \mathbb{Z}[t]$ with non-negative coefficients such that $f(E) \cong g(E)$.

Let C be a connected smooth projective curve over k. The *degree* of a vector bundle E on C is defined to be the number

$$\deg(E) := c_1(E) \cdot [C]\,.$$

A vector bundle E on C is said to be *semistable* if for any non-zero proper subbundle $F \subset E$, we have

$$\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)} \leq \frac{\deg(E)}{\mathrm{rank}(E)} =: \mu(E)\,.$$

Definition 2.1.1. Let X be a connected, projective and reduced k-scheme. Let $\mathcal{C}^{nf}(X)$ denote the full subcategory of $\mathrm{QCoh}(X)$, whose objects are coherent sheaves E on X satisfying the following two conditions:

(1) E is locally free, and
(2) for any smooth projective curve C over k and any morphism $f : C \longrightarrow X$, the vector bundle f^*E is semistable of degree 0.

We call the objects of the category $\mathcal{C}^{nf}(X)$ *numerically flat vector bundles* on X. In the literature these are also referred to as semistable vector bundles, see [Lan11, Remark 5.2]. However, we reserve the term semistable to refer to slope semistable.

Definition 2.1.2. A vector bundle E on X is said to be *essentially finite* if there exist two numerically flat vector bundles V_1, V_2 and finitely many finite vector bundles $F_1, \ldots, F_n$ on X with $V_2 \subseteq V_1 \subseteq \bigoplus_{i=1}^{n} F_i$ such that $E \cong V_1/V_2$.

Let $\mathrm{EF}(X)$ be the full subcategory of $\mathrm{Vect}(X)$ whose objects are essentially finite vector bundles on X. Let Vect_k be the category of finite dimensional k-vector spaces. Fix a closed point $x \in X$, and let

$$T_x : \mathrm{EF}(X) \longrightarrow \mathrm{Vect}_k$$

be the fiber functor defined by sending an object $E \in \mathrm{EF}(X)$ to its fiber E_x at x. Then the quadruple $(\mathrm{EF}(X), \bigotimes, T_x, \mathcal{O}_X)$ is a neutral Tannakian category. The affine k-group scheme $\pi^N(X, x)$ representing the functor of k-algebras $\underline{\mathrm{Aut}}^{\otimes}(T_x)$ is called the *Nori's fundamental group scheme* of X based at x (see [DMOS82, Section 1] for definition of the functor $\underline{\mathrm{Aut}}^{\otimes}(T_x)$). It is shown in [Nor82, Proposition 4, p. 88] that $\pi^N(X, x) \cong \pi^N(X, y)$ for any two closed points $x, y \in X$.

2.2. S**-fundamental group scheme.** A coherent sheaf G is said to be *reflexive* if the natural $\mathcal{O}_X$-module homomorphism $G \to G^{\vee\vee}$ is an isomorphism.

Definition 2.2.1. Let X be a connected, smooth and projective variety over k of dimension d, and let H be an ample divisor on X. Let $\mathrm{Vect}_0^s(X)$ be the full subcategory of $\mathrm{QCoh}(X)$ whose objects are coherent sheaves G on X satisfying the following three conditions:

(1) G is reflexive,
(2) G is strongly H-semistable, and
(3) $\mathrm{ch}_1(G) \cdot H^{d-1} = \mathrm{ch}_2(G) \cdot H^{d-2} = 0$, where $\mathrm{ch}_i(G)$ denote the i-th Chern character of G, for all $i = 1, 2$.

Since X is smooth, it follows from [Lan11, Proposition 4.1] that the objects of $\mathrm{Vect}_0^s(X)$ are in fact locally free sheaves, and all of their Chern classes vanishes. Moreover, the category $\mathrm{Vect}_0^s(X)$ does not depend on choice of ample divisor H [Lan11, Proposition 4.5]. One of the main results in [Lan11, Theorem 5.1, p. 2094] is that the categories $\mathcal{C}^{nf}(X)$ and $\mathrm{Vect}_0^s(X)$ are the same when X is smooth. This is stated very precisely in [Lan12, Theorem 2.2].

Assume that X is smooth. Fix a k-valued point $x \in X$. Let $T_x : \mathrm{Vect}_0^s(X) \longrightarrow \mathrm{Vect}_k$ be the fiber functor defined by sending an object E of $\mathrm{Vect}_0^s(X)$ to its fiber $E_x \in \mathrm{Vect}_k$ at x. Then $(\mathrm{Vect}_0^s(X), \otimes, T_x, \mathcal{O}_X)$ is a neutral Tannaka category [Lan11, Proposition 5.5, p. 2096]. The affine k-group scheme $\pi^S(X, x)$ Tannaka dual to this category is called the *S-fundamental group scheme* of X with base point x [Lan11, Definition 6.1, p. 2097].

We could not find a precise reference for the following result, which maybe well-known to experts, and so we include a proof.

Lemma 2.2.2. *Let X be a connected, smooth and projective k-scheme. Then $\pi^S(X, x_1) \cong \pi^S(X, x_2)$, for all $x_1, x_2 \in X(k)$.*

Proof. Since $\pi^S(X, x)$ is the affine k-group scheme representing the functor of k-algebras $\underline{\mathrm{Aut}}^{\otimes}(T_x)$, where T_x is the fiber functor $T_x : \mathrm{Vect}_0^s(X) \longrightarrow \mathrm{Vect}_k$, it suffices to show that, for any two points $x_1, x_2 \in X(k)$, the fiber functors T_{x_1} and T_{x_2} are isomorphic. Given any object $\mathcal{V} \in \mathrm{Vect}_0^s(X)$, we need to define a *natural* k-linear isomorphism

$$\eta_{\mathcal{V}} : T_{x_1}(\mathcal{V}) = \mathcal{V}_{x_1} \longrightarrow \mathcal{V}_{x_2} = T_{x_2}(\mathcal{V})\,;$$

meaning that for any morphism $f : \mathcal{V} \to \mathcal{V}'$ of objects in $\mathrm{Vect}_0^s(X)$, the following diagram should commute.

(2.2.3)
$$\begin{tikzcd} T_{x_1}(\mathcal{V}) \arrow[r, "T_x(f)"] \arrow[d, "\eta_{\mathcal{V}}"] & T_{x_1}(\mathcal{V}') \arrow[d, "\eta_{\mathcal{V}'}"] \\ T_{x_2}(\mathcal{V}) \arrow[r, "T_y(f)"] & T_{x_2}(\mathcal{V}') \end{tikzcd}$$

For any group scheme H over k, denote by $\mathrm{Rep}_k(H)$ the category of representations of H into finite dimensional k-vector spaces. Let $G = \pi^S(X, x_1)$. Then there is an equivalence of categories $\zeta : \mathrm{Vect}_0^s(X) \xrightarrow{\sim} \mathrm{Rep}_k(G)$, and the inverse of this equivalence of categories defines a principal G-bundle $p : P \to X$, (see [Nor76, Proposition 2.9] for the construction), known as the *S-universal cover of X* (see [Lan11, p. 2097]). This associates to a G-module V an object $\mathcal{V} := P \times^G V$ in the category $\mathrm{Vect}_0^s(X)$; moreover, any morphism $\mathcal{V} \to \mathcal{V}'$ in the category $\mathrm{Vect}_0^s(X)$ comes from a G-module homomorphism $V \to V'$ in $\mathrm{Rep}_k(G)$.

