

Introduction to Bridgeland Stability

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ABSTRACT. In this expository article, we discuss the concept of stability conditions in triangulated category and derived category as introduced by Tom Bridgeland in his famous 2007 paper [Bri07]. After reviewing some basic theories, we discuss some of its interesting applications in algebraic geometry.

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Caution! I have started writing this note (around June 2020) for myself to learn derived category, triangulated category, Bridgeland stability conditions, and their applications in various areas of algebraic geometry. The present note is unorganized, incomplete, and may contains many inaccuracies to be fixed. Stable polished version (as an expository note) may be released publicly in my home page after a year. Any suggestions to improve the exposition are welcome.

Rolling version: <https://arjunpaul29.github.io/home/notes/stability.pdf>.

0. INTRODUCTION

One of my preliminary motivation to start this series of discussions is to understand *stability condition* in triangulated category as introduced by Tom Bridgeland in his celebrated 2007 paper published in Annals of Mathematics. I also would like to learn some of its application in some other areas, like birational geometry, mirror symmetry etc.

Bridgeland's original motivation was to mathematically formulate the concept of Π -*stability* in theoretical physics as formulated by Douglas. In physics, Π -stability is something to relate a *super-symmetric non-linear sigma model* with a $(2, 2)$ *Super Conformal Field Theory (SCFT)*. Let's have a quick tour into an interesting intersection of geometry and physics.

0.1. Motivation from modern physics. Let us start with a tailor of a largely speculating theory, known as *mirror symmetry*. A *super-symmetric non-linear sigma model* consists of a complex Calabi-Yau variety $X = (M, I)$ admitting a Ricci flat Kähler form ω and a “*B-field*”. Let us explain the terminologies:

- M is the underlined real manifold of X and I is a complex structure on it,
- the variety X is *Calabi-Yau* means that the canonical line bundle K_X is trivial,
- the Kähler form ω is *Ricci flat* means that the curvature $F_{\det(\nabla_\omega)} = 0$, where $\det(\nabla_\omega)$ is the connection on $\det(TX)$ induced by the Chern connection ∇_ω on TX with respect to the Kähler form ω , and
- that “ B -field” is something mysterious.

In the context of SYZ mirror symmetry (an attempt to understand mathematically original version of mirror symmetry in physics), a B -field should be a class of a *unitary flat gerbe*, as suggested by Hitchin.

It is expected from physical ground that such a *super-symmetric non-linear sigma model* should give us a $(2, 2)$ *Super Conformal Field Theory* (SCFT). However, we don’t know any precise mathematical formulation of $(2, 2)$ SCFT, except for few cases! Roughly, a $(2, 2)$ SCFT is some physical theory that depends on both complex and symplectic structures of varieties, and using *topological twists* one may separate its parts:

- *A-side*: depend only on symplectic structure, and
- *B-side*: depend only on complex structure.

In his famous ICM talk in 1994, Maxim Kontsevich proposed that the mathematical objects obtained from these topological twists should be in the derived category of coherent sheaves on the B -side (algebraic side), and in the derived Fukaya category of Lagrangian submanifolds on the A -side (symplectic side). Physically, objects of these categories are considered to be boundary conditions, known as *branes*. In this sense, Fukaya category is the category of A -branes and the derived category of coherent sheaves is the category of B -branes.

Conjecture 0.1.1 (Kontsevich). *If two super-symmetric non-linear sigma models (X, ω, B) and (X', ω', B') , as described above, defines mirror symmetric SCFTs, then there are equivalences of categories:*

$$D^b(X) \simeq D^b(\text{Fukaya}(X', \omega')) \quad \text{and} \quad D^b(\text{Fukaya}(X, \omega)) \simeq D^b(X').$$

This is mathematically quite vague because we don’t have precise mathematical formulation of SCFT!

From mathematical point of view, Kontsevich’s Conjecture may be considered as a definition of *homological mirror symmetry*. Two super symmetric non-linear sigma models (X, ω, B) and (X', ω', B') are said to be *homological mirror partner* to each other if there are equivalences of such derived categories.

Remark 0.1.2. I think, finding explicit examples of such homological mirror symmetric pairs of super-symmetric non-linear sigma models would be very difficult

problem. There is a notion of mirror symmetric varieties in SYZ sense, which identifies X and X' as dual to each other in an appropriate sense; see e.g., works of Hitchin, Hausel-Thaddeus, Donagi-Pantev etc. This notion is different from the notion of homological mirror symmetry.

We shall see from construction of $D^b(X)$ that the derived category $D^b(X)$ depends only on complex/algebraic structure of X , and so $D^b(X)$ keeps only half information of the SCFT. Douglas argued that for any Ricci flat Kähler metric ω on X , there is a subcategory of $D^b(X)$, whose objects are physical branes, and these subcategories changes as the Kähler class ω moves in the stringy Kähler moduli.

To get an intuitive idea what this mathematically means, instead of looking at whole $D^b(X)$, consider the abelian category $\mathcal{Coh}(X)$. Then a choice of Kähler class (or polarization) singles out semistable and stable objects of $\mathcal{Coh}(X)$, and as we change the polarization, the collection of stable/semistable objects changes. Thus, there might be some way to encode more information of SCFT purely in terms of triangulated category $D^b(X)$ together with some extra structure on it. In a series of papers, Bridgeland set out to put these ideas on a mathematical setting and introduced the notion of *stability conditions* on a triangulated category. He has shown that the space of such stability conditions forms an (infinite) dimensional manifold, and this can be thought of an approximation of the stringy Kähler moduli space.

