LIE ALGEBROID CONNECTIONS ON PRINCIPAL BUNDLES

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ABSTRACT. Let X be an irreducible smooth complex projective variety. Let G be a linear algebraic group over $\mathbb{C}$. We define the notion of Lie algebroid valued connection on holomorphic principal G–bundles on X, and study their basic properties under extension and reduction of structure group. Finally we investigate criterions for existence of a Lie algebroid connection on principal G–bundles over smooth complex projective curves.

CONTENTS

1. INTRODUCTION

A famous theorem of A. Weil [Wei38] says that a holomorphic vector bundle E on a compact connected Riemann surface X admits a holomorphic connection if and only if each indecomposible holomorphic direct summand of E has degree zero. In [Ati57] M. Atiyah generalizes the notion of holomorphic connections in the context of

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holomorphic principal G–bundles on compact Kähler manifolds, and gives an algebro-geometric proof and Weils' theorem for holomorphic vector bundles on compact connected Riemann surfaces. In [AB02] Azad and Biswas generalize Weils theorem for holomorphic principal G–bundles on compact connected Riemann surfaces. It is clear from these results that not every holomorphic vector bundles and principal G–bundles can admit holomorphic connections. This naturally leads one to consider the notion of meromorphic connections. One of the simplest kind of meromorphic connections is the notion of logarithmic connection, which are treated for holomorphic vector bundles and holomorphic principal G–bundles, for example in [BDP18], [BDPS17], [GP20] etc.

In the context of complex algebraic and differential geometry, the classical notion of holomorphic as-well-as singular connections has natural generalization by replacing tangent bundle with a Lie algebroid leading to the notion of *Lie algebroid connections*, which is more convenient to work in some setups like Poisson geometry, foliation theory etc. The notion of Lie algebroid connections also generalize the notion of holomorphic and logarithmic connections. It is an interesting problem to study Lie algebroid connections on holomorphic vector bundles and principal bundles.

Let X be a connected compact Riemann surface. Fix a holomorphic Lie algebroid $\mathcal{V} = (V, [\cdot\,,\,\cdot], \varphi)$ on X with V a stable vector bundle. In [BKS24] the authors shows that every holomorphic vector bundle on X admits a $\mathcal{V}$–valued Lie algebroid connection generalizing a result [AO24, Corollary 3.17] of Alfaya and Oliveire. In this paper we generalize the notion of $\mathcal{V}$–valued Lie algebroid connections in the context of principal G–bundles (see Definition 2.2.7), study their properties under extension and reduction of the structure group of the principal bundles (see § 3), and prove the following.

Theorem 1.0.1. *Let X be an irreducible smooth complex projective curve of genus $g \geq 2$. Fix a Lie algebroid $\mathcal{V} = (V, [\cdot\,,\,\cdot], \varphi)$ on X such that V is a stable vector bundle on X with the slope $\mu(V) \neq 2 - 2g$. Let G be a reductive linear algebraic group over $\mathbb{C}$. Then any holomorphic principal G–bundle E_G on X admits a $\mathcal{V}$–valued Lie algebroid connection.*

This generalize the main result of [BKS24] to the case of holomorphic principal G–bundles on X.

2. Lie Algebroid Connections

2.1. **The case of vector bundles.** Let X be an irreducible smooth projective variety over $\mathbb{C}$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, and let TX be the holomorphic tangent bundle of X.

Definition 2.1.1. [AO24, § 1.1] A *Lie algebroid* on X is a triple $\mathcal{V} := (V, [\cdot, \cdot], \varphi)$, where

(i) V is a holomorphic vector bundle on X,

(ii) $[\cdot,\cdot] : V \times V \to V$ is a $\mathbb{C}$–bilinear skew-symmetric morphism of sheaves such that for all locally defined sections u, v, w of V, the following Jacobi identity holds:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0;$$

(iii) $\varphi : V \to TX$ is a vector bundle homomorphism satisfying the following properties: for all locally defined sections s, t of V and locally defined section f of $\mathcal{O}_X$, we have

(a) *Compatibility of Lie algebra structures*: $\varphi([s,t]) = [\varphi(s), \varphi(t)]$, and

(b) *Leibniz rule*: $[fs, t] = f[s,t] - \varphi(t)(f)s$.