Fix two points $\tilde{x}_1, \tilde{x}_2 \in P$ such that $p(\tilde{x}_i) = x_i$, for $i = 1, 2$. Then we have natural isomorphisms

$$\xi_i : G \longrightarrow P_{x_i}, \qquad i = 1, 2\,.$$

Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional linear representation, and let $\mathcal{V} := P \times^G V$ be the associated vector bundle on X. Then we have canonical linear isomorphism

$$\mathcal{V}_{x_i} = P_{x_i} \times^G V \overset{\xi_i}{\cong} G \times^G V \longrightarrow V\,,\ \forall\ i = 1, 2.$$

This gives a k-linear isomorphism of the fibers

$$\eta_{\mathcal{V}} : \mathcal{V}_{x_1} \longrightarrow \mathcal{V}_{x_2}\,.$$

Since any homomorphism $f : \mathcal{V} \to \mathcal{V}'$ of objects in $\mathrm{Vect}_0^s(X)$ comes from a G-module homomorphism $\tilde{f} : V \to V'$, it follows from above construction that the above diagram in (2.2.3) commutes. ∎

3. Hilbert-Chow Morphism

3.1. **Definition.** From now on, we denote by X an irreducible smooth projective surface over k. For an integer $n \geq 1$, let S_n be the permutation group of n symbols. Then S_n acts on the product X^n, and the associated quotient $S^n(X) = X^n/S_n$ is a normal projective variety of dimension $2n$ over k. Note that $S^n(X)$ is not smooth, and its smooth locus $S^n(X)_{\text{sm}} \subset S^n(X)$ is the open dense subscheme consisting of reduced effective 0-cycles of length n in X. Since $\dim_k(X) = 2$, the singular locus $S^n(X)_{\text{sing}} := S^n(X) \setminus S^n(X)_{\text{sm}}$ is a closed subscheme of codimension 2 in $S^n(X)$.

Let $\mathcal{H}ilb^n_X$ be the Hilbert scheme parametrizing effective 0-cycles of length n in X. This is an irreducible smooth projective scheme of dimension $2n$ over k. Consider the Hilbert-Chow morphism

$$\varphi : \mathcal{H}ilb^n_X \longrightarrow S^n(X) \tag{3.1.1}$$

given by sending $Z \in \mathcal{H}ilb^n_X$ to

$$\sum_{p \in \text{Supp}(Z)} \ell(\mathcal{O}_{Z,p})[p] \in S^n(X),$$

where

$$\text{Supp}(Z) = \{p \in X : \mathcal{O}_{Z,p} \neq 0\}$$

denotes the support of the 0-cycle Z in X, and $\ell(\mathcal{O}_{Z,p})$ the length of the local ring $\mathcal{O}_{Z,p}$ as a module over itself.

3.2. **Stratification of $S^n(X)$.** A point $y \in S^n(X)$ can be written as

$$\sum_{j=1}^{r} n_j x_j,$$

where $x_1, \ldots, x_r \in X$ are distinct points with multiplicities

$$n_1 \geq n_2 \geq \cdots \geq n_r \in \mathbb{Z}_{>0}, \tag{3.2.1}$$

respectively, such that $\sum_{j=1}^{r} n_j = n$. The r-tuple of positive integers

$$\langle n_1, n_2, \ldots, n_r \rangle$$

is called the *type* of y. Let $Z_{\langle n_1, n_2, \ldots, n_r \rangle}$ denote the locus of points in $S^n(X)$ of type $\langle n_1, n_2, \ldots, n_r \rangle$. The fiber $\varphi^{-1}(y)$ has dimension $n - r$, for all $y \in Z_{\langle n_1, n_2, \ldots, n_r \rangle}$ (see [Fog73, p. 667]). The dimension of the locus of points of type $\langle n_1, n_2, \ldots, n_r \rangle$ is $2r$. From this the following lemma follows.

Lemma 3.2.2. *The dimension of the subset $\varphi^{-1}(Z_{\langle n_1, n_2, \ldots, n_r \rangle})$ is $n + r$.*

3.3. **Fibers of Hilbert-Chow morphism.** Let $W \subset S^n(X)$ denote the open subset consisting of points of type $\langle 1,1,1,\ldots,1\rangle$ and $\langle 2,1,1,\ldots,1\rangle$. Let V denote the open subset $\varphi^{-1}(W)$ and let

$$\varphi : V \longrightarrow W \tag{3.3.1}$$

be the restriction of the morphism φ in (3.1.1) to V. It follows from Lemma 3.2.2 that the dimension of $\mathcal{H}ilb^n_X \setminus V$ is $n+n-2 = 2n-2$, and hence $\mathrm{codim}_{\mathcal{H}ilb^n_X}(\mathcal{H}ilb^n_X \setminus V) = 2$.

It was shown in [Fog73, Lemma 4.3, p. 668] that for any point $q \in S^n(X)$ of type $\langle 2,1,1,\ldots,1\rangle$, the schematic fiber $\varphi^{-1}(q)$, with its reduced structure, is isomorphic to $\mathbb{P}^1_k$. We need that $\varphi^{-1}(q)$ is reduced. We could not find a precise reference for Proposition 3.3.3, which maybe well-known to experts, and so we include a proof.

First we recall the following well-known result.

Lemma 3.3.2. *Let I be an ideal of a commutative ring A with identity. Let $A[It] := \bigoplus_{i=0}^{\infty} I^i t^i \subset A[t]$ be the Rees algebra of I in the polynomial ring $A[t]$. Let $\pi : \mathrm{Proj}(A[It]) \to \mathrm{Spec}\, A$ be the associated projective A-scheme. For an A-algebra B, consider the graded A-algebra structure on $A[It] \otimes_A B$ given by $(A[It] \otimes_A B)_d := (I^d \otimes_A B)t^d$, for all $d \geq 0$. Then we have a canonical isomorphism of A-schemes*

$$\psi : \mathrm{Proj}(A[It] \otimes_A B) \xrightarrow{\simeq} \mathrm{Proj}(A[It]) \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B\,.$$

Proof. Follows from [Stk, Lemma 26.11.6., Tag 01MX]. □

Proposition 3.3.3. *Assume that $\mathrm{char}(k) \neq 2$. Let $q \in W$ be a point of type $\langle 2,1,1,\ldots,1\rangle$. The scheme theoretic fiber $\varphi^{-1}(q)$ is a reduced subscheme of V.*

Proof. Let $\tilde{q} \in X^n$ be a point such that $\tilde{q} \mapsto q$ under the natural map $\psi : X^n \to S^n(X)$. The formal neighbourhood of $\tilde{q}$ is given by the spectrum of the local ring

$$\widehat{\mathcal{O}}_{X^n,\tilde{q}} = k[[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]]\,.$$

There is an inclusion $\widehat{\mathcal{O}}_{W,q} \hookrightarrow \widehat{\mathcal{O}}_{X^n,\tilde{q}}$. By the discussion in the paragraph just before [FGI+05, Theorem 7.3.4, p. 170], we have

$$\widehat{\mathcal{O}}_{W,q} = k[[u, v, w, x', y', x_3, y_3, \ldots, x_n, y_n]]/(uw - v^2)\,,$$

where $x = x_1 - x_2$, $y = y_1 - y_2$, $x' = x_1 + x_2$, $y' = y_1 + y_2$, $u = x^2$, $v = xy$ and $w = y^2$. Here we are using the assumption $\mathrm{char}(k) \neq 2$.

Let $Z \subset W$ denote the irreducible closed subset consisting of points of type $\langle 2,1,1,\ldots,1\rangle$. Let J denote the stalk at q of the ideal sheaf of Z in the local ring $\mathcal{O}_{W,q}$, and let $\widehat{J}$ denote its image in $\widehat{\mathcal{O}}_{W,q}$. Now Z is contained in the image $\psi(X^{n-1}) \subset S^n(X)$, where the inclusion $X^{n-1} \hookrightarrow X^n$ is given by

$$(x, x_3, x_4, \ldots, x_n) \longmapsto (x, x, x_3, \ldots, x_n)\,.$$

Clearly, the ideal of X^{n-1} in $\widehat{\mathcal{O}}_{X^n,\tilde{q}}$ is given by $x_1 - x_2 = y_1 - y_2 = 0$. From this, we conclude that $\widehat{J}$ is the kernel of the composite homomorphism

$$\widehat{\mathcal{O}}_{W,q} \hookrightarrow \widehat{\mathcal{O}}_{X^n,\tilde{q}} \twoheadrightarrow \widehat{\mathcal{O}}_{X^n,\tilde{q}}/(x, y)\,,$$

where $x = x_1 - x_2$ and $y = y_1 - y_2$. This proves that $\widehat{J} = (u, v, w)$.

