A NOTE ON SEMISTABLE VECTOR BUNDLES AND HIGGS BUNDLES

ARJUN PAUL

ABSTRACT. We shall define the notion of stability and semistability for vector bundles and Higgs bundles on smooth complex projective variety X. Then we discuss some of the basic properties of semistable vector bundles, with outline of proofs (some good references will be indicated). We shall see that under certain conditions on the cotangent bundle Ω^1_X of the base space, underlying vector bundle of any semistable Higgs bundle become semistable.

1. INTRODUCTION

1.1. **Basic notations.** In this note, X will always be a smooth irreducible projective variety over an algebraically closed field k of characteristic 0. For simplicity, we may assume that $k = \mathbb{C}$. The structure sheaf of X will be denoted by $\mathcal{O}_X$. For $x \in X$, let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$, and $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, the residue field of at $x \in X$. Note that k being algebraically closed, for any closed point $x \in X$, the residue field $k(x) \cong k$. A polarization of X is a choice of an ample line bundle on X. We fix an ample line bundle on X. This is required to define the degree of line bundles on X.

1.2. **Vector bundle and locally free sheaf.** Let X be a smooth projective variety over k. A rank r vector bundle on X is a k–variety E together with a surjective morphism $p : E \longrightarrow X$ such that each fiber $E(x) := p^{-1}(x)$ has a structure of an r–dimensional k–vector space, for each $x \in X$; there is an open cover $\{U_\alpha\}_{\alpha\in I}$ of X and isomorphisms (local trivializations) $\psi_\alpha : p^{-1}(U_\alpha) = E \times_X U_\alpha \longrightarrow k^r \times_k U_\alpha$ which restricts to k–linear isomorphisms, when restricted to each fibers $E(x) = p^{-1}(x)$, for all $x \in U_\alpha$. The morphisms $\psi_\beta^{-1} \circ \psi_\alpha : (U_\alpha \cap U_\beta) \times k^r \longrightarrow (U_\alpha \cap U_\beta) \times k^r$ is of the form $(\mathrm{Id}, \psi_{\alpha\beta})$, where $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_r(k)$, called *transition map*, such that for $\alpha, \beta, \gamma \in I$, they satisfy the conditions: $\psi_{\alpha\alpha} = \mathrm{Id}$ on U_α and $\psi_{\alpha\beta} \cdot \psi_{\beta\gamma} \cdot \psi_{\gamma\alpha} = \mathrm{Id}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Let $\mathcal{E}$ be a locally free coherent sheaf of $\mathcal{O}_X$–modules of rank r over X. The stalk of $\mathcal{E}$ at $x \in X$ is an $\mathcal{O}_{X,x}$–module, denoted by $\mathcal{E}_x$. The fiber of $\mathcal{E}$ over a point $x \in X$ is the $k(x)$–vector space $\mathcal{E}_x \otimes_k k(x)$. If X is connected, then the dimension $\dim_k \mathcal{E}(x)$ of the fibers $\mathcal{E}(x)$ does not depend on the choices of $x \in X$, and is equal to the rank of $\mathcal{E}$. Algebraically, we can define a scheme $E := \mathbf{Spec}(\mathrm{Sym}^*(\mathcal{E}^\vee))$ together with a morphism of k–schemes $p : E \longrightarrow X$, which is locally trivial (in Zariski topology) over X, see [Har, p. 128]. One can check that $p : E \longrightarrow X$ satisfies the usual axioms of vector bundles as described above. We call it the geometric vector bundle associated to $\mathcal{E}$.

Date: November 21, 2017.
2010 *Mathematics Subject Classification.* 14J60.
Key words and phrases. Higgs bundle, semistable bundle, vector bundle.
Corresponding author: Arjun Paul.

A section of E over $U \subset X$ is a map $s : U \longrightarrow E|_U$ such that $p \circ s = \mathrm{Id}_U$. The sheaf $\mathcal{E}$ of germs of sections of E gives a locally free sheaf of $\mathcal{O}_X$–modules of rank r over X.

Conversely, given a locally free coherent sheaf of $\mathcal{O}_X$–modules $\mathcal{E}$ of rank r on X, choose an open cover $\{U_\alpha\}_{\alpha \in I}$ such that there are isomorphisms $g_\alpha : \mathcal{E}|_{U_\alpha} \longrightarrow \mathcal{O}_X^{\oplus r}|_{U_\alpha}$, for each $\alpha \in I$. These gives isomorphisms $g_{\alpha\beta} := g_\alpha \circ g_\beta^{-1} : \mathcal{O}_X^{\oplus r}|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{O}_X^{\oplus r}|_{U_\alpha \cap U_\beta}$, for each $\alpha, \beta \in I$. This defines the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_r$, which satisfies the cocycle condition due to sheaf conditions. Therefore, we get a rank r vector bundle E over X. One can check that these two constructions are inverse to each other, and gives an one-to-one correspondence between the isomorphism classes of rank r–vector bundles on X, and the isomorphism classes of locally free coherent sheaf of $\mathcal{O}_X$–modules of rank r on X; and therefore, we shall not make any distinction between them.

Remark 1.1. The above correspondence does not preserve the sub-objects nicely. If $\mathcal{F} \subset \mathcal{E}$ is a subsheaf of a locally free coherent sheaf of $\mathcal{O}_X$–modules $\mathcal{E}$ on X, then we have injective homomorphisms $\mathcal{F}_x \hookrightarrow \mathcal{E}_x$ of $\mathcal{O}_{X,x}$–modules at the level of stalks, but this may not remain injective at the level of fibers, because tensoring by $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, the resulting $k(x)$–linear homomorphism of vector spaces $\mathcal{F}(x) \longrightarrow \mathcal{E}(x)$ is no more injective in general. Therefore, the locally free subsheaf of $\mathcal{O}_X$–modules may not corresponds to subbundle in general. However, the inclusion $\mathcal{F} \subset \mathcal{E}$ give rise to a vector bundle homomorphism, which is not necessarily injective.

For example, take an effective divisor $D \subset X$ of an irreducible smooth projective curve X over $\mathbb{C}$. Then $\mathcal{O}_X(-D)$ is a locally free shubsheaf of $\mathcal{O}_X$ on X, but $\mathcal{O}_X(-D)$ does not corresponds to a subbundle of the trivial line bundle $X \times \mathbb{C}$ on X corresponding to the sheaf $\mathcal{O}_X$.

However, an $\mathcal{O}_X$–submodule $\mathcal{F} \subset \mathcal{E}$ on X corresponds to a subbundle of E if the quotient $\mathcal{E}/\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_X$–module on X.

1.3. **Higgs bundle.** Let Ω_X^1 be the cotangent bundle of X. A *Higgs bundle* on X is a pair (E, θ) consisting of a vector bundle E over X together with a $\mathcal{O}_X$–linear homomorphism

$$\theta : E \longrightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1 \,,$$

such that the $\theta \wedge \theta = 0$ in $H^0(X, \mathcal{E}nd(E) \otimes \Omega_X^2)$. More precisely, the following homomorphism vanishes identically.

$$\theta \wedge \theta : E \xrightarrow{\theta} E \otimes \Omega_X^1 \xrightarrow{\theta \otimes \mathrm{Id}_{\Omega_X^1}} E \otimes \Omega_X^1 \otimes \Omega_X^1 \xrightarrow{\mathrm{Id}_E \otimes (- \wedge -)} E \otimes \Omega_X^2 \,. \tag{1.1}$$

We call θ the Higgs field of (E, θ). Note that we can consider θ as a section $\theta \in H^0(X, \mathcal{E}nd(E) \otimes \Omega_X^1)$. Any vector bundle E over X is a Higgs bundle with zero Higgs field.