Mathematically, interesting point is that this new theory associates to a very algebraic object, like a triangulated category, a moduli space with meaningful geometric structure.

Roughly, a *stability condition* on a triangulated category \mathcal{A} is given by a heart \mathcal{H} of a bounded t -structure on \mathcal{A} and an additive group homomorphism $Z : K_0(\mathcal{H}) \rightarrow \mathbb{C}$, called the “central charge”, satisfying Harder-Narasimhan property.

Well, enough introduction, and we shall see these in detail! Let’s first set up some languages from category theory.

1. CATEGORY THEORY

Joke: Category theory is like Ramayana and Mahabharata — there are lots of arrows!

1.1. Abelian category.

Definition 1.1.1. A category \mathcal{A} consists of the following data:

- (i) a class of objects, denoted $\text{Ob}(\mathcal{A})$,
- (ii) for $X, Y \in \text{Ob}(\mathcal{A})$, a class of morphisms from X into Y , denoted $\text{Mor}_{\mathcal{A}}(X, Y)$,

(iii) for each $X, Y, Z \in \text{Ob}(\mathcal{A})$, a *composition map*

$$\text{Mor}_{\mathcal{A}}(X, Y) \times \text{Mor}_{\mathcal{A}}(Y, Z) \rightarrow \text{Mor}_{\mathcal{A}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

which satisfies associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \text{Mor}_{\mathcal{A}}(X, Y)$, $g \in \text{Mor}_{\mathcal{A}}(Y, Z)$ and $h \in \text{Mor}_{\mathcal{A}}(Z, W)$, for all $X, Y, Z, W \in \text{Ob}(\mathcal{A})$.

A category \mathcal{A} is said to be *locally small* if $\text{Mor}_{\mathcal{A}}(X, Y)$ is a set, for all $X, Y \in \text{Ob}(\mathcal{A})$. A category \mathcal{A} is said to be *small* if it is locally small and the class of objects $\text{Ob}(\mathcal{A})$ is a set.

Example 1.1.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathcal{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \rightarrow A_2$ and $g : A_2 \rightarrow A_1$ in \mathcal{A} such that $g \circ f = \text{Id}_{A_1}$ and $f \circ g = \text{Id}_{A_2}$.

Let \mathcal{A} and \mathcal{B} be two categories. A functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is given by the following data:

- (i) for each $X \in \mathcal{A}$ there is an object $\mathcal{F}(X) \in \mathcal{B}$,
- (ii) for $X, Y \in \mathcal{A}$ and $f \in \text{Hom}_{\mathcal{A}}(X, Y)$, there is $\mathcal{F}(f) \in \text{Mor}_{\mathcal{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, which are compatible with the composition maps.

A functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathcal{A}$, the induced map

$$\mathcal{F} : \text{Mor}_{\mathcal{A}}(A_1, A_2) \longrightarrow \text{Mor}_{\mathcal{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$$

is injective (resp., surjective). We say that \mathcal{F} is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathcal{A} \rightarrow \mathcal{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is given by the following data: for each object $A \in \mathcal{A}$, a map $\varphi_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \rightarrow A'$ in \mathcal{A} , the following diagram commutes.

$$(1.1.3) \quad \begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(A') \\ \varphi_A \downarrow & & \downarrow \varphi_{A'} \\ \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(A') \end{array}$$

Definition 1.1.4. A morphism $f \in \text{Mor}_{\mathcal{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathcal{A}$ and two morphisms $g, h \in \text{Hom}_{\mathcal{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \text{Mor}_{\mathcal{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathcal{A}$ and two morphisms $g, h \in \text{Mor}_{\mathcal{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories \mathcal{A} and \mathcal{B} , we can define a category $\mathcal{F}un(\mathcal{A}, \mathcal{B})$, whose objects are functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathcal{A}, \mathcal{B})$, there is a morphism set $\text{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \rightarrow \mathcal{G}$, as defined above.

Proposition 1.1.5. *Let \mathcal{A} and \mathcal{B} be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathcal{A}, \mathcal{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ such that for any object $A \in \mathcal{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ is an isomorphism in \mathcal{B} .*

Definition 1.1.6. A category \mathcal{A} is said to be *pre-additive* if for any two objects $X, Y \in \mathcal{A}$, the set $\text{Mor}_{\mathcal{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\text{Mor}_{\mathcal{A}}(X, Y) \times \text{Mor}_{\mathcal{A}}(Y, Z) \longrightarrow \text{Mor}_{\mathcal{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is \mathbb{Z} -bilinear, for all $X, Y, Z \in \mathcal{A}$.

Notation. For any pre-additive category \mathcal{A} , we denote by $\text{Hom}_{\mathcal{A}}(X, Y)$ the abelian group $\text{Mor}_{\mathcal{A}}(X, Y)$, for all $X, Y \in \text{Ob}(\mathcal{A})$.

