MA5202: Algebraic Geometry

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This note will be updated from time to time.
If there are any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\bigsqcup$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
A := ...	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Basic Theory of Schemes

1.1 Affine algebraic set

Let k be a field and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ and coefficients from the field k. Given a subset $E \subseteq k[x_1, \ldots, x_n]$, let

$$\mathcal{Z}(E) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0, \ \forall\, f \in E\}$$

be the subset of all common zeros of the polynomials in E. We are interested to study geometry of $\mathcal{Z}(E)$. If $f \in E$ is a linear polynomial with zero constant term, i.e., $f(0, \ldots, 0) = 0$, the map $T_f : k^n \to k$ defined by

$$T_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n), \ \ \forall\, (a_1, \ldots, a_n) \in k^n,$$

is a k-linear map, and that $\mathcal{Z}(f) = \mathrm{Ker}(T_f)$ is a k-linear subspace of k^n. Therefore, if all the polynomials in E are linear with zero constant terms, then

$$\mathcal{Z}(E) = \bigcap_{f \in E} \mathrm{Ker}(T_f)$$

is a k-linear subspace of k^n, and in this case standard linear algebra machinery could be used to study the space $\mathcal{Z}(E)$. However, when $f \in A$ is not a linear polynomial, $\mathcal{Z}(f)$ is no longer a liner space, and hence we cannot use linear algebra machinery to study geometry of $\mathcal{Z}(f)$. In this situation, the techniques from commutative algebra comes into the picture.

Proposition 1.1.1. *Let E be a non-empty subset of the polynomial ring $k[x_1, \ldots, x_n]$. Then $\mathcal{Z}(E) = \mathcal{Z}(\langle E \rangle)$, where $\langle E \rangle$ is the ideal of $k[x_1, \ldots, x_n]$ generated by E.*

Proof. Since $E \subseteq \langle E \rangle$, it follows from the definition of $\mathcal{Z}(E)$ that $\mathcal{Z}(\langle E \rangle) \subseteq \mathcal{Z}(E)$. Conversely, let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{Z}(E)$. Let $f \in \langle E \rangle$ be arbitrary. Then $f = \sum\limits_{j=1}^{m} \phi_j f_j$, for some $\phi_j \in k[x_1, \ldots, x_n]$ and $f_j \in E$, for all $j \in \{1, \ldots, m\}$. Since $f_j(\mathbf{a}) = 0$, for all j, we have $f(\mathbf{a}) = \sum\limits_{j=1} \phi_j(\mathbf{a}) f_j(\mathbf{a}) = 0$. Therefore, $\mathbf{a} \in \mathcal{Z}(\langle E \rangle)$. □

We now introduce a class of commutative rings with identity for which every ideals are finitely generated. Such a ring is called Noetherian. We show that polynomial ring

$k[x_1, \ldots, x_n]$ and its quotient rings are Noetherian. One of the advantage to work with such rings is that all of its ideals being finitely generated, zero locus of a given family of possibly infinitely many polynomials is determined by a finite number of polynomials among them.

Let A be a commutative ring with identity.

Definition 1.1.2. An A-module M is said to be *finitely generated* if there exists finitely many elements $x_1, \ldots, x_n \in M$ such that given any $x \in M$ there exists $a_1, \ldots, a_n \in A$ such that $x = a_1x_1 + \cdots + a_nx_n$.

Example 1.1.3. For each $n \in \mathbb{N}$, the A-module $A^{\oplus n}$ is finitely generated. Indeed, it is generated by $\{e_1, \ldots, e_n\}$, where $e_j \in A^{\oplus n}$ is the ordered n-tuple of elements of A whose j-th coordinate is $1 \in A$, and all other coordinates are $0 \in A$.

Example 1.1.4. Let $f : M \to N$ be a surjective A-module homomorphism. If M is a finitely generated A-module, so is N. Indeed, if M is generated by $\{x_1, \ldots, x_n\}$ as an A-module, then N is generated by $\{f(x_1), \ldots, f(x_n)\}$ as an A-module.

Corollary 1.1.5. *An A-module M is finitely generated if and only if there exists a surjective A-module homomorphism $f : A^{\oplus n} \to M$, for some $n \in \mathbb{N}$.*

Note that, an A-submodule of a finitely generated A-module need not be finitely generated. For example, take $A = k[X_1, X_2, \ldots]$ be the polynomial ring over a field k with countably infinitely many variables $\{X_n : n \in \mathbb{N}\}$. Clearly, A is a commutative ring with identity. Let $\mathfrak{a}$ be the ideal of A generated by its variables $(X_n : n \in \mathbb{N})$. Clearly $\mathfrak{a}$ is a non-zero proper ideal of A (and hence an A-module), which is clearly not finitely generated.

Definition 1.1.6. An A-module M is said to be *noetherian* if every A-submodule of M is finitely generated. We say that A is *noetherian* if it is noetherian as a module over itself.

In particular, a noetherian A-module is a finitely generated A-module. However, the converse may not be true (see above Example).

Proposition 1.1.7. *A is noetherian if and only if every ideal of A is finitely generated.*

Proof. Since any A-submodule of A is an ideal of A, the result follows. □

Lemma 1.1.8. *Let*

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

be a short exact sequence of A-modules. Then M is noetherian if and only if both M' and M'' are noetherian.

Proof. Suppose that M is noetherian. Let N' be an A-submodule of M'. Then N is isomorphic to the A-submodule $\phi(N')$ of M, and hence is finitely generated. Therefore, M' is noetherian. Let N'' be an A-submodule of M''. Then $N := \psi^{-1}(N'')$ is an A-submodule of M, and so is finitely generated. Since the A-module homomorphism $\psi|_N : N \to N''$ is surjective and N is finitely generated, N'' is finitely generated. Therefore, M'' is noetherian.

Conversely, suppose that both M' and M'' are noetherian A-modules. Let N be an A-submodule of M. Since $N' := \phi^{-1}(N)$ and $N'' := \psi(N)$ are A-submodule of noetherian A-modules, they are finitely generated. Then we have an exact sequence of A-modules

$$0 \to N' \xrightarrow{\phi} N \xrightarrow{\psi} N'' \to 0.$$

Suppose that $\phi^{-1}(N)$ and $\psi(N)$ are generated as A-modules by $x_1, \ldots, x_m \in \phi^{-1}(N)$ and $y_1, \ldots, y_n \in \psi(N)$, respectively. Fix an element $z_j \in \psi^{-1}(y_j) \subseteq N$, for each $j \in \{1, \ldots, n\}$. Let $x \in N$ be given. Then $\psi(x) = b_1y_1 + \cdots + b_ny_n$, for some $b_1, \ldots, b_n \in A$. Consider the element $w = x - (b_1z_1 + \cdots + b_nz_n) \in N$. Since $\psi(w) = 0$, there exists $a_1, \ldots, a_m \in A$ such that $w = a_1x_1 + \cdots + a_mx_m$. Then we have $x = a_1x_1 + \cdots + a_mx_m + b_1z_1 + \cdots + b_nz_n$. Therefore, N is generated as an A-module by $\{x_1, \ldots, x_m\} \cup \{z_1, \ldots, z_n\}$. Therefore, M is noetherian. □

Corollary 1.1.9. *If M and N are noetherian A-modules, so is $M \oplus N$.*

Proof. Follows from Lemma 1.1.8. □

Corollary 1.1.10. *Any finitely generated module over a noetherian ring is noetherian.*

Proof. Let A be a noetherian ring and let M be a finitely generated A-module. Then there exists a surjective A-module homomorphism

$$\varphi : A^{\oplus n} \to M,$$

for some $n \in \mathbb{N}$. Since $A^{\oplus n}$ is noetherian by Corollary 1.1.9, that M is noetherian by Lemma 1.1.8. □

Theorem 1.1.11 (Hilbert's basis theorem)**.** *If A is a noetherian ring, the polynomial ring $A[x_1, \ldots, x_n]$ is noetherian.*

Proof. Since the polynomial ring $A[x_1, \ldots, x_n]$ is isomorphic to the polynomial ring $B[x_n]$, where $B = A[x_1, \ldots, x_n]$, using induction it suffices to prove the result for $n = 1$. Consider the polynomial ring $A[x]$. Let $I \subset A[x]$ be an ideal of $A[x]$. Since the cases $I = 0$ and $I = A[x]$ are trivial, we assume that $I \neq 0$ and $I \neq A[x]$. Let

$$J = \{0\} \cup \{\text{set of all leading coefficients of non-zero polynomials in } I\}.$$