2. Semistable bundles

2.1. **Definitions.** Let X be a connected smooth projective variety over $k = \mathbb{C}$. Fix an ample line bundle $\mathcal{O}_X(1) = H$ on X. For any vector bundle (locally free coherent sheaf of $\mathcal{O}_X$–modules) on X, we define $\det(E) := \bigwedge^r E$, where $r = rk(E)$. This is a line bundle on X. If E is a coherent sheaf of $\mathcal{O}_X$–modules over X, and X being smooth projective, E admits a finite locally free resolution

$$0 \longrightarrow E_m \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow E \longrightarrow 0\,.$$

Then we define a line bundle $\det(E) := \bigotimes_{j=0}^{m} \det(E_j)^{(-1)^j}$. One can see that this definition is independent of choices of above resolution (see [HL, p. 9] and [Ko, Proposition 5.6.10] for details). We call $\det(E)$ the *determinant bundle* of E. The degree of E is then defined to be the degree of the line bundle $\det(E)$ on (X, H). That means,

$$\deg(E) = c_1(\det(E)) \cdot H^{n-1} \in \mathbb{Z}\,,$$

where $c_1(\det(E)) \in H^2(X, \mathbb{Z})$ is the first Chern class of the line bundle $\det(E)$, and $n = \dim X$. More explicitly, consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \stackrel{\exp(2\pi i \bullet)}{\longrightarrow} \mathcal{O}_X^{\times} \longrightarrow 0$$

of sheaves of abelian groups on X. The associated long exact sequence of cohomologies gives a group homomorphism

$$c_1 : \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^{\times}) \longrightarrow H^2(X, \mathbb{Z})\,,$$

called the first Chern class map. Now take $c_1(H) \in H^2(X, \mathbb{Z})$ of the fixed ample line bundle H, and take the cup product of this cohomology class of it with itself $n-1$ times to get an element $c_1(H)^{n-1} \in H^{2n-2}(X, \mathbb{Z})$; then take the cup product of this cohomology class with $c_1(\det(E))$ to get an element $c_1(\det(E)) \cdot H^{n-1} := c_1(\det(E)) \cup c_1(H)^{n-1} \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$. This gives an integer, called the degree of the vector bundle E on (X, H). Geometrically, this says that fix an embedding $X \hookrightarrow \mathbb{CP}^N$ of X into some complex projective space, and consider the intersections of $n-1$ number of general hyperplanes in $\mathbb{CP}^N$ and then further intersect this with X. This gives a smooth projective curve C inside X. Then restrict the line bundle $\det(E)$ to C to get a line bundle on C. Now the degree of this line bundle $\det(E)|_C$ is an integer, which we define as the degree of E.

The *slope* of a torsion free coherent sheaf of $\mathcal{O}_X$–modules E is defined to be the rational number $\mu(E) = \frac{\deg(E)}{rk(E)}$, where $rk(E)$ denotes the rank of E. The following are useful in computing degree of $E \otimes F$:

- $c_1(E) = c_1(\det(E))$
- $\mathrm{ch}(E) = rk(E) + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \ldots$
- $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$,

where $\mathrm{ch}(E)$ is the *Chern character* of E, and $c_j(E)$ is the j^{th} Chern class of E.

Let E be a coherent sheaf of $\mathcal{O}_X$–modules E on X. The *singularity set* of E is defined to be the subset

$$\mathrm{Sing}(E) := \{x \in X : E_x \text{ is not free } \mathcal{O}_{X,x}\text{–module}\} \subset X\,.$$

This is a closed subset of codimension ≥ 1 in X. The fiber of E at x is a $k(x)$–vector space $E(x) := E_x \otimes_{\mathcal{O}_{X,x}} k(x)$, where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. We define the *rank* of E to be $rk(E) := \dim_{k(x)} E(x)$, the dimension of the fiber $E(x)$ as a $k(x)$–vector space, for any $x \in X(k) \setminus \mathrm{Sing}(E)$.

Since X is connected, this definition does not depend on the choice of $x \in X \setminus \mathrm{Sing}(E)$. A coherent sheaf of $\mathcal{O}_X$–modules E on X is said to be *torsion free* if every stalks E_x are torsion free $\mathcal{O}_{X,x}$–modules. If E is torsion free coherent sheaf of $\mathcal{O}_X$–modules on X, then its singularity set $\mathrm{Sing}(S)$ has codimension ≥ 2. Note that every coherent $\mathcal{O}_X$–submodule of a torsion free coherent $\mathcal{O}_X$–module is again torsion free.

Definition 2.1. A torsion free coherent sheaf of $\mathcal{O}_X$–modules E over X is said to be

- *μ–semistable* if for every coherent subsheaf $F \subset E$ of E, with $0 < rk(F) \leq rk(E)$, we have $\mu(F) \leq \mu(E)$.
- *μ–stable* if for every coherent subsheaf $F \subset E$ of E, with $0 < rk(F) < rk(E)$, we have $\mu(F) \leq \mu(E)$.

Remark 2.2. (1) If E and F are torsion free coherent sheaves of $\mathcal{O}_X$–modules on X with $F \subset E$ and $rk(F) = rk(E)$, then we have an injective $\mathcal{O}_X$–linear homomorphism $\det(F) \hookrightarrow \det(E)$, and hence, $\deg(F) \leq \deg(E)$. Therefore, if E is μ–stable, then it is μ–semistable.

(2) In the definition of semistable and stable sheaves, it suffices to check the slope inequalities for non-zero subsheaves $F \subset E$, with $0 < rk(F) < rk(E)$ such that the quotient E/F is torsion free.

Example 2.1. Let E be a torsion free coherent sheaf of $\mathcal{O}_X$–modules of rank 1 on X. Let $F \subset E$ be any non-zero coherent $\mathcal{O}_X$–submodule of E on X. Then $0 < rk(F) = rk(E)$. So, there is nothing to check for stability, i.e., E is automatically μ–stable.

Definition 2.3. [Si, p. 18] A Higgs bundle (E, θ) on X is said to be *μ–semistable* (resp. *μ–stable*) if for every subsheaf $F \subset E$, with $0 < rk(F) \leq$ (resp. $<$) $rk(E)$, and $\theta(F) \subseteq F \otimes \Omega^1_X$, we have $\mu(F) \leq$ (resp. $<$)$\mu(E)$.

Example 2.2.

(1) Take any holomorphic line bundle L on a compact Riemann surface X, together with a holomorphic 1–form $\omega \in H^0(X, \Omega^1_X)$. Then the pair (L, ω) is a stable Higgs bundle on X.

(2) If E is a semistable vector bundle on X, then for any Higgs field θ on E, the Higgs bundle (E, θ) is semistable. The converse is not true (see Example 3.2).