Let \mathcal{A} and \mathcal{B} be pre-additive categories. A functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is said to be *additive* if for all objects $X, Y \in \mathcal{A}$, the induced map

$$\mathcal{F}_{X,Y} : \text{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.1.7 (Additive category). A category \mathcal{A} is said to be *additive* if for any two objects $A, B \in \mathcal{A}$, the set $\text{Hom}_{\mathcal{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

- (i) The composition map $\text{Hom}_{\mathcal{A}}(A, B) \times \text{Hom}_{\mathcal{A}}(B, C) \longrightarrow \text{Hom}_{\mathcal{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is \mathbb{Z} -bilinear, for all $A, B, C \in \mathcal{A}$.
- (ii) There is a zero object 0 in \mathcal{A} , i.e., $\text{Hom}_{\mathcal{A}}(0, 0)$ is the trivial group with one element.
- (iii) For any two objects $A_1, A_2 \in \mathcal{A}$, there is an object $B \in \mathcal{A}$ together with morphisms $j_i : A_i \rightarrow B$ and $p_i : B \rightarrow A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in \mathcal{A} .

Definition 1.1.8. Let k be a field. A *k-linear category* is an additive category \mathcal{A} such that for any $A, B \in \mathcal{A}$, the abelian groups $\text{Hom}_{\mathcal{A}}(A, B)$ are k -vector spaces such that the composition morphisms

$$\text{Hom}_{\mathcal{A}}(A, B) \times \text{Hom}_{\mathcal{A}}(B, C) \longrightarrow \text{Hom}_{\mathcal{A}}(A, C), \quad (f, g) \mapsto g \circ f$$

are k -bilinear, for all $A, B, C \in \mathcal{A}$.

Remark 1.1.9. Additive functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ between two k -linear additive categories \mathcal{A} and \mathcal{B} over the same base field k are assumed to be k -linear, i.e., for any two objects $A_1, A_2 \in \mathcal{A}$, the map $\mathcal{F}_{A_1, A_2} : \text{Hom}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Hom}_{\mathcal{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k -linear.

Let \mathcal{A} be an additive category. Then there is a unique object $0 \in \mathcal{A}$, called the *zero object* such that for any object $A \in \mathcal{A}$, there are unique morphisms $0 \rightarrow A$ and $A \rightarrow 0$ in \mathcal{A} . For any two objects $A, B \in \mathcal{A}$, the *zero morphism* $0 \in \text{Hom}_{\mathcal{A}}(A, B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B.$$

In particular, taking $A = 0$, we see that, the set $\text{Hom}_{\mathcal{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in \mathcal{A} .

Definition 1.1.10. Let $f : A \rightarrow B$ be a morphism in \mathcal{A} . Then *kernel* of f is a pair $(\iota, \text{Ker}(f))$, where $\text{Ker}(f) \in \mathcal{A}$ and $\iota \in \text{Hom}_{\mathcal{A}}(\text{Ker}(f), A)$ such that

- (i) $f \circ \iota = 0$ in $\text{Hom}_{\mathcal{A}}(\text{Ker}(f), B)$, and
- (ii) given any object $C \in \mathcal{A}$ and a morphism $g : C \rightarrow A$ with $f \circ g = 0$, there is a unique morphism $\tilde{g} : C \rightarrow \text{Ker}(f)$ such that $\iota \circ \tilde{g} = g$.

$$\begin{array}{ccccc} & & C & & \\ & \swarrow \exists! \tilde{g} & \downarrow g & \searrow 0 & \\ \text{Ker}(f) & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

The *cokernel* of $f \in \text{Hom}_{\mathcal{A}}(A, B)$ is defined by reversing the arrows of the above diagram.

Definition 1.1.11. The *cokernel* of $f : A \rightarrow B$ is a pair $(\pi, \text{Coker}(f))$, where $\text{Coker}(f)$ is an object of \mathcal{A} together with a morphism $\pi : B \rightarrow \text{Coker}(f)$ in \mathcal{A} such that

- (i) $\pi \circ f = 0$ in $\text{Hom}_{\mathcal{A}}(A, \text{Coker}(f))$, and
- (ii) given any object $C \in \mathcal{A}$ and a morphism $g : B \rightarrow C$ with $g \circ f = 0$ in $\text{Hom}_{\mathcal{A}}(A, C)$, there is a unique morphism $\tilde{g} : \text{Coker}(f) \rightarrow C$ such that $\tilde{g} \circ \pi = g$.

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\pi} & \text{Coker}(f) \\ & \searrow 0 & \downarrow g & \swarrow \exists! \tilde{g} & \\ & & C & & \end{array}$$

Definition 1.1.12. The *coimage* of $f \in \text{Hom}_{\mathcal{A}}(A, B)$, denoted by $\text{Coim}(f)$, is the cokernel of $\iota : \text{Ker}(f) \rightarrow A$ of f , and the *image* of f , denoted $\text{Im}(f)$, is the kernel of the cokernel $\pi : B \rightarrow \text{Coker}(f)$ of f .

Lemma 1.1.13. Let \mathcal{C} be a preadditive category, and $f : X \rightarrow Y$ a morphism in \mathcal{C} .

- (i) If a kernel of f exists, then it is a monomorphism.
- (ii) If a cokernel of f exists, then it is an epimorphism.
- (iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.
- (iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \text{Ker}(f) \rightarrow X$ of f exists. Let $\alpha, \beta \in \text{Hom}_{\mathcal{C}}(Z, \text{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\text{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \text{Hom}(Z, \text{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). \square

Exercise 1.1.14. Let \mathcal{A} be an additive category. Let $f \in \text{Hom}_{\mathcal{A}}(X, Y)$ be such that $\text{Ker}(f) \xrightarrow{\iota} X$ exists in \mathcal{A} . Then the kernel of $\iota : \text{Ker}(f) \rightarrow X$ is the unique morphism $0 \rightarrow \text{Ker}(f)$ in \mathcal{A} .