By [Fog73, Lemma 4.4] the map φ is the blowup of W along Z. Let $\mathcal{O}_{W,q}[tJ]$ denote the Rees algebra of the ideal J. By Lemma 3.3.2, the schematic fiber $\varphi^{-1}(q)$ is

$$\text{Proj}\ (\mathcal{O}_{W,q}[tJ]) \times_{\text{Spec}(\mathcal{O}_{W,q})} \text{Spec}(\mathcal{O}_{W,q}/\mathfrak{m}_q) \cong \text{Proj}\ (\mathcal{O}_{W,q}[tJ] \otimes_{\mathcal{O}_{W,q}} (\mathcal{O}_{W,q}/\mathfrak{m}_q))\,,$$

where $\mathfrak{m}_q$ is the maximal ideal of the local ring $\mathcal{O}_{W,q}$ at q. It follows from the isomorphism

$$\begin{aligned}\mathcal{O}_{W,q}[tJ] \otimes_{\mathcal{O}_{W,q}} (\mathcal{O}_{W,q}/\mathfrak{m}_q) &\cong \mathcal{O}_{W,q}[tJ] \otimes_{\mathcal{O}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q)\\ &\cong \widehat{\mathcal{O}}_{W,q}[t\widehat{J}] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q)\end{aligned}$$

that the schematic fiber $\varphi^{-1}(q)$ is

$$\text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[t\widehat{J}] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q))\,.$$

Denote $I := (u, w) \subset \widehat{\mathcal{O}}_{W,q}$ and observe that $\widehat{J}^2 \subset I \subset \widehat{J} \subset \widehat{\mathfrak{m}}_q$. Then we have the following isomorphisms (see Lemma 3.3.2)

$$\begin{aligned}\text{Proj}(\mathcal{O}_{W,q}[tJ] \otimes_{\mathcal{O}_{W,q}} (\mathcal{O}_{W,q}/\mathfrak{m}_q)) &\cong \text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[t\widehat{J}] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q))\\ &\cong \text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[t\widehat{J}]) \times_{\text{Spec}(\widehat{\mathcal{O}}_{W,q})} \text{Spec}(\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q)\\ &\cong \text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[tI]) \times_{\text{Spec}(\widehat{\mathcal{O}}_{W,q})} \text{Spec}(\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q)\\ &\cong \text{Proj}(\widehat{\mathcal{O}}_{W,q}[tI] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/I)) \times_{\text{Spec}(\widehat{\mathcal{O}}_{W,q}/I)} \text{Spec}(\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q).\end{aligned}$$

The ring $\widehat{\mathcal{O}}_{W,q}$ is Cohen-Macaulay by [Har77, Chapter II, Theorem 8.21A (d)]. Now (u, w) is a regular sequence for $\widehat{\mathfrak{m}}_q$. Applying [Har77, Chapter II, Theorem 8.21A (e)], we have

$$\text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[tI] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/I)) \cong \mathbb{P}^1_{\mathbb{Z}} \times \text{Spec}\ (\widehat{\mathcal{O}}_{W,q}/I)\,.$$

This proves that

$$\text{Proj}\ (\widehat{\mathcal{O}}_{W,q}[t\widehat{J}] \otimes_{\widehat{\mathcal{O}}_{W,q}} (\widehat{\mathcal{O}}_{W,q}/\widehat{\mathfrak{m}}_q)) \cong \mathbb{P}^1_{\mathbb{Z}} \times \text{Spec}\ (\widehat{\mathcal{O}}_{W,q}/\widehat{m}_q) \cong \mathbb{P}^1_k\,.$$

Thus we have proved that the scheme theoretic fiber $\varphi^{-1}(q) \cong \mathbb{P}^1_k$, which proves that it is reduced. $\square$

4. Homomorphism of S-fundamental Group Schemes

In this section we construct a homomorphism of S-fundamental group schemes

$$\pi^S(X, x)_{\text{ab}} \longrightarrow \pi^S(\mathcal{H}ilb^n_X, n[x])\,,$$

where $\pi^S(X, x)_{\text{ab}}$ is the abelianization of $\pi^S(X, x)$.

4.1. **A group theoretic lemma.** We need the following group theoretic result for later use. See also [Lan12, Lemma 5.7].

Lemma 4.1.1. *Let G and H be two group schemes over k. We denote by G^n the group scheme $G \times \cdots \times G$ (= the n-fold product of G with itself). The symmetric group of n symbols S_n acts on G^n by*

$$(g_1, \cdots, g_n) \overset{\sigma}{\longmapsto} (g_{\sigma(1)}, \cdots, g_{\sigma(n)}), \quad \forall\, \sigma \in S_n,$$

for all $(g_1, \cdots, g_n) \in G^n$. Let $f : G^n \longrightarrow H$ be a homomorphism of k-group schemes such that $f \circ \sigma = f$, for all $\sigma \in S_n$. Then the homomorphism f factors as

$$\begin{tikzcd} G^n \arrow[rr, "f"] \arrow[dr, "f_0"'] & & H \\ & G_{\mathrm{ab}} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

where f_0 is the composite group homomorphism

$$f_0 : G^n \xrightarrow{\alpha^n} (G_{\mathrm{ab}})^n \xrightarrow{m} G_{\mathrm{ab}},$$

where $\alpha : G \to G_{\mathrm{ab}} := G/[G, G]$ denotes the abelianization homomorphism and m denotes the multiplication homomorphism.

Proof. For any k-group scheme G, we denote by

- $m_G : G \times G \longrightarrow G$ the multiplication morphism of G,
- $i_G : G \longrightarrow G$ the inversion morphism of G, and
- $e_G \in G(k)$ the identity of G.

We sketch the proof for $n = 2$; the general case is similar and left to the reader as an exercise. We have a homomorphism $f : G \times G \longrightarrow H$ such that $f \circ \sigma = f$, where $\sigma : G \times G \to G \times G$ is the homomorphism switching the factors. Let $p_1, p_2 : G \times G \to G$ denote the projections onto the first and second factors, respectively. Then one can easily check that

$$\begin{aligned} f \circ (m_G, e_G) &= f \circ m_{G\times G} \circ ((p_1, e), (p_2, e)) \\ &= m_H \circ (f \circ (p_1, e), f \circ (p_2, e)) \\ &= m_H \circ (f \circ (p_1, e), f \circ (e, p_2)) \\ &= f \circ (p_1, p_2) \\ &= f \circ (m_G \circ \sigma, e_G)\,. \end{aligned}$$

Using this it easily follows that

$$f \circ m_{G\times G}((m_G, e_G), (i_G \circ m_G \circ \sigma, e_G)) = e_H\,. \tag{4.1.2}$$

Now one easily concludes that f factors through the map $G \times G \to G_{\mathrm{ab}} \times G_{\mathrm{ab}}$. Let $\Delta' : G \to G \times G$ denote the map $g \mapsto (g, g^{-1})$. Then one checks easily that $f \circ \Delta' = e_H$. From these the lemma follows. □

4.2. **Construction of the homomorphism.** Given a numerically flat vector bundle E on $\mathcal{H}ilb^n_X$, we want to associate to it a numerically flat vector bundle $\mathcal{G}$ on X^n. We first associate to E a reflexive sheaf $\mathcal{G}$ on X^n, and then use the criterion in [Lan11, Theorem 5.1] to show that $\mathcal{G}$ is locally free and numerically flat.

Proposition 4.2.1. *Let E be a numerically flat vector bundle of rank r on $\mathcal{H}ilb^n_X$. Then $\varphi_*(E|_V)$ is a locally free coherent sheaf on W. Moreover, the natural map*

$$\varphi^*\varphi_*(E|_V) \longrightarrow E|_V \tag{4.2.2}$$

is an isomorphism.

