MA3201: Topology

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List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\bigsqcup$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
$A := \ldots$	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Syllabus

MA3201 (Topology)

- **Metric Spaces:** Metric space topology, equivalent metrics, sequences, complete metric spaces, limits and continuity, uniform continuity, extension of uniformly continuous functions. [1 week]
- **Topological Spaces:** Definition, examples, bases, sub-bases, product topology, subspace topology, metric topology, second countability and separability. [2 weeks]
- **Continuity:** Continuous functions on topological spaces, homeomorphisms, quotient topology. [1 week]
- **Connectedness:** Definition, example, path connectedness and local connectedness. [2 weeks]
- **Compactness:** Definition, limit point compactness, sequential compactness, net and directed set, local compactness, Tychonoff theorem, Stone-Weierstrass theorem, ArzelaAscoli theorem. [3 weeks]
- **Separation Axioms:** Hausdorff, regular and normal spaces; Urysohn lemma and Tietze extension theorem; compactification. [2 weeks]
- **Metrizability:** Urysohn metrization theorem. [1 week]

Chapter 1

Metric Space

1.1 Definition and Examples

A *metric* on a set X is a map

$$d : X \times X \to [0, \infty) := \{t \in \mathbb{R} : t \geq 0\}$$

such that

(i) $d(x_1, x_2) \geq 0$, $\forall\, x_1, x_2 \in X$, with equality holds if and only if $x_1 = x_2$;

(ii) $d(x_1, x_2) = d(x_2, x_1)$, for all $x_1, x_2 \in X$, and

(iii) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$, for all $x_1, x_2, x_3 \in X$.

A *metric space* is a pair (X, d) consisting of a set X and a metric d on it.

Example 1.1.1. The *absolute value* of a real number $x \in \mathbb{R}$ is a non-negative real number $|x|$, defined by

$$|x| := \begin{cases} x, & \text{if} \quad x \geq 0, \\ -x, & \text{if} \quad x < 0. \end{cases}$$

Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x, y) = |x - y|,\ \forall\, x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, d)$ is a metric space.

Example 1.1.2 (Euclidean metric on $\mathbb{R}^n$)**.** Fix an integer $n \geq 1$, and let

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

be the map defined by

$$d\left((x_1, \ldots, x_n),\, (y_1, \ldots, y_n)\right) := \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2},$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{R}^n$, called the *Euclidean metric* on $\mathbb{R}^n$.

Example 1.1.3 (Euclidean metric on $\mathbb{C}^n$)**.** Fix an integer $n \geq 1$, and let

$$d : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$$

be the map defined by

$$d\left((z_1, \dots, z_n), (w_1, \dots, w_n)\right) := \left(\sum_{j=1}^{n} |z_j - w_j|^2\right)^{1/2},$$

for all $(z_1, \dots, z_n), (w_1, \dots, w_n) \in \mathbb{C}^n$. It is straight-forward to check that d is a metric on $\mathbb{C}^n$, called the *Euclidean metric* on $\mathbb{C}^n$.

Example 1.1.4 (Taxicab/rectilinear metric)**.** Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{j=1}^{n} |x_j - y_j|,$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$. Verify that d is a metric on $\mathbb{R}^n$.

Example 1.1.5. Given a non-empty set X, let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) := \begin{cases} 1, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Then d is a metric on X, called the *discrete metric* on X.

Exercise 1.1.6. Let (X, d) be a metric space. Show that

$$d'(x, y) := \min\{1, d(x, y)\}, \ \forall\, x, y \in X$$

defines a metric on X.

Exercise 1.1.7. Let (X, d) be a metric space. Show that the map $d' : X \times X \to \mathbb{R}$ defined by

$$d'(x_1, x_2) := \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}, \ \ \forall\, x_1, x_2 \in X,$$

is a metric on X.

Exercise 1.1.8 (Subspace)**.** Let (X, d) be a metric space. For any non-empty subset Y of X, show that the restriction map

$$d_Y : Y \times Y \to \mathbb{R}, \ \ (y_1, y_2) \mapsto d(y_1, y_2),$$

is a metric on Y, called the *induced metric* on Y from (X, d). Then the pair (Y, d_Y) is called the *subspace* of the metric space (X, d).

Example 1.1.9. Let d be the Euclidean metric on $\mathbb{R}$. Then

- $([0, 1], d)$ is a subspace of $(\mathbb{R}, d)$.

- $(\mathbb{Q}, d)$ is a subspace of $(\mathbb{R}, d)$.

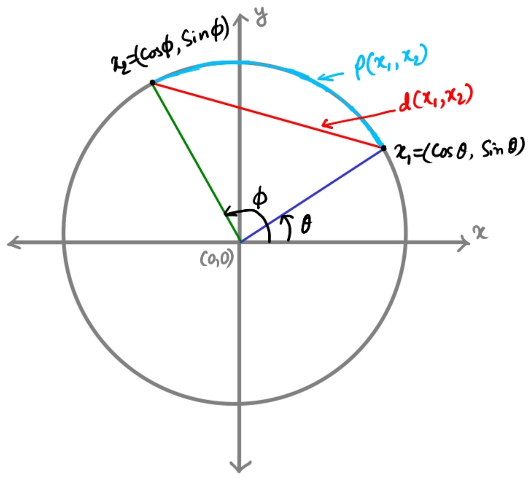
Exercise 1.1.10. Consider the unit circle

$$S^1 := \{(\cos t, \sin t) \in \mathbb{R}^2 : 0 \leq t < 2\pi\}$$

in $\mathbb{R}^2$. Given two points $\mathbf{x}_1 := (\cos\theta, \sin\theta), \mathbf{x}_2 := (\cos\phi, \sin\phi) \in S^1$, where $0 \leq \theta, \phi < 2\pi$, define

$$\rho(\mathbf{x}_1, \mathbf{x}_2) := \min\{|\theta - \phi|, 2\pi - |\theta - \phi|\}.$$

Show that ρ is a metric on S^1 that is not induced from the Euclidean metric d on $\mathbb{R}^2$.



Exercise 1.1.11. Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that the rule

$$d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, \forall\, (x_1, y_1), (x_2, y_2) \in X \times Y$$

defines a metric on $X \times Y$, called the *product metric* on $X \times Y$. (*Caution: This is a non-standard terminology, and has nothing to do with product in general sense*).

Definition 1.1.12. Let $\mathbb{k}$ be the field of real numbers or the field of complex numbers with the Euclidean metric on it. A *norm* on a $\mathbb{k}$-vector space X is a map $\|\cdot\| : X \to \mathbb{R}$ satisfying the following properties:

(i) $\|x\| \geq 0$, $\forall\, x \in V$, and $\|x\| = 0$ if and only if $x = 0$ in X.

(ii) $\|\alpha x\| = |\alpha|\|x\|$, $\forall\, \alpha \in \mathbb{k}$, $x \in X$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall\, x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *normed linear space*.

Example 1.1.13. Let $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ be defined by

(i) $\|(x_1, \ldots, x_n)\|_1 := \sum\limits_{j=1}^{n} |x_j|$, $\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$,

(ii) $\|(x_1,\ldots,x_n)\|_2 := \left(\sum\limits_{j=1}^{n} x_j^2\right)^{1/2}$, $\forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n$, and

(iii) $\|(x_1,\ldots,x_n)\|_\infty := \max\limits_{1\le j\le n}|x_j|$, $\forall\,(x_1,\ldots,x_n)\in\mathbb{R}^n$.

Then $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$.

Example 1.1.14. Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ together with the Euclidean metric on it. Let X be a non-empty set and let $\mathcal{B}(X)$ be the set of all $\mathbb{k}$-valued bounded functions defined on X. Define

$$\|f\|_\infty := \sup\{|f(x)| : x\in X\},\ \forall\, f\in\mathcal{B}(X).$$

Then $\|\cdot\|_\infty$ is a norm on $\mathcal{B}(X)$.

Proposition 1.1.15. *Let $(X,\|\cdot\|)$ be a normed linear space. Then the map $d : X\times X\to\mathbb{R}$ defined by*

$$d(x,y) := \|x-y\|,\ \ \forall\, x,y\in X,$$

is a metric on X, called the norm-induced metric on $(X,\|\cdot\|)$.

Proof. Let $x,y\in X$ be arbitrary. Then by definition of norm, we have $d(x,y)=\|x-y\|\ge 0$, for all $x,y\in X$, with equality holds if and only if $x-y=0$, i.e., $x=y$. Note that, $d(y,x)=\|y-x\|=|-1|\|x-y\|=d(x,y)$. Moreover, given any $z\in X$, we have

$$\begin{aligned} d(x,y) &= \|x-y\| = \|(x-z)+(z-y)\| \\ &\le \|x-z\|+\|z-y\| \\ &= d(x,z)+d(z,y) \\ &= d(x,z)+d(y,z). \end{aligned}$$

Therefore, d is a metric on X. $\square$

Example 1.1.16. Given any $(x_1,\ldots,x_n),(y_1,\ldots,y_n)\in\mathbb{R}^n$, the following formulae

- $d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := |x_1-y_1|+\cdots+|x_n-y_n|$,
- $d_2((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$, and
- $d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := \max\{|x_1-y_1|,\ldots,|x_n-y_n|\}$,

define metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ induced by the norms $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively.

Exercise 1.1.17. Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ together with the standard Euclidean norm on it. Fix an integer $n\ge 1$. For any real number $p\ge 1$, show that the map $\|\cdot\|_p : \mathbb{k}^n\to\mathbb{R}$ defined by

$$\|(x_1,\ldots,x_n)\|_p := \left(|x_1|^p+\cdots+|x_n|^p\right)^{\frac{1}{p}},\ \forall\,(x_1,\ldots,x_n)\in\mathbb{k}^n$$

is a norm on $\mathbb{k}^n$, for all $n\ge 1$. The normed linear space $(\mathbb{k}^n,\|\cdot\|_p)$ is denoted by $\ell_p^n(\mathbb{k})$.

Exercise 1.1.18. Fix a real number p with $0<p<1$, and an integer $n\ge 2$.

(i) Show that the the map $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := (|x_1 - y_1|^p + \cdots + |x_n - y_n|^p)^{\frac{1}{p}},$$

for all $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \mathbb{R}^n$, is not a metric on $\mathbb{R}^n$. (*Hint:* Show that the triangle inequality fails for $x = (1,1,0,\ldots,0), y = (0,1,0,\ldots,0)$ and $z = (0,0,\ldots,0)$ in $\mathbb{R}^n$).

(ii) Verify if the map $d'_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d'_p((x_1,\ldots,x_n),(y_1,\ldots,y_n)) := |x_1 - y_1|^p + \cdots + |x_n - y_n|^p,$$

for all $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$.

Exercise 1.1.19 (ℓ_p space)**.** Let $\Bbbk$ be the field of real numbers or the field of complex numbers together with the Euclidean metric on it. A *sequence* in $\Bbbk$ is a map $f : \mathbb{N} \to \Bbbk$; we generally denote it by $(a_n)_{n=1}^{\infty}$, where $a_n := f(n)$, $\forall\, n \in \mathbb{N}$. Fix a natural number $p \geq 1$. Let

$$\ell_p(\Bbbk) := \left\{ (a_n)_{n=1}^{\infty} : a_n \in \Bbbk,\ \forall\, n \in \mathbb{N},\ \text{and}\ \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}.$$

Given $a = (a_n)_{n=1}^{\infty}$ let

$$\|a\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}.$$

Show that $\|\cdot\|_p$ is a norm on $\ell_p(\Bbbk)$, and hence $\ell_p(\Bbbk)$ is a metric space.

Exercise 1.1.20. Fix real numbers a and b with $a < b$, and let

$$C[a,b] := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

be the set of all real-valued continuous maps defined on $[a,b]$. Show that the map $\|\cdot\| : C[a,b] \to \mathbb{R}$ defined by

$$\|f\| := \int_a^b |f(t)|dt,\ \forall\, f \in C[a,b],$$

is a norm on the $\mathbb{R}$-vector space $C[a,b]$, which makes $C[a,b]$ a metric space.

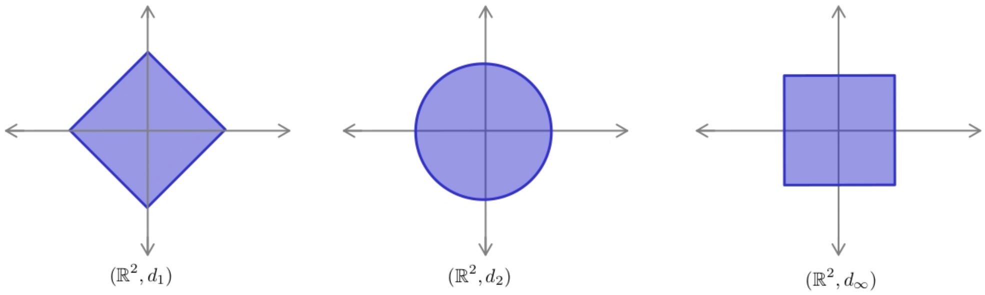
1.2 Topological properties

Let (X,d) be a metric space. Given a point $x_0 \in X$ and a real number $\delta > 0$, the *open ball in (X,d) with center at x_0 and radius δ* is the subset

$$B_d(x_0,\delta) := \{x \in X : d(x,x_0) < \delta\}.$$

Example 1.2.1. (i) In the real line $\mathbb{R}$ with the Euclidean metric d, the open ball with center at $0 \in \mathbb{R}$ and radius $r > 0$ is the open interval $(-r,r)$.

(ii) On $\mathbb{R}^2$, the open balls with center at the origin $(0,0) \in \mathbb{R}^2$ and radius 1 with respect to the metrics d_1, d_2 and d_∞ (see Example 1.1.16) are given as follow:



Let U be a non-empty subset of X. A point $x \in U$ is said to be an *interior point* of U if there exists a real number $\delta_x > 0$ such that $B(x, \delta_x) \subseteq U$. A subset $U \subseteq X$ is said to be *open* in (X, d) if either $U = \emptyset$ or each point of U is an interior point of U. A subset $Z \subseteq X$ is said to be *closed* if its complement $X \setminus Z$ is open in (X, d).

Example 1.2.2. Given $a, b \in \mathbb{R}$ with $a \leq b$, show that each of the intervals listed below are open with respect to the Euclidean metric on $\mathbb{R}$.

- $(a, b) := \{t \in \mathbb{R} : a < t < b\}$,
- $(-\infty, a) := \{t \in \mathbb{R} : t < a\}$,
- $(a, \infty) := \{t \in \mathbb{R} : a < t\}$, and
- $(-\infty, \infty) := \mathbb{R}$.

Lemma 1.2.3. *Let (X, d) be a metric space.*

(i) X and $\emptyset$ are both open and closed in (X, d).

(ii) Arbitrary union of open subsets of X is open.

(iii) Finite intersection of open subsets of X is open.

Proof. (i) Clear.

(ii) Let $\{U_\alpha : \alpha \in I\}$ be an indexed family of open subsets of (X, d). Let $x \in \bigcup_{\alpha \in I} U_\alpha$. Then there exists $\alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Then there exists a real number $\delta > 0$ such that $B(x, \delta) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$. Thus $\bigcup_{\alpha \in I} U_\alpha$ is open in (X, d).

(iii) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of (X, d). Let $x_0 \in \bigcap_{j=1}^{n} U_j$. Since $x_0 \in U_j$ and U_j is open in (X, d), there exists a $\delta_j > 0$ such that $B(x_0, \delta_j) \subseteq U_j$, for each $j = 1, \ldots, n$. Let $\delta := \min\{\delta_1, \ldots, \delta_n\} > 0$. Then $B(x_0, \delta) \subseteq B(x_0, \delta_j) \subseteq U_j$, for all $j = 1, \ldots, n$, and hence $B(x_0, \delta) \subseteq \bigcap_{j=1}^{n} U_j$.

□

Corollary 1.2.4. *Let (X, d) be a metric space. Then arbitrary intersections of closed subsets are closed, and a finite unions of closed subsets are closed.*

Example 1.2.5. Given $a, b \in \mathbb{R}$ with $a \le b$, let

$$[a, b] := \{t \in \mathbb{R} : a \le t \le b\}.$$

Since $[a, b] = \mathbb{R} \setminus ((-\infty, a) \bigcup (b, \infty))$, it is closed in $\mathbb{R}$.

Definition 1.2.6. A point $x_0 \in X$ is said to be a *limit point* of a subset $A \subseteq X$ if for each real number $\delta > 0$ we have

$$(B(x_0, \delta) \setminus \{x_0\}) \bigcap A \neq \emptyset.$$

Example 1.2.7. (i) Consider the Euclidean space $\mathbb{R}$. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then $0 \in \mathbb{R}$ is a limit point of A.

(ii) Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Equip X with the discrete metric d. Then 0 is not a limit point of (X, d).

(iii) Let $X = [0, 1] \cup \{2\}$ equipped with the metric d' induced from the Euclidean space $\mathbb{R}$. Then (X, d) is a metric subspace of $\mathbb{R}$. Let $A = B_{d'}(1, 1) = \{x \in X : d'(x, 1) < 1\}$. Then A is an open ball in X with center 1 and radius 1. However, $2 \in X$ is not a limit point of A.

Proposition 1.2.8. *Let (X, d) be a metric space. A subset Z of X is closed in (X, d) if and only if Z contains all of its limit points.*

Proof. Suppose that Z is closed in (X, d). If $x_0 \in U := X \setminus Z$, then U being open in (X, d), there exists a $\delta > 0$ such that $B(x_0, \delta) \subseteq U$, and so $B(x_0, \delta) \cap Z = \emptyset$. Therefore, x_0 cannot be a limit point of Z.

Conversely, suppose that Z contains all of its limit points in (X, d). Let $U := X \setminus Z$, and $x_0 \in U$. Since $x_0 \notin Z$ and x_0 is not a limit point of Z, there exists a $\delta > 0$ such that $B(x_0, \delta) \bigcap Z = \emptyset$. Therefore, $B(x_0, \delta) \subseteq U$. Since $x_0 \in U$ is arbitrary, U is open in (X, d) and hence Z is closed. $\square$

Lemma 1.2.9. *Let (X, d) be a metric space and let $A \subseteq X$. Let $\mathscr{C}_A$ be the collection of all closed subsets of (X, d) containing A. Then $\bigcap_{Z \in \mathscr{C}_A} Z$ is the smallest closed subset of X containing A, called the* ***closure of** A **in** (X, d).*

Proof. It follows from Lemma 1.2.3 that $\overline{A} := \bigcap_{Z \in \mathscr{C}_A} Z$ is a closed subset of X. Clearly $A \subseteq \overline{A}$. Let W be any closed subset of X containing A. Then $W \in \mathscr{C}_A$, and hence $\bigcap_{Z \in \mathscr{C}_A} \subseteq W$. $\square$

Proposition 1.2.10. *Let (X, d) be a metric space. A subset $A \subseteq X$ is closed if and only if $\overline{A} = A$.*

Proof. Suppose that A is closed. Let $\mathscr{C}_A$ be the collection of all closed subsets of (X, d) containing A. If A is closed, then $A \in \mathscr{C}_A$, and hence $A \subseteq \overline{A} = \bigcap_{Z \in \mathscr{C}_A} Z \subseteq A$ shows that $A = \overline{A}$. Converse is obvious since $\overline{A} = \bigcap_{Z \in \mathscr{C}_A} Z$ is closed. $\square$

Proposition 1.2.11. *Let (X, d) be a metric space. Given any two distinct points $x, y \in X$ there exist positive real numbers r_x and r_y such that $B(x, r_x) \cap B(y, r_y) = \emptyset$.*

Proof. Let $x, y \in X$ with $x \neq y$. Then $r := d(x,y) > 0$. Then the open balls $B(x, r/2)$ and $B(y, r/2)$ do not intersect each others. Indeed, if there were $z \in B(x,r/2) \cap B(y,r/2)$, then $d(x,z) < r/2$ and $d(z,y) < r/2$ gives $r = d(x,y) \leq d(x,z) + d(z,y) < r/2 + r/2 = r$, which is not possible. □

Definition 1.2.12 (Equivalent Metrics)**.** Two metrics d_1 and d_2 on a non-empty set X are said to be *topologically equivalent* if for any subset $U \subseteq X$, U is open in (X, d_1) if and only if U is open in (X, d_2).

Proposition 1.2.13. *Let d_1 and d_2 be two metrics on a non-empty set X. Then the following are equivalent.*

(i) d_1 and d_2 are topologically equivalent.

(ii) given any point $x \in X$ and a real number $r > 0$, there exists real numbers $r', r'' > 0$ such that

$$B_{d_2}(x, r'') \subseteq B_{d_1}(x, r) \quad \text{and} \quad B_{d_1}(x, r') \subseteq B_{d_2}(x, r).$$

Proof. Suppose that d_1 and d_2 are topologically equivalent metrics on X. Let $x \in X$ and $r > 0$ be given. Since $B_{d_1}(x,r)$ is open in (X, d_2), there exists $r'' > 0$ such that $B_{d_2}(x, r'') \subseteq B_{d_1}(x,r)$. Similarly, since $B_{d_2}(x,r)$ is open in (X, d_1), there exists $r' > 0$ such that $B_{d_1}(x,r') \subseteq B_{d_2}(x,r)$.

Conversely, suppose that given any point $x \in X$ and a real number $r > 0$, there exists real numbers $r', r'' > 0$ such that

$$B_{d_2}(x, r'') \subseteq B_{d_1}(x, r) \quad \text{and} \quad B_{d_1}(x, r') \subseteq B_{d_2}(x, r).$$

Let $U \subseteq X$. Suppose that U is open in (X, d_1). Then for given $x \in U$, there exists $r_x > 0$ such that $B_{d_1}(x, r_x) \subseteq U$. Then by assumption, there exists $r''_x > 0$ such that $B_{d_2}(x, r''_x) \subseteq B_{d_1}(x, r_x) \subseteq U$, and hence x is an interior point of U with respect to the d_2-metric on X. Therefore, U is open in (X, d_2). Similarly, if U is open in (X, d_2), then for each $x \in U$ there exists $s_x > 0$ such that $B_{d_2}(x, s_x) \subseteq U$. But then by our assumption, there exists $s'_x > 0$ such that $B_{d_1}(x, s'_x) \subseteq B_{d_2}(x, s_x) \subseteq U$, and hence x is an interior point of U with respect to the d_1 metric on X. Therefore, U is open in (X, d_1). □

Lemma 1.2.14. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Let d_1 and d_2 be the metrics on X induced by the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then d_1 is topologically equivalent to d_2 if and only if there exist positive real numbers $\alpha, \beta > 0$ such that*

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1, \quad \forall\, x \in X.$$

Proof. Suppose that there exist positive real numbers $\alpha, \beta > 0$ such that

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1, \quad \forall\, x, y \in X.$$

Let $U \subseteq X$ be open in (X, d_1). Then for given any $x \in U$, there exists a real number $r_x > 0$ such that

$$B_{d_1}(x, r_x) = \{y \in X : \|x - y\|_1 < r_x\} \subseteq U.$$

Since
$$B_{d_2}(x, \alpha r_x) = \{y \in X : \|x - y\|_2 < \alpha r_x\} \subseteq B_{d_1}(x, r_x) \subseteq U,$$
we see that U is open in (X, d_2). Now suppose that U is open in (X, d_2). Then given $x \in U$, there exists a real number $s_x > 0$ such that
$$B_{d_2}(x, s_x) = \{y \in X : \|x - y\|_2 < s_x\} \subseteq U.$$

Since
$$B_{d_1}(x, s_x/\beta) = \{y \in X : \|x - y\|_1 < s_x/\beta\} \subseteq \{y \in X : \|x - y\|_2 < \beta\} = B_{d_2}(x, s_x) \subseteq U,$$
we see that U is open in (X, d_1). Therefore, d_1 and d_2 are equivalent metrics on X.

Conversely, suppose that d_1 and d_2 are topologically equivalent metrics on X. Since $B_{d_1}(0, 1) = \{x \in X : \|x\|_1 < 1\}$ is open in (X, d_2), there exists a real number $r > 0$ such that $B_{d_2}(0, r) \subseteq B_{d_1}(0, 1)$. In other words,
$$\|x\|_1 < 1 \quad \text{whenever} \quad \|x\|_2 < r.$$
Now given any $x \in X$ with $x \neq 0$, let $y = (r/\|x\|_1)x \in X$ so that $\|y\|_1 = r$. Then we have $\|(r/\|x\|_1)x\|_2 < 1$, i.e., $\|x\|_2 < \frac{1}{r}\|x\|_1$. So we set $\beta = 1/r > 0$ to get $\|x\|_2 \leq \beta\|x\|_1$, $\forall\, x \in X$. Similarly, since $B_{d_2}(0, 1) = \{x \in X : \|x\|_2 < 1\}$ is open in (X, d_1), we can find a positive real number $\alpha > 0$ such that $\alpha\|x\|_1 \leq \|x\|_2$, $\forall\, x \in X$. This completes the proof. □

Exercise 1.2.15. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are said to be *equivalent* if there exist positive real numbers $\alpha, \beta > 0$ such that
$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \ \forall\, x \in X.$$
Show that norm equivalence on X is an equivalence relation.

Example 1.2.16. For any real number $p \geq 1$, we show that the ℓ_p-metric on $\mathbb{R}^n$ is equivalent to the ℓ_∞-metric on it. Indeed, given any point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ note that
$$\begin{aligned} \|(x_1, \ldots, x_n)\|_\infty &= \max\{|x_1|, \ldots, |x_n|\} \\ &\leq (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \\ &= \|(x_1, \ldots, x_n)\|_p \\ &\leq n^{\frac{1}{p}} \max\{|x_1|, \ldots, |x_n|\} \\ &= n^{\frac{1}{p}} \|(x_1, \ldots, x_n)\|_\infty. \end{aligned}$$
Therefore, ℓ_p-norm on $\mathbb{R}^n$ is equivalent to the ℓ_∞-norm on it, and hence the metrics induced by them on $\mathbb{R}^n$ are topologically equivalent. As a result, for any real numbers $p, q \geq 1$, the metrics on $\mathbb{R}^n$ induced by the ℓ_p-norm and the ℓ_q-norms on it are topologically equivalent.

The following results shows that any two norm induced topologies on a finite dimensional vector space are the same. We need notion of compact set, continuous maps and some related results to prove the following lemma. So you may skip it for the first reading.

Lemma 1.2.17. *Any two norm-induced metrics on a finite dimensional vector space are equivalent.*

Proof. Let X be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Without loss of generality we may assume that $X = \mathbb{C}^n$, for some $n \in \mathbb{N}$. Let $\|\cdot\|$ be a norm on X. It suffices to show that $\|\cdot\|$ is equivalent to the ℓ_2-norm $\|\cdot\|_2$ on $\mathbb{C}^n$. Let $\mathbf{z} = (z_1, \ldots, z_n) = z_1 e_1 + \cdots + z_n e_n \in \mathbb{C}^n$ be given. Then

$$\|\mathbf{z}\| \leq \sum_{j=1}^{n} |z_j| \|e_j\| \leq \left(\sum_{j=1}^{n} \|e_j\| \right) \|x\|_2.$$

Setting $M = \sum\limits_{j=1}^{n} \|e_j\| > 0$, we have

$$\|\mathbf{z}\| \leq M \|\mathbf{z}\|_2, \ \ \forall\, \mathbf{z} \in \mathbb{C}^n.$$

Since

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq M \|x - y\|_2, \ \forall\, x, y \in \mathbb{X},$$

the map $x \mapsto \|x\|$ from the $\mathbb{C}^n$ equipped with the ℓ_2-norm into $\mathbb{R}$ is continuous.

Let

$$\mathbb{S}^1 = \{x \in X : \|x\|_2 = 1\}$$

be the unit sphere in X with respect to the ℓ_2-norm. Note that $\mathbb{S}^1$ is closed and bounded, and so it is compact. Then the continuous map $x \mapsto \|x\|$ attains a minimum value at some point, say $x_0 \in \mathbb{S}^1$. Then $\|x\| \geq \|x_0\|, \forall\, x \in \mathbb{S}^1$. Let $K := \|x_0\| > 0$. Since $\|x_0\|_2 = 1$, it follows that $x_0 \neq 0$ and that $K > 0$. Now for given $x \in X \setminus \{0\}$, we have $\|x\| / \|x\|_2 = \|x / \|x\|_2\| \geq K$, and so

$$K \|x\|_2 \leq \|x\|, \ \forall\, x \in X.$$

This completes the proof. ☐

1.3 Sequence

Let (X, d) be a metric space.

Definition 1.3.1. A *sequence* in X is a map $f : \mathbb{N} \to X$. We generally denote a sequence $f : \mathbb{N} \to X$ by its image $\{x_n\}_{n \in \mathbb{N}}$, where $x_n := f(n), \ \forall\, n \in \mathbb{N}$.

Definition 1.3.2. A sequence $\{x_n\}$ in (X, d) is said to be a *Cauchy sequence* if given any real number $\epsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon, \ \forall\, m, n \geq n_0$.

A sequence $\{x_n\}$ in (X, d) is said to be *convergent* if there exists $x_0 \in X$ such that given any $\epsilon > 0$ there exists a natural number n_0 such that $d(x_n, x) < \epsilon, \ \forall\, n \geq n_0$. In this case, we say that x_0 is a *limit point* of the sequence $\{x_n\}$ and we denote this symbolically as $x_0 = \lim\limits_{n \to \infty} x_n$.

Lemma 1.3.3. *Any convergent sequence in*

Definition 1.3.4. A metric space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) is convergent in (X, d).

Example 1.3.5. (i) The open interval $(0,1) \subset \mathbb{R}$ with the Euclidean metric induced from $\mathbb{R}$ is not complete. Indeed, the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, is a Cauchy sequence in $(0,1)$, but it does not converges to a point of $(0,1)$.

(ii) The metric subspace $\mathbb{Q}$ of the Euclidean space $\mathbb{R}$ is not complete. Indeed, given any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can always find a Cauchy sequence of rational numbers $(x_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} x_n = \alpha$. For example, taking $x_n = \sum\limits_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$, $\forall\, n \in \mathbb{N}$, we see that the sequence (x_n) converges to $e \in \mathbb{R}$, which is not a rational number.

(iii) The real line $\mathbb{R}$ with the Euclidean metric on it is complete. This is a standard result from basic real analysis course.

(iv) Any discrete metric space is complete. Indeed, if d is a discrete metric on a non-empty set X, then any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) is *eventually constant* (i.e., there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n+1}$, $\forall\, n \geq n_0$).

Exercise 1.3.6. Let $C[0,1]$ be the set of all real valued continuous functions defined on $[0,1]$. Note that $C[0,1]$ is a metric space with respect to the metric defined by

$$d(f,g) := \int_0^1 |f(t) - g(t)|dt, \ \forall\, f,g \in C[0,1].$$

Is $(C[0,1],d)$ complete?

1.4 Limit and Continuity

1.5 Uniform continuity

1.6 Extension of uniformly continuous functions

Chapter 2

Point Set Topology

2.1 Topological space

A topology on a set X is given by specifying which subsets of X are 'open'. Naturally those subsets should satisfies certain properties as we are familiar from basic analysis and metric space courses.

Definition 2.1.1. A *topology* is on a set X is a collection τ of subsets of X satisfying the following properties:

(i) $\emptyset$ and X are in τ,

(ii) for any collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of objects of τ, their union $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \tau$,

(iii) for a finite collection of objects $U_1, \ldots, U_n \in \tau$, their intersection $\bigcap\limits_{i=1}^{n} U_i \in \tau$.

The pair (X, τ) is called a *topological space*, and the objects of τ are called *open subsets* of (X, τ). For notational simplicity, we suppress τ and denote a topological space (X, τ) simply by X.

Joke: An empty set may contain some air since it is open!

Remark 2.1.2. One can also define a topology on a set X by considering a collection τ_c of subsets of X such that

(i) both $\emptyset$ and X are in τ_c,

(ii) τ_c is closed under arbitrary intersections, and

(iii) τ_c is closed under finite unions.

This is known as the *closed set axioms for a topology*. In this settings, objects of τ_c are called *closed subsets* of X. It is easy to switch between these two definitions by taking complements of objects of τ and τ_c in X. However, unless explicitly mentioned, we usually work with open set axioms for topology.

Example 2.1.3. (i) If $X = \emptyset$ then $\tau = \{\emptyset\}$ is the only topology on $\emptyset$.

(ii) For any set X, $\tau_{\text{disc}} := \mathcal{P}(X)$ and $\tau_{\text{triv}} := \{\emptyset, X\}$ are topologies on X, called the *discrete topology* and the *indiscrete topology* on X, respectively. Note that, $\tau_{\text{disc}}(X)$ and $\tau_{\text{triv}}(X)$ are different if X has at least two elements.

(iii) Let $X \neq \emptyset$, and let $\tau = \{A \in \mathcal{P}(X) : X \setminus A \text{ is finite}\}$. Then (X, τ) is a topological space; such a topology is called the *cofinite topology* on X.

(iv) Consider the set $\mathbb{R}^n$. Let $\tau_E(\mathbb{R}^n)$ be the set of all subsets $U \subseteq \mathbb{R}^n$ such that given any $x \in U$ there exists a real number $r > 0$ such that

$$B(x, r) := \{y \in \mathbb{R}^n : \|x - y\| < r\} \subseteq U.$$

Then the set $\tau_E(\mathbb{R}^n)$ is a topology on $\mathbb{R}^n$ (verify!), called the *standard topology* or the *Euclidean topology* on $\mathbb{R}^n$.

(v) Any metric space (X, d) is a topological space where the topology on X is given by the collection of all open subsets of (X, d).

Let (X, τ) be a topological space. Given a subset Y of X, let

$$\tau_Y := \{U \cap Y : U \in \tau\}.$$

Clearly $\emptyset, Y \in \tau_Y$. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of elements of τ_Y. Then for each $\alpha \in \Lambda$, we have $A_\alpha = U_\alpha \cap Y$, for some $U_\alpha \in \tau$. Then $\bigcup\limits_{\alpha \in \Lambda} A_\alpha = \bigcup\limits_{\alpha \in \Lambda} (U_\alpha \cap Y) = \left(\bigcup\limits_{\alpha \in \Lambda} U_\alpha\right) \cap Y \in \tau_Y$. If $V_1, V_2 \in \tau_Y$, then $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$, for some $U_1, U_2 \in \tau$. Then $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y \in \tau_Y$. Therefore, τ_Y is a topology on Y, called the ***subspace topology*** on Y induced from (X, τ).

