1. Model Answer

Question. Use Plancherel's identity to find a relation between the solution and the initial data in terms of mean square norm called L^2 norm of the Cauchy problem:

$$u_t + u_{xxxx} + 2u = 0, \ -\infty < x < \infty, \ t > 0$$

with initial data $u(x, 0) = u_0(x), \ -\infty < x < \infty$; and also find $\lim\limits_{t \to \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 \, dx$.

Model Answer: We have given

(1.1) $$u_t + u_{xxxx} + 2u = 0, \ -\infty < x < \infty, \ t > 0,$$

with initial data $u(x, 0) = u_0(x), \ -\infty < x < \infty$. Taking Fourier transform with respect to x, from equation (1.1), we get

$$\widehat{u}_t(w, t) + (iw)^4 \widehat{u} + 2\widehat{u} = 0$$

$$\implies \widehat{u}_t(w, t) + (w^4 + 2)\widehat{u} = 0$$

(1.2) $$\implies \widehat{u}(w, t) = \widehat{u}_0(w) e^{-t(w^4 + 2)}$$

Using Plancherel's identity and (1.1), we get

$$\int_{-\infty}^{\infty} |u(x, t)|^2 dx = \int_{-\infty}^{\infty} |\widehat{u}(w, t)|^2 dw = e^{-4t} \int_{-\infty}^{\infty} |e^{-2tw^4}| \cdot |\widehat{u}_0(w)|^2 dw.$$

Since $|e^{-2tw^4}| \leq 1, \forall t > 0$, from above equation we have

(1.3) $$\int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq e^{-4t} \int_{-\infty}^{\infty} |\widehat{u}_0(w)|^2 dw.$$

Since $e^{-4t} \leq 1$, for all $t > 0$, the relation between the solution and the initial data is

$$\int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq \int_{-\infty}^{\infty} |\widehat{u}_0(w)|^2 dw, \ \forall t > 0.$$

Taking limit as $t \to \infty$ in (1.3), we get

$$0 \leq \lim_{t \to \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq \lim_{t \to \infty} e^{-4t} \int_{-\infty}^{\infty} |\widehat{u}_0(w)| dw = 0.$$

$$\implies \lim_{t \to \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 dx = 0.$$

$\square$