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# MA5202: Algebraic Geometry

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*This note will be updated from time to time.  
If there are any potential mistakes, please bring it to my notice.*



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# List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exist
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\Sigma$	Sum
$\Pi$	Product
$\pm$	Plus and/or minus
$\infty$	Infinity
$\sqrt{a}$	Square root of $a$
$\cup$	Union
$\sqcup$	Disjoint union
$\cap$	Intersection
$A \rightarrow B$	$A$ mapping into $B$
$a \mapsto b$	$a$ maps to $b$
$\hookrightarrow$	Inclusion map
$A \setminus B$	$A$ setminus $B$
$\cong$	Isomorphic to
$A := \dots$	$A$ is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
$\alpha$	alpha	$\beta$	beta
$\gamma$	gamma	$\delta$	delta
$\pi$	pi	$\phi$	phi
$\varphi$	var-phi	$\psi$	psi
$\epsilon$	epsilon	$\varepsilon$	var-epsilon
$\zeta$	zeta	$\eta$	eta
$\theta$	theta	$\iota$	iota
$\kappa$	kappa	$\lambda$	lambda
$\mu$	mu	$\nu$	nu
$\upsilon$	upsilon	$\rho$	rho
$\varrho$	var-rho	$\xi$	xi
$\sigma$	sigma	$\tau$	tau
$\chi$	chi	$\omega$	omega
$\Omega$	Capital omega	$\Gamma$	Capital gamma
$\Theta$	Capital theta	$\Delta$	Capital delta
$\Lambda$	Capital lambda	$\Xi$	Capital xi
$\Sigma$	Capital sigma	$\Pi$	Capital pi
$\Phi$	Capital phi	$\Psi$	Capital psi

Some of the useful Greek letters

## Chapter 1

# Basic Theory of Schemes

### 1.1 Classical variety

Let  $k$  be a field and let  $k[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables  $x_1, \dots, x_n$  and coefficients from the field  $k$ . Given a subset  $E \subseteq k[x_1, \dots, x_n]$ , let

$$\mathcal{Z}(E) := \{(a_1, \dots, a_n) \in k^n : f(a_1, \dots, a_n) = 0, \forall f \in E\}$$

be the subset of all common zeros of the polynomials in  $E$ . We are interested to study geometry of  $\mathcal{Z}(E)$ . If  $f \in E$  is a linear polynomial with zero constant term, i.e.,  $f(0, \dots, 0) = 0$ , the map  $T_f : k^n \rightarrow k$  defined by

$$T_f(a_1, \dots, a_n) = f(a_1, \dots, a_n), \quad \forall (a_1, \dots, a_n) \in k^n,$$

is a  $k$ -linear map, and that  $\mathcal{Z}(f) = \text{Ker}(T_f)$  is a  $k$ -linear subspace of  $k^n$ . Therefore, if all the polynomials in  $E$  are linear with zero constant terms, then

$$\mathcal{Z}(E) = \bigcap_{f \in E} \text{Ker}(T_f)$$

is a  $k$ -linear subspace of  $k^n$ , and in this case standard linear algebra machinery could be used to study the space  $\mathcal{Z}(E)$ . However, when  $f \in E$  is not a linear polynomial,  $\mathcal{Z}(f)$  is no longer a linear space, and hence we cannot use linear algebra machinery to study geometry of  $\mathcal{Z}(f)$ . In this situation, the techniques from commutative algebra comes into the picture.

**Proposition 1.1.1.** *Let  $E$  be a non-empty subset of the polynomial ring  $k[x_1, \dots, x_n]$ . Then  $\mathcal{Z}(E) = \mathcal{Z}(\langle E \rangle)$ , where  $\langle E \rangle$  is the ideal of  $k[x_1, \dots, x_n]$  generated by  $E$ .*