Proof. Let $q \in W$ be a point of type $\langle 2, 1, 1, \ldots, 1\rangle$. Let $\mathcal{I} \subset \mathcal{O}_V$ denote the reduced sheaf of ideals of the closed subscheme $\varphi^{-1}(q)$. Let $\mathscr{I}_q$ be the ideal sheaf of the closed point $q \in W$, and for each integer $n \geq 1$, let $\mathscr{I}_q^n$ be the ideal sheaf of the n-th order thickening of q in W. By Proposition 3.3.3 we have

$$\mathcal{I} = \mathscr{I}_q \mathcal{O}_V .$$

For each integer $n \geq 1$, let Y_n denote the closed subscheme of V corresponding to the sheaf of ideals $\mathcal{I}^n$. Since E is numerically flat and $Y_1 \cong \mathbb{P}^1_k$ (see Proposition 3.3.3), it follows that the restriction of E to Y_1 is trivial.

Consider the following short exact sequence of sheaves on V

$$0 \longrightarrow \mathcal{I} \otimes E \longrightarrow E \longrightarrow E|_{Y_1} \longrightarrow 0. \tag{4.2.3}$$

Applying φ_* to it we get the following exact sequence of sheaves on W.

$$\varphi_*(E) \longrightarrow H^0(Y_1, E|_{Y_1}) \longrightarrow R^1\varphi_*(\mathcal{I} \otimes E). \tag{4.2.4}$$

We claim that the completion of $R^1\varphi_*(\mathcal{I} \otimes E)$ at the maximal ideal $\mathfrak{m}_q$ of q is 0. By the Theorem on Formal Functions (see [Har77, Chapter III, Theorem 11.1]), we have

$$(R^1\varphi_*(\mathcal{I} \otimes E))\hat{} \cong \varprojlim H^1(Y_n, \mathcal{I} \otimes E \otimes \mathcal{O}_V/\mathcal{I}^n). \tag{4.2.5}$$

We will prove by induction on n that $H^1(Y_n, \mathcal{I} \otimes E \otimes \mathcal{O}_V/\mathcal{I}^n) = 0$. Since $\mathcal{I} = \mathscr{I}_q\mathcal{O}_V$, it follows that there is a surjection

$$(\mathfrak{m}_q^n/\mathfrak{m}_q^{n+1}) \otimes_{\mathcal{O}_{W,q}} \mathcal{O}_V \cong \mathscr{I}_q^n/\mathscr{I}_q^{n+1} \otimes_{\mathcal{O}_W} \mathcal{O}_V \twoheadrightarrow \mathcal{I}^n/\mathcal{I}^{n+1}.$$

The locally free sheaf $\mathcal{I}^n/\mathcal{I}^{n+1}$ on $Y_1 \cong \mathbb{P}^1$ is a direct sum of line bundles. It follows that each of these line bundle has degree ≥ 0. For $n = 1$, the base case of induction, we have

$$H^1(Y_1, \mathcal{I} \otimes E \otimes \mathcal{O}_V/\mathcal{I}) = H^1(Y_1, \mathcal{I}/\mathcal{I}^2 \otimes E_1) = 0\,.$$

Assume that we have proved the assertion for n. Then the assertion for $n+1$ follows from the long exact cohomology sequence attached to the short exact sequence of sheaves on Y_{n+1}

$$0 \longrightarrow (\mathcal{I}^{n+1}/\mathcal{I}^{n+2}) \otimes E \longrightarrow (\mathcal{I}/\mathcal{I}^{n+2}) \otimes E \longrightarrow (\mathcal{I}/\mathcal{I}^{n+1}) \otimes E \longrightarrow 0\,.$$

This proves the claim that $R^1\varphi_*(\mathcal{I} \otimes E)$ at the maximal ideal $\mathfrak{m}_q$ of q is 0.

This proves that the natural map

$$\varphi_*(E) \longrightarrow H^0(Y_1, E|_{Y_1}) \tag{4.2.6}$$

in (4.2.4) is surjective in a neighborhood around q. Let $s_1, s_2 \dots, s_r$ be a basis for $H^0(Y_1, E|_{Y_1})$. Let $\mathrm{Spec}(A)$ be an affine neighborhood where the map in (4.2.6) is surjective. Choosing lifts $\tilde{s}_i \in \Gamma(\mathrm{Spec}(A), \varphi_*(E))$ of s_i, we get a homomorphism

$$\mathcal{O}_V^{\oplus r} \longrightarrow E \tag{4.2.7}$$

over $\varphi^{-1}(\mathrm{Spec}(A))$, which is a surjection over the fiber Y_1. Since φ is proper, it follows that there is a smaller affine neighborhood W_0 of q over which there is an isomorphism $\mathcal{O}_{V_0}^{\oplus r} \xrightarrow{\sim} E$, where $V_0 = \varphi^{-1}(W_0)$. Applying φ_*, using normality of $S^n(X)$ and that φ is birational, the Proposition follows. ☐

Corollary 4.2.8. *Let F denote the absolute Frobenius morphism. With the above notations, we have an isomorphism $F^*\varphi_*(E|_V) \xrightarrow{\sim} \varphi_*(F^*E|_V)$.*

Proof. Since F^*E is numerically flat, it follows that both these sheaves are locally free of the same rank. It suffices to show that the natural map

$$F^*\varphi_*(E|_V) \longrightarrow \varphi_*(F^*E|_V) \tag{4.2.9}$$

is surjective. This is clear over the smooth locus of $S^n(X)$ since F is faithfully flat over the smooth locus. Let $q \in W$ be a point of type $\langle 2, 1, 1 \dots, 1\rangle$. It follows from Proposition 3.3.3 that the restriction of $F^*\varphi_*(E|_V)$ to q is naturally isomorphic to $H^0(Y_1, E_1)$, and the restriction of $\varphi_*(F^*E|_V)$ at q is naturally isomorphic to $H^0(Y_1, F^*E_1)$. The restriction to q of the natural homomorphism in (4.2.9) is the map

$$F^* : H^0(Y_1, E_1) \longrightarrow H^0(Y_1, F^*E_1)\,,$$

which is a surjection. From this the Corollary follows. ☐

Recall the quotient map $\psi : X^n \longrightarrow S^n(X)$ defined in (3.1.1). Let $j : \psi^{-1}(W) \hookrightarrow X^n$ denote the inclusion. Recall the definition of the category $\mathcal{C}^{nf}(X)$ from Definition 2.1.1.

Proposition 4.2.10. *If E is an object of $\mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$, then*

$$\mathscr{G}(E) := (j_*\psi^*\varphi_*(E|_V))^{\vee\vee}$$

is an object of $\mathcal{C}^{nf}(X^n)$.

Proof. It is proved in Proposition 4.2.1 that $\varphi_*(E)$ is locally free on W. For notational simplicity, here we denote by $\mathcal{G}$ the sheaf $\mathscr{G}(E)$. Since $X^n \setminus \psi^{-1}(W)$ has codimension ≥ 4, it follows that

$$\mathcal{G} := (j_*\psi^*\varphi_*(E|_V))^{\vee\vee} \tag{4.2.11}$$

is a coherent reflexive sheaf on X^n. Note that $\mathcal{G}|_{\psi^{-1}(W)} = \psi^*\mathcal{E}$ is locally free.