The homomorphism φ is called the *anchor map* of the Lie algebroid $\mathcal{V}$. The *degree* and the *rank* of $\mathcal{V}$ is defined to be the degree and the rank, respectively, of the underlying vector bundle V of $\mathcal{V}$.

The dual of the anchor map gives a holomorphic vector bundle homomorphism

$$\varphi^* : \Omega^1_X \longrightarrow V^*,$$

where Ω^1_X is the holomorphic cotangent bundle of X. Fix a Lie algebroid $\mathcal{V} := (V, [\cdot,\cdot], \varphi)$ on X. Let $\mathcal{E}$ be a holomorphic vector bundle on X.

Definition 2.1.2. A *$\mathcal{V}$–valued Lie algebroid connection* on $\mathcal{E}$ on X is a $\mathbb{C}$–linear homomorphism of sheaves

$$D : \mathcal{E} \longrightarrow \mathcal{E} \otimes V^*$$

satisfying the *φ^*–twisted Leibniz rule*:

$$D(f \cdot s) = fD(s) + s \otimes \varphi^*(df), \tag{2.1.3}$$

for all locally defined section s of $\mathcal{E}$ and for all locally defined section f of $\mathcal{O}_X$.

2.2. **The case of principal G–bundles.** Now we extend the definition of Lie algebroid connection to the case of principal bundles following a construction given in [BP17]. Let G be a linear algebraic group over $\mathbb{C}$ with the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. Let $p : E_G \to X$ be a holomorphic principal G–bundle on X. The adjoint representation

$$\mathrm{ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$$

of G on its Lie algebra $\mathfrak{g}$ gives rise to a vector bundle

$$\mathrm{ad}(E_G) := E_G \times^{\mathrm{ad}} \mathfrak{g}$$

on X, called the *adjoint vector bundle* of E_G. If E is the frame bundle of a vector bundle $\mathcal{E}$ of rank n on X, then we have $\mathrm{ad}(E) \cong \mathrm{End}(\mathcal{E})$, the endomorphism bundle of $\mathcal{E}$. The surjective submersion $p : E_G \to X$ gives rise to an exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0 \tag{2.2.1}$$

called the Atiyah exact sequence of E_G. A connection on the principal G–bundle E_G on X is an $\mathcal{O}_X$–linear homomorphism $\nabla : TX \to \mathrm{At}(E_G)$ such that $d'p \circ \nabla = \mathrm{Id}_{TX}$.

Fix a Lie algebroid $\mathcal{V} = (V, [\cdot\,,\,\cdot], \varphi)$ on X, and consider the map

$$\rho : \mathrm{At}(E_G) \oplus V \longrightarrow TX$$

defined by

$$\rho(\xi, v) = d'p(\xi) - \varphi(v), \tag{2.2.2}$$

for all locally defined section ξ of $\mathrm{At}(E_G)$ and locally defined section v of V. Note that ρ is a vector bundle homomorphism and

$$\mathrm{At}_\varphi(E_G) := \rho^{-1}(0) \tag{2.2.3}$$

is a vector bundle on X. The restriction of the second projection map gives rise to a vector bundle homomorphism

$$\widetilde{\rho} : \mathrm{At}_\varphi(E_G) \longrightarrow V \tag{2.2.4}$$

with kernel

$$\mathrm{Ker}(\widetilde{\rho}) = \mathrm{ad}(E_G).$$

Thus we have the following short exact sequence

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}_\varphi(E_G) \xrightarrow{\widetilde{\rho}} V \longrightarrow 0 \tag{2.2.5}$$

of vector bundles on X, which fits into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r] \arrow[d, equal] & \mathrm{At}_\varphi(E_G) \arrow[r, "\widetilde{\rho}"] \arrow[d] & V \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r] & \mathrm{At}(E_G) \arrow[r, "d'p"] & TX \arrow[r] & 0
\end{tikzcd} \tag{2.2.6}$$

of vector bundle homomorphisms with all rows exact.