2.2. A cohomological criterion for semistability.

Theorem 2.4. *Let E be a vector bundle over a smooth projective curve X of genus $g \geq 2$ over a field $k = \bar{k}$, $\mathrm{char}(k) = 0$. Then E is μ–semistable if (and only if) there is a vector bundle F over X such that $H^i(X, E \otimes F) = 0$, for all $i \geq 0$.*

Proof. Suppose that there is a vector bundle F over X with $H^i(X, E \otimes F) = 0\,,\ \forall\, i \geq 0$. If E is not μ–semistable, then there is a subsheaf $E' \subset E$ with $0 < rk(E')$ and $\mu(E') > \mu(E)$. Now by Riemann-Roch theorem for a vector bundle $\mathcal{E}$ over X, we have

$$\mu(\mathcal{E}) = \frac{h^0(\mathcal{E}) - h^1(\mathcal{E})}{rk(\mathcal{E})} + g - 1\,,$$

where $h^i(\mathcal{E}) = \dim_k H^i(X, \mathcal{E})\,,\ \forall\, i \geq 0$. Now, $H^0(X, E \otimes F) = H^1(X, E \otimes F) = 0$ gives $\mu(E) + \mu(F) = \mu(E \otimes F) = g - 1$. Therefore,

$$\mu(E' \otimes F) = \mu(E') + \mu(F) > \mu(E) + \mu(F) = \mu(E \otimes F) = g - 1 \geq 0\,. \tag{2.1}$$

Again by Riemann-Roch theorem, we have

$$h^0(E' \otimes F) = h^1(E' \otimes F) + \big[\mu(E' \otimes F) + 1 - g\big]\, rk(E' \otimes F) > 0\,.$$

But $E' \otimes F \subset E \otimes F$ gives $H^0(X, E' \otimes F) \subset H^0(X, E \otimes F) = 0$, which contradicts (2.1).

The converse part is not easy. See [F, Theorem 1.2] for algebraically closed field of characteristic zero, and [BHH] for arbitrary field. □

Question 2.1. Is there any cohomological criterion of semistability of vector bundles on higher dimensional varieties?

3. Properties of semistable bundles

Lemma 3.1. *Let*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of coherent sheaves on X. Then

$$rk(E')\,\big[\mu(E) - \mu(E')\big] + rk(E'')\,\big[\mu(E) - \mu(E'')\big] = 0\,. \tag{3.1}$$

Proof. Since $\det(E) \simeq \det(E') \otimes \det(E'')$, we have $c_1(E) = c_1(E') + c_1(E'')$, and hence $\deg(E) = \deg(E') + \deg(E'')$. Again, $rk(E) = rk(E') + rk(E'')$. Then the above equality (3.1) follows from the definition of slope. □

The above Lemma 3.1 enable us to rephrase the definition of semistability (resp. stablity) as follows.

Proposition 3.2. *Let E be a torsion free coherent sheaf of $\mathcal{O}_X$–modules on X. Then the following are equivalent.*

(1) *E is semistable (resp. stable).*
(2) *for all torsion free quotient sheaf of $\mathcal{O}_X$–modules $E \twoheadrightarrow Q$ of E, with $rk(Q) > 0$ (resp. $0 < rk(Q) < rk(E)$), we have $\mu(E) \leq$ (resp. $<$)$\mu(Q)$.*

Proof. Let E be semistable (resp. stable) and $q : E \longrightarrow Q$ be a torsion free quotient sheaf of E. Let $F = \ker(q : E \longrightarrow Q)$. Then we have a short exact sequence of coherent sheaves of $\mathcal{O}_X$–modules on X

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0\,.$$

Then (2) follows from semistability (resp. stability) of E and Lemma 3.1. Converse is also similar. □

Proposition 3.3. *Let E and F be two μ–semistable torsion free coherent sheaves of $\mathcal{O}_X$–modules on X. If $\mu(E) > \mu(F)$, then $\mathrm{Hom}_{\mathcal{O}_X}(E, F) = 0$.*

Proof. Let $f : E \longrightarrow F$ be any nonzero $\mathcal{O}_X$–module homomorphism. Let $G = \mathrm{image}(f)$. Then from μ–semistability of E and F, using Proposition 3.2, we have $\mu(E) \leq \mu(G) \leq \mu(F)$. This contradics our assumption that $\mu(E) > \mu(F)$. So we must have $f = 0$. $\square$

Corollary 3.4. *Let E be a semistable vector bundle over X, with $\deg(E) < 0$. Then $H^0(X, E) = 0$.*

Proof. Suppose on the contrary that there is a nonzero global section $s \in H^0(X, E)$. Then s defines a nonzero (injective) $\mathcal{O}_X$–module homomorphism $\tilde{s} : \mathcal{O}_X \longrightarrow E$, with $1 = \mu(\mathcal{O}_X) > \mu(E)$, contradicting Proposition 3.3. $\square$

Corollary 3.5. *If E is a stable vector bundle on X, then $\mathcal{E}nd(E) = H^0(X, \mathcal{E}nd(E)) \cong k$, i.e., all the $\mathcal{O}_X$–linear endomorphisms of E are scalar multiplication of the identity endomorphism of E.*

Proposition 3.6. *If E is a μ–semistable sheaf on X, with $\deg(E)$ and $rk(E)$ coprime, then E is μ–stable.*

Proof. Suppose that E is not μ–stable. Then there is a subsheaf $F \subset E$, with $0 < rk(F) < rk(E)$, such that $\mu(F) = \mu(E)$. Then $\deg(F)\, rk(E) - \deg(E)\, rk(F) = 0$, which contradics the fact that $\gcd(\deg(E), rk(E)) = 1$. $\square$

We now state some properties of semistable and stable vector bundles without proof.

Proposition 3.7. [Ko, Section 5.7] *Let E and F be two vector bundles over X. Then*

(1) $E \oplus F$ is μ–semistable if and only if both E and F are μ–semistable with slope $\mu(E) = \mu(F) = \mu(E \oplus F)$.
(2) Any line bundle is μ-stable.
(3) E is μ–semistable if and only if $E^* = \mathcal{H}om(E, \mathcal{O}_X)$ is μ–semistable.
(4) E is μ–(semi)stable if and only if $E \otimes L$ is μ–(semi)stable, for any line bundle L over X.

Proof. (2), (3) and (4) are obvious.
(1): Suppose that $E\oplus$ be μ–semistable. If $\mu(E) \neq \mu(F)$, then $\mu(E) > \mu(F)$, say. Then

$$\mu(E \oplus F) = \left(\frac{rk(E)}{rk(E) + rk(F)}\right) \mu(E) + \left(\frac{rk(F)}{rk(E) + rk(F)}\right) \mu(F) < \mu(E)\,. \tag{3.2}$$

This contradicts the fact that $E \oplus F$ is μ–semistable. So $\mu(E) = \mu(F)$. Now any subsheaf G of E (or F) with $rk(G) > 0$ is also a subsheaf of $E \oplus F$. Then by semistability of $E \oplus F$, we have $\mu(G) \leq \mu(E)$ and $\mu(G) \leq \mu(F)$. Therefore, both E and F are semistable with slope μ.