Lemma 1.1.15. Let $f : X \rightarrow Y$ be a morphism in a preadditive category \mathcal{C} such that the kernel, cokernel, image and coimage all exist in \mathcal{C} . Then f uniquely factors as $X \rightarrow \text{Coim}(f) \rightarrow \text{Im}(f) \rightarrow Y$ in \mathcal{C} .

Proof. Since $\text{Ker}(f) \rightarrow X \rightarrow Y$ is zero, there is a canonical morphism $\text{Coim}(f) \rightarrow Y$ such that the composite morphism $X \rightarrow \text{Coim}(f) \rightarrow Y$ is f . The composition $\text{Coim}(f) \rightarrow Y \rightarrow \text{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \rightarrow Y \rightarrow \text{Coker}(f)$, which is zero. Hence $\text{Coim}(f) \rightarrow Y$ factors uniquely through $\text{Im}(f) = \text{Ker}(\pi_f)$ (see Lemma 1.1.13 (iii)). This completes the proof.

$$(1.1.16) \quad \begin{array}{ccccc} \text{Ker}(f) & \xrightarrow{\iota} & X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\pi_f} & \text{Coker}(f) \\ & & \searrow \pi_{\iota} & & \nearrow j & & \\ & & \text{Coim}(f) & \longrightarrow & \text{Im}(f) & & \end{array}$$

\square

Definition 1.1.17. An abelian category \mathcal{A} is an additive category such that for any morphism $f : A \rightarrow B$ in \mathcal{A} , its kernel $\iota : \text{Ker}(f) \rightarrow A$ and cokernel $p : B \rightarrow \text{Coker}(f)$ exists in \mathcal{A} , and the natural morphism $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism in \mathcal{A} (c.f. Definition 1.1.12).

Example 1.1.18. (1) For any commutative ring A with identity, the category Mod_A of A -modules is an abelian category.

- (2) Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of \mathcal{O}_X -modules on X . Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{Qcoh}(X)$ (resp., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of \mathcal{O}_X -modules on X , are also abelian. However, the full subcategory $\mathcal{Vect}(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of \mathcal{O}_X -modules on X , is not abelian, because kernel of a morphism in $\mathcal{Vect}(X)$ may not be in $\mathcal{Vect}(X)$.

1.2. Triangulated category. Let \mathcal{A} be an additive category. A *shift functor* is an additive functor

$$(1.2.1) \quad T : \mathcal{A} \longrightarrow \mathcal{A},$$

which is an equivalence of categories. A *triangle* in (\mathcal{A}, T) is given by a diagram

$$(1.2.2) \quad A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A),$$

with objects and arrows in \mathcal{A} . A *morphism of triangles* in (\mathcal{A}, T) is given by a commutative diagram

$$(1.2.3) \quad \begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

where $f[1] := T(f) \in \text{Hom}_{\mathcal{A}}(A[1], A'[1])$. If, in addition, f, g, h are isomorphisms in \mathcal{A} , we say that (1.2.3) is an isomorphism of triangles. We denote by $A[n]$ the object $T^n(A) \in \mathcal{A}$, and denote by $f[n]$ the morphism $T^n(f) \in \text{Hom}_{\mathcal{A}}(A[n], B[n])$, for $f \in \text{Hom}_{\mathcal{A}}(A, B)$.

Definition 1.2.4. A *triangulated category* is an additive category \mathcal{A} together with an additive equivalence (*shift functor*)

$$(1.2.5) \quad T : \mathcal{A} \longrightarrow \mathcal{A},$$

and a set of *distinguished triangles*

$$(1.2.6) \quad A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A),$$

satisfying the following axioms (TR1) – (TR4) below.

(TR1) (i) Any triangle of the form

$$A \xrightarrow{\text{Id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is a distinguished triangle.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is a distinguished triangle.

(TR3) Any commutative diagram of distinguished triangles with vertical arrows f and g

$$(1.2.7) \quad \begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ f \downarrow & & g \downarrow & & \exists h \downarrow & & f[1] \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

can be completed to a commutative diagram (not necessarily in a unique way).

(TR4) (Octahedral axiom) Given any three distinguished triangles

$$\begin{array}{l} A \xrightarrow{u} B \longrightarrow C' \longrightarrow A[1] \\ B \xrightarrow{v} C \longrightarrow A' \longrightarrow B[1] \\ A \xrightarrow{w} C \longrightarrow B' \longrightarrow A[1] \end{array}$$

there is a distinguished triangle $C' \longrightarrow B' \longrightarrow A' \longrightarrow C'[1]$ such that the following diagram is commutative.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \longrightarrow & C' & \longrightarrow & A[1] \\ \downarrow \text{Id} & & \downarrow v & & \downarrow & & \downarrow \text{Id} \\ A & \xrightarrow{w} & C & \longrightarrow & B' & \longrightarrow & A[1] \\ \downarrow u & & \downarrow \text{Id} & & \downarrow & & \downarrow u[1] \\ B & \xrightarrow{v} & C & \longrightarrow & A' & \longrightarrow & B[1] \\ \downarrow & & \downarrow & & \downarrow \text{Id} & & \downarrow \\ C' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & C'[1] \end{array}$$

This axiom is called “*octahedron axiom*” because of its original formulation: given composable morphisms $A \xrightarrow{u} B \xrightarrow{v} C$, with $w := v \circ u$, we have the

following octahedron diagram.



where any triangle of the form $\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & \longrightarrow & Y \end{array}$ are distinguished triangles, and the arrow $Z \dashrightarrow X$ stands for $Z \rightarrow X[1]$.

