MA4104: Algebraic Topology

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Disclaimer: This note will be updated from time to time.
If you find any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\coprod$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
$A := \ldots$	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Algebraic Topology

MA4104 Syllabus

Homotopy theory: Review of quotient topology, path homotopy, definition of fundamental group, covering spaces, path and homotopy lifting, fundamental group of S^1, deformation retraction, Brouwer's fixed point theorem, Borsuk-Ulam theorem, Van-Kampen's theorem, fundamental group of surfaces, universal covering space, correspondence between covering spaces and subgroups of fundamental group.

Homology Theory: Simplicial complexes and maps, homology groups, computation for surfaces.

References:

1. Algebraic Topology by Allen Hatcher.
2. Topology by J. R. Munkres.
3. Algebraic Topology by J. R. Munkres.
4. Algebraic Topology by Tammo Tom Dieck.
5. A Concise Course in Algebraic Topology by J. P. May.

[[This is a preliminary draft! We mainly follow Hatcher's book.]]

[[It will be expanded with more details, additional results, examples and exercises.]]

1.1 Review of quotient spaces

1.1.1 Examples

1.1.2 CW Complex

1.1.3 Grassmanians

1.2 Homotopy of maps

Let I be the closed interval $[0,1] \subset \mathbb{R}$. Let X and Y be topological spaces.

Definition 1.2.1. Let $f_0, f_1 : X \to Y$ be continuous maps. We say that f_0 *is homotopic to* f_1, written as $f_0 \simeq f_1$, if there is a continuous map

$$F : X \times I \longrightarrow Y$$

such that $F(x,0) = f_0(x)$, $\forall\, x \in X$, and $F(x,1) = f_1(x)$, $\forall\, x \in X$. In this case, the continuous map F is called the *homotopy* from f_0 to f_1. A continuous map $f : X \to Y$ is said to be *null homotopic* if f is homotopic to a constant map from X into Y.

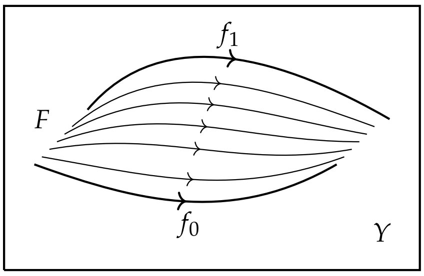


FIGURE 1.1: Homotopy

Example 1.2.2. 1. Let X be a space. Then any two continuous maps $f, g : X \to \mathbb{R}^2$ are homotopic. To see this, note that the map $F : X \times I \to \mathbb{R}^2$ defined by

$$F(x,t) = (1-t)f(x) + tg(x), \;\; \forall\, (x,t) \in X \times I,$$

is continuous and satisfies $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in X$. Thus F is a homotopy from f to g; such a homotopy is called a *straight-line homotopy*, because for each $x \in X$, it movies $f(x)$ to $g(x)$ along the straight-line segment joining them.

Before proceeding further, let us recall the following useful result from basic topology course, that we need frequently in this course.

Lemma 1.2.3 (Joining continuous maps)**.** *Let A and B be two closed subsets of topological space X such that $X = A \cup B$. Let Y be any topological space. Let $f : A \to Y$ and $g : B \to Y$ be continuous*

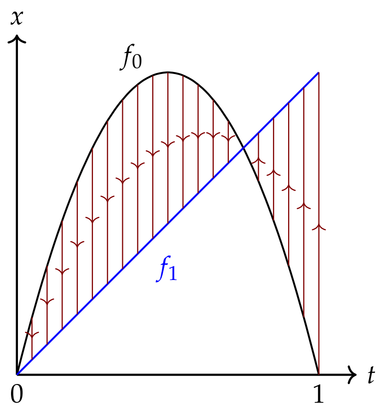


FIGURE 1.2: Example of a straight-line homotopy

maps such that $f(x) = g(x)$, for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if} \quad x \in A, \\ g(x), & \text{if} \quad x \in B, \end{cases}$$

is continuous.

Proof. Let $Z \subseteq Y$ be a closed subset. It is enough to check that $h^{-1}(Z)$ is closed in X. Note that

$$\begin{aligned} h^{-1}(Z) &= (h^{-1}(Z) \cap A) \bigcup (h^{-1}(Z) \cap B) \\ &= f^{-1}(Z) \bigcup g^{-1}(Z). \end{aligned}$$

Since f and g are continuous, $f^{-1}(Z)$ is closed in A and $g^{-1}(Z)$ is closed in B. Since A and B are closed in X, both $f^{-1}(Z)$ and $g^{-1}(Z)$ are closed in X, and so is their union $h^{-1}(Z)$. This completes the proof. □

Lemma 1.2.4. *The relation "being homotopic maps" is an equivalence relation on the set $C(X, Y)$ of all continuous maps from X into Y.*

Proof. For any $f \in C(X, Y)$, taking

$$F : X \times I \to Y, \ \ (x, t) \mapsto f(x)$$

we see that f is homotopic to itself, and hence "being homotopic maps" is a reflexive relation. Let $f_0, f_1 \in C(X, Y)$ be such that f_0 is homotopic to f_1 with homotopy F. Then the continuous map

$$G : X \times I \to Y, \ \ (x, t) \mapsto F(x, 1 - t)$$

is a homotopy from f_1 to f_0. So "being homotopic maps" is a symmetric relation. Let $f_0, f_1, f_2 \in C(X, Y)$ be such that $f_0 \simeq f_1$ with a homotopy F, and $f_1 \simeq f_2$ with a homotopy G. Consider the map $H : X \times I \to Y$ defined by

$$H(x,t) := \begin{cases} F(x,2t), & \text{if} \quad t \in [0, \frac{1}{2}], \\ G(x,2t-1), & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

Since at $t = \frac{1}{2}$, we have $F(x,2t) = F(x,1) = f_1(x) = G(x,0) = G(x,2t-1)$, for all $x \in X$, we see that H is a well-defined continuous map (c.f., Lemma 1.2.3). Clearly H satisfies $H(x,0) = f_0(x)$ and $H(x,1) = f_2(x)$, for all $x \in X$. Therefore, H is a homotopy from f_0 to f_2. Thus "being homotopic maps" is a transitive relation, and hence is an equivalence relation on $C(X,Y)$. $\square$

Exercise 1.2.5. Let $f, g \in C(X,Y)$, and $F : X \times I \to Y$ be a homotopy from f to g. Use F to construct a homotopy G from f to g with $G \neq F$. Therefore, homotopy between two maps need not be unique. (*Hint: take* $G(x,t) = F(x,t^2)$).

Lemma 1.2.6. *Let $f_0, f_1 : X \to Y$ be two continuous maps such that f_0 is homotopic to f_1. Then for any spaces Z and W, and continuous maps $g : Z \to X$ and $h : Y \to W$, we have $f_0 \circ g \simeq f_1 \circ g$ and $h \circ f_0 \simeq h \circ f_1$.*

$$\begin{tikzcd} Z \arrow[r, "g"] & X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, "h"] & W. \end{tikzcd}$$

Proof. Let $F : X \times I \to Y$ be a continuous map such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$, for all $x \in X$. Define $G : Z \times I \to Y$ by setting

$$G(z,t) = F(g(z),t), \ \forall\, (z,t) \in Z \times I.$$

Clearly G is a continuous function with $G(z,0) = F(g(z),0) = (f_0 \circ g)(z)$, and $G(z,1) = F(g(z),1) = (f_1 \circ g)(z)$, for all $z \in Z$. Therefore, G gives a homotopy $f_0 \circ g \simeq f_1 \circ g$. Similarly, taking

$$H : X \times I \to W, \ \ (x,t) \mapsto h(F(x,t)),$$

we see that H is a continuous map satisfying $H(x,0) = h(F(x,0)) = (h \circ f_0)(x)$ and $H(x,1) = h(F(x,0)) = (h \circ f_1)(x)$, for all $x \in X$. Therefore, H gives a homotopy $h \circ f_0 \simeq h \circ f_1$. $\square$

Definition 1.2.7. Let $f_0, f_1 : (X,x_0) \to (Y,y_0)$ be continuous maps of pointed topological spaces. A *homotopy* from f_0 to f_1 is a continuous map $F : X \times I \to Y$ such that

(i) $F(x,0) = f_0(x)$, $\forall\, x \in X$,

(ii) $F(x,1) = f_1(x)$, $\forall\, x \in X$, and

(iii) $F(x_0,t) = y_0$, $\forall\, t \in [0,1]$.

When we talk about homotopy of continuous maps of pointed topological spaces, we always mean that the homotopy preserve the marked points in the sense of (iii) mentioned above.

Exercise 1.2.8. Let $f_0, f_1 : (X,x_0) \to (Y,y_0)$ be two continuous maps of pointed topological spaces. If f_0 is homotopic to f_1 in the sense of Definition 1.2.7, show that for any spaces Z and W, and continuous maps $g : (Z,z_0) \to (X,x_0)$ and $h : (Y,y_0) \to (W,w_0)$, we have

(i) $f_0 \circ g$ is homotopic to $f_1 \circ g$ in the sense of Definition 1.2.7, and

(ii) $h \circ f_0$ is homotopic to $h \circ f_1$ in the sense of Definition 1.2.7.

Definition 1.2.9. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there exist a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. In this case, we say that X *is homotopy equivalent to* Y (or, X *and* Y *have the same homotopy type*), and write it as $X \simeq Y$.

Lemma 1.2.10. *Being homotopy equivalent spaces is an equivalence relation.*

Proof. For any space X, we can take $f = g = \mathrm{Id}_X$ to get $f \circ g = \mathrm{Id}_X = g \circ f$ so that X is homotopy equivalent to itself (verify!). It follows from the Definition 1.2.9 that the relation "being homotopy equivalent spaces" is symmetric. Let X, Y and Z be topological spaces such that $X \simeq Y$ and $Y \simeq Z$. Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be homotopy equivalences. Then there are continuous maps $g_1 : Y \to X$ and $g_2 : Z \to Y$ such that $g_1 \circ f_1 \simeq \mathrm{Id}_X$, $f_1 \circ g_1 \simeq \mathrm{Id}_Y$, $g_2 \circ f_2 \simeq \mathrm{Id}_Y$ and $f_2 \circ g_2 \simeq \mathrm{Id}_Z$. Now using Lemma 1.2.6 we have

$$\begin{aligned}(f_2 \circ f_1) \circ (g_1 \circ g_2) &= f_2 \circ (f_1 \circ g_1) \circ g_2 \\ &\simeq f_2 \circ \mathrm{Id}_Y \circ g_2 \\ &= f_2 \circ g_2 \simeq \mathrm{Id}_Z \,.\end{aligned}$$

Similarly, we have $(g_1 \circ g_2) \circ (f_2 \circ f_1) \simeq \mathrm{Id}_X$. Therefore, $f_2 \circ f_1 : X \to Z$ is a homotopy equivalence, and hence $X \simeq Z$. Thus "being homotopy equivalent spaces" is a transitive relation, and hence is an equivalence relation. □

Definition 1.2.11. A space X is said to be *contractible* if the identity map $\mathrm{Id}_X : X \to X$ is null homotopic.

Exercise 1.2.12. Show that a contractible space is path-connected.

Corollary 1.2.13. *A space X is contractible if and only if given any topological space T, any two continuous maps $f, g : T \to X$ are homotopic.*

Proof. Suppose that X is contractible. Let T be any topological space, and let $f, g : T \to X$ be any two continuous maps. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ of X is homotopic to a constant map $c_{x_0} : X \to X$ given by $c_{x_0}(x) = x_0$, $\forall\, x \in X$. Then $f = \mathrm{Id}_X \circ f$ is homotopic to the constant map $c_{x_0} \circ f : T \to X$. Similarly, g is homotopic to the constant map $c_{x_0} \circ g : T \to X$. Since $c_{x_0} \circ f = c_{x_0} \circ g$, and being homotopic maps is an equivalence relation by Lemma 1.2.4, we see that f is homotopic to g. Converse part is obvious. □

1.3 Fundamental group

1.3.1 Construction

A *path* in X is a continuous map $\gamma : I \to X$; the point $\gamma(0) \in X$ is called the *initial point* of γ, and $\gamma(1) \in X$ is called the *terminal point* or the *final point* of γ.

Definition 1.3.1. Fix two points $x_0, x_1 \in X$. Two paths $f, g : I \to X$ with the same initial point x_0 and terminal point x_1 are said to be *path homotopic* if

(i) $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, and

(ii) there is a continuous map $F : I \times I \to X$ such that for each $t \in I$, the map

$$\gamma_t : I \to X, \;\; s \mapsto F(s, t)$$

is a path in X from x_0 to x_1, and that $\gamma_0 = f$ and $\gamma_1 = g$.

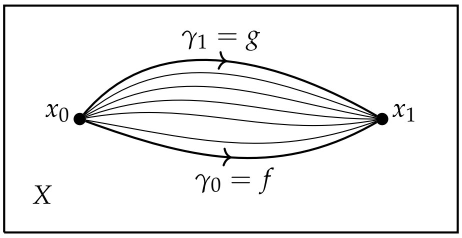


FIGURE 1.3: Path homotopy

Exercise 1.3.2. Given $x_0, x_1 \in X$, let

$$Path(X; x_0, x_1) := \{f : I \to X \,|\, f(0) = x_0, \, f(1) = x_1\}$$

be the set of all paths in X starting at x_0 and ending at x_1. Show that being path homotopic is an equivalence relation on $Path(X; x_0, x_1)$. (*Hint: Follow the proof of Lemma 1.2.4*).

Remark 1.3.3. If $\gamma, \delta : I \to X$ are two paths in X, we use the symbol $\gamma \simeq \delta$ to mean γ and δ are path-homotopic in X in the sense of Definition 1.3.1. Unless explicitly mentioned, by a *homotopy between two paths* we always mean a path-homotopy between them.

A *loop* in X is a path $\gamma : I \to X$ with the same initial and terminal point: i.e., $\gamma(0) = \gamma(1) = x_0 \in X$; the point x_0 is called the base point of the loop γ. For a loop $\gamma : I \to X$ based at $x_0 \in X$, let

$$[\gamma] := \{\delta : I \to X \,|\, \delta(0) = \delta(1) = x_0 \text{ and } \delta \simeq \gamma\},$$

the homotopy equivalence class of γ. Fix a base point $x_0 \in X$, and let

$$\pi_1(X, x_0) := \{[\gamma] \;\big|\; \gamma : I \to X \text{ with } \gamma(0) = \gamma(1) = x_0\}$$

be the set of all equivalence classes of loops in X based at x_0. Next we define a binary operation on $\pi_1(X, x_0)$ and show that it is a group.

Given any two loops $\gamma_1, \gamma_2 : I \to X$ in X with the base point $x_0 \in X$, we define the *product of* γ_1 *with* γ_2 to be the map $\gamma_1 \star \gamma_2 : I \to X$ defined by

$$(\gamma_1 \star \gamma_2)(t) := \begin{cases} \gamma_1(2t), & \text{if} \quad t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1), & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases} \tag{1.3.4}$$

That is, we first travel along γ_1 with double speed from $t = 0$ to $t = \frac{1}{2}$, and then along γ_2 from $t = \frac{1}{2}$ to $t = 1$. Clearly $\gamma_1 \star \gamma_2$ is a continuous map with $(\gamma_1 \circ \gamma_2)(0) = (\gamma_1 \circ \gamma_2)(1) = x_0$, and hence $\gamma_1 \circ \gamma_2$ is a loop in X with the base point x_0. Note that $\gamma_1 \star \gamma_2 \neq \gamma_2 \star \gamma_1$, in general (Find such an example).

Remark 1.3.5. In fact, we shall see later examples of topological spaces X admitting loops $\gamma_1, \gamma_2 : I \to X$ with the same base point $x_0 \in X$ such that $\gamma_1 \star \gamma_2$ is not homotopic to $\gamma_2 \star \gamma_1$.

Proposition 1.3.6. *Let γ_1, γ_2, δ_1, δ_2 be loops in X based at x_0. If $\gamma_1 \simeq \delta_1$ and $\gamma_2 \simeq \delta_2$, then $(\gamma_1 \star \gamma_2) \simeq (\delta_1 \star \delta_2)$. Consequently, the map*

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0), \ \ ([\gamma_1], [\gamma_2]) \mapsto [\gamma_1 \star \gamma_2] \tag{1.3.7}$$

is well-defined, and hence is a binary operation on the set $\pi_1(X, x_0)$.

Proof. Let $F : I \times I \to X$ be a homotopy from $F(-, 0) = \gamma_1$ to $F(-, 1) = \delta_1$, and let $G : I \times I \to X$ be a homotopy from $G(-, 0) = \gamma_2$ to $G(-, 1) = \delta_2$. Define a map $F \star G : I \times I \to X$ by sending $(s, t) \in I \times I$ to

$$(F \star G)(s, t) := \begin{cases} F(2s, t), & \text{if} \quad 0 \leq s \leq 1/2, \\ G(2s - 1, t), & \text{if} \quad 1/2 \leq s \leq 1. \end{cases}$$

Clearly $F \star G$ is a continuous map with $(F \star G)(-, 0) = \gamma_1 \star \gamma_2$ and $(F \star G)(-, 1) = \delta_1 \star \delta_2$. Therefore, $\gamma_1 \star \gamma_2 \simeq \delta_1 \star \delta_2$. $\square$

Theorem 1.3.8. *The set $\pi_1(X, x_0)$ together with the binary operation (1.3.7) defined in Proposition 1.3.6 is a group, known as the fundamental group of X with base point $x_0 \in X$.*

To prove this theorem, we use the following technical tool (Lemma 1.3.10).

Definition 1.3.9. A *reparametrization* of a path $\gamma : I \to X$ is defined to be a composition $\gamma \circ \varphi$, where $\varphi : I \to I$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Lemma 1.3.10. *A reparametrization of a path preserves its homotopy class.*

Proof. Let $\gamma : I \to X$ be a path in X. Let

$$\gamma \circ \varphi : I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

be a reparametrization of γ in X, for some continuous map $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Consider the straight-line homotopy from φ to the identity map of I given by

$$\varphi_t(s) := (1 - t)\varphi(s) + ts, \ \ \forall\, s, t \in I.$$

Now it is easy to check that the map

$$F : I \times I \to X, \ \ (s, t) \mapsto \gamma(\varphi_t(s)),$$

is continuous and satisfies $F(s, 0) = (\gamma \circ \varphi)(s)$ and $F(s, 1) = \gamma(s)$, for all $s \in I$. Therefore, $\gamma \circ \varphi \simeq \gamma$ via the homotopy F. $\square$

Proof of Theorem 1.3.8. We need to verify group axioms.