Example 2.1.4. (i) Consider the real line $\mathbb{R}$ with the Equipped with the Euclidean topology on it. Then the set $\mathbb{Q}$ inherits a subspace topology where a subset $U \subseteq Q$ is open if and only if $U = V \cap \mathbb{Q}$, for some open subset V of $\mathbb{R}$.

(ii) The subspace topology on $\mathbb{Z}$ induced from the Euclidean topology on $\mathbb{R}$ is discrete topology on $\mathbb{Z}$.

(iii) Consider $Y = [0, 1) \subset \mathbb{R}$. Note that open subsets of Y in the subspace topology induced from $\mathbb{R}$ are of the form $U \cap [0, 1)$, for some open subset U of $\mathbb{R}$. Note that $[0, 1/2)$ is open in Y, but not in $\mathbb{R}$.

Exercise 2.1.5. Are the subspace topology on the unit circle S^1 in the Euclidean plane $\mathbb{R}^2$ and the metric subspace topology on S^1 induced from the Euclidean metric on $\mathbb{R}^2$ the same?

Proposition 2.1.6. *Let Y be a subspace of a topological space X. If $U \subseteq Y$ is open in Y, and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, $U = Y \cap V$, for some open subset V of X. Since Y is open in X, $U = Y \cap V$ is open in X. □

Proposition 2.1.7. *Let Y be a closed subspace of X. If $Z \subseteq Y$ is closed in Y, then Z is closed in X.*

Proof. Note that, $X \setminus Z = (X \setminus Y) \cup (Y \setminus Z)$. Since Y is closed in X, $X \setminus Y$ is open in X. Since Z is closed in Y, $Y \setminus Z$ is open in Y, and hence $Y \setminus Z = U \cap Y$, for some open subset U of X. We claim that $X \setminus Z = (X \setminus Y) \cup U$. Since $Y \setminus Z = U \cap Y \subseteq U$, we have $X \setminus Z \subseteq (X \setminus Y) \cup U$. Again, since $Z \subseteq Y$ and $Y \setminus Z = U \cap Y$, we must have $U \subseteq X \setminus Z$. Therefore, $X \setminus Z = (X \setminus Y) \cup U$, and hence $X \setminus Z$ is open in X, which in turn gives that Z is closed in X. □

Definition 2.1.8 (Basis)**.** Let (X, τ) be a topological space. A subset $\mathcal{B} \subseteq \tau$ is said to be a *basis* for the topology τ on X if given any $U \in \tau$ and any $x \in U$, there exists an element $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$. The elements of $\mathcal{B}$ are called *basic open subsets* of X.

Remark 2.1.9. For some technical reason we include the empty subset $\emptyset$ of X in a basis for X.

Example 2.1.10. (i) If τ is the discrete topology on X, then $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for (X, τ).

(ii) Let $\mathcal{B}$ be the set of all open intervals $(a, b) \subset \mathbb{R}$, where $a < b$. Then $\mathcal{B}$ is a basis for the Euclidean topology on $\mathbb{R}$.

(iii) Let (X, d) be a metric space. Then $\mathcal{B} = \{B(x, r) : x \in X,\ r > 0\}$ is a basis for the metric topology on (X, d).

Lemma 2.1.11. *Let $\mathcal{B}$ be a basis for a topological space (X, τ). Then*

(i) $\bigcup\limits_{V \in \mathcal{B}} V = X$, *and*

(ii) any non-empty open subset of X is a unions of members from $\mathcal{B}$.

Proof. (i) Follows from the Definition 2.1.8 by taking $X = U \in \tau$.
(ii) Since $\mathcal{B} \subseteq \tau$ and τ is closed under arbitrary union, it remains to show that any $U \in \tau$ can be written as a union of members of $\mathcal{B}$. Let $U \in \tau$ be arbitrary. Since $\mathcal{B}$ is a basis for the topology τ on X, for each $x \in U$ there is a basic open subset $V_x \in \mathcal{B}$ with $x \in V_x$ such that $V_x \subseteq U$. Then $U = \bigcup\limits_{x \in U} V_x$. This completes the proof. □

Proposition 2.1.12. *Let $\mathcal{B}$ be a basis for a topological space (X, τ). Let $Y \subseteq X$. Then $\mathcal{B}_Y := \{V \cap Y : V \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof. Since $\mathcal{B}$ is a basis for (X, τ), we have $\bigcup\limits_{V \in \mathcal{B}} V = X$. Then $\bigcup\limits_{V \in \mathcal{B}} (V \cap Y) = Y$. Let $U \cap Y \in \tau_Y$ and $y \in U \cap Y$. Then there exists $V \in \mathcal{B}$ such that $y \in V \subseteq U$. Then $y \in V \cap Y \subseteq U \cap Y$. Therefore, $\mathcal{B}_Y$ is a basis for the subspace topology τ_Y on Y. □

Proposition 2.1.13 (Topology generated by a basis)**.** *Let X be a set. Let $\mathcal{B}$ be a collection of subsets of X satisfying the following properties.*

(i) $\bigcup\limits_{V \in \mathcal{B}} V = X$, *and*

(ii) given any $V_1, V_2 \in \mathcal{B}$ and a point $x \in V_1 \cap V_2$, there exists a $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.

Then there is a unique topology $\tau_{\mathcal{B}}$ on X such that $\mathcal{B}$ is a basis for $\tau_{\mathcal{B}}$. Such a topology $\tau_{\mathcal{B}}$ on X is called the topology generated by $\mathcal{B}$.

Proof. Take

$$\tau_{\mathcal{B}} := \{U \in \mathcal{P}(X) : \text{for each } x \in U,\ \exists\, V_x \in \mathcal{B} \text{ such that } x \in V_x \subseteq U\}.$$

Clearly $\emptyset$ and X are in $\tau_{\mathcal{B}}$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any collection of objects from $\tau_{\mathcal{B}}$, and let $x \in \bigcup\limits_{\alpha\in\Lambda} U_\alpha$. Then $x \in U_\lambda$, for some $\lambda \in \Lambda$. Then by construction of $\tau_{\mathcal{B}}$ there exists a $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U_\lambda$, and hence $x \in V_x \subseteq \bigcup\limits_{\alpha\in\Lambda} U_\alpha$. Thus $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \tau_{\mathcal{B}}$. Let $U_1, U_2 \in \tau_{\mathcal{B}}$. Let $x \in U_1 \cap U_2$ be arbitrary. Then there exist $V_1, V_2 \in \mathcal{B}$ such that $x \in V_1 \subseteq U_1$ and $x \in V_2 \subseteq U_2$. Then by property (ii) of $\mathcal{B}$, there exists a $W \in \mathcal{B}$ such that $x \in W \subseteq V_1 \cap V_2 \subseteq U_1 \cap U_2$. Therefore, $U_1 \cap U_2 \in \tau_{\mathcal{B}}$. Thus, $\tau_{\mathcal{B}}$ is topology on X. It follows from the definition of $\tau_{\mathcal{B}}$ that $\mathcal{B}$ is a basis for $(X, \tau_{\mathcal{B}})$.

Let τ be a topology on X such that $\mathcal{B}$ is a basis for τ. Let $U \in \tau$ be arbitrary. Then given any $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V \subseteq U$. Then $U \in \tau_{\mathcal{B}}$ by construction of $\tau_{\mathcal{B}}$. Therefore, $\tau \subseteq \tau_{\mathcal{B}}$. Conversely, let $U \in \tau_{\mathcal{B}}$ be arbitrary. Then $U = \bigcup\limits_{x\in U} V_x$, where $V_x \in \mathcal{B}$ with $x \in V_x \subseteq U$, by Lemma 2.1.11. Since $\mathcal{B} \subseteq \tau$ and τ is closed under arbitrary union of its elements, $U \in \tau$. Therefore, $\tau_{\mathcal{B}} \subseteq \tau$, and hence $\tau_{\mathcal{B}} = \tau$. This proves uniqueness part. □

Proposition 2.1.14. *Let $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies τ and τ' on X. Then the following are equivalent.*

(i) $\tau \subseteq \tau'$,

(ii) given $V \in B$ and $x \in V$, there exists $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\tau \subseteq \tau'$. Let $V \in \mathcal{B}$ and $x \in V$ be given. Since $\mathcal{B} \subseteq \tau \subseteq \tau'$ and $\mathcal{B}'$ is a basis for τ', there exists a $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V$.

(ii) $\Rightarrow$ (i): Let $U \in \tau$ be arbitrary. Since $\mathcal{B}$ is a basis for τ, by Lemma 2.1.11 we have $U = \bigcup\limits_{x\in U} V_x$, where $V_x \in \mathcal{B}$ with $x \in V_x \subseteq U$. Then by (ii) there exists $W_x \in \mathcal{B}'$ such that $x \in W_x \subseteq V_x$, for all $x \in U$. Then $U = \bigcup\limits_{x\in U} W_x \in \tau'$. □

Example 2.1.15. Given $a, b \in \mathbb{R}$ with $a < b$, let $[a, b) := \{t \in \mathbb{R} : a \leq t < b\}$. Let $\mathcal{B}_\ell = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R} \text{ with } a < b\}$. Note that $\mathcal{B}_\ell$ is a basis for a topology τ_ℓ on $\mathbb{R}$, called the *lower limit topology* on $\mathbb{R}$. We denote by $\mathbb{R}_\ell$ the topological space $(\mathbb{R}, \tau_\ell)$. Let $a, b \in \mathbb{R}$ with $a < b$ and let $c \in (a, b)$. Then $c \in [c, b) \subset (a, b)$. Then it follows from Proposition 2.1.14 that $\tau_E \subseteq \tau_\ell$, where τ_E is the Euclidean topology on $\mathbb{R}$. Fix $a, b \in \mathbb{R}$ with $a < b$. Then $a \in [a, b)$, but there is no open interval (c, d) in $\mathbb{R}$ such that $a \in (c, d) \subseteq [a, b)$. Therefore, τ_ℓ is strictly finer than the Euclidean topology on $\mathbb{R}$.

Example 2.1.16 (K-topology)**.** Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $\mathcal{B}_K$ be the set of all open intervals in $\mathbb{R}$ along with the subsets of the form $(a, b) \setminus K$, where $a, b \in \mathbb{R}$ with $a < b$. It is easy to check that $\mathcal{B}_K$ is a basis for a topology τ_K on $\mathbb{R}$ which is strictly finer than the Euclidean topology on

$\mathbb{R}$. The topology τ_K on $\mathbb{R}$ generated by $\mathcal{B}_K$ is called the *K-topology* on $\mathbb{R}$. Note that $(-1,1) \setminus K$ is open in the K-topology on $\mathbb{R}$, but not in the Euclidean topology on $\mathbb{R}$ because there is no open interval (a,b) containing 0 and contained in $(-1,1) \setminus K$.

Exercise 2.1.17. Show that the lower limit topology and the K-topology on $\mathbb{R}$ are not comparable in the sense that neither $\tau_\ell \subseteq \tau_K$ nor $\tau_K \subseteq \tau_\ell$.

Let X be a non-empty set and let $\mathcal{P}(X)$ be the power set of X. Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be such that $\bigcup_{V \in \mathcal{S}} V = X$. Then we can use $\mathcal{S}$ to construct a topology on X as follow: let $\mathcal{B} \subseteq \mathcal{P}(X)$ be the set of all finite intersections of elements from $\mathcal{S}$. Note that $\mathcal{S} \subseteq \mathcal{B}$ and so $\bigcup_{V \in \mathcal{B}} V = X$. Let $V, W \in \mathcal{B}$ be arbitrary. Then $V = \bigcap_{j=1}^{m} V_j$ and $W = \bigcap_{k=1}^{n} W_k$, for some $V_1, \ldots, V_m, W_1, \ldots, W_n \in \mathcal{S}$. Then their intersection $V \cap W = \left(\bigcap_{j=1}^{m} V_j \right) \cap \left(\bigcap_{k=1}^{n} W_k \right)$ is again a finite intersection of elements from $\mathcal{S}$, and hence is an element of $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis for a topology $\tau_\mathcal{B}$ on X, called the ***topology generated by the subbasis*** $\mathcal{S}$. This motivates us to define the notion of subbasis for a topological space as follow.

Definition 2.1.18. Let X be a topological space. A set $\mathcal{S} \subseteq \mathcal{P}(X)$ of subsets of X is said to be a ***subbasis for the topology on*** X if $\bigcup_{V \in \mathcal{S}} V = X$ and the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on X.

Example 2.1.19. (i) The collection $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ is a subbasis for the Euclidean topology on $\mathbb{R}$.

(ii) The collection $\mathcal{S}_\ell := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{[b, \infty) : b \in \mathbb{R}\}$ is a subbasis for the lower limit topology on $\mathbb{R}$.

(iii) The collection $\mathcal{S} := \{[a,b] : a, b \in \mathbb{R},\ a < b\}$ is a subbasis for a topology τ on $\mathbb{R}$, where τ is strictly finer than the Euclidean topology on $\mathbb{R}$. Is τ discrete?

2.1.1 Order topology

Let $(X, \leq)$ be a simply ordered (i.e., totally ordered) set. Given $a, b \in X$ with $a < b$, there are four types of subset of X that are called intervals determined by a and b, namely

$$\begin{aligned}
(a,b) &:= \{x \in X : a < x < b\}, \\
[a,b] &:= \{x \in X : a \leq x \leq b\}, \\
[a,b) &:= \{x \in X : a \leq x < b\}, \\
(a,b] &:= \{x \in X : a < x \leq b\}.
\end{aligned}$$

Let $\mathcal{B}$ be the set of all subsets of X of the following forms

(i) all open intervals (a,b) in X,

(ii) all intervals of the form $[a_0, b)$, where a_0 is the smallest element, if exists, of X,

(iii) all intervals of the form $(a, b_0]$, where b_0 is the largest element, if exists, of X.

Then the collection $\mathcal{B}$ is a basis for a topology on X, called the *order topology* on $(X, \leq)$.

Example 2.1.20. The Euclidean topology on $\mathbb{R}$ is the order topology on it.

Exercise 2.1.21. Consider the set $\mathbb{R} \times \mathbb{R}$ and give a partial order relation on it by setting $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. Draw intervals $((a, b), (c, d))$ in $\mathbb{R} \times \mathbb{R}$, for the case $a < c$, and the case $a = c$ with $b < d$.

2.2 Interior point and limit point

Let X be a topological space, and let $A \subseteq X$.

Definition 2.2.1. A point $x \in A$ is said to be an interior point of A if there exists an open subset U of X such that $x \in U$ and $U \subseteq A$.

Proposition 2.2.2. *A subset $A \subseteq X$ is open if and only if either $A = \emptyset$ or each of the points of A are interior points.*

Proof. Since empty subset is open by definition, we may assume that $A \neq \emptyset$. If A is open in X, given any $x \in A$, we can take $U = A$ so that $x \in U \subseteq A$ holds, so that x is an interior point of A. Conversely, suppose that each point $x \in A$ is an interior point. Then given $a \in A$, there exists an open subset V_a of X such that $a \in V_a \subseteq A$. Then $A = \bigcup\limits_{a \in A} V_a$. Since arbitrary union of open subsets of X is open in X, that A is open in X. □

Example 2.2.3. The subset $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ is open in the Euclidean space $\mathbb{R}$, and hence all of its points are interior points.

Example 2.2.4. Consider the subset $A = [0, 1)$ of $\mathbb{R}$. Note that any point of A other than 0 is an interior point of it when the real line is equipped with the Euclidean topology or the lower limit topology. However, $0 \in A$ is not an interior point of A if $\mathbb{R}$ is equipped with the Euclidean topology, but if we equip $\mathbb{R}$ with the lower limit topology, then 0 is an interior point of A in $\mathbb{R}_\ell$.

Definition 2.2.5. A point $x \in X$ is said to be a *limit point* of $A \subseteq X$ if given any open subset U of X containing x, there exists an element $a \in A$ such that $a \neq x$ and $a \in U$; in other words, $(U \setminus \{x\}) \cap A \neq \emptyset$.

Proposition 2.2.6. *Let X be a topological space, and let Z be a non-empty subset of X. Then Z is closed in X if and only if it contains all of its limit points.*

Proof. Suppose that Z is closed in X. If $x \notin Z$, then x is in the open subset $U = X \setminus Z$, and that $(U \setminus \{x\}) \cap Z = \emptyset$. Then x cannot be a limit point of Z. Conversely, suppose that Z contain all of its limit points in X. Let $U = X \setminus Z$. If $U = \emptyset$, then $Z = X$, and hence it is closed in X. Assume that $U \neq \emptyset$. Then given any $x \in U$, x is not a limit point of Z. Then there exists an open subset V_x of X such that

$$(V_x \setminus \{x\}) \cap Z = \emptyset.$$

Since $x \notin Z$, we have $V_x \cap Z = \emptyset$. Then $V_x \subseteq U = X \setminus Z$. Therefore, x is an interior point of U. Thus, U is open in X, and hence Z is closed in X. $\square$

Proposition 2.2.7. *Let X be a topological space. Given a subset A of X, let*

$$\mathscr{C}_A := \{Z \subseteq X : A \subseteq Z \text{ and } Z \text{ is closed in } X\}$$

be the set of all closed subsets of X containing A. Then $\bigcap\limits_{Z \in \mathscr{C}_A} Z$ is the smallest closed subset of X containing A. We call $\bigcap\limits_{Z \in \mathscr{C}_A} Z$ the closure of A in X, and denote it by $\overline{A}$.

Proof. Clearly $A \subseteq \bigcap\limits_{Z \in \mathscr{C}_A} Z$. Since $X \setminus \left(\bigcap\limits_{Z \in \mathscr{C}_A} Z \right) = \bigcup\limits_{Z \in \mathscr{C}_A} (X \setminus Z)$ is open in X by definition of a topological space, the subset $\bigcap\limits_{Z \in \mathscr{C}_A} Z$ is closed in X. If W is any closed subset of X with $A \subseteq W$, then $W \in \mathscr{C}_A$, and hence $\bigcap\limits_{Z \in \mathscr{C}_A} Z \subseteq W$. This completes the proof. $\square$

2.3 Continuity

Definition 2.3.1 (Continuous map)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous at* $x_0 \in X$ if for each open subset V of Y containing $f(x_0)$, there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$. We say that f is continuous if it is continuous at every point of X.

Lemma 2.3.2. *Let X and Y be topological spaces. A map $f : X \to Y$ is continuous if and only if given any open subset V of Y, the subset $f^{-1}(V)$ is open in X.*

Proof. Suppose that f is continuous. Let V be any open subset of Y. Since $f^{-1}(\emptyset) = \emptyset$, we may assume that $V \neq \emptyset$. Let $x_0 \in f^{-1}(V)$ be given. Then $f(x_0) \in V$. Since f is continuous at x_0, there exists an open subset U of X with $x_0 \in U$ such that $f(U) \subseteq V$. Then $U \subseteq f^{-1}(V)$ with $x_0 \in U$, and so x_0 is an interior point of $f^{-1}(V)$. Since $x_0 \in f^{-1}(V)$ is chosen arbitrarily, $f^{-1}(V)$ is open in X by Proposition 2.2.2. Converse is obvious. $\square$

The following result shows that to check continuity of a map it suffices to show that inverse image of every basic open subset is open.

Corollary 2.3.3. *Let X and Y be topological spaces. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Then a map $f : X \to Y$ is continuous if and only if given any basic open subset $V \in \mathcal{B}_Y$ of Y, its inverse image $f^{-1}(V)$ is open in X.*

Proof. If f is continuous, then $f^{-1}(B)$ is open in X, for any basic open subset B of Y. To show the converse, let U be any open subset of Y. Then by Lemma 2.1.11, $U = \bigcup\limits_{\alpha \in \Lambda} B_\alpha$, for some collection $\{B_\alpha : \alpha \in \Lambda\}$ of basic open subsets of Y. Since $f^{-1}(B_\alpha)$ is open in X by assumption, the subset $f^{-1}(U) = \bigcup\limits_{\alpha \in \Lambda} f^{-1}(B_\alpha)$ is open in X. Therefore, f is continuous. $\square$

As an immediate consequence of Lemma 2.3.2, we have the following.

Corollary 2.3.4. *Let X be a non-empty set together with two topologies τ_1 and τ_2. For each $j = 1, 2$, let $X_j = (X, \tau_j)$ be the topological space whose underlying set is X and the topology is τ_j. Then τ_2 is finer than τ_1 (i.e., $\tau_1 \subseteq \tau_2$) if and only if the identity map $\mathrm{Id}_X : X_2 \to X_1$ is continuous.*

Example 2.3.5. Let $\mathbb{R}$ be the real line with the Euclidean topology on it, and let $\mathbb{R}_\ell$ be the real line with the lower limit topology on it. Since any open interval (a, b) in $\mathbb{R}$ can be written as

$$(a, b) = \bigcup_{n \in \mathbb{N}} [a - \frac{1}{n}, b),$$

and each of $[a - \frac{1}{n}, b)$ are open in $\mathbb{R}_\ell$, it follows that (a, b) is open in $\mathbb{R}_\ell$. However, $[a, b)$ is open in $\mathbb{R}_\ell$ but not in $\mathbb{R}$. Therefore, the lower limit topology on the real line is strictly finer than the Euclidean topology on it. Then by Corollary 2.3.4, the identity map $\mathbb{R}_\ell \to \mathbb{R}$ is continuous, while the identity map $\mathbb{R} \to \mathbb{R}_\ell$ is not continuous.

Lemma 2.3.6. *Let X and Y be topological spaces and let $f : X \to Y$ be a map. Then the following are equivalent.*

(i) f is continuous.

(ii) for every closed subset Z of Y, $f^{-1}(Z)$ is closed in X.

(iii) $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \subseteq X$.

Proof. (i) $\Rightarrow$ (ii): Let Z be a closed subset of Y. Then $Y \setminus Z$ is open in Y. Since f is continuous by assumption (i), $f^{-1}(Y \setminus Z)$ is open in X. Since $f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z)$ (verify!), we conclude that $f^{-1}(Z)$ is closed in X.

(ii) $\Rightarrow$ (iii): Let $A \subseteq X$. Since $\overline{f(A)}$ is closed in Y by Proposition 2.2.7, $f^{-1}(\overline{f(A)})$ is closed in X by assumption (ii). Since

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}),$$

and $\overline{A}$ is the smallest closed subset of X containing A, we have $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Therefore, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

(iii) $\Rightarrow$ (i): Let V be a non-empty open subset of Y. Since $Z := Y \setminus V$ is closed in Y, we have $\overline{Z} = Z$ by Proposition 2.2.7. Apply assumption (iii) to the subset $A := f^{-1}(Z) \subseteq X$ to get $f(\overline{A}) \subseteq \overline{f(A)}$. Since $f(A) = f(f^{-1}(Z)) \subseteq Z$, we have

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{Z} = Z.$$

Then $\overline{A} \subseteq f^{-1}(Z) = A$, and hence $\overline{A} = A$. Then A is closed in X by Proposition 2.2.7. Since

$$A = f^{-1}(Z) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),$$

we see that $f^{-1}(V)$ is open in X. Therefore, f is continuous. $\square$

Exercise 2.3.7. Let A be a non-empty subset of a topological space. Show that the subspace topology on A induced from X is the smallest topology on A such that the inclusion map

$$\iota_A : A \hookrightarrow X, \; a \mapsto a,$$

is continuous.

Answer: Since the subspace topology τ_A on A induced from X is given by $\tau_A = \{U \cap A : U \text{ is open in } X\}$, given an open subset U of X, the subset $\iota_A^{-1}(U) = U \cap A$ is in τ_A. If τ' is any topology on A such that the inclusion map $\iota_A : A \hookrightarrow X$ is continuous, then $U \cap A \in \tau'$, for all open subset U of X, and hence $\tau_A \subseteq \tau'$. □

Exercise 2.3.8. Show that the direct image of open (resp., closed) subset under a continuous map of topological spaces need not be open (resp., closed). (*Hint:* Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, $\forall\, x \in \mathbb{R}$, and note that $f((-1,1)) = [0,1)$. Take $g : [1,\infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$, $\forall\, x$. Note that the image of the closed subset $[1,\infty)$ under g is not closed in $\mathbb{R}$.)

Let X and Y be topological spaces. A map $f : X \to Y$ is said to be a *constant map* if there exists $y_0 \in Y$ such that $f(x) = y_0$, $\forall\, x \in X$.

Corollary 2.3.9. *Let X and Y be two topological spaces. Then any constant map $f : X \to Y$ is continuous.*

Proof. Let $f(x) = y_0$, for all $x \in X$. Let V be any open subset of Y. If $y_0 \in Y$, then $f^{-1}(V) = X$, and if $y_0 \notin V$, then $f^{-1}(V) = \emptyset$. Therefore, $f^{-1}(V)$ is open in X, and hence f is continuous. □

Proposition 2.3.10. *Let $\Bbbk$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the Euclidean topology on it. Let X be a topological space and $f, g : X \to \Bbbk$ be two continuous maps. Then the maps $f+g, fg : X \to \Bbbk$ defined by point-wise addition and multiplication of real numbers,*

$$\begin{aligned} (f+g)(x) &:= f(x) + g(x), \; \forall\, x \in X, \\ (fg)(x) &:= f(x)g(x), \; \forall\, x \in X, \end{aligned}$$

are continuous. Moreover, if $f(x) \neq 0$, $\forall\, x \in X$, then the map

$$g : X \to \mathbb{R}, \; x \mapsto \frac{1}{f(x)},$$

is continuous. The set $C(X, \Bbbk)$ of all $\Bbbk$-valued continuous functions on X forms an $\Bbbk$-algebra.

Proof. Suppose that $f, g \in C(X, \Bbbk)$ be given. Let $x_0 \in X$ be arbitrary. Since f and g are continuous at $x_0 \in X$, given a real number $r > 0$ there exists open subsets U and V of X containing x_0 such that

$$\begin{aligned} |f(x) - f(x_0)| &< r/2, \; \forall\, x \in U, \; \text{ and} \\ |g(x) - g(x_0)| &< r/2, \; \forall\, x \in V. \end{aligned}$$

Then for any $x \in U \cap V$, we have

$$\begin{aligned} |(f+g)(x) - (f+g)(x_0)| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Therefore, $f+g$ is continuous at x_0. Since $x_0 \in X$ is chosen arbitrarily, $f+g$ is continuous on X. Similarly, one can show that $f \cdot g$ is continuous on X (verify!). □

Corollary 2.3.11. *Let X be a topological space and $f, g : X \to \mathbb{R}$ be two continuous maps. Then*

(i) $A := \{x \in X : f(x) < g(x)\}$ is open in X, and

(ii) $B := \{x \in X : f(x) \leq g(x)\}$ is closed in X.

Proof. Since $g - f : X \to \mathbb{R}$ is continuous by Proposition 2.3.10 and since $\mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ is open in $\mathbb{R}$ and $\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : t \geq 0\}$ is closed in $\mathbb{R}$, the subset $A = (g-f)^{-1}(\mathbb{R}^+)$ is open in X, and the subset $B = (g-f)^{-1}(\mathbb{R}_{\geq 0})$ is closed in X. □

Lemma 2.3.12. *Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f$.*

Proof. Let V be an open subset of Z. Since g is continuous, $g^{-1}(V)$ is open in Y, and since f is continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. Therefore, $g \circ f$ is continuous. □

Lemma 2.3.13. *Let $f : X \to Y$ be continuous map. Given any non-empty subset $A \subseteq X$, equip A with the subspace topology induced from X. Then the restriction map $f\big|_A : A \to Y$ is continuous.*

Proof. Since inclusion map $\iota_A : A \hookrightarrow X$ and $f : X \to Y$ are continuous, so is their composition map $f\big|_A : A \to Y$. □

Exercise 2.3.14. Give example of maps $f : X \to Y$ and $g : Y \to Z$ of topological spaces such that both g and $g \circ f$ are continuous, but f is not continuous. (*Hint:* Consider a constant function).

Definition 2.3.15. Let X and Y be topological spaces. A map $f : X \to Y$ is said to be a *homeomorphism* if f is continuous and there exists a continuous map $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

In other words, a homeomorphism is a continuous bijective map of topological spaces whose inverse is also continuous. If there is a homeomorphism $f : X \to Y$ then we say that X is homeomorphic to Y, and express it as $X \cong Y$.

Example 2.3.16. (i) For any topological space X, the identity map $\mathrm{Id}_X : X \to X$ is a homeomorphism.

(ii) Let X be the real line $\mathbb{R}$ with the usual topology on it. Then for any $a \in \mathbb{R}$, the translation by a map

$$t_a : \mathbb{R} \to \mathbb{R}, \;\; x \mapsto x + a,$$

is a homeomorphism. Indeed, t_a is a continuous bijective map with the continuous inverse t_{-a} (verify!).

(iii) Equip $\mathbb{C}$ and $\mathbb{R}^2$ with the usual Euclidean topologies. Then the map $f : \mathbb{C} \to \mathbb{R}^2$ given by

$$f(a+ib) = (a,b),\ \forall\, a+ib \in \mathbb{C},$$

is a homeomorphism.

(iv) The map $f : \mathbb{R} \to \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ defined by $f(x) = e^x$, $\forall\, x \in \mathbb{R}$, is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$, where $\mathbb{R}^+$ is equipped with the subspace topology induced from the usual topology on $\mathbb{R}$.

(v) The map $f : (-1,1) \to \mathbb{R}$ defined by

$$f(t) = \frac{t}{t^2-1},\ \forall\, t \in (-1,1),$$

is a homeomorphism (verify!).

Remark 2.3.17. Note that "being homeomorphic topological spaces" is an equivalence relation on the collection of all topological spaces. Indeed, any topological space X is homeomorphic to itself via the identity map $\mathrm{Id}_X : X \to X$. If $f : X \to Y$ is a homeomorphism of topological spaces, then $f^{-1} : Y \to X$ is a homeomorphism from Y into X. If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms of topological spaces, then $g \circ f : X \to Z$ is a homeomorphism.

2.4 Product topology

Before proceeding further, let us introduce a terminology, namely *category*, that is a systematic common framework to study for various mathematical objects.

Definition 2.4.1. A *category* $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X,Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X,Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X,Y,Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X,Y) \times \mathrm{Mor}_{\mathscr{C}}(Y,Z) \to \mathrm{Mor}_{\mathscr{C}}(X,Z),\ \ (f,g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X,Y,Z,W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y,Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z,W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

Example 2.4.2. (i) In the category of sets (Set), we consider the collection of all sets as $\mathrm{ob}(Set)$ and given any two objects (i.e., sets) X and Y, we have a collection $\mathrm{Mor}_{(Set)}(X, Y)$ which consists of all set theoretic maps from X into Y.

(ii) In the category of groups $(\mathcal{G}rp)$, we take $\mathrm{ob}(\mathcal{G}rp)$ to be the collection of all groups and given any two objects (groups) G and H, we take $\mathrm{Mor}_{(\mathcal{G}rp)}(G, H) = \mathrm{Hom}(G, H)$, the set of all group homomorphisms from G into H.

(iii) Let k be a field. In the category of k-vector spaces $(\mathcal{V}ect_k)$, we take $\mathrm{ob}(\mathcal{V}ect_k)$ to be the collection of all k-vector spaces, and given any two objects (k-vector spaces) V and W, we take $\mathrm{Mor}_{(\mathcal{V}ect_k)}(V, W) = \mathrm{Hom}_k(V, W)$, the set of all k-linear maps from V into W.

(iv) In the category of topological spaces $\mathcal{T}op$, we take $\mathrm{ob}(\mathcal{T}op)$ to be the collection of all topological spaces, and given any two objects (topological spaces) X and Y, we take $\mathrm{Mor}_{\mathcal{T}op}(X, Y)$ to be the set of all continuous maps from X into Y.

One can easily verify that all the axioms of the above Definition 2.4.3 are satisfied for each of the above mentioned examples.

Let $\mathscr{C}$ be a category (think of any one from the above examples).

Definition 2.4.3. The *product* of an indexed family of objects $\{X_\alpha : \alpha \in \Lambda\}$ in $\mathscr{C}$ is a pair $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha \in \Lambda})$, consisting of an object P in $\mathscr{C}$ and a family of morphisms $\{\pi_\alpha : P \to X_\alpha\}_{\alpha \in \Lambda}$ in $\mathscr{C}$, satisfying the following universal property: given any object T of $\mathscr{C}$ and a family of morphisms $\{f_\alpha : T \to X_\alpha\}_{\alpha \in \Lambda}$ in $\mathscr{C}$, there exists a unique morphism $f : T \to P$ in $\mathscr{C}$ such that $\pi_\alpha \circ f = f_\alpha$, for all $\alpha \in \Lambda$.

$$\begin{tikzcd} T \arrow[r, dashed, "f"] \arrow[dr, "f_\alpha"'] & P \arrow[d, "\pi_\alpha"] \\ & X_\alpha \end{tikzcd}$$

It follows from the universal property of product that, if it exists, then it is unique upto a unique isomorphism making the above diagram commutative. Indeed, if $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha \in \Lambda})$ and $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha \in \Lambda})$ are two products of an indexed family of objects $\{X_\alpha : \alpha \in \Lambda\}$ in $\mathscr{C}$, then applying universal property of $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha \in \Lambda})$ for the test object $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha \in \Lambda})$, we get a unique morphism $f : P' \to P$ such that $\pi_\alpha \circ f = \pi'_\alpha$, $\forall\, \alpha \in \Lambda$. Similarly, applying universal property of $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha \in \Lambda})$ for the test object $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha \in \Lambda})$ we have a unique morphism $g : P \to P'$ such that $\pi'_\alpha \circ g = \pi_\alpha$, $\forall\, \alpha \in \Lambda$.

$$\begin{tikzcd} P' \arrow[r, dashed, "f"] \arrow[rr, bend left, "\mathrm{Id}_{P'}"] \arrow[dr, "\pi'_\alpha"'] & P \arrow[r, dashed, "g"] \arrow[d, "\pi_\alpha"] & P' \arrow[dl, "\pi'_\alpha"] \\ & X_\alpha & \end{tikzcd}$$

Since both the morphisms $g \circ f, \mathrm{Id}_{P'} \in \mathrm{Mor}_{\mathscr{C}}(P', P')$ make the following diagram commutative,

$$\begin{tikzcd} P' \arrow[rr, dashed, "g\circ f", "\mathrm{Id}_{P'}"'] \arrow[dr, "\pi'_\alpha"'] & & P' \arrow[dl, "\pi'_\alpha"] \\ & X_\alpha & \end{tikzcd}$$

for all $\alpha \in \Lambda$, it follows from the uniqueness assertion in Definition 2.4.3 that $g \circ f = \mathrm{Id}_{P'}$. Similarly one can show that $f \circ g = \mathrm{Id}_P$. Therefore, the product of $\{X_\alpha\}_{\alpha \in \Lambda}$ in $\mathscr{C}$, if it exists, is unique upto a unique isomorphism making the diagram in Definition 2.4.3 commutative. We denote the product of $\{X_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}$ by $\prod_{\alpha\in\Lambda} X_\alpha$.