*Proof.* Since  $E \subseteq \langle E \rangle$ , it follows from the definition of  $\mathcal{Z}(E)$  that  $\mathcal{Z}(\langle E \rangle) \subseteq \mathcal{Z}(E)$ . Conversely, let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{Z}(E)$ . Let  $f \in \langle E \rangle$  be arbitrary. Then  $f = \sum_{j=1}^m \phi_j f_j$ , for some  $\phi_j \in k[x_1, \dots, x_n]$  and  $f_j \in E$ , for all  $j \in \{1, \dots, m\}$ . Since  $f_j(\mathbf{a}) = 0$ , for all  $j$ , we have  $f(\mathbf{a}) = \sum_{j=1}^m \phi_j(\mathbf{a}) f_j(\mathbf{a}) = 0$ . Therefore,  $\mathbf{a} \in \mathcal{Z}(\langle E \rangle)$ .  $\square$

We now introduce a class of commutative rings with identity for which every ideals are finitely generated. Such a ring is called Noetherian. We show that polynomial ring

$k[x_1, \dots, x_n]$  and its quotient rings are Noetherian. One of the advantage to work with such rings is that all of its ideals being finitely generated, zero locus of a given family of possibly infinitely many polynomials is determined by a finite number of polynomials among them.

Let  $A$  be a commutative ring with identity.

**Definition 1.1.2.** An  $A$ -module  $M$  is said to be *finitely generated* if there exists finitely many elements  $x_1, \dots, x_n \in M$  such that given any  $x \in M$  there exists  $a_1, \dots, a_n \in A$  such that  $x = a_1x_1 + \dots + a_nx_n$ .

**Example 1.1.3.** For each  $n \in \mathbb{N}$ , the  $A$ -module  $A^{\oplus n}$  is finitely generated. Indeed, it is generated by  $\{e_1, \dots, e_n\}$ , where  $e_j \in A^{\oplus n}$  is the ordered  $n$ -tuple of elements of  $A$  whose  $j$ -th coordinate is  $1 \in A$ , and all other coordinates are  $0 \in A$ .

**Example 1.1.4.** Let  $f : M \rightarrow N$  be a surjective  $A$ -module homomorphism. If  $M$  is a finitely generated  $A$ -module, so is  $N$ . Indeed, if  $M$  is generated by  $\{x_1, \dots, x_n\}$  as an  $A$ -module, then  $N$  is generated by  $\{f(x_1), \dots, f(x_n)\}$  as an  $A$ -module.

**Corollary 1.1.5.** An  $A$ -module  $M$  is finitely generated if and only if there exists a surjective  $A$ -module homomorphism  $f : A^{\oplus n} \rightarrow M$ , for some  $n \in \mathbb{N}$ .

Note that, an  $A$ -submodule of a finitely generated  $A$ -module need not be finitely generated. For example, take  $A = k[X_1, X_2, \dots]$  be the polynomial ring over a field  $k$  with countably infinitely many variables  $\{X_n : n \in \mathbb{N}\}$ . Clearly,  $A$  is a commutative ring with identity. Let  $\mathfrak{a}$  be the ideal of  $A$  generated by its variables  $(X_n : n \in \mathbb{N})$ . Clearly  $\mathfrak{a}$  is a non-zero proper ideal of  $A$  (and hence an  $A$ -module), which is clearly not finitely generated.

**Definition 1.1.6.** An  $A$ -module  $M$  is said to be *noetherian* if every  $A$ -submodule of  $M$  is finitely generated. We say that  $A$  is *noetherian* if it is noetherian as a module over itself.

In particular, a noetherian  $A$ -module is a finitely generated  $A$ -module. However, the converse may not be true (see above Example).

**Proposition 1.1.7.**  $A$  is noetherian if and only if every ideal of  $A$  is finitely generated.

*Proof.* Since any  $A$ -submodule of  $A$  is an ideal of  $A$ , the result follows.  $\square$

**Lemma 1.1.8.** Let

$$0 \rightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \rightarrow 0$$

be a short exact sequence of  $A$ -modules. Then  $M$  is noetherian if and only if both  $M'$  and  $M''$  are noetherian.