Clearly J is an ideal of A, and hence is finitely generated because A is noetherian. Let $c_1, \ldots, c_r \in A$ be non-zero elements of A that generates J as an ideal of A. Each c_j is a leading coefficient of a non-zero element, say f_j, of I. Let $J' = (f_1, \ldots, f_r)$ be the ideal of $A[x]$ generated by $f_1, \ldots, f_r$. Let $m := \max\limits_{1 \leq j \leq r} \deg(f_j)$, and let

$$M := I \cap \left(A + Ax + \cdots + Ax^{m-1}\right).$$

Then M is an A-submodule of $A[x]$. We claim that $I = M + J'$. Since both M and J' are subsets of I and I is an ideal, $M + J' \subseteq I$. To show the reverse inclusion, we need to show that every $f \in I$ is in $M + J'$. We show this by induction on $d = \deg(f)$. If $d \leq m - 1$, then $f \in M \subseteq M + J'$. Suppose that $d := \deg(f) \geq m$, and assume that for given any $g \in I$ with $\deg(g) < d$ we have $g \in M + J'$. Let c be the leading coefficient of f. Since $J = (c_1, \dots, c_r)$ and $c \in J$, we have $c = \sum_{j=1}^{r} a_j c_j$, for some $a_1, \dots, a_r \in A$. Since $g := f - \sum_{j=1}^{r} a_j x^{d-\deg(f_j)} f_j \in I$ with $\deg(g) \le d - 1$, by induction hypothesis $g \in M + J'$. Then $f = g + \sum_{j=1}^{r} a_j x^{d-\deg(f_j)} f_j \in M + J'$, as required. Therefore, by induction $I = M + J'$. Since A is noetherian and $A + Ax + \cdots + Ax^{m-1}$ is a finitely generated A-module, that $A + Ax + \cdots + Ax^{m-1}$ is a noetherian A-module by Corollary 1.1.10. Since M is an A-submodule of $A + Ax + \cdots + Ax^{m-1}$, M is a finitely generated A-module, generated by, say $g_1, \dots, g_n$. Then $I = M + J'$ is generated as an $A[x]$-module by $f_1, \dots, f_r, g_1, \dots, g_n$. This completes the proof. □

By Hilbert basis theorem, every ideal of $A = k[x_1, \dots, x_n]$ are finitely generated. Then every generating subset E of a finitely generated ideal $\mathfrak{a}$ of A contains a finite subset that generates the ideal $\mathfrak{a}$. Therefore, for every $E \subseteq A$, there exists finitely many elements $f_1, \dots, f_n \in E$ such that $\mathcal{Z}(E) = \mathcal{Z}(f_1, \dots, f_n) = \bigcap_{j=1}^{n} \mathcal{Z}(f_j)$. Note that, given $E_1 \subseteq E_2 \subseteq A$, we have $\mathcal{Z}(E_2) \subseteq \mathcal{Z}(E_1)$.

Proposition 1.1.12. *The sets $\mathcal{Z}(\mathfrak{a})$, where $\mathfrak{a}$ runs over the set of all ideals of $k[x_1, \dots, x_n]$ satisfy axioms for closed subsets for a topology on k^n, called the Zariski topology.*

Proof. The proposition follows from the following observations.

(i) $\mathcal{Z}(1) = \emptyset$ and $\mathcal{Z}(0) = k^n$;

(ii) Given any family of ideals $\{\mathfrak{a}_j : j \in I\}$ of $k[x_1, \dots, x_n]$, we have

$$\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) = \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big);$$

(iii) given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $k[x_1, \dots, x_n]$, we have

$$\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{Z}(\mathfrak{a}\mathfrak{b}).$$

The first point is obvious. To see the second, note that

$$\begin{aligned}
\bigcap_{j \in I} \mathcal{Z}(\mathfrak{a}_j) &= \bigcap_{j \in I} \{x \in k^n : f(x) = 0,\ \forall\, f \in \mathfrak{a}_j\} \\
&= \{x \in k^n : f(x) = 0,\ \forall\, f \in \mathfrak{a}_j,\ \forall\, j \in I\} \\
&= \mathcal{Z}\big(\bigcup_{j \in I} \mathfrak{a}_j\big) \\
&= \mathcal{Z}\big(\sum_{j \in I} \mathfrak{a}_j\big).
\end{aligned}$$

To see the third point, note that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, which is a subset of both $\mathfrak{a}$ and $\mathfrak{b}$. Therefore, $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{Z}(\mathfrak{a}\mathfrak{b})$. Conversely, if $x \in \mathcal{Z}(\mathfrak{a}\mathfrak{b})$ and $x \notin \mathcal{Z}(\mathfrak{a})$, then there exists $f \in \mathfrak{a}$ such that $f(x) \neq 0$, and for any $g \in \mathfrak{b}$ that $f(x)g(x) = (fg)(x) = 0$, since $fg \in \mathfrak{ab}$ and $x \in \mathcal{Z}(\mathfrak{ab})$. Since k is an integral domain, we must have $g(x) = 0$, for all $g \in \mathfrak{b}$. Therefore, $x \in \mathcal{Z}(\mathfrak{b})$. This completes the proof. □

Definition 1.1.13. The set k^n together with the Zariski topology on it is called the *affine n-space over k* and is denoted by $\mathbb{A}^n(k)$. A closed subspace of $\mathbb{A}^n(k)$ is called an *algebraic set*.

Given a point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$, consider the evaluation map

$$ev_a : k[x_1, \ldots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a_1, \ldots, a_n), \ \ \forall f \in k[x_1, \ldots, x_n].$$

Note that ev_a a surjective ring homomorphism with kernel

$$\mathrm{Ker}(ev_a) = \mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n).$$

Therefore, $\{a\} = \mathcal{Z}(\mathfrak{m}_a)$ is a closed subset of $\mathbb{A}^n(k)$. As a result, any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.

Example 1.1.14. For $n = 1$, the polynomial ring $k[x]$ is a principal ideal domain. So every ideal of $k[x]$ is generated by a single polynomial. Since a polynomial in $k[x]$ has only finite number of roots in k, any closed subset of $\mathbb{A}^1(k)$ is either finite or $\mathbb{A}^1(k)$ itself.

Example 1.1.15. For $n = 2$, the situation is more complicated. Here is an obvious list of closed subsets of $\mathbb{A}^2(k)$.

- $\emptyset$ and $\mathbb{A}^2(k)$.
- any finite subset of $\mathbb{A}^2(k)$.
- $\mathcal{Z}(f)$, where $f \in k[x_1, x_2]$ is an irreducible polynomial.

In fact, we shall see later that the no-empty closed subsets of $\mathbb{A}^2(k)$ listed above are of the form $\mathcal{Z}(\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of $k[x_1, x_2]$. Moreover, any closed subsets of $\mathbb{A}^2(k)$ is a finite union of the closed subsets of the form listed above.

Connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz and its corollaries.

Theorem 1.1.16 (Hilbert's Nullstellensatz)**.** *Let K be a field that is not necessarily algebraically closed, and let A be a finitely generated K-algebra. Then A is Jacobson; i.e., every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is an intersection of all maximal ideals of A containing $\mathfrak{p}$.*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{p})} \mathfrak{m},$$

where $V_{\max}(\mathfrak{p})$ is the set of all maximal ideals of A containing $\mathfrak{p}$. Moreover, if $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite degree field extension of K.

Before proving Hilbert's Nullstellensatz, lets discuss some of its consequences.

Corollary 1.1.17. *Let k be an algebraically closed field, and let A be a finitely generated k-algebra.*

(i) Then $A/\mathfrak{m} = k$, for every maximal ideal $\mathfrak{m}$ of A.

(ii) Let $\mathfrak{m}$ be a maximal ideal of the polynomial ring $k[x_1, \ldots, x_n]$. Then there exists $(a_1, \ldots, a_n) \in \mathbb{A}^n(k)$ such that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

Proof. (i) Since the ring homomorphism $k \to A/\mathfrak{m}$ is a finite degree field extension of k by Hilbert's Nullstellensatz (Theorem 1.1.16), $A/\mathfrak{m}$ is an algebraic field extension of k. Since k is algebraically closed, we have $k \to A/\mathfrak{m}$ is an isomorphism of rings.

(ii) Let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is a finitely generated k-algebra and k is algebraically closed, by part (i) the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is isomorphic to the field k. Note that the quotient map

$$\varphi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/\mathfrak{m} = k$$

is a k-algebra homomorphism. For each $i \in \{1, \ldots, n\}$, let $a_i = \varphi(x_i) \in k$. Then $\varphi(x_j - a_j) = 0,\ \forall\, j \in \{1, \ldots, n\}$. Therefore, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is contained in the maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$. Since $(x_1 - a_1, \ldots, x_n - a_n)$ is also maximal ideal of $k[x_1, \ldots, x_n]$, it follows that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. □

Corollary 1.1.18. *Let k be an algebraically closed field. Let $\mathfrak{a}$ be an ideal of $k[x_1, \ldots, x_n]$, and let $\mathcal{Z}(\mathfrak{a})$ be the algebraic subset of $\mathbb{A}^n(k)$ defined by $\mathfrak{a}$. Then there is a one-to-one correspondence between the points of $\mathcal{Z}(\mathfrak{a})$ and the set of all maximal ideals of $k[x_1, \ldots, x_n]/\mathfrak{a}$.*