Lemma 2.4.4. *The product $\prod\limits_{\alpha\in\Lambda} X_\alpha$ of a family of sets $\{X_\alpha : \alpha \in \Lambda\}$ exists in the category of sets.*

Proof. Let

$$\prod_{\alpha\in\Lambda} X_\alpha := \left\{ f : \Lambda \to \bigsqcup_{\alpha\in\Lambda} X_\alpha \mid f(\alpha) \in X_\alpha,\ \forall\, \alpha \in \Lambda \right\}.$$

Define a map $\pi_\beta : \prod\limits_{\alpha\in\Lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f_\beta := f(\beta)$, $\forall\, \beta \in \Lambda$. Suppose that we are given a set T together with maps $f_\alpha : T \to X_\alpha$, for each $\alpha \in \Lambda$. Define a map $F : T \to \prod\limits_{\alpha\in\Lambda} X_\alpha$ which sends $t \in T$ to $F(t) \in \prod\limits_{\alpha\in\Lambda} X_\alpha$ defined by $F(t)(\alpha) = f_\alpha(t)$, $\forall\, \alpha \in \Lambda$. Then $(\pi_\alpha \circ F)(t)(\alpha) = \pi_\alpha(F(t)) = F(t)(\alpha) = f_\alpha(t)$, $\forall\, t \in T$. Therefore, $\pi_\alpha \circ F = f_\alpha$, $\forall\, \alpha \in \Lambda$. To show uniqueness of F, suppose that $G : T \to \prod\limits_{\alpha\in\Lambda} X_\alpha$ be any map such that $\pi_\alpha \circ G = f_\alpha$, $\forall\, \alpha \in \Lambda$. Then for any $t \in T$, we have

$$G(t)(\alpha) = \pi_\alpha(G(t)) = f_\alpha(t) = F(t)(\alpha),\ \ \forall\, \alpha \in \Lambda,$$

and hence $G(t) = F(t)$, $\forall\, t \in T$. This proves that $G = F$. $\square$

Theorem 2.4.5. *Let $\mathcal{T}op$ be the category of topological spaces. The categorical product of a family of topological spaces $\{X_\alpha : \alpha \in \Lambda\}$ exists in $\mathcal{T}op$, and is **unique upto a unique homeomorphism** in the sense that if $(P, \{\pi_\alpha : P \to X_\alpha\}_{\alpha\in\Lambda})$ and $(P', \{\pi'_\alpha : P' \to X_\alpha\}_{\alpha\in\Lambda})$ are products of the family of topological spaces $\{X_\alpha : \alpha \in \Lambda\}$, then there exists a unique homeomorphism $\Phi : P' \to P$ such that $\pi_\alpha \circ \Phi = \pi'_\alpha$, for all $\alpha \in \Lambda$.*

Proof. Uniqueness follows from the universal property of product in a category. We only prove existence. Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Let τ_α be the topology on X_α, for all $\alpha \in \Lambda$. If the product $\prod\limits_{\alpha\in\Lambda} X_\alpha$ exists in $\mathcal{T}op$, its underlying set of points may be described as in Lemma 2.4.4. We just need to give a suitable topology on the set $\prod\limits_{\alpha\in\Lambda} X_\alpha$ that makes it the product in $\mathcal{T}op$. First of all, we need the projection maps

$$\pi_\alpha : \prod_{\beta\in\Lambda} X_\beta \longrightarrow X_\alpha$$

to be continuous, for all $\alpha \in \Lambda$. Let

$$\mathcal{S} := \bigcup_{\alpha\in\Lambda} \{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \tau_\alpha\}.$$

Note that, $\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta\in\Lambda} U_\beta$, where $U_\beta = X_\beta$, for all $\beta \neq \alpha$ in Λ (here product is taken in the category of sets). Clearly the union of all elements of $\mathcal{S}$ is the set $X := \prod_{\alpha\in\Lambda} X_\alpha$, and hence $\mathcal{S}$ is a subbasis for a topology on X. Let $\mathcal{B}$ be the set of all finite intersections of elements from $\mathcal{S}$. Then $\mathcal{B}$ is a basis for the topology on X generated by the subbasis $\mathcal{S}$. A typical element of $\mathcal{B}$ is of the form $\bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j}) = \prod_{\beta\in\Lambda} U_\beta$, where $U_\beta = X_\beta$, for all $\beta \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$, for some $n \in \mathbb{N}$. Clearly the maps $\pi_\alpha : \prod_{\beta\in\Lambda} X_\beta \to X_\alpha$ are continuous by construction of topology on $\prod_{\beta\in\Lambda} X_\beta$.

Let T be a topological space, and consider a family $\mathcal{F} := \{f_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ of continuous maps from X into X_α, for all $\alpha \in \Lambda$. By universal property of product set $X := \prod_{\beta\in\Lambda} X_\beta$, there exists a unique set map $f : T \to \prod_{\beta\in\Lambda} X_\beta$ such that $\pi_\alpha \circ f = f_\alpha$, $\forall\, \alpha \in \Lambda$. To check continuity of f, it suffices to check that $f^{-1}(B)$ is open in T, for all $B \in \mathcal{B}$. Now a basic open subset $B \in \mathcal{B}$ is of the form $B = \bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j})$, for some $n \in \mathbb{N}$. Then

$$f^{-1}(B) = \bigcap_{j=1}^{n} f^{-1}\left(\pi_{\alpha_j}^{-1}(U_{\alpha_j})\right) = \bigcap_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j})$$

is open in T. Therefore, f is continuous. This completes the proof. □

Let τ_α be the topology on X_α, for all $\alpha \in \Lambda$. It is clear from the above construction of product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ that the collection

$$\mathcal{B} := \left\{ \prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \in \tau_{X_\alpha} \text{ and } U_\alpha = X_\alpha, \text{ for all but finitely many } \alpha \in \Lambda \right\}$$

is a basis for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$. However, we can further cut down $\mathcal{B}$ to construct another basis for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ by looking at basic open subsets of X_α's. Indeed, fixing a basis $\mathcal{B}_\alpha$ for each X_α, one can easily verify that the collection

$$\mathcal{B}' := \left\{ \prod_{\alpha\in\Lambda} V_\alpha : V_\alpha \in \mathcal{B}_\alpha \cup \{X_\alpha\} \text{ and } V_\alpha = X_\alpha, \text{ for all but finitely many } \alpha \in \Lambda \right\}$$

forms a basis for the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$.

Proposition 2.4.6. *Let X and Y be two topological spaces. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for X and Y, respectively. Then the collection*

$$\mathcal{B} := \{U \times V : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$$

is a basis for the product topological space $X \times Y$.

Proof. We first show that $\mathcal{B}_X \times \mathcal{B}_Y$ forms a basis for some topology on $X \times Y$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, given any $(x, y) \in X \times Y$, there exist $U \in \mathcal{B}_X$ and

$V \in \mathcal{B}_Y$ such that $x \in U$ and $y \in V$, so that $(x,y) \in U \times V$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and $(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ be given. Since $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, there exist $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ such that $x \in U \subseteq U_1 \cap U_2$ and $y \in V \subseteq V_1 \cap V_2$. Then $(x,y) \in U \times V \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$. Therefore, $\mathcal{B}$ is a basis for a topology on $X \times Y$. Let $A \subseteq X \times Y$ be any non-empty open subset in the product topological space $X \times Y$. Let $(x,y) \in A$ be given. Then there exist $U \in \tau_X$ and $V \in \tau_Y$ such that $(x,y) \in U \times V \subseteq A$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for (X, τ_X) and (Y, τ_Y), respectively, there exist $U_1 \in \mathcal{B}_X$ and $V_1 \in \mathcal{B}_Y$ such that $x \in U_1 \subseteq U$ and $y \in V_1 \subseteq V$. Then $(x,y) \in U_1 \times V_1 \subseteq \mathcal{A}$. Therefore, $\mathcal{B}$ is a basis for the product topology on $X \times Y$. $\square$

Exercise 2.4.7. Show that the product topology on $\mathbb{R}^n$ coincides with the Euclidean topology on $\mathbb{R}^n$.

Let X and Y be two sets. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the maps defined by

$$\pi_X(x,y) = x, \ \forall\, (x,y) \in X \times Y,$$
$$\text{and } \pi_Y(x,y) = y, \ \forall\, (x,y) \in X \times Y.$$

The maps π_X and π_Y are called the projection maps onto X and Y, respectively. Note that, $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, for any subsets $U \subseteq X$ and $V \subseteq Y$, respectively.

Theorem 2.4.8. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then the collection*

$$\mathcal{S} := \{\pi_X^{-1}(U) : U \in \tau_X\} \cup \{\pi_Y^{-1}(V) : V \in \tau_Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Since $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, for any subsets $U \subseteq X$ and $V \subseteq Y$, respectively, we see that $\mathcal{S} \subseteq \tau_{X\times Y}$, where $\tau_{X\times Y}$ is the product topology on $X \times Y$. Since the topology $\tau_{\mathcal{S}}$ generated by $\mathcal{S}$ consists of arbitrary unions of finite intersections of elements from $\mathcal{S}$, we have $\tau_{\mathcal{S}} \subseteq \tau_{X\times Y}$. On the other hand, every basic open subset $U \times V$ for the product topology $\tau_{X\times Y}$ can be written as finite intersection

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V).$$

Therefore, these two topologies coincides. $\square$

Theorem 2.4.9. *Let A be a subspace of X and B be a subspace of Y. Then the product topology on $A \times B$ coincides with the subspace topology on $A \times B$ induced from $X \times Y$.*

Proof. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies on X and Y, respectively. Then $\mathcal{B}_A = \{U \cap A : U \in \mathcal{B}_X\}$ and $\mathcal{B}_B = \{U \cap A : U \in \mathcal{B}_Y\}$ are bases for the subspace topologies on A and B, respectively. Then $\mathcal{B} = \{(U \cap A) \times (V \cap B) : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the product topology on $A \times B$. Note that $\mathcal{B}_{X\times Y} = \{U \times V : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$. Then $\mathcal{B}' = \{(U \times V) \cap (A \times B) : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the subspace topology on $A \times B$ induced from $X \times Y$. Since $(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B)$, for all $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$, we have $\mathcal{B} = \mathcal{B}'$. Hence the result follows. $\square$

Lemma 2.4.10. *Let X, Y and Z be topological spaces. Equip $Y \times Z$ with the product topology, and let $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ be the projection maps onto the first and the second factors, respectively. Then a map $f : X \to Y \times Z$ is continuous if and only if both $\pi_Y \circ f : X \to Y$ and $\pi_Z \circ f : X \to Z$ are continuous.*

Proof. If f is continuous, then $\pi_Y \circ f$ and $\pi_Z \circ f$ are continuous by Lemma 2.3.12. Conversely, assume that both $\pi_Y \circ f : X \to Y$ and $\pi_Z \circ f : X \to Z$ are continuous. Then for given any open subsets $U \subseteq Y$ and $V \subseteq Z$, the subsets $(\pi_Y \circ f)^{-1}(U)$ and $(\pi_Z \circ f)^{-1}(V)$ are open in X. Then the subset

$$\begin{aligned} f^{-1}(U \times V) &= \{x \in X : f(x) \in U \times V\} \\ &= \{x \in X : (\pi_Y \circ f)(x) \in U \text{ and } (\pi_Z \circ f)(x) \in V\} \\ &= (\pi_Y \circ f)^{-1}(U) \cap (\pi_Z \circ f)^{-1}(V) \end{aligned}$$

is open in X. Since a basic open subset of $Y \times Z$ is of the form $U \times V$, where U and V are open subsets of Y and Z, respectively, it follows from Corollary 2.3.3 that f is continuous. $\square$

Corollary 2.4.11. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be maps of topological spaces. Equip $X_1 \times X_2$ and $Y_1 \times Y_2$ with the product topologies. Then the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by*

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)), \ \forall\, (x_1, x_2) \in X_1 \times X_2,$$

is continuous if and only if both f_1 and f_2 are continuous.

Proof. Let $\pi_1 : Y_1 \times Y_2 \to Y_1$ and $\pi_2 : Y_1 \times Y_2 \to Y_2$ be the projection maps onto the first and the second factors, respectively. Then in view of Lemma 2.4.10, it suffices to show that

$$\begin{aligned} &\pi_1 \circ (f_1 \times f_2) : X_1 \times X_2 \to Y_1, \\ \text{and } \ &\pi_2 \circ (f_1 \times f_2) : X_1 \times X_2 \to Y_2 \end{aligned}$$

are continuous. Let V_1 and V_2 be open subsets of Y_1 and Y_2, respectively. Then

$$\begin{aligned} (\pi_1 \circ (f_1 \times f_2))^{-1}(V_1) &= \{(x_1, x_2) \in X_1 \times X_2 : \pi_1(f_1(x_1), f_2(x_2)) \in V_1\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) \in V_1\} \\ &= f_1^{-1}(V_1) \times X_2 \end{aligned}$$

is open in $X_1 \times X_2$. Similarly, $(\pi_2 \circ (f_1 \times f_2))^{-1}(V_2) = X_1 \times f_2^{-1}(V_2)$ is open in $X_1 \times X_2$. This completes the proof. $\square$

2.5 Hausdorff space

Definition 2.5.1. A topological space X is said to be *Hausdorff* or, *T2* or, *separated** if each pair of distinct points of X can be separated by a pair of disjoint open neighbourhoods of them. In

*Not to be confused with the notion of a *separable* topological space.

other words, give $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open subsets V_1, V_2 of X with $x_1 \in V$, $x_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$.

Lemma 2.5.2. *A topological space is Hausdorff if and only if the image of the diagonal map*

$$\Delta_X : X \to X \times X, \ \ x \mapsto (x, x)$$

is closed in $X \times X$.

Proof. Suppose that X is Hausdorff. It is enough to show that $U := (X \times X) \setminus \Delta_X(X)$ is open in $X \times X$. Since any point of U is of the form $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$, there are open neighbourhoods $x_j \in V_j \subset X$, $j = 1, 2$, such that $V_1 \cap V_2 = \emptyset$. Then $(x_1, x_2) \in V_1 \times V_2 \subseteq U$, and hence U is open.

Conversely suppose that X is separated. If $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $(x_1, x_2) \in U := (X \times X) \setminus \Delta_X(X)$. Since U is open in $X \times X$, there exist open subsets $V_1, V_2 \subset X$ with $x_j \in V_j$, $j = 1, 2$, such that $(x_1, x_2) \in V_1 \times V_2 \subseteq U = (X \times X) \setminus \Delta_X(X)$. Then $(V_1 \times V_2) \cap \Delta_X(X) = \emptyset$, and hence $V_1 \cap V_2 = \emptyset$. □

Exercise 2.5.3. Let $f, g : X \to Y$ be continuous maps of topological spaces. If Y is Hausdorff, show that the subset $\{x \in X : f(x) = g(x)\}$ is closed in X. (*Hint:* Look at the inverse image of $\Delta_Y(Y) \subset Y \times Y$ under the map $(f, g) : X \to Y \times Y$ given by $x \mapsto (f(x), g(x))$.)

Definition 2.5.4. Fix a topological space Z. A *Z-topological space* (or, a *topological space over Z*) is a pair (X, f), where X is a topological space and $f : X \to Z$ is a continuous maps. Given two Z-topological spaces (X, f) and (Y, g), a *morphism from (X, f) to (Y, g)* is given by a continuous map $\varphi : X \to Y$ such that $g \circ \varphi = f$. In other words, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Definition 2.5.5. The fiber product of a family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha \in \Lambda}$ is a pair $(F, \{\pi_\alpha : F \to X_\alpha\}_{\alpha \in \Lambda})$, where F is a topological space and $\{\pi_\alpha : F \to X_\alpha\}_{\alpha \in \Lambda}$ is a family of continuous maps indexed by Λ such that

(FP1) $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta, \ \forall\, \alpha, \beta \in \Lambda$, and

(FP2) given any topological space T and a family of continuous maps $\{g_\alpha : T \to X_\alpha\}_{\alpha \in \Lambda}$ satisfying $f_\alpha \circ g_\alpha = f_\beta \circ g$, there exists a unique continuous map $g : T \to F$ such that $\pi_\alpha \circ g = g_\alpha$, for all $\alpha \in \Lambda$.

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "g_\alpha"] \arrow[ddr, bend right, "g_\beta"'] & & \\
& F \arrow[r, "\pi_\alpha"] \arrow[d, "\pi_\beta"'] & X_\alpha \arrow[d, "f_\alpha"] \\
& X_\beta \arrow[r, "f_\beta"] & Z
\end{tikzcd}$$

Proposition 2.5.6. *Fiber product of a family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha\in\Lambda}$ exists, and is unique up to a unique homeomorphism in the sense that if $(F, \{\pi_\alpha : F \to X_\alpha\}_{\alpha\in\Lambda})$ and $(F', \{\pi'_\alpha : F' \to X_\alpha\}_{\alpha\in\Lambda})$ are fiber products of the family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha\in\Lambda}$, then there exists a unique homeomorphism $\Phi : F' \to F$ such that the following diagram commutes, for all $\alpha, \beta \in \Lambda$.*

$$\begin{tikzcd}
F' \arrow[dr, dashed, "\exists!\Phi"] \arrow[drr, bend left, "\pi'_\alpha"] \arrow[ddr, bend right, "\pi'_\beta"'] & & \\
& F \arrow[r, "\pi_\alpha"] \arrow[d, "\pi_\beta"'] & X_\alpha \arrow[d, "f_\alpha"] \\
& X_\beta \arrow[r, "f_\beta"] & Z
\end{tikzcd}$$

Proof. Uniqueness follows from the universal property of fiber product. We only show its existence. Consider the subset

$$F = \left\{ (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} : f_\alpha(x_\alpha) = f_\beta(x_\beta),\ \forall\, \alpha, \beta \in \Lambda \right\}.$$

of the product $\prod\limits_{\alpha\in\Lambda} X_\alpha$, and equip it with the subspace topology induced from the product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$. For each $\alpha \in \Lambda$, let $\pi_\alpha : F \to X_\alpha$ be the restriction of the projection map onto the α'th factor; clearly this is continuous. By construction of F, we have $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta$, for all $\alpha, \beta \in \Lambda$. Let T be a topological space and $\{g_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ be a family of continuous maps such that $f_\alpha \circ g_\alpha = f_\beta \circ g_\beta$, for all $\alpha, \beta \in \Lambda$. Define a map $g : T \to F$ by

$$g(t) = (g_\alpha(t))_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} X_\alpha.$$

Since $f_\alpha \circ g_\alpha = f_\beta \circ g_\beta$, for all $\alpha, \beta \in \Lambda$, it follows that $g(t) \in F$, $\forall\, t \in T$, and that $\pi_\alpha \circ g = g_\alpha$, $\forall\, \alpha \in \Lambda$. Uniqueness of g is clear due to the condition that $\pi_\alpha \circ g = g_\alpha$, $\forall\, \alpha \in \Lambda$. It remains to show that g is continuous. For that, we take a basic open subset of F of the form

$$V := F \cap \prod_{\alpha\in\Lambda} U_\alpha,$$

where U_α is an open subset of X_α, and that $U_\alpha = X_\alpha$, for all but finitely many $\alpha \in \Lambda$, say $U_{\alpha_1}, \ldots, U_{\alpha_n}$ are only proper open subsets. Then $g^{-1}(V) = \bigcap\limits_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$, and hence is open in T. This completes the proof. □

Exercise 2.5.7. Let $f : X \to Z$ and $g : Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, show that the subset

$$FP(f,g) := \{(x,y) \in X \times Y : f(x) = g(y)\}$$

is closed in the product topological space $X \times Y$. (*Hint:* Note that $FP(f,g) = (f \times g)^{-1}(\Delta_Z(Z))$, where $f \times g : X \times Y \to Z \times Z$ is the continuous map defined by $(f \times g)(x,y) = (f(x), g(y))$, for all $(x,y) \in X \times Y$; see Corollary 2.4.11).

Exercise 2.5.8. Let Z be a Hausdorff topological space. Given a finite family of Z-topological spaces $\{f_k : X_k \to Z \mid k = 1, \ldots, n\}$, show that the subset

$$FP(f_1, \ldots, f_n) := \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n : f_1(x_1) = \cdots = f_n(x_n)\}$$

is closed in the product topological space $X_1 \times \cdots \times X_n$. (*Hint:* We only prove for the case $n = 3$; general case is similar. For $n = 3$, we have $FP(f_1, f_2, f_3) = (FP(f_1, f_2) \times X_3) \cap (X_1 \times FP(f_2, f_3))$. Since both $FP(f_1, f_2)$ and $FP(f_2, f_3)$ are closed in $X_1 \times X_2$ and $X_2 \times X_3$, respectively, by previous Exercise 2.5.7, the result follows.)

Exercise 2.5.9. Given a Hausdorff space Z, can you generalize Exercise 2.5.8 to arbitrary family of Z-topological spaces?

Definition 2.5.10. A continuous map $f : X \to Y$ is said to be *separated* if the image of the induced map $\Delta_f : X \to X \times_Y X$ is closed in $X \times_Y X$.

Exercise 2.5.11. Let X be a non-empty set together with a pseudo metric d on it. Show that the topology on X generated by d is Hausdorff if and only if d is a metric on X.

2.6 Quotient space

Let's recall the notion of quotients from algebra course. Let G be a group and H a normal subgroup of G. Then we have a relation $\sim$ on G defined by

$$g_1 \sim g_2 \quad \text{if } g_1^{-1} g_2 \in H.$$

Clearly this is an equivalence relation on G, and we have a partition of G into a disjoint union of its subsets (equivalence classes)

$$G = \bigcup_{g \in G} gH,$$

where $gH = \{g' \in G : g \sim g'\}$ is the equivalence class of g in G, for all $g \in G$.

Now question is does there exists a pair (Q, q) consisting of a group Q and a map $q : G \to Q$ such that

(QG1) $q : G \to Q$ is a surjective group homomorphism satisfying $q(g) = q(g')$ whenever $g \sim g'$, and

(QG2) given any group G' and a group homomorphism $f : G \to G'$ with $H \subseteq \mathrm{Ker}(f)$, there should exists a unique group homomorphism $\widetilde{f} : Q \to G'$ such that $\widetilde{f} \circ q = f$?

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "q"'] & G' \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd} \tag{2.6.1}$$

Interesting point is that, without knowing existence of such a pair (Q, q), it follows immediately from the properties (QG1) and (QG2) that such a pair (Q, q), if it exists, must be unique up to a unique isomorphism of groups in the sense that, given another such pair (Q', q') satisfying the above two conditions, there is a unique group isomorphism $\phi : Q \to Q'$ such that $\phi \circ q = q'$).

Exercise 2.6.2. Prove the above mentioned uniqueness statement.

Now question is about its existence. The condition (QG1) suggests that the elements of Q should be the fibers of the map q, which are nothing but the $\sim$-equivalence classes

$$[g]_\sim = \{g' \in G : g' \sim g\} = gH, \ \forall g \in G.$$

This suggests us to consider $\{gH : g \in G\}$ as a possible candidate for the set Q. Now question is what should be the appropriate group structure on it? Take any group homomorphism $f : G \to G'$ such that $H \subseteq \mathrm{Ker}(f)$. This says that $f(g_1) = f(g_2)$ if $g_1 \sim g_2$ (equivalently, $g_1^{-1}g_2 \in H$). The commutativity of the diagram (2.6.1) tells us to send $gH \in Q$ to $f(g) \in G'$ to define the map $\widetilde{f} : Q \to G'$ (note that this is well-defined!), and since we want $\widetilde{f} : Q \to G'$ to be a group homomorphism, we should define a binary operation on $Q = \{gH : g \in G\}$ in such a way that $(g_1H) * (g_2H) \xmapsto{\widetilde{f}} f(g_1)f(g_2) = f(g_1g_2)$, for all $g_1, g_2 \in G$. So the obvious choice is to define

$$(g_1H) * (g_2H) := (g_1g_2)H, \ \forall g_1, g_2 \in G. \tag{2.6.3}$$

Clearly this is a well-defined binary operation on $Q = \{gH : g \in G\}$, since H is normal.

Exercise 2.6.4. Verify that (2.6.3) makes Q a group such that the pair (Q, q) satisfies the condition (QG1) and (QG2).

Exercise 2.6.5. Verify analogous stories for the cases rings and vector spaces.

We are going to witness the same phenomenon in topology! Let X be a topological space.

Definition 2.6.6. Given an equivalence relation $\sim$ on X, the associated *quotient topological space* (or, *identification space*) $X/\!\sim$ is a pair (Q, q) consisting of a topological space Q and a continuous map $q : X \to Q$ such that

(QT1) q is surjective and satisfies $q(x) = q(x')$ whenever $x \sim x'$ in $(X, \sim)$; and

(QT2) given any topological space Y and a continuous map $f : X \to Y$ satisfying $f(x) = f(x')$ whenever $x \sim x'$ in $(X, \sim)$, there is a unique continuous map $\widetilde{f} : Q \to Y$ such that $\widetilde{f} \circ q = f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

The map q is called the *quotient map* (or, *identification map*) for $(X, \sim)$.

As an immediate corollary to the Definition 2.6.6, we have the following.

Corollary 2.6.7. *If (Q, q) is a quotient space for $(X, \sim)$, then the topology on Q is the largest topology on the set Q such that the map $q : X \to Q$ is continuous.*

Proof. By a topology on a set S we mean a collection τ of subsets of S that satisfies axioms for open subsets in S. Suppose on the contrary that the statement in Corollary 2.6.7 is false. Then there is a topology τ' on the set Q finer than the quotient topology on Q such that the set map $q : X \to Q' := (Q, \tau')$ is continuous. Then by property (QT2) of (Q, q) in Definition 2.6.6, there is a unique (continuous) map $f : Q \to Q'$ such that $f \circ q = q$. Since $q : X \to Q$ is surjective, it admits a right inverse (set theoretically). This forces $f : Q \to Q'$ to be the identity map. This is not possible because f is continuous and the topology on Q' is finer than that of Q by our assumption (see Corollary 2.3.4). Hence the result follows. $\square$

Remark 2.6.8. In Definition 2.6.6, the first condition suggests what should be the underlying set of points of Q and the map $q : X \to Q$, and the second condition suggests what should be the topology on the set Q.

Theorem 2.6.9. *Given a topological space X and an equivalence relation $\sim$ on X, the associated quotient space (Q, q) for $(X, \sim)$ exists, and is unique up to a unique homeomorphism (i.e., if (Q, q) and (Q', q') are two quotient spaces for $(X, \sim)$, then there is a unique homeomorphism $\varphi : Q \to Q'$ such that $\varphi \circ q = q'$).*

Proof. We first prove uniqueness of the pair (Q, q), up to a unique homeomorphism. Let (Q', q') be another quotient space for the pair $(X, \sim)$. Since q' is continuous and $q'(x) = q'(y)$ whenever $x \sim y$ in $(X, \sim)$, we have a unique continuous map $\widetilde{q} : Q' \to Q$ such that $\widetilde{q} \circ q' = q$.

$$\begin{tikzcd} & X \arrow[dl, "q'"'] \arrow[d, "q"] \arrow[dr, "q'"] & \\ Q' \arrow[r, dashed, "\widetilde{q}"'] & Q \arrow[r, dashed, "\widetilde{q}'"'] & Q' \end{tikzcd} \tag{2.6.10}$$

Similarly, interchanging the role of (Q, q) and (Q', q') we get a unique continuous map $\widetilde{q}' : Q \to Q'$ such that $\widetilde{q}' \circ q = q'$. Then we have $(\widetilde{q}' \circ \widetilde{q}) \circ q' = q'$. Since the identity map $\mathrm{Id}_{Q'} : Q' \to Q'$ is continuous and satisfies $\mathrm{Id}_{Q'} \circ q' = q'$, we must have $\widetilde{q}' \circ \widetilde{q} = \mathrm{Id}_{Q'}$. Similarly, we have $\widetilde{q} \circ \widetilde{q}' = \mathrm{Id}_Q$. Therefore, both $\widetilde{q}$ and $\widetilde{q}'$ are homeomorphisms. Thus the pair (Q, q) is unique, up to a unique homeomorphism.

Now (following Remark 2.6.8) we give an explicit construction of $(Q, \sim)$. For each $x \in X$, the *equivalence class of* x in $(X, \sim)$ is the subset

$$[x] := \{x' \in X : x \sim x'\} \subseteq X.$$

Let Q be the set of all distinct equivalence classes of elements of X. Consider the map $q : X \to Q$ defined by sending each point $x \in X$ to its equivalence class $[x] \in Q$. Note that, the map q is surjective. As suggested in Corollary 2.6.7, we define a topology on Q by declaring a subset $U \subseteq Q$ to be *open* if its inverse image $q^{-1}(U) \subseteq X$ is open in X. Clearly this makes $q : X \to Q$ continuous. It remains to check property (QT2) as in Definition 2.6.6. Let Y be any topological space and $f : X \to Y$ any continuous map satisfying $f(x) = f(x')$ for $x \sim x'$ in $(X, \sim)$. Define a map $\widetilde{f} : Q \to Y$ by $\widetilde{f}([x]) = f(x)$, for all $[x] \in Q$. Clearly $\widetilde{f}$ is well-defined, and by its construction it satisfies

$$\widetilde{f} \circ q = f. \tag{2.6.11}$$

Since f is continuous, for any open subset $V \subseteq Y$, the subset

$$q^{-1}(\widetilde{f}^{-1}(V)) = (\widetilde{f} \circ q)^{-1}(V) = f^{-1}(V)$$

is open in X, and hence $\widetilde{f}^{-1}(V)$ is open in Q by definition of the topology on Q. Therefore, $\widetilde{f}$ is continuous. If $g : Q \to Y$ is any continuous map satisfying $g \circ q = f$, then $g([x]) = (g \circ q)(x) = f(x)$, for all $[x] \in Q$, and hence $g = \widetilde{f}$. □

Theorem 2.6.12. *Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective continuous map. Then the following are equivalent.*

(i) The pair (Y, p) is a quotient space for some equivalence relation $\sim$ on X.

(ii) A subset $U \subseteq Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

(iii) A subset $Z \subseteq Y$ is closed in Y if and only if $p^{-1}(Z)$ is closed in X.

Proof. (i) ⇒ (ii) and (i) ⇒ (iii): Follow from Corollary 2.6.7.

We show (ii) ⇒ (i) and (iii) ⇒ (i) together. Consider the relation $\sim$ on X defined by

$$x_1 \sim x_2 \text{ in } X, \text{ if } \; p(x_1) = p(x_2).$$

It is easy to see that $\sim$ is an equivalence relation on X. Clearly $p(x) = p(x')$ whenever $x \sim x'$ in X. Let T be a topological space and let $f : X \to T$ be a continuous map such that $f(x) = f(x')$ if $x \sim x'$ in X. Since $p : X \to Y$ is surjective, for each $y \in Y$ we can choose a point $x_y \in p^{-1}(y) \subseteq X$ by axiom of choice. Since $f(x) = f(x')$ for all $x, x' \in p^{-1}(y)$, we get a well-defined map $\widetilde{f} : Y \to T$ defined by

$$\widetilde{f}(y) = f(x_y), \; \forall \, y \in Y.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & T \\ Y \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Clearly $\widetilde{f} \circ p = f$. Let $V \subseteq T$ be an open (resp., closed) subset of T. Since f is continuous, $p^{-1}(\widetilde{f}^{-1}(V)) = f^{-1}(V)$ is open (resp., closed) in X. Then it follows from the assumption (ii) (resp., (iii)) that $\widetilde{f}^{-1}(V)$ is open (resp., closed) in Y. Therefore, $\widetilde{f} : Y \to T$ is continuous. If $g : Y \to T$ is any continuous map such that $g \circ p = f$, then $g(p(x)) = f(x)$, $\forall \, x \in X$ gives $g(y) = \widetilde{f}(y)$, for all $y \in Y$. Therefore, $\widetilde{f}$ is the unique continuous map such that $\widetilde{f} \circ p = f$. Thus (Y, p) is the quotient space of X by the equivalence relation $\sim$ on X (see Definition 2.6.6). □

Remark 2.6.13. It follows from construction of quotient space in Theorem 2.6.9, and the proof of Theorem 2.6.12 that if $f : X \to Y$ is a quotient map then the set of all fibers $\{f^{-1}(y) : y \in Y\}$

of f gives a partition of X, and hence defines an equivalence relation $\sim$ on X such that the associated quotient space $X/\sim$ is homeomorphic to (Y, f).

Definition 2.6.14. A map $f : X \to Y$ is said to be *open* (resp., *closed*) if $f(U)$ is open (resp., closed) in Y for any open (resp., closed) subset U of X.

Exercise 2.6.15. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two open maps of topological spaces. Show that the product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)(x_1, x_2) = (f(x_1), f_2(x_2)), \ \forall\, (x_1, x_2) \in X_1 \times X_2,$$

is an open map.

Corollary 2.6.16. *A surjective continuous open (or, closed) map is a quotient map.*

Proof. Let $f : X \to Y$ be a surjective map. Then for any $V \subseteq Y$ we have $f(f^{-1}(V)) = V$. Suppose that f is also continuous and open (resp., closed). Then for any $V \subseteq Y$ with $f^{-1}(V)$ open (resp., closed) in X, $V = f(f^{-1}(V))$ is open (resp., closed) in Y. Hence the result follows from Theorem 2.6.12. □

Remark 2.6.17. Corollary 2.6.16 fails without continuity assumption on f. For example, take a set X with at least two elements. Let τ_0 and τ_1 be the trivial topology and the discrete topology on X, respectively. Then the identity map $\mathrm{Id}_X : (X, \tau_0) \to (X, \tau_1)$ is a surjective open map, which is not continuous let alone be a quotient map.

Exercise 2.6.18. Let $q : X \to Q$ be a quotient map, and let $Z \subseteq Q$. Show by an example that the restriction map $q\big|_{q^{-1}(Z)} : q^{-1}(Z) \to Z$ need not be a quotient map, in general. If Z is open or q is an open map, show that the above restriction map become a quotient map.