Conversely, suppose that both E and F are semistable with the same slope μ. Then clearly $\mu(E \oplus F) = \mu$. Take any coherent subsheaf G of $E \oplus F$ with $rk(G) > 0$. Let $G' = G \cap (E \oplus 0)$ and let G'' be the image of G under the projection map $p : E \oplus F \longrightarrow F$. Then we have the following commutative diagram with two rows exact:

$$\begin{tikzcd}
0 \arrow[r] & G' \arrow[r] \arrow[d, hook] & G \arrow[r, "p"] \arrow[d, hook] & G'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & E \arrow[r] & E \oplus F \arrow[r, "p"] & F \arrow[r] & 0\,.
\end{tikzcd}$$

Since both E and F are semistable with slope μ, so

$$\deg(G') \leq \mu \cdot rk(G') \quad \text{and} \quad \deg(G'') \leq \mu \cdot rk(G'') \,. \tag{3.3}$$

Then

$$\mu(G) = \frac{\deg(G') + \deg(G'')}{rk(G') + rk(G'')} \leq \mu = \mu(E \oplus F) \,.$$

Therefore, $E \oplus F$ is semistable. □

Remark 3.8. If E_i are μ–stable vector bundles with the same slope μ, then $\bigoplus_{i=1}^{n} E_i$ is μ–semistable with the same slope μ, but it can never be μ–stable. Such vector bundles are called *μ–polystable.*

Theorem 3.9. [HL, Remark 1.5.12, p. 25] *The category $\mathcal{C}_\lambda^{ss}$ of μ–semistable sheaves on X with fixed slope λ, froms a full subcategory of the category $\mathrm{Coh}(X)$ of coherent sheaves on X. The category $\mathcal{C}_\lambda^{ss}$ is abelian.*

Theorem 3.10. *The tangent and cotangent bundles of the projective spaces $\mathbb{P}_k^n$, are stable.*

Proof. See [HL, p. 20] or [OSS, Theorem 1.3.2, p. 92] for details. □

Remark 3.11. The above Theorem 3.10 fails for arbitrary projective varieties.

Theorem 3.12. *Any vector bundle E over X admits a filtration by torsion free subsheaves*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E \,, \tag{3.4}$$

such that each E_i/E_{i-1} is semistable, and their slopes $\mu_i = \mu(E_i/E_{i-1})$ satisfies

$$\mu_{\max}(E) := \mu_1 > \mu_2 > \cdots > \mu_l =: \mu_{\min}(E) \,. \tag{3.5}$$

Such a filtration is unique upto isomorphism.

Proof. For any non-zero subsheaf F of E with $0 < rk(F) < rk(E)$, and E/F torsion free, we have $\mu(F) \leq q_0$, where q_0 is a fixed rational number depending only on E and X. We define a partial order relation "$\preceq$" on the set of all such non-zero subsheaves of E as follow: $F \preceq G$ if $F \subset G$ and $\mu(F) \leq \mu(G)$. Recall that, for any torsion free coherent sheaf F on X, $\mathrm{Sing}(F)$ has codimension ≥ 2, and $F|_U$ is locally free, where $U = X \setminus \mathrm{Sing(F)}$. Note that, any increasing chain of such subsheaves of E terminates. So for any subsheaf $F \subset E$, we have a maximal subsheaf $\widetilde{F}$ with $F \subset \widetilde{F} \subset E$. Consider the set of all such $\preceq$–maximal subsheaves $\widetilde{F}$ of E, and choose $E_1 \subset E$ to be of minimal rank among them. Then one can check that for any subsheaf F of E, we have $\mu(F) \leq \mu(E_1)$, and if $\mu(F) = \mu(E_1)$ then $F \subset E_1$. Moreover, this subsheaf E_1 of E is unique and μ–semitsable. We call it the *maximal destabilizing subsheaf* of E. Now the theorem follows by induction on rank of E by applying the above method on E/E_1. □

We call the filtration (3.4) the *Harder–Narasimhan filtration* of E. For detail proof of Theorem 3.12, see [HL, Theorem 1.3.4, p. 17] and [Ko, Theorem 5.7.15, p. 159].

Corollary 3.13. *Let E and F be any two vector bundles (or torsion free coherent sheaves of $\mathcal{O}_X$–modules) on X. If $\mu_{\min}(E) > \mu_{\max}(F)$, then $\mathrm{Hom}_{\mathcal{O}_X}(E, F) = 0$.*

Proposition 3.14. *Let E be a vector bundle on a polarized smooth projective variety (X, H). Let $D \overset{\iota}{\hookrightarrow} X$ be a smooth divisor on X numerically equivalent to H (i.e., $D \in |H|$) such that $E|_D := \iota^* E$ is torsion free and μ–semistable with respect to the induced polarization $H_D := \iota^* H$ on D. Then E is semistable.*

Proof. Suppose that E is not μ–semistable on (X, H). Then there is a (saturated) maximal destabilizing subsheaf $F \subset E$ of E, i.e., $\mu(F) > \mu(E)$. Then with respect to the induced polarization $H_D := \iota^* H$ on D, we have

$$\mu(\iota^* F) = \mu(F) + \deg(\mathcal{O}_D) > \mu(E) + \deg(\mathcal{O}_D) = \mu(\iota^* E)\,,$$

contradicting semistability of $E|_D = \iota^* E$ on (D, H_D). Therefore, E is semistable. □

Note that, the converse of the above proposition is not true in general. For example, let (X, H) be a polarized irreducible smooth complex projective variety of dimension $n \geq 2$. Take any smooth divisor D on X which is not numerically equivalent to H. Then for any semistable vector bundle E on X, $E|_D$ is not semistable with respect to the induced polarization $H|_D$ on D.

Example 3.1. Restriction of a semistable vector bundle to a divisor is not semistable in general. For example, let $X = \mathbb{CP}^n$, with $n \geq 2$, and let $E = T_{\mathbb{CP}^n}$. Then E is semistable with respect to the polarization $L = \mathcal{O}_{\mathbb{CP}^n}(1)$. Now for any hyperplane $H \overset{\iota}{\hookrightarrow} \mathbb{CP}^n$, we have $(T_{\mathbb{CP}^n})|_H = T_H \bigoplus \mathcal{O}_H(1)$, where $\mathcal{O}_H(1) = \iota^* \mathcal{O}_{\mathbb{CP}^n}(1)$. Now $\mu_{L|_H}(T_H) = \frac{n}{n-1}$ and $\mu_{L|_H}(\mathcal{O}_H(1)) = 1$. Therefore, $(T_{\mathbb{CP}^n})|_H$ is not $L|_H$–semistable.

Theorem 3.15. [MR1, MR2] *Let $(X, \mathcal{O}_X(1))$ be a polarized smooth projective variety of dimension $n \geq 2$, and E is a semistable (resp. stable) vector bundle over X. Then there is a positive integer $a_0 > 0$ such that for all $a \geq a_0$, there is an open dense subset U_a of the linear system $|\mathcal{O}_X(a)|$ such that for any divisor $D \in U_a$, the restriction $E|_D$ is semistable (resp. stable).*

Remark 3.16. The above theorem is true in positive characteristic also.