Definition 2.2.7. A *$\mathcal{V}$–valued Lie algebroid connection* on E_G is a vector bundle homomorphism

$$\nabla : V \longrightarrow \mathrm{At}_\varphi(E_G)$$

such that $\widetilde{\rho} \circ \nabla = \mathrm{Id}_V$, where $\widetilde{\rho}$ is defined in (2.2.4).

The short exact sequence (2.2.5) defines a cohomology class

$$\Phi_{\mathcal{V}}(E_G) \in H^1(X, \mathrm{ad}(E_G) \otimes V^*), \tag{2.2.8}$$

such that the exact sequence (2.2.5) splits holomorphically if and only if $\Phi_{\mathcal{V}}(E_G) = 0$.

Proposition 2.2.9. *A holomorphic principal G–bundle E_G on X admits a $\mathcal{V}$–valued holomorphic Lie algebroid connection if and only if $\Phi_{\mathcal{V}}(E_G) = 0$. We call $\Phi_{\mathcal{V}}(E_G)$ the $\mathcal{V}$–valued Atiyah class of E_G.*

Let $\nabla : V \to \mathrm{At}_\varphi(E_G)$ be a $\mathcal{V}$–valued Lie algebroid connection on E_G over X. For all locally defined holomorphic sections s and t of V, let

$$\kappa_\nabla(s,t) := [\nabla(s), \nabla(t)] - \nabla([s,t]).$$

Since the homomorphism $\widetilde{\rho} : \mathrm{At}_\varphi(E_G) \to V$ respects the Lie algebra structures on the sheaves of sections, $\kappa_\nabla(s,t)$ defines a holomorphic local section of $\mathrm{ad}(E_G)$. Thus we obtain a section

$$\kappa_\nabla \in H^0(X, \mathrm{ad}(E_G) \otimes \bigwedge^2 V^*),$$

called the *curvature* of the $\mathcal{V}$–valued Lie algebroid connection ∇ on E_G. The section κ_∇ can be considered as an obstruction for ∇ to be a Lie algebra homomorphism.

Definition 2.2.10. A $\mathcal{V}$–valued Lie algebroid connection ∇ on a principal G–bundle E_G on X is said to be *flat* if $\kappa_\nabla = 0$.

Proposition 2.2.11. *If* $\mathrm{rank}(\mathcal{V}) = 1$, *any $\mathcal{V}$–valued Lie algebroid connection on E_G is flat.*

Proof. If $\mathrm{rank}(\mathcal{V}) = 1$, then $\bigwedge^2 V^* = 0$ and so for any $\mathcal{V}$–valued connection ∇ on E_G, its curvature κ_∇, being an element of $H^0(X, \mathrm{ad}(E_G) \otimes \bigwedge^2 V^*) = 0$, vanishes identically. This completes the proof. $\square$

3. Basic Properties

3.1. **Extension of structure groups.** Let G and H be linear algebraic groups over $\mathbb{C}$ with their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Given a homomorphism of algebraic groups $f : G \to H$ over $\mathbb{C}$, let $df : \mathfrak{g} \to \mathfrak{h}$ be the Lie algebra homomorphism induced by f. Let $p : E_G \to X$ be a holomorphic principal G–bundle over X, and let

$$p' : E_H := E_G \times^f H \to X$$

be the associated principal H–bundle on X obtained by extending the structure group of E_G along f. Let

$$\begin{aligned} \mathrm{ad}(f) : \mathrm{ad}(E_G) &\longrightarrow \mathrm{ad}(E_H) \\ \text{and} \quad \mathrm{At}(f) : \mathrm{At}(E_G) &\longrightarrow \mathrm{At}(E_H) \end{aligned}$$

be the homomorphisms of the adjoint bundles and the Atiyah bundles of E_G and E_H, respectively, induced by f. Then we have the following commutative diagram of vector bundle homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r, "\iota_G"] \arrow[d, "\mathrm{ad}(f)"] & \mathrm{At}(E_G) \arrow[r, "d'p"] \arrow[d, "\mathrm{At}(f)"] & TX \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{ad}(E_H) \arrow[r, "\iota_H"] & \mathrm{At}(E_H) \arrow[r, "d'p'"] & TX \arrow[r] & 0. \end{tikzcd} \tag{3.1.1}$$