*Proof.* Suppose that  $M$  is noetherian. Let  $N'$  be an  $A$ -submodule of  $M'$ . Then  $N'$  is isomorphic to the  $A$ -submodule  $\phi(N')$  of  $M$ , and hence is finitely generated. Therefore,  $M'$  is noetherian. Let  $N''$  be an  $A$ -submodule of  $M''$ . Then  $N := \psi^{-1}(N'')$  is an  $A$ -submodule of  $M$ , and so is finitely generated. Since the  $A$ -module homomorphism  $\psi|_N : N \rightarrow N''$  is surjective and  $N$  is finitely generated,  $N''$  is finitely generated. Therefore,  $M''$  is noetherian.



Conversely, suppose that both  $M'$  and  $M''$  are noetherian  $A$ -modules. Let  $N$  be an  $A$ -submodule of  $M$ . Since  $N' := \phi^{-1}(N)$  and  $N'' := \psi(N)$  are  $A$ -submodule of noetherian  $A$ -modules, they are finitely generated. Then we have an exact sequence of  $A$ -modules

$$0 \rightarrow N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \rightarrow 0.$$

Suppose that  $\phi^{-1}(N)$  and  $\psi(N)$  are generated as  $A$ -modules by  $x_1, \dots, x_m \in \phi^{-1}(N)$  and  $y_1, \dots, y_n \in \psi(N)$ , respectively. Fix an element  $z_j \in \psi^{-1}(y_j) \subseteq N$ , for each  $j \in \{1, \dots, n\}$ . Let  $x \in N$  be given. Then  $\psi(x) = b_1 y_1 + \dots + b_n y_n$ , for some  $b_1, \dots, b_n \in A$ . Consider the element  $w = x - (b_1 z_1 + \dots + b_n z_n) \in N$ . Since  $\phi(w) = 0$ , there exists  $a_1, \dots, a_m \in A$  such that  $w = a_1 x_1 + \dots + a_m x_m$ . Then we have  $x = a_1 x_1 + \dots + a_m x_m + b_1 z_1 + \dots + b_n z_n$ . Therefore,  $N$  is generated as an  $A$ -module by  $\{x_1, \dots, x_m\} \cup \{z_1, \dots, z_n\}$ . Therefore,  $M$  is noetherian.  $\square$

**Corollary 1.1.9.** *If  $M$  and  $N$  are noetherian  $A$ -modules, so is  $M \oplus N$ .*

*Proof.* Follows from Lemma 1.1.8.  $\square$

**Corollary 1.1.10.** *Any finitely generated module over a noetherian ring is noetherian.*

*Proof.* Let  $A$  be a noetherian ring and let  $M$  be a finitely generated  $A$ -module. Then there exists a surjective  $A$ -module homomorphism

$$\varphi : A^{\oplus n} \rightarrow M,$$

for some  $n \in \mathbb{N}$ . Since  $A^{\oplus n}$  is noetherian by Corollary 1.1.9, that  $M$  is noetherian by Lemma 1.1.8.  $\square$

**Theorem 1.1.11** (Hilbert's basis theorem). *If  $A$  is a noetherian ring, the polynomial ring  $A[x_1, \dots, x_n]$  is noetherian.*

*Proof.* Since the polynomial ring  $A[x_1, \dots, x_n]$  is isomorphic to the polynomial ring  $B[x_n]$ , where  $B = A[x_1, \dots, x_{n-1}]$ , using induction it suffices to prove the result for  $n = 1$ . Consider the polynomial ring  $A[x]$ . Let  $I \subset A[x]$  be an ideal of  $A[x]$ . Since the cases  $I = 0$  and  $I = A[x]$  are trivial, we assume that  $I \neq 0$  and  $I \neq A[x]$ . Let

$$J = \{0\} \cup \{\text{set of all leading coefficients of non-zero polynomials in } I\}.$$