Remark 1.2.8. A triangulated category need not be abelian, in general. In triangulated category, distinguished triangles play the roles of exact sequences in abelian categories. Examples of triangulated categories, we will be interested in, are derived categories of abelian categories.

Definition 1.2.9. Let \mathcal{A} and \mathcal{B} be two triangulated categories. An *exact functor* of triangulated categories \mathcal{A} to \mathcal{B} is a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ such that for any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in \mathcal{A} , there is an isomorphism $F(A[1]) \xrightarrow{\phi} F(A)[1]$ such that

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{\phi \circ F(h)} F(A)[1]$$

is a distinguished triangle in \mathcal{B} . By a *morphism of triangulated categories*, we always mean an exact functor between them.

Definition 1.2.10 (Adjoint functors). Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor between any two categories. A functor $G : \mathcal{B} \rightarrow \mathcal{A}$ is said to be *right adjoint to F* , written as $F \dashv G$, if there is an isomorphism

$$(1.2.11) \quad \text{Hom}_{\mathcal{B}}(F(A), B) \cong \text{Hom}_{\mathcal{A}}(A, G(B)), \quad \forall A \in \mathcal{A}, B \in \mathcal{B},$$

which is functorial in both A and B .

Similarly, a functor $H : \mathcal{B} \rightarrow \mathcal{A}$ is said to be *left adjoint to F* , written as $H \dashv F$, if there is an isomorphism

$$\text{Hom}_{\mathcal{B}}(B, F(A)) \cong \text{Hom}_{\mathcal{A}}(H(B), A), \quad \forall A \in \mathcal{A}, B \in \mathcal{B},$$

which is functorial in both A and B .

Remark 1.2.12. (1) Note that, G is right adjoint to F if and only if F is left adjoint to G .

(2) If $F \dashv G$, then $\text{Id}_{F(A)} \in \text{Hom}_{\mathcal{B}}(F(A), F(A)) \cong \text{Hom}_{\mathcal{A}}(A, (G \circ F)(A))$ induces a morphism $A \rightarrow (G \circ F)(A)$, for all $A \in \mathcal{A}$. The naturality of this morphism gives us a morphism of functors

$$\text{Id}_{\mathcal{A}} \rightarrow G \circ F.$$

Similarly, taking $A = G(B)$ in (1.2.11), we get a morphism of functors

$$F \circ G \rightarrow \text{Id}_{\mathcal{B}}.$$

In particular, if F and G are quasi-inverse to each other (in case of equivalence of categories), then one is both left and right adjoint to the other one.

(3) Using Yoneda lemma, one can check that, left (resp., right) adjoint of a functor, if it exists, is unique up to isomorphisms.

Proposition 1.2.13. Let $F : \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor of triangulated categories. Let $G : \mathcal{D}' \rightarrow \mathcal{D}$ be a functor. If $F \dashv G$, or $G \dashv F$, then G is also exact.

1.3. Semi-orthogonal decomposition. Let \mathcal{D} be a k -linear triangulated category. An object $E \in \mathcal{D}$ is said to be *exceptional* if

$$(1.3.1) \quad \text{Hom}_{\mathcal{D}}(E, E[\ell]) = \begin{cases} k & \text{if } \ell = 0, \\ 0 & \text{if } \ell \neq 0. \end{cases}$$

An *exceptional sequence* in \mathcal{D} is a sequence of exceptional objects E_1, E_2, \dots, E_n of \mathcal{D} such that $\text{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = 0$, for all $i > j$ and all ℓ . In other words, if

$$(1.3.2) \quad \text{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if } i = j \text{ and } \ell = 0, \\ 0 & \text{if } i > j, \text{ or if } \ell \neq 0 \text{ and } i = j. \end{cases}$$

An exceptional sequence $\{E_i\}_{i=1}^n$ is said to be *full* if, as a triangulated category, \mathcal{D} is generated by $\{E_i\}_{i=1}^n$; i.e., if \mathcal{D}' is a triangulated full subcategory of \mathcal{D} containing E_i , for all $i = 1, \dots, n$, then the inclusion morphism $\mathcal{D}' \hookrightarrow \mathcal{D}$ is an equivalence of categories.

1.4. t -structure and heart. Let (\mathcal{A}, T) be a triangulated category. Let \mathcal{B} be a subcategory of \mathcal{A} . For an integer n , we denote by $\mathcal{B}[n]$ the full subcategory of \mathcal{A} , whose objects are of the form $X[n]$, with $X \in \mathcal{B}$. In other words, $\mathcal{B}[n] = T^n(\mathcal{B}) \subset \mathcal{A}$.

Definition 1.4.1. Let $\mathcal{A}^{\leq 0}$ and $\mathcal{A}^{\geq 0}$ be two full subcategories of \mathcal{A} . For an integer n , let $\mathcal{A}^{\leq n} := \mathcal{A}^{\leq 0}[-n]$ and $\mathcal{A}^{\geq n} := \mathcal{A}^{\geq 0}[-n]$. A *t -structure* on \mathcal{A} is given by a pair $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ of full subcategories of \mathcal{A} satisfying the following axioms.

(t1) $\mathcal{A}^{\leq -1} \subset \mathcal{A}^{\leq 0}$ and $\mathcal{A}^{\geq 1} \subset \mathcal{A}^{\geq 0}$.

