

1. MODEL ANSWER

Question. Use Plancherel's identity to find a relation between the solution and the initial data in terms of mean square norm called L^2 norm of the Cauchy problem:

$$u_t + u_{xxxx} + 2u = 0, \quad -\infty < x < \infty, \quad t > 0$$

with initial data $u(x, 0) = u_0(x)$, $-\infty < x < \infty$; and also find $\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 dx$.

Model Answer: We have given

$$(1.1) \quad u_t + u_{xxxx} + 2u = 0, \quad -\infty < x < \infty, \quad t > 0,$$

with initial data $u(x, 0) = u_0(x)$, $-\infty < x < \infty$. Taking Fourier transform with respect to x , from equation (1.1), we get

$$\begin{aligned} \hat{u}_t(w, t) + (iw)^4 \hat{u} + 2\hat{u} &= 0 \\ \implies \hat{u}_t(w, t) + (w^4 + 2)\hat{u} &= 0 \\ (1.2) \quad \implies \hat{u}(w, t) &= \hat{u}_0(w) e^{-t(w^4+2)} \end{aligned}$$

Using Plancherel's identity and (1.1), we get

$$\int_{-\infty}^{\infty} |u(x, t)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(w, t)|^2 dw = e^{-4t} \int_{-\infty}^{\infty} e^{-2tw^4} \cdot |\hat{u}_0(w)|^2 dw.$$

Since for each real number $t > 0$, we have $e^{-2tw^4} \leq 1, \forall w \in \mathbb{R}$, from above equation we have

$$(1.3) \quad \int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq e^{-4t} \int_{-\infty}^{\infty} |\hat{u}_0(w)|^2 dw.$$

Since $e^{-4t} \leq 1$, for all $t > 0$, again using Plancherel's identity, we find that the relation between the solution and the initial data is

$$\int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq \int_{-\infty}^{\infty} |\hat{u}_0(w)|^2 dw = \int_{-\infty}^{\infty} |u_0(x)|^2 dx, \quad \forall t > 0.$$

Taking limit as $t \rightarrow \infty$ in (1.3), we get

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 dx \leq \lim_{t \rightarrow \infty} e^{-4t} \int_{-\infty}^{\infty} |\hat{u}_0(w)|^2 dw = 0. \\ \implies \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} |u(x, t)|^2 dx &= 0. \end{aligned}$$

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