It is clear from the above diagram that a holomorphic connection on E_G induces a holomorphic connection on $E_H := E_G \times^f H$. Let

$$\rho' : \mathrm{At}(E_H) \oplus V \to TX$$

be the homomorphism defined by

$$\rho'(\xi, v) = d'p'(\xi) - \varphi(v),$$

for all locally defined sections ξ of $\mathrm{At}(E_H)$ and v of V, respectively. Let

$$\widetilde{\rho'} : \mathrm{At}_\varphi(E_H) := \mathrm{Ker}(\rho') \longrightarrow V$$

be the restriction of the second projection map. Then we have a vector bundle homomorphism

$$\mathrm{At}_\varphi(f) : \mathrm{At}_\varphi(E_G) \longrightarrow \mathrm{At}_\varphi(E_H)$$

such that $\widetilde{\rho'} \circ \mathrm{At}_\varphi(f) = \widetilde{\rho}$. Thus, the above commutative diagram (3.1.1) induces the following commutative diagram of vector bundles and homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r] \arrow[d, "\mathrm{ad}(f)"] & \mathrm{At}_\varphi(E_G) \arrow[r, "\widetilde{\rho}"] \arrow[d, "\mathrm{At}_\varphi(f)"] & V \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{ad}(E_H) \arrow[r] & \mathrm{At}_\varphi(E_H) \arrow[r, "\widetilde{\rho'}"] & V \arrow[r] & 0\,. \end{tikzcd} \tag{3.1.2}$$

From this, we have a natural homomorphism of cohomologies

$$H^1(f) : H^1(X, \mathrm{ad}(E_G) \otimes V^*) \longrightarrow H^1(X, \mathrm{ad}(E_H) \otimes V^*) \tag{3.1.3}$$

such that $H^1(f)(\Phi_{\mathcal{V}}(E_G)) = \Phi_{\mathcal{V}}(E_H)$. As an immediate consequence of it, we have the following result.

Proposition 3.1.4. *Let $f : G \to H$ be a homomorphism of linear algebraic groups over $\mathbb{C}$. Let E_G be a holomorphic principal G–bundle on X, and let E_H be the holomorphic principal H–bundle on X obtained from E_G by extension of its structure group along f. Then any $\mathcal{V}$–valued Lie algebroid connection on E_G induces a $\mathcal{V}$–valued Lie algebroid connection on E_H.*

Proof. If E_G admits a $\mathcal{V}$–valued Lie algebroid connection, then $\Phi_{\mathcal{V}}(E_G) = 0$. Since $\Phi_{\mathcal{V}}(E_H) = H^1(f)(\Phi_{\mathcal{V}}(E_G)) = 0$, the result follows from Proposition 2.2.9 □

3.2. **Reduction of structure group.** Now it is interesting to ask the following question: Suppose that $f : G \to H$ be a homomorphism of linear algebraic groups over $\mathbb{C}$. If E_H admits a $\mathcal{V}$–valued connection, does E_G admits a $\mathcal{V}$–valued connection? We give partial answers to this question.

Proposition 3.2.1. *Let $f : G \to H$ be an injective homomorphism of linear algebraic groups over $\mathbb{C}$ with G reductive. Let E_G be a principal G–bundle on X, and let*

$$E_H = E_G \times^f H$$

be the principal H–bundle on X obtained by extending the structure group of E_G along the homomorphism f. If E_H admits a $\mathcal{V}$–valued Lie algebroid connection, then E_G admits a $\mathcal{V}$–valued Lie algebroid connection.

Proof. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$ be the Lie algebras of G and H, respectively. Let $df : \mathfrak{g} \to \mathfrak{h}$ be the Lie algebra homomorphism induced by f, and let

$$\mathrm{ad}(f) : \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_H)$$

be the vector bundle homomorphism induced by df. Let $\alpha : G \to \mathrm{End}(\mathfrak{g})$ and $\beta : H \to \mathrm{End}(\mathfrak{h})$ be the adjoint actions of G and H, respectively, on their Lie algebras. Then the composite map

$$\beta \circ f : G \to \mathrm{End}(\mathfrak{h})$$

gives an adjoint action of G on $\mathfrak{h}$. Since df is a G–module homomorphism and G is reductive, there is a G–submodule W of $\mathfrak{h}$ such that