Clearly  $J$  is an ideal of  $A$ , and hence is finitely generated because  $A$  is noetherian. Let  $c_1, \dots, c_r \in A$  be non-zero elements of  $A$  that generates  $J$  as an ideal of  $A$ . Each  $c_j$  is a leading coefficient of a non-zero element, say  $f_j$ , of  $I$ . Let  $J' = (f_1, \dots, f_r)$  be the ideal of  $A[x]$  generated by  $f_1, \dots, f_r$ . Let  $m := \max_{1 \leq j \leq r} \deg(f_j)$ , and let

$$M := I \cap (A + Ax + \dots + Ax^{m-1}).$$

Then  $M$  is an  $A$ -submodule of  $A[x]$ . We claim that  $I = M + J'$ . Since both  $M$  and  $J'$  are subsets of  $I$  and  $I$  is an ideal,  $M + J' \subseteq I$ . To show the reverse inclusion, we need to show that every  $f \in I$  is in  $M + J'$ . We show this by induction on  $d = \deg(f)$ . If  $d \leq m - 1$ , then  $f \in M \subseteq M + J'$ . Suppose that  $d := \deg(f) \geq m$ , and assume that for

given any  $g \in I$  with  $\deg(g) < d$  we have  $g \in M + J'$ . Let  $c$  be the leading coefficient of  $f$ . Since  $J = (c_1, \dots, c_r)$  and  $c \in J$ , we have  $c = \sum_{j=1}^r a_j c_j$ , for some  $a_1, \dots, a_r \in A$ . Since  $g := f - \sum_{j=1}^r a_j x^{d-\deg(f_j)} f_j \in I$  with  $\deg(g) \leq d-1$ , by induction hypothesis  $g \in M + J'$ . Then  $f = g + \sum_{j=1}^r a_j x^{d-\deg(f_j)} f_j \in M + J'$ , as required. Therefore, by induction  $I = M + J'$ . Since  $A$  is noetherian and  $A + Ax + \dots + Ax^{m-1}$  is a finitely generated  $A$ -module, that  $A + Ax + \dots + Ax^{m-1}$  is a noetherian  $A$ -module by Corollary 1.1.10. Since  $M$  is an  $A$ -submodule of  $A + Ax + \dots + Ax^{m-1}$ ,  $M$  is a finitely generated  $A$ -module, generated by, say  $g_1, \dots, g_n$ . Then  $I = M + J'$  is generated as an  $A[x]$ -module by  $f_1, \dots, f_r, g_1, \dots, g_n$ . This completes the proof.  $\square$

By Hilbert basis theorem, every ideal of  $A = k[x_1, \dots, x_n]$  are finitely generated. Then every generating subset  $E$  of a finitely generated ideal  $\mathfrak{a}$  of  $A$  contains a finite subset that generates the ideal  $\mathfrak{a}$ . Therefore, for every  $E \subseteq A$ , there exists finitely many elements  $f_1, \dots, f_n \in E$  such that  $\mathcal{Z}(E) = \mathcal{Z}(f_1, \dots, f_n) = \bigcap_{j=1}^n \mathcal{Z}(f_j)$ . Note that, given  $E_1 \subseteq E_2 \subseteq A$ , we have  $\mathcal{Z}(E_2) \subseteq \mathcal{Z}(E_1)$ .

**Proposition 1.1.12.** *The sets  $\mathcal{Z}(\mathfrak{a})$ , where  $\mathfrak{a}$  runs over the set of all ideals of  $k[x_1, \dots, x_n]$  satisfy axioms for closed subsets for a topology on  $k^n$ , called the Zariski topology.*

*Proof.* The proposition follows from the following observations.

- (i)  $\mathcal{Z}(1) = \emptyset$  and  $\mathcal{Z}(0) = k^n$ ;
- (ii) Given any family of ideals  $\{\mathfrak{a}_j : j \in I\}$  of  $k[x_1, \dots, x_n]$ , we have

$$\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) = \mathcal{Z}\left(\sum_{j \in I} \mathfrak{a}_j\right);$$

- (iii) given any two ideals  $\mathfrak{a}$  and  $\mathfrak{b}$  of  $k[x_1, \dots, x_n]$ , we have

$$\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{Z}(\mathfrak{a}\mathfrak{b}).$$