Proof. Let A be the quotient ring $k[x_1, \ldots, x_n]/\mathfrak{a}$. By correspondence theorem, the maximal ideals of A are in one-to-one correspondence with the maximal ideals of $k[x_1, \ldots, x_n]$ containing $\mathfrak{a}$. Let $a = (a_1, \ldots, a_n) \in \mathcal{Z}(\mathfrak{a})$ be given. Since $\mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$ is the kernel of the surjective ring homomorphism

$$ev_a : k[x_1, \ldots, x_n] \longrightarrow k$$

defined by

$$ev_a(f) = f(a),\ \ \forall\, f \in k[x_1, \ldots, x_n],$$

$\mathfrak{m}_a$ is a maximal ideal of $k[x_1, \ldots, x_n]$. Since $a \in \mathcal{Z}(\mathfrak{a})$, we have $ev_a(f) = f(a) = 0$, $\forall\, f \in \mathfrak{a}$. Therefore, $\mathfrak{a} \subseteq \mathrm{Ker}(ev_a) = \mathfrak{m}_a$. Let $\mathrm{MaxSpec}(A)$ be the set of all maximal ideals of $A = k[x_1, \ldots, x_n]/\mathfrak{a}$. Thus we get a map

$$\psi : \mathcal{Z}(\mathfrak{a}) \longrightarrow \mathrm{MaxSpec}(A)$$

defined by sending $a \in \mathcal{Z}(\mathfrak{a})$ to the maximal ideal $\overline{\mathfrak{m}_a}$ of A associated to $\mathfrak{m}_a$. Clearly ψ is injective by construction. To see ψ is surjective, note that a maximal ideal of A is given by a maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ containing $\mathfrak{a}$. Since k is algebraically closed, by Hilbert's Nullstellensatz (Corollary 1.1.17) we have $\mathfrak{m} = \mathfrak{m}_a$ for some $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$. Since $\mathfrak{a} \subseteq \mathfrak{m}_a$, given any $f \in \mathfrak{a}$, there exists $g_1, \ldots, g_n \in k[x_1, \ldots, x_n]$ such that

$f = \sum_{i=1}^{n}(x_i - a_i)g_i$. Then $f(a) = 0$. Therefore, $a \in \mathcal{Z}(\mathfrak{a})$, and hence ψ is surjective. This completes the proof. □

We now give a proof of Theorem 1.1.16 (Hilbert's Nullstellensatz). The main ingredient is Noether's normalization lemma (for a proof, see e.g. Basic Commutative Algebra book by Balwant Singh).

Definition 1.1.19. Let $\phi : A \to B$ be a ring homomorphism. An element $\beta \in B$ is said to be *integral over* A if there exists a non-constant monic polynomial $x^n + a_1x^{n-1} + \cdots + a_n \in A[x]$ such that $\beta^n + \phi(a_1)\beta^{n-1} + \cdots + \phi(a_n) = 0$. If every element of B is integral over A, then B is called *integral* over A.

Lemma 1.1.20. *Let $\phi : A \to B$ be a ring homomorphism. Then B is a finitely generated A-algebra that is integral over A if and only if B is a finitely generated A-module.*

Definition 1.1.21. Let K be a field and let A be a finitely generated K-algebra. A finite subset $\{a_1, \ldots, a_n\}$ of A is said to be *algebraically dependent over* K if there exists a non-zero polynomial $f \in K[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$. If $\{a_1, \ldots, a_n\} \subset A$ is said to be *algebraically independent over* K if it is not algebraically dependent over K.

Theorem 1.1.22 (Noether's Normalization lemma)**.** *Let K be a field (not necessarily algebraically closed), and let $A \neq 0$ be a finitely generated K-algebra. Then there exists an integer $n \geq 0$ and $t_1, \ldots, t_n \in A$ such that the K-algebra homomorphism*

$$K[x_1, \ldots, x_n] \longrightarrow A, \;\; x_j \mapsto t_j, \; \forall\, j,$$

is injective, and A is a finitely generated $K[x_1, \ldots, x_n]$-algebra that is integral over $K[x_1, \ldots, x_n]$.

To deduce Hilbert's Nullstellensatz from Noether's normalization lemma, we need the following two lemmas.

Lemma 1.1.23. *Let A and B be integral domains and let $A \to B$ be an injective ring homomorphism. If B is integral over A, then A is a field if and only if B is a field.*

Proof. Suppose that A is a field. Let $b \in B \setminus \{0\}$. Since b is integral over A, $A[b]$ is a finite dimensional A-vector space. Since B is an integral domain, the multiplication by b map

$$\mu_b : A[b] \to A[b], \;\; f(b) \mapsto bf(b)$$

is injective. Clearly μ_b is A-linear, and hence is bijective. Then $bf(b) = 1$, for some $f(b) \in A[b]$, and hence b is a unit in $A[b]$. Thus B is a field.

Conversely, suppose that B is a field. Let $a \in A \setminus \{0\}$. Let $b = a^{-1}$ in B. Since B is integral over A, there exists $a_1, \ldots, a_n \in A$ such that $b^n + a_1b^{n-1} + \cdots + a_n = 0$. Multiplying both sides by $a^{n-1} \neq 0$ and using the fact that $ab = 1$, we see that

$$b = -(a_1 + a_2b + \cdots + a_na^{-1}) \in A.$$

Therefore, A is a field. □

Lemma 1.1.24. *Let K and L be a fields such that L is a finitely generated K-algebra. Then L is a finite degree field extension of K.*

Proof. Since L is a finitely generated K-algebra, by Noether's normalization lemma (Theorem 1.1.22) there exists an injective K-algebra homomorphism $\varphi : K[x_1, \ldots, x_n] \to L$ that makes L integral over $K[x_1, \ldots, x_n]$, for some integer $n \geq 0$. Since L is a field, by Lemma 1.1.23 we conclude that $n = 0$. Then L is a finitely generated K-algebra that is integral over K, and hence L is a finitely generated K-vector space. Therefore, L is a finite degree field extension of K. □

Proof of Hilbert's Nullstellensatz (Theorem 1.1.16). Let A be a finitely generated K-algebra. To see the second part, let $\mathfrak{m}$ be a maximal ideal of A. Since the composite map

$$K \longrightarrow A \xrightarrow{\pi} A/\mathfrak{m}$$

is a non-zero K-algebra homomorphism, $A/\mathfrak{m}$ is a field extension of K. Since $A/\mathfrak{m}$ is a finitely generated K-algebra, it follows from Lemma 1.1.24 that $A/\mathfrak{m}$ is a finite degree field extension of K.

For the first part, we begin with the following remark. If L is a finite degree field extension of K and $\varphi : A \to L$ is a K-algebra homomorphism, the image of φ is an integral domain that is finitely generated K-vector space. Then $\varphi(A)$ is a field by Lemma 1.1.24. Therefore, $\mathrm{Ker}(\varphi)$ is a maximal ideal of A.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Replacing A by $A/\mathfrak{p}$, if required, it suffices to show that if A is an integral domain that is a finitely generated K-algebra then intersection of all maximal ideals of A is the zero ideal. For this we show that, given any $\alpha \in A \setminus \{0\}$ there is a maximal ideal $\mathfrak{m}$ of A such that $\alpha \notin \mathfrak{m}$. Note that, for $\alpha \neq 0$ in A, the ring $A[\alpha^{-1}] \subseteq Q(A)$ is a non-zero finitely generated K-algebra. Let $\mathfrak{n}$ be a maximal ideal of $A[\alpha^{-1}]$. Then $L := A[\alpha^{-1}]/\mathfrak{n}$ is a finite degree field extension of K by the second assertion. Then the kernel of the composite map

$$\varphi : A \to A[\alpha^{-1}] \to L$$

is a maximal ideal, say $\mathfrak{m}$, of A by above remark. Clearly $\alpha \notin \mathfrak{m}$. □

Let A be a commutative ring with identity. Given an ideal $\mathfrak{a}$ of A, the subset

$$\mathrm{rad}(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a}, \text{ for some } n \in \mathbb{N}\}$$

forms an ideal of A, called the *radical of* $\mathfrak{a}$ *in* A. Note that, $\mathfrak{a} \subseteq \mathrm{rad}(\mathfrak{a})$. If $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$, then we call $\mathfrak{a}$ a *radical ideal* of A. For example, prime ideals are radical ideals.

Exercise 1.1.25. Let A be a commutative ring with identity. Given an ideal $\mathfrak{a}$ of A, let $V(\mathfrak{a})$ be the set of all prime ideals of A containing $\mathfrak{a}$. Show that $\mathrm{rad}(\mathfrak{a}) = \bigcap\limits_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

Exercise 1.1.26. Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients for a field k. Let $\mathfrak{a}$ be an ideal of A. Use Hilbert's Nullstellensatz to show that

$$\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{m} \in V_{\max}(\mathfrak{a})} \mathfrak{m},$$

where $V_{\max}(\mathfrak{a})$ is the set of all maximal ideals of A containing $\mathfrak{a}$.

Exercise 1.1.27. Given an ideal $\mathfrak{a}$ of the polynomial ring $k[x_1, \ldots, x_n]$, show that $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}(\mathrm{rad}(\mathfrak{a}))$.

Definition 1.1.28. Given an algebraic subset Z of $\mathbb{A}^n(k)$, the subset

$$\mathcal{I}(Z) := \{f \in k[x_1, \ldots, x_n] : f(a) = 0, \forall\, a \in Z\}$$

is an ideal of $k[x_1, \ldots, x_n]$, called the *ideal of polynomials vanishing on Z*.

Let Z be an algebraic subset of $\mathbb{A}^n(k)$. Since for given any $f \in k[x_1, \ldots, x_n]$ and $a \in \mathbb{A}^n(k)$, we have $f(a) = 0$ if and only if $f \in \mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$, it follows that $\mathcal{I}(Z) = \bigcap\limits_{a \in Z} \mathfrak{m}_a$.