Choose $m \gg 0$ so that mH is very ample. Choose general hyperplanes $H_1, \dots, H_{d-1} \in |mH|$ so that $C = H_1 \cap H_2 \cap \dots \cap H_{d-1} \xhookrightarrow{i} \psi^{-1}(W)$ is a smooth complete intersection curve whose image $\psi(C)$ lies in the smooth locus of $S^n(X)$. We can lift i to a morphism

$\tilde{i}$ which makes the following diagram commute.

$$\begin{tikzcd} & & V \arrow[r, hook] \arrow[d, "\varphi"] & \mathcal{H}ilb^n_X \arrow[d, "\varphi"] \\ C \arrow[r, hook, "i"] \arrow[urr, bend left, "\tilde{i}"] & \psi^{-1}(W) \arrow[r, "\psi"] & W \arrow[r, hook] & S^n(X) \end{tikzcd}$$

It follows from Proposition 4.2.1 that

$$i^*\mathcal{G} \cong \tilde{i}^*\varphi^*\varphi_*(E|_V) \cong \tilde{i}^*(E|_V)\,.$$

Since E is in $\mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$ it follows that $i^*\mathcal{G}$ is semistable of degree 0. This shows that $\mathcal{G}$ is H-semistable.

It follows from Corollary 4.2.8 that the locally free sheaves $F^*\varphi_*(E|_V)$ and $\varphi_*(F^*E|_V)$ are isomorphic. Since X^n is smooth, the Frobenius is faithfully flat and so $F^*\mathcal{G}$ is reflexive. The restriction of $F^*\mathcal{G}$ on $\psi^{-1}(W)$ is

$$F^*\psi^*\varphi_*(E|_V) \cong \psi^*F^*\varphi_*(E|_V) \cong \psi^*\varphi_*(F^*E|_V).$$

Since the reflexive extension on X^n is unique (see [Har80, Proposition 1.6, p. 126]), we conclude that

$$F^*\mathcal{G} \cong (j_*(\psi^*\varphi_*(F^*E|_V)))^{\vee\vee}\,.$$

Since $E \in \mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$, we have $F^*E \in \mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$; then following the arguments in the preceding paragraph, we see that $F^*\mathcal{G}$ is H-semistable. This shows that $\mathcal{G}$ is strongly H-semistable.

It is clear from above that $\mathrm{ch}_1(\mathcal{G})\cdot H^{d-1} = 0$. Choose general hyperplanes $H_1,\ldots,H_{d-2}$ in the linear system $|mH|$ so that

$$S = H_1 \cap H_2 \cap \cdots \cap H_{d-2} \subset \psi^{-1}(W)$$

is a smooth surface. We can do this since $X^n \setminus \psi^{-1}(W)$ has codimension ≥ 4. It suffices to show that $\mathrm{ch}_2(\mathcal{G}|_S) = 0$. Now $\mathcal{G}|_S$ is locally free as $S \subset \psi^{-1}(W)$ and $\mathcal{G}$ is locally free on $\psi^{-1}(W)$. Therefore, in view of [Lan12, Theorem 2.2], it suffices to show that $\mathcal{G}|_S \in \mathcal{C}^{nf}(S)$. But this follows from the arguments as in the second paragraph of this proof. Therefore, we have $\mathcal{G} \in \mathrm{Vect}^s_0(X^n)$, and hence by [Lan11, Theorem 5.1] $\mathcal{G}$ is locally free and is in $\mathcal{C}^{nf}(X^n)$. This proves the Proposition. □

Proposition 4.2.12. *With the above notations,*

$$\mathscr{G} : \mathcal{C}^{nf}(\mathcal{H}ilb^n_X) \longrightarrow \mathcal{C}^{nf}(X^n)$$

is a additive tensor functor.

Proof. Let $f : E \to E'$ be a morphism in the category $\mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$. We need to find a canonical morphism $\mathscr{G}(f) : \mathscr{G}(E) \to \mathscr{G}(E')$ in $\mathcal{C}^{nf}(X^n)$. There is a morphism $\psi^*\varphi_*(f) : \mathscr{G}(E)|_{\psi^{-1}(W)} \to \mathscr{G}(E')|_{\psi^{-1}(W)}$. Since $X^n \setminus \psi^{-1}(W)$ has codimension ≥ 4 and $\mathscr{G}(E)$, $\mathscr{G}(E')$ are locally free, it follows that this morphism extends uniquely to give a morphism $\mathscr{G}(E) \to \mathscr{G}(E')$.

The bundles $\mathscr{G}(E \oplus E')$ and $\mathscr{G}(E) \oplus \mathscr{G}(E')$ are naturally isomorphic on $\psi^{-1}(W)$ and so they are naturally isomorphic. Similarly, $\mathscr{G}(E \otimes E')$ is naturally isomorphic to $\mathscr{G}(E) \otimes \mathcal{G}(E')$. $\square$

Fix distinct k-valued points $x_1, \dots, x_n \in X(k)$ of X. Let $\tilde{x} \in \mathcal{H}ilb_X^n(k)$ be such that $\varphi(\widetilde{x}) = \psi(x_1, \cdots, x_n) \in S^n(X)_{\mathrm{sm}}$. For any locally free sheaf E on $\mathcal{H}ilb_X^n$, there are natural isomorphisms

$$E_{\tilde{x}} \cong (\varphi_* E)_{\varphi(\tilde{x})} \cong (\psi^* \varphi_*(E))_{(x_1, x_2, \dots, x_n)}.$$

Consider the following diagram.

$$\begin{array}{ccc}
(\mathcal{C}^{nf}(\mathcal{H}ilb_X^n), \otimes, T_{\tilde{x}}, \mathcal{O}_{\mathcal{H}ilb_X^n}) & \longrightarrow & (\mathcal{C}^{nf}(X^n), \otimes, T_{(x_1,\dots,x_n)}, \mathcal{O}_{X^n}) \\
\uparrow & & \downarrow \\
(\mathcal{C}^{nf}(\mathcal{H}ilb_X^n), \otimes, T_{n[x]}, \mathcal{O}_{\mathcal{H}ilb_X^n}) & & (\mathcal{C}^{nf}(X^n), \otimes, T_{(x,\dots,x)}, \mathcal{O}_{X^n})
\end{array}$$

The horizontal arrow is a morphism of Tannakian categories coming from Proposition 4.2.10. The two vertical arrows are due to Lemma 2.2.2. Thus, we get a homomorphism of S-fundamental group schemes

$$f : \pi^S(X^n, (x, \dots, x)) \longrightarrow \pi^S(\mathcal{H}ilb_X^n, n[x]) .$$

For $\sigma \in S_n$ we get an automorphism σ_* of $\pi^S(X^n, (x, \dots, x))$. It is easily checked that $f \circ \sigma_* = f$. By [Lan12, Theorem 4.1, p. 842] there is an isomorphism

$$\pi^S(X^n, (x, \dots, x)) \longrightarrow \pi^S(X, x) \times_k \cdots \times_k \pi^S(X, x).$$

By abuse of notation, denote the composite of f and the inverse of this isomorphism by f. Thus, we have a homomorphism

$$f : \pi^S(X, x) \times_k \cdots \times_k \pi^S(X, x) \to \pi^S(\mathcal{H}ilb_X^n, n[x]) \tag{4.2.13}$$

which satisfies $f \circ \sigma_* = f$. It follows from Lemma 4.1.1 that the homomorphism of the S-fundamental group schemes in (4.2.13) factors through a homomorphism

$$\tilde{f} : \pi^S(X, x)_{\mathrm{ab}} \longrightarrow \pi^S(\mathcal{H}ilb_X^n, n[x]) . \tag{4.2.14}$$

This completes the construction of our homomorphism of k-group schemes.

Corollary 4.2.15. *Any vector bundle in the category* $\mathrm{Vect}_0^s(X^n)$ *associated to a representation of* $\pi^S(X, x)_{\mathrm{ab}}$ *admits an* S_n*-equivariance structure over* X^n.

5. Isomorphism of Group Schemes

5.1. **Faithfully flatness.** In this section we use [DMOS82, Proposition 2.21] to show that the homomorphism $\tilde{f}$ in (4.2.14) is an isomorphism. We begin by recalling this result for the convenience of the reader. Let $\theta : G \longrightarrow G'$ be a homomorphism of affine group schemes over k, and let

$$\widetilde{\theta} : \mathrm{Rep}_k(G') \longrightarrow \mathrm{Rep}_k(G) \tag{5.1.1}$$

be the functor given by sending $\rho' : G' \to \mathrm{GL}(V)$ to $\rho' \circ \theta : G \to \mathrm{GL}(V)$. An object $\rho : G \to \mathrm{GL}(V)$ in $\mathrm{Rep}_k(G)$ is said to be a *subquotient* of an object $\eta : G \to \mathrm{GL}(W)$ in $\mathrm{Rep}_k(G)$ if there are two G-submodules $V_1 \subset V_2$ of W such that $V \cong V_2/V_1$ as G-modules.