Corollary 2.6.19. *Let $f : X \to Y$ be a continuous surjective map. If X is compact and Y is Hausdorff, then f is a quotient map.*

Proof. Let Z be a closed subset of X. Since X is compact, Z is compact. Since f is continuous, $f(Z)$ is a compact subset of Y. Since Y is Hausdorff, $f(Z)$ is closed in Y. Hence the result follows from Corollary 2.6.16. □

Proposition 2.6.20. *Let $\sim$ be an equivalence relation on a topological space X, and let (Q, q) be the associated quotient space. Given a topological space Y, a map $\phi : Q \to Y$ is continuous if and only if the composite map $\phi \circ q : X \to Y$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\phi\circ q"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof. Since the quotient map q is continuous, the composite map $\phi \circ q$ is continuous whenever ϕ is continuous. Conversely, let $\phi \circ q$ be continuous. Since for any open subset $V \subseteq Y$, we have $q^{-1}(\phi^{-1}(V)) = (\phi \circ q)^{-1}(V)$ is open in X, by construction of topology of Q, the subset $\phi^{-1}(V)$ is open in Q. Thus ϕ is continuous. □

Example 2.6.21. (i) **Circle**: Let $I = [0, 1] \subset \mathbb{R}$ be the unit closed interval in $\mathbb{R}$. Define a map

$$f : [0, 1] \to S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

by

$$f(t) = (\cos 2\pi i t, \sin 2\pi i t), \ \forall \, t \in [0, 1].$$

Clearly f is a surjective continuous map. Since $[0, 1]$ is compact and S^1 is Hausdorff, it follows from Corollary 2.6.19 that $f : I \to S^1$ is a quotient map. Note that $f^{-1}(1, 0) = \{0, 1\}$ and $f^{-1}(x, y)$ is singleton for $(x, y) \in S^1 \setminus \{(1, 0)\}$. Therefore, S^1 is the quotient space of $[0, 1]$ for the equivalence relation on $[0, 1]$ which only identify the end points of $[0, 1]$ to a single point.

(ii) **Cylinder**: Let $I = [0,\, 1] \subset \mathbb{R}$. Consider the unit square

$$I \times I = \{(x,\, y) \in \mathbb{R}^2 \,:\, 0 \leq \, x,\, y \, \leq 1\}$$

in $\mathbb{R}^2$. Define an equivalence relation $\sim_1$ on $I \times I$ by setting

$$(x, y) \sim_1 (x', y'), \ \text{ if } \ x' = x + 1 = 1 \ \text{ and } \ y = y'.$$

This identifies points of two vertical sides of $I \times I$ (see Figure 2.1 below), and the associated quotient space $(I \times I)/\sim_1$ is homeomorphic to the cylinder

$$S^1 \times [0,\, 1] = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \ 0 \leq z \leq 1\}.$$

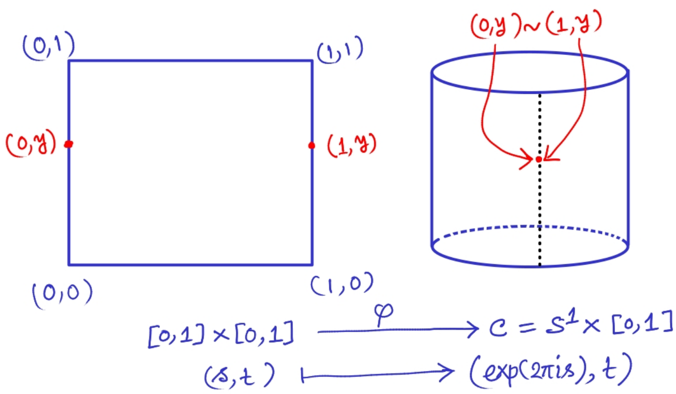


FIGURE 2.1

Indeed, we can define a (continuous) map

$$\phi : I \times I \to S^1 \times [0,\, 1]$$

by $\phi(s,\, t) = (\exp(2\pi i s),\, t)$, for all $(s,\, t) \in I \times I$. Then the set $\{\phi^{-1}(z, t) : (z,\, t) \in S^1 \times [0,\, 1]\}$ of all fibers of ϕ is precisely the partition of $I \times I$ given by the equivalence relation

$\sim_1$ on $I \times I$. It follows from Corollary 2.6.19 that ϕ is a quotient map, and by Remark 2.6.13 the associated quotient space $(I \times I)/\sim$ is homeomorphic to $S^1 \times I$.

(iii) **Torus**: Consider an equivalence relation $\sim_2$ on the cylinder $S^1 \times I$ defined by

$$(z, t) \sim_2 (z', t') \text{ if } z = z', \text{ and } t' = t + 1 = 1.$$

This identifies each point of the bottom circle of $S^1 \times I$ with the corresponding point of the top circle on $S^1 \times I$ (see Figure 2.2 below).

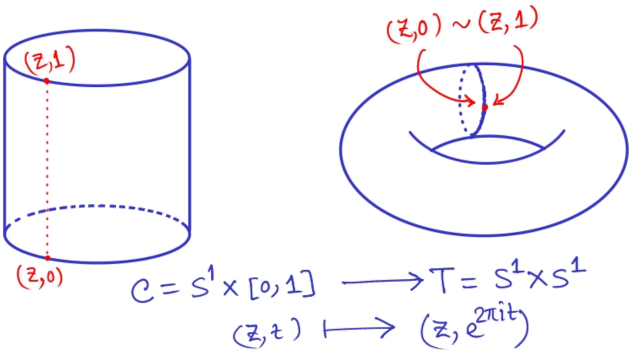


FIGURE 2.2

Then the associated quotient space $(S^1 \times I)/\sim_2$ is homeomorphic to the torus $T := S^1 \times S^1$ in $\mathbb{R}^3$. Indeed, we can define a (continuous) map

$$\psi : S^1 \times I \to S^1 \times S^1$$

by $\psi(z, t) = (z, \exp(2\pi i t))$, for all $(z, t) \in S^1 \times I$. As before, it is easy to see that the set of all fibers of the map ψ is precisely the partition of $S^1 \times I$ defined by the equivalence relation $\sim_2$ on the cylinder $S^1 \times I$. As before, it follows from Corollary 2.6.19 that ψ is a quotient map, and by Remark 2.6.13 the associated quotient space $(S^1 \times I)/\sim$ is homeomorphic to $S^1 \times S^1$.

(iv) Define a relation $\rho \subset \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$ by $(x, y) \in \rho$ if $x - y \in \mathbb{Z}$. Note that this is an equivalence realtion on $\mathbb{R}$. Let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Define a map $f : \mathbb{R} \to S^1$ by

$$f(t) = (\cos 2\pi i t, \sin 2\pi i t), \ \forall\, t \in \mathbb{R}.$$

Clearly f is a surjective continuous map. We show that f is an open map. For this, it suffices to show that image of an open interval $(a, b) \subset \mathbb{R}$ is open in S^1. Let $a, b \in \mathbb{R}$ with $a < b$. Since $f(t + 1) = f(t)$, $\forall\, t \in \mathbb{R}$, it follows that $f((a, b)) = S^1$ if $b - a \geq 1$. Suppose that $b - a < 1$. Let $p := (x, y) \in f((a, b))$ be arbitrary. Then $(x, y) = (\cos 2\pi i t, \sin 2\pi i t)$, for some $t \in (a, b)$. Taking $r = \min\{t - a, b - t\} > 0$, we see that $B(p, r) \cap S^1 \subseteq f((a, b))$ (verify). Thus, $f((a, b))$ is open in S^1, and hence f is an open map. Then by Corollary 2.6.16 f is a quotient map. Since the fibers of f can be identified with $\mathbb{Z}$ (verify!), we may denote the associated quotient space by $\mathbb{R}/\mathbb{Z}$. Then f induces a homeomorphism of $\mathbb{R}/\mathbb{Z}$ onto S^1.

(v) **Cone**: Let $I = [0, 1] \subseteq \mathbb{R}$. The *cone* of a topological space X is the quotient space $CX := (X \times I)/\sim$ of $X \times I$ for the equivalence relation $\sim$ on $X \times I$ defined by

$$(x,t) \sim (x',t'), \text{ if } t = t' = 1. \tag{2.6.22}$$

The associated set of all partitions of $X \times I$ is the set

$$\big\{X \times \{1\}, \{(x,t)\} : x \in X, 0 \leq t < 1\big\}.$$

Thus we identify all points of $X \times \{1\} \subseteq X \times I$ into a single point, called the *vertex* of the

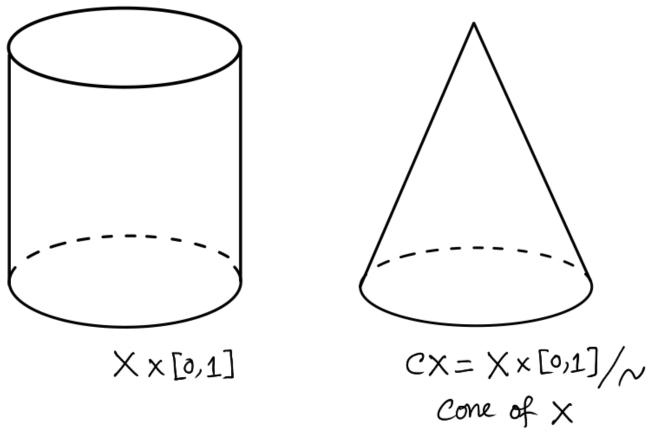


FIGURE 2.3

cone CX, and the remaining points of $X \times [0,1)$ remains as they are.

If X is a compact subset of an Euclidean space $\mathbb{R}^n$, then we can construct CX more geometrically as follow. Embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$, and fix a point $v \in \mathbb{R}^{n+1}$ which lies outside the image of this embedding; for example take $v = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Note that $\ell_{[v,x]} := \{tv + (1-t)x : 0 \leq t \leq 1\} \subseteq \mathbb{R}^{n+1}$ is the straight line segment in $\mathbb{R}^{n+1}$ joining v and $x \in X$. The subset

$$\bigcup_{x \in X} \ell_{[v,x]} \subseteq \mathbb{R}^{n+1}$$

with the subspace topology induced from $\mathbb{R}^{n+1}$ is called the *geometric cone* of X. We show that the geometric cone of X is homeomorphic to the cone of X, i.e.,

$$CX \cong \bigcup_{x \in X} \ell_{[v,x]}.$$

Define a map

$$f : X \times I \to \bigcup_{x \in X} \ell_{[v,x]}$$

by $f(x,t) = tv + (1-t)x$, for all $(x,t) \in X \times I$. Clearly f is a surjective continuous map, and $f(x,t) = f(x,t')$ if and only if either $x = x'$ and $t = t'$, or $t = t' = 1$. Since X is compact and its image is Hausdorff (being a subspace of $\mathbb{R}^{n+1}$), it follows from Corollary

2.6.19 that f is a quotient map. Since the fibers of the map f are precisely the equivalence classes for the equivalence relation on $X \times I$ defined in (2.6.22), it follows from Remark 2.6.13 that $CX = (X \times I)/\sim$ is homeomorphic to the geometric cone of X.

(vi) **The space** X/A: Let A be a subset of a topological space X. Define an equivalence relation $\sim$ on X by

$$x \sim x' \text{ if both } x \text{ and } x' \text{ are in } A.$$

We denote by X/A the associated quotient space $X/\sim$. Here we collapse the subspace A into a single point, and the remaining points of $X \setminus A$ remains as they were. For example, $CX = (X \times I)/(X \times \{1\})$.

(vii) **The space** B^n/S^{n-1}: Consider the closed unit ball

$$B^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1\}$$

in $\mathbb{R}^n$, and its boundary

$$\partial B^n = \{(x_1, \ldots, x_n) \in \mathbb{B}^n : \sum_{j=1}^{n} x_j^2 = 1\} = S^{n-1}.$$

Then the associated quotient space is denoted by B^n/S^{n-1} is homeomorphic to S^n. This is quite easy to visualize for $n = 1$ and 2. For $n = 1$, $B^1 = [-1, 1] \subseteq \mathbb{R}$, and $S^0 = \{-1, 1\}$ is its boundary. If we identify all points of $S^0 = \{-1, 1\}$ into a single point and keep all other points of B^1 as they were, we get a circle S^1 in $\mathbb{R}^2$; see Figure 2.4 below.

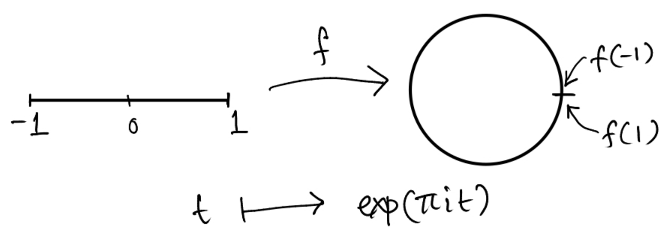


FIGURE 2.4

The case $n = 2$ is explained in the Figure 2.5 below.

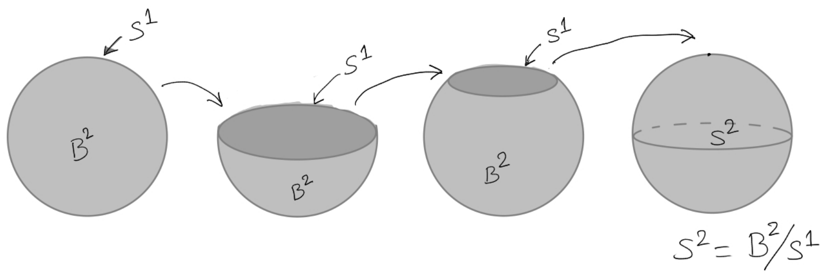


FIGURE 2.5

In general, it suffices to construct a surjective continuous map

$$f : B^n \to S^n$$

such that $f\big|_{B^n \setminus S^{n-1}}$ is injective and $f(S^{n-1})$ is a singleton subset of S^n. Then by Corollary 2.6.19, f become a quotient map producing a homeomorphism of B^n/S^{n-1} onto S^n. To construct such a map f, note that $\mathbb{R}^n$ is homeomorphic to $B^n \setminus S^{n-1}$ and $S^n \setminus \{p\}$, for any $p \in S^n$. Fix two homeomorphisms $h_1 : B^n \setminus S^{n-1} \to \mathbb{R}^n$ and $h_2 : \mathbb{R}^n \to S^n \setminus \{p\}$, and define

$$f(x) := \begin{cases} h_2(h_1(x)), & \text{if } \ x \in B^n \setminus S^{n-1}, \\ p, & \text{if } \ x \in S^{n-1}. \end{cases} \tag{2.6.23}$$

It is easy to check that f has desired properties (verify).

Example 2.6.24 (Attaching spaces along a map)**.** Let X and Y be two topological spaces. Suppose we wish to attach X by identifying points of a subspace $A \subseteq X$ with points of Y in a continuous way. This can be done by using a continuous map $f : A \to Y$. Indeed, we identify $x \in A$ with its image $f(x) \in Y$. This defines an equivalence relation on $X \sqcup Y$, and we denote the associated quotient space by $X \sqcup_f Y$, and call it the *space Y with X attached along A via f*. Let us discuss some examples.

(i) Let $I = [0, 1] \subset \mathbb{R}$. Given a space X, we call $X \times I$ the *cylinder over* X. Let $f : X \to Y$ be a continuous map of topological spaces. Then f induces a continuous map $\widetilde{f} : X \times \{0\} \to Y$ given by

$$\widetilde{f}(x, 0) = f(x), \ \forall\, (x, 0) \in X \times \{0\}.$$

If we attach Y with the *cylinder* $X \times I$ of X along its base $X \times \{0\} \subset X \times I$ via the map $\widetilde{f}$, by identifying $(x, 0) \sim f(x)$, then the associated quotient space $M_f = (X \times I) \sqcup_{\widetilde{f}} Y$ is called the *mapping cylinder of f* (see Figure 2.6).

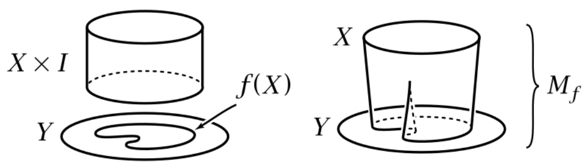


FIGURE 2.6: Mapping Cylinder

(ii) Let $CX = (X \times I)/(X \times \{1\})$ be the *cone over* X obtained by collapsing the subspace $X \times \{1\}$ of the cylinder $X \times I$ over X to a single point. Let $f : X \to Y$ be a continuous map. If we attach this cone CX with Y along its base $X \times \{0\} \subset CX$ by identifying $(x, 0) \in X \times \{0\}$ with $f(x) \in Y$, then the resulting quotient space $C_f = Y \sqcup_f CX$ is called the *mapping cone of f* (see Figure 2.7). Note that, the mapping cone C_f can also be obtained as a quotient space of the mapping cylinder M_f by collapsing $X \times \{1\} \subset M_f$ to a point.

(iii) Let X be a topological space. The *suspension* SX of X is the quotient space of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

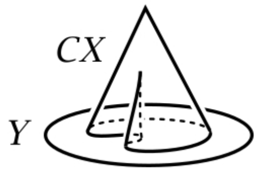


FIGURE 2.7: Mapping Cone

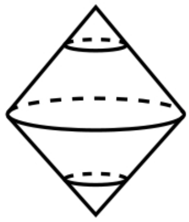


FIGURE 2.8: Suspension

For example, if we take X to be the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then $X \times I$ is a cylinder $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\, 0 \leq z \leq 1\}$, and then we collapse two circular edges of C to two points to get SX, which is homeomorphic to the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

We can think of SX as a *double cone on* X: take disjoint union of two cones $C_1 := (X \times I)/(X \times \{1\})$ and $C_2 := (X \times I)/(X \times \{0\})$, and then attach C_1 with C_2 via the continuous map $f : X \times \{0\} \to C_2$ given by $f(x,0) = (x,0) \in C_2$, $\forall\, (x,0) \in X \times \{0\} \subset C_1$.

The following exercise shows that a quotient of a Hausdorff space need not be Hausdorff in general.

Exercise 2.6.25. Consider the double real line $X = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$, and the equivalence relation $\sim$ on X defined by $(t,0) \sim (t,1)$, for all $t \in \mathbb{R} \setminus \{0\}$. The associated quotient space $X/{\sim}$ is called the *real line with double origin*. Show that $X/{\sim}$ is not Hausdorff.

Exercise 2.6.26. Define an equivalence relation $\sim$ on the Euclidean line $\mathbb{R}$ by $x \sim y$ if either $x = y$ or $|x| = |y|$ and $|x| > 1$, and let $Y := X/\sim$ be the associated quotient space. Show that every point $z \in Y$ has an open neighbourhood homeomorphic to $(-1,1)$. Show that Y is not Hausdorff.

Proposition 2.6.27. *Let $\rho \subseteq X \times X$ be an equivalence relation on a topological space X, and let $q : X \to Q := X/\rho$ be the quotient map. Then we have the following.*

(i) Q is a T1 space if and only if every ρ-equivalence class is closed in X.

(ii) If Q is Hausdorff then ρ is a closed subspace of the product space $X \times X$. The converse holds if $q : X \to Q$ is an open map.

Proof. (i) Let Q be T1. Let $x \in X$. Choose a $y \in X \setminus [x]$. Then $[x] \neq [y]$ in Q. Since Q is T1, there is an open subset $V_y \subseteq Q$ such that $[y] \in V_y$ and $[x] \notin V_y$. Then $q^{-1}(V_y)$ is an open neighbourhood of y with $q^{-1}(V_y) \cap [x] = \emptyset$. Therefore, y is an interior point of $X \setminus [x]$. Therefore, $X \setminus [x]$ is open, and hence $[x] \subseteq X$ is closed.

Conversely, suppose that $[x] \subseteq X$ is closed, for all $x \in X$. To show Q is T1, we need to show that $\{[x]\}$ is closed in Q, for all $x \in X$. Since $q^{-1}(Q \setminus \{[x]\}) = \{y \in X : q(y) \neq [x]\} = X \setminus [x]$ is open, $Q \setminus \{[x]\}$ is open in Q, for all $[x] \in Q$. Therefore, Q is a T1 space.

(ii) Consider the commutative diagram of continuous maps

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\Delta_X"'] & Q \arrow[d, "\Delta_Q"] \\ X \times X \arrow[r, "q \times q"] & Q \times Q, \end{tikzcd}$$

where $q \times q : X \times X \to Q \times Q$ is the product map given by

$$(q \times q)(x, y) = (q(x), q(y)), \ \forall\, (x, y) \in X \times X.$$

Note that, $(q \times q)^{-1}(\Delta_Q(Q)) = \{(x, y) \in X \times X : q(x) = q(y)\} = \rho$. If Q is Hausdorff, then $\Delta_Q(Q)$ is closed by Lemma 2.5.2. Since $q \times q$ is continuous, ρ is closed in $X \times X$.

Now we assume that q is an open map, and that ρ is closed in $X \times X$. Since $q \times q$ is a continuous surjective open map (verify!), it is a quotient map by Corollary 2.6.16. Since $(q \times q)^{-1}(\Delta_Q(Q)) = \rho$ is closed in $X \times X$, the diagonal $\Delta_Q(Q)$ is closed in $Q \times Q$ by Theorem 2.6.12 (iii). Therefore, Q is Hausdorff by Lemma 2.5.2.

□

Now we give an example to show that even if X is Hausdorff and the equivalence relation ρ is closed in $X \times X$, the associated quotient space $Q = X/\rho$ need not be Hausdorff without the assumption that q is an open map. For this, we first recall the following.

Definition 2.6.28. A topological space X is said to be *normal* if any two disjoint closed subsets can be separated by a pair of disjoint open subsets containing them. In other words, given two closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there are open subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Most of the familiar examples of topological spaces are generally normal (e.g., $\mathbb{R}^n$), and a closed subspace of a normal space is normal. The following example shows that a Hausdorff space need not be normal.

Example 2.6.29. Let $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Consider the topology τ_K on $\mathbb{R}$ whose basis for open subsets is given by the collection

$$\mathscr{B} = \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R} \text{ with } a < b\}.$$

Clearly this topology on $\mathbb{R}$ is strictly finer than the Euclidean topology on $\mathbb{R}$, and hence $(\mathbb{R}, \tau_K)$ is a Hausdorff space. Note that in this topology, K and $\{0\}$ are disjoint closed subsets that cannot be separated by a pair of disjoint open subsets containing them. Therefore, $(\mathbb{R}, \tau_K)$ is not normal.

Exercise 2.6.30. Start with a Hausdorff space X that is not normal. Choose two disjoint closed subsets $A, B \subset X$ that cannot be separated by two disjoint open subsets containing them. Take $\rho = \Delta_X(X) \cup (A \times A) \cup (B \times B)$. Note that ρ is an equivalence relation on X, and is closed in $X \times X$ (why?). Show that the associated quotient space X/ρ is T1 but not Hausdorff.

2.7 Projective space and Grassmannian

2.7.1 Real and complex projective spaces

Fix an integer $n \geq 0$. Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$v \sim v' \text{ if } v' = \lambda \cdot v, \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}. \tag{2.7.1}$$

In other words, identify all points lying on the same straight-line in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. Then the associated quotient space

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

is called the *real projective n-space*. As a set, $\mathbb{RP}^n$ consists of all straight-lines in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. So an element of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \big\{\lambda \cdot (a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R} \setminus \{0\}\big\}. \tag{2.7.2}$$

Let $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map for the projective n-space. Note that the unit n-sphere $S^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} a_j^2 = 1\}$ is a compact connected subspace of $\mathbb{R}^{n+1} \setminus \{0\}$. Since the restriction map $q\big|_{S^n} : S^n \longrightarrow \mathbb{RP}^n$ is continuous and surjective, $\mathbb{RP}^n$ is compact and connected.

Exercise 2.7.3. Prove the following.

(i) Show that the map $f := q\big|_{S^n} : S^n \to \mathbb{RP}^n$ is a quotient map.

(ii) For each $\ell \in \mathbb{RP}^n$, show that $f^{-1}(\ell) = \{v, -v\}$, for some $v \in S^n$.

(iii) Show that $\mathbb{RP}^n$ is Hausdorff.

Outline of solution. Note that, the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is given by sending $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ to the straight line

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) : \lambda \in \mathbb{R}\} \in \mathbb{RP}^n.$$

Since $\mathbb{RP}^n$ consists of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin, given a straight-line $\ell \in \mathbb{RP}^n$, choosing any non-zero point $v := (a_0, \ldots, a_n) \in \ell$, we find an element $v/\|v\| \in S^n$

with $f(v/\|v\|) = \ell$, where $\|v\| := \big(\sum\limits_{j=1}^{n} a_j^2\big)^{1/2}$. Thus, f is surjective.

$$f : S^n \overset{\iota}{\hookrightarrow} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{q} \mathbb{RP}^n.$$

Note that given any subset $V \subseteq \mathbb{RP}^n$, we have $f^{-1}(V) = q^{-1}(V) \cap S^n$. So continuity of f follows from that of q. To see that f is a quotient map, suppose that $f^{-1}(V)$ is open in S^n. To show that V is open in $\mathbb{RP}^n$, fix a point $\ell \in V$. Its fiber $f^{-1}(\ell) = \{v, -v\}$ consists of the two antipodal points of S^n obtained by intersecting the line ℓ with S^n. Since the points v and $-v$ lies on two hemispheres separated by a great circle on S^n, we can find a small enough (connected) open neighbourhood $U \subset f^{-1}(V)$ of v such that $-U := \{-u : u \in U\} \subset S^n$ is an open neighbourhood of $-v$ in S^n, and $U \cap (-U) = \emptyset$. Note that, $-U \subseteq f^{-1}(V)$. Then $f\big|_U$ is a homeomorphism of U onto the open neighbourhood $f(U) \subseteq V$ of ℓ in $\mathbb{RP}^n$. Thus f is a quotient map. □

Next we show that $\mathbb{RP}^n$ can be covered by $n+1$ open subsets each homeomorphic to $\mathbb{R}^n$. Let $p_j : \mathbb{R}^{n+1} \to \mathbb{R}$ be the j-th projection map defined by

$$p_j(x_0, \ldots, x_n) = x_j, \ \ \forall\, (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}.$$

For each $j \in \{0, 1, \ldots, n\}$, consider the *hyperplane*

$$H_j := \{[a_0 : \cdots : a_n] \in \mathbb{RP}^n : a_j = 0\} \subset \mathbb{RP}^n.$$

Since q is a quotient map and

$$\begin{aligned} q^{-1}(H_j) &= \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : a_j = 0\} \\ &= p_j^{-1}(0) \cap (\mathbb{R}^{n+1} \setminus \{0\}), \end{aligned}$$

we conclude that H_j is a closed subset of $\mathbb{RP}^n$. Let $U_j := \mathbb{P}^n \setminus H_j$, $\forall\, j = 0, 1, \ldots, n$. Since any point of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R}\},$$

with $a_j \neq 0$, for some j, we see that $\{U_0, U_1, \ldots, U_n\}$ is an open cover of $\mathbb{RP}^n$.

Proposition 2.7.4. *The open subset $U_j \subset \mathbb{RP}^n$ is homeomorphic to $\mathbb{R}^n$, for all j.*

Proof. Consider the map $\phi_j : U_j \longrightarrow \mathbb{R}^n$ given by

$$[a_0 : \cdots : a_n] \overset{\phi_j}{\longmapsto} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right).$$

Note that ϕ_j is a well-defined bijective map with its inverse $\psi_j : \mathbb{R}^n \to U_j$ given by

$$(b_0, \ldots, b_{n-1}) \mapsto \big[b_0 : \cdots : b_{j-1} : 1 : b_j : \cdots : b_n\big].$$

Note that $V_j = q^{-1}(U_j) = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : a_j \neq 0\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$, and the map $f_j : V_j \to \mathbb{R}^n$ given by $(a_0, \ldots, a_n) \xmapsto{f_j} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right)$ is continuous (why?). Since $q^{-1}(\phi_j^{-1}(V)) = f_j^{-1}(V), \forall\ V \subseteq \mathbb{R}^n$, and q is a quotient map, we conclude that ϕ_j is continuous, for all j (c.f. Proposition 2.6.20).

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "q"'] & V_j \arrow[l, hook'] \arrow[r, "f_j"] \arrow[d, "q_j"'] & \mathbb{R}^n \arrow[dl, dashed, bend left, "\psi_j"] \\
\mathbb{RP}^n & U_j \arrow[l, hook'] \arrow[ur, "\phi_j"] &
\end{tikzcd}$$

Since f_j is a quotient map by Corollary 2.6.16, as before we see that $\psi_j = \phi_j^{-1}$ is also continuous. This completes the proof. $\square$

Corollary 2.7.5. *$\mathbb{RP}^n$ is a compact connected Hausdorff space.*

Exercise 2.7.6. Define an equivalence relation $\sim$ on S^n by

$$v \sim v' \text{ if } v' = -v.$$

Show that the associated quotient space $S^n/\sim$ is homeomorphic to $\mathbb{RP}^n$. Conclude that $\mathbb{RP}^n$ is a compact connected Hausdorff space.

The *complex projective n-space* $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim$ defined by

$$v \sim v' \text{ if } v' = \lambda v, \text{ for some } \lambda \in \mathbb{C}.$$

So the points of $\mathbb{CP}^n$ are precisely one dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^{n+1}$ (i.e., complex lines in $\mathbb{C}^{n+1}$ passing through the origin $0 \in \mathbb{C}^{n+1}$).

Exercise 2.7.7. Show that $\mathbb{CP}^n$ is a compact connected Hausdorff space.

Remark on notations: The real projective n-space $\mathbb{RP}^n$ is also denoted by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n(\mathbb{R})$. Similar notations $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^n(\mathbb{C})$ are also used for complex projective n-space $\mathbb{CP}^n$.

2.7.2 Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$

Fix two positive integers k and n, with $k < n$. Let

$$(\mathbb{R}^n)^k := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}}$$

be k-fold product of $\mathbb{R}^n$ together with the product topology. A typical element of $(\mathbb{R}^n)^k$ is of the form $(v_1, \ldots, v_k)$, where $v_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$, for all $j = 1, \ldots, k$. Note that, we can identify

$(\mathbb{R}^n)^k$ with $M_{k,n}(\mathbb{R})$ using the bijective map

$$(v_1, \dots, v_k) \longmapsto \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}.$$

Consider the subset

$$X := \{(v_1, \dots, v_k) \in (\mathbb{R}^n)^k \mid \{v_1, \dots, v_k\} \text{ is } \mathbb{R}\text{-linearly independent}\}$$

with the subspace topology induced from $(\mathbb{R}^n)^k$. Given $A := (v_1, \dots, v_k)$ and $A' := (v'_1, \dots, v'_k)$ in X, we define $A \sim A'$ if

$$\mathrm{Span}_{\mathbb{R}}\{v_1, \dots, v_k\} = \mathrm{Span}_{\mathbb{R}}\{v'_1, \dots, v'_k\}.$$

Clearly $\sim$ is an equivalence relation on X. The associated quotient topological space $X/\sim$ is known as the *Grassmannian of k-dimensional $\mathbb{R}$-linear subspaces of* $\mathbb{R}^n$, and is denoted by $\mathrm{Gr}(k, \mathbb{R}^n)$. As a set, $\mathrm{Gr}(k, \mathbb{R}^n)$ consists of all $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$ of dimension k.

Corollary 2.7.8. $\mathrm{Gr}(1, \mathbb{R}^n)$ *is homeomorphic to* $\mathbb{RP}^{n-1}$.

Remark 2.7.9 (Plücker embedding)**.** Given a n-dimensional $\mathbb{R}$-vector space V, its k-th exterior power $\bigwedge^k V$ is a $\mathbb{R}$-vector space of dimension $\binom{n}{k}$. Sending $W \in \mathrm{Gr}(k, \mathbb{R}^n)$ to its k-th exterior power $\bigwedge^k W \subset \bigwedge^k \mathbb{R}^n$, we get a continuous map

$$\Phi : \mathrm{Gr}(k, \mathbb{R}^n) \longrightarrow \mathbb{RP}^N,$$

where $N = \binom{n}{k} - 1$. It turns out that Φ is a closed embedding (homeomorphism onto a closed subspace of $\mathbb{RP}^N$). From this, one can conclude that $\mathrm{Gr}(k, \mathbb{R}^n)$ is a compact Hausdorff space. We shall not go into detailed proofs of the above statements.

2.8 Topological group

Definition 2.8.1. A *topological group* is a topological space G which is also a group G such that the binary map (group operation)

$$m : G \times G \to G, \ \ (x, y) \mapsto xy,$$

and the inversion map

$$\mathrm{inv} : G \to G, \ \ x \mapsto x^{-1},$$

involved in its group structure, are continuous. Here we consider $G \times G$ as the product topological space.

We recast the above definition of topological group in more formal language, without using points of G. This formalism, with appropriate type of spaces and maps between them, defines

Lie group, *algebraic group*, *group-scheme* and more generally, a *group object* in a category (for curious readers!). Denote by $*$ the topological space whose underlying set is singleton. This space is unique up to a unique homeomorphism. Given any topological space X, any map $* \to X$ is continuous, and they are in bijection with the underlying set of points of X. On the other hand, the space $*$ is the *final object* in the category of topological spaces in the sense that, given any topological space X, there is a unique continuous map $X \to *$. Clearly the product space $X \times *$ is homeomorphic to X, and the set of all such homeomorphisms are in bijection with the set of all *automorphisms of* X (i.e., homeomorphisms of X onto itself). Unless explicitly specified, we consider the homeomorphism $X \times * \to X$ given by the identity map $\mathrm{Id}_X : X \to X$ of X.

Now the above Definition 2.8.1 essentially says that, a topological group is a pair (G, m), where G is a topological space and $m : G \times G \to G$ is a continuous map such that the following axioms holds.

(TG1) *Associativity:* The following diagram is commutative.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m\times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

(TG2) *Existence of neutral element:* There is a continuous map $e : * \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
* \times G \arrow[r, "e\times \mathrm{Id}_G"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{Id}_G \times e"'] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd}$$

(TG3) *Existence of inverse:* There is a continuous map $\mathrm{inv} : G \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
G \arrow[r, "(\mathrm{Id}_G, \mathrm{inv})"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{inv}, \mathrm{Id}_G)"'] \arrow[d] \\
* \arrow[r, "e"] & G & * \arrow[l, "e"']
\end{tikzcd}$$

Example 2.8.2. (i) Any abstract group is a topological group with respect to the discrete topology on it.

(ii) $(\mathbb{R}, +)$, the real line with usual addition of real numbers, is a topological group.

(iii) $(\mathbb{R}^*, \cdot)$, the subspace of non-zero real numbers with usual multiplication is a topological group.

(iv) $(\mathbb{Z}, +)$ is a topological group, where the topology on $\mathbb{Z}$ is discrete.

(v) For any integer $n \geq 1$, the Euclidean space $\mathbb{R}^n$ with the component wise addition, i.e.,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n), \ \ \forall\, a_i, b_i \in \mathbb{R},$$

is a topological group.