Proposition 1.1.29. *(i) Let $\mathfrak{a}$ be an ideal of $k[x_1, \ldots, x_n]$. Then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.*

(ii) Let $Z \subseteq \mathbb{A}^n(k)$ be a subset (not necessarily Zariski closed), and let $\overline{Z}$ be its closure in $\mathbb{A}^n(k)$. Then $\mathcal{Z}(\mathcal{I}(Z)) = \overline{Z}$.

Proof. (i) Since k is algebraically closed and $k[x_1, \ldots, x_n]$ is a finitely generated k-algebra, it follows from Hilbert's Nullstellensatz (Corollary 1.1.17) that every prime ideal $\mathfrak{p}$ of $A := k[x_1, \ldots, x_n]$ is the intersection of all maximal ideals of A containing $\mathfrak{p}$. Since $\mathrm{rad}(\mathfrak{a}) = \bigcap\limits_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$, it follows from Corollary 1.1.17 that $\mathrm{rad}(\mathfrak{a}) = \bigcap\limits_{\mathfrak{m} \in V_{\max}(\mathfrak{a})} \mathfrak{m}$, where $V_{\max}(\mathfrak{a})$ is the set of all maximal ideals of A containing $\mathfrak{a}$.

Let $a \in Z$ be given. Then for any $f \in \mathcal{I}(Z)$ we have $f(a) = 0$. and so $a \in \mathcal{Z}(\mathcal{I}(Z))$. Thus $Z \subseteq \mathcal{Z}(\mathcal{I}(Z))$. Since $\mathcal{Z}(\mathcal{I}(Z))$ is closed in $\mathbb{A}^n(k)$, we have $\overline{Z} \subseteq \mathcal{Z}(\mathcal{I}(Z))$. To show the converse, it suffices to show that any closed subset of $\mathbb{A}^n(k)$ containing Z contains $\mathcal{Z}(\mathcal{I}(Z))$. Let W be any closed subset of $\mathbb{A}^n(k)$ containing Z. Then $W = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. Then $\mathrm{rad}(\mathfrak{a}) = \mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subseteq \mathcal{I}(Z)$. Then $\mathcal{Z}(\mathcal{I}(Z)) \subseteq \mathcal{Z}(\mathrm{rad}(\mathfrak{a})) = \mathcal{Z}(\mathfrak{a}) = W$. Therefore, $\overline{Z} = \mathcal{Z}(\mathcal{I}(Z))$. □

Definition 1.1.30. Given an affine algebraic subset $X \subseteq \mathbb{A}^n(k)$ the quotient ring

$$k[X] := k[x_1, \ldots, x_n]/\mathcal{I}(X)$$

is called the *affine coordinate ring* of X.

Definition 1.1.31. A commutative ring A with identity $1 \neq 0$ is said to be *reduced* if $\mathrm{Nil}(A) = 0$.

Exercise 1.1.32. Let A be a commutative ring with identity. Let $\mathfrak{a}$ be an ideal of A. Show that, $\mathfrak{a} = \mathrm{rad}(\mathfrak{a})$ if and only if A/I is reduced.

Corollary 1.1.33. *Let X be an affine algebraic subset of $\mathbb{A}^n(k)$. Then its coordinate ring $k[X]$ is a reduced finitely generated k-algebra.*

Proof. Since X is a closed subset of $\mathbb{A}^n(k)$, the ideal of polynomials $\mathcal{I}(X) \subseteq k[x_1, \ldots, x_n]$ is a radical ideal by Proposition 1.1.29 (i). Then by Exercise 1.1.32 the coordinate ring $k[X] = k[x_1, \ldots, x_n]/\mathcal{I}(X)$ of X is reduced, and hence the result follow. □

Lemma 1.1.34. *Let $\mathfrak{a}$ be an ideal of a commutative ring A with identity, and let $A/\mathfrak{a}$ be the associated quotient ring. Then there is a one-to-one correspondence between the set of all radical ideals of A containing $\mathfrak{a}$ and the set of all radical ideals of $A/\mathfrak{a}$.*

Proof. Let $\pi : A \to A/\mathfrak{a}$ be the natural surjective ring homomorphism. Given an ideal I of A containing $\mathfrak{a}$, its image $I/\mathfrak{a}$ is an ideal of $A/\mathfrak{a}$. This gives a one-to-one correspondence between the set of all ideals of A containing $\mathfrak{a}$ and the set of all ideals of $A/\mathfrak{a}$. By third isomorphism theorem we have an isomorphism of quotient rings $(A/\mathfrak{a})/(I/\mathfrak{a}) \cong A/I$. Then $\mathrm{Nil}\left((A/\mathfrak{a})/(I/\mathfrak{a})\right) \cong \mathrm{Nil}(A/I)$. Therefore, I is a radical ideal of A if and only if $I/\mathfrak{a}$ is a radical ideal of $A/\mathfrak{a}$. Hence the result follows. □

Corollary 1.1.35. *Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic subset with affine coordinate ring $k[X]$. Equip X with the subspace topology induced from the Zariski topology on $\mathbb{A}^n(k)$. Then there is a one-to-one correspondence between the set of all closed subsets of X and the set of all radical ideals of $k[X]$.*

Proof. Let Z be a closed subset of X. Since X is closed in $\mathbb{A}^n(k)$, so is Z. Then $Z = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. Since $\mathcal{Z}(\mathrm{rad}(\mathfrak{a})) = \mathcal{Z}(\mathfrak{a})$, we may assume that $\mathfrak{a} = \mathrm{rad}(\mathfrak{a})$. Since $Z \subseteq X$ if and only if $\mathcal{I}(X) \subseteq \mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$, using Lemma 1.1.34 we have an one-to-one correspondence between the set of all closed subsets of X and the set of all radical ideals of $k[X]$. □

Definition 1.1.36. Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be two affine algebraic subsets. A *morphism* from X into Y is a map $\varphi : X \to Y$ such that there exists polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ such that $\varphi(a) = (f_1(a), \ldots, f_n(a)),\ \forall\, a \in X$. The set of all morphisms from X into Y is denoted by $\mathrm{Hom}(X, Y)$.

Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic subsets. Let $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ be two ordered n-tuples of polynomials in $k[x_1, \ldots, x_m]$ defining morphisms φ and ψ from X into Y. Then $\varphi = \psi$ if and only if for all $j \in \{1, \ldots, n\}$ we have

$$\begin{aligned} &(f_j - g_j)(a) = 0, \quad \forall\, a \in X \\ \Leftrightarrow\ & f_j - g_j \in \mathcal{I}(X) \\ \Leftrightarrow\ & \overline{f_j} = \overline{g_j} \text{ in } \ k[X]. \end{aligned}$$

Proposition 1.1.37. *Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be two affine algebraic subsets. Then there is a one-to-one correspondence between $\mathrm{Hom}(X, Y)$ and the set of all k-algebra homomorphisms from $k[Y]$ into $k[X]$.*

Proof. Let $\varphi = (f_1, \ldots, f_n) \in \mathrm{Hom}(X, Y)$ be given. Define a k-algebra homomorphism

$$\widetilde{\varphi} : k[y_1, \ldots, y_n] \to k[X] = k[x_1, \ldots, x_m]/\mathcal{I}(X)$$

by sending y_j to the image of f_j in the coordinate ring $k[X]$ under the quotient map $k[x_1, \ldots, x_m] \to k[X]$, for all $j = 1, \ldots, n$. Note that $g \in \mathrm{Ker}(\widetilde{\varphi})$ if and only if $g(f_1, \ldots, f_n) \in \mathcal{I}(X)$ if and only if $g(\varphi(a)) = g(f_1(a), \ldots, f_n(a)) = 0,\ \forall\, a \in X$ if and only if $g \in \mathcal{I}(\varphi(X))$. In particular, $\mathrm{Ker}(\widetilde{\varphi}) = \mathcal{I}(Y)$ if and only if φ is surjective. Since $\varphi(X) \subseteq Y$,

we have $\mathcal{I}(Y) \subseteq \mathcal{I}(\varphi(X))$. Therefore, $\mathcal{I}(Y) \subseteq \mathrm{Ker}(\widetilde{\varphi})$. Then there is a unique k-algebra homomorphism
$$\varphi^{\#} : k[Y] \to k[X]$$
such that the following diagram commutes.

$$\begin{tikzcd} k[y_1, \ldots, y_n] \arrow[r, "\widetilde{\varphi}"] \arrow[d] & k[X] \\ k[Y] \arrow[ur, "\varphi^{\#}"'] & \end{tikzcd}$$

Conversely, given any k-algebra homomorphism
$$f : k[y_1, \ldots, y_n]/\mathcal{I}(Y) \to k[x_1, \ldots, x_m]/\mathcal{I}(X),$$
fix a polynomial $f_j \in k[x_1, \ldots, x_m]$ whose image in the coordinate ring $k[X]$ is $f(\overline{y_j})$, where $\overline{y_j}$ is the image of y_j in $k[Y]$, for all $j = 1, \ldots, n$. Then f defines a morphism of affine algebraic sets
$$\varphi : X \to Y, \ a \mapsto (f_1(a), \ldots, f_n(a))$$
such that $\varphi^{\#} = f$. This completes the proof. $\square$

Let $(\mathfrak{Alg}\,/k)$ be the category whose objects are k-algebras and morphisms are k-algebra homomorphisms. Let $(\mathfrak{Alg}\,/k)_{red,fg}$ be the full subcategory of $(\mathfrak{Alg}\,/k)$ whose objects are finitely generated reduced k-algebras. Let $(\mathfrak{Alg}\,/k)^{\mathrm{op}}_{red,fg}$ be the opposite category of $(\mathfrak{Alg}\,/k)_{red,fg}$. Let $(\mathfrak{Aff}\,/k)$ be the category of affine algebraic sets defined over k. As usual, we assume that k is an algebraically closed field.