Associativity: Given any three loops $\gamma_1, \gamma_2, \gamma_3 : I \to X$ based at x_0, it is enough to show that $(\gamma_1 \star \gamma_2) \star \gamma_3 \simeq \gamma_1 \star (\gamma_2 \star \gamma_3)$.

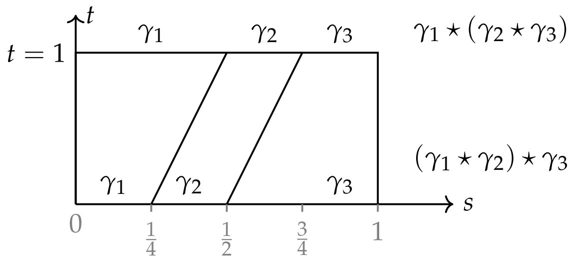


FIGURE 1.4: Homotopy for associativity

Note that

$$((\gamma_1 \star \gamma_2) \star \gamma_3)(t) = \begin{cases} \gamma_1(4t), & \text{if} \quad 0 \le t \le 1/4, \\ \gamma_2(4t-1), & \text{if} \quad 1/4 \le t \le 1/2, \\ \gamma_3(2t-1), & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

and

$$(\gamma_1 \star (\gamma_2 \star \gamma_3))(t) = \begin{cases} \gamma_1(2t), & \text{if} \quad 0 \le t \le 1/2, \\ \gamma_2(4t-2), & \text{if} \quad 1/2 \le t \le 3/4, \\ \gamma_3(4t-3), & \text{if} \quad 3/4 \le t \le 1. \end{cases}$$

It's an easy exercise to check that $\gamma_1 \star (\gamma_2 \star \gamma_3)$ is a reparametrization of $(\gamma_1 \star \gamma_2) \star \gamma_3$ by a piece-wise linear function (hence, continuous) $\varphi : I \to I$ defined by

$$\varphi(t) = \begin{cases} t/2, & \text{if} \quad 0 \le t \le 1/2, \\ t - \frac{1}{4}, & \text{if} \quad 1/2 \le t \le 3/4, \\ 2t-1, & \text{if} \quad 3/4 \le t \le 1, \end{cases}$$

(see Figure 1.5). Then using Lemma 1.3.10 we conclude that $\gamma_1 \star (\gamma_2 \star \gamma_3) \simeq (\gamma_1 \star \gamma_2) \star \gamma_3$.

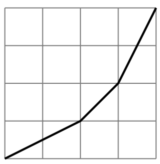


FIGURE 1.5: Graph of φ

Existence of identity: Let $e \in \pi_1(X, x_0)$ be the homotopy class of *constant loop*,

$$c_{x_0} : I \to X, \;\; t \mapsto x_0,$$

at x_0. Let $\gamma : I \to X$ be any loop in X based at x_0. Since $\gamma \star c_{x_0}$ is a reparametrization of γ via the function

$$\psi(t) := \begin{cases} 2t, & \text{if} \quad 0 \le t \le 1/2, \\ 1, & \text{if} \quad 1/2 \le t \le 1, \end{cases}$$

by Lemma 1.3.10 we have $\gamma \star c_{x_0} \simeq \gamma$. Similarly, $c_{x_0} \star \gamma$ is a reparametrization of γ by via the function

$$\eta(t) := \begin{cases} 0, & \text{if } 0 \leq t \leq 1/2, \\ 2t-1, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

by Lemma 1.3.10 we have $c_{x_0} \star \gamma \simeq \gamma$.

Existence of inverse: Given any loop γ in X based at x_0, we can define its *inverse loop* or *opposite loop* $\overline{\gamma} : I \to X$ by setting $\overline{\gamma}(t) = \gamma(1-t)$, for all $t \in I$. We need to show that $\gamma \star \overline{\gamma} \simeq c_{x_0}$ and $\overline{\gamma} \star \gamma \simeq c_{x_0}$. To show $\gamma \star \overline{\gamma} \simeq c_{x_0}$, consider the map $H : I \times I \to X$ given by

$$H(s,t) := f_t(s) \star g_t(s), \ \forall \ (s,t) \in I \times I,$$

where $f_t : I \to X$ is the path defined by

$$f_t(s) = \begin{cases} \gamma(s), & \text{for } 0 \leq s \leq 1-t, \\ \gamma(1-t), & \text{for } 1-t \leq s \leq 1, \end{cases}$$

and $g_t : I \to X$ is the inverse path of f_t, i.e., $g_t(s) = f_t(1-s)$, $\forall \, s \in I$. It is an easy exercise to check that H is a continuous map satisfying

$$H(s,0) = \gamma \star \overline{\gamma}, \ \text{and} \ \ H(s,1) = c_{x_0}, \ \forall \, s \in I.$$

The homotopy H can be understood using the Figure 1.6. In the bottom line $t = 0$, we have

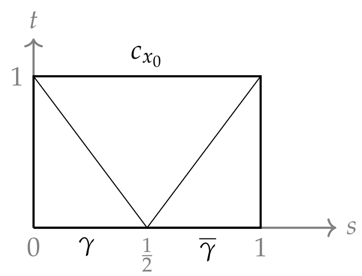


FIGURE 1.6: Homotopy H

$\gamma \star \overline{\gamma}$ while on the top line $t = 1$ we have the constant loop c_{x_0}. And below the 'V' shape we let $H(s,t)$ be independent of t while above the 'V' shape we let $H(s,t)$ be independent of s. Therefore, we have $\gamma \star \overline{\gamma} \simeq c_{x_0}$. Interchanging the roles of γ and $\overline{\gamma}$ in the above construction, we see that $\overline{\gamma} \star \gamma \simeq c_{x_0}$. Therefore, $\pi_1(X, x_0)$ is a group. $\square$

1.3.2 Functoriality

By a *pointed topological space* we mean a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. In the above construction, given a pointed topological space (X, x_0) we attached a group $\pi_1(X, x_0)$, known as the *fundamental group of X with the base point at x_0*. Next we see how fundamental group of a pointed space behaves under continuous maps and their compositions.

Let $f : (X, x_0) \to (Y, y_0)$ be a *continuous map of pointed spaces* (this means, $f : X \to Y$ is a continuous map with $f(x_0) = y_0$). Let $\gamma : I \to X$ be a loop in X based at x_0. Then the composition $f \circ \gamma$,

$$I \xrightarrow{\gamma} X \xrightarrow{f} Y,$$

is a loop in Y based at $f(x_0) = y_0$. Let $\gamma, \delta : I \to X$ be loops in X based at x_0. If $F : I \times I \to X$ is a homotopy from γ to δ, then $f \circ F : I \times I \to Y$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$ (see Lemma 1.2.6). Thus we have a well-defined map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]. \tag{1.3.11}$$

Proposition 1.3.12. *The map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is a group homomorphism.*

Proof. Note that for any two loops $\gamma, \delta : I \to X$ based at x_0, we have

$$\begin{aligned} f_*([\gamma \star \delta]) &= [f \circ (\gamma \star \delta)] \\ &= [(f \circ \gamma) \star (f \circ \delta)] \\ &= [f \circ \gamma] \cdot [f \circ \delta] \\ &= f_*([\gamma]) \cdot f_*([\delta]). \end{aligned}$$

□

Remark 1.3.13. If $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the homomorphism of fundamental group of a pointed topological space (X, x_0) induced by the identity map of (X, x_0) onto itself, then it follows from the construction of the map f_* given in (1.3.11) that $f_* = \mathrm{Id}_{\pi_1(X, x_0)}$, the identity map of $\pi_1(X, x_0)$ onto itself.

Proposition 1.3.14. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps of pointed spaces. Then $g_* \circ f_* = (g \circ f)_*$. In other words, the following diagram commutes.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \pi_1(Y, y_0) \arrow[d, "g_*"] \\ & \pi_1(Z, z_0) \end{tikzcd}$$

Proof. Left as an exercise. □

Corollary 1.3.15. *If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism of pointed spaces with its inverse $g : (Y, y_0) \to (X, x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups with its inverse $g_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.*

Proof. Since $g \circ f = \mathrm{Id}_{(X, x_0)}$ and $f \circ g = \mathrm{Id}_{(Y, y_0)}$, applying Proposition 1.3.14 we have $g_* \circ f_* = \mathrm{Id}_{\pi_1(X, x_0)}$ and $f_* \circ g_* = \mathrm{Id}_{\pi_1(Y, y_0)}$. □

Lemma 1.3.16. *Homotopic continuous maps of pointed topological spaces induces the same homomorphism of fundamental groups.*

Proof. Let $f, g : (X, x_0) \to (Y, y_0)$ be two continuous maps of pointed space. If f is homotopic to g in the sense of Definition 1.2.7, then for any loop $\gamma : I \to X$ based at x_0, using Exercise 1.2.8 (c.f. Lemma 1.2.6) we have $f \circ \gamma$ is homotopic to $g \circ \gamma$ in the sense of Definition 1.2.7, and hence $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma])$. Hence the result follows. □

Definition 1.3.17. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\text{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\text{ob}(\mathscr{C})$, there is a collection $\text{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \text{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \text{Mor}_{\mathscr{C}}(X, Y) \times \text{Mor}_{\mathscr{C}}(Y, Z) \to \text{Mor}_{\mathscr{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \text{ob}(\mathscr{C})$, and $f \in \text{Mor}_{\mathscr{C}}(X, Y)$, $g \in \text{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \text{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \text{ob}(\mathscr{C})$, there exists a morphism $\text{Id}_X \in \text{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \text{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \text{Id}_Y = f$ and $\text{Id}_Z \circ f = f$.

Example 1.3.18. (i) Let (Set) be the category of sets; its objects are sets and arrows are map of sets.

(ii) Let $(\mathcal{G}rp)$ be the category of groups; its objects are groups and arrows are group homomorphisms.

(iii) Let $(\mathcal{T}op)$ be the category of topological spaces; its objects are topological spaces and arrows are continuous maps.

(iv) Let $(\mathcal{R}ing)$ be the category of rings; its objects are rings and morphisms are ring homomorphisms.

Definition 1.3.19. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be an isomorphism if there is a morphism $g : Y \to X$ in $\mathscr{C}$ such that $g \circ f = \text{Id}_X$ and $f \circ g = \text{Id}_Y$.

Definition 1.3.20. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *covariant functor* (resp., a *contravariant functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a rule

$$\mathcal{F} : \mathscr{C} \to \mathscr{D}$$

which associate to each object $X \in \mathscr{C}$ an object $\mathcal{F}(X) \in \mathscr{D}$, and to each morphism $f \in \text{Mor}_{\mathscr{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \text{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ (resp., a morphism $\mathcal{F}(f) \in \text{Mor}_{\mathscr{D}}(\mathcal{F}(Y), \mathcal{F}(X))$) such that

(i) $\mathcal{F}(\text{Id}_X) = \text{Id}_{\mathcal{F}(X)}$, for all $X \in \mathscr{C}$, and

(ii) given any objects $X, Y, Z \in \mathscr{C}$ and morphisms $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$, we have $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (resp., $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$).

If $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is a functor, for each ordered pair of objects $X, Y \in \mathscr{C}$ we denote by $\mathcal{F}_{X,Y}$ the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Mor}_{\mathscr{C}}(X, Y) \to \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y)),$$

defined by $\mathcal{F}_{X,Y}(f) := \mathcal{F}(f)$. The same notation is used for contravariant functor.

Definition 1.3.21. A functor $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is said to be

(i) *faithful* if $\mathcal{F}_{X,Y}$ is injective, $\forall\, X, Y \in \mathscr{C}$.

(ii) *full* if $\mathcal{F}_{X,Y}$ is surjective, $\forall\, X, Y \in \mathscr{C}$.

(iii) *fully faithful* if $\mathcal{F}_{X,Y}$ is bijective, $\forall\, X, Y \in \mathscr{C}$.

(iv) *essentially surjective* if given any object $Y \in \mathscr{D}$, there is an object $X \in \mathscr{C}$ and an isomorphism $\varphi : \mathcal{F}(X) \xrightarrow{\simeq} Y$ in $\mathscr{D}$.

(v) *equivalence of categories* if there is a functor $\mathcal{G} : \mathscr{D} \to \mathscr{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathscr{D}}$. This is equivalent to say that $\mathcal{F}$ is fully faithful and essentially surjective.

Remark 1.3.22. Let $\mathcal{T}op_0$ be the category of pointed topological spaces; its objects are pointed topological space, and given any two pointed topological spaces (X, x_0) and (Y, y_0), a morphism $f : (X, x_0) \to (Y, y_0)$ in $\mathcal{T}op_0$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. Then it follows from Propositions 1.3.12 and 1.3.14 and the Remark 1.3.13 that

$$\begin{aligned} \pi_1 \ : \mathcal{T}op_0 &\longrightarrow (\mathcal{G}rp) \\ (X, x_0) &\mapsto \pi_1(X, x_0) \\ f &\mapsto f_* \end{aligned}$$

is a *covariant functor* from the category of pointed topological spaces to the category of groups. It follows from Lemma 1.3.16 that the functor π_1 is not faithful. It is a non-trivial fact that π_1 is not full. (i.e., there exist pointed topological spaces (X, x_0) and (Y, y_0), and a group homomorphism $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that $\varphi \neq f_*$, for all continuous map $f : (X, x_0) \to (Y, y_0)$.) However, π_1 is *essentially surjective* (i.e., given any group G there is a pointed topological space (X, x_0) such that $\pi_1(X, x_0) \cong G$).

1.3.3 Dependency on base point

Now we investigate relation between fundamental groups of X for different choices of base point. Let $x_0, x_1 \in X$. Let $f : I \to X$ be a path in X joining x_0 to x_1, i.e., f is a continuous map satisfying $f(0) = x_0$ and $f(1) = x_1$. We define the *opposite path* of f to be the map

$$\overline{f} : I \to X, \ \ t \mapsto f(1-t); \tag{1.3.23}$$

note that $\overline{f}(0) = x_1$ and $\overline{f}(1) = x_0$, hence $\overline{f}$ is a path from x_1 to x_0.

Exercise 1.3.24. Show that $f \star \overline{f} \simeq c_{x_0}$ and $\overline{f} \star f \simeq c_{x_1}$, where $\simeq$ stands for path-homotopy relation (see Definition 1.3.1).

Given a loop γ in X based at x_1, we can define $\widetilde{\gamma} := f \star \gamma \star \overline{f}$. Note that $\widetilde{\gamma} : I \to X$ is a continuous map satisfying $\widetilde{\gamma}(0) = f(0) = x_0 = \overline{f}(1) = \widetilde{\gamma}(1)$, and hence is a loop in X based at x_1. Strictly speaking, we have two choices to define this product $\widetilde{\gamma}$, namely $(f \star \gamma) \star \overline{f}$ or $f \star (\gamma \star \overline{f})$, but we are interested in only homotopy classes of paths, and following the proof of associativity as in Theorem 1.3.8 one can easily verify that $(f \star \gamma) \star \overline{f} \simeq f \star (\gamma \star \overline{f})$, therefore, we just fix one ordering of taking products to define $\widetilde{\gamma}$.

If γ and γ' are two loops in X based at x_1 with $\gamma \simeq \gamma'$ via a homotopy $\{h_t\}_{t \in I}$, then $\{f \star h_t \star \overline{f}\}_{t \in I}$ is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma'}$ (*Exercise: Write down the homotopy explicitly and check details*). Thus, we have a well-defined map

$$\beta_f : \pi_1(X, x_1) \to \pi_1(X, x_0), \ \ [\gamma] \mapsto [(f \star \gamma) \star \overline{f}]. \tag{1.3.25}$$

Proposition 1.3.26. *The map β_f defined in (1.3.25) is a group isomorphism.*

Proof. Since $\overline{f} \star f \simeq c_{x_0}$ for any two loops γ and δ in X based at x_1, using Exercise 1.3.24, we have

$$\begin{aligned} f \star (\gamma \star \delta) \star \overline{f} &\simeq f \star \gamma \star c_{x_0} \star \delta \star \overline{f} \\ &\simeq (f \star \gamma \star \overline{f}) \star (f \star \delta \star \overline{f}). \end{aligned}$$

Therefore, $\beta_f([\gamma \star \delta]) = [f \star (\gamma \star \delta) \star \overline{f}] = [f \star \gamma \star \overline{f}][f \star \delta \star \overline{f}] = \beta_f([\gamma])\beta_f([\delta])$, and hence β_f is a group homomorphism. To show β_f an isomorphism of groups, it is enough to show that the group homomorphism

$$\beta_{\overline{f}} : \pi_1(X, x_0) \to \pi_1(X, x_1), \ \ [\gamma] \mapsto [\overline{f} \star \gamma \star f]$$

is the inverse of β_f. Indeed, for any $\gamma \in \pi_1(X, x_0)$ we have

$$\begin{aligned} \beta_f(\beta_{\overline{f}}([\gamma])) &= \beta_f([\overline{f} \star \gamma \star f]) \\ &= [f \star \overline{f} \star \gamma \star f \star \overline{f}] \\ &= [c_{x_0} \star \gamma \star c_{x_0}] = [\gamma], \end{aligned}$$

and similarly, for any $\delta \in \pi_1(X, x_1)$ we have

$$\begin{aligned} \beta_{\overline{f}}(\beta_f([\delta])) &= \beta_{\overline{f}}([f \star \delta \star \overline{f}]) \\ &= [\overline{f} \star f \star \delta \star \overline{f} \star f] \\ &= [c_{x_1} \star \delta \star c_{x_1}] = [\delta]. \end{aligned}$$

Therefore, $\beta_{\overline{f}}$ is the inverse homomorphism of β_f, and hence both of them are isomorphisms. $\square$

Remark 1.3.27. Thus if X is a path connected space, up to isomorphism its fundamental group is independent of choice of base point, and so we may denote it by $\pi_1(X)$ without specifying its base point.