Theorem 3.17. [NS1, NS2] *Let X be a compact connected Riemann surface.*

(1) *A holomorphic vector bundle E_ρ defined by a unitary representation of the fundamental group $\rho : \pi_1(X) \longrightarrow U_r(\mathbb{C})$ is μ–semistable of degree 0.*
(2) *A holomorphic vector bundle E on X is μ–stable if and only if E is given by an irreducible unitary representation ρ of $\pi_1(X)$.*

Theorem 3.18. *Let E and F be two vector bundles over X. Then $E \otimes F$ is μ–semistable if and only if both E and F are μ–semistable.*

Proof. Suppose that $E \otimes F$ is μ–semistable. If E were not μ–semistable, then there is a destabilizing quotient sheaf E' of E, i.e., $E \twoheadrightarrow E'$ with $\mu(E) > \mu(E')$. Since the functor $- \otimes F$ is right exact, $E' \otimes F$ is a (destabilizing) quotient sheaf of $E \otimes F$, with

$$\mu(E' \otimes F) = \mu(E') + \mu(F) < \mu(E) + \mu(F) = \mu(E \otimes F)\,,$$

contradicting μ–semistability of $E \otimes F$. Therefore, E is μ–semistable. Similarly, F is semistable.

To prove the converse, we use induction on $n = \dim_k X$. Let us first assume that $n \geq 2$. Since both E and F are semistable, by Theorem 3.15 we may assume that $E|_D$ and $F|_D$ are semistable, for general $D \in |\mathcal{O}_X(a_0)|$, for $a_0 \gg 0$. Then by induction hypothesis $(E \otimes F)|_D = E|_D \otimes F|_D$ is semistable on D. Therefore, $E \otimes F$ is semistable by Proposition 3.14. Therefore, it is enough to prove the result for $\dim_k X = 1$. We may assume that $k = \mathbb{C}$, and X is a compact connected Riemann surface. By last part of the Proposition 3.7, we may assume that $\deg(E) = 0 = \deg(F)$. Now any (degree 0) semistable vector bundle E over X admits a Jordan–Hölder filtration (not necessarily unique) by its subbundles

$$\mathcal{E}^\bullet \;:\; 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E\,,$$

such that each E_i/E_{i-1} is a stable vector bundle (of degree 0) over X, and the associated graded vector bundle $\mathrm{gr}(\mathcal{E}^\bullet) := \bigoplus_{i=1}^{n} E_i/E_{i-1}$ is semistable and independent of choice of the filtration (see [HL, Proposition 1.5.2]). Therefore, any such filtration of E and F induces a filtration of $E \otimes F$ such that the associated graded vector bundle is a direct summ of tensor products of stable vector bundles of degree 0 over X. Therefore, we may assume that both E and F are stable vector bundles of degree 0 over X. Then the result follows from the theorem of Narasimhan–Seshadri (Theorem 3.17). Because, if E and F are given by irreducible unitary representations ρ_E and ρ_F, respectively, of $\pi_1(X)$, then $E \otimes F$ is given by the unitary representation $\rho_E \otimes \rho_F$, and hence is semistable by Theorem 3.17. □

Remark 3.19. (1) The above theorem fails in positive characteristic.

(2) There are purely algebraic proofs of this theorem (without using the results of [NS1, NS2]). See [HL, Theorem 3.1.4] or [RR].

(3) Note that if $rk(E), rk(F) \geq 2$, then $E \otimes F$ may not be μ–stable even if both E and F are μ–stable. For example, take E to be any μ–stable vector bundle of $rk(E) \geq 2$, and consider the vector bundle $\mathcal{E}nd(E) = E \otimes E^*$. Then the (nonzero) global section of $\mathcal{E}nd(E)$ corresponding to the identity homomorphism of E gives a nonzero $\mathcal{O}_X$–module homomorphism $f : \mathcal{O}_X \longrightarrow \mathcal{E}nd(E)$. But $\mu(\mathcal{E}nd(E)) = 0 = \deg(\mathcal{O}_X)$. Therefore, $\mathcal{E}nd(E)$ is not μ–stable.

Lemma 3.20. *Let X be a smooth projective surface together with an ample line bundle H on X. Let E be a torsion free coherent sheaf of $\mathcal{O}_X$–modules on X. Then*

$$\frac{h^0(X,E)}{rk(E)} \leq \max\{\mu_{\max}(E)\,, 0\}\,,$$

where $h^0(X,E) = \dim H^0(X,E)$.

Proof. If $H^0(X,E) = 0$, then we have nothing to prove. Assume that $h^0(X,E) \geq 1$. Consider the Harder-Narasimhan filtration for E:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\,,$$

where E_i/E_{i-1} are semistable and their slopes $\mu_i = \mu(E_i/E_{i-1})$ satisfies

$$\mu_{\max}(E) := \mu_1 > \cdots > \mu_l =: \mu_{\min}(E)\,.$$

Now look at the short exact sequences

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow E_i/E_{i-1} \longrightarrow 0\,.$$

Then the corresponding long exact sequence of cohomologies gives

$$h^0(X, E_i) \leq h^0(X, E_{i-1}) + h^0(X, E_i/E_{i-1})\,, \ \ \forall\, i \geq 1\,.$$

Therefore, we have $h^0(X, E) \leq \sum_{i=1}^{l} h^0(X, E_i/E_{i-1}) \leq rk(E)\mu_{\max}(E)$. $\square$

3.1. **Applications.** Let (X, H) be a polarized smooth projective variety of dimension $n \geq 2$ over $\mathbb{C}$. The *discriminant* of a vector bundle E on X is the characteristic class

$$\Delta(E) := c_2(E) - \frac{r-1}{2r} c_1(E)^2 \,\in\, H^4(X\,,\,\mathbb{R})\,,$$

where $c_i(E)$ is the i^{th} Chern class of E, and $r = rk(E)$. Note that, for any two vector bundles E and F on X,

- $c_1(E) = c_1(\det(E))$
- $\mathrm{ch}(E) = rk(E) + c_1(E) + \frac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big) + \dots$
- $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$,

where $\mathrm{ch}(E)$ is the *Chern character* of E, and $c_j(E)$ is the j^{th} Chern class of E. Therefore, $\Delta(E \otimes F) = rk(F)\Delta(E) + rk(E)\Delta(F)$. In particular, $\Delta(\mathcal{E}nd(E)) = 2\,rk(E)\Delta(E)$, since $\Delta(E^\vee) = \Delta(E)$. Since $c_1(E^\vee) = -c_1(E)$, so $c_1(\mathcal{E}nd(E)) = 0$. Therefore, it follows directly from the definition that $\Delta(\mathcal{E}nd(E)) = c_2(\mathcal{E}nd(E))$.

Theorem 3.21. *(Bogomolov) Let (X, H) be a poloarized smooth projective variety of dimension $n \geq 2$ over $\mathbb{C}$. If E is a H–semistable vector bundle on X, then $\Delta(E) \cdot H^{n-2} \geq 0$.*

Proof. Using Mehta-Ramanathan's restriction theorem 3.15, we can find an integer $n_0 \geq 1$, such that for any integer $m \geq m_0$, there is a non-empty open dense subset U of the linear system $|mH|$ such that for any $D_1, \dots, D_{n-2} \in U$ with $Y = D_1 \cap \dots \cap D_{n-2}$ a complete intersection, the vector bundle $E|_Y$ is semistable. Again, $m^{n-2}\Delta(E) \cdot H^{n-2} = \Delta(E|_Y)$. Therefore, it is enough to prove the theorem for X a smooth projective surface together with an ample line bundle H on it.