Theorem 1.1.38. *With the above notations, the functor*
$$\Gamma : (\mathfrak{Aff}\,/k) \to (\mathfrak{Alg}\,/k)^{\mathrm{op}}_{red,fg},$$
which sends $X \in \mathrm{ob}(\mathfrak{Aff}\,/k)$ to its coordinate ring $k[X]$ and a morphism of affine algebraic sets $\varphi : X \to Y$ to the corresponding k-algebra homomorphism $\varphi^{\#} : k[Y] \to k[X]$ (see Proposition 1.1.37), is an equivalence of categories.

Proof. Let $X \subseteq \mathbb{A}^m(k)$, $Y \subseteq \mathbb{A}^n(k)$ and $Z \subseteq \mathbb{A}^r(k)$ be affine algebraic sets with coordinate rings $k[X]$, $k[Y]$ and $k[Z]$, respectively. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of affine algebraic sets. Then $\varphi = (f_1, \ldots, f_n)$ and $\psi = (g_1, \ldots, g_r)$, for some polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ and $g_1, \ldots, g_r \in k[y_1, \ldots, y_n]$. Then it follows from the proof of Proposition 1.1.37 that the morphisms φ, ψ and $\psi \circ \varphi$ uniquely correspond to k-algebra homomorphisms $\varphi^{\#} : k[Y] \to k[X]$, $\psi^{\#} : k[Z] \to k[Y]$ and $(\psi \circ \varphi)^{\#} : k[Z] \to k[X]$, respectively, that are given by
$$\begin{aligned} \varphi^{\#}(\overline{y_j}) &= f_j, \ \forall\, j \in \{1, \ldots, n\}, \\ \psi^{\#}(\overline{z_j}) &= g_j, \ \forall\, j \in \{1, \ldots, r\}, \\ \text{and} \quad (\psi \circ \varphi)^{\#}(\overline{z_j}) &= g_j(f_1, \ldots, f_n), \ \forall\, j \in \{1, \ldots, r\}. \end{aligned}$$

Then it follows that

$$(\varphi^{\#} \circ \psi^{\#})(\overline{z_j}) = g_j(f_1, \dots, f_n) = (\psi \circ \varphi)^{\#}(\overline{z_j}), \ \ \forall\, j \in \{1, \dots, r\},$$

and hence $\varphi^{\#} \circ \psi^{\#} = (\psi \circ \varphi)^{\#}$. Then it follows that Γ is a functor from the category of affine algebraic sets over k to the category of reduced finitely generated k-algebras. Since for any two affine algebraic subsets X and Y defined over k, the natural map

$$\text{(1.1.39)} \qquad \mathrm{Hom}(X, Y) \to \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X]), \ \ \varphi \mapsto \varphi^{\#},$$

is bijective (see Proposition 1.1.37), the functor Γ is fully faithful. To show that Γ is essentially surjective, let A be a finitely generated reduced k-algebra. Then A is isomorphic to the quotient ring $k[x_1, \dots, x_n]/\mathfrak{a}$, for some radical ideal $\mathfrak{a}$ of $k[x_1, \dots, x_n]$. Then $X := \mathcal{Z}(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ is an affine algebraic subset of $\mathbb{A}^n(k)$ with $\mathcal{I}(X) = \mathfrak{a}$, and hence its coordinate ring $k[X]$ is equal to $k[x_1, \dots, x_n]/\mathfrak{a} = A$. Therefore, Γ is essentially surjective, and hence is an equivalence of category. $\square$

Corollary 1.1.40. *Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic subsets, and let $\varphi : X \to Y$ be a morphism of affine algebraic sets. Then φ is continuous for the subspace Zariski topologies on X and Y.*

Proof. It suffices to assume that $Y = \mathbb{A}^n(k)$ and show that $\varphi^{-1}(Z)$ is closed in X, for every closed subset Z of $\mathbb{A}^n(k)$. Let $f_1, \dots, f_n \in k[x_1, \dots, x_m]$ be such that $\varphi = (f_1, \dots, f_n)$. Let Z be a closed subset of $\mathbb{A}^n(k)$. Then $Z = \mathcal{Z}(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of $k[y_1, \dots, y_n]$. Let

$$\varphi^{\#} : k[y_1, \dots, y_m] \to k[X], \ \ g \mapsto g(f_1, \dots, f_n),$$

be the k-algebra homomorphism induced by φ. Let $\mathfrak{b} = \langle \varphi^{\#}(\mathfrak{a}) \rangle$ be the ideal of $k[X]$ generated by the image of $\mathfrak{a}$ under $\varphi^{\#}$. Then it is easy to check (verify!) that

$$\varphi^{-1}(\mathcal{Z}(\mathfrak{a})) = \{a \in X : (f_1(a), \dots, f_n(a)) \in \mathcal{Z}(\mathfrak{a})\} = \mathcal{Z}(\mathfrak{b}).$$

This completes the proof. $\square$

Let X be an affine algebraic subset of $\mathbb{A}^n(k)$. Given an element $f \in k[x_1, \dots, x_n]$, let

$$D(f) = X \setminus \mathcal{Z}(f) = \{a \in X : f(a) \neq 0\}.$$

Then $D(f)$ is an open subset of X, called a *principal open subset* of X.

Proposition 1.1.41. *Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic subset. The collection of all principal open subsets of X forms a basis for the subspace topology on X induced from $\mathbb{A}^n(k)$.*

Proof. Note that, $D(f) \cap D(g) = D(fg)$, for all $f, g \in k[x_1, \dots, x_n]$. Let $\mathfrak{a} = \mathcal{I}(X)$. Since X is an affine algebraic subset of $\mathbb{A}^n(k)$, $\mathfrak{a}$ is a radical ideal of $k[x_1, \dots, x_n]$ and $X = \mathcal{Z}(\mathfrak{a})$. Let U be an open subset of X. Since X is closed in $\mathbb{A}^n(k)$, so is $Z := X \setminus U$. Then it follows from Corollary 1.1.35 that $Z = \mathcal{Z}(\mathfrak{b})$, for some radical ideal $\mathfrak{b}$ of $k[x_1, \dots, x_n]$ containing $\mathfrak{a}$. Since $k[x_1, \dots, x_n]$ is a noetherian ring, $\mathfrak{b} = (f_1, \dots, f_r)$, for some $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Then $Z = \mathcal{Z}(\mathfrak{b}) = \mathcal{Z}(f_1, \ldots, f_r) = \bigcap_{j=1}^{r} \mathcal{Z}(f_j)$, and hence $U = X \setminus Z = \bigcup_{j=1}^{r} D(f_j)$. This completes the proof. □

Proposition 1.1.42. *Let X be an affine algebraic subset of $\mathbb{A}^n(k)$. Then there is a k-algebra isomorphism $\Phi : \mathrm{Hom}(X, \mathbb{A}^1(k)) \to k[X]$.*

Proof. Note that each element $\varphi \in \mathrm{Hom}(X, \mathbb{A}^1(k))$ is defined by a polynomial $f \in k[x_1, \ldots, x_n]$. Two polynomials $f, g \in k[x_1, \ldots, x_n]$ give rise to the same morphism $\varphi : X \to \mathbb{A}^1(k)$ if and only if $f - g \in \mathcal{I}(X)$. Therefore, we have a well-defined injective map

$$\Phi : \mathrm{Hom}(X, \mathbb{A}^1(k)) \to k[X] = k[x_1, \ldots, x_n]/\mathcal{I}(X)$$

defined by sending $\varphi = f \in \mathrm{Hom}(X, \mathbb{A}^1(k))$ to $\overline{f} \in k[X]$, where $\overline{f}$ is the image of f under the natural surjective homomorphism $\pi : k[x_1, \ldots, x_n] \to k[X]$. The map Φ is clearly surjective. We can use the ring structure on k to define a k-algebra structure on $\mathrm{Hom}(X, \mathbb{A}^1(k))$ as follow. Let $\varphi, \psi \in \mathrm{Hom}(X, \mathbb{A}^1(k))$ be two morphisms defined by two polynomials $f, g \in k[x_1, \ldots, x_n]$, respectively. Let $\alpha \in k$ be given. Define morphisms $\varphi + \psi, \varphi \cdot \psi, \alpha \cdot \varphi \in \mathrm{Hom}(X, \mathbb{A}^1(k))$ by setting

$$\begin{aligned} (\varphi + \psi)(a) &= f(a) + g(a), \ \ \forall\, a \in X, \\ (\varphi \cdot \psi)(a) &= f(a)g(a), \ \ \forall\, a \in X, \\ \text{and} \ \ (\alpha \cdot \varphi)(a) &= \alpha f(a), \ \ \forall\, a \in X. \end{aligned}$$