Proposition 1.3.28. *Let $f, g : X \to Y$ be two continuous maps of topological spaces. Fix a point $x_0 \in X$, and let $y_0 = f(x_0)$ and $y_1 = g(x_0)$. Let $F : X \times I \to Y$ be a continuous map such that $F(-, 0) = f$ and $F(-, 1) = g$. Then for any loop γ in X based at x_0, the loop $f \circ \gamma$ is path-homotopic to the loop $F_0 \star (g \circ \gamma) \star \overline{F_0}$ in Y, where $F_0 : I \to Y$ is the path in Y defined by $F_0(t) = F(x_0, t)$, $\forall\, t \in I$.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & \pi_1(Y, y_1) \arrow[dl, "\beta_{F_0}"] \\ & \pi_1(Y, y_0) & \end{tikzcd}$$

Proof. Left as an exercise. $\square$

Corollary 1.3.29. *Let $f, g : X \to Y$ be two homotopic continuous maps of topological spaces. Let $x_0 \in X$ be such that $f(x_0) = g(x_0) = y_0 \in Y$. Then the homomorphisms of fundamental groups f_* and g_*, induced by f and g, respectively, are conjugate by an element of $\pi_1(Y, y_0)$. In other words, there exists an element $[\eta] \in \pi_1(Y, y_0)$ such that $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$.*

Proof. Let $F : X \times I \to X$ be a continuous map such that

$$F(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ g(x), & \text{if} \quad t = 1. \end{cases}$$

Then by Proposition 1.3.28 we have $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$, where $\eta : I \to Y$ is the loop defined by $\eta(t) := F(x_0, t)$, $\forall\, t \in I$. $\square$

Corollary 1.3.30. *If $f, g : (X, x_0) \to (Y, y_0)$ are two homotopic continuous maps of pointed topological spaces (see Definition 1.2.7), then $f_* = g_*$.*

Proof. Follows from Corollary 1.3.29. $\square$

Definition 1.3.31. A space X is said to be *simply connected* if X is path connected and $\pi_1(X)$ is trivial.

Corollary 1.3.32. *A contractible space (see Definition 1.2.11) is simply connected.*

Proof. Let X be a contractible space. Then X is path-connected by Exercise 1.2.12. Fix a point $x_0 \in X$, and let $c_{x_0} : X \to X$ be the constant map sending all points to x_0. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ is homotopic to the constant map c_{x_0} in X. Then by Corollary 1.3.29 the identity homomorphism $\mathrm{Id}_{\pi_1(X, x_0)} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is conjugate to the trivial homomorphism $(c_{x_0})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by an element of $\pi_1(X, x_0)$. Therefore, the image of the identity homomorphism $\mathrm{Id}_{\pi_1(X, x_0)}$ is trivial, and hence $\pi_1(X, x_0)$ is trivial. $\square$

Corollary 1.3.33. *A space X is simply connected if and only if there is a unique path-homotopy class of paths connecting any two points of X.*

Proof. Suppose that X is simply connected. Fix $x_0, x_1 \in X$. Since X is path-connected, there is a path in X joining x_0 to x_1. Let $f, g : I \to X$ be any two paths in X from x_0 to x_1. Let $\overline{f}$ and $\overline{g}$ be the opposite paths of f and g, respectively. Since $f \star \overline{g}$ is a loop in X based at x_0 and $\pi_1(X, x_0)$ is trivial, we have $f \star \overline{g}$ is path-homotopic to the constant loop c_{x_0} in X based at x_0. Since $\overline{g} \star g$ is path-homotopic to the constant loop c_{x_1} by Exercise 1.3.24, we have

$$f \simeq f \star c_{x_1} \simeq f \star \overline{g} \star g \simeq c_{x_0} \star g \simeq g.$$

To see the converse part, note that path connectedness of X means any two points of X can be joined by a path in X. Since there is a unique homotopy class of paths connecting any two points of X, path connectedness of X is automatic, and any loop in X based at a given point $x_0 \in X$ is homotopically trivial. Thus, X is path connected with $\pi_1(X, x_0)$ trivial, and hence is simply connected. □

Proposition 1.3.34. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Note that $X \times Y$ naturally acquires product topology induced from X and Y, and the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$, for all $(x, y) \in X \times Y$, are continuous. Moreover, given any space Z and a map $f : Z \to X \times Y$, we have $f = (p_1 \circ f, p_2 \circ f)$. From this, it follows that f is continuous if and only if its components $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to Y$ are continuous. Therefore, to give a loop $\gamma : I \to X \times Y$ based at $(x_0, y_0) \in X \times Y$ is equivalent to give a pair of loops $(p_1 \circ \gamma, p_2 \circ \gamma)$ in the pointed spaces (X, x_0) and (Y, y_0), respectively. Similarly, to give a homotopy $F : I \times I \to X \times Y$ of loops $\gamma, \delta : I \to X \times Y$ based at $(x_0, y_0))$ is equivalent to give a pair of homotopies $(p_1 \circ F, p_2 \circ F)$ of the corresponding loops $p_j \circ \gamma$ with $p_j \circ \delta$, where $j \in \{1, 2\}$. Thus we have a bijection

$$\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0), \quad [\gamma] \mapsto ([p_1 \circ \gamma], [p_2 \circ \gamma]).$$

To see ϕ is an isomorphism, note that for any two loops γ and δ in $X \times Y$ based at $(x_0, y_0) \in X \times Y$, we have

$$\begin{aligned}
\phi([\gamma] \cdot [\delta]) &= \phi([\gamma \star \delta]) \\
&= ([p_1 \circ (\gamma \star \delta)], [p_2 \circ (\gamma \star \delta)]) \\
&= ([(p_1 \circ \gamma) \star (p_1 \circ \delta)], [(p_2 \circ \gamma) \star (p_2 \circ \delta)]) \\
&= ([(p_1 \circ \gamma)] \cdot [(p_1 \circ \delta)], [(p_2 \circ \gamma)] \cdot [(p_2 \circ \delta)]) \\
&= ([p_1 \circ \gamma], [p_2 \circ \gamma]) \cdot ([p_1 \circ \delta], [p_2 \circ \delta]) \\
&= \phi([\gamma]) \cdot \phi([\delta]).
\end{aligned}$$

This completes the proof. □

Example 1.3.35. As an immediate application of Proposition 1.3.34 we see that the fundamental group of the 1-torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

We end this subsection with the following useful remark.

Remark 1.3.36. A loop in X based at x_0 can equivalently be defined as a continuous map of pointed spaces $\gamma : (S^1, 1) \to (X, x_0)$. Indeed, since a loops in X based at $x_0 \in X$ is a continuous map $\gamma : I = [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, and since S^1 is homeomorphic to the quotient space $[0, 1]/\sim$, where only the end points 0 and 1 of the interval I are identified, $\gamma : I \to X$ uniquely factors as

$$\begin{tikzcd} I \arrow[r, "\gamma"] \arrow[d, "q"'] & X \\ S^1 \arrow[ur, dashed, "\gamma'"'] & \end{tikzcd}$$

where $q : I \to S^1$ is the quotient map given by $q(t) = e^{2\pi i t}$, for all $t \in I$. Therefore, $\pi_1(X, x_0)$ is the group of all homotopy classes of continuous maps $(S^1, 1) \to (X, x_0)$.

1.3.4 Fundamental group of some spaces

Proposition 1.3.37. $\pi_1(\mathbb{R}, 0) = \{1\}$.

Proof. Consider the continuous map $F : \mathbb{R} \times I \to \mathbb{R}$ defined by

$$F(x, t) = (1 - t)x, \ \forall\, (x, t) \in \mathbb{R} \times I.$$

Note that, for all $x \in \mathbb{R}$ and $t \in I$ we have

- $F(x, 0) = x$,
- $F(x, 1) = 0$, and
- $F(0, t) = 0$.

Therefore, F "contracts" whole $\mathbb{R}$ to the point 0 leaving the point 0 intact at all times. Let $\gamma : I \to \mathbb{R}$ be a loop based at 0. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \xrightarrow{\gamma \times \mathrm{Id}_I} \mathbb{R} \times I \xrightarrow{F} \mathbb{R}$$

is a homotopy from γ to the constant loop $0 : I \to \mathbb{R}$ which sends all points of S^1 to $0 \in \mathbb{R}$. This completes the proof. □

Proposition 1.3.38. *Let*

$$D^2 := \{z \in \mathbb{C} \,:\, |z| \leq 1\} = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\}$$

be the closed unit disk in the plane. Then $\pi_1(D^2, 1) = \{1\}$.

Proof. Consider the map $F : D^2 \times I \to D^2$ defined by

$$F(z, t) = (1 - t)z + t, \ \forall\, (z, t) \in D^2 \times I.$$

Note that F is continuous and for all $z \in D^2$ and $t \in I$ we have

- $F(z, 0) = z$,
- $F(z, 1) = 1$, and
- $F(1, t) = 1$.

Therefore, F contracts D^2 to the point 1 leaving 1 intact at all times. Let $\gamma : I \to D^2$ be a loop based at 1. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \xrightarrow{\gamma \times \mathrm{Id}_I} D^2 \times I \xrightarrow{F} D$$

is a homotopy from γ to the constant loop $1 : I \to D^2$ which sends all points of I to $1 \in D^2$. This completes the proof. □

1.4 Covering Space

1.4.1 Covering map

We begin this section with an aim to compute fundamental group of the unit circle in plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and we'll see how the idea of a 'covering map' could help us.

Let $\omega : I \to S^1$ be the map defined by $\omega(t) = e^{2\pi i t}$, $\forall\, t \in I$, where $i = \sqrt{-1}$. Then ω is a loop in S^1 based at $x_0 := 1 \in S^1$. For each integer n, let $\omega_n : I \to S^1$ be the loop based at x_0 defined by $\omega_n(t) = e^{2\pi i n t}$, $\forall\, t \in I$. So ω_n winds around the circle $|n|$-times in the anti-clockwise direction if $n > 0$, and in the clockwise direction if $n < 0$. We shall see later that $[\omega]^n = [\omega_n]$ in $\pi_1(S^1, 1)$, for all $n \in \mathbb{Z}$. The following is the main theorem of this section.

Theorem 1.4.1. *$\pi_1(S^1, x_0)$ is the infinite cyclic group $\mathbb{Z}$ generated by the loop ω.*

To prove this theorem, we compare paths in S^1 with paths in $\mathbb{R}$ via the map

$$p : \mathbb{R} \to S^1 \text{ given by } p(s) = (\cos 2\pi s, \sin 2\pi s),\ \forall\, s \in \mathbb{R}.$$

We can visualize this map geometrically by embedding $\mathbb{R}$ inside $\mathbb{R}^3$ as the helix parametrized as

$$s \mapsto (\cos 2\pi s, \sin 2\pi s, s),$$

and then p is the restriction of the projection map

$$\mathbb{R}^3 \to \mathbb{R}^2,\ \ (x, y, z) \mapsto (x, y)$$

from this helix onto $S^1 \subset \mathbb{R}^2$, as shown in the Figure 1.7.

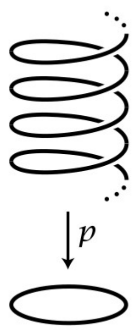


FIGURE 1.7

In this setup, the loop

$$\omega_n : I \to S^1, \;\; s \mapsto (\cos 2n\pi s, \sin 2n\pi s)$$

is the composition $p \circ \widetilde{\omega}_n$, where

$$\widetilde{\omega}_n : I \to \mathbb{R}, \;\; s \mapsto ns$$

is the path in $\mathbb{R}$ starting at 0 and ending at n, winding around the helix $|n|$-times, upward direction if $n > 0$, and downward direction if $n < 0$.

Before proceeding further, we introduce notion of a *covering map*, and discuss some of its useful properties.

Definition 1.4.2. Let $f : X \to Y$ be a continuous map. An open subset $V \subseteq Y$ is said to be *evenly covered by* f if $f^{-1}(V)$ is a union of pairwise disjoint open subsets of X each of which are homeomorphic to V by f (meaning that, $f^{-1}(V) = \bigcup\limits_{i \in I} U_i$, where $U_i \subseteq X$ is an open subset of X with $U_i \cap U_j = \emptyset$, for all $i \neq j$ in I, and $f\big|_{U_i} : U_i \to V$ is a homeomorphism, for all $i \in I$).

Example 1.4.3. (i) Let $f : \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ be the map defined by $f(t) = e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$, for all $t \in \mathbb{R}$. For $a, b \in \mathbb{R}$ with $a < b$, we define an open subset

$$V_{a,b} := \{f(t) : a \leq t \leq b\} \subseteq S^1.$$

If $b - a < 1$, then $V_{a,b}$ is evenly covered by f. In fact, in this case, we have $f^{-1}(V_{a,b}) =$

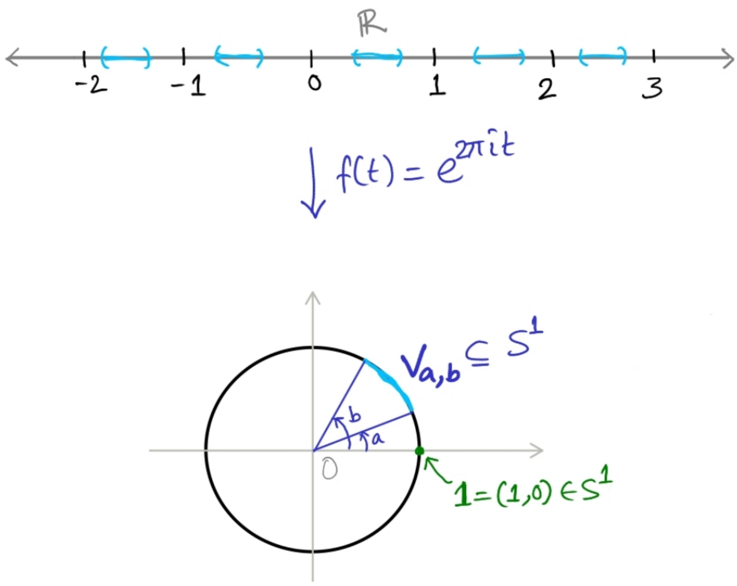


FIGURE 1.8

$\bigsqcup\limits_{n \in \mathbb{Z}} (a+n, b+n)$, and $f : (a+n, b+n) \xrightarrow{\simeq} V_{a,b}$ is a homeomorphism, $\forall\, n \in \mathbb{Z}$. See Figure 1.8.

If $b - a \geq 1$, then $V_{a,b} = S^1$, and hence $f^{-1}(V_{a,b}) = \mathbb{R}$. In this case, $V_{a,b}$ is not evenly covered by f, for otherwise we would have $\mathbb{R} = \bigsqcup_{i \in I} U_i$ with each U_i open subset of $\mathbb{R}$ and $f\big|_{U_i} : U_i \to S^1$ is a homeomorphism, which is not possible because S^1 is compact, whereas an open subset of $\mathbb{R}$ cannot be compact.

(ii) Let $\mathbb{R}_{>0} := \{t \in \mathbb{R} : t > 0\}$ be the positive part of the real line. Let

$$f : \mathbb{R}_{>0} \to S^1, \ \ t \mapsto e^{2\pi i t}. \tag{1.4.4}$$

For any point $x \in S^1$ with $x \neq \mathbb{1} := (1, 0) \in S^1$, we can choose a small enough open neighbourhood V of x in S^1 with $\mathbb{1} \notin V$. Then it is easy to see that V is evenly covered by f. However, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$. To see this, note that

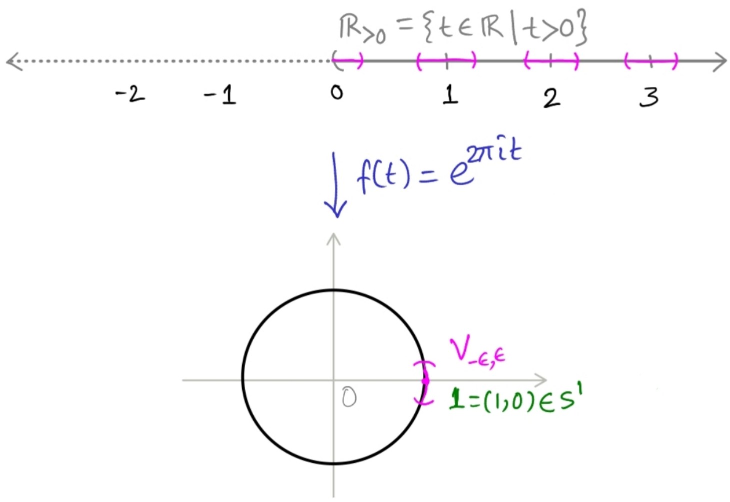


FIGURE 1.9

if $U \subseteq V$ is an open subset of an evenly covered neighbourhood V, then U is also evenly covered. Thus, if there is a neighbourhood V of $\mathbb{1}$ which is evenly covered, then we may find $\epsilon \in (0, 1/2)$ small enough such that $V_{-\epsilon,\epsilon} \subseteq V$, and hence $V_{-\epsilon,\epsilon}$ is evenly covered. Then we must have $f^{-1}(V_{-\epsilon,\epsilon}) = \bigsqcup_{i \in I} U_i$, with $f\big|_{U_i} : U_i \to V_{-\epsilon,\epsilon}$ homeomorphism, for all $i \in I$. In particular, each U_i is connected and are path components of $f^{-1}(V_{-\epsilon,\epsilon})$. Let U_0 be the path component of $\epsilon/2 \in \mathbb{R}_{>0}$. Since

$$f^{-1}(V_{-\epsilon,\epsilon}) = (0, \epsilon) \bigcup \left(\bigcup_{n \geq 1} (n - \epsilon, n + \epsilon) \right),$$

we must have $U_0 = (0, \epsilon)$. But $f\big|_{(0,\epsilon)} : (0, \epsilon) \to V_{-\epsilon,\epsilon}$ cannot be surjective because only possible preimage of $\mathbb{1} \in V_{-\epsilon,\epsilon}$ in $\mathbb{R}^+$ could be positive integers, and none of which are in the domain of $f\big|_{(0,\epsilon)}$. Thus we get a contradiction. See Figure 1.9. Therefore, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$ for the map f in (1.4.4).

Definition 1.4.5. A continuous map $f : X \to Y$ is called a *covering map* if each point $y \in Y$ has an open neighbourhood $V_y \subseteq Y$ that is evenly covered by f.

Note that, a covering map is always surjective. This follows immediately from the Definition 1.4.5.

Example 1.4.6. (i) Let F be a non-empty discrete topological space, and let X be any topological space. Give $X \times F$ the product topology. Then the projection map $pr_1 : X \times F \to X$ defined by $pr_1(x, v) = x$, $\forall\ (x, v) \in X \times F$, is a covering map. Such a covering map is called a *trivial cover* of X.

(ii) The continuous map
$$f : \mathbb{R} \to S^1, \ \ t \mapsto e^{2\pi i t},$$
as discussed in Example 1.4.3 (i), is a covering map, while its restriction $f\big|_{\mathbb{R}^+} : \mathbb{R}^+ \to S^1$, in Example 1.4.3 (ii), is not a covering map.

(iii) The map $f : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by $f(z) = e^z$, for all $z \in \mathbb{C}$, is a covering map.

(iv) Fix an integer $n \geq 1$. Then the map $f : \mathbb{C}^* \to \mathbb{C}^*$ defined by $f(z) = z^n$, for all $z \in \mathbb{C}$, is a covering map, known as the *n-sheeted covering map* of $\mathbb{C}^*$.