Proposition 5.1.2 (Proposition 2.21, [DMOS82]). *Let $\theta : G \longrightarrow G'$ be a homomorphism of affine algebraic groups over k. Then*

(a) θ is faithfully flat if and only if the functor $\widetilde{\theta}$ in (5.1.1) is fully faithful and given any subobject $W \subset \widetilde{\theta}(V')$, with $V' \in \mathrm{Rep}_k(G')$, there is a subobject $W' \subset V'$ in $\mathrm{Rep}_k(G')$ such that $\widetilde{\theta}(W') \cong W$ in $\mathrm{Rep}_k(G)$.
(b) f is a closed immersion if and only if every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of an object of the form $\widetilde{\theta}(V')$, for some $V' \in \mathrm{Rep}_k(G)$.

Now we have the following.

Proposition 5.1.3. *The homomorphism*

$$\tilde{f} : \pi^S(X, x)_{\mathrm{ab}} \to \pi^S(\mathcal{H}ilb^n_X, n[x])$$

defined in (4.2.14) is faithfully flat.

Proof. We will apply [DMOS82, Proposition 2.21 (a)]. Let E_1 be an object in the category $\mathrm{Vect}^s_0(\mathcal{H}ilb^n_X) = \mathcal{C}^{nf}(\mathcal{H}ilb^n_X)$. Let $\mathcal{G}_1$ be the vector bundle on X^n associated to E as defined in (4.2.11). Clearly $\mathcal{G}_1$ has the same rank as that of E_1. If $\mathcal{G}_2 \subset \mathcal{G}_1$ is a subbundle corresponding to a representation of $\pi^S(X, x)_{\mathrm{ab}}$, we need to show that there is a subbundle $E_2 \subset E_1$ such that the bundle associated to E_2 is $\mathcal{G}_2$. For this, we use induction on the rank of E_1. If $\mathrm{rank}(E_1) = 1$, there is nothing to prove. Assume that $\mathrm{rank}(E_1) = r + 1$, with $r \geq 1$.

The vector bundles $\mathcal{G}_i$ corresponds to a representation

$$\pi^S(X^n, (x, \dots, x)) \xrightarrow{f_0} \pi^S(X, x)_{\mathrm{ab}} \xrightarrow{\rho_i} \mathrm{GL}(V_i).$$

Since $\pi^S(X, x)_{\mathrm{ab}}$ is an abelian k-group scheme, it follows from [Wat79, Theorem 9.4, p. 70], that we can find a surjective $\pi^S(X, x)_{\mathrm{ab}}$-module homomorphism $V_1 \to L_1$, where L_1 is one dimensional and V_2 is a $\pi^S(X, x)_{\mathrm{ab}}$-submodule of the kernel of this homomorphism. Let $\mathcal{L}$ be the line bundle on X^n corresponding to the representation L_1. Then it is clear that $\mathcal{L}$ is S_n-equivariant (see Corollary 4.2.15) and there is an S_n-equivariant exact sequence of bundles

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{L} \longrightarrow 0$$

on X^n such that $\mathcal{G}_2 \subset \mathcal{K}$.

Since every S_n-equivariant line bundle on X^n is the pullback of a line bundle from $S^n(X)$, it follows that $L' := \psi_*(\mathcal{L})^{S_n}$ is a line bundle on all of $S^n(X)$. Let $L := \varphi^* L'$. Then L is a line bundle on $\mathcal{H}ilb^n_X$. Let us check that L is numerically flat on $\mathcal{H}ilb^n_X$. Given a morphism $C \longrightarrow \mathcal{H}ilb^n_X$ from a curve C into $\mathcal{H}ilb^n_X$, we can find a curve $\widetilde{C}$ and a morphism $\widetilde{C} \longrightarrow C$ making the following diagram commutative.

$$\begin{tikzcd} & & & X^n \arrow[d, "\psi"] \\ \widetilde{C} \arrow[r] \arrow[rrru, bend left] & C \arrow[r] & \mathcal{H}ilb^n_X \arrow[r, "\varphi"] & S^n(X) \end{tikzcd}$$

Since $\mathcal{L}$ is numerically flat on X^n, it follows from $\psi^*L' = \mathcal{L}$ and $L = \varphi^*L'$, that L is numerically flat.

We claim that

$$0 \to \psi_*(\mathcal{K})^{S_n}\Big|_W \to \psi_*(\mathcal{G}_1)^{S_n}\Big|_W \to \psi_*(\mathcal{L})^{S_n}\Big|_W \to 0 \tag{5.1.4}$$

is exact. The sequence (5.1.4) can fail to be exact only on the right. Note that $\psi_*(\mathcal{G}_1)^{S_n}$ restricted to W is $\varphi_*(E_1|_V)$. Let J be the cokernel:

$$\varphi_*(E_1|_V) \to L'\Big|_W \to J \to 0\,.$$

Pulling this back by ψ we get the following commutative diagram on $\psi^{-1}(W)$ with exact rows.

$$\begin{tikzcd} \psi^*\varphi_*(E_1|_V) \arrow[r] \arrow[d, equal] & \psi^*L'\Big|_{\psi^{-1}(W)} \arrow[r] \arrow[d, equal] & \psi^*J \arrow[r] & 0 \\ \mathcal{G}_1\Big|_{\psi^{-1}(W)} \arrow[r] & \mathcal{L}\Big|_{\psi^{-1}(W)} \arrow[r] & 0 & \end{tikzcd}$$

This shows that $\psi^*J = 0$. It is easy to conclude that $J = 0$, since ψ is surjective. This proves the exactness of (5.1.4). It follows that $K' := \psi_*(\mathcal{K})^{S_n}$ is locally free on W. Applying φ^* to (5.1.4), we get the following short exact sequence of locally free sheaves on V.

$$0 \longrightarrow \varphi^*K' \longrightarrow E_1|_V \longrightarrow L|_V \longrightarrow 0\,.$$

If $i : V \hookrightarrow \mathcal{H}ilb^n_X$ denotes the inclusion, applying i_* and taking double dual we get a diagram

$$\begin{tikzcd} E_1 \arrow[d, equal] & L \arrow[d, equal] & \\ (i_*E_1|_V)^{\vee\vee} \arrow[r] & (i_*L|_V)^{\vee\vee} \arrow[r] & 0 \end{tikzcd} \tag{5.1.5}$$

The bottom row of the above diagram (5.1.5) is exact because this is a nonzero homomorphism of numerically flat vector bundles and the target is a line bundle.

It is clear that on X^n, $\mathscr{G}(L) = \mathcal{L}$. Let K denote the kernel of the homomorphism $E_1 \longrightarrow L$ in (5.1.5). It is clear that on X^n, $\mathscr{G}(K) = \mathcal{K}$. Since $\mathcal{G}_2 \subset \mathcal{K}$ the assertion that there is $E_2 \subset E_1$ such that $\mathcal{G}_2$ is the vector bundle on X^n associated to E_2 follows by induction on rank.