It follows from the definition of the map Φ that it is a k-algebra isomorphism. □

Corollary 1.1.43. *Let $\varphi : X \to Y$ be a morphism of affine algebraic sets defined over k, and let $\varphi^{\#} : k[Y] \to k[X]$ be the associated k-algebra homomorphism of their affine coordinate rings induced by φ. Then for each $x \in X$, we have $(\varphi^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)}$, where $\mathfrak{m}_x$ and $\mathfrak{m}_{\varphi(x)}$ are the maximal ideals of $k[X]$ and $k[Y]$, respectively, corresponding to the closed subsets $\{x\} \subseteq X$ and $\{\varphi(x)\} \subseteq Y$, respectively.*

Proof. Suppose that X and Y are closed subsets of $\mathbb{A}^m(k)$ and $\mathbb{A}^n(k)$, respectively. Then $\mathfrak{m}_x = \{f \in k[X] : f(x) = 0\}$ and $\mathfrak{m}_{\varphi(x)} = \{g \in k[Y] : g(\varphi(x)) = 0\}$. Then

$$\begin{aligned} (\varphi^{\#})^{-1}(\mathfrak{m}_x) &= \{g \in k[Y] : \varphi^{\#}(g) \in \mathfrak{m}_x\} \\ &= \{g \in k[Y] : \varphi^{\#}(g)(x) = 0\} \\ &= \{g \in k[Y] : g(\varphi(x)) = 0\} \\ &= \mathfrak{m}_{\varphi(x)}. \end{aligned}$$

This completes the proof. □

Definition 1.1.44. Let X be a non-empty topological space. Then X is said to be *reducible* if there exists two non-empty proper closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$. If X is not reducible, then we call it *irreducible*.

Definition 1.1.45. An *affine algebraic variety over k* is an irreducible closed subset of $\mathbb{A}^n(k)$, for some $n \in \mathbb{N}$.

1.2 Variety

In Section §1.1 we have discussed topology of affine algebraic sets and its relationships with its affine coordinate ring. In fact, we have shown that the category of affine algebraic sets is equivalent to the category of finitely generated reduced k-algebras. This equivalence essentially captures topology of an affine algebraic set in terms of radical ideals of its affine coordinate ring. In this section, we introduce the notion of a "space with functions" which encodes geometric behavior of affine algebraic sets. For this purpose, and also for later use, we introduce the notion of presheaf and sheaf on a topological space (more generally, on a category admitting a topology in an appropriate sense).

1.2.1 Presheaf and Sheaf

Given a category $\mathscr{C}$, let $\mathscr{C}^{\mathrm{op}}$ be the category whose objects are the same as objects of $\mathscr{C}$, but the arrows are reversed (i.e., given any two objects $X, Y \in \mathscr{C}$, we set $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathscr{C}}(Y, X)$).

Definition 1.2.1 (Presheaf)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *presheaf* on $\mathscr{C}$ with objects from $\mathscr{D}$ is a functor

$$F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}\,,$$

where $\mathscr{C}^{\mathrm{op}}$ is the opposite category of $\mathscr{C}$. More explicitly,

(i) for each object $U \in \mathscr{C}$, we have an object $F(U) \in \mathscr{D}$, and

(ii) for each morphism $f : U \to V$ in $\mathscr{C}$, we have a morphism $F(f) : F(V) \to F(U)$ such that

  (a) $F(\mathrm{Id}_U) = \mathrm{Id}_{F(U)}, \ \forall\, U \in \mathscr{C}$, and

  (b) if we have arrows $f : U \to V$ and $g : V \to W$ in $\mathscr{C}$, then $F(f) \circ F(g) = F(g \circ f)$.

For our purpose, we take $\mathcal{C}$ to be the category τ_X of open subsets of a topological space X; its objects are open subsets of X, and given any two open subsets U and V of X, if $U \subseteq V$, then we define $\mathrm{Hom}(U, V)$ to be the singleton set consisting of the inclusion map $\iota_{U,V} : U \hookrightarrow V$; and if U is not a subset of V, then we take $\mathrm{Hom}(U, V) = \emptyset$. This defines a category τ_X, called the category of open subsets of X. The examples of the category $\mathscr{D}$ we are interested in are as follow.

- (Set), the category of sets. Its objects are sets and morphisms are set maps.
- $(\mathcal{G}rp)$, the category of groups. Its objects are groups and morphisms are group homomorphisms.
- $(\mathcal{R}ing)$, the category of rings. Its objects are rings and morphisms are ring homomorphisms.
- $(\mathfrak{Alg}\,/K)$, the category of K-algebras. Its objects are K-algebras and morphisms are K-algebra homomorphisms.

Definition 1.2.2. Let $\mathscr{C}$ be a category. Given objects $X, Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphisms $f \in \mathrm{Hom}(X, Z)$ and $g \in \mathrm{Hom}(Y, Z)$, the *fiber product* of $\varphi : X \to Z$ and $\psi : Y \to Z$ in $\mathscr{C}$ is a triple $(X \times_Z Y, p_X, p_Y)$, where $X \times_Z Y$ is an object of $\mathscr{C}$ and $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ are morphisms in $\mathscr{C}$ such that

(i) $\varphi \circ p_X = \psi \circ p_Y$, and

(ii) given any object T of $\mathscr{C}$ and morphisms $f : T \to X$ and $g : T \to Y$ in $\mathscr{C}$ satisfying $\varphi \circ f = \psi \circ g$, there exists a unique morphism $\xi : T \to X \times_Z Y$ in $\mathscr{C}$ such that $p_X \circ \xi = p_Y \circ \xi$.

Example 1.2.3. Let $\mathscr{C}$ be the category whose objects are sets and morphisms are set maps. Given sets X, Y and Z, and set maps $\varphi : X \to Z$ and $\psi : Y \to Z$, the set

$$X \times_Z Y := \{(x, y) \in X \times Y : \varphi(x) = \psi(y)\}$$

together with the projection maps

$$\begin{aligned} &p_X : X \times_Z Y \to X, \quad (x, y) \mapsto x, \\ \text{and} \quad &p_Y : X \times_Z Y \to Y, \quad (x, y) \mapsto y, \end{aligned}$$

makes the triple $(X \times_Z Y, p_X, p_Y)$ the fiber product of the morphisms $\varphi : X \to Z$ and $\psi : Y \to Z$ in the category $\mathscr{C}$.

We can define the notion of a topology on a category $\mathscr{C}$ that admits fiber products of finitely many morphisms with the same target object, called a *Grothendieck topology* on $\mathscr{C}$. For this purpose, we need a good notion of "covering" of objects of $\mathscr{C}$ which behaves like open subsets of a topological space.

Definition 1.2.4. Let $\mathscr{C}$ be a category that admit finite fiber products. A *Grothendieck topology* on a category $\mathscr{C}$ consists of the following data: for each object $X \in \mathrm{ob}(\mathscr{C})$ there is a set $\mathcal{C}ov_{\mathscr{C}}(X)$ of families of morphisms in $\mathscr{C}$ with target X, called *covers of X in $\mathscr{C}$*, which satisfies the following axioms:

- If $\phi : U \longrightarrow X$ is an isomorphism in $\mathscr{C}$, then $\{\phi\} \in \mathcal{C}ov_{\mathscr{C}}(X)$.
- If $\{f_i : U_i \longrightarrow X\}_{i \in I} \in \mathcal{C}ov_{\mathscr{C}}(X)$, and $\{g_{ij} : U_{ij} \longrightarrow U_i\}_{j \in J} \in \mathcal{C}ov_{\mathscr{C}}(U_i)$, for each $i \in I$, then $\{f_i \circ g_{ij} : U_{ij} \longrightarrow X\}_{(i,j) \in I \times J} \in \mathcal{C}ov_{\mathscr{C}}(X)$.
- If $\{f_i : U_i \longrightarrow X\}_{i \in I} \in \mathcal{C}ov_{\mathscr{C}}(X)$ and if $g : Y \longrightarrow X$ is a morphisms in $\mathscr{C}$, then the collection of morphisms (obtained by taking fiber products) $\{U_i \times_X Y \longrightarrow Y\}_{i \in I} \in \mathcal{C}ov_{\mathscr{C}}(Y)$.

A *site* is a category with a Grothendieck topology.

For example, given a topological space X, the category τ_X of open subsets of X is a category with Grothendieck topology on it in an obvious way, and so it can be considered as a site.

Definition 1.2.5. Let $\mathscr{C}$ and $\mathscr{D}$ be categories admitting finite fiber products. Assume that $\mathscr{C}$ is a site. A *sheaf* on $\mathscr{C}$ with objects from $\mathscr{D}$ (in short, a $\mathscr{D}$-sheaf on $\mathscr{C}$) is a presheaf

$$F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$$

such that given any object $U \in \mathscr{C}$ and any cover $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ of U in the site $\mathscr{C}$, the following diagram of morphisms in $\mathscr{D}$ is equalizer.

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(j,k) \in I \times I} F(U_j \times_U U_k)$$

More explicitly, the following holds:

(i) If $s, t \in F(U)$ satisfy $F(f_i)(s) = F(f_i)(t),\ \forall\, i \in I$, then $s = t$ in $F(U)$.