(vi) Given integers $m, n \geq 1$, the set of all $(m \times n)$-matrices with real entries $\mathrm{M}_{m,n}(\mathbb{R})$, considered as the Euclidean topological space $\mathbb{R}^{mn}$, is a topological group with respect to the usual matrix addition.

(vii) $\mathrm{GL}_n(\mathbb{R})$, the subspace of all invertible $(n \times n)$-matrices with real entries, is a topological group with respect to multiplication of matrices.

(viii) Circle group: The space $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$, together with the multiplication of complex numbers, is a topological group.

(ix) Any abstract subgroup of a topological group is a topological group with respect to the subspace topology.

(x) Product of two topological groups is a topological group.

Exercise 2.8.3. Let G be a topological group. If $U \subseteq G$ is an open neighbourhood of identity $e \in G$, show that there is an open neighbourhood $V \subset G$ of identity such that $V^2 := \{ab : a, b \in V\} \subseteq U$. (Hint: Use continuity of the multiplication map m.)

Exercise 2.8.4. Show that for any $a \in G$, the *right translation by a map*

$$R_a : G \to G, \ \ g \mapsto ga,$$

is a homeomorphism. Prove the same statement for the *left translation by a map* given by $L_a(g) = ga$, for all $g \in G$. (*Hint:* Note that R_a is the composite map $g \mapsto (g, a) \overset{m}{\mapsto} ga$ with inverse $R_{a^{-1}}$.)

Exercise 2.8.5. Show that a topological group G is Hausdorff if and only if it is a T1 space. (*Hint:* $\Delta_G(G)$ is precisely the inverse image of $\{e\} \subseteq G$ under the map $(x, y) \mapsto x^{-1}y$.)

Lemma 2.8.6. *Let G be a topological group. Let H be the connected component of G containing the neutral element $e \in G$. Then H is a closed normal subgroup of G.*

Proof. Since connected components are closed, H is closed. Since for any $a \in H$, the set $Ha^{-1} = \{ha^{-1} : h \in H\} = R_{a^{-1}}(H)$ contains e, and is homeomorphic to H, we must have $Ha^{-1} \subseteq H$. Since this holds for all $a \in H$, we see that H is a subgroup of G. To see that H is normal, note that, for any $g \in G$, the set $gHg^{-1} = L_g(R_{g^{-1}}(H))$ is a connected subset of G containing e, and hence $gHg^{-1} \subseteq H$. This completes the proof. □

Definition 2.8.7. A *right action* of a topological group G on a topological space X is a continuous map $\sigma : X \times G \to X$ such that $\sigma(x, e) = x$, and $\sigma(\sigma(x, g_1), g_2) = \sigma(x, m(g_1, g_2))$, for all $x \in X$ and $g_1, g_2 \in G$, where $m : G \times G \to G$ is the product operation (multiplication map) on G. Similarly, one can left action of G on X.

Without using points, a right G-action σ on X can be defined by commutativity of the following diagrams.

(i)

$$\begin{tikzcd} X \times * \arrow[r, "\mathrm{Id}_X \times e"] \arrow[dr, "\cong"'] & X \times G \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

(ii)

$$\begin{tikzcd} X \times G \times G \arrow[r, "\sigma \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_X \times m"'] & X \times G \arrow[d, "\sigma"] \\ X \times G \arrow[r, "\sigma"] & X \end{tikzcd}$$

A right G-action σ on X induces an equivalence relation on X, which gives a partition of X as a disjoint union of equivalence classes. A typical equivalence class is of the form

$$\mathrm{orb}_G(x) := \{x' \in X : x' = xg, \text{ for some } g \in G\} = xG,$$

and is called the *G-orbit* of $x \in X$. The associated quotient space, denoted by X/σ or X/G, consists of all G-orbits of elements of X as its points. For this reason, X/G is also called *orbit space*. If the G-action on X is *transitive* (i.e., given any $x, x' \in X$, there exists $g \in G$ such that $x' = xg$), then X is called a *homogeneous space*. In this case, the associated quotient space X/G is singleton.

Exercise 2.8.8. Given a subgroup H of a topological group G, the H-action on G defined by

$$G \times H \mapsto G, \ \ (g, h) \mapsto gh$$

gives a partition of G into all right cosets of H in G. Show that the orbit space G/H is a homogeneous space.

Exercise 2.8.9. Let $I = [0, 1] \subset \mathbb{R}$. Define the $\mathbb{Z}_2$-action on $I \times I$ which gives identifications $(0, t) \sim (1, 1 - t)$, for each $t \in I$. Convince yourself that the associated quotient space is homeomorphic to the *Möbius strip* (see Figure 2.9). Note that, Möbius strip has only one side!

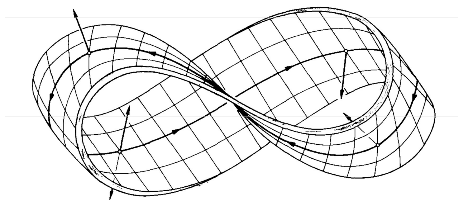


FIGURE 2.9: Möbius strip

Exercise 2.8.10. Define a $\mathbb{Z}_2$-action on the n-sphere $S^n \subset \mathbb{R}^{n+1}$ which identifies $v \in S^n$ with its *antipodal point* $-v \in S^n$. Show that the associated quotient space $S^n/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^n$.

Exercise 2.8.11. Show that $\mathbb{R}/\mathbb{Q}$ is a non-Hausdorff topological group. Hint: Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, and right cosets are just translates of $\mathbb{Q}$.

Exercise 2.8.12. Let $\sigma : X \times G \to X$ be a right action of a topological group on a space X. For each $g \in G$, show that the induced map

$$\sigma_g : X \to X, \;\; x \mapsto xg := \sigma(x, g)$$

is a homeomorphism.

Exercise 2.8.13. Let X be a topological space together with an action of a topological group G. Show that the quotient map $q : X \to X/G$ is open. (Hint: For $V \subseteq X$ open, show that $q^{-1}(q(V)) = \bigcup_{g \in G} Vg$ is open by Exercise 2.8.12, where $Vg = \{vg : v \in V\}, \forall g \in G$.)

Proposition 2.8.14. *Let H be a subgroup of a topological group G. Then the orbit space G/H is Hausdorff if and only if H is closed in G. (Here G/H is not necessarily a group because H need not be a normal subgroup of G.)*

Proof. If G/H is Hausdorff, then it is a T1 space so that $H = \mathrm{orb}_H(e) \in G/H$ is a closed point. Since H is the inverse image of this point under the quotient map $q : G \to G/H$ (continuous), H is closed in G. Conversely, suppose that H is closed in G. Since the equivalence relation given by the H-action on G is precisely the inverse image of H under the continuous map

$$G \times G \longrightarrow G, \;\; (g_1, g_2) \mapsto g_1^{-1} g_2,$$

and the quotient map $q : G \to G/H$ is open by Exercise 2.8.13, the converse part follows from Proposition 2.6.27 because H is closed in G. $\square$

Corollary 2.8.15. *The topological group $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.*

Definition 2.8.16. A *homomorphism* of topological groups is a continuous group homomorphism. An *isomorphism* of topological groups is a bijective bi-continuous homomorphism of topological groups.

Exercise 2.8.17. If $f : G \to H$ is a homomorphism of topological groups with H Hausdorff, show that $\mathrm{Ker}(f) := \{g \in G : f(g) = e_H\}$ is a closed normal subgroup of G.

Exercise 2.8.18. If $f : G \to H$ is a homomorphism of topological groups, show that the induced map $G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism of topological groups.

Exercise 2.8.19. Show that $f : \mathbb{R} \to S^1$ defined by $f(t) = e^{2\pi i t}$, for all $t \in \mathbb{R}$, is a surjective homomorphism of topological groups. Use Exercise 2.8.18 to show that $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

Exercise 2.8.20. Let $f : G \to H$ be a continuous bijective homomorphism of topological groups. Show that $f^{-1} : H \to G$ is continuous (Hint: Use Exercise 2.8.13).

Corollary 2.8.21. *A bijective homomorphism $f : G \to H$ of topological groups is an isomorphism.*

Exercise 2.8.22. Consider the $\mathbb{Z}$-action on $\mathbb{R}$ given by $\sigma(t, n) = t + n$, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Show that the associated quotient space $\mathbb{R}/\sigma$ is homeomorphic to S^1.

Exercise 2.8.23. Show that $\mathrm{GL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$ as topological groups.

Exercise 2.8.24. Show that $\mathrm{GL}_n(\mathbb{R})$ is disconnected, and has precisely two connected components, whereas $\mathrm{GL}_n(\mathbb{C})$ is path-connected. (*Hint:* For the first part, use determinant map. For the second part, given $A \in \mathrm{GL}_n(\mathbb{C})$ use left and right translation homeomorphisms to move it to an upper triangular matrix, and then use convex combination map for its entries to move it to the identity matrix in $\mathrm{GL}_n(\mathbb{C})$.)

Exercise* 2.8.25. Show that the group

$$SO_n = \{A \in \mathrm{GL}_n(\mathbb{R}) : AA^t = A^tA = I_n \text{ and } \det(A) = 1\}$$

is compact and connected.

Exercise* 2.8.26 (Universal property of product)**.** Let G_1 and G_2 be two topological groups. Let P be a topological group together with homomorphisms of topological groups $p_1 : P \to G_1$ and $p_2 : P \to G_2$ such that given given any topological group H and homomorphisms of topological groups $f_1 : H \to G_1$ and $f_2 : H \to G_2$, there is a unique homomorphism of topological groups $f : H \to P$ such that the following diagram commutes.

$$\begin{tikzcd} & H \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\ G_1 & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_2. \end{tikzcd}$$

Prove that there is a unique isomorphism of topological groups $\phi : P \to G_1 \times G_2$.

2.9 Connectedness

Let X be a topological space. A *separation* of X is a pair of open subsets $U, V \subseteq X$ such that $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$.

Definition 2.9.1. A topological space X is said to be *connected* if there is no separation of X by non-empty pair of disjoint open subsets of X that covers X. If X is not connected, it is called disconnected. A subset $A \subseteq X$ is said to be connected if the topological space A, with the subspace topology induced from X, is connected.

Example 2.9.2. (i) The empty subset $\emptyset \subseteq X$ is always connected because there is no separation of it.

(ii) The punctured real line $\mathbb{R} \setminus \{0\}$ is disconnected in $\mathbb{R}$ since it has a separation given by the open subsets $(-\infty, 0)$ and $(0, \infty)$.

(iii) The subset $[0, 1] \setminus \{1/2\} \subset \mathbb{R}$ is disconnected, since it has a separation given by the subsets $[0, 1/2), (1/2, 1]$ open in $[0, 1] \setminus \{1/2\}$.

(iv) Let $L_{m,c} := \{(x, mx + c) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be a straight-line in the Euclidean plane $\mathbb{R}^2$. Then the subset $\mathbb{R}^2 \setminus L_{m,c}$ is disconnected, since it has a separation given by the subsets $U = \{(x, y) : y < mx + c\}$ and $V = \{(x, y) : y > mx + c\}$ open in $\mathbb{R}^2 \setminus L_{m,c}$.

Proposition 2.9.3. *A topological space X is connected if and only if the only subsets of X that are both open and closed are the empty subset of X and X itself.*

Proof. Suppose that X is connected. Suppose on the contrary that there is a non-empty proper subset $U \subset X$ of X that is both open and closed in X. Then $V := X \setminus U$ is a non-empty proper open subset of X such that $U \cap V = \emptyset$ and $U \cup V = X$. This is not possible since X is connected.

Conversely, suppose that the only non-empty subset of X that is both open and closed in X is X itself. Suppose that $X = U \cup V$ for some open subsets U and V of X with $U \cap V = \emptyset$. If $U \neq \emptyset$, then $V = X \setminus U$ is both closed and also open in X, and hence it must be empty set. Thus X has no separation in X, and hence is connected. □

Lemma 2.9.4. *Let X be a topological space. Let $U, V \subset X$ be two non-empty disjoint open subsets of X such that $X = U \cup V$. If A is a connected subset of X, then either $A \subseteq U$ or $A \subseteq V$.*

Proof. Since $A \subseteq X = U \cup V$, A intersects at least one of U and V. Suppose that $A \cap U \neq \emptyset$. Then $A \cap U$ and $A \cap V$ are open subsets of A with $(A \cap U) \cup (A \cap V) = A$. Since A is connected and $A \cap U \neq \emptyset$, we must have $A \cap V = \emptyset$, and hence $A \subseteq U$. □

Lemma 2.9.5. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of connected subsets of X having a common point $x_0 \in \bigcap\limits_{\alpha \in \Lambda} A_\alpha$. Suppose on the contrary that $A := \bigcup\limits_{\alpha \in \Lambda} A_\alpha$ is not connected. Then there exists a pair of disjoint non-empty subsets $U, V \subseteq A$ open in A such that $A = U \cup V$. Since A_α is connected, A_α is entirely contained in exactly one of U and V. Suppose that $A_{\alpha_0} \subseteq U$. Since $\bigcap\limits_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, we see that $A_\alpha \subseteq U, \ \forall\, \alpha \in \Lambda$. Then $A = \bigcup\limits_{\alpha \in \Lambda} A_\alpha \subseteq U$. This forces V to be an empty set, which is a contradiction. Therefore, A must be connected. □

Lemma 2.9.6. *Let $f : X \to Y$ be a continuous map of topological spaces, and let A be a connected subset of X. Then $f(A)$ is a connected subset of Y.*

Proof. First proof: Suppose on the contrary that $f(A)$ is disconnected. Then there is a pair of non-empty subsets $U', V' \subseteq f(A)$ open in $f(A)$ with $U' \cap V' = \emptyset$ and $U' \cup V' = f(A)$. Now $U' = U \cap f(A)$ and $V' = V \cap f(A)$, for some open subsets U and V of Y. Since f is continuous, $\widetilde{U} := f^{-1}(U) \cap A$ and $\widetilde{V} := f^{-1}(V) \cap A$ are open subsets of A. Note that $\widetilde{U}$ and $\widetilde{V}$ are non-empty. Indeed, $U \cap f(A) \neq \emptyset$ implies that $f(x) \in U$, for some $x \in A$. Then $x \in f^{-1}(U) \cap A = \widetilde{U}$. Similarly, $\widetilde{V} \neq \emptyset$. Since $U \cap V \cap f(A) = U' \cap V' = \emptyset$, we have

$$\widetilde{U} \cap \widetilde{V} = (f^{-1}(U) \cap A) \bigcap (f^{-1}(V) \cap A) = f^{-1}(U \cap V \cap f(A)) = \emptyset.$$

Since $f(A) = U' \cup V' \subseteq U \cup V$, we have $A \subseteq f^{-1}(U) \cup f^{-1}(V)$, and hence $A = \widetilde{U} \cup \widetilde{V}$. This contradicts our assumption that A is connected. Therefore, $f(A)$ must be connected.

Second proof: Suppose on the contrary that $f(A)$ is disconnected. Then $f(A) = U \cup V$, for a pair of non-empty disjoint subsets $U, V \subset f(A)$ opne in $f(A)$. Since $g := f\big|_A : A \to f(A)$ is continuous, $g^{-1}(U)$ and $g^{-1}(V)$ are open subsets of A. Since both U and V are non-empty and g is surjective, both $g^{-1}(U)$ and $g^{-1}(V)$ are non-empty. Since $f(A) = U \cup V$, it follows that $A = f^{-1}(U) \cup f^{-1}(V)$. But this is not possible since A is connected. Therefore, $f(A)$ must be connected. $\square$

Corollary 2.9.7. *Let X and Y be homeomorphic topological spaces. Then X is connected if and only if Y is connected.*

Theorem 2.9.8 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is a totally ordered set together with the order topology. Let $a, b \in X$ and $y \in Y$ be such that $f(a) < y < f(b)$ in Y. Then there exists $c \in X$ such that $f(c) = y$.*

Proof. Note that $U := f(X) \cap (-\infty, y)$ and $V := f(X) \cap (y, \infty)$ are disjoint open subsets of $f(X)$ containing $f(a)$ and $f(b)$, respectively. If $f^{-1}(y) = \emptyset$, then $f(X) = U \cup V$. Which is not possible since $f(X)$ is connected by Lemma 2.9.6. Therefore, $f^{-1}(y) \neq \emptyset$. $\square$

Theorem 2.9.9. *If X and Y are connected topological spaces, so is their product space $X \times Y$.*

Proof. Let X and Y be connected topological spaces, and let $X \times Y$ be their product space. Fix a point $b \in Y$. Since $X \times \{b\}$ is homeomorphic to X, it follows from Lemma 2.9.6 that $X \times \{b\}$ is connected. Similarly, since Y is connected, $\{x\} \times Y$ is connected, for all $x \in X$. Since $(X \times \{b\}) \cap (\{x\} \times Y) = \{(x, b)\} \neq \emptyset$, it follows from Lemma 2.9.5 that

$$T_{x,b} := (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected. Since $\bigcup_{x \in X} T_{x,b} = X \times Y$ and $\bigcap_{x \in X} T_{x,b} = X \times \{b\} \neq \emptyset$, it follows from Lemma 2.9.5 that $X \times Y$ is connected. $\square$

Corollary 2.9.10. *Let X and Y be two topological spaces. Then $X \times Y$ is connected if and only if both X and Y are connected.*

Proof. One direction is already proved in Theorem 2.9.9. Since both of the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous and surjective, the converse part follows from Lemma 2.9.6. $\square$

Corollary 2.9.11. *A finite Cartesian product of connected spaces is connected.*

Proof. Let $X_1, \ldots, X_n$ be connected topological spaces. For $n = 1$, the result holds trivially. Assume that $n > 1$, and the result holds for any $n - 1$ number of connected topological spaces. Then $Y := X_1 \times \cdots \times X_{n-1}$ is connected by induction hypothesis. Since $X_1 \times \cdots \times X_n$ is homeomorphic to $Y \times X_n$, and that $Y \times X_n$ is connected by Corollary 2.9.9, it follows from Lemma 2.9.6 that $X_1 \times \cdots \times X_n$ is connected. $\square$

Lemma 2.9.12. *Let X be a topological space. Let A be a connected subset of X. If $B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$, then B is connected in X.*

Proof. If possible suppose that B is not connected in X. Then there exists a pair of non-empty subsets $V_1, V_2 \subseteq B$ open in B such that $V_1 \cup V_2 = B$ and $V_1 \cap V_2 = \emptyset$. Now $V_1 = U_1 \cap B$ and $V_2 = U_2 \cap B$, for some open subsets U_1, U_2 of X. Since $A \subseteq B$, $W_1 := U_1 \cap A = V_1 \cap A$ and $W_2 := U_2 \cap A = V_2 \cap A$ are open subsets of A with $W_1 \cup W_2 = V_1 \cap V_2 \cap A = \emptyset$ and $W_1 \cup W_2 = (V_1 \cap A) \cup (V_2 \cap A) = (V_1 \cup V_2) \cap A = B \cap A = A$. We claim that both W_1 and W_2 are non-empty. Indeed, since $U_1 \cap B = V_1 \neq \emptyset$, there exists a point $b \in B \subseteq \overline{A}$ such that $b \in U_1$. Since $b \in \overline{A}$, we must have $W_1 = U_1 \cap A \neq \emptyset$. Similarly, we have $W_2 \neq \emptyset$. Thus we get a separation of A, which contradicts our assumption that A is connected in X. Therefore, B must be connected. □

Definition 2.9.13. A subset $I \subseteq \mathbb{R}$ is said to be an *interval* in $\mathbb{R}$ if given any two points $a, b \in I$ with $a < b$, we have $(a, b) := \{x \in \mathbb{R} : a < x < b\} \subseteq I$.

Proposition 2.9.14. *A connected subset of* $\mathbb{R}$ *is an interval.*

Proof. Let $I \subseteq \mathbb{R}$ be a connected subset of $\mathbb{R}$. If I is an empty set or a singleton subset of $\mathbb{R}$, the result holds trivially. Assume that I contains at least two distinct points. Let $a, b \in I$ be arbitrary. Let $x \in \mathbb{R}$ be such that $a < x < b$. Let $U_x = (-\infty, x) \cap I$ and $V_x = (x, \infty) \cap I$. Note that $a \in U_x$ and $b \in V_x$. Then U_x and V_x are non-empty open subsets of I. If $x \notin I$, then $I = U_x \cup V_x$. This is not possible since I is connected. Therefore, I must be an interval in $\mathbb{R}$. □

Theorem 2.9.15. *Any non-empty interval in* $\mathbb{R}$ *is connected.*

Proof. In view of Lemma 2.9.12, it suffices to show that any open interval in $\mathbb{R}$ is connected. Let I be an open interval in $\mathbb{R}$. If possible suppose that I is not connected. Then there exists a pair of non-empty subsets $U, V \subseteq I$ open in I (and hence in $\mathbb{R}$) such that $U \cup V = I$ and $U \cap V = \emptyset$. Fix two points $a \in U$ and $b \in V$. Without loss of generality, we may assume that $a < b$. Let

$$A := \{x \in \mathbb{R} : [a, x) \subseteq U\}.$$

Since $b \in V$ and $V \subseteq I$ is open, there exists a $\delta > 0$ such that $(b - \delta, b + \delta) \subseteq V$. Since $[a, b) \cap (b - \delta, b + \delta) \neq \emptyset$ and $U \cap V = \emptyset$, we must have $b \notin A$. Then $x < b$, $\forall\ x \in A$. Therefore, A is a bounded above subset of $I \subseteq \mathbb{R}$, and so it has a least upper bound, say $\ell := \sup(A) \in \mathbb{R}$. Clearly $a \leq \ell \leq b$. Since I is an interval, $\ell \in I$. Since $I = U \cup V$, either $\ell \in U$ or $\ell \in V$.

Case 1: Suppose that $\ell \in U$. Then U being an open set, $(\ell - \epsilon, \ell + \epsilon) \subseteq U$, for some $\epsilon > 0$. On the other hand, since $\ell = \sup(A)$ and $\epsilon/2 > 0$, there exists $x_0 \in A$ such that $\ell - \frac{\epsilon}{2} < x_0$. Then $[a, \ell + \epsilon) = [a, x_0) \cup (\ell - \epsilon, \ell + \epsilon) \subseteq U$, and so $\ell + \epsilon \in A$, which is not possible since $\ell = \sup(A)$.

Case 2: Suppose that $\ell \in V$. Since V is open, there exists $\epsilon' > 0$ such that $(\ell - \epsilon', \ell + \epsilon') \subseteq V$. Since $\ell = \sup(A)$, there exists $x_1 \in A$ such that $\ell - \frac{\epsilon'}{2} < x_1$. Then $\ell - \frac{\epsilon'}{2} \in [a, x_1) \cap (\ell - \epsilon', \ell + \epsilon') \subseteq U \cap V$, which is not possible since $U \cap V = \emptyset$.

Since we are getting contradictions in both cases, I must be connected. □

Corollary 2.9.16. *The Euclidean space* $\mathbb{R}^n$ *is connected, for all* $n \geq 1$.

Proof. For $n = 1$, since $\mathbb{R}$ is an interval in itself, it is connected by Theorem 2.9.15. Suppose that $n > 1$. Since $\mathbb{R}^n$ is homeomorphic to the product topological space $X_1 \times \cdots \times X_n$, where $X_j = \mathbb{R}$, $\forall\, j = 1, \ldots, n$, the result follows from Corollary 2.9.11. $\square$

Theorem 2.9.17. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of connected topological spaces. Then the product space $\prod\limits_{\alpha\in\Lambda} X_\alpha$ is connected.*

Proof. Let $X := \prod\limits_{\alpha\in\Lambda} X_\alpha$ be equipped with the product topology. Fix a point $\mathbf{a} = (a_\alpha)_{\alpha\in\Lambda} \in X$. Let $\mathscr{F}(\lambda)$ be the collection of all finite subsets of Λ. Given a finite subset $K \in \mathscr{F}(\Lambda)$ of Λ, let

$$X_K := \{\mathbf{x} = (x_\alpha)_{\alpha\in\Lambda} \in X : x_\alpha = a_\alpha,\ \ \forall\, \alpha \in \Lambda \setminus K\}.$$

Then the subset $X_K \subseteq X$, with the subspace topology induced from X, is homeomorphic to $X_{\alpha_1} \times \cdots \times X_{\alpha_n}$, where $K = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Lambda$. Since X_α is connected, for each $\alpha \in \Lambda$, it follows from Corollary 2.9.11 that X_K is connected. Let $Y = \bigcup\limits_{F\in\mathscr{F}(\Lambda)} X_F$. Since X_F is connected, for all $F \in \mathscr{F}(\Lambda)$, and since $\mathbf{a} \in \bigcap\limits_{F\in\mathscr{F}(\Lambda)} X_F$, it follows from Lemma 2.9.5 that Y is connected.

We now show that the closure of Y in X is X itself. Let $\mathbf{b} = (b_\alpha)_{\alpha\in\Lambda} \in X$ be arbitrary. Let $U = \prod\limits_{\alpha\in\Lambda} U_\alpha$ be a non-empty basic open subset of X containing $\mathbf{b}$. Then $U_\alpha \subseteq X_\alpha$ is an open neighbourhood of b_α in X_α, $\forall\, \alpha \in \Lambda$, and there is a finite subset $G \in \mathscr{F}(\Lambda)$ such that $U_\alpha = X_\alpha,\ \ \forall\, \alpha \in \Lambda \setminus G$. Consider the point $\mathbf{c} = (c_\alpha)_{\alpha\in\Lambda} \in X$ defined by

$$c_\alpha = \begin{cases} b_\alpha, & \text{if} \quad \alpha \in G, \\ a_\alpha, & \text{if} \quad \alpha \in \Lambda \setminus G. \end{cases}$$

Then $\mathbf{c} \in U \cap X_G \subseteq U \cap Y$. Therefore, $\mathbf{c} \in \overline{Y}$, and hence $\overline{Y} = X$. Since closure of a connected set is connected (see Lemma 2.9.6), the result follows. $\square$

However, the following example shows that the conclusion of the Theorem 2.9.17 fails if we equip $X = \prod\limits_{\alpha\in\Lambda} X_\alpha$ with the box topology instead of the product topology.

Example 2.9.18. For each $n \in \mathbb{N}$, let X_n be the Euclidean space $\mathbb{R}$. Let $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ be the set $\prod\limits_{n\in\mathbb{N}} X_n$ of all sequences of real numbers. Equip the set $\prod\limits_{n\in\mathbb{N}} X_n$ with the box topology. Consider the subsets

$$\begin{aligned} U &= \{(a_n)_{n\in\mathbb{N}} : (a_n)_{n\in\mathbb{N}} \text{ is a bounded sequence}\}, \\ \text{and} \quad V &= \{(a_n)_{n\in\mathbb{N}} : (a_n)_{n\in\mathbb{N}} \text{ is an unbounded sequence}\}. \end{aligned}$$

of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Clearly both U and V are non-empty subsets of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ with $U \cap V = \emptyset$. Given a point $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{\text{box}}$, the subset $W := \prod\limits_{n\in\mathbb{N}} U_n$, where $U_n = (a_n - 1, a_n + 1) \subset \mathbb{R}$, $\forall\, n \in \mathbb{N}$, is an open neighbourhood of $\mathbf{a}$ in the box topological space $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Note that all sequences in W are bounded (resp., unbounded) if $\mathbf{a}$ is bounded (resp., unbounded). Therefore, both U and V are open in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Thus we get a separation of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, and hence the space $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is disconnected.

Let X be a topological space. Let

$$\rho := \{(a,b) \in X \times X : \text{there is a connected subspace of } X \text{ containing } a \text{ and } b\} \subseteq X \times X.$$

Clearly ρ is reflexive and symmetric. If $(a,b),(b,c) \in \rho$, then there are connected subspaces A and B of X with $a,b \in A$ and $b,c \in B$. Then $A \cap B \neq \emptyset$, and hence $A \cup B$ is connected by Lemma 2.9.5. Therefore, $(a,c) \in \rho$. Thus, ρ is an equivalence relation on X. The ρ-equivalence classes in X are called the *connected components* of X. Clearly, connected components of X are precisely maximal connected subsets of X, and gives a partition of X. Moreover, any non-empty connected subspace of X is contained in exactly one of the connected components of X. Since closure of a connected subspace is connected, it follows that connected components of X are closed in X. Therefore, if X has only finitely many connected components, then the connected components of X are both open and closed in X.

Example 2.9.19. All connected components of $\mathbb{Q}$ are one-point space. Indeed, if A is a subspace of $\mathbb{Q}$ containing at least two points, say $a,b \in A$, then choosing an irrational number $\alpha \in \mathbb{R}$ with $a < \alpha < b$ we get a separation of A by two non-empty disjoint open subsets $(-\infty,\alpha) \cap A$ and $(\alpha,\infty) \cap A$ of A. Moreover, any one-point subspace of $\mathbb{Q}$ does not admit any separation, and hence is connected. Therefore, the only connected subspaces of $\mathbb{Q}$ are one-point subspaces.

Lemma 2.9.20. *Let X be a topological space. Let $A \subseteq X$ be a non-empty connected subset of A that is both open and closed in X. Then A is a connected component of X.*

Proof. Since A is a non-empty connected subspace of A, there is a unique connected component of X, say C such that $A \subseteq C$. Since A is closed in X, its complement $U := X \setminus A$ is open in X. Since $C = A \cup (C \cap U)$ and C is connected, $C \cap U$ must be an empty set. Since $A \subseteq C$, this forces $A = C$. □

2.10 Path-connectedness

Definition 2.10.1. A *path* in X from $x_0 \in X$ to $x_1 \in X$ is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A topological space X is said to be *path-connected* if given any two points x_0 and x_1 of X, there is a path in X from x_0 to x_1.

Remark 2.10.2. A path in X joining $x_0 \in X$ to $x_1 \in X$ can equivalently be defined to be a continuous map $\gamma : [a,b] \to X$ with $\gamma(a) = x_0$ and $\gamma(b) = x_1$, where $a,b \in \mathbb{R}$ with $a < b$. To see this, note that given $a,b \in \mathbb{R}$ with $a < b$, we have continuous maps

$$\sigma : [0,1] \to [a,b] \text{ and } \sigma' : [a,b] \to [0,1]$$

defined by

$$\sigma(t) = (1-t)a + tb, \ \forall\, t \in [0,1],$$
$$\text{and } \sigma'(s) = \frac{s-a}{b-a}, \ \forall\, s \in [a,b].$$

Note that both σ and σ' are continuous with $\sigma \circ \sigma' = \mathrm{Id}_{[a,b]}$ and $\sigma' \circ \sigma = \mathrm{Id}_{[0,1]}$.

Then given a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, the composite map $\sigma' \circ \gamma : [a,b] \to X$ is continuous and $(\sigma' \circ \gamma)(a) = x_0$ and $(\sigma' \circ \gamma)(b) = x_1$. Conversely, given a continuous map $\delta : [a,b] \to X$ with $\delta(a) = x_0$ and $\delta(b) = x_1$, the composite map $\delta \circ \sigma$ is continuous and satisfies $(\delta \circ \sigma)(0) = x_0$ and $(\delta \circ \sigma)(1) = x_1$.

Example 2.10.3. (i) Any interval I in $\mathbb{R}$ is path-connected. Indeed, given any two points $a, b \in I$, the map $\gamma : [0,1] \to \mathbb{R}$ defined by

$$\gamma(t) = (1-t)a + tb, \ \ \forall\, t \in [0,1],$$

is continuous with $\gamma(0) = a$ and $\gamma(1) = b$. Note that, $\gamma(t) = a + t(b-a) \in I$, $\forall\, t \in [0,1]$. Therefore, γ is a path in I from a to b.

(ii) Consider the subspace $X_n = \mathbb{R}^n \setminus \{0\}$ of the Euclidean space $\mathbb{R}^n$. For $n = 1$, it follows from the intermediate value theorem that there is no path in $\mathbb{R} \setminus \{0\}$ joining a negative real number to a positive real number. Therefore, $\mathbb{R} \setminus \{0\}$ is not path-connected. In fact, it is not connected. Assume that $n \geq 2$. Let $a, b \in \mathbb{R}^n \setminus \{0\}$ be arbitrary. Consider the convex combination of a and b, namely

$$\gamma(t) := (1-t)a + tb, \ \ t \in [0,1].$$

If $\gamma(t) \neq 0$, $\forall\, t \in [0,1]$, then $t \mapsto \gamma(t)$ gives a path in $\mathbb{R}^n \setminus \{0\}$ from a to b, and we are done. If $\gamma(t) = 0$, for some $t \in [0,1]$, then choose a point $c \in \mathbb{R}^n \setminus \{0\}$ that does not lie on the straight-line passing through a and b in $\mathbb{R}^n$. Then the straight-lines in $\mathbb{R}^n$ joining a to c and c to b do not pass through the origin, and thus we get a path in $\mathbb{R}^n \setminus \{0\}$ from a to b.

(iii) Given a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ be the Euclidean norm of x in $\mathbb{R}^n$. The subspace $B(0,r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ of $\mathbb{R}^n$ is path-connected. Indeed, let $a, b \in B(0,r)$ be arbitrary. Consider the convex combination

$$\gamma_{a,b}(t) := (1-t)a + tb, \ \ \forall\, t \in [0,1].$$

Since $\|\gamma_{a,b}(t)\| \leq (1-t)\|a\| + t\|b\| < (1-t)r + tr = r$, the map $t \mapsto \gamma_{a,b}(t)$ gives a path in $B(0,r)$ joining a to b. Therefore, $B(0,r)$ is path-connected.

Proposition 2.10.4. *Continuous image of a path-connected space is path-connected.*

Proof. Let $f : X \to Y$ be a surjective continuous map of topological spaces. Assume that X is path-connected. Let $y_0, y_1 \in Y$ be given. Since f is surjective, there exists $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Since X is path-connected, there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $f \circ \gamma : [0,1] \to Y$ is a continuous map with $(f \circ \gamma)(0) = y_0$ and $(f \circ \gamma)(1) = y_1$. Thus, Y is path-connected. $\square$

Example 2.10.5. For each integer $n \geq 2$, the subspace $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ of the Euclidean space $\mathbb{R}^n$ is path-connected. To see this, note that $\mathbb{R}^n \setminus \{0\}$ is path-connected and the map $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by

$$f(x) = \frac{x}{\|x\|}, \ \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

is continuous and surjective. Therefore, it follows from Proposition 2.10.4 that S^{n-1} is path-connected, for all $n \geq 2$.