Exercise 1.4.7. If $f_i : X_i \to Y_i$ is a covering map, for $i = 1, 2$, show that the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by sending $(x_1, x_2) \in X_1 \times X_2$ to $(f_1(x_1), f_2(x_2)) \in Y_1 \times Y_2$, is a covering map.

Exercise 1.4.8. If $f : X \to Y$ is a covering map, for any subspace $Z \subseteq Y$, the restriction of f on $f^{-1}(Z) \subseteq X$ is a covering map.

Definition 1.4.9. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covering maps. A *morphism of covering maps* from p_1 to p_2 is a continuous map $\phi : Y_1 \to Y_2$ such that $p_2 \circ \phi = p_1$. In other words, the following diagram commutes.

$$\begin{tikzcd}
Y_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

A morphism of covering maps $\phi : Y_1 \to Y_2$ is said to be an *isomorphism of covering maps* if there is a covering map $\psi : Y_2 \to Y_1$ such that $\phi \circ \psi = \mathrm{Id}_{Y_2}$ and $\psi \circ \phi = \mathrm{Id}_{Y_1}$. In other words, an isomorphism of covering spaces is a homeomorphism of the covers compatible with the base. An isomorphism of a covering map $p : Y \to X$ to itself is called a *Deck transformation* or a *covering transformation*.

Exercise 1.4.10. Show that any covering map $p : Y \to X$ is locally trivial (i.e., each point $x \in X$ has an open neighbourhood $U_x \subseteq X$ such that the restriction map $p : p^{-1}(U_x) \to U_x$ is isomorphic to a trivial covering map over U_x).

A continuous map $f : X \to Y$ is said to be an *open map* if for any open subset U of X, $f(U)$ is open in Y.

Proposition 1.4.11. *If $f : X \to Y$ is a covering map, then f is an open map.*

Proof. Let $U \subseteq X$ be an open subset of X, and let $y \in f(U)$. Then there is $x_0 \in U$ such that $f(x_0) = y$. Since f is a covering map, there is an open neighbourhood $V \subseteq Y$ of y such that $f^{-1}(V) = \bigcup_{j \in J} W_i$ is a union of pairwise disjoint open subsets $W_j \subseteq X$, and that $f|_{W_j} : W_j \to V$ is a homeomorphism, for all $j \in J$. Then $x_0 \in U \cap W_{j_0}$, for some unique $i_0 \in I$. Since $f|_{W_i}$ is a homeomorphism, $f(U \cap W_{j_0}) \subseteq V$ is an open neighbourhood of $f(x_0) = y$. Since V is open in Y, $f(U \cap W_{j_0})$ is open in Y. Thus $f(U)$ is open in Y, and hence f is an open map. □

Theorem 1.4.12 (Lifting path to a cover)**.** *Let $f : X \to Y$ be a covering map. Let $\gamma : [0,1] \to Y$ be a path in Y. Fix a point $x_0 \in X$ such that $f(x_0) = y_0 := \gamma(0)$. Then there is a unique path $\widetilde{\gamma} : [0,1] \to X$ with $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$.*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ {[0,1]} \arrow[ur, dashed, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & Y \end{tikzcd}$$

The path $\widetilde{\gamma}$ is called a lift of γ in X starting at x_0.

Proof. We first prove uniqueness of lift of γ, if it exists. Let $\eta_1, \eta_2 : [0,1] \to X$ be any two continuous maps such that $\eta_1(0) = x_0 = \eta_2(0)$ and $f \circ \eta_1 = \gamma = f \circ \eta_2$. We need to show that $\eta_1 = \eta_2$ on $[0,1]$. Let

$$S = \{t \in [0,1] \,:\, \eta_1(t) = \eta_2(t)\}.$$

Since both η_1 and η_2 are continuous, S is a closed subset of $[0, 1]$. Note that $S \neq \emptyset$ since $0 \in S$. Since $[0, 1]$ is connected, it is enough to show that S is both open and closed in $[0,1]$, so that S is a connected component of $[0, 1]$, and hence $S = [0, 1]$.

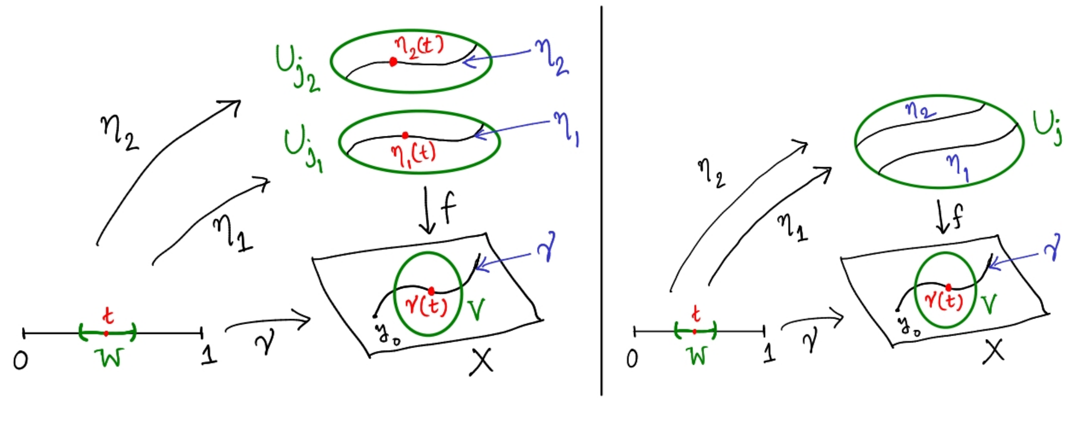


FIGURE 1.10

Fix a $t \in S$, and let $V \subseteq Y$ be an open neighbourhood of $y := \gamma(t)$ that is evenly covered by f. So $f^{-1}(V) = \bigcup_{j \in J} U_j$, where $\{U_j\}_{j \in J}$ is a collection of pairwise disjoint open subsets of X each of which gets mapped homeomorphically onto V by f. Then there are $j_1, j_2 \in J$ such that $\eta_1(t) \in U_{j_1}$ and $\eta_2(t) \in U_{j_2}$. Since η_1 and η_2 are continuous at $t \in [0, 1]$, there is an open neighbourhood $W \subseteq [0,1]$ of t such that $\eta_1(W) \subseteq U_{j_1}$ and $\eta_2(W) \subseteq U_{j_2}$. Since $U_{j_1} \cap U_{j_2} = \emptyset$

for $j_1 \neq j_2$, and since $\eta_1(t) = \eta_2(t)$ by assumption, we must have $j_1 = j_2$ and $U_{j_1} = U_{j_2}$. Since $f\big|_{U_j} : U_j \to V$ is injective (in fact, homeomorphism), for all $j \in J$, and $f \circ \eta_1 = f \circ \eta_2$, we must have $\eta_1\big|_W = \eta_2\big|_W$. Therefore, $W \subseteq S$. Thus S is both open and closed in $[0, 1]$, and hence is the connected component of $[0, 1]$. Therefore, $S = [0, 1]$, and hence $\eta_1 = \eta_2$ on $[0, 1]$.

Remark 1.4.13. Note that, by replacing $[0,1]$ with any connected topological space T in the above proof of uniqueness of lift of γ, we get the following result:

Lemma 1.4.13. *Let $f : X \to Y$ be a covering map. Let $\eta_1, \eta_2 : T \to X$ be any continuous maps such that $f \circ \eta_1 = f \circ \eta_2$. If T is connected and $\eta_1(t) = \eta_2(t)$, for some $t \in T$, then $\eta_1 = \eta_2$ on whole T.*

To complete the proof of Theorem 1.4.12, it remains to construct an explicit lift of γ to the cover $f : X \to Y$ starting at x_0. For this we use a result from basic topology course, called *Lebesgue number lemma*.

Lemma 1.4.14 (Lebesgue number lemma)**.** *Let $\{U_j\}_{j \in J}$ be an open cover of a compact metric space (X, d). Then there is a $\delta > 0$ such that for each $x_0 \in X$, the open ball $B_\delta(x_0)$ is contained in U_{j_0}, for some $j_0 \in J$.*

Since $f : X \to Y$ is a covering map, we can write $Y = \bigcup\limits_{y \in Y} V_y$, where $V_y \subseteq Y$ is an open neighbourhood of y that is evenly covered by f, for all $y \in Y$. Since $[0,1] = \bigcup\limits_{y \in Y} \gamma^{-1}(V_y)$, by Lebesgue covering lemma (c.f. Lemma 1.4.14) we can find a $\delta > 0$ such that for each $t \in (0,1)$ there is a $y_t \in Y$ such that $\gamma\big([t - \frac{\delta}{2}, t + \frac{\delta}{2}] \cap [0, 1]\big) \subseteq V_{y_t}$. Choose $n \gg 0$ such that $\frac{1}{n} < \delta$, and write

$$[0,1] = \bigcup_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Now $\gamma([0, 1/n]) \subseteq V_0$, for some open subset $V_0 \subset Y$ evenly covered by f, and $y_0 = \gamma(0) \in V_0$. Write

$$f^{-1}(V_0) = \bigsqcup_{j \in J} U_{0,j}\,,$$

where $\{U_{0,j}\}_{j \in J}$ is a collection of pair-wise disjoint open subsets of X each of which are homeomorphic to V_0 via the restriction of f onto them. Since $x_0 \in f^{-1}(V_0)$, there is a unique $j_0 \in J$ such that $x_0 \in U_{0,j_0}$. Let $s_0 : V_0 \to U_{0,j_0}$ be the inverse of the homeomorphism $f\big|_{U_{0,j_0}}$. Clearly $s_0(y_0) = x_0$. Consider the map $\widetilde{\gamma}_0 : [0, \frac{1}{n}] \to U_{0,j_0}$ defined by

$$\widetilde{\gamma}_0(t) := s_0(\gamma(t)), \ \ \forall \ t \in [0, 1/n]\,.$$

Then $\widetilde{\gamma}_0$ satisfies $\widetilde{\gamma}_0(0) = x_0$ and $f \circ \widetilde{\gamma}_0 = \gamma$ on $[0, \frac{1}{n}]$.

Let $x_1 = \widetilde{\gamma}_0(\frac{1}{n})$ and $y_1 = \gamma(\frac{1}{n}) = (f \circ \widetilde{\gamma}_0)(\frac{1}{n})$. Then there is an open subset $V_1 \subseteq Y$ which is evenly covered by f and $\gamma([\frac{1}{n}, \frac{2}{n}]) \subseteq V_1$. Proceeding in the same way as above, we can write

$$f^{-1}(V_1) = \bigsqcup_{j \in J} U_{1,j},$$

where $U_{1,j}$ are pairwise disjoint open subsets of X each of which are homeomorphic to V_1 by the restriction of f onto them. Since $x_1 = \widetilde{\gamma}_0(\frac{1}{n}) \in f^{-1}(V_1)$, there is a $j_1 \in J$ such that $x_1 \in U_{1,j_1}$. Let $s_1 : V_1 \to U_{1,j_1}$ be the inverse of the homeomorphism $f : U_{1,j_1} \to V_1$. Clearly $s_1(y_1) = x_1$. Then the continuous map $\widetilde{\gamma}_1 : [\frac{1}{n}, \frac{2}{n}] \to U_{1,j_1}$ defined by

$$\widetilde{\gamma}_1(t) = s_1(\gamma(t)), \ \forall\, t \in [1/n, 2/n]$$

satisfies $\widetilde{\gamma}_1(\frac{1}{n}) = x_1$ and $f \circ \widetilde{\gamma}_1 = \gamma$ on $[\frac{1}{n}, \frac{2}{n}]$. Since the maps $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ agrees on $[0, \frac{1}{n}] \cap [\frac{1}{n}, \frac{2}{n}] = \{\frac{1}{n}\}$, by Lemma 1.2.3 we can join them to get a continuous map $\widetilde{\gamma} : [0, \frac{2}{n}] \to X$ such that $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$ on $[0, \frac{2}{n}]$. Proceeding in this way we can construct a lift $\widetilde{\gamma}$ of γ to the whole $[0, 1]$ as required. □

Next we lift homotopy from a base to its cover.

Lemma 1.4.15 (Glueing continuous maps)**.** *Let X and Y be two topological spaces. Let $\{U_j\}_{j\in J}$ be an open covering of X. Then given a family of continuous maps $\{f_j : U_j \to Y\}_{j\in J}$ satisfying $f_j\big|_{U_j\cap U_k} = f_k\big|_{U_j\cap U_k}$, for all $j, k \in J$, there is a unique continuous map $f : X \to Y$ such that $f\big|_{U_j} = f_j$, for all $j \in J$.*

Proof. Left as an exercise. □

Theorem 1.4.16 (Lifting homotopy to covers)**.** *Let $I := [0, 1] \subset \mathbb{R}$. Let $f : X \to Y$ be a covering map. Let $F : I \times I \to Y$ be a continuous map. Let $y_0 := F(0, 0)$ and fix a point $x_0 \in f^{-1}(y_0)$. Then there is a unique continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F$.*

Proof. Since $I \times I$ is connected, uniqueness of $\widetilde{F}$, if it exists, follows from Remark 1.4.13. We only show a construction of such a lift $\widetilde{F}$.

It is enough to show that, for each $s \in I$ there is a connected open neighbourhood $U_s \subseteq I$ of $s \in I$ such that $\widetilde{F}$ can be constructed on $U_s \times I$. Indeed, since $\{U_s \times I : s \in I\}$ is a connected open covering of $I \times I$ and those $\widetilde{F}$'s agree on their intersections $(U_s \times I) \cap (U_{s'} \times I) = (U_s \cap U_{s'}) \times I$, which are connected (because U'_s are open intervals), uniqueness of liftings $\widetilde{F}$'s defined on connected domains ensures that they can be glued together to get a well-defined continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F$ on $I \times I$.

Now we construct such a lift $\widetilde{F} : U \times I \to X$, for some open neighbourhood $U \subseteq I$ of a given point $s_0 \in I$. Since F is continuous, each point $(s_0, t) \in I \times I$ has an open neighbourhood $U_t \times (a_t, b_t) \subset I \times I$ such that $F(U_t \times (a_t, b_t))$ is contained in some open neighbourhood of $F((s_0, t)) \in Y$ that is evenly covered by f. Since $\{s_0\} \times I$ is compact, finitely many such open subsets $U_t \times (a_t, b_t)$ cover $\{s_0\} \times I$. Taking intersection of those finitely many open subsets $U_t \subseteq I$, we can find a single open neighbourhood $U \subset I$ of s_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $I = [0, 1]$ such that for each $i \in \{0, 1, \ldots, m\}$, $F(U \times [t_i, t_{i+1}]) \subseteq V_i$, for some open subset $V_i \subset Y$ that is evenly covered by f.

By Theorem 1.4.12 (Lifting paths to a cover), we can find a unique continuous function $\widetilde{F} : I \times \{0\} \to X$ with $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F\big|_{I\times\{0\}}$. Assume inductively that $\widetilde{F}$ has been constructed on $U \times [0, t_i]$, starting with the given $\widetilde{F}$ on $U \times \{0\} \subseteq I \times \{0\}$. Since $F(U \times$

$[t_i, t_{i+1}]) \subseteq V_i$, and V_i is evenly covered by f, there is an open subset $W_i \subseteq X$ such that $\widetilde{F}(s_0, t_i) \in W_i$ and $f|_{W_i} : W_i \to V_i$ is a homeomorphism. Replacing U by a smaller open neighbourhood of $s_0 \in I$, if required, we may assume that $\widetilde{F}(U \times \{t_i\}) \subseteq W_i$; for instance, it is enough to replace $U \times \{t_i\}$ with $(U \times \{t_i\}) \bigcap \big(\widetilde{F}|_{U\times\{t_i\}}\big)^{-1}(W_i)$. Then we can define $\widetilde{F}$ on $U \times [t_i, t_{i+1}]$ to be the composition $\varphi \circ F$, where $\varphi : V_i \to W_i$ is the inverse of the homeomorphism $f|_{W_i} : W_i \to V_i$. Continuing in this way, after a finite number of steps, we get a continuous map $\widetilde{F} : U \times I \to X$ with $\widetilde{F}(0,0) = x_0$ and $f \circ \widetilde{F} = F|_{U\times I}$, as required. □

Lemma 1.4.17. *Let $f : X \to Y$ be a covering map, and let $\gamma : I \to X$ be a continuous map. If $f \circ \gamma$ is a constant map, so is γ.*

Proof. Suppose that $(f \circ \gamma)(t) = y_0$, for all $t \in I$. Let $V \subseteq Y$ be an open neighbourhood of y_0 that is evenly covered by f. Then $f^{-1}(V) = \bigsqcup_{\alpha\in\Lambda} U_\alpha$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is a family of pairwise disjoint open subsets of X with $f|_{U_\alpha} : U_\alpha \to V$ a homeomorphism, for all $\alpha \in \Lambda$. Since $\gamma(t) \in f^{-1}(V)$, for all $t \in I$, and I is connected, there is a unique $\alpha_0 \in \Lambda$ such that $\gamma(t) \in U_{\alpha_0}$, for all $t \in I$. Since $f|_{U_{\alpha_0}}$ is a homeomorphism, its restriction on the image of γ must be a homeomorphism; this is not possible since $f \circ \gamma$ is a constant map. □

Corollary 1.4.18 (Lifting of path-homotopy)**.** *Let $f : X \to Y$ be a covering map. Let $\gamma_0, \gamma_1 : I \to Y$ be two paths in Y with $\gamma_0(0) = \gamma_1(0) = y_0$ and $\gamma_0(1) = \gamma_1(1) = y_1$. Let $F : I \times I \to Y$ be a path-homotopy from γ_0 to γ_1 in X. If $\widetilde{F} : I \times I \to X$ is a lifting of F on X, then $\widetilde{F}$ is a path-homotopy.*

Proof. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{F} : I \times I \to X$ be the lifting of F on X with $\widetilde{F}(0,0) = x_0$. Then by Theorem 1.4.16, $\widetilde{F}$ is a homotopy of maps from $\widetilde{\gamma}_0 := \widetilde{F}(-,0)$ to $\widetilde{\gamma}_1 := \widetilde{F}(-,1)$. Let $x_1 := \widetilde{\gamma}_0(1) = \widetilde{F}(1,0)$. To show $\widetilde{F}$ is a path-homotopy, we need to ensure that $\widetilde{F}(0,t) = x_0$ and $\widetilde{F}(1,t) = x_1$, for all $t \in I$. This follows from the Lemma 1.4.17 applied to the paths $t \mapsto \widetilde{F}(0,t)$ and $t \mapsto \widetilde{F}(1,t)$. □

Corollary 1.4.19. *Let $f : X \to Y$ be a covering map. Let $y_0 \in Y$ and fix a point $x_0 \in f^{-1}(y_0)$. Then the group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is injective. The image subgroup $f_*(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$ consists of the homotopy classes of loops in Y based at y_0 whose lifts to X starting at x_0 are loops.*

Proof. Let $[\gamma], [\delta] \in \pi_1(X, x_0)$ be such that $f_*([\gamma]) = f_*([\delta])$. Then $f \circ \gamma$ is homotopic to $f \circ \delta$. Let $F : I \times I \to Y$ be a path homotopy from $f \circ \gamma$ to $f \circ \delta$. Then by Theorem 1.4.16 and its Corollary 1.4.18, we can lift F to a path-homotopy $\widetilde{F} : I \times I \to X$ with $\widetilde{F}(0,0) = x_0$. By uniqueness of path-lifting (see Theorem 1.4.12), $\widetilde{F}$ must be a path-homotopy from γ to δ (verify!). Therefore, f_* is injective.