$$\mathfrak{h} = df(\mathfrak{g}) \bigoplus W \tag{3.2.2}$$

as G–modules. Since df is injective, from the direct sum decomposition of G–modules in (3.2.2) projecting to the first factor we get a G–module homomorphism $\pi_{\mathfrak{g}} : \mathfrak{h} \to \mathfrak{g}$ such that $\pi_{\mathfrak{g}} \circ df = \mathrm{Id}_{\mathfrak{g}}$. Then $\pi_{\mathfrak{g}}$ induces a vector bundle homomorphism

$$\widetilde{\pi_{\mathfrak{g}}} : \mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G) \tag{3.2.3}$$

such that $\widetilde{\pi_{\mathfrak{g}}} \circ \mathrm{ad}(f) = \mathrm{Id}_{\mathrm{ad}(E_G)}$.

Suppose that E_H admits a $\mathcal{V}$–valued Lie algebroid connection. Then there exists a $\mathcal{O}_X$–module homomorphism

$$\eta : \mathrm{At}_{\varphi}(E_H) \to \mathrm{ad}(E_H)$$

such that $\eta \circ \iota_H = \mathrm{Id}_{\mathrm{ad}(E_H)}$, where $\iota_H : \mathrm{ad}(E_H) \to \mathrm{At}_{\varphi}(E_H)$ is the homomorphism in (3.1.1). Then it follows from the commutativity of the diagram (3.1.1) that the composition

$$\widetilde{\pi_{\mathfrak{g}}} \circ \eta \circ \mathrm{At}(f) : \mathrm{At}_{\varphi}(E_G) \to \mathrm{ad}(E_G)$$

gives an $\mathcal{O}_X$–linear splitting of the top exact sequence in (3.1.1). Thus E_G admits a $\mathcal{V}$–valued Lie algebroid connection. □

Now we consider the case when the structure group of a principal bundle is not reductive. Let G be a reductive linear algebraic group over $\mathbb{C}$. A closed subgroup P of G is said to be *parabolic* if G/P is a complete $\mathbb{C}$–variety. Let P be a parbolic subgroup of G. Let $\mathfrak{R}_u(P)$ be the unipotent radical of P, and let

$$q : P \longrightarrow P/\mathfrak{R}_u(P)$$

be the associated quotient map. Let $L \subseteq P$ be a *Levi factor of* P; a closed connected subgroup of P such that $q\big|_L : L \to P/\mathfrak{R}_u(P)$ is an isomorphism of algebraic groups over $\mathbb{C}$. Note that L is reductive. Given a principal P–bundle E_P on X, let $E_L := E_P \times^{q'} L$ be the principal L–bundle on X obtained by extending the structure group of E_P along the homomorphism

$$q' := (q\big|_L)^{-1} \circ q : P \to L.$$

The action of P on the nilpotent radical $\mathfrak{n} := \mathrm{Lie}(\mathfrak{R}_u(P))$ of the Lie algebra $\mathfrak{p} := \mathrm{Lie}(P)$ gives rise to a subbundle $E_P(\mathfrak{n}) := E_P \times^P \mathfrak{n}$ of the adjoint bundle $\mathrm{ad}(E_P)$ of E_P, and the associated quotient bundle $\mathrm{ad}(E_P)/E_P(\mathfrak{n}) \cong E_P(\mathfrak{l}) = \mathrm{ad}(E_L)$, where $\mathfrak{l} = \mathrm{Lie}(L)$ is the Lie algebra of L. Then we have the following commutative diagram of vector bundle homomorphisms with all rows and columns exact (c.f. (3.1.2)):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E_P(\mathfrak{n}) \arrow[r, equal] \arrow[d] & E_P(\mathfrak{n}) \arrow[d] & & \\
0 \arrow[r] & \mathrm{ad}(E_P) \arrow[r] \arrow[d, "\mathrm{ad}(q')"] & \mathrm{At}_\varphi(E_P) \arrow[r, "\widetilde{\rho}_P"] \arrow[d, "\mathrm{At}(q')"] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{ad}(E_L) \arrow[r] \arrow[d] & \mathrm{At}_\varphi(E_L) \arrow[r, "\widetilde{\rho}_L"] \arrow[d] & V \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.2.4}$$