To complete the proof of the Proposition, we need to show that if E_1 and E_2 are numerically flat vector bundles on $\mathcal{H}ilb^n_X$ then the natural map

$$\mathrm{Hom}_{\mathcal{H}ilb^n_X}(E_1, E_2) \longrightarrow \mathrm{Hom}_{X^n}(\mathcal{G}_1, \mathcal{G}_2)$$

is bijective. It is clear that this natural map is injective (faithful). Therefore, it suffices to show the following. If $\mathcal{G}$ is the vector bundle on X^n associated to a numerically flat vector bundle E on $\mathcal{H}ilb^n_X$, then any nonzero homomorphism $\phi : \mathcal{O}_{X^n} \longrightarrow \mathcal{G}$ comes from a nonzero homomorphism $\widetilde{\phi} : \mathcal{O}_{\mathcal{H}ilb^n_X} \longrightarrow E$. Since the homomorphism $\pi^S(X^n, x) \longrightarrow \pi^S(X, x)_{\mathrm{ab}}$ is faithfully flat, and $\mathcal{G}$ arises from a representation of $\pi^S(X, x)_{\mathrm{ab}}$, it follows that ϕ is a map between two representations of $\pi^S(X, x)_{\mathrm{ab}}$. This shows that ϕ is S_n-equivariant on X^n. Now from the preceding discussion it follows that ϕ arises from a morphism $\mathcal{O}_{\mathcal{H}ilb^n_X} \longrightarrow E$. □

5.2. **Closed immersion.** In this subsection we show that the homomorphism $\tilde{f}$ in (4.2.14) is a closed immersion. For this, we will apply [DMOS82, Proposition 2.21 (b)].

Let $q \in S^n(X)$ be a point of type $\langle n_1, n_2, \ldots, n_r \rangle$. Let $\tilde{q}_i$, for $i = 1, 2, \ldots, m$, denote the points in the fiber $\psi^{-1}(q)$. The stabilizer of $\tilde{q}_i$, denoted $\mathrm{St}(\tilde{q}_i)$, is isomorphic to $S_{n_1} \times S_{n_2} \times \ldots \times S_{n_r}$. Let A denote the local ring $\mathcal{O}_{S^n(X),q}$ and let B denote the semilocal ring $\mathcal{O}_{X^n} \otimes A$. Then B is a finite A module and $A = B^{S_n}$.

Let M be a B module such that the action of S_n on B lifts to an action of S_n on M. There is a short exact sequence of A modules

$$0 \to M^{S_n} \to M \to \bigoplus_{g \in S_n} M\,,$$

where the last map is given by $m \mapsto (g \cdot m - m)_{g \in S_n}$. Let $\widehat{A}$ be the completion of A with respect to its maximal ideal. Applying the functor $- \otimes_A \widehat{A}$, we conclude that the following natural map is an isomorphism.

$$\widehat{M^{S_n}} \xrightarrow{\sim} \widehat{M}^{S_n}\,.$$

The ring $\widehat{B} = B \otimes_A \widehat{A}$ decomposes as

$$\widehat{B} \cong \bigoplus_{i=1}^{m} \widehat{B}_i\,, \tag{5.2.1}$$

where $\widehat{B}_i$ denotes the completion of B at the maximal ideal corresponding to the point $\tilde{q}_i$, for all $i = 1, \ldots, m$. Applying the functor $M \otimes_B -$ to the above isomorphism (5.2.1) and using the isomorphism

$$\widehat{M} \cong M \otimes_A \widehat{A} \cong M \otimes_B \widehat{B}\,,$$

we see that

$$\widehat{M} \cong \bigoplus_{i=1}^{m} \widehat{M}_i\,, \tag{5.2.2}$$

where M_i is the localization of M at the maximal ideal corresponding to the point $\tilde{q}_i$. Taking S_n-invariants in (5.2.2), it easily follows that

$$\widehat{M}^{S_n} \cong \widehat{M}_i^{\mathrm{St}(\tilde{q}_i)}\,,\ \ \forall\, i\,.$$

Proposition 5.2.3. *With notation as above, whenever* $\mathrm{char}(k) > n_1$*, any* S_n*-equivariant surjective* B*-module homomorphism* $f : M \longrightarrow N$ *of finitely generated* B*-modules descends to surjective* A*-module homomorphism of their* S_n*-invariants* $M^{S_n} \longrightarrow N^{S_n}$*.*

Proof. Suppose we have an S_n-equivariant exact sequence of B-modules

$$M \longrightarrow N \longrightarrow 0\,.$$

Taking S_n-invariants we get a homomorphism of A-modules

$$M^{S_n} \longrightarrow N^{S_n}\,. \tag{5.2.4}$$

To check this is surjective, it suffices to check that the map (5.2.4) is surjective after passing to the completion. From the preceding discussion, it follows that it suffices to check that

$$\widehat{M}_i^{\mathrm{St}(\tilde{q}_i)} \longrightarrow \widehat{N}_i^{\mathrm{St}(\tilde{q}_i)} \tag{5.2.5}$$

is surjective for one (and hence any) i. We know that $\widehat{M} \to \widehat{N}$ is surjective. Thus, the above map in (5.2.5) will be surjective if we can lift a section of $\widehat{N}_i^{\mathrm{St}(\tilde{q}_i)}$ to $\widehat{M}$ and average it, that is, apply the operator

$$\frac{1}{\#\mathrm{St}(\tilde{q}_i)} \sum_{g \in \mathrm{St}(\tilde{q}_i)} g.$$

This is possible if $\mathrm{char}(k) = p > n_1$ (c.f. inequalities (3.2.1)). □

Proposition 5.2.6. *Let* $\mathcal{G}$ *be a numerically flat* S_n*-invariant locally free sheaf on* X^n*.*

(1) *Let* $q \in S^n(X)$ *be a point of type* $\langle n_1, n_2, \ldots, n_r\rangle$*. Assume that* $\mathrm{char}(k) = p > n_1$*. Then the sheaf* $\psi_*(\mathcal{G})^{S_n}$ *is locally free in a neighborhood of* q*.*
(2) *Let* U_0 *denote the largest open subset where* $\psi_*(\mathcal{G})^{S_n}$ *is locally free. Then on* $\psi^{-1}(U_0)$ *the natural homomorphism*

$$\psi^*(\psi_*(\mathcal{G})^{S_n}) \longrightarrow \mathcal{G} \tag{5.2.7}$$

is an isomorphism.

Proof. If $\mathcal{G}$ has rank 1 then $\psi_*(\mathcal{G})^{S_n}$ is a line bundle on all of $S^n(X)$. Since $\mathcal{G}$ corresponds to a representation of an abelian group scheme, it follows that there is an S_n-equivariant exact sequence of locally free sheaves on X^n

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G} \longrightarrow \mathcal{L} \longrightarrow 0\,,$$

with $\mathrm{rank}(\mathcal{L}) = 1$. By induction on rank of $\mathcal{G}$, it suffices to show that the homomorphism on the right of the following exact sequence

$$0 \longrightarrow \psi_*(\mathcal{K})^{S_n} \longrightarrow \psi_*(\mathcal{G})^{S_n} \longrightarrow \psi_*(\mathcal{L})^{S_n}$$

is surjective in a neighborhood of q. This surjection can be checked after passing to a formal neighborhood of q. Now the first assertion of the Proposition follows from the above Proposition 5.2.3.

To prove the second assertion, note that both sheaves are locally free of the same rank over $\psi^{-1}(U_0)$. The locus where the natural homomorphism (5.2.7) is not an isomorphism is either empty or a closed subset of codimension 1 in $\psi^{-1}(U_0)$. However, we know that the morphism ψ is finite étale over the smooth locus of $S^n(X)$, and hence the homomorphism (5.2.7) is an isomorphism on the inverse image of the smooth locus of $S^n(X)$. Since the complement of the smooth locus of $S^n(X)$ has codimension 2, it follows that the natural map in (5.2.7) is an isomorphism over $\psi^{-1}(U_0)$. □

Proposition 5.2.8. *Let* $\mathrm{char}(k) > 3$*. Then the homomorphism* $\tilde{f}$ *in* (4.2.14) *is a closed immersion.*

Proof. By [DMOS82, Proposition 2.21(b)] it suffices to show that every S_n-invariant numerically flat bundle $\mathcal{G}$ on X^n arises in the way described in Proposition 4.2.10. In other words, we have to show that there is a numerically flat bundle E on $\mathcal{H}ilb^n_X$ such that $\mathcal{G} = j_*(\psi^*\varphi_*(E|_V))^{\vee\vee}$.