To see the second part, note that any element of $f_*(\pi_1(X, x_0))$ is of the form $[f \circ \gamma]$, for some loop $\gamma : I \to X$ in X based at x_0. By Theorem 1.4.12 (Path-lifting) we can lift $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ starting at $\gamma(0)$. Then by uniqueness of path-lifting, we have $\widetilde{f \circ \gamma} = \gamma$. Conversely, if δ is a loop in Y based at x_0 such that its lift $\widetilde{\delta}$ in X is a loop in X based at x_0, then $f_*([\widetilde{\delta}]) = [f \circ \widetilde{\delta}] = [\delta]$. This completes the proof. □

Exercise 1.4.20 (Lifting of opposite path)**.** Let $f : X \to Y$ be a covering map. Let $\gamma : I \to Y$ be a path in Y from y_0 to y_1. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{\gamma}$ be the lift of γ in X starting at x_0. Let $\overline{\gamma}$ be the opposite path of γ. If $\widetilde{\overline{\gamma}}$ is the lift of $\overline{\gamma}$ in X starting at $\widetilde{\gamma}(1)$, then show that $\overline{\widetilde{\gamma}} = \widetilde{\overline{\gamma}}$.

Exercise 1.4.21 (Lifting of product of paths)**.** Let $f : X \to Y$ be a covering map. Let $\gamma, \delta : I \to Y$ be two paths in Y such that $\gamma(1) = \delta(0)$. Fix a point $x_0 \in f^{-1}(\gamma(0))$, and let $\widetilde{\gamma}$ and $\widetilde{\gamma \star \delta}$ be the liftings of the paths γ and $\gamma \star \delta$, respectively, in X starting at x_0. If $\widetilde{\delta}$ is the lifting of δ in X starting at $x_1 := \widetilde{\gamma}(1)$, show that $\widetilde{\gamma} \star \widetilde{\delta} = \widetilde{\gamma \star \delta}$.

Lemma 1.4.22. *Let $f : X \to Y$ be a covering space. If both X and Y are path-connected, then the cardinality of the fiber $f^{-1}(y)$ is independent of $y \in Y$.*

Proof. Fix a point $y_0 \in Y$, and a point $x_0 \in f^{-1}(y_0) \subseteq X$. Let $G = \pi_1(Y, y_0)$ and $H = f_*\big(\pi_1(X, x_0)\big)$. Let $H\backslash G := \{Hg : g \in G\}$ be the set of all right cosets of H in G. Since both X and Y are path-connected, the cardinality of the set $H\backslash G$ is independent of choices of $y_0 \in Y$ and $x_0 \in f^{-1}(y_0)$. Therefore, to show the cardinality of the fibers $f^{-1}(y)$ is independent of $y \in Y$, it is enough to construct a bijective map

$$\Phi : H\backslash G \longrightarrow f^{-1}(y_0). \tag{1.4.23}$$

Given a loop γ in Y based at y_0, let $\widetilde{\gamma}$ be the lifting of γ in X starting at x_0. Note that, $x_1 := \widetilde{\gamma}(1) \in f^{-1}(y_0)$. Then we define

$$\Phi(H[\gamma]) := \widetilde{\gamma}(1). \tag{1.4.24}$$

We need to show that x_1 is independent of choice of γ. Let δ be a loop in Y based at y_0 with $H[\gamma] = H[\delta]$. Then $[\gamma \star \overline{\delta}] = [\gamma][\delta]^{-1} \in H = f_*\big(\pi_1(X, x_0)\big)$, where $\overline{\delta}$ is the opposite path of δ. Then by Corollary 1.4.19 the loop $\gamma \star \overline{\delta}$ lifts to a unique loop $(\widetilde{\gamma \star \overline{\delta}})$ in X based at x_0. Let $\widetilde{\overline{\delta}}$ be the lifting of $\overline{\delta}$ in X starting at $x_1 := \widetilde{\gamma}(1)$. Then by Exercises 1.4.21 we have $\widetilde{\gamma \star \overline{\delta}} = \widetilde{\gamma} \star \widetilde{\overline{\delta}}$. Since $\widetilde{\gamma \star \overline{\delta}}$ is a loop in X based at x_0, we have $\widetilde{\overline{\delta}}(1) = x_0$. Let η be the opposite path of $\widetilde{\overline{\delta}}$ in X. Since

$$\begin{aligned}(f \circ \eta)(t) = f(\eta(t)) &= f\big(\widetilde{\overline{\delta}}(1-t)\big)\\ &= \overline{\delta}(1-t) = \delta(t), \quad \forall\, t \in I,\end{aligned}$$

η is a lift of δ in X starting at $\eta(0) = \widetilde{\overline{\delta}}(1) = x_0$. Then by uniqueness of path-lifting (Theorem 1.4.12) we have $\eta = \widetilde{\delta}$. Then $\widetilde{\delta}(1) = \eta(1) = \widetilde{\overline{\delta}}(0) = x_1$. Therefore, the map Φ in (1.4.24) is well-defined. Since X is path connected, given any $x_1 \in f^{-1}(y_0)$, there is a path φ in X from x_0 to x_1. Then $f \circ \varphi$ is a loop in Y based at y_0 whose lift $\widetilde{f \circ \varphi}$ starting at x_0 is the unique path φ ending at $x_1 = \varphi(1)$. Therefore, Φ is surjective. Let $[\gamma], [\delta] \in \pi_1(Y, y_0) = G$ be such that $\Phi(H[\gamma]) = \Phi(H[\delta])$. Let $\widetilde{\gamma}$ and $\widetilde{\delta}$ be the lifts of γ and δ, respectively, in X starting at x_0. Let $\overline{\widetilde{\delta}}$ be the opposite path of $\widetilde{\delta}$ in X. Since $\Phi(H[\gamma]) = \Phi(H[\delta])$, we have $\widetilde{\gamma}(1) = \widetilde{\delta}(1)$, and hence $\widetilde{\gamma} \star \overline{\widetilde{\delta}}$ is a loop in X based at x_0. Since $f \circ \big(\widetilde{\gamma} \star \overline{\widetilde{\delta}}\big) = \gamma \circ \overline{\delta}$, by uniqueness of path-lifting and Corollary 1.4.19, we conclude that $[\gamma \star \overline{\delta}] \in f_*\big(\pi_1(X, x_0)\big) = H$. Therefore, $H[\gamma] = H[\delta]$, and hence Φ is injective. Therefore, $\Phi : H\backslash G \to f^{-1}(y_0)$ is a bijection. □

Exercise 1.4.25. Give an example to show that the Lemma 1.4.22 fails if X and Y are not path-connected.

Theorem 1.4.26 (General Lifting Criterion). *Let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let T be a path-connected and locally path-connected space. A continuous map $g : (T, t_0) \to (Y, y_0)$ lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ if and only if $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Note that, such a lift $\widetilde{g}$ of g, if it exists, is unique by Lemma 1.4.13.*

Proof. If g lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ such that $f \circ \widetilde{g} = g$, then $g_*\big(\pi_1(T, t_0)\big) = f_*\big(\widetilde{g}_*(\pi_1(T, t_0))\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$.

To see the converse, suppose that $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Since T is path-connected, given a point $t_1 \in T$, there is a path $\gamma : I \to T$ with $\gamma(0) = t_0$ and $\gamma(1) = t_1$. Then $g \circ \gamma : I \to Y$ is a path in Y from $g(t_0) = y_0$ to $y_1 := g(t_1) = (g \circ \gamma)(1)$. Since $f : (X, x_0) \to (Y, y_0)$ is a covering map, by Theorem 1.4.12 (Path-lifting) the path $g \circ \gamma$ lifts to a unique path $\widetilde{g \circ \gamma}$ in X starting at x_0. Define a map

$$\widetilde{g} : T \to X \tag{1.4.27}$$

by sending t_1 to $x_1 := \widetilde{g \circ \gamma}(1) \in X$. To show the map $\widetilde{g}$ is independent of choice of a path γ in T from t_0 to t_1, note that given any path $\delta : I \to T$ from t_0 to t_1, the product path $\gamma \star \overline{\delta}$ is a loop in T based at t_0. Since $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$, by the second part of the Corollary 1.4.19 the loop $g \circ (\gamma \star \overline{\delta}) = (g \circ \gamma) \star (g \circ \overline{\delta})$ lifts to a unique loop, say φ, in X based at x_0. Let $\widetilde{(g \circ \overline{\delta})}$ be the lifting of $g \circ \overline{\delta}$ in X starting at $x_1 := \widetilde{g \circ \gamma}(1)$. Then by Exercises 1.4.21 we have $\varphi = \widetilde{(g \circ \gamma)} \star \widetilde{(g \circ \overline{\delta})}$. Since φ is a loop in X based at x_0, we have $\widetilde{(g \circ \overline{\delta})}(1) = x_0$. Let η be the opposite path of $\widetilde{(g \circ \overline{\delta})}$. Since

$$\begin{aligned}
(f \circ \eta)(t) &= f\big(\widetilde{(g \circ \overline{\delta})}(1 - t)\big) \\
&= (g \circ \overline{\delta})(1 - t) \\
&= (g \circ \delta)(t), \ \forall\, t \in I,
\end{aligned}$$

and $\eta(0) = \widetilde{(g \circ \overline{\delta})}(1) = x_0$, by uniqueness of path-lifting, we have $\eta = \widetilde{(g \circ \delta)}$. Then $\widetilde{(g \circ \delta)}(1) = \eta(1) = \widetilde{(g \circ \overline{\delta})}(0) = \widetilde{(g \circ \gamma)}(1) = x_1$. Therefore, the map $\widetilde{g}$ in (1.4.27) is well-defined. It follows from the construction of $\widetilde{g}$ that $f \circ \widetilde{g} = g$. It remains to show that $\widetilde{g}$ is continuous. Here we need to use local path-connectedness of T.

Fix a point $t_1 \in T$ and let $y_1 = g(t_1) \in Y$ and $x_1 := \widetilde{g}(t_1) \in f^{-1}(y_1)$. Since f is a covering map, there is an open neighbourhood $U \subseteq Y$ of y_1 and an open neighbourhood $\widetilde{U} \subseteq X$ of x_1 such that

$$f\big|_{\widetilde{U}} : \widetilde{U} \to U \tag{1.4.28}$$

is a homeomorphism. Since T is locally path-connected and g is continuous, there is a path-connected neighbourhood $V \subseteq T$ of t_1 such that $g(V) \subseteq U$. To show $\widetilde{g} : T \to X$ continuous, it is enough to show that $\widetilde{g}(V) \subseteq \widetilde{U}$. Given $t' \in V$, choose a path α inside V joining t_1 to t'. Then $\gamma \star \alpha$ is a path in T joining t_0 to t', and its image $g \circ (\gamma \star \alpha)$ has a lifting, say β, in X starting at x_0. Let $\widetilde{\alpha} := s \circ (g \circ \alpha)$, where $s : U \to \widetilde{U}$ is the inverse of the homeomorphism $f\big|_{\widetilde{U}}$ given in (1.4.28). Since $\widetilde{\gamma}(1) = (s \circ g \circ \alpha)(0)$, by uniqueness of path-lifting, β coincides with $\widetilde{\gamma} \star (s \circ g \circ \alpha)$. Then $\widetilde{g}(t') = \beta(1) = (\widetilde{\gamma} \star (s \circ g \circ \alpha))(1) = (s \circ g \circ \alpha)(1) \in \widetilde{U}$. Therefore, $\widetilde{g}(V) \subseteq \widetilde{U}$, and hence $\widetilde{g}$ is continuous. □

1.4.2 Fundamental group of S^1

Now we are in a position to compute fundamental group of the unit circle

$$S^1 := \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Assuming that the reader has forgotten the statement of Theorem 1.4.1 by now, let's recall it once again.

Theorem 1.4.1. *The fundamental group $\pi_1(S^1, 1)$ of the unit circle S^1 with the base point $1 \in S^1$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ generated by the loop $\omega : I \to S^1$ defined by $\omega(t) = e^{2\pi it}$, for all $t \in I = [0, 1]$.*

Proof. Let $\gamma : I \to S^1$ be a loop based at $x_0 = 1 \in S^1$. Since

$$p : \mathbb{R} \to S^1, \ \ t \mapsto e^{2\pi it}$$

is a covering map (c.f. Example 1.4.6 (i)), there is a unique continuous map $\widetilde{\gamma} : I \to \mathbb{R}$ such that $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} = \gamma$. Since $p^{-1}(\gamma(1)) = \mathbb{Z}$, the path $\widetilde{\gamma}$ ends at some integer, say n. Note that, we have a path

$$\widetilde{\omega}_n : I \to \mathbb{R}, \ \ s \mapsto ns,$$

starting at 0 and ending at n. Clearly the path $\widetilde{\gamma}$ is homotopic to $\widetilde{\omega}_n$ by the linear homotopy

$$F : I \times I \to \mathbb{R}, \ \ (s, t) \mapsto (1 - t)\widetilde{\gamma}(s) + t\,\widetilde{\omega}_n(s).$$

Then the composition $p \circ F : I \times I \to S^1$ is a homotopy from γ to ω_n, where $\omega_n : I \to S^1$ is the loop based at $1 \in S^1$ defined by

$$\omega_n(s) = e^{2\pi ins}, \ \ \forall\, s \in I.$$

Therefore, $[\gamma] = [\omega_n]$ in $\pi_1(S^1, 1)$.

Define a map

$$\varphi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1), \ \ n \mapsto [\omega_n].$$

It follows from the above construction that φ is surjective. To show that φ is a group homomorphism, we need to show that $\omega_m \star \omega_n \simeq \omega_{m+n}$, for all $m, n \in \mathbb{Z}$. To see this, consider the *"translation by m"* map

$$\tau_m : \mathbb{R} \to \mathbb{R}, \ \ x \mapsto x + m.$$

Note that $\tau_m \circ \widetilde{\omega}_n$ is a path in $\mathbb{R}$ starting at m and ending at $m + n$, and hence the path $\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)$ in $\mathbb{R}$ starts at 0 and ends at $m + n$. Then it follows from the first paragraph that $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n))$ is homotopic to ω_{m+n}. Since $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)) = \omega_m \star \omega_n$, we conclude that φ is a group homomorphism.

To show that φ is injective, it is enough to show if a loop $\gamma : I \to S^1$ based at 1 is homotopic to both ω_n and ω_m, for some $m, n \in \mathbb{Z}$, then $m = n$. Indeed, if $\gamma \simeq \omega_m$ and $\gamma \simeq \omega_n$, then $\omega_m \simeq \omega_n$ by Lemma 1.2.4. Let $G : I \times I \to S^1$ be a homotopy from ω_m to ω_n in S^1. By

Theorem 1.4.16 there is a unique continuous map $\widetilde{G} : I \times I \to \mathbb{R}$ such that $p \circ \widetilde{G} = G$ and $\widetilde{G}(0,0) = 0$. Then by uniqueness of path lifting (c.f. Theorem 1.4.12) we have $\widetilde{G}|_{\{0\}\times I} = \widetilde{\omega}_n$ and $\widetilde{G}|_{\{1\}\times I} = \widetilde{\omega}_m$. Since $\{\widetilde{G}|_{\{t\}\times I} : I \to \mathbb{R}\}_{t\in I}$ is a homotopy of paths, the end points $\widetilde{G}|_{\{t\}\times I}(1)$ are independent of t. Thus, $m = \widetilde{G}|_{\{0\}\times I}(1) = \widetilde{G}|_{\{1\}\times I}(1) = n$, and hence φ is injective. This completes the proof. □

1.4.3 Fundamental group of S^n, for $n \geq 2$

In this subsection we show that S^n is simply connected, for $n \geq 2$. First we need the following.

Lemma 1.4.29. *Let (X, x_0) be a pointed topological space. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X such that*

1. *each U_α is path-connected,*
2. *$x_0 \in U_\alpha$, for all $\alpha \in \Lambda$,*
3. *$U_\alpha \cap U_\beta$ is path-connected, for all $\alpha, \beta \in \Lambda$.*

Then any loop in X based at x_0 is homotopic to a finite product of loops each of which is contained in a single U_α, for finitely many α's.

Proof. Let $\gamma : I \to X$ be a loop based at x_0. Since γ is continuous, each $s \in I$ is contained in an open neighbourhood $V_s := (s - \delta_s, s + \delta_s) \subseteq I$ of s such that $\gamma(\overline{V}_s) \subseteq U_{\alpha_s}$, for some $\alpha_s \in \Lambda$. Since I is compact, we can choose finitely many such open neighbourhoods V_s's to cover I. Thus we get a finite partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $I = [0, 1]$ such that $\gamma([s_{j-1}, s_j]) \subseteq U_{\alpha_j}$, for some $\alpha_j \in \Lambda$, for all $j = 1, \ldots, m$. Therefore, the restriction

$$\gamma_j := \gamma|_{[s_{j-1}, s_j]} : [s_{j-1}, s_j] \to U_{\alpha_j} \subseteq X$$

is a path in U_{α_j}, for each $j = 1, \ldots, m$, and that $\gamma = \gamma_1 \star \cdots \star \gamma_m$. Since $U_j \cap U_{j+1}$ is path-

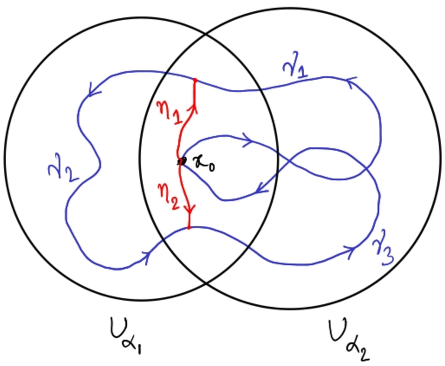


FIGURE 1.11

connected, we may choose a path η_i in $U_{\alpha_j} \cap U_{\alpha_{j+1}}$ from the base point x_0 to the point $\gamma(s_j) \in$

$U_{\alpha_j} \cap U_{\alpha_{j+1}}$, for all j (see Figure 1.11). Denote by $\overline{\eta}_j$ the opposite path of η_j, for all j (see definition (1.3.23) in §1.3.3). Then the product loop

$$(\gamma_1 \star \overline{\eta_1}) \star (\eta_1 \star \gamma_2 \star \overline{\eta_2}) \star (\eta_2 \star \gamma_3 \star \overline{\eta_3}) \star \cdots \star (\eta_{m-1} \star \gamma_m) \tag{1.4.30}$$

is homotopic to γ (see Exercise 1.3.24). Clearly this loop is a composition of the loops $\gamma_1 \star \overline{\eta_1}$, $\eta_1 \star \gamma_2 \star \overline{\eta_2}$, $\eta_2 \star \gamma_3 \star \overline{\eta_3}, \cdots, \eta_{m-1} \star \gamma_m$ based at x_0, each lying inside a single U_{α_j}, for all $j = 1, \ldots, m$. This completes the proof. $\square$

Exercise 1.4.31. Fix an integer $n \geq 1$.