Now by Theorem 3.18, E is semistable if and only if $\mathcal{E}nd(E)$ is semistable. Therefore, by replacing E with $\mathcal{E}nd(E)$, we may assume that E is semistable with $c_1(E) = 0$. Then $E^{\otimes m}$ is also semistable with $c_1(E^{\otimes m}) = 0$. Note that $rk(E^{\otimes m}) = rk(E)^m$ and $\Delta(E^{\otimes m}) = m\,rk(E)^{m-1}\Delta(E)$, for all $m \geq 1$. Now again using Mehta-Ramanathan's restriction theorem 3.15, we can find an integer $q \gg 0$ and a general curve $C \in |qH|$ such that both $E|_C$ and $(E^{\otimes m})|_C$ are semistable.

Since $\mu_{\max}(E \otimes F) = \mu_{\max}(E) + \mu_{\max}(F)$, so we have $h^0(X, E^{\otimes m}) \leq m\,rk(E)^m\gamma(E)$, where $\gamma(E) = \max\{\mu_{\max}(E), 0\}$. Using Serre duality and enlarging $\gamma(E)$ if necessary, we have $h^2(X, E^{\otimes m}) \leq m\,rk(E)^m\gamma(E)$. Therefore,

$$\chi(E^{\otimes m}) = h^0(E^{\otimes m}) - h^1(E^{\otimes m}) + h^2(E^{\otimes m}) \leq 2m\,rk(E)^m\gamma(E)\,. \tag{3.6}$$

On the other hand, by Hirzebruch-Riemann-Roch formula for surface, we have

$$\chi(E) = rk(E)\deg(X) + \frac{1}{2}[c_1(E)^2 - c_1(E)c_1(K_X)] - c_2(E)\,,$$

where $\deg(X) = \chi(\mathcal{O}_X)$. Since $c_1(E) = 0$ by our assumption, so $\chi(E) = rk(E)\deg(X) - c_2(E) = rk(E)\deg(X) - \Delta(E)$. Replacing E with $E^{\otimes m}$ above, we have

$$\chi(E^{\otimes m}) = rk(E)^m \deg(X) - m\, rk(E)^{m-1}\Delta(E)\,. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we have

$$rk(E)^m \deg(X) - m\, rk(E)^{m-1}\Delta(E) \leq 2m\, rk(E)^m \gamma(E)\,,\ \ \forall\, m \geq 1\,.$$

If $\Delta(E) < 0$, then taking m sufficiently large, we get a contradiction. Therefore, $\Delta(E) \geq 0$. □

What can we say if we assume $\Delta(E)\cdot H^{n-2} = 0$ in addition to semistability of E?

Theorem 3.22. [Mi, BG] *Let (X, H) be a poloarized smooth projective variety over $\mathbb{C}$. Let E be a vector bundle on X. Then the following are equivalent:*

(1) E is H–semistable and $\Delta(E)\cdot H^{n-2} = 0$.
(2) for any smooth projective curve C and any morphism $f : C \longrightarrow X$, the pullback F^*E is semistable on C.
(3) E admits a filtration by subsheaves

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that the quotients E_i/E_{i-1} are projectively flat bundles and $\mu(E_i/E_{i-1}) = \mu(E)$, for all $i = 1, \ldots, l$.

Note that, E is semistable if and only if $\mathcal{E}nd(E)$ is semistable. Since $c_1(\mathcal{E}nd(E)) = 0$, so $\Delta(\mathcal{E}nd(E)) = c_2(\mathcal{E}nd(E))$. Now, condition (1) of the above theorem is equivalent to $\mathcal{E}nd(E)$ is H–semistable with $\Delta(\mathcal{E}nd(E))\cdot H^{n-2} = c_2(\mathcal{E}nd(E))\cdot H^{n-2} = 0$ and $c_1(\mathcal{E}nd(E)) = 0$; which in turns, is equivalent to the fact that all Chern classes $c_i(\mathcal{E}nd(E))$ vanishes. Therefore, we may replace the condition (1) above with E is semistable with $\Delta(E) = 0$.

Definition 3.23. A projective variety X over k is said to be a Higgs variety if one of the following holds:

- $\dim X = 1$ or,
- if $\dim X \geq 2$, then the following holds: if (E, ϕ) is a Higgs bundle over X such that $f^*(E, \phi)$ is semistable on C, for every morphism $f : C \longrightarrow X$ of a smooth projective curve C into X, then $\Delta(E) = 0$.

3.2. **An Example.** We now give an example of a μ–stable Higgs bundle on X, whose underlying vector bundle is not μ–semistable.

Example 3.2. Let X be an irreducible smooth complex projective algebraic curve of genus $g \geq 2$. Let $K = \Omega^1_X$ be the canonical line bundle over X. A square root of K is a line bundle L over X such that $L^{\otimes 2} = K$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^{\times} \longrightarrow 0\,.$$

Look at the long exact sequence of cohomologies associated to it:

$$\deg\ :\ \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \simeq \mathbb{Z} \longrightarrow 0\,.$$

Since $\deg(K) = 2g-2$ is even, there is a line bundle L such that $L \otimes L \cong K$, because $\mathrm{Pic}^0(X)$ is an abelian variety which is a divisible group. Any two square roots L and L' of K are differ by tensor product of a 2–torsion line bundle (i.e., a square root of the trivial line bundle $\mathcal{O}_X$). Note that, $\mathrm{Pic}^0(X)$ is a divisible group, and the number of 2–torsion points in $\mathrm{Pic}^0(X)$ is 2^{2g}. So, there are 2^{2g} number of isomorphism classes of line bundles L on X such that $L^{\otimes 2} = K$. Fix such a line bundle, and denote it by $K^{1/2}$. Let $K^{-1/2}$ be the dual line bundle of $K^{1/2}$. Consider the vector bundle $E = K^{1/2} \oplus K^{-1/2}$ over X. Choose a nonzero section $\omega \in H^0(X, K^2) \cong H^0(X, \mathcal{H}om(K^{-1/2}, K^{1/2} \otimes K))$. Define an $\mathcal{O}_X$–linear homomorphism

$$\theta : E = K^{1/2} \oplus K^{-1/2} \longrightarrow (K^{1/2} \oplus K^{-1/2}) \otimes K\,, \tag{3.8}$$

by setting

$$\theta = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}\,,$$

where $1 \in H^0(X, \mathcal{H}om(K^{1/2}, K^{-1/2} \otimes K)) \simeq H^0(X, \mathcal{O}_X)$ is the identity section of the trivial line bundle $\mathcal{O}_X$. Note that $\deg(K^{1/2}) = g-1$, and so $\deg(E) = \deg(K^{1/2}) + \deg(K^{-1/2}) = 0$. If (E, θ) were not stable, then there is a destabilizing torsion free subsheaf $F \subset E$ preserved by θ with $\mu(F) > 0 = \mu(E)$. Since $\deg(K^{-1/2}) = 1-g < 0$, that F cannot be a subsheaf of $0 \oplus K^{-1/2}$. Then F should be a subsheaf of $K^{1/2}$, which is not possible, since $K^{1/2}$ is not preserved by θ. Therefore, (E, θ) is μ–stable as Higgs bundle. Since $\mu(K^{1/2}) = g-1 = -\mu(K^{-1/2})$, where g is the genus of X, it follows from the above Proposition 3.7 that, E is not even μ–semistable as a vector bundle.