Let $T \supset W$ be the open subset of $S^n(X)$ containing W and points of the type $\langle 3,1,1,\ldots,1\rangle$ and $\langle 2,2,1,\ldots,1\rangle$. Then $\varphi^{-1}(T)$ is an open subset of $\mathcal{H}ilb^n_X$ such that $\mathcal{H}ilb^n_X \setminus \varphi^{-1}(T)$ has codimension at least 3 in $\mathcal{H}ilb^n_X$. Let $i : \varphi^{-1}(T) \hookrightarrow \mathcal{H}ilb^n_X$ denote the inclusion. Define

$$E := (i_*\varphi^*((\psi_*(\mathcal{G})|_T)^{S_n}))^{\vee\vee}.$$

By Proposition 5.2.6 we see that $\psi_*(\mathcal{G})^{S_n}$ is locally free on T and on $\psi^{-1}(T)$ the natural homomorphism $\psi^*(\psi_*(\mathcal{G})^{S_n}) \to \mathcal{G}$ is an isomorphism. Consider the natural homomorphisms

$$F^*\left(\psi_*(\mathcal{G})^{S_n}\right) \longrightarrow (F^*\psi_*(\mathcal{G}))^{S_n} \longrightarrow (\psi_*(F^*\mathcal{G}))^{S_n}\,, \tag{5.2.9}$$

where F denotes the absolute Frobenius morphism. We claim that the above composite homomorphism is an isomorphism over T. If not, then let J denote the cokernel of the composite map (5.2.9).

$$F^*\left(\psi_*(\mathcal{G})^{S_n}\right) \longrightarrow (\psi_*(F^*\mathcal{G}))^{S_n} \longrightarrow J \longrightarrow 0\,.$$

Applying ψ^* to the above exact sequence, we get the following commutative diagram with exact rows.

$$\begin{tikzcd}
\psi^*F^*\left(\psi_*(\mathcal{G})^{S_n}\right) \arrow[r] \arrow[d, "\wr"] & \psi^*\left(\psi_*(F^*\mathcal{G})\right)^{S_n} \arrow[r] \arrow[d, "\wr"] & \psi^*J \arrow[r] & 0 \\
F^*\mathcal{G} \arrow[r, equal] & F^*\mathcal{G} & &
\end{tikzcd}$$

The two vertical arrows are isomorphisms on T because of Proposition 5.2.6. It follows that $\psi^*J = 0$. Then we easily conclude that $J = 0$ since ψ is surjective. This proves that the composite homomorphism in (5.2.9) is an isomorphism over T. It follows that $F^*E \cong \left(i_*\varphi^*(\psi_*(F^*\mathcal{G})^{S_n})\right)^{\vee\vee}$. Now imitating the proof of Proposition 4.2.10 we see that E is locally free and numerically flat on $\mathcal{H}ilb^n_X$. It is clear that $\mathscr{G}(E) = \mathcal{G}$ (see the construction in the proof of Proposition 4.2.10). This proves the Proposition. □

Theorem 5.2.10. *Let* $\mathrm{char}(k) > 3$. *Then the homomorphism*

$$\widetilde{f} : \pi^S(X, x)_{\mathrm{ab}} \longrightarrow \pi^S(\mathcal{H}ilb^n_X, n[x])$$

in (4.2.14) *is an isomorphism.*

Proof. Since $\widetilde{f}$ is faithfully flat by Proposition 5.1.3 and closed immersion by Proposition 5.2.8, it is an isomorphism. □

5.3. **Nori's fundamental group scheme of $\mathcal{H}ilb^n_X$.** Let E be an essentially finite vector bundle over a connected, reduced and proper k-scheme X. Then there is a finite k-group scheme G, a principal G-bundle $p : P \to X$ and a finite dimensional k-linear representation $\rho : G \to \mathrm{GL}(V')$ such that E is the vector bundle associated to the representation ρ. It follows from the proof of [Nor76, Proposition 3.8] that there is a finite vector bundle $\mathcal{V}$ on X such that E is a subbundle of $\mathcal{V}$.

As before, let X be an irreducible smooth projective surface over k, and $\mathcal{H}ilb^n_X$ the Hilbert scheme of n points on X. It is clear that the functor $\mathscr{G}$ in Proposition 4.2.10 takes a finite vector bundle to a finite vector bundle. Thus, $\mathscr{G}(E) \subset \mathscr{G}(\mathcal{V})$, which shows that $\mathscr{G}$ takes essentially finite vector bundles to essentially finite vector bundles. It is easily checked, using [Lan11, Lemma 6.2], that there is a commutative diagram

$$\begin{tikzcd}
\pi^S(X, x)_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d] & \pi^S(\mathcal{H}ilb^n_X, n[x]) \arrow[d] \\
\pi^N(X, x)_{\mathrm{ab}} \arrow[r, "\widetilde{f}^N"] & \pi^N(\mathcal{H}ilb^n_X, n[x])
\end{tikzcd}$$

where the vertical arrows are faithfully flat. It follows that $\widetilde{f}^N$ is faithfully flat.

Now let $\mathcal{G}$ be an essentially finite S_n-invariant vector bundle on X^n. It is easy to find a finite and S_n-invariant bundle $\mathcal{V}$ on X^n and an S_n-equivariant inclusion $\mathcal{G} \subset \mathcal{V}$. Then following the proof of Proposition 5.2.8, we define E and V by

$$E := (i_*\varphi^*((\psi_*(\mathcal{G})|_T)^{S_n}))^{\vee\vee} \subset (i_*\varphi^*((\psi_*(\mathcal{V})|_T)^{S_n}))^{\vee\vee} =: V\,.$$

It is clear that V is a finite vector bundle, and so E is essentially finite, and $\mathscr{G}(E) = \mathcal{G}$. This shows, imitating the proof of the Proposition 5.2.8, that $\widetilde{f}^N$ is a closed immersion whenever $\mathrm{char}(k) > 3$. Thus, we have the following.

Theorem 5.3.1. *Let* $\mathrm{char}(k) > 3$. *There is a natural isomorphism of affine k-group schemes*

$$\tilde{f}^N : \pi^N(X, x)_{\mathrm{ab}} \longrightarrow \pi^N(\mathcal{H}ilb^n_X, n[x]).$$

5.4. **Étale Fundamental Group Scheme of $\mathcal{H}ilb^n_X$.** In this subsection we sketch how to deduce from Theorem 5.3.1 the same assertion for $\pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$. This result is already contained in [BH15]. Note that there is a commutative diagram

$$\begin{tikzcd}
\pi^N(X,x) \arrow[r, two heads] \arrow[d] & \pi^N(X,x)_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d] & \pi^N(\mathcal{H}ilb^n_X, n[x]) \arrow[d, "d"] \\
\pi^{\text{ét}}(X,x) \arrow[r, two heads] & \pi^{\text{ét}}(X,x)_{\mathrm{ab}} \arrow[r] & \pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])\,.
\end{tikzcd}$$

From this it follows that $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \longrightarrow \pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$ is faithfully flat. Consider a homomorphism $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \to \mathrm{GL}(V)$. It follows using [Nor76, Proposition 3.10] that this homomorphism factors through a finite and reduced group scheme G. Now consider the diagram

$$\begin{tikzcd}
\pi^N(X,x)_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d] & \pi^N(\mathcal{H}ilb^n_X, n[x]) \arrow[r, two heads, "d"] \arrow[d, dashed] & \pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x]) \arrow[dl, dashed] \\
\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \arrow[r] & G \arrow[r] & \mathrm{GL}(V)\,.
\end{tikzcd}$$

The right vertical arrow is the unique map which makes the square commute. It factors through d since G is finite and reduced. Now it follows from [DMOS82, Proposition 2.21 (b)] that $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \to \pi^{\text{ét}}(\mathcal{H}ilb^n_X, n[x])$ is a closed immersion.

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