(ii) Given $s_i \in F(U_i)$, for each $i \in I$, if $F(\iota_{ij})(s_i) = F(\iota_{ji}(s_j)),\ \forall\, i, j \in I$, then there exists an object $s \in F(U)$ such that $F(f_i)(s) = s_i$, for all $i \in I$.

Definition 1.2.6. Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, the presheaf $f_*\mathcal{F}$ on Y which associate to each open subset V of Y the set $\mathcal{F}(f^{-1}(V))$, is a sheaf on Y, called the *direct image sheaf of* $\mathcal{F}$ or the *push-forward of* $\mathcal{F}$ *along* f.

Definition 1.2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a topological space X. A morphism of sheaves from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of functors $\Phi : \mathcal{F} \to \mathcal{G}$. In other words, for each object $U \in \tau_X$, we have a morphism $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that given any objects U and V of τ_X and any morphism $f : U \to V$ in τ_X, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\Phi_U"] \arrow[d, "\mathcal{F}(f)"'] & \mathcal{G}(U) \arrow[d, "\mathcal{G}(f)"] \\
\mathcal{F}(V) \arrow[r, "\Phi_V"] & \mathcal{G}(V).
\end{tikzcd}$$

A morphism of sheaves $\Phi : \mathcal{F} \to \mathcal{G}$ on X is said to be an *isomorphism* of sheaves on X if there exists a morphism of sheaves $\Psi : \mathcal{G} \to \mathcal{F}$ on X such that $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{G}}$ and $\Psi \circ \Phi = \mathrm{Id}_{\mathcal{F}}$.

Definition 1.2.8. Let K be a field.

1. A *space with K-valued functions* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on X.

2. Given two spaces with K-valued functions $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y.

1.3 Schemes

Let A be a commutative ring with identity. Let $\operatorname{Spec}(A)$ be the set of all prime ideals of A. Given a subset $E \subseteq A$, let

$$V(E) := \{\mathfrak{p} \in \operatorname{Spec}(A) : E \subseteq \mathfrak{p}\}.$$

Note that, $V(E) = V(\langle E \rangle)$, where $\langle E \rangle$ is the ideal of A generated by E. Moreover, if $E_1 \subseteq E_2 \subseteq A$, then $V(E_2) \subseteq V(E_1)$.

Proposition 1.3.1. *With the above notations the following holds.*

(i) $V(0) = \operatorname{Spec}(A)$ *and* $V(1) = \emptyset$;

(ii) Given a collection $\{\mathfrak{a}_j : j \in I\}$ *of ideals of* A*, we have* $\bigcap\limits_{j\in I} V(\mathfrak{a}_j) = V(\sum\limits_{j\in I} \mathfrak{a}_j)$*; and*

(iii) Given any two ideals $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* A*, we have* $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

Consequently, the collection $\tau_c := \{V(\mathfrak{a}) : \mathfrak{a}$ *is an ideal of* $A\}$*, satisfies axioms for closed subsets of a topology on* $\operatorname{Spec}(A)$*, called the Zariski topology on* $\operatorname{Spec}(A)$*. The set* $\operatorname{Spec}(A)$ *together with the Zariski topology on it is called an affine scheme.*

Proof. (i) is obvious. To see (ii), note that, $\mathfrak{a}_j \subseteq \sum\limits_{i\in I} \mathfrak{a}_i$ gives $V(\sum\limits_{i\in I} \mathfrak{a}_i) \subseteq V(\mathfrak{a}_j),\ \forall\, j \in I$, and hence $V(\sum\limits_{i\in I} \mathfrak{a}_i) \subseteq \bigcap\limits_{j\in I} V(\mathfrak{a}_j)$. Conversely, if $\mathfrak{p} \in \bigcap\limits_{j\in I} V(\mathfrak{a}_j)$, then $\mathfrak{a}_j \subseteq \mathfrak{p},\ \forall\, j \in I$, and so $\sum\limits_{j\in I} \mathfrak{a}_j \subseteq \mathfrak{p}$. Therefore, $\mathfrak{p} \in V(\sum\limits_{i\in I} \mathfrak{a}_i)$. This proves (ii). To see (iii), note that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b}$ gives $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$. Conversely, suppose that $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$. If $\mathfrak{p} \notin V(\mathfrak{a})$, then there exists $f \in \mathfrak{a}$ such that $f \notin \mathfrak{p}$. Let $g \in \mathfrak{b}$ be arbitrary. Then $fg \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal of A and $f \notin \mathfrak{p}$, we have $g \in \mathfrak{p}$. Therefore, $\mathfrak{b} \subseteq \mathfrak{p}$, and hence $\mathfrak{p} \in V(\mathfrak{b})$. This proves (iii). □

Exercise 1.3.2. Show that $V(\mathfrak{a}) = V(\operatorname{rad}(\mathfrak{a}))$, for any ideal $\mathfrak{a}$ of A.

Exercise 1.3.3. Given a point $\mathfrak{p} \in \operatorname{Spec}(A)$, show that $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}}$ if and only if $\mathfrak{p}$ is a maximal ideal of A.

Example 1.3.4. Let R be a commutative ring with identity, and let $R[x_1, \ldots, x_n]$ be the polynomial ring in n-variables $x_1, \ldots, x_n$ over the ring R. The set $\operatorname{Spec}(R[x_1, \ldots, x_n])$ together with the Zariski topology on it is called the *affine n-space over R*, and is denoted by $\mathbb{A}^n_R$. For $\mathbb{R} = \mathbb{Z}$, we simply denote $\mathbb{A}^n_{\mathbb{Z}}$ by $\mathbb{A}^n$. When the base ring R is an algebraically closed field k, the set of all closed points of $\mathbb{A}^n_k$ is precisely the affine n-space $\mathbb{A}^n(k)$ defined in the last section.

Given a closed subset Z of an affine scheme $X = \operatorname{Spec}(A)$, consider the subset

$$\mathcal{I}(Z) := \{f \in A : f \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z\}.$$

Clearly $0 \in \mathcal{I}(Z)$. Let $f, g \in \mathcal{I}(Z)$ and $h \in A$ be arbitrary. Then $f, g \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z$ implies that $f + gh \in \mathfrak{p},\ \forall\, \mathfrak{p} \in Z$. Therefore, $f + gh \in \mathcal{I}(Z)$. Therefore, $\mathcal{I}(Z)$ is an ideal of A. By definition of $\mathcal{I}(Z)$, we have

$$\mathcal{I}(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Proposition 1.3.5. *(i) Let $\mathfrak{a}$ be an ideal of A. Then $\mathcal{I}(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.*

(ii) Given any subset Z of $X = \mathrm{Spec}(A)$, we have $V(\mathcal{I}(Z)) = \overline{Z}$.

Proof. (i) Note that, $\mathfrak{p} \in V(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$. Therefore, we have

$$\mathcal{I}(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p} = \mathrm{rad}(\mathfrak{a}).$$

(ii) Since $\mathcal{I}(Z) = \bigcap\limits_{\mathfrak{q} \in Z} \mathfrak{q}$, we have $\mathcal{I}(Z) \subseteq \mathfrak{p}$, $\forall\, \mathfrak{p} \in Z$. Therefore, $\mathfrak{p} \in V(\mathcal{I}(Z))$, $\forall\, \mathfrak{p} \in Z$, and hence $Z \subseteq V(\mathcal{I}(Z))$. Since $V(\mathcal{I}(Z))$ is closed in $X = \mathrm{Spec}(A)$, we have $\overline{Z} \subseteq V(\mathcal{I}(Z))$. To show the reverse inclusion, it suffices to show that $V(\mathcal{I}(Z)) \subseteq V$, for any closed subset V of X containing Z. Let $\mathfrak{a}$ be an ideal of A be such that $Z \subseteq V(\mathfrak{a})$. Then for any $\mathfrak{p} \in Z$ we have $\mathfrak{a} \subseteq \mathfrak{p}$. Therefore, $\mathfrak{a} \subseteq \bigcap\limits_{\mathfrak{p} \in Z} \mathfrak{p} = \mathcal{I}(Z)$, and hence $V(\mathcal{I}(Z)) \subseteq V(\mathfrak{a})$, as required. Thus, $V(\mathcal{I}(Z)) = \overline{Z}$. $\square$

Definition 1.3.6. Let X be a non-empty topological space. Then X is said to be *reducible* if there exists two non-empty proper closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$. If X is not reducible, then we call it *irreducible*.

Example 1.3.7. The affine line $\mathbb{A}^1(k)$ over a field k is irreducible if and only if k is infinite.