(i) For any $x_0 \in S^n$, show that $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbb{R}^n$.

(ii) For a pair of antipodal points $x_1, x_2 \in S^n$, let $U_j := S^n \setminus \{x_j\}$, for $j = 1, 2$. Show that $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Proposition 1.4.32. *For an integer $n \geq 2$, we have $\pi_1(S^n) = \{1\}$.*

Proof. Fix a pair of antipodal points x_1, x_2 in S^n. Then we have two open subsets $U_1 = S^n \setminus \{x_1\}$ and $U_2 = S^n \setminus \{x_2\}$ each homeomorphic to $\mathbb{R}^n$. Clearly $S^n = U_1 \cup U_2$ and $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. Then by Exercise 1.4.31 we have $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, which is path-connected because $n \geq 2$. Fix a base point $x_0 \in U_1 \cap U_2$. Let γ be a loop in S^n based at x_0. Then by Lemma 1.4.29 γ is homotopic to a product of finitely many loops in S^n based at x_0 each of which are contained in either U_1 or U_2. Since both U_1 and U_2 are homeomorphic to $\mathbb{R}^n$ by Exercise 1.4.31, we have $\pi_1(U_j) = \pi_1(\mathbb{R}^n) = \{1\}$, for $j = 1, 2$. Therefore, γ is homotopic to a finite product of loops based at x_0 each of which are null-homotopic, and hence γ is null-homotopic. $\square$

Corollary 1.4.33. *S^n is simply connected, for $n \geq 2$.*

Exercise 1.4.34. For a point $x_0 \in \mathbb{R}^n$, show that the space $\mathbb{R}^n \setminus \{x_0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Corollary 1.4.35. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for $n \neq 2$.*

Proof. If possible let $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. For $n = 1$, since $\mathbb{R}^2 \setminus \{0\}$ is path-connected while $\mathbb{R} \setminus \{f(0)\}$ is disconnected, there is no such homeomorphism in this case. Suppose that $n > 2$. In this case, we cannot distinguish $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{R}^n \setminus \{f(0)\}$ in terms of number of path-components; but we can distinguish them by their fundamental groups.

Since for any point $x \in \mathbb{R}^n$ the space $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ by Exercise 1.4.34, we have

$$\begin{aligned}
\pi_1(\mathbb{R}^n \setminus \{x\}) &\cong \pi_1(S^{n-1} \times \mathbb{R}) \\
&\cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \\
&\cong \pi_1(S^{n-1}),
\end{aligned}$$

because $\pi_1(\mathbb{R})$ is trivial. Since $\pi_1(S^1) \cong \mathbb{Z}$ by Theorem 1.4.1 while $\pi_1(S^{n-1}) \cong \{1\}$, for $n > 2$, by Proposition 1.4.32, such a homeomorphism cannot exists. $\square$

Remark 1.4.36. A more general result that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$ can be proved in a similar fashion using higher homotopy groups or homology groups. In fact, using homology groups one can show that *non-empty open subsets of* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *can be homeomorphic if and only if* $m = n$.

1.4.4 Some applications

Theorem 1.4.37 (Fundamental theorem of algebra)**.** *Every non-constant polynomial with coefficients from* $\mathbb{C}$ *has a root in* $\mathbb{C}$.

Proof. Take a non-constant polynomial $p(z) \in \mathbb{C}[z]$. Diving $p(z)$ by its leading coefficient, if required, we may assume that

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z].$$

If $p(z)$ has no roots in $\mathbb{C}$, then for each real number $r \geq 0$, the map $\gamma_r : I \to S^1 \subset \mathbb{C}$ defined by

$$\gamma_r(s) := \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \quad \forall\, s \in I, \tag{1.4.38}$$

is a loop in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ with the base point $1 \in \mathbb{C}$. As r varies, the collection $\{\gamma_r\}_{r \geq 0}$ defines a homotopy of loops in S^1 based at 1. Since γ_0 is the constant loop 1 in S^1, we see that the homotopy class $[\gamma_r] \in \pi_1(S^1, 1)$ is trivial, for all $r \geq 0$.

Choose any $r \in \mathbb{R}$ with $r > \max\{1, |a_1| + \cdots + |a_n|\}$. Then for $|z| = r$ we have

$$\begin{aligned} |z^n| = r^n = r \cdot r^{n-1} &> \big(|a_1| + \cdots + |a_n|\big)|z^{n-1}| \\ &\geq |a_1 z^{n-1} + \cdots + a_n| \end{aligned}$$

From this inequality, it follows that for each $t \in [0, 1]$, the polynomial

$$p_t(z) := z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = r$. Replacing $p(z)$ with $p_t(z)$ in the expression of γ_r in (1.4.38) and letting t vary from 1 to 0, we get a homotopy from the loop γ_r to the loop

$$\omega_n : I \to S^1, \;\; s \mapsto e^{2\pi i n s}.$$

Since the loop ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1, 1) \cong \mathbb{Z}$, and that $[\omega_n] = [\gamma_t] = 0$, we must have $n = 0$. Thus the only polynomials without roots in $\mathbb{C}$ are constants. $\square$

Definition 1.4.39. A *deformation retraction* of X onto its subspace A is a continuous map $F : X \times I \to X$ such that the associated family of continuous maps

$$\Big\{ f_t := F|_{X \times \{t\}} : X \to X \Big\}_{t \in I}$$

obtained by restricting F on the slices $X \times \{t\} \hookrightarrow X \times I$, for each $t \in I$, satisfies $f_0 = \mathrm{Id}_X$, $f_1(X) = A$, and $f_t\big|_A = \mathrm{Id}_A$, $\forall\, t \in I$. In this case, we say that A is a deformation retract of X.

Example 1.4.40. (i) Let $D = \{re^{i\theta} \in \mathbb{C} : 0 < r \leq 1,\ 0 \leq \theta < 2\pi\}$ be the punctured disk of radius 1 in the plane $\mathbb{C}$, and let $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subseteq X$ be the unit circle. For each $t \in I = [0,1]$, we define a map

$$f_t : D \longrightarrow D$$

by sending $re^{i\theta} \in D$ to $\big(t + (1-t)r\big)e^{i\theta} \in D$. It is easy to verify that $\{f_t\}_{t\in I}$ is a family of continuous maps from D into itself, and satisfies $f_0 = \mathrm{Id}_D$, $f_1(D) = S^1$ and $f_t\big|_{S^1} = \mathrm{Id}_{S^1}$. Therefore, $\{f_t\}_{t\in I}$ is a deformation retraction of D onto S^1.

(ii) Let X be the Möbius strip (see Figure 1.12) and $A \subset X$ be the central simple loop of X. Then there is a deformation retraction of X onto A.

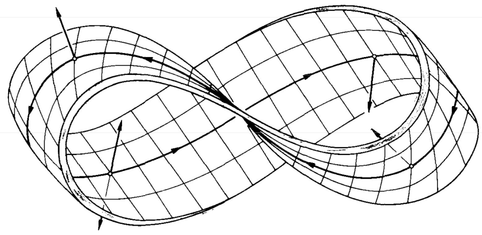


FIGURE 1.12: Möbius strip

Definition 1.4.41. A *retraction* of X onto a subspace $A \subset X$ is a continuous map $f : X \to X$ such that $f(X) = A$ and $f\big|_A = \mathrm{Id}_A$. A subspace $A \subseteq X$ is said to be a *retract* of X if there is a retraction of X onto A.

Note that a retraction $f : X \to X$ of X onto a subspace $A \subseteq X$ can be characterized by its property $f \circ f = f$, and hence we can think of it as a topological analogue of a *projection operator* in algebra.

Lemma 1.4.42. *If $A \subseteq X$ is a retract of X, for any $a_0 \in A$ the homomorphism of fundamental groups*

$$\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0),$$

induced by the inclusion map $\iota : A \hookrightarrow X$, is injective.

Proof. Let $f : X \to X$ be a retraction of X onto A. Then $f \circ \iota = \mathrm{Id}_A$, the identity map of A. Then by Proposition 1.3.12 and Remark 1.3.13 we have $f_* \circ \iota_* = \mathrm{Id}_{\pi_1(A,a_0)}$. Thus ι_* admits a left inverse, and hence is injective. □

Proposition 1.4.43. *If $A \subseteq X$ is a deformation retract of X, then X is homotopically equivalent to A (see Definition 1.2.9).*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto its subspace A. Since

$$f_0 : X \to X,\ \ x \mapsto F(x,\, 0)$$

is the identity map $\mathrm{Id}_X : X \to X$, and

$$f_1 : X \to X,\ x \mapsto F(x,1)$$

is a retraction of X onto A, we conclude that F is a homotopy from Id_X to a retraction of X onto A. Since $f_1 \circ \iota = \mathrm{Id}_A$ and $\iota \circ f_1$ is homotopic to the identity map of X, we conclude that X and A are homotopically equivalent. □

Corollary 1.4.44. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$ we have an isomorphism of fundamental groups $\pi_1(A,a_0) \cong \pi_1(X,a_0)$.*

Proof. Follows from Lemma 1.3.16. □

Remark 1.4.45. Note that the constant map $X \to \{x_0\} \subseteq X$ being continuous, every space X admits a retraction onto a point of it. However, the next Proposition 1.4.46 and Lemma 1.4.47 produce examples of topological spaces that do not admit any deformation retract onto a point of it.

Proposition 1.4.46. *If there is a deformation retract of X onto a point $x_0 \in X$, then X is path connected.*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto a point $x_0 \in X$. Since for any point $x \in X$, the continuous map

$$\phi_x : I \to X,\ t \mapsto F(x,t)$$

is a path joining $F(x,0) = x$ and $F(x,1) = x_0$, X is path connected. □

Lemma 1.4.47. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$, the homomorphism of fundamental groups $\iota_* : \pi_1(A,a_0) \to \pi_1(X,a_0)$ induced by the inclusion map $\iota : A \hookrightarrow X$ is an isomorphism.*

Proof. Let $F : X \times I \to X$ be a deformation retraction of X onto X. Then $f_1 := F\big|_{X\times\{1\}} : X \to X$ is a retraction of X onto A. Then by Lemma 1.4.42 the homomorphism $\iota_* : \pi_1(A,a_0) \to \pi_1(X,a_0)$ is injective. To show ι_* is an isomorphism, it enough to show that it is surjective. Note that, given any loop $\gamma : I \to X$ in X based at a_0, the composite map

$$G : I \times I \xrightarrow{\gamma\times\mathrm{Id}_I} X \times I \xrightarrow{F} X$$

is a path-homotopy from $G\big|_{I\times\{0\}} = \gamma$ to a loop $g := G\big|_{I\times\{1\}} : I \to A$ based at a_0. Thus, $\iota_*([g]) = [g] = [\gamma]$, and hence ι_* is surjective. □

Remark 1.4.48. The notion of deformation retraction of a space X onto a subspace $A \subseteq X$ is a way to continuously deform X onto A in a very strong sense, while the notion of homotopy equivalence seems to be a weaker notion of being able to deform a space into another space. However, if two spaces X and Y are homotopically equivalent, then there is a space Z such that both X and Y are deformation retracts of Z. Such a space Z can be constructed as a mapping cylinder

$$M_f := \big((X \times I) \sqcup Y\big)/(x,1) \sim f(x)$$

of a homotopy equivalence $f : X \to Y$. We shall not go into details for its proof in this course.

Exercise 1.4.49. Show that the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ do not admit any deformation retraction onto a point of it.

Exercise 1.4.50. Show that $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \mathbb{Z}$.

For an integer $n \geq 1$, let

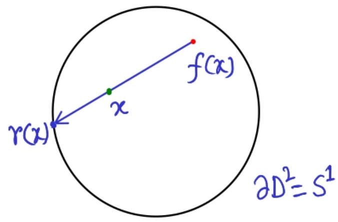
$$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1\}$$

be the *closed unit disk* in $\mathbb{R}^n$. Its boundary ∂D^n is the *unit sphere* in $\mathbb{R}^n$ given by

$$S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 = 1\}.$$

Theorem 1.4.51 (Brouwer's fixed point theorem)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Suppose on the contrary that $f : D^2 \to D^2$ has no fixed point, i.e., $f(x) \neq x$, $\forall\, x \in D^2$. Then for each $x \in D^2$, the ray in $\mathbb{R}^2$ starting at $f(x)$ and passing through x hits a unique point, say $r(x) \in S^1$. This defines a map $r : D^2 \to S^1$. Since f is continuous, small perturbations of



x produce small perturbations of $f(x)$, and hence small perturbations of the ray starting from $f(x)$ and passing through x, it follows that the function $x \mapsto r(x)$ is continuous. Explicit proof of continuity could be given by writing down the explicit expression for $r(x)$ in terms of $f(x)$. Note that $r(x) = x$, for all $x \in S^1$. Therefore, $r : D^2 \to S^1$ is a retraction of D^2 onto its subspace $S^1 = \partial D^2$. Then by Lemma 1.4.42 the homomorphism of fundamental groups

$$\iota_* : \pi_1(S^1, (1,0)) \longrightarrow \pi_1(D^2, (1,0))$$

induced by the inclusion map $\iota : S^1 \hookrightarrow D^2$, is injective. Since $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1,0))$ is trivial, we get a contradiction. $\square$

Remark 1.4.52. The corresponding statement for Brouwer's fixed point theorem holds, more generally, for a closed unit disk $D^n \subset \mathbb{R}^n$, for all $n \geq 2$. If time permits, we shall give a proof of it using homology. However, the original proof of it, due to Brouwer, neither uses homology nor uses homotopy groups, which was not invented at that time. Instead, Brouwer's proof uses the notion of degree of maps $S^n \to S^n$, which could be defined later using homology, but Brouwer defined it more directly in a geometric way.

Definition 1.4.53. For $x = (x_1, \ldots, x_{n+1}) \in S^n$, we define its *antipodal point* to be the point $-x := (-x_1, \ldots, -x_{n+1}) \in S^n$.

Theorem 1.4.54 (Borsuk-Ulam)**.** *Let $n \in \{1,2\}$. Then for every continuous map $f : S^n \to \mathbb{R}^n$, there is a pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$.*

Proof. The case $n = 1$ is easy. Indeed, since the function

$$g : S^1 \to \mathbb{R}, \quad x \mapsto f(x) - f(-x)$$

changes its sign after the point $x \in S^1$ moves half way along the circle S^1, there must be a pint $x \in S^1$ such that $f(x) = f(-x)$.

Assume that $n = 2$. We use the same technique used to compute the fundamental group of S^1. Suppose on the contrary that there is a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$, for all $x \in S^2$. Then we can define a map $g : S^2 \to \mathbb{R}^2$ by

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}, \ \forall\, x \in S^2,$$

where $\|(y_1, y_2)\| := \sqrt{y_1^2 + y_2^2}$ is the *norm* of $(y_1, y_2) \in \mathbb{R}^2$. Since $\|g(x)\| = 1$, the image of the map g lands inside $S^1 \subset \mathbb{R}^2$. Note that the map $g : S^2 \to S^1$ is continuous. Define a loop $\eta : I = [0, 1] \to S^2$ by

$$\eta(s) = (\cos 2\pi s, \, \sin 2\pi s, \, 0), \ \forall\, s \in I. \tag{1.4.55}$$

Then η circles around the equator of the sphere $S^2 \subset \mathbb{R}^3$. Let $h : I \to S^1$ be the composite map $h := g \circ \eta$.

$$h : I \xrightarrow{\eta} S^2 \xrightarrow{g} S^1.$$

Since $g(x) = -g(-x)$, we have

$$h(s + \frac{1}{2}) = -h(s), \ \forall\, s \in [0, 1/2]. \tag{1.4.56}$$

Now consider the covering map

$$p : \mathbb{R} \to S^1, \ \ s \mapsto e^{2\pi i s} = (\cos 2\pi s, \, \sin 2\pi s).$$

Lift the loop $h : I \to S^1$ to this cover to get a unique path $\widetilde{h} : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$ (see Theorem 1.4.12). Then it follows from the relation (1.4.56) that

$$\widetilde{h}(s + \frac{1}{2}) = \widetilde{h}(s) + \frac{q(s)}{2}, \tag{1.4.57}$$

for some odd integer $q(s)$ depending on $s \in [0, \frac{1}{2}]$. Since $\widetilde{h}$ is continuous, it follows from the equation (1.4.57) that the map

$$I \to \mathbb{R}, \ \ s \mapsto q(s),$$

is continuous on $[0, \frac{1}{2}]$. Since q is a discrete function taking values in odd integers, we must have $q(s) = q$, for some odd integer q, for all $s \in [0, \frac{1}{2}]$. In particular, putting $s = 1/2$ and 0 in (1.4.57) we have

$$\widetilde{h}(1) = \widetilde{h}(1/2) + \frac{q}{2} = \widetilde{h}(0) + q.$$

This means that the loop h represents q times a generator of $\pi_1(S^1)$. Since q is an odd integer, h cannot be null homotopic. But this cannot happen because the loop $\eta : I \to S^2$ being null-homotopic, the loop $h := g \circ \eta : I \to S^2 \to S^1$ should be null-homotopic. Thus we get a contradiction. This completes the proof. $\square$

Remark 1.4.58. (i) Borsuk-Ulam theorem (Theorem 1.4.54) holds for all integer $n \geq 1$. A general proof could be given using homology theory later.

(ii) Theorem 1.4.54 says that there is no one-to-one continuous map from S^n into $\mathbb{R}^n$. As a result, S^n cannot be homeomorphic to a subspace of $\mathbb{R}^n$.