(t2) For any $X \in \mathcal{A}^{\leq 0}$ and $Y \in \mathcal{A}^{\geq 1}$, we have $\text{Hom}_{\mathcal{A}}(X, Y) = 0$.

(t3) For any $X \in \mathcal{A}$, there is a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1],$$

with $X_0 \in \mathcal{A}^{\leq 0}$ and $X_1 \in \mathcal{A}^{\geq 1}$.

In this case, the full subcategory $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ of \mathcal{A} is called the *heart* (or, *core*) of the t -structure $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$.

Example 1.4.2 (Standard t -structure on $D^b(X)$). Consider the full subcategories

$$\mathcal{A}^{\leq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0, \forall i > 0\} \text{ and}$$

$$\mathcal{A}^{\geq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0, \forall i < 0\}$$

of $D^b(X)$. The axiom (T1) is easy to see. To check axiom (T2), we need some notations. For an integer $n \in \mathbb{Z}$, let $D^{\leq n}(X)$ (resp., $D^{\geq n}$) be the full subcategory of $D^b(X)$, whose objects are $E^\bullet \in D^b(X)$ satisfying $E^i = 0$, for all $i \leq n$ (resp., for all $i \geq n$). Consider the *truncation functors*

$$(1.4.3) \quad \tau^{\leq n} : D^b(X) \longrightarrow D^{\leq n}(X) \quad \text{and} \quad \tau^{\geq n} : D^b(X) \longrightarrow D^{\geq n}(X)$$

defined by

$$\tau^{\leq n}(E^\bullet) := (\cdots \rightarrow E^{n-2} \rightarrow E^{n-1} \rightarrow \text{Ker}(d_{E^\bullet}^n) \rightarrow 0 \rightarrow \cdots), \text{ and}$$

$$\tau^{\geq n}(E^\bullet) := (\cdots \rightarrow 0 \rightarrow \text{Ker}(d_{E^\bullet}^n) \rightarrow E^n \rightarrow E^{n+1} \rightarrow \cdots).$$

where $d_{E^\bullet}^n : E^n \rightarrow E^{n+1}$, and $E^\bullet \in D^b(X)$. Now take $E^\bullet \in \mathcal{A}^{\leq 0}$ and $F^\bullet \in \mathcal{A}^{\geq 1}$. If $f \in \text{Hom}_{D^b(X)}(E^\bullet, F^\bullet)$, then f factors as

$$\begin{array}{ccc} E^\bullet & \xrightarrow{f} & F^\bullet \\ \searrow \cong & & \nearrow \\ & \tau^{\leq 0}(E^\bullet) \xrightarrow{\tau^{\leq 0}(f)} \tau^{\leq 0}(F^\bullet) & \end{array}.$$

Since $\tau^{\leq 0}(F^\bullet) = 0$ in $D^b(X)$, we conclude that $f = 0$. Axiom (T3) follows from the exact triangle

$$\tau^{\leq 0}(E^\bullet) \longrightarrow E^\bullet \longrightarrow \tau^{\geq 1}(E^\bullet) \longrightarrow \tau^{\leq 0}(E^\bullet)[1], \quad \forall E^\bullet \in D^b(X).$$

Thus $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a t -structure on $D^b(X)$, and the associated heart $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ is isomorphic to $\mathcal{Coh}(X)$.

The above mentioned t -structure on $D^b(X)$ is not interesting, and somehow useless. We shall be interested in some non-trivial t -structures on $D^b(X)$ giving more interesting and useful hearts different from $\mathcal{Coh}(X)$.

Lemma 1.4.4. *The heart of a t -structure on \mathcal{A} is an abelian category.*

Remark 1.4.5. If $D^b(\mathcal{A}) \cong D^b(X)$ for some abelian subcategory \mathcal{A} of $D^b(X)$, then \mathcal{A} is a heart of a t -structure on $D^b(X)$. However, the converse is not true, in general.

1.5. Derived category of an abelian category. Let \mathcal{A} be an abelian category. A complex in \mathcal{A} is given by

$$(1.5.1) \quad A^\bullet : \dots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \dots,$$

where A^i are objects of \mathcal{A} and d_A^i are morphisms in \mathcal{A} such that $d_A^i \circ d_A^{i-1} = 0$, for all $i \in \mathbb{Z}$. A complex A^\bullet in \mathcal{A} is said to be *bounded above* (resp., *bounded below*) if there is an integer i_0 such that $A^i = 0$, for all $i \geq i_0$ (resp., if there is an integer j_0 such that $A^j = 0$, for all $j \leq j_0$). If A^\bullet is both bounded above and bounded below, we say that A^\bullet is *bounded*.

A morphism $f^\bullet : A^\bullet \rightarrow B^\bullet$ between two complexes A^\bullet and B^\bullet of objects and morphisms from \mathcal{A} is given by a collection of morphisms $\{f^i : A^i \rightarrow B^i\}_{i \in \mathbb{Z}}$ in \mathcal{A} such that the following diagram commutes.

$$(1.5.2) \quad \begin{array}{ccccccc} \dots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} \xrightarrow{d_A^{i+1}} \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} \\ \dots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} \xrightarrow{d_B^{i+1}} \dots \end{array}$$

Let $\text{Kom}(\mathcal{A})$ be the category, whose objects are complexes of objects and morphisms from \mathcal{A} , and morphisms are given by morphism of complexes, as defined in (1.5.2). Denote by $\text{Kom}^-(\mathcal{A})$, $\text{Kom}^+(\mathcal{A})$ and $\text{Kom}^b(\mathcal{A})$ the full subcategories of $\text{Kom}(\mathcal{A})$, whose objects are bounded above complexes, resp., bounded below complexes, resp., bounded complexes. Then we have the following.