4. Semistable Pairs

In this section, we shall see that under some conditions on Ω^1_X, underlying vector bundle of any μ–semistable Higgs bundle over X become μ–semistable.

Definition 4.1. Let E and G be two vector bundles over X, and $\phi : E \longrightarrow E \otimes G$ be a vector bundle homomorphism. The pair (E, ϕ) is said to be *semistable* (resp. *stable*), if for any torsion free coherent subsheaf F of E with $0 < rk(F) \leq$ (resp. $<$) $rk(E)$ and $\phi(F) \subseteq F \otimes G$, we have $\mu(F) \leq$ (resp. $<$)$\mu(E)$.

Theorem 4.2. [BG, p. 403] *Let X be a smooth projective variety over an algebraically closed field k of characteristic 0. Let E and F be a vector bundle over X, with a vector bundle homomorphism $\phi : E \longrightarrow E \otimes F$. Assume that F semistable and $\deg(F) \leq 0$. If (E, ϕ) is a semistable pair, then E is μ–semistable.*

Proof. Suppose that E is not μ–semistable as vector bundle. Then we have a nontrivial Harder–Narasimhan filtration of E

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\,,$$

with E_i/E_{i-1} semistable of slope μ_i satisfying

$$\mu_{\max}(E) := \mu_1 > \mu_2 > \cdots > \mu_l =: \mu_{\min}(E)\,. \tag{4.1}$$

Since the pair (E, ϕ) is semistable, and $\mu(E_1) > \mu(E)$, so E_1 is not ϕ–invariant; i.e, $\phi(E_1) \nsubseteq E_1 \otimes F$. Let $j > 1$ be the smallest integer such that $\phi(E_1) \subset E_j \otimes F$. Then the induced homomorphism

$$\phi : E_1 \longrightarrow (E_j/E_{j-1}) \otimes F$$

is nonzero. Since both E_j/E_{j-1} and F are semistable, their tensor product $(E_j/E_{j-1}) \otimes F$ is also μ–semistable (see Theorem 3.18), with slope

$$\mu((E_j/E_{j-1}) \otimes F) = \mu(E_j/E_{j-1}) + \mu(F)\,. \tag{4.2}$$

Then using Proposition 3.3, equality (4.2), and the condition $\deg(F) \leq 0$, we have

$$\mu_1 = \mu(E_1) \leq \mu(E_j/E_{j-1}) + \mu(F) \leq \mu(E_j/E_{j-1}) = \mu_j\,, \text{ with } j > 1\,.$$

This contradicts (4.1). □

Corollary 4.3. *Let X be a smooth projective variety over k, whose cotangent bundle Ω^1_X is μ–semistable with $\deg(\Omega^1_X) \leq 0$. Then for any μ-semistable Higgs bundle (E, θ) over X, the underlying vector bundle E is μ-semistable.*

Example 4.1. The following are some examples of smooth projective varieties over k whose cotangent bundles Ω^1_X are μ–semistable and have non-positive degree. Then the above Corollary 4.3 applies to them.

(1) Let X be a smooth projective curve of genus 1 over k. Then $\deg(\Omega^1_X) = 0$.
(2) Let $X = \mathbb{P}^n_k$, $n \geq 1$. Then Ω^1_X is μ–semistable by Theorem 3.10, and $\deg(\Omega^1_X) = -n - 1 \leq 0$.
(3) An *abelian variety* is a complete connected group variety over a field k. It is a projective variety with trivial tangent bundle (see [M]). Hence Corollary 4.3 applies.
(4) A smooth projective variety X over k is said to be *Fano* (resp. *Calabi–Yau*) if its anti-canonical line bundle K^*_X is ample (resp. trivial). For such a variety X, the degree of K_X is negative (resp. zero). A large class of Fano (resp. Calabi–Yau) varieties have semistable cotangent bundles (see [BG, St]).

5. Principal Higgs bundle

5.1. **Definitions.** Let X be a polarized smooth projective variety over a field $k\,(= \mathbb{C})$. Let Ω^1_X be the cotangent bundle of X. Let G be an affine algebraic group over k. The Lie algebra of G will be denoted by $\mathfrak{g}$. A *principal G–bundle* on X is a variety E_G together with a surjective submersion $p : E_G \longrightarrow X$ and a right G–action $\sigma : E_G \times G \longrightarrow E_G$ such that

(1) $p \circ \sigma = p \circ pr_1$, where $pr_1 : E_G \times G \longrightarrow E_G$ is the projection onto the first factor,
(2) the map $(\sigma, pr_1) : E_G \times G \longrightarrow E_G \times_X E_G$ is an isomorphism, and
(3) there is an étale open cover $\mathcal{U} = \{U_i \xrightarrow{f_i} X\}_{i \in I}$ of X such that there is a G–equivariant isomorphism $\phi_i : U_i \times_X E_G \longrightarrow U_i \times G$, for all $i \in I$; here the G–action on $U_i \times_X E_G$ (resp. $U_i \times G$) is given by the trivial G–action on U_i, and the given G–action on E_G (resp. the multiplication action of G on itself).

The first condition says that the action of G is along the fibers of p, the second condition says that the G–action on E_G is free and transitive, and the third condition says that the map p is locally trivial in the étale topology on X.

The action σ on E_G and the adjoint action of G on $\mathfrak{g}$ give rise to a G–action on $E_G \times \mathfrak{g}$, and the resulting quotient space is denote by $E_G \times^G \mathfrak{g}$. One can check that the natural projection map $p_{\text{ad}} : E_G \times^G \mathfrak{g} \longrightarrow X$ induced by p, makes it a vector bundle, whose fiber over a point $x \in X$ is non-canonically isomorphic to the Lie algebra $\mathfrak{g}$. We call $E_G \times^G \mathfrak{g}$ the adjoint vector bundle of E_G, and denote it by $\text{ad}(E_G)$. The Lie algebra structure on the fibers of $p_{\text{ad}} : \text{ad}(E_G) \longrightarrow X$ produces a Lie algebra structure

$$[\, , \,]_U : \text{ad}(E_G)(U) \times \text{ad}(E_G)(U) \longrightarrow \text{ad}(E_G)(U)$$

on the sections of $\text{ad}(E_G)$ over an open subset $U \subset X$.

Definition 5.1. A *principal Higgs G–bundle* on X is a pair (E_G, θ), where $p : E_G \longrightarrow X$ is a principal G–bundle on X, and $\theta \in H^0(X, \text{ad}(E_G) \otimes \Omega^1_X)$ such that $\theta \wedge \theta = 0$ in $H^0(X, \text{ad}(E_G) \otimes \Omega^2_X)$. The product $\theta \wedge \theta$ is constructed using the Lie algebra structure on the sections of $\text{ad}(E_G)$ and the exterior algebra structure on the sections of Ω^1_X. In terms of a local coordinate $z = (z_1, \cdots, z_n)$ on an open neighbourhood U of a point $x \in X$, if $s = \sum_{i=1}^{n} s_i \otimes dz_i, t = \sum_{j=1}^{n} t_j \otimes dz_j \in \Gamma(U, \text{ad}(E_G) \otimes \Omega^1_X)$, then $s \wedge t = \sum_{i,j} [s_i, t_j] \otimes (dz_i \wedge dz_j)$.