1.5 Galois theory for covering spaces

1.5.1 Universal cover

Since we shall work with paths in X, and a locally path-connected space is connected if and only if it is path-connected, and path-connected components of X are the same as connected components of X, there is no harm in assuming that X is connected or equivalently path-connected. Unless explicitly mentioned, in this section, we always assume that X is path-connected and locally path-connected.

Proposition 1.5.1. *Let X be a connected and locally path-connected topological space. Fix a point $x_0 \in X$. Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a simply connected covering. Then for any connected covering $f : (Y, y_0) \to (X, x_0)$, there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $p \circ F = f$.*

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x_0}) \arrow[dl, dashed, "F"'] \arrow[dd, "p"] \\ (Y, y_0) \arrow[dr, "f"'] & \\ & (X, x_0) \end{tikzcd}$$

Proof. Since X is locally path-connected and $\widetilde{X}$ is a simply connected covering of X, $\widetilde{X}$ is path-connected and locally path-connected. Since $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial and $f : (Y, y_0) \to (X, x_0)$ is a covering map, by general lifting criterion (see Theorem 1.4.26) there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $f \circ F = p$. $\square$

Proposition 1.5.2. *Let (X, x_0) be a locally path-connected and path-connected topological space. Let $p_1 : (\widetilde{X_1}, \widetilde{x_1}) \to (X, x_0)$ and $p_2 : (\widetilde{X_2}, \widetilde{x_2}) \to (X, x_0)$ be two simply connected covering spaces of (X, x_0). Then there is a unique homeomorphism of pointed topological spaces $F : (\widetilde{X_1}, \widetilde{x_1}) \to (\widetilde{X_2}, \widetilde{x_2})$ such that $p_2 \circ F = p_1$.*

$$\begin{tikzcd} (\widetilde{X_1}, \widetilde{x_1}) \arrow[rr, "F"] \arrow[dr, "p_1"'] & & (\widetilde{X_2}, \widetilde{x_2}) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

Proof. Follows from Proposition 1.5.1. □

Definition 1.5.3. A simply connected covering space of a path-connected locally path-connected topological space (X, x_0) is called the *universal cover* of (X, x_0). This name is due to its universal property (c.f. Proposition 1.5.1) and uniqueness upto a unique homeomorphism (c.f. Proposition 1.5.2).

It is not yet clear if universal cover of a path-connected locally path-connected topological space exists or not, however if it exists, it is unique up to a unique homeomorphism of pointed topological space by Proposition 1.5.2. The following Lemma 1.5.4 gives a necessary condition on (X, x_0) for existence of a universal covering space.

Lemma 1.5.4. *Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be the universal cover of (X, x_0). Then each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.*

Proof. Fix $x \in X$. Then there is a path-connected open neighbourhood $U \subseteq X$ which is evenly covered by the covering map p. Let $\widetilde{U} \subseteq \widetilde{X}$ be the path-connected open subset such that $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Let γ be a loop in U based at x. Using the homeomorphism $p|_{\widetilde{U}}$, we can lift it to a loop $\widetilde{\gamma}$ in $\widetilde{X}$ based at the point $\widetilde{x} \in \widetilde{U} \cap p^{-1}(x)$. Since $\widetilde{X}$ is simply-connected, we have a path-homotopy $F : I \times I \to \widetilde{X}$ from $\widetilde{\gamma}$ to the constant loop $c_{\widetilde{x}}$ at $\widetilde{x}$ in $\widetilde{X}$. Composing F with p we get a path-homotopy $p \circ F$ from γ to the constant loop c_x at x in X. This shows that the homomorphism $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $\iota : U \hookrightarrow X$ is trivial. □

Definition 1.5.5. A path-connected and locally path-connected topological space X is said to be *semi-locally simply connected* if each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.

1.5.2 Construction of universal cover

The following theorem shows that the condition on (X, x_0) for existence of its universal covering space given in Lemma 1.5.4 is, in fact, sufficient.

Theorem 1.5.6. *Let X be a path-connected, locally path-connected topological space. Fix a point $x_0 \in X$. Then a simply connected covering space of (X, x_0) exists if and only if X is semi-locally simply connected.*

Proof. If a simply connected covering space for X exists, then X is semi-locally simply connected by Lemma 1.5.4.

Suppose that X is semi-locally simply connected. We give an explicit construction of a simply connected covering space of X. Note that, if $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a simply connected covering space for (X, x_0), then for each $\widetilde{x} \in \widetilde{X}$, there is a unique path-homotopy class of paths in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x}$ (see Corollary 1.3.33). Thus, points of $\widetilde{X}$ can be thought of as homotopy classes of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$, and hence can be thought of as the homotopy classes of paths in X starting at x_0 thanks to the homotopy lifting property. This motivates us to construct the underlined set of points of $\widetilde{X}$ as

$$\widetilde{X} := \{[\gamma] : \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

where $[\gamma]$ denotes the path-homotopy class of a path γ in X. Define

$$p : \widetilde{X} \to X \tag{1.5.7}$$

by sending a $[\gamma] \in \widetilde{X}$ to the end point $\gamma(1) \in X$ of γ; this map is well-defined because of the definition of path-homotopy (see Definition 1.3.1). Since X is path-connected, given any $x_1 \in X$ there is a path γ in X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $[\gamma] \in \widetilde{X}$ with $p([\gamma]) = x_1$. Thus, p is surjective. If we set $\widetilde{x_0} \in \widetilde{X}$ to be the path-homotopy class of the constant path $c_{x_0} : I \to X$ given by $c_{x_0}(t) = x_0$, $\forall\, t \in I$, then $p(\widetilde{x_0}) = x_0$.

It remains to give a suitable topology on $\widetilde{X}$ to make $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ a simply connected covering space of (X, x_0). Let

$$\begin{aligned}\mathscr{U} := \{V \overset{\iota}{\hookrightarrow} X \mid\ & V \text{ is a path-connected open subset of } X \text{ such that} \\ & \text{the homomorphism } \iota_* : \pi_1(V) \to \pi_1(X) \text{ is trivial } \}.\end{aligned}$$

Note that, if the homomorphism $\iota_* : \pi_1(V, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : V \hookrightarrow X$, is trivial for some $x \in V$, then it is trivial for all points of V, whenever V is path-connected. Moreover, if U and V are two path-connected open subsets of X with $V \subseteq U$ and $U \in \mathscr{U}$, then it follows from the following commutative diagram

$$\begin{tikzcd} \pi_1(V) \arrow[rr, "\iota_{V,U*}"] \arrow[dr, "\iota_{V*}"'] & & \pi_1(U) \arrow[dl, "\iota_{U*}"] \\ & \pi_1(X) & \end{tikzcd}$$

that $V \in \mathscr{U}$, where $\iota_U : U \hookrightarrow X$, $\iota_V : V \hookrightarrow X$ and $\iota_{V,U} : V \hookrightarrow U$ are inclusion maps. Since X is locally-path-connected, path-connected and semi-locally simply connected, now it follows that $\mathscr{U}$ is a basis for the topology on X (verify!).

We now use the collection $\mathscr{U}$ to construct a collection $\mathscr{B}$ of subsets of $\widetilde{X}$ which forms a basis for the desired topology on $\widetilde{X}$. Given $U \in \mathscr{U}$ and a path γ in X starting at x_0 and ending at a point in U, consider the subset

$$U_{[\gamma]} := \{[\gamma \star \eta] : \eta \text{ is a path in } U \text{ starting at } \gamma(1)\} \subseteq \widetilde{X}.$$

Note that, if γ is path-homotopic to γ' in X, then $\gamma(1) = \gamma'(1)$, and hence for any path η in U starting at $\gamma(1) = \gamma'(1)$, we have $[\gamma \star \eta] = [\gamma' \star \eta]$. Therefore, the subset $U_{[\gamma]} \subseteq \widetilde{X}$ depends only on U and the path-homotopy class of γ in X.

Observation 1: The restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U \tag{1.5.8}$$

is bijective. Indeed, it is surjective because U is path-connected. To see it is injective, note that if $p([\gamma \star \eta]) = p([\gamma \star \eta'])$, then $\eta(1) = \eta'(1)$ and so the loop $\eta \star \overline{\eta'}$ is path-homotopic to the constant path $c_{\eta(0)}$ inside X, because the homomorphism $\iota_* : \pi_1(U) \to \pi_1(X)$ is trivial. Then it follows that $[\gamma \star \eta] = [\gamma \star \eta']$. Therefore, the restriction of p on $U_{[\gamma]}$ (see (1.5.8)) is injective, and hence is bijective.

Observation 2: Given $U \in \mathscr{U}$ and any two paths γ and δ in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1), \delta(1) \in U$, if $[\delta] \in U_{[\gamma]}$, then we must have $U_{[\gamma]} = U_{[\delta]}$. Indeed, if $[\delta] \in U_{[\gamma]}$, then $[\delta] = [\gamma \star \eta]$, for some path η in U with $\eta(0) = \gamma(1)$. Then for any path α in U with $\alpha(0) = \delta(1)$, we have $[\delta \star \alpha] = [(\gamma \star \eta) \star \alpha] = [\gamma \star (\eta \star \alpha)] \in U_{[\gamma]}$. Thus $U_{[\delta]} \subseteq U_{[\gamma]}$. Conversely, given any $[\gamma \star \alpha] \in U_{[\gamma]}$ we have $[\gamma \star \alpha] = [\gamma \star \eta \star \overline{\eta} \star \alpha] = [\delta \star (\overline{\eta} \star \alpha)] \in U_{[\delta]}$, which shows that $U_{[\gamma]} \subseteq U_{[\delta]}$. Therefore, we conclude that $U_{[\gamma]} = U_{[\delta]}$ if $[\delta] \in U_{[\gamma]}$.

Now we use the above two observations to show that the collection

$$\mathscr{B} := \{U_{[\gamma]} : U \in \mathscr{U} \text{ and } \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

forms a basis for a topology on $\widetilde{X}$. Note that, X being path-connected, we have $\widetilde{X} = \bigcup\limits_{U_{[\gamma]} \in \mathscr{B}} U_{[\gamma]}$. To check the second property for $\mathscr{B}$ to be a basis for a topology on $\widetilde{X}$, suppose that we are given two objects $U_{[\gamma]}, V_{[\delta]} \in \mathscr{B}$ and an element

$$[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}. \tag{1.5.9}$$

Now $U, V \in \mathscr{U}$, and γ and δ are paths in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) \in U$, $\delta(1) \in V$. We claim that

$$U_{[\gamma]} = U_{[\alpha]} \quad \text{and} \quad V_{[\delta]} = V_{[\alpha]}. \tag{1.5.10}$$

Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, we have $[\alpha] = [\gamma \star \eta] = [\delta \star \eta']$, for some paths η and η' in U and V respectively, with $\eta(0) = \gamma(1)$ and $\eta'(0) = \delta(1)$. Since $\gamma \star \eta$ is path-homotopic to $\delta \star \eta'$, both of them have the same end point, and hence $\alpha(1) = \eta(1) = \eta'(1) \in U \cap V$. Then the claim in (1.5.10) follows from the Observation 2. Since $\mathscr{U}$ is a basis for the topology on X, and $\alpha(1) \in U \cap V$, there is an object $W \in \mathscr{U}$ such that $\alpha(1) \in W$ and $W \subseteq U \cap V$. Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, the argument given in Observation 2 shows that

$$W_{[\alpha]} \subseteq U_{[\alpha]} \cap V_{[\alpha]} = U_{[\gamma]} \cap V_{[\delta]},$$

where the equality of sets on the right side is by (1.5.10). Clearly $[\alpha] \in W_{[\alpha]}$. Therefore, $\mathscr{B}$ is a basis for a topology on $\widetilde{X}$. Give $\widetilde{X}$ the topology generated by this basis $\mathscr{B}$.

Now it remains to show that $p : \widetilde{X} \to X$ in (1.5.7) is a covering map and that $\widetilde{X}$ is simply connected. We first show that, for each $U_{[\gamma]} \in \mathscr{B}$, the restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U$$

is a homeomorphism. We already have shown that $p|_{U_{[\gamma]}}$ is bijective. Note that, for any $V'_{[\delta]} \in \mathscr{B}$ with $V'_{[\delta]} \subseteq U_{[\gamma]}$ we have $p(V'_{[\delta]}) = V' \subseteq U$. Since both $\mathscr{U}$ and $\mathscr{B}$ are basis for the topologies of X and $\widetilde{X}$, respectively, this shows that the restriction map $p|_{U_{[\gamma]}}$ is open. To show that $p|_{U_{[\gamma]}}$ is continuous, it suffices to show that for any $V \in \mathscr{U}$ with $V \subseteq U$, we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma]}$. Indeed, for any $[\alpha] \in p^{-1}(V) \cap U_{[\gamma]}$, we have $\alpha(1) \in V \cap U$, and so $V_{[\alpha]} \subseteq U_{[\alpha]} = U_{[\gamma]}$ by Observation 2. Since $p(V_{[\alpha]}) = V$, it follows that $p^{-1}(V) \cap U_{[\gamma]} = V_{[\alpha]}$.

Since $\mathscr{B}$ is a basis for the topologies on $\widetilde{X}$, it follows that $p^{-1}(V)$ is open in $\widetilde{X}$, for all $V \in \mathscr{U}$. Since $\mathscr{U}$ is a basis for the topology on X, it follows that $p : \widetilde{X} \to X$ is continuous. Given a point $x \in X$, choose an object $U \in \mathscr{U}$ with $x \in U$. We claim that the collection

$$\mathscr{C}_U := \{U_{[\gamma]} : \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

is a partition of $p^{-1}(U)$. Since $p^{-1}(U) = \bigcup_{U_{[\gamma]} \in \mathscr{C}_U} U_{[\gamma]}$, it suffices to show that objects of the collection $\mathscr{C}_U$ are either disjoint or identical. If $[\alpha] \in U_{[\gamma]} \cap U_{[\delta]}$, then α is a path in X with $\alpha(0) = x_0$ and $\alpha(1) \in U$, and hence by Observation 2 we have $U_{[\gamma]} = U_{[\alpha]} = U_{[\delta]}$. Since the restriction of p on each of $U_{[\gamma]}$ is a homeomorphism, $p : \widetilde{X} \to X$ is a covering map.

It remains to show that $\widetilde{X}$ is simply connected. Given a point $[\gamma] \in \widetilde{X}$ and $t \in I$, consider the map $\gamma_t : I \to X$ defined by

$$\gamma_t(s) := \begin{cases} \gamma(s), & \text{if } \quad 0 \leq s \leq t, \text{ and} \\ \gamma(t), & \text{if } \quad t \leq s \leq 1. \end{cases} \tag{1.5.11}$$

Note that, each γ_t is a path in X starting at x_0, and hence its path-homotopy class is an element of $\widetilde{X}$. Then the map $\phi_{[\gamma]} : I \to \widetilde{X}$ defined by

$$\phi_{[\gamma]}(t) = [\gamma_t], \ \ \forall\, t \in I,$$

is a path (why it is continuous?) in $\widetilde{X}$ starting at $\widetilde{x_0} = [c_{x_0}] \in \widetilde{X}$ and ending at $[\gamma] \in \widetilde{X}$. Therefore, $\widetilde{X}$ is path-connected. Since $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a covering map, the homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ induced by the map p is injective by Corollary 1.4.19. Therefore, to show $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial it suffices to show that $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big)$ is the trivial subgroup of $\pi_1(X, x_0)$. By Corollary 1.4.19 elements of $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big) \subseteq \pi_1(X, x_0)$ are given by loops γ in X based at x_0 whose lift to the cover $p : \widetilde{X} \to X$ starting at $\widetilde{x_0}$ is a loop in $\widetilde{X}$ based at $\widetilde{x_0}$. Since $\phi_{[\gamma]}$ is a path in $\widetilde{X}$ starting at $\widetilde{x_0}$ and $p \circ \phi_{[\gamma]} = \gamma$, we must have $[\gamma] = \phi_{[\gamma]}(1) = \widetilde{x_0} = [c_{x_0}]$. In other words, γ is path-homotopic to the constant loop c_{x_0} in X. This completes the proof. □

We now go towards establishing Galois correspondence for covering spaces. Whenever we talk about simply connected covering space of X, we assume that X is semi-locally simply connected in addition to be it path-connected and locally path-connected.

1.5.3 Group action and covering map

Before proceeding further, let's recall some standard terminologies related to group action. Let G be a group, and let $\sigma : G \times X \to X$ be a left G-action on X. For notational simplicity, we denote by $g \cdot x$ the element $\sigma(g, x) \in X$, for all $(g, x) \in G \times X$. Given $x \in X$, the subset

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\} \subseteq G$$

is a subgroup of G, known as the *stabilizer of* x or the *isotropy subgroup* for x. The G-action σ is said to be *free* if $\mathrm{Stab}_G(x) = \{e\}$, for all $x \in X$. This means that, for each $x \in X$, given $g_1, g_2 \in G$, we have $g_1 \cdot x = g_2 \cdot x$ if and only if $g_1 = g_2$. Note that the G-action σ on X defines an equivalence relation on X; for $x \in X$, its equivalence class is the subset

$$\mathrm{Orb}_G(x) := \{g \cdot x : g \in G\} \subseteq X,$$

called the *G-orbit* of x in X. The G-action σ is said to be *transitive* if there is exactly one G-orbit in X. In other words, given any two points $x_1, x_2 \in X$, there exists $g \in G$ such that $x_2 = g \cdot x_1$.

Definition 1.5.12. Let G be a group. A G-action $\sigma : G \times X \to X$ on X is said to be *even* (or, *properly discontinuous* according to old texts) if the G-action map σ is continuous, and each point $x_0 \in X$ has an open neighbourhood $V \subseteq X$ such that $(g \cdot V) \cap V = \emptyset$, for all $g \neq e$ in G, where $g \cdot V := \{g \cdot x : x \in V\} \subseteq X$.

Remark on old notation: Most of the old texts uses the term *properly discontinuous G-action* to mean an even G-action. This terminology is awkward because the G-action on X itself is a continuous map.

Proposition 1.5.13. *If a group G is acting evenly on a path-connected and locally path-connected topological space Y, then the associated quotient map $q : Y \to Y/G$ is a covering map.*

Proof. Clearly the quotient map $q : Y \to Y/G$ is continuous. Note that, for any subset $V \subseteq Y$ we have

$$q^{-1}(q(V)) = \bigcup_{g \in G} g \cdot V, \tag{1.5.14}$$

where $g \cdot V = \{g \cdot v : v \in V\} \subseteq X$, for all $g \in G$. Since the left translation map $L_g : Y \to Y$ given by

$$L_g(y) = g \cdot y := \sigma(g, y), \ \forall\, y \in Y$$

is a homeomorphism, V is open in Y if and only if $g \cdot V = L_g(V)$ is open in Y, for all $g \in G$. Since q is a quotient map, it follows that $q(V)$ is open in Y/G if V is open in Y. Therefore, q is an open map.