Proposition 1.5.3. *For any abelian category \mathcal{A} , the categories $\text{Kom}(\mathcal{A})$, $\text{Kom}^-(\mathcal{A})$, $\text{Kom}^+(\mathcal{A})$ and $\text{Kom}^b(\mathcal{A})$ are abelian.*

Definition 1.5.4. For any complex $A^\bullet \in \text{Kom}(\mathcal{A})$ and $k \in \mathbb{Z}$, we define its k^{th} -shift to be the complex $A[k]^\bullet \in \text{Kom}(\mathcal{A})$ satisfying

- (i) $A[k]^i := A^{k+i}$, for all $i \in \mathbb{Z}$, and
- (ii) $d_{A[k]}^i := (-1)^k d_{A^\bullet}^{i+k} : A[k]^i \rightarrow A[k]^{i+1}$, for all $i \in \mathbb{Z}$.

Proposition 1.5.5. *For any integer k , the k^{th} -shift functor*

$$\text{Kom}(\mathcal{A}) \longrightarrow \text{Kom}(\mathcal{A}), \quad A^\bullet \longmapsto A[k]^\bullet$$

is an equivalence of categories.

Proof. Clearly, the $(-k)^{\text{th}}$ -shift functor $A^\bullet \mapsto A[-k]^\bullet$ defines the inverse functor of the k^{th} -shift functor. \square

Remark 1.5.6. We shall see later that the category $\text{Kom}(\mathcal{A})$ together with the shift functor do not form a triangulated category, in general. However, we shall construct

the derived category $D^b(\mathcal{A})$ from $\text{Kom}(\mathcal{A})$, which will turn out to be a triangulated category.

Given a complex $A^\bullet \in \text{Kom}(\mathcal{A})$, we define its i^{th} cohomology sheaf

$$(1.5.7) \quad \mathcal{H}^i(A^\bullet) := \frac{\text{Ker}(d_{A^\bullet}^i)}{\text{Im}(d_{A^\bullet}^{i-1})} \in \mathcal{A}, \quad \forall i \in \mathbb{Z}.$$

A complex $A^\bullet \in \text{Kom}(\mathcal{A})$ is said to be *acyclic* if $\mathcal{H}^i(A^\bullet) = 0$, for all $i \in \mathbb{Z}$. Any morphism $f^\bullet : A^\bullet \rightarrow B^\bullet$ of complexes gives rise to natural homomorphisms

$$(1.5.8) \quad \mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet), \quad \forall i \in \mathbb{Z}.$$

Let \mathcal{A} and \mathcal{B} be two abelian categories. Let

$$(1.5.9) \quad \mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$$

be an additive functor. Then \mathcal{F} induces a functor, also denoted by the same symbol,

$$(1.5.10) \quad \mathcal{F} : \text{Kom}(\mathcal{A}) \rightarrow \text{Kom}(\mathcal{B})$$

defined by sending $A^\bullet \in \text{Kom}(\mathcal{A})$ to the complex $\mathcal{F}(A^\bullet)$, defined by

- (i) $\mathcal{F}(A^\bullet)^i := \mathcal{F}(A^i)$, for all $i \in \mathbb{Z}$, and
- (ii) $d_{\mathcal{F}(A^\bullet)}^i : \mathcal{F}(A^i) \xrightarrow{F(d_{A^\bullet}^i)} \mathcal{F}(A^{i+1})$, for all $i \in \mathbb{Z}$,

and for any morphism $f^\bullet : A^\bullet \rightarrow B^\bullet$, we have a natural morphism of complexes

$$\mathcal{F}(f^\bullet) : \mathcal{F}(A^\bullet) \rightarrow \mathcal{F}(B^\bullet)$$

defined by $\mathcal{F}(f^\bullet)^i := \mathcal{F}(f^i) : \mathcal{F}(A^i) \rightarrow \mathcal{F}(B^i)$, for all $i \in \mathbb{Z}$.

Definition 1.5.11. An additive functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is said to be *exact* if it takes exact sequence to exact sequence.

Remark 1.5.12. Note that \mathcal{F} is exact if and only if for any acyclic complex $A^\bullet \in \text{Kom}(\mathcal{A})$, its image $\mathcal{F}(A^\bullet) \in \text{Kom}(\mathcal{B})$ is acyclic.

Since $\text{Kom}(\mathcal{A})$ is abelian for \mathcal{A} abelian, we can talk about short exact sequences in $\text{Kom}(\mathcal{A})$. Then by standard techniques from homological algebra, any short exact sequence

$$(1.5.13) \quad 0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$$

gives rise to a long exact sequence of cohomologies (which are objects of \mathcal{A})

$$(1.5.14) \quad \cdots \rightarrow \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet) \rightarrow \mathcal{H}^i(C^\bullet) \rightarrow \mathcal{H}^{i+1}(A^\bullet) \rightarrow \cdots, \quad \forall i \in \mathbb{Z}.$$

Definition 1.5.15. A morphism of complexes $f^\bullet : A^\bullet \rightarrow B^\bullet$ in $\text{Kom}(\mathcal{A})$ is called *quasi-isomorphism* if the induced morphism

$$(1.5.16) \quad \mathcal{H}^i(f^\bullet) : \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet)$$

is an isomorphism, for all $i \in \mathbb{Z}$.