Suppose that E_L admits a $\mathcal{V}$–valued Lie algebroid connection $\nabla : V \to \mathrm{At}_\varphi(E_L)$. Then $\widetilde{\rho}_L \circ \nabla = \mathrm{Id}_V$. Then the subsheaf

$$\mathcal{E}_\nabla := \mathrm{At}(q')^{-1} (\nabla(V)) \subseteq \mathrm{At}_\varphi(E_P)$$

fits into the following short exact sequence of $\mathcal{O}_X$–modules

$$0 \to E_P(\mathfrak{n}) \to \mathcal{E}_\nabla \to V \to 0 \tag{3.2.5}$$

on X whose splitting gives rise to a $\mathcal{V}$–valued connection on E_P. Note that the above short exact sequence (3.2.5) defines a cohomology class

$$\Phi(E_P, L, \nabla) \in H^1(X, E_P(\mathfrak{n}) \otimes V^*), \tag{3.2.6}$$

which vanishes if and only if the exact sequence in (3.2.5) splits $\mathcal{O}_X$–linearly. From this, we have the following result.

Proposition 3.2.7. *With the above notations, if $H^1(X, E_P(\mathfrak{n}) \otimes V^*) = 0$, then a $\mathcal{V}$–valued Lie algebroid connection on E_L gives rise to a $\mathcal{V}$–valued Lie algebroid connection on E_P.*

4. Existence of Lie Algebroid Connections

In this section we assume that X is an irreducible smooth complex projective curve of genus $g \geq 2$. The *degree* of a coherent sheaf of $\mathcal{O}_X$–modules E on X is defined by

$$\deg(E) := \int_X c_1(E) \in \mathbb{Z},$$

where $c_1(E)$ stands for the first Chern class of E. The rational number

$$\mu(E) := \frac{\deg(E)}{\mathrm{rank}(E)}$$

is called the *slope* of E.

Definition 4.0.1. A vector bundle E on X is said to be *stable* (resp., *semistable*) if for any non-zero proper subsheaf F of E we have $\mu(F) < \mu(E)$ (resp., $\mu(F) \leq \mu(E)$).

The notion of slope semistablity and stability has a natural generalization to the case of principal G–bundles on X. Let G be a reductive linear algebraic group over $\mathbb{C}$. If a principal G–bundle E_G on X admits a holomorphic reduction $E_P \subseteq E_G$ of its structure group to a parabolic subgroup $P \subseteq G$, for any character $\chi : P \to \mathbb{G}_m$ of P, we get a holomorphic line bundle

$$\chi_* E_P := E_P \times^{\chi} \mathbb{G}_a$$

on X.

Definition 4.0.2. [Ram96, Ram75] A principal G–bundle E_G on X is said to be *semistable* (resp., *stable*) if for any reduction $E_P \subseteq E_G$ of the structure group of E_G to a proper parabolic subgroup $P \subseteq G$, and any nontrivial dominant character $\chi : P \to \mathbb{G}_m$, we have $\deg(\chi_* E_P) \leq 0$(resp., < 0).

Fix a Lie algebroid $\mathcal{V} = (V, [\cdot, \cdot], \varphi)$ on X such that the underlying vector bundle V of $\mathcal{V}$ is stable. Let

$$\mu(\mathcal{V}) := \frac{\deg(V)}{\mathrm{rank}(V)}$$

be the *slope* of the underlying vector bundle V of the Lie algebroid $\mathcal{V}$. Note that TX is a line bundle on X with the slope $\mu(TX) = 2 - 2g$.

If $\mu(\mathcal{V}) > 2 - 2g = \mu(TX)$, then both V and TX being stable vector bundles on X we have $H^0(X, \mathrm{Hom}(V, TX)) = 0$ (see [HL10, Proposition 1.2.7]), and hence $\varphi = 0$ in this case. Then a $\mathcal{V}$–valued Lie algebroid connection on E_G is just a global section of $\mathrm{ad}(E_G) \otimes V^*$; so we may take the zero section in $H^0(X, \mathrm{ad}(E_G) \otimes V^*)$, in particular.