The first point is obvious. To see the second, note that

$$\begin{aligned} \bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) &= \bigcap_{j \in I} \{x \in k^n : f(x) = 0, \forall f \in \mathfrak{a}_j\} \\ &= \{x \in k^n : f(x) = 0, \forall f \in \mathfrak{a}_j, \forall j \in I\} \\ &= \mathcal{Z}\left(\bigcup_{j \in I} \mathfrak{a}_j\right) \\ &= \mathcal{Z}\left(\sum_{j \in I} \mathfrak{a}_j\right). \end{aligned}$$

To see the third point, note that  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ , which is a subset of both  $\mathfrak{a}$  and  $\mathfrak{b}$ . Therefore,  $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a}\mathfrak{b})$ . Conversely, if  $x \in \mathcal{Z}(\mathfrak{a}\mathfrak{b})$  and  $x \notin \mathcal{Z}(\mathfrak{a})$ , then there

exists  $f \in \mathfrak{a}$  such that  $f(x) \neq 0$ , and for any  $g \in \mathfrak{b}$  that  $f(x)g(x) = (fg)(x) = 0$ , since  $fg \in \mathfrak{ab}$  and  $x \in \mathcal{Z}(\mathfrak{ab})$ . Since  $k$  is an integral domain, we must have  $g(x) = 0$ , for all  $g \in \mathfrak{b}$ . Therefore,  $x \in \mathcal{Z}(\mathfrak{b})$ . This completes the proof.  $\square$

**Definition 1.1.13.** The set  $k^n$  together with the Zariski topology on it is called the *affine  $n$ -space over  $k$*  and is denoted by  $\mathbb{A}^n(k)$ . A closed subspace of  $\mathbb{A}^n(k)$  is called an *algebraic set*.

Given a point  $a = (a_1, \dots, a_n) \in \mathbb{A}^n(k)$ , consider the evaluation map

$$ev_a : k[x_1, \dots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a_1, \dots, a_n), \quad \forall f \in k[x_1, \dots, x_n].$$

Note that  $ev_a$  is a surjective ring homomorphism with kernel

$$\text{Ker}(ev_a) = \mathfrak{m}_a := (x_1 - a_1, \dots, x_n - a_n).$$

Therefore,  $\{a\} = \mathcal{Z}(\mathfrak{m}_a)$  is a closed subset of  $\mathbb{A}^n(k)$ . As a result, any finite subset of  $\mathbb{A}^n(k)$  is an algebraic set.

**Example 1.1.14.** For  $n = 1$ , the polynomial ring  $k[x]$  is a principal ideal domain. So every ideal of  $k[x]$  is generated by a single polynomial. Since a polynomial in  $k[x]$  has only finite number of roots in  $k$ , any closed subset of  $\mathbb{A}^1(k)$  is either finite or  $\mathbb{A}^1(k)$  itself.

**Example 1.1.15.** For  $n = 2$ , the situation is more complicated. Here is an obvious list of closed subsets of  $\mathbb{A}^2(k)$ .

- $\emptyset$  and  $\mathbb{A}^2(k)$ .
- any finite subset of  $\mathbb{A}^2(k)$ .
- $\mathcal{Z}(f)$ , where  $f \in k[x_1, x_2]$  is an irreducible polynomial.

In fact, we shall see later that the non-empty closed subsets of  $\mathbb{A}^2(k)$  listed above are of the form  $\mathcal{Z}(\mathfrak{p})$ , for some prime ideal  $\mathfrak{p}$  of  $k[x_1, x_2]$ . Moreover, any closed subsets of  $\mathbb{A}^2(k)$  is a finite union of the closed subsets of the form listed above.

Connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz and its corollaries.

**Theorem 1.1.16** (Hilbert's Nullstellensatz). *Let  $k$  be a field that is not necessarily algebraically closed, and let  $A$  be a finitely generated  $k$ -algebra. Then  $A$  is Jacobson; i.e., every prime ideal  $\mathfrak{p} \in \text{Spec}(A)$  is an intersection of all maximal ideals of  $A$  containing  $\mathfrak{p}$ .*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{p})} \mathfrak{m},$$

where  $V_{\max}(\mathfrak{p})$  is the set of all maximal ideals of  $A$  containing  $\mathfrak{p}$ .

*Proof.*

$\square$