For any closed subgroup $P \subset G$ of G, we have a principal G/P–bundle $E_G(G/P)$ over X by extending the structure group of E_G by the homomorphism $pr : G \longrightarrow G/P$. Note that $E_G(G/P) \simeq E_G/P$. A *reduction of the structure group* of E_G to P is given by a principal P–bundle E_P over X together with an isomorphism of principal G–bundles

$$\phi : E_P \times^P G \xrightarrow{\simeq} E_G$$

over X. This is equivalent to giving a section $\tau : X \longrightarrow E_G/P \simeq E_G(G/P)$ of the principal G/P–bundle $p_1 : E_G/P \longrightarrow X$. Because, $E_G \longrightarrow E_G/P$ is a principal P–bundle over E_G, and then the pullback $\tau^* E_G \longrightarrow X$ is the right choice for E_P over X.

5.2. **Semistable principal Higgs bundle.** From now on, we assume that G is reductive. Let P be a parabolic subgroup of G; meaning that P is a closed subgroup of G such that G/P is a complete k–variety. Given a character $\chi : P \longrightarrow \mathbb{G}_m$ of P, and a principal P–bundle E_P on X, we can associate a line bundle $\chi_* E_P := E_P \times^\chi \mathbb{G}_a = (E_P \times \mathbb{G}_a)/P$ on X, where P acts on $\mathbb{G}_a$ by the character χ.

Definition 5.2. [Ra1, Ra2] A principal G–bundle $p : E_G \longrightarrow X$ on X is said to be *semistable* (resp. *stable*), if for given a reduction $E_P \subset E_G$ of the structure group of E_G to any proper parabolic subgroup $P \subset G$ over any open subset $U \subset X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$, and any nontrivial dominant character $\chi : P \longrightarrow \mathbb{G}_m$, we have $\deg(\chi_* E_P) \leq 0$ (resp. < 0).

Definition 5.3. A principal Higgs G–bundle (E_G, θ) on X is said to be *semistable* (resp. *stable*), if for given a reduction $E_P \subset E_G$ of the structure group of E_G to any proper parabolic subgroup $P \subset G$ over any open subset $U \subset X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$, such that $\theta \in H^0(X, \text{ad}(E_P) \otimes$

Ω^1_X), and for any nontrivial dominant character $\chi : P \longrightarrow \mathbb{G}_m$, we have $\deg(\chi_* E_P) \leq 0$ (resp. < 0).

It follows from Theorem 3.18 that a vector bundle E on X is semistable if and only if $\mathcal{E}nd(E)$ is semistable. This has the following generalization to the case of principal bundles.

Proposition 5.4. [RR, Theorem 3.18][AB, Proposition 2.10] *A principal G–bundle E_G on X is semistable if and only if the associate adjoint vector bundle* $\mathrm{ad}(E_G)$ *is μ–semistable.*

If (E_G, θ) is a principal Higgs G–bundle over X, then the Higgs field $\theta \in H^0(X, \mathrm{ad}(E_G) \otimes \Omega^1_X)$ defines a Higgs field $\theta_{\mathrm{ad}} \in H^0(X, \mathcal{E}nd(\mathrm{ad}(E_G)) \otimes \Omega^1_X)$ on the associated adjoint vector bundle $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ by Lie algebra operation on the sections of $\mathrm{ad}(E)$. Then Proposition 5.4 has the following generalization in case of principal Higgs G–bundles over X.

Lemma 5.5. [AB, Lemma 4.7, p. 226] *A principal Higgs G–bundle (E_G, θ) on X is semistable if and only if the associated adjoint Higgs vector bundle* $(\mathrm{ad}(E_G), \theta_{\mathrm{ad}})$ *is μ–semistable.*

Corollary 5.6. *Let X be a smooth projective variety over k, such that Ω^1_X is a μ–semistable vector bundle of degree ≤ 0. If (E_G, θ) is a semistable principal Higgs G–bundle over X, then E_G is semistable.*

Proof. Follows from Corollary 4.3 and Lemma 5.5. □

References

[AB] B. Anchouche, I. Biswas, Einstein-Hermitian Connections on Polystable Principal Bundles over a Compact Kähler Manifold, *Amer. Jour. Math.* **123** (2001), 207–228.

[BG] Ugo Bruzzo, Alessio Lo Giudice, Restricting Higgs bundles to curves, *Asian J. Math.* **20** (2016), no 3, 399–408.

[BHH] I. Biswas, G. Hein, N. Hoffmann, On semistable vector bundles over curves, *C. R. Math. Acad. Sci. Paris* **346** (2008), no. 17-18, 981–984.

[F] G. Faltings, Stable G–bundles and projective connections, *J. Algebraic Geom.* **2** (1993) 507–568.

[Har] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. *Springer-Verlag, New York-Heidelberg*, 1977. xvi+496 pp.

[HL] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Second edition, Cambridge University Press, Cambridge (2010).

[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Reprint of the 1987 edition, Princeton University Press, Princeton, NJ, (2014) [MR0909698].

[M] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London (1970) [MR0282985].

[MR1] V. Mehta, A. Ramanathan, Semistable sheaves on projective varieties and the restriction to curves, *Math. Ann.* **258** (1982), 213–226.

[MR2] V. Mehta, A. Ramanathan, Restriction of stable sheaves and representations of the fundamental group. *Invent. Math.* **77** (1984), 163–172.

[Mi] Yoichi Miyaoka, The Chern classes and Kodaira dimension of a minimal variety, *Algebraic geometry, Sendai*, 1985, 449–476, Adv. Stud. Pure Math., **10**, *North-Holland, Amsterdam*, 1987.

[NS1] M.S. Narasimhan, C.S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math. (2)* **82** (1965), 540–567.

[NS2] M.S. Narasimhan, C.S. Seshadri, Holomorphic vector bundles on a compact Riemann surface. *Math. Annalen* **155** (1964), 69–80.

[OSS] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, (2011) [MR2815674].

[Ra1] A. Ramanathan, Stable principal bundles on a compact Riemann surface, *Math. Ann.* **213** (1975), 129–152.

[Ra2] A. Ramanathan, Moduli for principal bundles over algebraic curves. I, *Proc. Indian Acad. Sci. Math. Sci.* **106** (1996), 301–328.

[RR] S. Ramanan, A. Ramanathan, Some remarks on the instability flag, *Tohoku Math. Journal* **36** (1984), 269–291.

[Si] C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

[St] A. Steffens, On the stability of the tangent bundle of Fano manifolds, *Math. Ann.* **304** (1996), no. 4, 635–643.

ARJUN PAUL
SCHOOL OF MATHEMATICS,
TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
HOMI BHABHA ROAD, MUMBAI 400005, MAHARASHTRA, INDIA.
EMAIL: apmath90@math.tifr.res.in