Proposition 2.10.6. *A path-connected space X is connected.*

Proof. First proof: Suppose on the contrary that $X = U \cup V$, where U and V are non-empty open subsets of X with $U \cap V = \emptyset$. Choose points $x_0 \in U$ and $x_1 \in V$. Since X is path-connected, there is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since γ is continuous and $[0,1] \subset \mathbb{R}$ is connected, the subset $\gamma([0,1])$ is connected in X, and hence $\gamma([0,1])$ lies entirely either in U or in V by Lemma 2.9.4. This gives a contradiction because $x_0 = \gamma(0) \in U$, $x_1 = \gamma(1) \in V$, and $U \cap V = \emptyset$ by assumption. Thus X must be connected.

Second proof: Fix a point $a \in X$. Since X is path-connected, given any $x \in X$, there is a continuous map $\gamma_x : [0,1] \to X$ with $\gamma_x(0) = a$ and $\gamma_x(1) = x$. For each $x \in X$, the subset $\gamma_x([0,1]) \subseteq X$ is connected by Lemma 2.9.6, and $a \in \bigcap\limits_{x \in X} \gamma_x([0,1])$. Then $X = \bigcup\limits_{x \in X} \gamma_x([0,1])$ is connected by Lemma 2.9.5. ☐

However, the following example shows that a connected space need not be path-connected.

Example 2.10.7 (Topologist's sine curve). Consider the subspace $S = \{(x, \sin\frac{1}{x}) : x \in (0,\infty)\}$ of the Euclidean space $\mathbb{R}^2$. Since $(0,\infty) \subset \mathbb{R}$ is path-connected and the map

$$f : (0,\infty) \to S$$

defined by

$$f(x) = \left(x, \sin\frac{1}{x}\right), \ \forall\, x \in (0,\infty),$$

is continuous and surjective, it follows from Proposition 2.10.4 that S is path-connected, and hence S is connected. It follows from Lemma 2.9.12 that the subspace $\overline{S}$ is connected. The subspace $\overline{S}$ is called *topologist's sine curve*. Note that, $\overline{S} = S \cup A$, where $A = \{0\} \times [-1,1] \subset \mathbb{R}^2$. To see this, fix a point $(0,t) \in A$ and a real number $r > 0$. Consider the open neighbourhood $U = (-r,r) \times (t-r, t+r) \subset \mathbb{R}^2$ of $(0,t)$. By Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{4n\pi} < \frac{1}{2n\pi} < r$. Then it follows from the intermediate value theorem (Theorem 2.9.8) that, there exists a real number $x \in (\frac{1}{4n\pi}, \frac{1}{2n\pi}) \subseteq (0,r)$ such that $\sin\frac{1}{x} = t$. Therefore, $(0,t) \in \overline{S}$, and hence $\overline{S} = S \cup A$. We claim that there is no path in $\overline{S}$ joining $(0,0)$ to $(x, \sin\frac{1}{x}) \in S$, where $x > 0$, and hence $\overline{S}$ is not path-connected. Suppose on the contrary that there is a path $\gamma : [0,1] \to \overline{S}$ in $\overline{S}$ joining the origin $(0,0)$ to $(x, \sin\frac{1}{x}) \in S$, for some $x > 0$. By Archimedean property of $\mathbb{R}$, there exists $m \in \mathbb{N}$ such that $\frac{1}{2m\pi} < x$. For each $n \in \mathbb{N}$, let $t_n := \frac{1}{(2mn+1)\pi}$. Note that, $(t_n)_{n \in \mathbb{N}}$ is a sequence of points of $[0,1]$ converging to 0. Since γ is continuous, the sequence $(\gamma(t_n))_{n \in \mathbb{N}}$ should converge to $\gamma(0) = (0,0)$. But

$$\gamma(t_n) = \left(t_n, \sin\frac{1}{t_n}\right) = \left(\frac{1}{(2mn+1)\pi}, 1\right) \ \forall\, n \in \mathbb{N}.$$

So $(\gamma(t_n))_{n\in\mathbb{N}}$ does not converges to $(0,0)$, contradicting continuity of γ. Therefore, there is no path in $\overline{S}$ joining $(0,0)$ to a point of S.

Definition 2.10.8. Given two paths $\gamma : [0,1] \to X$ and $\delta : [0,1] \to X$ with $\alpha(1) = \beta(0)$, we denote by $\gamma \star \delta : [0,1] \to X$ the *composite path* defined by

$$(\gamma \star \delta)(t) := \begin{cases} \gamma(2t), & \text{if} \quad 0 \leq t \leq 1/2, \\ \delta(2t-1), & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Note that $\gamma \star \delta$ is a continuous map, and hence is a path in X from $\gamma(0)$ to $\delta(1)$.

Proposition 2.10.9. *Let X be a topological space. The relation "being path-connected" is an equivalence relation on X.*

Proof. Let

$$\rho := \{(x,y) \in X \times X : \text{ there is a path in } X \text{ from } x \text{ to } y\}.$$

For each $x \in X$, the constant map $c_x : [0,1] \to X$ sending all points of $[0,1]$ to x is a path from x to itself in X. Therefore, $(x,x) \in \rho$, for all $x \in X$. Thus ρ is reflexive. Let $(x,y) \in \rho$. Then there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then the map $\overline{\gamma} : [0,1] \to X$ defined by $\overline{\gamma}(t) = \gamma(1-t)$, $\forall\, t \in [0,1]$, is a path in X from y to x, and hence $(y,x) \in \rho$. Thus ρ is symmetric. Let $(a,b), (b,c) \in \rho$. Let $\gamma, \delta : [0,1] \to X$ be two continuous maps with $\gamma(0) = a$, $\gamma(1) = b = \delta(0)$ and $\delta(1) = c$. Then the map $\gamma \star \delta : [0,1] \to X$ as defined in Definition 2.10.8 is a path in X joining a to c. Thus $(a,c) \in \rho$, and hence ρ is transitive. Therefore, ρ is an equivalence relation on X. The ρ-equivalence classes in X are called *path-components* of X, and X can be written as a disjoint union of its path-componenets. □

Proposition 2.10.10. *Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of path-connected subspaces of a topological space X. If $\bigcap\limits_{\alpha\in\Lambda} A_\alpha \neq \emptyset$, then $\bigcup\limits_{\alpha\in\Lambda} A_\alpha$ is path-connected.*

Proof. Choose a point $p \in \bigcap\limits_{\alpha\in\Lambda} A_\alpha$. Let $a, b \in \bigcup\limits_{\alpha\in\Lambda} A_\alpha$ be arbitrary. Then $a \in A_\alpha$ and $b \in A_\beta$, for some $\alpha, \beta \in \Lambda$. Since A_α and A_β are path-connected, there exist continuous maps (paths)

$$\gamma_{a,p} : [0,1] \to A_\alpha \quad \text{and} \quad \gamma_{p,b} : [0,1] \to A_\beta$$

joining a to p in A_α and p to b in A_β, respectively. Then the map

$$\gamma_{a,p} \star \gamma_{p,b} : [0,1] \to A_\alpha \cup A_\beta \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$$

as defined in Definition 2.10.8 is a path in $\bigcup\limits_{\alpha\in\Lambda} A_\alpha$ joining a to b. This completes the proof. □

Exercise 2.10.11. A *hyperplane* in the Euclidean space $\mathbb{R}^n$ is the zero locus of a non-constant linear polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$. In other words, a hyperplane in $\mathbb{R}^n$ is a subspace of the form

$$H = \{(a_1, \ldots, a_n) : f(a_1, \ldots, a_n) = 0\},$$

where f is a non-constant linear polynomial over $\mathbb{R}$, i.e., f is of the form $f = c_0 + c_1x_1 + \cdots + c_nx_n$ with coefficients $c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that not all of $c_1, \ldots, c_n$ are simultaneously zero. Show that a hyperplane in $\mathbb{R}^n$ is path-connected.

Proposition 2.10.12. *Assume that $n \geq 2$. For any countable subset A of $\mathbb{R}^n$, the subspace $\mathbb{R}^n \setminus A$ of $\mathbb{R}^n$ is path-connected.*

Proof. Given a point $p \in \mathbb{R}^n$, let $\mathscr{L}_p$ be the set of all straight-lines in $\mathbb{R}^n$ passing through p. Since for each $q \in \mathbb{R}^n \setminus \{p\}$ there is a unique straight-line in $\mathbb{R}^n$ passing through p and q, the set $\mathscr{L}_p$ is uncountable. To show $\mathbb{R}^n \setminus A$ is path-connected, we fix any two points $a, b \in \mathbb{R}^n$, and join them by a path in $\mathbb{R}^n \setminus A$.

Let $a, b \in \mathbb{R}^n \setminus A$ be arbitrary. Since A is countable, there exists a straight-line $L \in \mathscr{L}_a$ passing through a that does not intersect A. For each point $p \in L$, we have a unique straight-line $L_{p,b}$ in $\mathbb{R}^n$ passing through p and b. Since there are uncountably many points on L and A is a countable set, we can choose a straight-line $L_{p,b}$ in $\mathbb{R}^n$ that passes through b and $p \in L$ and does not intersect A. Since L and $L_{p,b}$ are path-connected (as being continuous image of the Euclidean line $\mathbb{R}$) and $L \cap L_{p,b} \neq \emptyset$, it follows from Prposition 2.10.10 that $L \cup L_{p,b}$ is path-connected. Since $a, b \in Y_{a,b} := L \cup L_{p,b}$, we have a path in $Y_{a,b} \subseteq \mathbb{R}^n \setminus A$ joining a to b. This completes the proof. □

Exercise 2.10.13. Assume that $n \geq 2$. Let W be a $\mathbb{R}$-linear subspace of the Euclidean space $\mathbb{R}^n$. If $\dim_{\mathbb{R}}(W) \leq n - 2$, show that $\mathbb{R}^n \setminus W$ is path-connected.

Proof. If $n = 2$, then $\dim_{\mathbb{R}}(W) = 0$. Then $W = \{(0,0)\} \subset \mathbb{R}^2$, and hence $\mathbb{R}^2 \setminus W$ is path-connected. Assume that $n \geq 3$. Choose an ordered basis, say $\{v_1, \ldots, v_{n-2}\}$ for W and extend it to an ordered basis $\{v_1, \ldots, v_{n-2}, v_{n-1}, v_n\}$ for $V = \mathbb{R}^n$. Fix two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus W$. Write $\mathbf{a} = \sum\limits_{i=1}^{n} a_iv_i$ and $\mathbf{b} = \sum\limits_{i=1}^{n} b_iv_i$, where $a_i, b_i \in \mathbb{R}$, $\forall\, i = 1, \ldots, n$. Since $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus W$, we have $(a_{n-1}, a_n), (b_{n-1}, b_n) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since $\mathbb{R}^2 \setminus \{(0,0)\}$ is path-connected, there is a path, say $\delta : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ from (a_{n-1}, a_n) to (b_{n-1}, b_n). Since $(a_1, \ldots, a_{n-2}), (b_1, \ldots, b_{n-2}) \in \mathbb{R}^{n-2}$ and $n - 2 \geq 1$, there is a path, say $\gamma : [0,1] \to \mathbb{R}^{n-2}$ joining $(a_1, \ldots, a_{n-2})$ to $(b_1, \ldots, b_{n-2})$. Let us denote by $p_j^k : \mathbb{R}^k \to \mathbb{R}$, the projection map onto the j-th coordinate of $\mathbb{R}^k$. More precisely,

$$p_j^k(x_1, \ldots, x_n) = x_j, \;\; \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^k.$$

Note that, p_j^k is continuous, for all $k \in \mathbb{N}$ and $j \leq k$. Let $\gamma_i = p_i^{n-2} \circ \gamma$ and $\delta_j = p_j^2 \circ \delta$, for all $i \in \{1, \ldots, n-2\}$ and $j \in \{1, 2\}$. Then the map $\alpha : [0,1] \to \mathbb{R}^n$ defined by

$$\alpha(t) = \sum_{j=1}^{n-2} \gamma_j(t)v_j + \delta_1(t)v_{n-1} + \delta_2(t)v_n, \;\; \forall\, t \in [0,1],$$

is continuous with $\alpha(0) = \mathbf{a}$ and $\alpha(1) = \mathbf{b}$. Since δ is a path in $\mathbb{R}^2 \setminus \{(0,0)\}$, the image of α lands in $\mathbb{R}^n \setminus W$. Therefore, $\mathbb{R}^n \setminus W$ is path-connected. □

Exercise 2.10.14. Let $\{W_j : j \in \mathbb{N}\}$ be a countable family of $\mathbb{R}$-linear subspaces of $\mathbb{R}^3$ such that $\dim_{\mathbb{R}}(W_j) \leq 1$, $\forall\, j \in \mathbb{N}$. Is $\mathbb{R}^3 \setminus \bigcup\limits_{j \in \mathbb{N}} W_j$ connected? Is it path-connected?

Hint: Note that, W_j is either a point or a straight-line in $\mathbb{R}^n$ passing through the origin. Let $a, b \in \mathbb{R}^3 \setminus \bigcup_{j\in\mathbb{N}} W_j$ be given. Since there are uncountably many planes in $\mathbb{R}^3$ passing through a and b, there is at least one such plane P whose intersection with $\bigcup_{j\in\mathbb{N}} W_j$ is at most countable. Since a plane is homeomorphic to $\mathbb{R}^2$, we see that $P \setminus \big(\bigcup_{j\in\mathbb{N}} W_j\big)$ is path-connected. Thus $\mathbb{R}^3 \setminus \bigcup_{j\in\mathbb{N}} W_j$ is path-connected. □

Proposition 2.10.15. *A connected open subspace of $\mathbb{R}^n$ is path-connected.*

Proof. Note that, given $a \in \mathbb{R}^n$ and any real number $r > 0$, the Euclidean open ball

$$B(a, r) := \{x \in \mathbb{R}^n : \|a - x\| < r\}$$

in $\mathbb{R}^n$ is path-connected. This follows by observing that given any point $x \in B(a, r)$ there is a path

$$t \mapsto \gamma_x(t) := (1-t)a + tx, \;\; \forall\, t \in [0, 1],$$

in $B(a, r)$ joining a to x.

Let A be a non-empty connected open subset of $\mathbb{R}^n$. Fix a point $a \in A$, and let

$$U_a = \{x \in A : \text{there is a path in } A \text{ joining } a \text{ to } x\}.$$

Note that $U_a \neq \emptyset$, since $a \in U_a$. Clearly U_a is path-connected. Let $x \in U_a$ be arbitrary. Let $\gamma_{a,x} : [0, 1] \to A$ be a path in A joining a to x. Since $x \in A$ and A is open in $\mathbb{R}^n$, there exists a real number $r > 0$ such that

$$B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\} \subseteq A,$$

where $d(x, y)$ is the Euclidean distance between x and y in $\mathbb{R}^n$. Since $B(x, r)$ is path-connected, for each $y \in B(x, r)$ there is a path, say

$$\gamma_{x,y} : [0, 1] \to B(x, r)$$

in $B(x, r)$ joining x to y. Then the map $\gamma_{a,x} \star \gamma_{x,y} : [0, 1] \to A$ as defined in Definition 2.10.8 is a path in A joining a to y. Therefore, $B(x, r) \subseteq U_a$, and hence U_a is open in A.

If $A \setminus U_a \neq \emptyset$, choose a point $y \in A \setminus U_a$. Then there is no path in A joining a to y. Since A is open in $\mathbb{R}^n$, there is a real number $r_0 > 0$ such that $B(y, r_0) \subseteq A$. We show that $B(y, r_0) \cap U_a = \emptyset$. For otherwise, if $x \in B(y, r_0) \cap U_a$, then there is a path, say $\gamma_x : [0, 1] \to A$, in A joining a to x. Since $B(y, r_0)$ is also path-connected, there is a path, say $\gamma_{x,y} : [0, 1] \to B(y, r_0) \subseteq A$ in $B(y, r_0)$ joining x to y. Then $(\gamma_{a,x} \star \gamma_{x,y}) : [0, 1] \to A$ is a path in A joining a to y. This contradicts the fact that $y \in A \setminus U_a$. Therefore, $A \setminus U_a$ is also open in A. Thus, U_a is both open and closed in the connected space A, and hence is equal to A by Lemma 2.9.20. □

Definition 2.10.16. A topological space X is said to be *locally connected at* $x \in X$ if each open neighbourhood of x contains a connected open neighbourhood of x. If X is locally connected at each point of it, then X is said to be locally connected. Similarly, X is said to be *locally path-connected at* $x \in X$ if each open neighbourhood of x contains a path-connected open neighbourhood of x; and X is said to be *locally path-connected* if it is locally path-connected at each of its points.

Example 2.10.17. The Euclidean line $\mathbb{R}$ is both locally connected and locally path-connected. The subspace $[0,1) \cup (1,2] \subset \mathbb{R}$ is locally connected and locally path-connected, but neither connected nor path-connected. The topologist's sine curve is connected but not locally connected.

Proposition 2.10.18. *Let X be a topological space. Then X is locally connected (resp., locally path-connected) if and only if for each open subset U of X, any connected component (resp., path-component) of U is open in X.*

Proof. Suppose that X is locally connected (resp., locally path-connected). Fix an open subset U of X. Let $C \subseteq U$ be a connected component (resp., path-component) of U. Let $x \in C$ be given. Since X is locally connected (resp., locally path-connected), there is a connected (resp., path-connected) open subset $V_x \subseteq X$ such that $x \in V_x$ and $V_x \subseteq U$. Since C is a connected component (resp., path-component) of U containing x and $V_x \cap C \neq \emptyset$, we have $V_x \subseteq C$. Therefore, C is open in X.

Conversely, suppose that any connected component (resp., path-component) of an open subset of X is open in X. Let $x_0 \in X$ and let U be an open neighbourhood of x_0 in X. Let $C \subseteq U$ be a connected component (resp., path-component) of U containing x_0. Then C is open in X. Thus, X is locally connected (resp., locally path-connected). $\square$

Theorem 2.10.19. *Let X be a topological space. Each path-component of X lies in a connected component of X. If X is locally path-connected, then the path-components and the connected components of X are the same.*

Proof. Let C be a path-component of X. Then C is path-connected, and hence is connected. Then C is contained in a connected component of X.

Assume that X is locally path-connected. It suffices to show that connected components of X are path-connected. Let C be a connected component of X. Fix a point $x_0 \in C$, and consider the subset

$$U := \{x \in C : \text{there is a path in } C \text{ joining } x_0 \text{ to } x\}.$$

Clearly U is non-empty subset of C since $x_0 \in U$. Note that U is path-connected. Indeed, given any two points $x, y \in U$, we have paths γ and δ in C joining x_0 to x and y, respectively. Let

$$\overline{\gamma} : [0,1] \to C, \ t \mapsto \gamma(1-t),$$

be the inverse path in C joining x to x_0. Then $\delta \star \overline{\gamma}$ is a path in C from x to y.

We show that U is both open and closed in X, and hence coincides with C. Let $x \in U$ be given. Since X is locally path-connected, there is a path-connected open neighbourhood $V_x \subseteq X$ of x. Since V_x is a connected subspace of X containing x, it must be contained in the connected component C of X containing x. Since V_x is path-connected, given a point of $y \in V_x$, there is a path, say $\delta : [0,1] \to V_x$ in $V_x \subseteq C$ from x to y. Since $x \in U$, there is a path, say $\gamma : [0,1] \to C$ from x_0 to x. Then $\gamma \star \delta$ is a path in C joining x_0 to y. Thus, $V_x \subseteq U$, and hence U is open in X.

We claim that U is closed in X. Let $y \in X \setminus U$ be arbitrary. Since X is locally path-connected, there is a path-connected open neighbourhood, say $V_y \subseteq X$ of y. We claim that $V_y \cap U = \emptyset$. Suppose on the contrary that there is a point $z \in V_y \cap U$. Since $z \in U \subseteq C$ and V_y is connected, we must have $V_y \subseteq C$. Choose a paths γ in C from x_0 to z, and a path δ in $V_y \subseteq C$ from z to y. Then $\gamma \star \delta$ is a path in C joining x_0 to y. Then $y \in U$, which contradicts our choice of y as a point of $X \setminus U$. Therefore, we must have $V_y \cap U = \emptyset$. Thus, no point of $X \setminus U$ can be a limit point of U. Therefore, U is closed in X. Since $U \subseteq C$ is both open and closed in X, it must be a connected component of X by Lemma 2.9.20, and hence $U = C$. $\square$

2.11 Compactness

Let X be a topological space. Let $A \subseteq X$. A collection $\mathcal{F} = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a *cover* of A if $A \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. If all members V_α of $\mathcal{F}$ are open subsets of X, then $\mathcal{F}$ is called an *open cover* of A in X. A *subcover* of $\mathcal{F}$ is a subcollection of $\mathcal{F}$ such that union of all its members cover A.

Definition 2.11.1. A topological space X is said to be *compact* if every open cover of X has a finite subcover. In other words, given a family of open subsets $\mathcal{F} = \{U_\alpha : \alpha \in \Lambda\}$ such that $X = \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\mathcal{F}$ such that $X = \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. A subset K of a topological space X is said to be *compact* if it is compact with respect to the subspace topology on K induced from X.

Example 2.11.2. The Euclidean space $\mathbb{R}$ is not compact, as it has an open cover $\{(n, n+2) : n \in \mathbb{N}\}$ which has no finite subcover.

Example 2.11.3. The subspace $K := \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact. To see this, for each $n \in \mathbb{N}$ we consider the open interval $U_n := (\frac{1}{n} - r_n, \frac{1}{n} + r_n)$, where $r_n := \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+1}\right)$. Then U_n is an open neighbourhood of $\frac{1}{n}$ and that $U_n \cap U_m = \emptyset$, for all $n \neq m$ in $\mathbb{N}$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of K, which has no finite subcover. However, $K \cup \{0\}$ is compact. To see this, consider an open cover $\mathcal{F} = \{U_\alpha : \alpha \in \Lambda\}$ of $K \cup \{0\}$. Then $0 \in U_{\alpha_0}$, for some $\alpha_0 \in \Lambda$. Since U_{α_0} is an open subset of $\mathbb{R}$, there exists a real number $r > 0$ such that $(-r, r) \subseteq U_{\alpha_0}$. Then by Archimedean property of $\mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < r$, $\forall\, n \geq n_0$. For each $j \in \{1, \ldots, n_0\}$, we can choose an open subset $U_{\alpha_j} \in \mathcal{F}$ such that $\frac{1}{j} \in U_{\alpha_j}$, $\forall\, j = 1, \ldots, n_0$. Then $K \cup \{0\} \subseteq U_{\alpha_0} \bigcup \big(\bigcup\limits_{j=1}^{n_0} U_{\alpha_j}\big)$. Therefore, $K \cup \{0\}$ is compact.

Lemma 2.11.4. *Let X be a topological space. A subspace $K \subseteq X$ is compact if and only if every open cover of K in X has a finite subcover.*

Proof. Let $K \subseteq X$. Assume that K is compact. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets U_α of X such that $K \subseteq \bigcup\limits_{\alpha \in \Lambda} U_\alpha$. Then $\mathscr{F}_K := \{U_\alpha \cap K : \alpha \in \Lambda\}$ is a collection of open subsets of K such that $K = \bigcup\limits_{\alpha \in \Lambda} (U_\alpha \cap K)$. Since K is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $K = \bigcup\limits_{j=1}^{n} (U_{\alpha_j} \cap K)$. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a required finite subcover for K.

Conversely, suppose that every open cover of K has a finite subcover. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of K such that $K = \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. Note that, for each $\alpha \in \Lambda$, we have $V_\alpha = U_\alpha \cap K$, for some open subset U_α of X. Since $K \subseteq \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, the collection $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of K in X. Then there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. Then $K = \bigcup\limits_{j=1}^{n} V_{\alpha_j}$. This completes the proof. □

Proposition 2.11.5. *A closed subspace of a compact space is compact.*

Proof. Let K be a closed subspace of a compact space X. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of K in X. Then $\mathscr{F} \cup \{X \setminus K\}$ is an open cover of X. Since X is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = (X \setminus K) \bigcup \big(\bigcup\limits_{j=1}^{n} U_{\alpha_j} \big)$. Since K does not intersect $X \setminus K$, we have $K \subseteq \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. Thus, K is compact. □

Proposition 2.11.6. *Let K be a compact subspace of a Hausdorff space X. Assume that $X \setminus K \neq \emptyset$. Then given $x \in X \setminus K$, there exists a pair of open subsets U and V of X such that $x \in U$, $K \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Since X is Hausdorff, for each $y \in K$ there exists a pair of open subsets U_y and V_y of X containing x and y, respectively, such that $U_y \cap V_y = \emptyset$. Then $\{V_y : y \in K\}$ is an open cover of K. Since K is compact, there exists finite number of points $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} V_{y_j}$. Let $U := \bigcap\limits_{j=1}^{n} U_{y_j}$ and $V := \bigcup\limits_{j=1}^{n} V_{y_j}$. Then U and V are open subsets of X containing x and K, respectively. Since $U_{y_j} \cap V_{y_j} = \emptyset$, for all $j = 1, \ldots, n$, it follows that $U \cap V = \emptyset$. □

Corollary 2.11.7. *A compact subspace of a Hausdorff space is closed.*

Proof. Let K be a compact subspace of a Hausdorff space X. It follows from Proposition 2.11.6 that a point of $X \setminus K$ cannot be a limit point of K. Therefore, K is closed in X. □

Proposition 2.11.8. *Continuous image of a compact space is compact.*

Proof. Let $f : X \to Y$ be a continuous map of topological spaces with X compact. We show that $f(X)$ is a compact subspace of Y. Let $\mathcal{F} = \{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of Y such that $f(X) \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. Since f is continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = \bigcup\limits_{j=1}^{n} f^{-1}(V_{\alpha_j})$. Then $f(X) \subseteq \bigcup\limits_{j=1}^{n} V_{\alpha_j}$. Therefore, $f(X)$ is compact. □

Corollary 2.11.9. *Let $f : X \to Y$ be a continuous bijective map. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. To show f is a homeomorphism, we show that $f^{-1} : Y \to X$ is continuous. Let $g = f^{-1}$. Let $Z \subseteq X$ be a closed subset of X. Since X is compact, Z is compact by Proposition 2.11.5. Then $g^{-1}(Z) = f(Z)$ is compact by Proposition 2.11.8. Since Y is Hausdorff, $g^{-1}(Z) = f(Z)$ is closed by Proposition 2.11.5. Therefore, f^{-1} is continuous, and hence f is a homeomorphism. $\square$

Lemma 2.11.10 (Tube lemma)**.** *Let X and Y be topological spaces. If B is a compact subspace of Y, given a point $x \in X$ and an open subset $W \subseteq X \times Y$ containing the slice $\{x\} \times B$, there exists an open neighbourhood $U \subseteq X$ of x and an open subset $V \subseteq Y$ containing B such that $U \times V \subseteq W$.*

Proof. Note that the product topology on $X \times Y$ has a basis consisting of subsets of the form $U \times V$, where U and V are open subsets of X and Y, respectively. Since $W \subseteq X \times Y$ is an open subset containing the slice $\{x\} \times B$, for each $y \in B$ we can choose open neighbourhoods $U_y \subseteq X$ and $V_y \subseteq Y$ of x and y, respectively, such that $(x, y) \in U_y \times V_y \subseteq W$. Then $\{V_y : y \in B\}$ is an open cover of B. Since B is compact, there are finite number of open subsets $V_{y_1}, \ldots, V_{y_n} \subseteq Y$ such that $B \subset \bigcup\limits_{j=1}^{n} V_{y_j}$. Let $U = \bigcap\limits_{j=1}^{n} U_{y_j}$ and $V = \bigcup\limits_{j=1}^{n} V_{y_j}$. Then U and V are open subsets of X and Y, respectively, and that $x \in U$ and $B \subseteq V$. Clearly, $\{x\} \times B \subseteq U \times V \subseteq W$, as required. $\square$

Example 2.11.11. Tube Lemma 2.11.10 fails if B is not compact. For example, let $X = Y = B = \mathbb{R}$ with the Euclidean topology on them. Then the slice $\{0\} \times \mathbb{R}$ (i.e., the y-axis in $\mathbb{R}^2$) has an open neighbourhood

$$W = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{1}{y^2 + 1} \right\}$$

in $\mathbb{R}^2$ which contains no open neighbourhood of $\{0\} \times \mathbb{R}$ of the form $U \times \mathbb{R}$, where U is an open neighbourhood of 0 in $\mathbb{R}$ (verify!).

Corollary 2.11.12 (Generalized tube lemma)**.** *Let A and B be compact subspaces of X and Y, respectively. Let $N \subseteq X \times Y$ be an open subset containing $A \times B$. Then there exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq N$.*

Proof. For each $a \in A$, the slice $\{a\} \times B$ is contained in N. Then by Tube lemma 2.11.10 there is an open neighbourhood $U_a \subseteq X$ of a and open subset $V_a \subseteq Y$ containing B such that $\{a\} \times B \subseteq U_a \times V_a \subseteq N$. Since $\{U_a : a \in A\}$ is an open cover of the compact space A, there is a finite subcover, say $\{U_{a_1}, \ldots, U_{a_n}\}$ for A. Let $U = \bigcup\limits_{j=1}^{n} U_{a_j}$ and $V = \bigcap\limits_{j=1}^{n} V_{a_j}$. Then U and V are open subsets of X and Y, respectively. Clearly, $A \subseteq U$ and $B \subseteq V$. Then $A \times B \subseteq U \times V \subseteq N$. $\square$

Theorem 2.11.13. *Finite product of compact spaces is compact.*

Proof. It suffices to show that product of two compact spaces is compact. Then the general case follows by induction on the number of compact spaces. Let X and Y be compact topological spaces. Let $\mathcal{F} = \{W_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of $X \times Y$ such that $\bigcup\limits_{\alpha \in \Lambda} W_\alpha = X \times Y$. For each $x \in X$, since the slice $\{x\} \times Y$ is a compact subspace of $X \times Y$, there exists

$\alpha(x,1),\ldots,\alpha(x,n(x)) \in \Lambda$ such that $\{x\} \times Y \subseteq \bigcup\limits_{j=1}^{n(x)} W_{\alpha(x,j)}$. Then by tube lemma (Lemma 2.11.10) there exists an open neighbourhood $U_x \subseteq X$ of x such that

$$\{x\} \times Y \subseteq U_x \times Y \subseteq \bigcup_{j=1}^{n(x)} W_{\alpha(x,j)}.$$

Since $\{U_x : x \in X\}$ is an open cover of X and X is compact, there exists $x_1,\ldots,x_m \in X$ such that $X = \bigcup\limits_{i=1}^{m} U_{x_i}$. Then

$$X \times Y \subseteq \bigcup_{i=1}^{m} U_{x_i} \times Y \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n(x_i)} W_{\alpha(x_i,j)} \subseteq X \times Y.$$

Therefore, $\{W_{\alpha(x_i,j)} : 1 \leq j \leq n(x_i),\ 1 \leq i \leq m\} \subseteq \mathcal{F}$ is a required finite subcover for $X \times Y$, and hence $X \times Y$ is compact. $\square$

Definition 2.11.14. A collection $\mathscr{C}$ of subsets of X is said to have *finite intersection property* if for any finite subcollection $\{C_1,\ldots,C_n\}$ of $\mathscr{C}$, we have $\bigcap\limits_{j=1}^{n} C_j \neq \emptyset$.

Theorem 2.11.15. *A topological space X is compact if and only if given any collection $\mathscr{C}$ of closed subsets of X having finite intersection property, the intersection $\bigcap\limits_{C\in\mathscr{C}} C$ is non-empty.*

Proof. Suppose that X is compact. Let $\mathscr{C}$ be a collection of closed subsets of X having finite intersection property. Suppose on the contrary that $\bigcap\limits_{Z\in\mathscr{C}} Z = \emptyset$. Then taking complement in X, we have $\bigcup\limits_{Z\in\mathscr{C}} (X \setminus Z) = X$. Then the collection

$$\mathscr{U} := \{X \setminus Z : Z \in \mathscr{C}\}$$

is an open cover of X. Since X is compact, there exists a finitely many elements $Z_1,\ldots,Z_n \in \mathscr{C}$ such that $\bigcup\limits_{j=1}^{n} (X \setminus Z_j) = X$. Taking complement in X, we have $\bigcap\limits_{j=1}^{n} Z_j = \emptyset$, which contradicts the fact that $\mathscr{C}$ has finite intersection property. Therefore, $\bigcap\limits_{Z\in\mathscr{C}} Z \neq \emptyset$.

Conversely, suppose that given any collection $\mathscr{C}$ of closed subsets of X having finite intersection property, the intersection $\bigcap\limits_{Z\in\mathscr{C}} Z$ is non-empty. Suppose on the contrary that X is not compact. Then there exists an open cover, say $\mathscr{U}$ of X that has no finite subcover. Then $\mathscr{C} := \{X \setminus U : U \in \mathscr{U}\}$ is a collection of closed subsets of X. Since $\mathscr{U}$ has no finite subcover, given any finite collection $\{X \setminus U_j : j = 1,\ldots,n\}$ of elements of $\mathscr{C}$, we have

$$\bigcap_{j=1}^{n} (X \setminus U_j) = X \setminus \left(\bigcup_{j=1}^{n} U_j \right) \neq \emptyset.$$

In other words, $\mathscr{C}$ has finite intersection property. Then by assumption, we have $\bigcap\limits_{Z\in\mathscr{C}} Z \neq \emptyset$. Taking complement in X, we see that $\mathscr{U}$ is not an open cover of X, which is a contradiction. Therefore, X must be compact. $\square$

Remark 2.11.16. Let X be a topological space. A sequence of subsets $\{Z_n : n \in \mathbb{N}\}$ of X is said to be *nested* if $Z_{n+1} \subseteq Z_n$, $\forall\, n \in \mathbb{N}$. Let $\{Z_n : n \in \mathbb{N}\}$ be a nested sequence of closed subsets of a compact topological space X. If $Z_n \neq \emptyset$, $\forall\, n \in \mathbb{N}$, then the collection $\{Z_n : n \in \mathbb{N}\}$ satisfies finite intersection property, and hence $\bigcap\limits_{n\in\mathbb{N}} Z_n \neq \emptyset$.

Definition 2.11.17. A collection $\mathscr{C}$ of closed subsets of X is said to be a *closed basis* for the topology on X if the collection $\{X \setminus Z : Z \in \mathscr{C}\}$ obtained by taking complement of all elements of $\mathscr{C}$ in X forms a basis for open subsets of X. A collection $\mathscr{C}$ of closed subsets of X is said to be a *closed subbasis* for the topology on X if the collection $\{X \setminus Z : Z \in \mathscr{C}\}$ forms a subbasis for the open subsets of X.