Proposition 1.3.8. *Let A be a non-zero commutative ring with identity, and let Z be a closed subset of* $\mathrm{Spec}(A)$*. Then Z is irreducible if and only if $\mathcal{I}(Z)$ is a prime ideal of A. In particular,* $\mathrm{Spec}(A)$ *is irreducible if and only if the nil radical* $\mathrm{Nil}(A)$ *of A is prime.*

Proof. Let $Z = V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of A. Then $\mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$. Suppose that Z is irreducible. Let $f, g \in A$ be such that $fg \in \mathrm{rad}(\mathfrak{a})$ and $f \notin \mathrm{rad}(\mathfrak{a})$. Then

$$Z = V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a})) \subseteq V(fg) = V(f) \cup V(g).$$

Since Z is irreducible, either $Z \subseteq V(f)$ or $Z \subseteq V(g)$. If $Z \subseteq V(f)$, then $f \in \mathrm{rad}(f) = \mathcal{I}(V(f)) \subseteq \mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$. But $f \notin \mathrm{rad}(\mathfrak{a})$. Then we must have $Z \subseteq V(g)$, which gives $g \in \mathrm{rad}(\mathfrak{a})$. Therefore, $\mathcal{I}(Z) = \mathrm{rad}(\mathfrak{a})$ is a prime ideal of A. Conversely, suppose that $\mathcal{I}(Z)$ is a prime ideal of A. Suppose that $Z \subseteq Y_1 \cup Y_2$, for some non-empty proper closed subsets Y_1 and Y_2 of $\mathrm{Spec}(A)$. Let $Y_1 = V(\mathfrak{a})$ and $Y_2 = V(\mathfrak{b})$, for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A. Since $Y_1 \cup Y_2 = V(\mathfrak{a}\mathfrak{b})$, we have $\mathfrak{a}\mathfrak{b} \subseteq \mathcal{I}(Z)$. Since $\mathcal{I}(Z)$ is prime, either $\mathfrak{a} \subseteq \mathcal{I}(Z)$ or $\mathfrak{b} \subseteq \mathcal{I}(Z)$. Since $Z = \overline{Z}$, the above two conditions gives either $Z \subseteq V(\mathfrak{a}) = Y_1$ or $Z \subseteq V(\mathfrak{b}) = Y_2$. Therefore, Z is irreducible.

Since $\mathrm{Spec}(A) = V(0)$, it follows that $\mathrm{Spec}(A)$ is irreducible if and only if $\mathrm{rad}(0) = \mathrm{Nil}(A)$ is a prime ideal of A. Note that, this happens if and only if A has a unique minimal prime ideal. $\square$

1.4 Sheaves

Let X be a topological space. A *presheaf* of sets on X is a contravariant functor $\mathcal{F} : \tau_X \to (\text{Set})$ from the category of open subsets of X to the category of sets. A presheaf $\mathcal{F} : \tau_X \to (\text{Set})$ is said to be a *sheaf* if for each open subset V of X and a collection of open subsets $\{V_\alpha : \alpha \in \Lambda\}$ of X with $V = \bigcap\limits_{\alpha\in\Lambda} V_\alpha$, the following diagram of sets is equilizer.

$$\mathcal{F}(V) \longrightarrow \prod_{\alpha\in\Lambda} \mathcal{F}(V_\alpha) \rightrightarrows \prod_{(\alpha,\beta)\in\Lambda\times\Lambda} \mathcal{F}(V_\alpha \cap V_\beta)$$

1.5 Appendix: Category Theory

Joke: Category theory is like Ramayana and Mahabharata — there are lots of arrows!

— *Nitin Nitsure*

Definition 1.5.1. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\text{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\text{ob}(\mathscr{C})$, there is a collection $\text{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \text{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \text{Mor}_{\mathscr{C}}(X, Y) \times \text{Mor}_{\mathscr{C}}(Y, Z) \to \text{Mor}_{\mathscr{C}}(X, Z), \ \ (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \text{ob}(\mathscr{C})$, and $f \in \text{Mor}_{\mathscr{C}}(X, Y)$, $g \in \text{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \text{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \text{ob}(\mathscr{C})$, there exists a morphism $\text{Id}_X \in \text{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \text{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \text{Id}_Y = f$ and $\text{Id}_Z \circ f = f$.

A category $\mathscr{A}$ is said to be *locally small* if $\text{Mor}_{\mathscr{A}}(X, Y)$ is a set, for all $X, Y \in \text{ob}(\mathscr{A})$. A category $\mathscr{A}$ is said to be *small* if it is locally small and the class of objects $\text{ob}(\mathscr{A})$ is a set.

Example 1.5.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{A}$ such that $g \circ f = \text{Id}_{A_1}$ and $f \circ g = \text{Id}_{A_2}$.

Definition 1.5.3. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,

(ii) for $X, Y \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, there is a morphism $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, such that the following conditions hold.

(a) $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$, where Id_X and $\mathrm{Id}_{\mathcal{F}(X)}$ are the identity morphisms of X and $\mathcal{F}(X)$, respectively.

(b) Given objects X, Y, Z in $\mathscr{A}$ and morphisms $f \in \mathrm{Mor}_{\mathscr{A}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathscr{A}}(Y, Z)$, the following diagram commutes in $\mathscr{B}$.

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[dr, "\mathcal{F}(g\circ f)"'] & \mathcal{F}(Y) \arrow[d, "\mathcal{F}(g)"] \\ & \mathcal{F}(Z). \end{tikzcd}$$

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$$

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful. We say that $\mathcal{F}$ is *essentially surjective* if given an object Y of $\mathscr{B}$, there exists an object X of $\mathscr{A}$ such that $\mathcal{F}(X) \cong Y$.

Definition 1.5.4. Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A *morphism of functors* $\Phi : \mathcal{F} \to \mathcal{G}$ is given by the following data:

(i) for each object $A \in \mathscr{A}$, there is a morphism $\Phi_A \in \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(A), \mathcal{G}(A))$, and

(ii) given any two objects $A, A' \in \mathscr{A}$ and any morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, A')$, the following diagram commutes.

(1.5.5)
$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(A') \arrow[d, "\Phi_{A'}"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A') \end{tikzcd}$$

A morphism of functors $\Phi : \mathcal{F} \to \mathcal{G}$ is said to be an *isomorphism* if there exists a morphism of functors $\Psi : \mathcal{G} \to \mathcal{F}$ such that $\Psi \circ \Phi = \mathrm{Id}_{\mathcal{F}}$ and $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{G}}$, where $\mathrm{Id}_{\mathcal{F}}$ and $\mathrm{Id}_{\mathcal{G}}$ are the identity morphisms of the functors $\mathcal{F}$ and $\mathcal{G}$, respectively. If $\mathcal{F}$ and $\mathcal{G}$ are isomorphic functors, then we express it as $\mathcal{F} \cong \mathcal{G}$.

Definition 1.5.6. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be an *equivalence of categories* if there exists a functor $\mathcal{G} : \mathscr{B} \to \mathscr{A}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathscr{A}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathscr{B}}$.

Lemma 1.5.7. *A functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is fully faithful and essentially surjective. Then using axiom of choice, choose for each object Y of $\mathcal{B}$ an object $j(Y)$ of $\mathcal{A}$ and an isomorphism $\varphi_Y : Y \to F(j(Y))$ in $\mathcal{B}$. Since F is fully faithful, given any two objects Y_1 and Y_2 of $\mathcal{B}$, the natural map $t : \mathrm{Hom}_{\mathcal{A}}(j(Y_1), j(Y_2)) \to \mathrm{Hom}_{\mathcal{B}}(F(j(Y_1)), F(j(Y_2)))$ is bijective. Then given any morphism $g \in \mathrm{Hom}_{\mathcal{B}}(Y_1, Y_2)$, we get a morphism

$$\widetilde{g} : F(j(Y_1)) \xrightarrow{\varphi_{Y_1}} Y_1 \xrightarrow{g} Y_2 \xrightarrow{\varphi_{Y_2}^{-1}} F(j(Y_2)).$$

Then $t^{-1}(\widetilde{g}) : j(Y_1) \longrightarrow j(Y_2)$ is the required morphism in $\mathrm{Hom}_{\mathcal{A}}(j(Y_1), j(Y_2))$. One can check that this construction of morphism preserves composition of morphisms, and hence defines a functor $j : \mathcal{B} \to \mathcal{A}$ such that $F \circ j \cong \mathrm{Id}_{\mathcal{B}}$ and $j \circ F \cong \mathrm{Id}_{\mathcal{A}}$. The converse part is easy to verify. □

Definition 1.5.8. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Mor}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathrm{Func}(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 1.5.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 1.5.10. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Mor}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Notation. For any pre-additive category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ the abelian group $\mathrm{Mor}_{\mathscr{A}}(X, Y)$, for all $X, Y \in \mathrm{ob}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.5.11 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.

(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0,0)$ is the trivial group with one element.

(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $j_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 1.5.12. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A,B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A,B) \times \mathrm{Hom}_{\mathscr{A}}(B,C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A,C)\,, \ \ (f,g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 1.5.13. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1,A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0,B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 1.5.14. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
 & C \arrow[dl, dashed, "\exists ! \widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A,B)$ is defined by reversing the arrows of the above diagram.

Definition 1.5.15. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists ! \widetilde{g}"] \\ & C & \end{tikzcd}$$

Definition 1.5.16. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 1.5.17. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.

(ii) If a cokernel of f exists, then it is an epimorphism.

(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.

(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). □

Exercise 1.5.18. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 1.5.19. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$ (see Lemma 1.5.17 (iii)). This completes the proof.

(1.5.20)
$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\ & & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & & \end{tikzcd}$$

□

Definition 1.5.21. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 1.5.16).

Example 1.5.22.

1. For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.
2. Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules on X. Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{QCoh}(X)$ (reps., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of $\mathcal{O}_X$-modules on X, are also abelian. However, the full subcategory $\mathcal{V}ect(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of $\mathcal{O}_X$-modules on X, is not abelian, because kernel of a morphism in $\mathcal{V}ect(X)$ may not be in $\mathcal{V}ect(X)$.