To see $q : Y \to Y/G$ is a covering map, let's fix a point $v \in Y/G$, and a point $y \in q^{-1}(v)$. Since the G-action on Y is even, y has an open neighbourhood $U_y \subseteq Y$ such that $(g \cdot U_y) \cap U_y = \emptyset$, for all $g \neq e$ in G. Take $V_y := q(U_y)$. Then it follows that

$$q^{-1}(V_y) = \bigsqcup_{g \in G} g \cdot U_y.$$

It remains to show that the restriction map

$$q\big|_{g\cdot U_y} : g\cdot U_y \to V_y = q(U_y)$$

is a homeomorphism, for all $g \in G$. Since q is continuous and open, it suffices to show that $q\big|_{g\cdot U_y}$ is bijective, for all $g \in G$.

If $q\big|_{g\cdot U_y}$ were not injective, then there exist $y_1, y_2 \in g\cdot U_y$ with $y_1 \neq y_2$ such that $q(y_1) = q(y_2)$. Then there exists $h \in G$ such that $y_2 = h\cdot y_1$. Then $y_2 = h\cdot y_1 \in U_y \cap (h\cdot U_1)$ implies $h = e$ because the G-action on Y is even. This contradicts our assumption that $y_1 \neq y_2 = h\cdot y_1$. Therefore, $q\big|_{g\cdot U_y}$ must be injective. To show $q\big|_{g\cdot U_y}$ is surjective, note that a typical element of $V_y = q(U_y)$ is of the form $q(y_1)$, for some $y_1 \in U_y$. Since $q(y_1) = \mathrm{Orb}_G(y_1) = \{a\cdot y_1 : a \in G\}$, we see that $g\cdot y_1 \in g\cdot U_y$ satisfies $q\big|_{g\cdot U_y}(g\cdot y_1) = q(y_1)$. Therefore, $q\big|_{g\cdot U_y}$ is surjective. □

Proposition 1.5.13 allow us to construct a lot of examples of covering maps.

1.5.4 Group of Deck transformations

Let $f : Y \to X$ be a covering map. An *automorphism of* $f : Y \to X$ is a homeomorphisms $\phi : Y \to Y$ satisfying $f \circ \phi = f$. The set

$$\mathrm{Aut}(Y/X) := \{\phi : Y \to Y \mid \phi \text{ is a homeomorphism satisfying } f\circ\phi = f\}$$

of all automorphisms of $f : Y \to X$ forms a group with respect to the binary operation on $\mathrm{Aut}(Y/X)$ given by composition of homeomorphisms. The group $\mathrm{Aut}(Y/X)$ is also known as the group of *Deck transformations* or *covering transformations* of $f : Y \to X$. Note that, $\mathrm{Aut}(Y/X)$ acts on Y from the left by automorphisms:

$$a : \mathrm{Aut}(Y/X) \times Y \to Y,\ \ (\phi, y) \mapsto \phi(y). \tag{1.5.15}$$

We shall show in Proposition 1.5.19 that if we equip $\mathrm{Aut}(Y/X)$ with discrete topology, then the action map in (1.5.15) become continuous.

Proposition 1.5.16. *Fix a point $x_0 \in X$, and a path-connected covering space $f : Y \to X$ of X. Then the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to give a free $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If Y is simply connected, then the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive.*

Proof. Let $y_0 \in f^{-1}(x_0)$ be given. Since $\phi \in \mathrm{Aut}(Y/X)$ satisfies $f\circ\phi = f$, we have $f(\phi(y_0)) = f(y_0) = x_0$, and hence $\phi(y_0) \in f^{-1}(x_0)$. Therefore, the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to an $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If $\phi(y_0) = y_0$, for some $\phi \in \mathrm{Aut}(Y/X)$, then by uniqueness of lifting of maps (see Theorem 1.4.26 or Lemma 1.4.13) we must have $\phi = \mathrm{Id}_Y$. Therefore, the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is free.

Now assume that Y is simply connected. To show that $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive, choose two points $y_0, y_1 \in f^{-1}(x_0)$. Since X is locally path-connected and $f : Y \to X$ is a covering map, Y is locally path-connected. Since by assumption Y is path-connected and locally path-connected (since X is so) with $\pi_1(Y)$ trivial, by general lifting criterion (Theorem 1.4.26) there is a unique continuous map $\phi : (Y, y_0) \to (Y, y_1)$ such that $f \circ \phi = f$. Similarly, there is a unique continuous map $\psi : (Y, y_1) \to (Y, y_0)$ such that $f \circ \psi = f$. Then by uniqueness of lifting (see Theorem 1.4.26), we must have $\phi \circ \psi = \mathrm{Id}_{(Y,y_1)}$ and $\psi \circ \phi = \mathrm{Id}_{(Y,y_0)}$. Therefore, both ϕ and ψ are homeomorphisms, and that $\phi(y_0) = y_1$. Thus, the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive. □

Let $f : Y \to X$ be a covering map. Fix a point $x_0 \in X$. Since X is locally path-connected, there is a path-connected open neighbourhood $U \subset X$ of x_0 which is evenly covered by f. Then we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y, \tag{1.5.17}$$

where $V_y \subset Y$ is the path-connected open neighbourhood of $y \in f^{-1}(y_0)$ such that $f|_{V_y} : V_y \to U$ is a homeomorphism. Note that, $\{V_y : y \in f^{-1}(x_0)\}$ is precisely the set of all path-components of $f^{-1}(U)$.

Proposition 1.5.18. *With the above notations,* $\mathrm{Aut}(Y/X)$ *acts freely on the set of all path-components* $\{V_y : y \in f^{-1}(x_0)\}$ *of* $f^{-1}(U)$*. Moreover, this action is transitive when* Y *is simply connected.*

Proof. Since $f : Y \to X$ is a covering map, the restricted map

$$f_U := f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

is a covering map. Since for any $\phi \in \mathrm{Aut}(Y/X)$ we have $f \circ \phi = f$, image of the restriction map $\phi|_{f^{-1}(U)} : f^{-1}(U) \to Y$ lands inside $f^{-1}(U)$, and hence gives rise to an automorphism of the covering space $f_U : f^{-1}(U) \to U$, i.e., $\phi|_{f^{-1}(U)} \in \mathrm{Aut}(f^{-1}(U)/U)$. Clearly $\phi \in \mathrm{Aut}(Y/X)$ takes path-components of $f^{-1}(U)$ to path-components of $f^{-1}(U)$. In particular, for each $y \in f^{-1}(x_0)$, the induced map

$$\phi : V_y \to V_{\phi(y)}$$

is a homeomorphism. Since $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is free by Proposition 1.5.16, if for some $y \in f^{-1}(x_0)$, the automorphism $\phi \in \mathrm{Aut}(Y/X)$ takes V_y to itself, then we must have $\Phi = \mathrm{Id}_Y$.

Now assume that Y is simply connected. Since the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive by Proposition 1.5.16, and the path-components of $f^{-1}(U)$ are uniquely determined by the conditions that $V_y \cap f^{-1}(x_0) = \{y\}$ and $V_{y_1} \cap V_{y_2} = \emptyset$ for $y_1 \neq y_2$ in $f^{-1}(x_0)$, given any two path-components V_{y_1}, V_{y_2} of $f^{-1}(U)$, there exists $\phi \in \mathrm{Aut}(Y/X)$ such that $\phi(y_1) = y_2$, and hence $\phi(V_{y_1}) = V_{y_2}$. Thus, the $\mathrm{Aut}(Y/X)$-action on the set of all path-components of $f^{-1}(U)$ is transitive. □

Proposition 1.5.19. *Let* $f : Y \to X$ *be a path-connected covering space of* X*. Equip* $\mathrm{Aut}(Y/X)$ *with discrete topology. Then there is a continuous map (action map)*

$$a : \mathrm{Aut}(Y/X) \times Y \to Y, \tag{1.5.20}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}(Y/X) \times Y \arrow[r, "a"] \arrow[d, "pr_2"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X, \end{tikzcd} \tag{1.5.21}$$

where $pr_2 : \mathrm{Aut}(Y/X) \times Y \to Y$ *is the projection map onto the second factor.*

Proof. Clearly $a : \mathrm{Aut}(Y/X) \times Y \to X$ is defined by

$$a(\phi, y) = \phi(y), \ \forall\, (\phi, y) \in \mathrm{Aut}(Y/X) \times Y,$$

makes the above diagram commutative. We only need to show that the action map a is continuous.

Let

$$\mathcal{B} := \{V \subseteq Y \ : V \text{ is path-connected, open and } \\ f(V) \text{ is evenly covered by } f\}.$$

Since Y is path-connected and locally path-connected covering space for X, it is easy to check that $\mathcal{B}$ is a basis for the topology on Y. Therefore, to show the action map a is continuous, it is enough to show that $a^{-1}(V)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $V \in \mathcal{B}$. Fix $V \in \mathcal{B}$. Since V is path-connected and $f : Y \to X$ is a covering map, $U := f(V)$ is path-connected and open in X. Fix a point $x_0 \in U$. Since $U = f(V)$ is evenly covered by f, we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y,$$

where $V_y \subseteq Y$ is an open neighbourhood of $y \in f^{-1}(x_0)$ such that $f\big|_{V_y} : V_y \to U$ is a homeomorphism. Since V is path-connected and $p(V) = U$, we have $V \subseteq V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Since $f\big|_{V_{y_0}} : V_{y_0} \to U$ is a homeomorphism, we must have $V = V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Therefore, it is enough to show that $a^{-1}(V_y)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $y \in f^{-1}(x_0)$.

Let $(\phi, y) \in a^{-1}(V_{y_0}) = \{(\psi, y') \in \mathrm{Aut}(Y/X) \times Y : \psi(y') \in V_{y_0}\}$ be arbitrary. Then $\phi(y) \in V_{y_0}$. Since ϕ is an automorphism of Y, there is a unique $y_1 \in Y$ such that $\phi(y_1) = y_0$. Then $\phi : V_{y_1} \to V_{y_0}$ is a homeomorphism. Since $\phi(y) \in V_{y_0}$, we must have $y \in V_{y_1}$. Then $\{\phi\} \times V_{y_1}$ is an open neighbourhood of (ϕ, y) in $\mathrm{Aut}(Y/X) \times Y$ such that $a\big(\{\phi\} \times V_{y_1}\big) \subseteq V_{y_0}$. Therefore, $a^{-1}(V_{y_0})$ is open in $\mathrm{Aut}(Y/X) \times Y$. This completes the proof. □

Corollary 1.5.22. *If* $f : Y \to X$ *is a connected cover of* X*, the action of* $\mathrm{Aut}(Y/X)$ *on* Y *is even (see Definition 1.5.12).*

Proof. Follows from Proposition 1.5.19 and 1.5.18. □

Proposition 1.5.23. *If a group* G *acts evenly on a connected topological space* Y*, then the automorphism group* $\mathrm{Aut}(Y/X)$ *of the covering map* $q : Y \to X := Y/G$ *is naturally isomorphic to* G*.*

Proof. Let $\sigma : G \times Y \to Y$ be the left G-action which is even. Since σ is continuous, for each $g \in G$, the induced map

$$\sigma_g : Y \to Y, \;\; y \mapsto g \cdot y := \sigma(g, y)$$

is a homeomorphism of Y onto itself. Since the quotient map $q : Y \to X := Y/G$ sends a point $y \in Y$ to its G-orbit $\mathrm{Orb}_G(y) \in Y/G$, it follows that $q(\sigma_g(y)) = q(y)$, for all $g \in G$. Therefore, $\sigma_g \in \mathrm{Aut}(Y/X)$. Thus we have a natural map

$$\Phi : G \longrightarrow \mathrm{Aut}(Y/X), \;\; g \longmapsto \sigma_g. \tag{1.5.24}$$

Note that, for any $g, h \in G$ we have

$$\sigma_{gh}(y) = (gh) \cdot y = g \cdot (h \cdot y) = \sigma_g(\sigma_h(y)), \; \forall\, y \in Y.$$

Therefore, Φ is a group homomorphism. Since the G-action on Y is even (see Definition 1.5.12), it follows that $\mathrm{Ker}(\Phi)$ is trivial, and hence Φ is injective. Let $\varphi \in \mathrm{Aut}(Y/X)$ be arbitrary. Fix a pint $y \in Y$, and let $x := q(y) \in X$. Since $\varphi(y) \in q^{-1}(x) = \mathrm{Orb}_G(y)$, we have $\varphi(y) = g \cdot y = \sigma_g(y)$, for some $g \in G$. Since both $\varphi, \sigma_g \in \mathrm{Aut}(Y/X)$ and they agree at a point of Y and Y is connected, by uniqueness of lifting (see Lemma 1.4.13) we have $\varphi = \sigma_g$. Therefore, Φ is surjective, and hence is an isomorphism. □

1.5.5 Galois covers

Let $f : Y \to X$ be a path-connected covering space of X. Then the natural left $\mathrm{Aut}(Y/X)$-action on Y gives rise to an equivalence relation on Y, where the equivalence classes are $\mathrm{Aut}(Y/X)$-orbits of points of Y. Given $y \in Y$, its $\mathrm{Aut}(Y/X)$-orbit is the subset

$$\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y) = \{\phi(y) : \phi \in \mathrm{Aut}(Y/X)\} \subseteq Y.$$

Fix $y_0 \in Y$, and let $x_0 = f(y_0)$. Clearly, $\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y_0) \subseteq f^{-1}(x_0)$, and equality holds if and only if the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive. By the universal property of quotient space, there is a unique continuous map

$$\widetilde{f} : Y/\mathrm{Aut}(Y/X) \to X \tag{1.5.25}$$

such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \\ & Y/\mathrm{Aut}(Y/X) \arrow[ur, dashed, "\exists!\,\widetilde{f}"'] & \end{tikzcd} \tag{1.5.26}$$

where $q : Y \to Y/\mathrm{Aut}(Y/X)$ is the quotient map.

Definition 1.5.27 (Galois cover)**.** A covering map $f : Y \to X$ is said to be a *Galois cover* of X if Y is path-connected and the continuous map $\widetilde{f} : Y/\mathrm{Aut}(Y/X) \to X$ in (1.5.25), induced by f, is a homeomorphism (see the diagram (1.5.26)).

Proposition 1.5.28. *A connected covering map $p : Y \to X$ is Galois if and only if* $\mathrm{Aut}(Y/X)$ *acts transitively on each fiber of the covering map p.*

Proof. Suppose that $p : Y \to X$ is Galois cover. Consider the commutative diagram.

$$\begin{tikzcd} Y \arrow[rr, "p"] \arrow[dr, "q"'] & & X \\ & Y/\operatorname{Aut}(Y/X) \arrow[ur, "\widetilde{p}"'] & \end{tikzcd}$$

Since the induced map $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is a homeomorphism (by definition), for each $x \in X$, the fiber $p^{-1}(x)$ coincides with the $\mathrm{Aut}(Y/X)$-orbit of a point of the fiber $p^{-1}(x)$. In other words, the $\mathrm{Aut}(Y/X)$-action on each of the fibers of p is transitive.

Conversely, if the $\mathrm{Aut}(Y/X)$-action on each of the fibers of p is transitive, then the induced continuous map $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is bijective. Therefore, to show that $p : Y \to X$ a Galois cover, it suffices to show that $\widetilde{p}$ is an open map. Let $U \subseteq Y/\operatorname{Aut}(Y/X)$ be an open subset. Since the quotient map $q : Y \to Y/\operatorname{Aut}(Y/X)$ is continuous, $q^{-1}(U)$ is open in Y. Since the covering map $p : Y \to X$ is an open map, $p\big(q^{-1}(U)\big)$ is open in X. Since q is surjective, we have $q(q^{-1}(U)) = U$. Since $p = \widetilde{p} \circ q$, we have

$$\widetilde{p}(U) = \widetilde{p}\big(q(q^{-1}(U))\big) = p(q^{-1}(U)).$$

Therefore, $\widetilde{p}(U)$ is open in X. This completes the proof. □

If Y is simply connected, as remarked above, the $\mathrm{Aut}(Y/X)$-orbit of y_0 is precisely the fiber $f^{-1}(x_0)$, for all $y_0 \in f^{-1}(x_0)$. Therefore, in that case, the map $\widetilde{f}$ is bijective. This leads to the following.

Corollary 1.5.29. *A simply-connected covering map $p : \widetilde{X} \to X$ is Galois cover.*

Proof. $\widetilde{X}$ being simply connected, $\mathrm{Aut}(\widetilde{X}/X)$ acts transitively on each fiber of p by Proposition 1.5.18. Therefore, the result follows from Proposition 1.5.28. □

Remark 1.5.30. If $p : Y \to X$ is a covering map with Y connected, then to show $p : Y \to X$ is a Galois cover it suffices to show that $\mathrm{Aut}(Y/X)$ acts transitively on one fibre. Indeed, since in this case $Y/\operatorname{Aut}(Y/X)$ is a connected cover of X where one of the fibres is singleton, it follows that $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is a homeomorphism.

1.5.6 Galois correspondence for covering spaces

Theorem 1.5.31. *Let $p : Y \to X$ be a Galois cover. For each subgroup H of the Galois group $G :=$* $\mathrm{Aut}(Y/X)$*, the projection map p induces a natural continuous map $\widetilde{p}_H : Y/H \to X$ which is a* *covering map. Conversely, if* $f : Z \to X$ *is a connected cover of* X *fitting into a commutative diagram*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "p"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

then $\phi : Y \to Z$ *is a Galois cover and* Z *is homeomorphic to* Y/H*. The maps* $H \mapsto Y/H$ *and* $Z \mapsto \mathrm{Aut}(Y/Z)$ *induces a natural one-to-one correspondence between the collection of subgroups of* G *and the intermediate covers of* $p : Y \to X$ *as above. Moreover, the cover* $f : Z := Y/H \to X$ *is Galois if and only if* H *is a normal subgroup of* G*; and in this case we have* $\mathrm{Aut}(Z/X) \cong G/H$.

[Need to be added!] □

1.5.7 Monodromy action

[[Need to be added]]

1.6 Homology

1.6.1 Simplicial Complex

1.6.2 Homology group

1.6.3 Homology group for surfaces

1.6.4 Applications

1.7 Cohomology

Chapter 2

Appendix

2.1 Category Theory

Definition 2.1.1. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

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