Lemma 2.11.18. *A topological space X is compact if and only if given any closed subbasis $\mathscr{C}$ for X having finite intersection property, the intersection $\bigcap\limits_{Z\in\mathscr{C}} Z$ is non-empty.*

Proof. If X is compact, then the conclusion of the lemma follows from Theorem 2.11.15. Conversely, suppose that given any closed subbasis $\mathscr{C}$ for X having finite intersection property, the intersection $\bigcap\limits_{Z\in\mathscr{C}} Z$ is non-empty. Let $\mathscr{Z} = \{Z_\alpha : \alpha \in \Lambda\}$ be a family of closed subsets of X having finite intersection property. Then $\mathscr{U} = \{X \setminus Z_\alpha : \alpha \in \Lambda\}$ is ... $\square$

Exercise 2.11.19. Let $f : X \to Y$ be a map of topological spaces, and let

$$G_f := \{(x, y) \in X \times Y : y = f(x)\}$$

be the *graph of f*.

(i) If f is continuous and Y is Hausdorff, then G_f is closed in $X \times Y$.

(ii) If Y is compact and G_f is closed, then f is continuous.

Proof. The map $f : X \to Y$ induces a map $(f \times \mathrm{Id}_Y) : X \times Y \to Y \times Y$ defined by

$$(f \times \mathrm{Id}_Y)(x, y) = (f(x), y), \ \forall\, (x, y) \in X \times Y,$$

where $\mathrm{Id}_Y : Y \to Y$ is defined by $\mathrm{Id}_Y(y) = y$, $\forall\, y \in Y$. Since both f and Id_Y are continuous, $(f \times \mathrm{Id}_Y)$ is continuous. Since Y is Hausdorff, $\Delta_Y(Y) = \{(y, y) : y \in Y\}$ is closed in $Y \times Y$. Since $G_f = (f \times \mathrm{Id}_Y)^{-1}(\Delta_Y(Y))$, we see that G_f is closed in $X \times Y$.

(ii) Let $Z \subset Y$ be any closed subset of Y. To show f is continuous, we show that $f^{-1}(Z)$ is closed in X. Since $X \times Z$ is closed in $X \times Y$, and since G_f is closed in $X \times Y$ by assumption, the subset

$$G_f \cap (X \times Z) = \{(x, y) \in X \times Y : f(x) \in Z\}$$

is closed in $X \times Y$. Note that $\pi_1(G_f \cap (X \times Z)) = f^{-1}(Z)$, where $\pi_1 : X \times Y \to X$ is the projection map onto the first component. Therefore, it suffices to show that π_1 is a closed map. Here we use compactness of Y. Let C be a closed subset of $X \times Y$. Let $x_0 \in X \setminus \pi_1(C)$ be arbitrary. Then the slice $\{x_0\} \times Y$ is contained in the open subset $(X \times Y) \setminus C$ of $X \times Y$. Since Y

is compact, by tube lemma there is an open neighbourhood W of x_0 in X such that

$$\{x_0\} \times Y \subset W \times Y \subseteq (X \times Y) \setminus C.$$

Then $x_0 \in W$ and $W \subseteq X \setminus \pi_1(C)$. Thus $X \setminus \pi_1(C)$ is open in X, and hence $\pi_1(C)$ is closed in X. Thus, π_1 is a closed map. This completes the proof. $\square$

Definition 2.11.20. Let (X, d) be a metric space. A subset A of X is said to be *bounded* if there exists a real number $M > 0$ such that $d(a, b) < M, \ \forall\, a, b \in A$. If A is a bounded subset of X, then

$$\operatorname{diam}(A) := \sup_{a,b \in X} d(a, b)$$

exists in $\mathbb{R}$, and is called the *diameter* of A.

Lemma 2.11.21. *Any closed and bounded interval in $\mathbb{R}$ is compact.*

Proof. Let $a, b \in \mathbb{R}$ with $a \leq b$, and consider the closed interval

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} \subset \mathbb{R}.$$

Let $\mathscr{F} := \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $[a, b]$ in $\mathbb{R}$. Let

$$K := \{x \in (a, b] : [a, x] \text{ can be covered by finitely many members of } \mathscr{F}\}.$$

Since $[a, b] \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$, there exists $\alpha_0 \in \lambda$ such that $a \in V_{\alpha_0}$. Since V_{α_0} is open in $\mathbb{R}$, there exists a real number $r > 0$ such that $(a - r, a + r) \subseteq V_{\alpha_0}$. Then for any $x \in (a, a + r)$, we have $[a, x) \subseteq V_{\alpha_0}$, and hence $x \in K$. Therefore, K is non-empty. Clearly K is bounded above by b. Then by the *least upper bound property* of $\mathbb{R}$, the least upper bound $c := \sup K$ exists in $\mathbb{R}$. Clearly $a < c \leq b$. We claim that $c \in F$. Since $c \in [a, b]$, there exists $\beta \in \Lambda$ such that $c \in V_\beta$. Since V_β is open in $\mathbb{R}$, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq V_\beta$. Since $c = \sup K$, there exists an element $x \in K$ such that $c - \delta < x < c$. Then $[a, x]$ can be covered by finitely many objects, say $V_{\alpha_1}, \ldots, V_{\alpha_n} \in \mathscr{F}$, and hence $[a, c]$ can be covered by $\{V_{\alpha_1}, \ldots, V_{\alpha_n}, V_\beta\} \subseteq \mathscr{F}$. Then $c \in K$. Now we show that $c = b$. If not, then $c < b$. Since $c \in V_\gamma$, for some $\gamma \in \Lambda$, there exists $t \in (c, b)$ such that $[c, t] \subseteq V_\gamma$. Then $[a, t] = [a, c] \cup [c, t]$ can be covered by finitely many elements from $\mathscr{F}$, and hence $t \in K$. Since $t > c = \sup K$, we get a contradiction. Therefore, $c = b$. This completes the proof. $\square$

Remark 2.11.22. Note that in proving Lemma 2.11.21, we have only used the fact that $\mathbb{R}$ is a simply ordered set having least upper bound property and the Euclidean topology on $\mathbb{R}$ is precisely the order topology on it. Therefore, the same proof gives the following general fact:

Theorem. *Let X be a simply ordered set having the least upper bound property. Then in the order topology on X, each closed and bounded interval in X is compact.*

Theorem 2.11.23. *A subspace K of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose that K is a compact subspace of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, K is closed in $\mathbb{R}^n$ by Corollary 2.11.7. Let $B_d(0,n) = \{x \in \mathbb{R}^n : \|x\| < n\}$, where $\|x\| \in \mathbb{R}$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. Since $\{B_d(0,n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$, and hence of A, and since $B(0,n) \subseteq B(0,n+1)$, $\forall\, n \in \mathbb{N}$, by compactness of K we can find $n_0 \in \mathbb{N}$ such that $A \subseteq B(0,n)$. Therefore, K is bounded.

Conversely, suppose that K is closed and bounded in $\mathbb{R}^n$. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Euclidean metric on $\mathbb{R}^n$ defined by

$$d(x,y) := \|x-y\| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2},$$

where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$. Since K is bounded, there exists a real number $M > 0$ such that $d(x,y) < M$, $\forall\, x,y \in K$. Fix a point $x_0 \in K$, and let $\ell := d(x_0,\mathbf{0})$ be the Euclidean distance of x_0 from the origin $\mathbf{0}$ of $\mathbb{R}^n$. Then by triangle inequality, we have

$$d(x,0) \leq d(x,x_0) + d(x_0,0) \leq M + \ell.$$

Let $r := M + \ell > 0$. Since the closed interval $[-r,r] \subset \mathbb{R}$ is compact by Lemma 2.11.21, its n-fold product $[-r,r]^n \subset \mathbb{R}^n$ is compact by Theorem 2.9.9. Note that, $K \subseteq [-r,r]^n$. Since K is closed in $\mathbb{R}^n$, it is closed in the compact space $[-r,r]^n$. Therefore, K is compact by Proposition 2.11.5. □

Theorem 2.11.24 (Extreme value theorem)**.** *Let X be a compact topological space and Y an ordered set together with the order topology on it. Let $f : X \to Y$ be a continuous map. Then there exist $a,b \in X$ such that $f(a) \leq f(x) \leq f(b)$, $\forall\, x \in X$.*

Proof. Since X is compact and f is continuous, $A := f(X)$ is a compact subspace of Y. We claim that A has a largest element and a smallest element (i.e., there exist $M,m \in A$ such that $m \leq a \leq M$, $\forall\, a \in A$). Suppose on the contrary that A has no largest element. Then for each $a \in A$ there exists $a' \in A$ such that $a < a'$ so that $a \in (-\infty,a') := \{y \in Y : y < a'\} \subseteq Y$. Then the collection $\mathscr{F} = \{(-\infty,a) : a \in A\}$ is an open cover of A. Since A is compact, $\mathscr{F}$ has a finite subcollection $\{(-\infty,a_1),\ldots,(-\infty,a_n)\}$, which covers A. Let $a_m = \max\{a_1,\ldots,a_n\} \in A$. Then $a_m \notin \bigcup\limits_{j=1}^{n}(-\infty,a_j) = A$, which contradicts the fact that $a_m \in A$. Therefore, A must have a largest element, say $M \in A$. A similar argument shows that A has a smallest element, say $m \in A$. Then $m = f(a)$ and $M = f(b)$, for some $a,b \in X$, and that $f(a) \leq f(x) \leq f(b)$, $\forall\, x \in X$. □

Let (X,d) be a metric space. Let A be a non-empty subset of X. The *distance from $x \in X$ to A* is the real number

$$d(x,A) := \inf\{d(x,a) : a \in A\}.$$

Proposition 2.11.25. *Let A be a non-empty subset of a metric space (X,d). Then the map $\phi_A : X \to \mathbb{R}$ defined by*

$$\phi_A(x) = d(x,A),\ \forall\, x \in X,$$

is continuous.

Proof. Given any $x, y \in X$, we have

$$\begin{aligned} \phi_A(x) = d(x, A) &\leq d(x, a), \; \forall\, a \in A, \\ &\leq d(x, y) + d(y, a), \; \forall\, a \in A, \\ \Rightarrow \; d(x, A) - d(x, y) &\leq d(y, a), \; \forall\, a \in A, \\ \Rightarrow \; d(x, A) - d(x, y) &\leq d(y, A) \\ \Rightarrow \; d(x, A) - d(y, A) &\leq d(x, y). \end{aligned}$$

Interchanging x and y, we have $d(y, A) - d(x, A) \leq d(x, y)$. Thus, ϕ_A is continuous. $\square$

A non-empty subset A of X is said to be *bounded* if there exists a real number $M > 0$ such that $d(a_1, a_2) \leq M$, $\forall\, a_1, a_2 \in A$. The *diameter* of a non-empty bounded subset A of X is the real number

$$\mathrm{diam}(A) := \sup\{d(a_1, a_2) : a_1, a_2 \in A\}.$$

A *Lebesgue number* of an open cover $\mathscr{F}$ of a metric space (X, d) is a real number $\delta > 0$ such that given any non-empty subset A of X of diameter $\mathrm{diam}(A) < \delta$, there exists an element $U \in \mathscr{F}$ such that $A \subseteq U$.

Lemma 2.11.26 (Lebesgue number lemma)**.** *If (X, d) is a compact metric space, every open cover of X has a Lebesgue number.*

Proof. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. If $X \in \mathscr{F}$, then every positive real number is a Lebesgue number for $\mathscr{F}$. Assume that $X \notin \mathscr{F}$. Since X is compact, there is a finite subfamily $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\} \subseteq \mathscr{F}$ that covers X. Consider the map $f : X \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{n} \sum_{j=1}^{n} d(x, Z_j), \; \forall\, x \in X,$$

where $Z_j := X \setminus U_{\alpha_j}$, $\forall\, j \in \{1, \ldots, n\}$. Since each of the maps $x \mapsto d(x, Z_j)$ is continuous by Proposition 2.11.25, it follows that f is continuous. We claim that $f(x) > 0$, for all $x \in X$. Let $x \in X$ be arbitrary. Since $X = \bigcup_{j=1}^{n} U_{\alpha_j}$, we have $x \in U_{\alpha_i}$, for some $i \in \{1, \ldots, n\}$. Since U_{α_i} is open in (X, d), there exists a real number $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U_{\alpha_i}$. Then $d(x, Z_i) \geq \epsilon$, and hence $f(x) = \frac{1}{n} \sum_{j=1}^{n} d(x, Z_j) \geq \epsilon/n > 0$. Since f is continuous and X is compact, there exists $x_0 \in X$ such that $f(x_0) \leq f(x)$, $\forall\, x \in X$. We claim that $\delta := f(x_0) > 0$ is a required Lebesgue number of $\mathscr{F}$. Let $A \subseteq X$ be such that $\mathrm{diam}(A) < \delta$. Fix a point $a \in A$. Then $A \subseteq B_d(a, \delta)$. Clearly $\delta \leq f(a)$. Let $\ell \in \{1, \ldots, n\}$ be such that $d(a, Z_\ell) = \max\{d(a, Z_j) : 1 \leq j \leq n\}$. Then

$$\delta \leq f(a) = \frac{1}{n} \sum_{j=1}^{n} d(a, Z_j) \leq d(a, Z_\ell),$$

and hence $B_d(a, \delta) \subseteq X \setminus Z_\ell = U_{\alpha_\ell}$. This completes the proof. $\square$

Definition 2.11.27. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be *uniformly continuous* if given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$d_Y(f(x_1), f(x_2)) < \epsilon, \text{ whenever } d_X(x_1, x_2) < \delta.$$

Note that a uniformly continuous map is continuous, but converse need not be true.

Theorem 2.11.28 (Uniform continuity theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces. If X is compact, then any continuous map $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous.*

Proof. Let $\epsilon > 0$ be given. Consider the open cover $\{B_{d_Y}(y, \epsilon/2) : y \in f(X)\}$ of $f(X) \subseteq Y$. Since f is continuous, $\mathscr{U} := \{f^{-1}(B_{d_Y}(y, \epsilon/2)) : y \in f(X)\}$ is an open cover of X. Since X is compact, there exists $y_1, \ldots, y_n \in f(X)$ such that $X = \bigcup_{j=1}^{n} f^{-1}(B_{d_Y}(y_j, \epsilon/2))$. Since X is compact, by Lebesgue number lemma 2.11.26 there exists a real number $\delta > 0$ such that given any subset A of X with $\operatorname{diam}(A) < \delta$, we have $A \subseteq f^{-1}(B_{d_Y}(y_{j_A}, \epsilon/2))$, for some $j_A \in \{1, \ldots, n\}$. Then given any $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, we can have $f(x_1), f(x_2) \in B_{d_Y}(y_{j_A}, \epsilon/2)$. Therefore, by triangle inequality we have

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq d_Y(f(x_1), y_{j_A}) + d_Y(f(x_2), y_{j_A}) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore, f is uniformly continuous. $\square$

Definition 2.11.29. Let X be a topological space. A point $x \in X$ is said to be an *isolated point* of X if the singleton subset $\{x\}$ is an open in X.

Theorem 2.11.30. *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Step 1: We first show that given a non-empty open subset U of X and a point $x \in X \setminus U$, there exists a non-empty open subset V of X such that $V \subseteq U$ and $x \notin \overline{V}$. To see this, note that if $x \notin U$, then U being non-empty we can choose a point $y \in U$ such that $y \neq x$; and if $x \in U$, since X has no isolated points, we can choose a point $y \in U$ such that $y \neq x$. Since X is Hausdorff, there exists a pair of disjoint open subsets W_1 and W_2 of X containing x and y, respectively. Take $V = U \cap W_2$. Since $x \in W_1$ and $W_1 \cap V = \emptyset$, we have $x \notin \overline{V}$.

Step 2: We now show that there is no surjective map $f : \mathbb{N} \to X$, which would imply that X is uncountable. To see this, let $f : \mathbb{N} \to X$ be any map. Let $x_n = f(n)$, $\forall\, n \in \mathbb{N}$. Applying step 1 to the non-empty open subset $U_1 = X$ and the point $x_1 = f(1)$, we can find a non-empty open subset $U_2 \subseteq U_1 = X$ such that $x_1 \notin \overline{U_2}$. Suppose that $n \geq 2$, and we have constructed non-empty open subsets $U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1$ such that $x_{n-1} \notin \overline{U_n}$. Again by Step 1 we can find a non-empty open subset $U_{n+1} \subseteq U_n$ such that $x_n \notin \overline{U_{n+1}}$. Thus we have a nested sequence of non-empty closed subsets

$$\overline{U_1} \supseteq \overline{U_2} \supseteq \overline{U_3} \supseteq \cdots$$

of X; clearly this has finite intersection property. Since X is compact, it follows from Theorem 2.11.15 that there is a point, say $x \in \bigcap_{n\in\mathbb{N}} \overline{U_n}$. Since $x_n \notin \overline{U_{n+1}}$, for all $n \in \mathbb{N}$, we see that $x \neq x_n$, for all $n \in \mathbb{N}$. Therefore, the map $f : \mathbb{N} \to X$ cannot be surjective. $\square$

Proposition 2.11.31. *Any interval in $\mathbb{R}$ having more than one points has no isolated points.*

Proof. Let I be an interval in $\mathbb{R}$ having at least two points. Let $a \in I$ be arbitrary. If possible suppose that the singleton subset $\{a\} \subseteq I$ is open in I. Then there exists an open subset V of $\mathbb{R}$ such that $\{a\} = V \cap I$. Then there exists a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq V$. Since I has at least two points, there exists $b \in I$ such that $b \neq a$. Then either $a < b$ or $b < a$. Assume that $a < b$. Then either $b < a + \delta$ or $a + \delta \leq b$. If $b < a + \delta$, then $b \in V \cap I = \{a\}$, which is not possible since $a \neq b$. Then $a + \delta \leq b$. Since $a < a + \frac{\delta}{2} < a + \delta \leq b$ and I is an interval, $a + \frac{\delta}{2} \in I$. Then $a + \frac{\delta}{2} \in (a - \delta, a + \delta) \cap I \subseteq V \cap I = \{a\}$, implies that $a + \frac{\delta}{2} = a$, which is not possible since $\delta > 0$. Similarly, if $b < a$ we get contradiction. Therefore, $\{a\}$ cannot be open in I, and hence I has no isolated points. $\square$

Corollary 2.11.32. *Any interval in $\mathbb{R}$ having more than one point is uncountable. In particular, the real line $\mathbb{R}$ is uncountable.*

Proof. Let I be an interval in $\mathbb{R}$ having at least two points, say $a, b \in I$. Without loss of generality, we may assume that $a < b$. Then $[a, b] \subseteq I$. Since $[a, b] \subset \mathbb{R}$ is compact and Hausdorff and has no isolated points by Proposition 2.11.31, it follows from Theorem 2.11.30 that $[a, b]$ is uncountable. Then I is uncountable, and hence $\mathbb{R}$ is uncountable. $\square$

Exercise 2.11.33. Show that a connected metric space having more than one point is uncountable.

Proof. Let (X, d) be a connected metric space with at least two points. Fix a point $a \in X$, and consider the map $f : X \to \mathbb{R}$ defined by

$$f(x) = d(a, x), \ \forall\, x \in X.$$

Note that f is continuous by Proposition 2.11.25. Since X is connected, $f(X)$ is a connected subspace of $\mathbb{R}$ by Lemma 2.9.6. Then $f(X)$ is an interval in $\mathbb{R}$ by Proposition 2.9.14. Let $b \in X$ be such that $a \neq b$. Since $f(a) = d(a, a) = 0$ and $f(b) = d(a, b) > 0$, the image set $f(X) \subset \mathbb{R}$ is an interval in $\mathbb{R}$ having more than one points, and hence is uncountable by Corollary 2.11.32. Then it follows that X is uncountable. $\square$

2.11.1 Limit point compactness

Definition 2.11.34. A topological space X is said to be *limit point compact* if every infinite subset of X has a limit point in X.

Proposition 2.11.35. *A compact space is limit point compact.*

Proof. Let X be a compact topological space. Let K be an infinite subset of X. Suppose on the contrary that K has no limit points in X. Then K is closed in X. Since X is compact, K is also compact. Since K has no limit points in X, for each $x \in K$, there exists an open neighbourhood V_x of x in X such that $K \cap (V_x \setminus \{x\}) = \emptyset$. Then $V_x \cap K = \{x\}$. Since $\{V_x : x \in K\}$ is an open cover of K in X and K is compact, there are finitely many points $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} V_{x_j}$. Since $K \cap V_{x_j} = \{x_j\}$, $\forall\, j = 1, \ldots, n$, it follows that $K = \{x_1, \ldots, x_n\}$ is a finite set, which is a contradiction to our assumption that K is infinite. Therefore, K must have a limit point in X. □

Example 2.11.36. Let X be the subspace $\mathbb{N}$ of $\mathbb{R}$ and let Y be the two points space with only open subsets $\emptyset$ and Y itself. Then $\mathbb{N} \times Y$ is limit point compact but not compact.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. A *subsequence* of $(x_n)_{n\in\mathbb{N}}$ is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ in X, where $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers.

Definition 2.11.37. A topological space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

Example 2.11.38. A compact topological space need not be sequentially compact. For example, for each $\alpha \in [0,1] \subset \mathbb{R}$ let $X_\alpha = [0,1] \subset \mathbb{R}$, and consider the product space $X := \prod_{\alpha\in[0,1]} X_\alpha$. Then X is compact by Tychonoff's theorem. As a set, X consists of all functions $f : [0,1] \to [0,1]$. For each $n \in \mathbb{N}$, consider the map $f_n : [0,1] \to [0,1]$ defined by sending $x \in [0,1]$ to the n-th place digit x_n of the binary representation of x. Then $f_n \in X$, $\forall\, n \in \mathbb{N}$, and thus we have a sequence $(f_n)_{n\in\mathbb{N}}$ in X. Note that (f_n) has no convergent subsequence. Indeed, given any subsequence $(f_{n_k})_{k\in\mathbb{N}}$, let $x \in [0,1]$ be the real number such that $x_k = 0$ if and only if k is even. Then $(f_{n_k}(x))_{k\in\mathbb{N}}$ is the sequence whose all even terms are 0 and odd terms are 1, and hence is not convergent. Therefore, the subsequence $(f_{n_k})_{k\in\mathbb{N}}$ cannot converges to a function $f : [0,1] \to [0,1]$ in the product space X. Therefore, X is not sequentially compact.

Theorem 2.11.39. *Let X be a metrizable topological space. Then the following are equivalent.*

(i) X is compact,

(ii) X is limit point compact,

(iii) X is sequentially compact.

Proof. (i) ⇒ (ii): Proved in Proposition 2.11.35.

(ii) ⇒ (ii): Let X be limit point compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Consider the set $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then there exists $x \in A$ such that $x_n = x$, for infinitely many $n \in \mathbb{N}$. This gives a constant subsequence of $(x_n)_{n\in\mathbb{N}}$ which is clearly convergent. If A is infinite, then X being limit point compact, there is a limit point, say $a \in X$ of A. Since X is metrizable, there is a metric d on X compatible with the topology on X. Then there exists a point, say $x_{n_1} \in (B_d(a,1) \setminus \{a\}) \cap A$. Let $k \geq 2$, and suppose that we have chosen points $x_{n_1}, \ldots, x_{n_{k-1}} \in A$ such that $n_1 < \cdots < n_{k-1}$ and $d(a, x_{n_j}) < 1/j$, $\forall\, j \in \{1, \ldots, k-1\}$. Since A intersects $B_d(a, 1/k)$ at infinitely many points, we can find an integer $n_k > n_{k-1}$ such that $x_{n_k} \in (B_d(a,1/k) \setminus \{a\}) \cap A$.

Thus we get a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $d(a, x_{n_k}) < 1/k$, $\forall\, k \in \mathbb{N}$. Therefore, $(x_{n_k})_{k\in\mathbb{N}}$ converges to a in X.

(iii) $\Rightarrow$ (i): Suppose that X is sequentially compact. We use the following two results to show that X is compact.

Lemma 2.11.40. *If (X, d) is a sequentially compact metric space, then every open cover of X has a positive Lebesgue number.*

Proof. Suppose on the contrary that there exists an open cover $\mathscr{F} = \{V_\alpha : \alpha \in \Lambda\}$ of X that has no positive Lebesgue number. Then for each $n \in \mathbb{N}$ there exists a subset $A_n \subseteq X$ of diameter $\mathrm{diam}(A_n) < 1/n$ such that A_n is not contained in any of the member of $\mathscr{F}$. Choose an element $x_n \in A_n$, for each $n \in \mathbb{N}$. Since X is sequentially compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$, converging to $a \in X$. Then $a \in V_\alpha$, for some $\alpha \in \Lambda$. Since V_α is open in X, there exists a real number $\epsilon > 0$ such that $B_d(a, \epsilon) \subseteq V_\alpha$. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{n_k} < \epsilon/2$ and that $d(a, x_{n_k}) < \epsilon/2$. Then we have

$$A_{n_k} \subseteq B_d(x_{n_k}, \epsilon/2) \subseteq V_\alpha.$$

This contradicts our assumption that $\mathscr{F}$ has no positive Lebesgue number. This completes the proof. $\square$

Lemma 2.11.41. *If (X, d) is a sequentially compact metric space, given a real number $\epsilon > 0$ there exists finitely many points $x_1, \ldots, x_n \in X$ such that $X = \bigcup\limits_{j=1}^{n} B_d(x_j, \epsilon)$.*

Proof. Suppose on the contrary that X cannot be covered by finitely many ϵ-balls. Start with a point $x_1 \in X$, and choose $x_2 \in X \setminus B_d(x_1, \epsilon)$ noting that $B_d(x_1, \epsilon)$ cannot cover X. Assume that $n \geq 2$, and $x_n \in X$ is chosen from $X \setminus \big(\bigcup\limits_{j=1}^{n-1} B_d(x_j, \epsilon)\big)$. Then choose $x_{n+1} \in X \setminus \big(\bigcup\limits_{j=1}^{n} B_d(x_j, \epsilon)\big)$. Thus we get a sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d). Since $d(x_i, x_n) \geq \epsilon$, for all $i \neq n$, we see that $(x_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence, which contradicts our assumption that (X, d) is sequentially compact. Therefore, X can be covered by finitely many ϵ-balls. $\square$

To complete the proof, we start with an open cover $\mathscr{F} = \{V_\alpha : \alpha \in \Lambda\}$ of X, and use Lemma 2.11.40 to find a positive Lebesgue number $\delta > 0$ for $\mathscr{F}$. Set $\epsilon = \frac{\delta}{3} > 0$, and use Lemma 2.11.41 to find a finite open cover $\{B_d(x_j, \epsilon) : x_1, \ldots, x_n \in X\}$ of X by ϵ-balls. Since $\mathrm{diam}(B_d(x_j, \epsilon)) = 2\epsilon = 2\delta/3 < \delta$, by definition of Lebesgue number, each of $B_d(x_j, \epsilon)$ is contained in $V_{\alpha_j} \in \mathscr{F}$, for some $\alpha_j \in \Lambda$. Then $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\} \subseteq \mathscr{F}$ is a required finite subcover for X. Therefore, X is compact. $\square$

2.11.2 Local compactness

Definition 2.11.42. A topological space X is said to be *locally compact at* $x \in X$ if there exists a compact subspace $K_x \subseteq X$ containing an open neighbourhood $V_x \subseteq X$ of x. If X is locally compact at each $x \in X$ then we say that X is locally compact.

Example 2.11.43. The real line $\mathbb{R}$ is locally compact, but not compact. Indeed, given a point $x \in \mathbb{R}$, we have a compact subspace $[x-1, x+1] \subset \mathbb{R}$ that contains an open neighbourhood $(x-1, x+1) \subset \mathbb{R}$ of x. Therefore, $\mathbb{R}$ is locally compact. Since the open cover $\{(n, n+2) : n \in \mathbb{Z}\}$ of $\mathbb{R}$ has no finite subcover for $\mathbb{R}$, it follows that $\mathbb{R}$ is not compact.

Exercise 2.11.44. Let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that the map $f : (0, 1) \to S^1$ defined by

$$f(t) = (\cos 2\pi t,\ \sin 2\pi t),\ \forall\, t \in \mathbb{R},$$

is a homeomorphism of $(0, 1)$ onto $S^1 \setminus \{(0, 0)\}$. Construct a homeomorphism $g : \mathbb{R} \to (0, 1)$, and precompose it with f to get a homeomorphism of $\mathbb{R}$ onto the subspace $S^1 \setminus \{(0, 0)\}$.

Example 2.11.45. Let X be the Euclidean space $\mathbb{R}^n$, where $n \geq 2$. Given a point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, for each $j \in \{1, \ldots, n\}$ there exists a compact subspace $[x_j - 1, x_j + 1] \subset \mathbb{R}$ that contains an open neighbourhood $(x_j - 1, x_j + 1)$ of x_j. Then $K = [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1] \subset \mathbb{R}^n$ is a compact subspace of $\mathbb{R}^n$ containing an open neighbourhood $(x_1 - 1, x_1 + 1) \times \cdot \times (x_n - 1, x_n + 1)$ of $\mathbf{x}$. Therefore, $\mathbb{R}^n$ is locally compact. However, $\mathbb{R}^n$ is not compact, because the open cover $\{B(0, n) : n \in \mathbb{N}\}$ of $\mathbb{R}^n$ has no finite subcover.

Exercise 2.11.46. Fix an integer $n \geq 2$. Consider the *unit n-sphere*

$$S^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1},$$

in $\mathbb{R}^{n+1}$. Show that S^n is compact and Hausdorff. Use stereographic projection map to construct a homeomorphism of $\mathbb{R}^n$ onto the punctured n-sphere $S^n \setminus \{*\}$, where $* \in S^n$ is a chosen point.

Theorem 2.11.47. *A topological space X is locally compact and Hausdorff if and only if there exists a topological space Y satisfying the following properties:*

(i) Y contains X as a subspace,

(ii) $Y \setminus X$ is singleton, and

(iii) Y is compact and Hausdorff.

If Y and Y' are two topological spaces satisfying the above three properties, then there exists a unique homeomorphism $f : Y \to Y'$ such that $f\big|_X = \mathrm{Id}_X$.

Proof. **Step 1 (Uniqueness):** Suppose that Y and Y' are two topological spaces satisfying the above three properties. Suppose that $Y \setminus X = \{*\}$ and $Y' \setminus X = \{*'\}$. Define a map $f : Y \to Y'$ by

$$f(y) = \begin{cases} y, & \text{if} \quad y \in X, \\ *', & \text{if} \quad y = *. \end{cases}$$

Clearly f is a bijective map such that $f\big|_X = \mathrm{Id}_X$. Let $U \subseteq Y$ be an open subset. If $* \notin U$, then $U \subseteq X$, and hence $f(U) = U$ is open in Y. If $* \in U$, then $Z := Y \setminus U \subseteq X$, and hence $f(U) = f(Y \setminus Z) = Y \setminus f(Z) = Y \setminus Z = U$. Therefore, f is open. By symmetry, f^{-1} is also open. Therefore, f is a homeomorphism.

Step 2 (Construction of Y): Let X be locally compact and Hausdorff. Let $*$ be any object that is not an element of X, and let $Y := X \cup \{*\}$. Let τ_X be the collection of all open subsets of X. Let

$$\tau_Y := \tau_X \cup \{Y \setminus K : K \text{ is a compact subspace of } X\}.$$

We show that τ_Y gives a topology on Y.

Let $U, V \in \tau_Y$ be arbitrary.
Case 1: If $U, V \in \tau_X$, then $U \cap V \in \tau_X \subseteq \tau_Y$.
Case 2: If $U \in \tau_X$ and $V = Y \setminus K$, for some compact subspace K of X, then $U \cap V = U \cap (Y \setminus K) = U \cap (X \setminus K)$. Since K is a compact subspace of X and X is **Hausdorff**, K is closed in X, and hence $X \setminus K$ is open in X. Then $U \cap V = U \cap (X \setminus K) \in \tau_X \subseteq \tau_Y$.
Case 3: If $U = X \setminus K_1$ and $V = Y \setminus K_2$, for some compact subspaces K_1 and K_2 of X, then $U \cap V = Y \setminus (K_1 \cup K_2) \in \tau_Y$, because $K_1 \cup K_2$ is a compact subspace of X. Thus, τ_Y is closed under finite intersection.

Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an indexed family of objects from τ_Y.
Case 1: If $V_\alpha \in \tau_X$, $\forall\, \alpha \in \Lambda$, then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha \in \tau_X \subseteq \tau_Y$.
Case 2: If each V_α is of the form $V_\alpha = Y \setminus K_\alpha$, for some compact subspace K_α of X, then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = Y \setminus \left(\bigcap\limits_{\alpha \in \Lambda} K_\alpha\right)$. Since K_α is a compact subspace of the **Hausdorff** space X, it is closed in X, and hence $\bigcap\limits_{\alpha \in \Lambda} K_\alpha$ is closed in X. Since $\bigcap\limits_{\alpha \in \Lambda} K_\alpha$ is a closed subset of a compact space K_β, where $\beta \in \Lambda$, it is compact. Therefore, $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = Y \setminus \left(\bigcap\limits_{\alpha \in \Lambda} K_\alpha\right) \in \tau_Y$.
Case 3: Let $\Lambda = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$, and suppose that $V_\alpha \in \tau_X$, $\forall\, \alpha \in \Lambda_1$ and $V_\beta = Y \setminus K_\beta$, for all $\beta \in \Lambda_2$. Then $U := \bigcup\limits_{\alpha \in \Lambda_1} V_\alpha \in \tau_X$ by case 1, and $V := \bigcup\limits_{\beta \in \Lambda_2} V_\beta = Y \setminus K$, for some compact subspace K of X, as discussed in case 2. Then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = U \cup V \in \tau_Y$ by case 2. Therefore, τ_Y is closed under arbitrary union. Therefore, τ_Y is a topology on Y.

Step 3 (X is a subspace of Y): Let $V \in \tau_Y$ be arbitrary. If $V \in \tau_X$, then $V \cap X = V \in \tau_X$. If $V \notin \tau_X$, then $V = Y \setminus K$, for some compact subspace K of X. Then $V \cap X = (Y \setminus K) \cap X = (X \setminus K) \cap X \in \tau_X$, since K being a compact subspace of the Hausdorff space, K is closed in X, and hence $X \setminus K$ is open in X. Therefore, the subspace topology on X induced from Y coincides with the topology on X. Thus, X is a subspace of Y.