If $\mu(\mathcal{V}) = 2 - 2g = \mu(TX)$, then any non-zero $\mathcal{O}_X$–module homomorphism $\varphi : V \to TX$ is an isomorphism (see [HL10, Proposition 1.2.7]). Then we may replace V with TX so that a $\mathcal{V}$–valued Lie algebroid connection on E_G is nothing but a holomorphic connection on E_G. This case is studied in detail in [AB02].

Now we assume that $\mu(\mathcal{V}) < 2 - 2g = \mu(TX)$. Then we have the following.

Proposition 4.0.3. *Let G be a reductive linear algebraic group over $\mathbb{C}$. With the above assumptions on $\mathcal{V}$, any semistable principal G–bundle E_G on X admits a $\mathcal{V}$–valued Lie algebroid connection.*

Proof. Let $\Phi_{\mathcal{V}}(E_G) \in H^1(X, \mathrm{ad}(E_G) \otimes V^*)$ be the $\mathcal{V}$–valued Atiyah class of E_G. By Serre duality, we have

$$H^1(X, \mathrm{ad}(E_G) \otimes V^*) \cong H^0(X, \mathrm{ad}(E_G)^* \otimes V \otimes K_X)^*,$$

where $K_X = \Omega^1_X$ is the canonical line bundle on X. Since E_G is semistable by assumption, its adjoint bundle $\mathrm{ad}(E_G)$ is semistable by [AB01, Proposition 2.10]. Then the tensor product bundle $\mathrm{ad}(E_G)^* \otimes V \otimes K_X$ is semistable (see [HL10, Theorem 3.1.4]). Since G is reductive, the adjoint bundle $\mathrm{ad}(E_G)$ is isomorphic to its dual, and hence $\deg(\mathrm{ad}(E_G)) = 0$. Then we have

$$\mu(\mathrm{ad}(E_G)^* \otimes V \otimes K_X) = \mu(K_X) + \mu(V) = 2g - 2 + \mu(V) < 0.$$

Then by [HL10, Proposition 1.2.7] $H^0(X, \mathrm{ad}(E_G)^* \otimes V \otimes K_X) = 0$, and hence $\Phi_{\mathcal{V}}(E_G) = 0$. Hence the result follows. □

Theorem 4.0.4. *Fix a Lie algebroid $\mathcal{V} = (V, [\cdot\,,\,\cdot], \varphi)$ on X such that V is stable with $\mu(V) < 2 - 2g = \deg(TX)$. Let G be a reductive linear algebraic group over $\mathbb{C}$. Let E_G be a principal G–bundle on X. Then E_G admits a $\mathcal{V}$–valued Lie algebroid connection.*

Proof. Let E_G be a principal G–bundle on X. Since G is reductive, by [AAB02, Theorem 1] E_G admits a canonical reduction $E_P \subseteq E_G$ of its structure group to a parabolic subgroup $P \subseteq G$ such that the associated principal L–bundle

$$E_L := E_P \times^q L$$

obtained by extension of the structure group of E_P by the quotient homomorphism

$$q : P \longrightarrow P/\mathfrak{R}_u(P) \cong L,$$

is semistable; here L is the *Levi factor* of P, a closed connected reductive subgroup of P such that the restriction of the quotient homomorphism $q : P \to P/\mathfrak{R}_u(P)$ to $L \subseteq P$ is an isomorphism of algebraic groups over $\mathbb{C}$. Then by Proposition 4.0.3 the principal L–bundle E_L admits a $\mathcal{V}$–valued Lie algebroid connection. Since $\mu_{\min}(E_P(\mathfrak{n})) \geq 0$ by [AAB02] and $V \otimes K_X$ is semistable with $\mu(V \otimes K_X) < 0$, it follows that $\mathrm{Hom}(E_P(\mathfrak{n}), V \otimes K_X) = 0$, and hence $H^1(X, E_P(\mathfrak{n}) \otimes V^*) = 0$ by Serre duality. Then by Proposition 3.2.7 that E_P admits a $\mathcal{V}$–valued Lie algebroid connection, and then by Proposition 3.1.4 E_G admits a $\mathcal{V}$–valued Lie algebroid connection. This completes the proof. □

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