Step 4 (Y is compact): Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then there exists $\alpha_0 \in \Lambda$ such that $* \in V_{\alpha_0}$. Clearly $V_{\alpha_0} = Y \setminus K$, for some compact subspace K of X. Then the collection of all V_α's that does not contain $*$ forms an open cover of X, and hence of the compact subspace K of X. Then we can choose finitely many such objects, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$ from $\mathscr{U}$ that covers K. Then $\{V_{\alpha_0}, V_{\alpha_1}, \ldots, V_{\alpha_n}\} \subseteq \mathscr{U}$ is a finite subcover form Y. Therefore, Y is compact.

Step 4 (Y is Hausdorff): Let $x, y \in Y$ be two distinct points. If $x, y \in X$, then X being **Hausdorff**

we can separate them by a pair of disjoint open neighbourhoods of them in X, and hence in Y. If $y = *$, then $x \in X$. Since X is **locally compact** we can find a compact subspace K of X containing an open neighbourhood $V \subseteq X$ of x. Then $U := Y \setminus K$ is an open neighbourhood of $*$ in Y disjoint from the open neighbourhood V of x. Therefore, Y is Hausdorff.

Step 5 (Converse part): Suppose that there is a compact Hausdorff topological space Y that contains X as a subspace of it and $Y \setminus X = \{*\}$ is singleton. Since Y is Hausdorff, it is a T1 space, and hence $Y \setminus X$ is closed in Y. Then X is open in Y. Clearly X is Hausdorff. Let $x \in X$ be arbitrary. Then Y being **Hausdorff**, there is a pair of disjoint open neighbourhoods U and V of x and $*$, respectively, in Y. Since $K := Y \setminus V$ is a closed subspace of the **compact** space Y, it is compact. Since $K \subseteq X$, K is a compact subspace of X. Clearly $x \in U \subseteq K$. Therefore, X is locally compact. This completes the proof. □

Definition 2.11.48. Let X be a topological space that is not compact. A *compactification* of X is a compact topological space Y containing X as its subspace such that the closure of X in Y is Y itself. If Y is a compactification of X such that $Y \setminus X$ is a singleton subset of Y, then Y is called an *one-point compactification* of X.

We say that a topological space X has property P locally if for each $x \in X$, every open neighbourhood $U_x \subseteq X$ of x contains a neighbourhood V_x of x that has property P. For example, local connectedness, local path-connectedness etc. are local property. However, the definition of local compactness given in Definition 2.11.42 is not "local" in nature, in general. Nevertheless, the following proposition says that local compactness is indeed a local property for a Hausdorff space.

Proposition 2.11.49. *Let X be a Hausdorff topological space. Then X is locally compact at $x \in X$ if and only if given any open subset $U \subseteq X$ containing x, there exists an open neighbourhood $V \subseteq X$ of x such that $\overline{V}$ is a compact subspace of X and that $\overline{V} \subseteq U$.*

Proof. Let X be a Hausdorff space. Suppose that X is locally compact. Let Y be the one-point compactification of X, and let $Y \setminus X = \{*\}$. Let $x \in X$ be arbitrary. Let U be an open neighbourhood of x in X. Since Y is Hausdorff, $X = Y \setminus \{*\}$ is open in Y, and therefore, U is open in Y. Then $K := Y \setminus U$ is a closed subspace of the compact space Y, and hence is compact. Since Y is a Hausdorff space, there exist open subsets V and W of Y such that $x \in V$, $K \subseteq W$ and $V \cap W = \emptyset$. Let $\overline{V}$ be the closure of V in Y. Then $\overline{V} \cap K = \emptyset$, and hence $\overline{V} \subseteq U$. Since Y is compact and $\overline{V}$ is closed in Y, $\overline{V}$ is compact.

Conversely, suppose that for each $x \in X$ and an open neighbourhood U of x in X there exists an open neighbourhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq U$. Then $K = \overline{V}$ is a compact subspace of X containing the open neighbourhood V of x in X. Therefore, X is locally compact. □

Corollary 2.11.50. *An open or a closed subspace of a locally compact Hausdorff space is locally compact and Hausdorff.*

Proof. Let X be a locally compact space. Let A be a closed subspace of X. Then for given $a \in A$, there exists a compact subspace K of X containing an open neighbourhood $V \subseteq X$ of x. Then

$A \cap V$ is an open neighbourhood of a in A contained in $A \cap K$. Since A is closed in X, $A \cap K$ is a closed subspace of the compact space K, and hence $A \cap K \subseteq A$ is compact. Thus, $A \cap V$ is an open neighbourhood of a in A contained in the compact subspace $A \cap K$ of A. Thus, A is locally compact. Note that, here we have not used Hausdorff property of X.

Assume that X is a locally compact Hausdorff space, and A is open in X. Let $a \in A$. Since A is an open neighbourhood of a in X, by Proposition 2.11.49 there exists an open neighbourhood V of a in X such that $\overline{V}$ is a compact subspace of X and $\overline{V} \subseteq A$. Then $K = \overline{V}$ is a required compact subspace of A containing an open neighbourhood V of a in A. Thus, A is locally compact. □

Corollary 2.11.51. *A topological space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.*

Proof. If X is locally compact and Hausdorff then we can take Y to be the one-point compactification of X. Then Y is a compact Hausdorff space that contains X as its open subspace.

Conversely, suppose that X is homeomorphic to an open subspace of a compact Hausdorff space Y. Then X is locally compact and Hausdorff by Corollary 2.11.50. □

2.11.3 Net and filter

A *directed set* is a partially ordered set $(I, \leq)$ such that given any two elements $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.11.52. A *net* in X is a map $f : I \to X$, where $(I, \leq)$ is a directed set. Like a sequence, we usually denote by x_α the image $f(\alpha)$ of $\alpha \in I$, and express a net as $(x_\alpha)_{\alpha \in I}$.

A net $(x_\alpha)_{\alpha \in I}$ in X is said to converge to a point $x \in X$, written as $(x_\alpha)_{\alpha \in I} \longrightarrow x$, if for given an open neighbourhood $U \subseteq X$ of x there exists an element $\alpha_U \in I$ such that $x_\beta \in U$, for all $\beta \in I$ satisfying $\alpha_U \leq \beta$.

Remark 2.11.53. Clearly, any sequence is a net but not the other way around. If $(I, \leq)$ is equal to $(\mathbb{N}, \leq)$, then the notion of net and its convergence coincides with the notion of a sequence and its convergence.

Exercise 2.11.54. Let X and Y be two topological spaces. Let $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ be two nets in X and Y, respectively, indexed by the same directed set $(I, \leq)$. If $(x_\alpha)_{\alpha \in I}$ converges to x in X and $(y_\alpha)_{\alpha \in I}$ converges to y in Y, show that the net $((x_\alpha, y_\alpha))_{\alpha \in I}$ converges to (x, y) in the product space $X \times Y$.

Proposition 2.11.55. *In a Hausdorff space X a net $(x_\alpha)_{\alpha \in I}$ can converge to at most one point of X.*

Proof. Suppose on the contrary that $(x_\alpha)_{\alpha \in I}$ converge to x and y in X, where $x \neq y$. Since X is Hausdorff, there exists a pair of open neighbourhoods V_x and V_y of x and y, respectively, such that $V_x \cap V_y = \emptyset$. Then by definition of convergence of a net in X, there exist $\alpha_0, \beta_0 \in I$ such that

$$\alpha_0 \leq \alpha \implies x_\alpha \in V_x, \quad \text{and} \quad \beta_0 \leq \beta \implies x_\beta \in V_y.$$

Since $(I, \leq)$ is a directed set, there exists $\gamma \in I$ such that $\alpha_0 \leq \gamma$ and $\beta_0 \leq \gamma$. Then $x_\gamma \in V_x \cap V_y$. But this is not possible, since $V_x \cap V_y = \emptyset$. $\square$

Theorem 2.11.56. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net $(x_\alpha)_{\alpha \in I}$ of points of A converging to x.*

Proof. Suppose that there is a net $(x_\alpha)_{\alpha \in I}$ of points of A that converges to $x \in X$. Let $U \subseteq X$ be an open neighbourhood of x. Since $(x_\alpha)_{\alpha \in I} \longrightarrow x$ in X, there exists $\alpha \in I$ such that $x_\alpha \in U$, and hence $A \cap U \neq \emptyset$. Thus, $x \in \overline{A}$.

Conversely, suppose that $x \in \overline{A}$. If $x \in A$, then we can take the constant sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n = x$, $\forall\, n \in \mathbb{N}$, that clearly converges to x in X. Suppose that $x \in \overline{A} \setminus A$. Let

$$I_x := \{V : V \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Given $V_1, V_2 \in I_x$, we define $V_1 \leq V_2$ if $V_2 \subseteq V_1$. Then $(I_x, \leq)$ is a partially ordered set. Given $V_1, V_2 \in I_x$, $V := V_1 \cap V_2 \in I_x$ and it satisfies $V_1 \leq V$ and $V_2 \leq V$. Therefore, $(I_x, \leq)$ is a directed set. Since $x \in \overline{A}$, for each open neighbourhood $V \in I_x$ of x in X, we can choose an element $x_V \in A \cap V$. Thus we have a net $(x_V)_{V \in I_x}$ of points of A. Given any open neighbourhood U of x in X, $U \in I_x$. Then if $V \in I_x$ with $U \leq V$, then $V \subseteq U$ and hence $x_V \in V$ implies $x_V \in U$. Thus the net $(x_V)_{V \in I_x}$ converges to x in X. $\square$

Corollary 2.11.57. *Let $A \subseteq X$. Then A is closed in X if and only if limit point of every convergent net of points of A is in A.*

Definition 2.11.58. Let $(I, \leq)$ be a directed set. A subset $J \subseteq I$ is said to be *cofinal* in $(I, \leq)$ if for each $i \in I$, there exists $j \in J$ such that $i \leq j$.

Proposition 2.11.59. *If J is a cofinal subset of $(I, \leq)$, then the partial order relation induced from $(I, \leq)$ makes $(J, \leq)$ a directed set.*

Proof. Clearly $(J, \leq)$ is a partially ordered set. Given $j_1, j_2 \in J$, there exists $i \in I$ such that $j_1 \leq i$ and $j_2 \leq i$. Since J is cofinal in $(I, \leq)$, there exists $j \in J$ such that $i \leq j$. Then by transitivity of the partial order relation we have $j_1 \leq j$ and $j_2 \leq j$. $\square$

Definition 2.11.60. Let $(J, \leq)$ be a directed set, and let $f : J \to X$ be a net in X. A *subnet* of points of X is a composite map $f \circ g : I \to X$, where $(I, \leq)$ is a directed set and $g : I \to J$ is a map satisfying the following properties:

(i) $\alpha \leq \beta$ in $(I, \leq)$ implies that $g(\alpha) \leq g(\beta)$ in $(J, \leq)$, and

(ii) the subset $g(I) = \{g(\alpha) : \alpha \in I\} \subseteq J$ is cofinal in $(J, \leq)$

Proposition 2.11.61. *If a net $(x_\alpha)_{\alpha \in I}$ in X converges to $x \in X$, so is any of its subnet.*

Proof. Let $f : J \to X$ be a net in X indexed by a directed set $(I, \leq)$. Let $f \circ g$ be a subnet of f, where $g : I \to J$ is a map of directed sets satisfying the conditions (i) and (ii) as in Definition 2.11.60. Let U be an open neighbourhood of x in X. Since f converges to x, there exists $\alpha \in J$ such that $f(\beta) \in U$, for all $\beta \in I$ satisfying $\alpha \leq \beta$. Since $g(I)$ is cofinal in $(J, \leq)$, there exists

$i \in I$ such that $\alpha \leq g(i)$. Then for any $j \in I$ with $i \leq j$ we have $\alpha \leq g(i) \leq g(j)$. Then we have $(f \circ g)(j)) \in U$, for all $j \in I$ satisfying $i \leq j$. Therefore, the subnet $f \circ g$ converges to x in X. $\square$

Definition 2.11.62. Let $(x_\alpha)_{\alpha\in I}$ be a net in X. A point $x \in X$ is said to be an *accumulation point* of $(x_\alpha)_{\alpha\in I}$ if for given any open neighbourhood $U \subseteq X$ of x the subset $I_U := \{\alpha \in I : x_\alpha \in U\}$ is cofinal in $(I, \leq)$.

Lemma 2.11.63. *A point $x \in X$ is an accumulation point of a net $(x_\alpha)_{\alpha\in I}$ in X if and only if there exists a subnet of $(x_\alpha)_{\alpha\in I}$ converging to x.*

Theorem 2.11.64. *A topological space X is compact if and only if every net in X has a convergent subnet.*

2.12 Second countability and separability

2.12.1 Regular and normal spaces

Proposition 2.12.1. *Subspace of a Hausdorff space is Hausdorff.*

Proof. Let A be a subspace of a Hausdorff space. Let $x, y \in A$ be two distinct points. Since X is Hausdorff, there exists a pair of disjoint open subsets U and V of X containing x and y, respectively. Then $U \cap A$ and $V \cap A$ are pair of disjoint open subsets of A containing x and y, respectively. $\square$

Proposition 2.12.2. *Product of Hausdorff spaces is Hausdorff.*

Proof. Let $\mathscr{F} = \{X_\alpha : \alpha \in \Lambda\}$ be a collection of Hausdorff spaces. Let $X = \prod\limits_{\alpha\in\Lambda} X_\alpha$ be the associated product space. Let $x = (x_\alpha)_{\alpha\in\Lambda}$ and $y = (y_\alpha)_{\alpha\in\Lambda}$ be two distinct points of X. Then there exists $\beta \in \Lambda$ such that $x_\beta \neq y_\beta$. Let $\pi_\beta : X = \prod\limits_{\alpha\in\Lambda} X_\alpha \to X_\beta$ be the projection map onto the β-th factor. Since X_β is Hausdorff, there exists a pair of disjoint open neighbourhoods U_β and V_β of x_β and y_β, respectively, in X_β. Then $\pi_\beta^{-1}(U_\beta)$ and $\pi_\beta^{-1}(V_\beta)$ are disjoint open neighbourhoods of x and y in X, respectively. Therefore, X is Hausdorff. $\square$

Definition 2.12.3. A topological space X is said to be *regular* if X is a T1 space and given any closed subset A of X and a point $x \in X \setminus A$, there exists a pair of disjoint open subsets U and V of X containing x and A, respectively.

Proposition 2.12.4. *A regular space is Hausdorff.*

Proof. Let X be a regular space. Then X is a T1 space, and hence all one-point subspaces of X are closed in X. Let $x, y \in X$ be such that $x \neq y$. Then $\{y\}$ is a closed subspace of X not containing x, and so by regularity of X, there exists a pair of disjoint open neighbourhoods U and V of x and y, respectively. Therefore, X is Hausdorff. $\square$

Proposition 2.12.5. *A topological space X is regular if and only if X is a T1 space such that given any $x \in X$ and an open neighbourhood U of x there exists an open neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.*

Proof. Let X be a regular space. Then X is a T1 space. Let $x \in X$ and U be an open neighbourhood of x in X. Then $A := X \setminus U$ is a closed subset of X such that $x \in X \setminus A$. Then by regularity of X, there exists a pair of disjoint open subsets V and W of X containing x and A, respectively. Then $\overline{V} \cap A = \emptyset$, and hence $\overline{V} \subseteq X \setminus A = U$.

For the converse part, let X be a T1 space such that given any $x \in X$ and an open neighbourhood U of x in X there exists an open neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. Let A be a closed subset of X and let $x \in X \setminus A$. Then $U := X \setminus A$ is an open neighbourhood of x in X. Then by assumption on X, there exists an open neighbourhood V of x in X such that $\overline{V} \subseteq U$. Then $W := X \setminus \overline{V}$ is an open subset of X containing $X \setminus U = A$ and $V \cap W = \emptyset$. □

Proposition 2.12.6. *Subspace of a regular space is regular.*

Proof. Let A be a subspace of a regular space X. Since X is Hausdorff, so is its subspace A, and hence A is a T1 space. Let $Z \subseteq A$ be a closed subspace of A and let $a \in Z \setminus A$. Let $\overline{Z}$ be the closure of Z in X. Then $Z = \overline{Z} \cap A$. Since $a \in A \setminus Z$, we have $a \notin \overline{Z}$. Since X is regular and $\overline{Z}$ is a closed subspace of X not containing a, there exists a pair of disjoint open subsets U and V of X containing a and $\overline{Z}$, respectively. Then $A \cap U$ and $A \cap V$ are open neighbourhoods of a and $Z = \overline{Z} \cap A$ in A, respectively. Clearly $(A \cap U) \cap (A \cap V) = \emptyset$. Therefore, A is regular. □

Exercise 2.12.7. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, and let $X = \prod\limits_{\alpha \in \Lambda} X_\alpha$ be the associated product space. Let $A_\alpha \subseteq X_\alpha$, for all $\alpha \in \Lambda$, and let $A = \prod\limits_{\alpha \in \Lambda} A_\alpha$. Then $\overline{\prod\limits_{\alpha \in \Lambda} A_\alpha} = \prod\limits_{\alpha \in \Lambda} \overline{A_\alpha}$, where $\overline{A_\alpha}$ is the closure of A_α in X_α, for each $\alpha \in \Lambda$.

Proposition 2.12.8. *Product of regular spaces is regular.*

Proof. Let $\mathscr{F} = \{X_\alpha : \alpha \in \Lambda\}$ be a collection of regular spaces, and let $X = \prod\limits_{\alpha \in \Lambda} X_\alpha$ be the associated product space. Let $x = (x_\alpha)_{\alpha \in \Lambda} \in X$ and let U be an open neighbourhood of x in X. Then there is a basic open neighbourhood $\prod\limits_{\alpha \in \Lambda} U_\alpha$ of x contained in U, where $U_\alpha = X_\alpha$, for all $\alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$. Since U_{α_j} is an open neighbourhood of x_{α_j} in the regular space X_{α_j}, there exists an open neighbourhood V_{α_j} of x_{α_j} such that $\overline{V_{\alpha_j}} \subseteq U_{\alpha_j}$, for all $j \in \{1, \ldots, n\}$. Set $V_\alpha = X_\alpha$, for $\alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$. Then $V = \prod\limits_{\alpha \in \Lambda} V_\alpha$ is an open neighbourhood of x in X such that $\overline{V} = \prod\limits_{\alpha \in \Lambda} \overline{V_\alpha} \subseteq U$. Therefore, X is regular. □

Definition 2.12.9. A topological space X is said to be *completely regular* if X is a T1 space and given a closed subset A and a point $x \in X \setminus A$, there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

Proposition 2.12.10. *A completely regular space is regular.*

Proof. Let X be a completely regular space. Let A be a closed subset of X and let $x \in X \setminus A$. Then there is a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$. Then $U := f^{-1}([0,1/2))$ and $V := f^{-1}((1/2,1])$ are pairwise disjoint open subsets of X containing x and A, respectively. Therefore, X is regular. □

Definition 2.12.11. A topological space X is said to be *normal* if X is a T1 space and given any two closed subsets A and B of X with $A \cap B = \emptyset$, there exist open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Proposition 2.12.12. *A normal space is regular.*

Proof. Let X be a normal space. Let A be a closed subset of X. Let $x \in X$ be such that $x \notin A$. Since X is a T1 space, $\{x\}$ is a closed subset of X. Then there exist open neighbourhoods U and V of $\{x\}$ and A, respectively, in X such that $U \cap V = \emptyset$. Therefore, X is regular. □

Theorem 2.12.13. *Every second countable regular space is normal.*

Proof. Let X be a second countable regular space. Let $\mathscr{B}$ be a countable basis for the topology on X. Let A and B be two non-empty closed subsets of X with $A \cap B = \emptyset$. Since X is regular, for each $a \in A$ there exist a pair of disjoint open subsets U_a and W_a containing a and B, respectively. Therefore,

$$\overline{U_a} \cap B = \emptyset, \ \forall\, a \in A. \tag{2.12.14}$$

Since $\mathscr{B}$ is a basis for the topology on X, for each $a \in A$ we can choose a basic open subset $V_a \in \mathscr{B}$ such that $a \in V_a \subseteq U_a$. Since $\{V_a : a \in A\} \subseteq \mathscr{B}$ and $\mathscr{B}$ is countable, we have a countable collection $\mathscr{V} = \{V_n : n \in \mathbb{N}\}$ of open subsets of X such that $A \subseteq V := \bigcup\limits_{n\in\mathbb{N}} V_n$ and $\overline{V_n} \cap B = \emptyset$, for all $n \in \mathbb{N}$. Similarly, we get a countable collection $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ of open subsets of X such that $B \subseteq W := \bigcup\limits_{n\in\mathbb{N}} W_n$ and $\overline{W_n} \cap A = \emptyset$, for all $n \in \mathbb{N}$. However, $V \cap W$ need not be empty. So we perform the following simple trick to modify them to get a pair of disjoint open neighbourhoods of A and B in X. For each $n \in \mathbb{N}$, let

$$V_n' = V_n \setminus \bigcup_{j=1}^{n} \overline{W_j} \ \text{ and } \ W_n' = W_n \setminus \bigcup_{j=1}^{n} \overline{V_j}. \tag{2.12.15}$$

Since V_n is open in X and $\bigcup\limits_{j=1}^{n} \overline{W_j}$ is closed in X, the set difference V_n' is open in X. Similarly, W_n' is open in X. Since $A \cap \overline{W_j} = \emptyset$, $\forall\, j$, the collection $\{V_n' : n \in \mathbb{N}\}$ is an open cover of A. Similarly, the collection $\{W_n' : n \in \mathbb{N}\}$ is an open cover of B. Finally, the open subsets $V' := \bigcup\limits_{n\in\mathbb{N}} V_n'$ and $W' := \bigcup\limits_{n\in\mathbb{N}} W_n'$ are disjoint. Indeed, if $x \in V' \cap W'$, then $x \in V_n' \cap W_m'$, for some $n, m \in \mathbb{N}$. Without loss of generality, we many assume that $m \leq n$. Then $x \in W_n' = W_n \setminus \bigcup\limits_{j=1}^{n} \overline{V_j}$ implies that $x \notin \overline{V_m}$, since $m \leq n$, and hence $x \notin V_m$, which contradicts the assumption that $x \in V_m' \subseteq V_m$. This completes the proof. □

Theorem 2.12.16. *Every metrizable space is normal.*

Proof. Let X be a metrizable space. Fix a metric d on X that induces the topology on X. Let A and B be two non-empty closed subsets of X with $A \cap B = \emptyset$. For each $a \in A$ we can choose a real number $r_a > 0$ such that $B_d(a, r_a) \cap B = \emptyset$. Similarly, for each $b \in B$ we can choose a real number $s_b > 0$ such that $B_d(b, s_b) \cap A = \emptyset$. Then $U := \bigcup_{a \in A} B_d(a, r_a/2)$ and $V := \bigcup_{b \in B} B_d(b, s_b/2)$ are open subsets of X containing A and B, respectively. If there exists a point $x \in U \cap V$, then $x \in B_d(a, r_a/2) \cap B_d(b, s_b/2)$, for some $a \in A$ and $b \in B$. Then $d(a, b) \leq d(a, x) + d(b, x) < r_a/2 + s_b/2$. If $r_a \leq s_b$, then $(r_a + s_b)/2 \leq s_b$, and hence $d(a, b) < s_b$. Then $a \in B_d(b, s_b)$, which is not possible since $A \cap B = \emptyset$. Similarly, if $s_b \leq r_a$, then $d(a, b) < r_a$, and hence $b \in B_d(a, r_a)$, which is not possible. Therefore, we must have $U \cap V = \emptyset$. Therefore, X is normal. $\square$

Lemma 2.12.17. *Let K be a compact subset of a Hausdorff space X. Given any $x \in X \setminus K$ there exists a pair of disjoint open subsets U and V of X containing x and K, respectively.*

Proof. Since X is Hausdorff, for each $y \in K$, there exists a pair of disjoint open subsets U_y and V_y of X containing x and y, respectively. Then $\mathscr{U} = \{V_y : y \in K\}$ is an open cover of K. Since K is compact, there exists finitely many points $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} V_{y_j}$. Then $U := \bigcap_{j=1}^{n} U_{y_j}$ and $V := \bigcup_{j=1}^{n} V_{y_j}$ are open neighbourhoods of x and K, respectively, such that $U \cap V = \emptyset$. This completes the proof. $\square$

Corollary 2.12.18. *A compact Hausdorff space is regular.*

Proof. Let X be a compact Hausdorff space. Let K be a closed subset of X and let $x \in X \setminus K$. Since closed subspace of a compact space is compact, K is compact. Then by Lemma 2.12.17 there exists a pair of disjoint open subsets U and V of X containing x and K, respectively. Thus X is regular. $\square$

Theorem 2.12.19. *A compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. Since X is compact, both A and B are compact. Since X is compact and Hausdorff, it is regular by Corollary 2.12.18. Then for each $a \in A$ there exists a pair of disjoint open subsets U_a and V_a of X containing a and B, respectively. Then $\{U_a : a \in A\}$ is an open cover of A in X. Since A is compact, there exists finitely many points $a_1, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{j=1}^{n} U_{a_j}$. Then $U := \bigcup_{j=1}^{n} U_{a_j}$ and $V := \bigcap_{j=1}^{n} V_{a_j}$ are pairwise disjoint open subsets of X containing A and B, respectively. Therefore, X is normal. $\square$

Lemma 2.12.20. *A topological space X is normal if and only if X is a T1 space such that given any closed subset A of X and an open neighbourhood U of A in X, there exists an open neighbourhood V of A whose closure $\overline{V}$ in X is contained in U.*

Proof. Suppose that X is a normal space. Then X is a T1 space. Let A be a closed subspace of X and let U be an open neighbourhood of A in X. Then $B := X \setminus U$ is a closed subset of X with

$A \cap B = \emptyset$. Then there exist open neighbourhoods V and W of A and B, respectively, in X such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \emptyset$. Then $\overline{V} \cap B = \emptyset$, and hence $\overline{V} \subseteq X \setminus B = U$, as required.

Conversely, suppose that X is a T1 space such that given any closed subset A of X and an open neighbourhood U of A in X, there exists an open neighbourhood V of A in X such that $\overline{V} \subseteq U$. Let A and B be two disjoint closed subsets of X. Then $U := X \setminus B$ is an open neighbourhood of A in X. Then by assumption there exists an open neighbourhood V of A in X such that $\overline{V} \subseteq X \setminus B$. Then $W := X \setminus \overline{V}$ is an open neighbourhood of B such that $V \cap W = \emptyset$. □

Theorem 2.12.21 (Urysohn's lemma)**.** *Given a pairwise disjoint closed subsets A and B of a normal space X, there exists a continuous map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Since $U := X \setminus B$ is an open neighbourhood of A in X and X is a normal space, by Lemma 2.12.20 we can find an open neighbourhood $U_{\frac{1}{2}}$ of A such that

$$A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U. \tag{2.12.22}$$

Since $U_{1/2}$ and U are open neighbourhoods of the closed subsets A and $\overline{U_{1/2}}$, respectively, applying normality we have open subsets $U_{1/4}$ and $U_{3/4}$ of X such that

$$A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U. \tag{2.12.23}$$

Continuing in this way, for each rational number

$$t \in T := \left\{ \frac{m}{2^n} \in \mathbb{Q} \mid m \in \{1, \ldots, 2^n - 1\} \text{ and } n \in \mathbb{N} \right\},$$

we have an open subset U_t containing A such that given $t_1, t_2 \in T$, we have

$$t_1 \leq t_2 \implies A \subseteq U_{t_1} \subseteq \overline{U_{t_1}} \subseteq U_{t_2} \subseteq \overline{U_{t_2}} \subseteq U = X \setminus B. \tag{2.12.24}$$

Define a map $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in \bigcap\limits_{t \in T} U_t, \\ \sup\{t \in T : x \notin U_t\}, & \text{if} \quad x \notin \bigcap\limits_{t \in T} U_t. \end{cases}$$

Clearly, $f(x) \in [0,1]$, $\forall\, x \in X$, and that $f(A) = \{0\}$ and $f(B) = \{1\}$. All it remains to show that f is continuous. Since the collection of all intervals of the form $[0, a)$ and $(a, 1]$, where $a \in (0,1)$, forms a subbasis for the subspace topology on $[0,1]$ induced from $\mathbb{R}$, to show f is continuous, it suffices to show that $f^{-1}([0,a))$ and $f^{-1}((a,1])$ are open in X, for all $a \in (0,1)$. Fix an element $a \in (0,1)$. Note that, *$f(x) < a$ if and only if $x \in U_t$, for some $t < a$*. Indeed, since T is dense in $[0,1]$, it follows from the definition of f that if $f(x) < a$, choosing an element $t_0 \in T$ with $f(x) < t_0 < a$ we have $x \in U_{t_0}$. Conversely, if $x \in U_{t_0}$ for some $t_0 < a$, then $f(x) := \sup\{t \in T : x \notin U_t\} < t_0$. Therefore, $f^{-1}([0,a)) = \{x \in X : f(x) < a\} = \bigcup\limits_{t<a} U_t$ is open in X. Now we show that $f^{-1}((a,1])$ is open in X. For this, note that *$f(x) > a$ if and only if $x \notin \overline{U_t}$, for some $t > a$*. Indeed, if $f(x) > a$, then T being dense in $[0,1]$, choosing $t_0, t_1 \in T$ with $a < t_0 < t_1 < f(x)$ we see that $x \notin U_{t_1}$. Since $\overline{U_{t_0}} \subseteq U_{t_1}$ by construction, it follows that $x \notin \overline{U_{t_0}}$. Conversely, if $x \notin \overline{U_t}$, for some $t > a$, then $x \notin U_t$ where $t > a$, and hence $f(x) = \sup\{s \in T : x \notin U_s\} > a$. Therefore, $f^{-1}((a, 1]) = \{x \in X : f(x) > a\} = \bigcup\limits_{t>a} (X \setminus \overline{U_t})$ is open in X. This completes the proof. □

Corollary 2.12.25 (Urysohn's lemma)**.** *Let A and B be two disjoint closed subsets of a normal space X. Then given $a, b \in \mathbb{R}$ with $a < b$, there exists a continuous function $f : X \to [a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

Proof. By Urysohn's lemma there exists a continuous map $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Let $\varphi : [0, 1] \to [a, b]$ be the map defined by

$$\varphi(t) = (1 - t)a + tb, \ \ \forall\, t \in [0, 1].$$

Then φ is a continuous map such that $\varphi(0) = a$ and $\varphi(1) = b$. Then $\varphi \circ f : X \to [a, b]$ is a continuous map such that $(\varphi \circ f)(A) = \{a\}$ and $(\varphi \circ f)(B) = \{b\}$. □

Corollary 2.12.26. *A normal space is completely regular.*

Proof. Follows from Urysohn's lemma. □

Theorem 2.12.27 (Tietze's extension theorem)**.** *Let Z be a closed subspace of a normal space X. Then given $a, b \in \mathbb{R}$ with $a \leq b$, any continuous map $f : Z \to [a, b]$ can be extended to a continuous map $\widetilde{f} : X \to [a, b]$ such that $\widetilde{f}\big|_Z = f$.*

Proof. If $a = b$, then f is a constant map, and hence we can extend it as a constant map on whole X which is clearly continuous. Assume that $a < b$. Without loss of generality, we may assume that $[a, b]$ is the smallest closed interval in $\mathbb{R}$ containing the image of f. Furthermore, by composing with a rescaling map, if required, it suffices to prove the result for $[a, b] = [-1, 1]$.

Let $f_0 = f : Z \to [-1, 1]$, and consider the subsets

$$A_0 := \left\{x \in Z : f_0(x) \leq -\frac{1}{3}\right\} \text{ and } B_0 := \left\{x \in Z : f_0(x) \geq \frac{1}{3}\right\}. \tag{2.12.28}$$

Then A_0 and B_0 are disjoint closed subspaces of Z. Since $[-1, 1]$ is the smallest closed interval in $\mathbb{R}$ containing the image of f_0 by assumption, both A_0 and B_0 are non-empty. Since Z is closed in X, both A_0 and B_0 are closed in X. Then by Corollary 2.12.25 (Urysohn's lemma) there exists a continuous map $g_0 : X \to [-1/3, 1/3]$ such that $g_0(A_0) = \{-1/3\}$ and $g_0(B_0) = \{1/3\}$. Then we define a map $f_1 : Z \to \mathbb{R}$ by

$$f_1(x) = f_0(x) - g_0(x), \ \forall\, x \in Z.$$

Then we have

$$|f_1(x)| = |f_0(x) - g_0(x)| \leq 2/3, \ \ \forall\, x \in Z.$$

Let

$$A_1 := \left\{x \in Z : f_1(x) \leq -\frac{1}{3} \cdot \frac{2}{3}\right\} \text{ and } B_1 := \left\{x \in Z : f_1(x) \geq \frac{1}{3} \cdot \frac{2}{3}\right\}. \tag{2.12.29}$$

Then both A_1 and B_1 are non-empty disjoint closed subsets of Z, and so by Uryshon's lemma there exists a continuous map

$$g_1 : X \to [(-1/3)(2/3), (1/3)(2/3)]$$

such that $g_1(A_1) = \{-(-1/3)(2/3)\}$ and $g_1(B_1) = \{(1/3)(2/3)\}$. Define a map $f_2 : Z \to [-1,1]$ by $f_2 = f_1 - g_1 = f_0 - (g_0 + g_1)$. Then $|f_2(x)| \leq (2/3)^2$, $\forall\, x \in Z$.

□

Proposition 2.12.30. *Let $\mathscr{A}$ be a collection of subsets of X satisfying finite intersection property. Then there exists unique maximal collection $\mathcal{D}$ of subsets of X such that $\mathcal{A} \subseteq \mathscr{D}$ and $\mathscr{D}$ satisfy finite intersection property.*

Proof. Let $\mathscr{C}$ be the collection of all collections of subsets of X satisfying finite intersection property. □

Theorem 2.12.31 (Tychonoff theorem)**.** *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a non-empty collection of compact topological spaces. Then the product space $X := \prod\limits_{\alpha \in \Lambda} X_\alpha$ is compact.*

Proof. □

Theorem 2.12.32 (Urysohn metrization theorem)**.**

2.12.2 Stone-Čech compactification

2.13 Metrizability

2.13.1 Stone-Weierstrass theorem

2.13.2 Arzela-Ascoli theorem

2.14 Metrization theorem

Theorem 2.14.1 (Nagata-Smrinov)**.**

2.15 Paracompactness

2.15.1 Partition of unity

2.16 Compactification

Chapter 3

Appendix

3.1 Category Theory

Definition 3.1.1. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from* X *to* Y *in* $\mathscr{C}$; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.