Example 1.5.17. Let X be a smooth projective k -variety and let E be a coherent sheaf on X . Then we can find a finite resolution

$$0 \rightarrow E^n \rightarrow E^{n-1} \rightarrow \cdots \rightarrow E^1 \rightarrow E^0 \rightarrow E \rightarrow 0.$$

of E with E^i projective (locally free) \mathcal{O}_X -modules. (We can use this to study many properties of E in terms of locally free coherent sheaves.) This gives rise to a morphism of complexes

$$f^\bullet : (0 \rightarrow E^n \rightarrow E^{n-1} \rightarrow \cdots \rightarrow E^1 \rightarrow E^0) \rightarrow (\cdots \rightarrow 0 \rightarrow E \rightarrow 0 \rightarrow \cdots),$$

which is a quasi-isomorphism.

The main idea for definition of derived category is: *quasi-isomorphism of complexes should become isomorphism in the derived category*. Therefore, the derived category $D(\mathcal{A})$ is the localization of $\text{Kom}(\mathcal{A})$ by quasi-isomorphisms. This can be done by passing to the appropriate homotopy category.

Theorem 1.5.18. Let \mathcal{A} be an abelian category, and $\text{Kom}(\mathcal{A})$ the category of complexes in \mathcal{A} . Then there is a category $D(\mathcal{A})$, known as the *derived category of \mathcal{A}* , together with a functor

$$(1.5.19) \quad Q : \text{Kom}(\mathcal{A}) \rightarrow D(\mathcal{A})$$

such that:

- (i) If $f^\bullet : A^\bullet \rightarrow B^\bullet$ in $\text{Kom}(\mathcal{A})$ is a quasi-isomorphism, then $Q(f^\bullet)$ is an isomorphism in $D(\mathcal{A})$,
- (ii) if a functor $\mathcal{F} : \text{Kom}(\mathcal{A}) \rightarrow \mathcal{D}$ satisfies property (i), there is a unique functor $\tilde{\mathcal{F}} : D(\mathcal{A}) \rightarrow \mathcal{D}$ such that $\tilde{\mathcal{F}} \circ Q \cong \mathcal{F}$.

$$(1.5.20) \quad \begin{array}{ccc} \text{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ & \searrow \mathcal{F} & \swarrow \exists! \tilde{\mathcal{F}} \\ & \mathcal{D} & \end{array}$$

Now we go ahead for construction of the derived category $D(\mathcal{A})$ of \mathcal{A} . Since we want any quasi-isomorphism $C^\bullet \rightarrow A^\bullet$ of complexes in $\text{Kom}(\mathcal{A})$ to become isomorphism in the derived category $D(\mathcal{A})$, any morphism of complexes $C^\bullet \rightarrow B^\bullet$

in $\text{Kom}(\mathcal{A})$ should give rise to a morphism $A^\bullet \rightarrow B^\bullet$ in $D^b(\mathcal{A})$. This leads to the definition of morphisms in $D^b(\mathcal{A})$ as diagrams of the form

$$\begin{array}{ccc} & C^\bullet & \\ \text{\textit{qis}} \swarrow & & \searrow \\ A^\bullet & & B^\bullet, \end{array}$$

where “*qis*” stands for “quasi-isomorphism” of complexes. To make this more precise, we need to define when two such roofs should be considered equal, and how to define their compositions. Then natural context for both problems is to consider the homotopy category $K(\mathcal{A})$ of complexes in $\text{Kom}(\mathcal{A})$, which is an intermediate step for going from $\text{Kom}(\mathcal{A})$ to $\mathcal{D}(\mathcal{A})$.

$$\begin{array}{ccc} \text{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ & \searrow & \nearrow \\ & K(\mathcal{A}) & \end{array}$$

Definition 1.5.21. Two morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \rightarrow B^\bullet$ in $\text{Kom}(\mathcal{A})$ are said to be *homotopically equivalent*, written as $f^\bullet \sim g^\bullet$, if there is a morphism of complexes $h^\bullet : A^\bullet \rightarrow B^\bullet[-1]$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

$$\begin{array}{ccccccc} A^\bullet & \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & \cdots \\ g^\bullet \downarrow & & & g^{i-1} \downarrow & & g^i \downarrow & & g^{i+1} \downarrow & & \\ B^\bullet & \cdots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \xrightarrow{d_B^{i+1}} & \cdots \end{array}$$

$\begin{array}{c} \xrightarrow{f^{i-1}} \\ \swarrow h^i \\ \xrightarrow{f^i} \end{array}$
 $\begin{array}{c} \xrightarrow{f^i} \\ \swarrow h^{i+1} \\ \xrightarrow{f^{i+1}} \end{array}$

Let $K(\mathcal{A})$ be the category, whose objects are the same as objects of $\text{Kom}(\mathcal{A})$ and morphisms are given by $\text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \text{Hom}_{\text{Kom}(\mathcal{A})}(A^\bullet, B^\bullet) / \sim$, for all $A^\bullet, B^\bullet \in \text{Kom}(\mathcal{A})$.

Following proposition is an easy consequence of the above definition.

- Proposition 1.5.22.** (i) Homotopy equivalence of morphisms $A^\bullet \rightarrow B^\bullet$ of complexes is an equivalence relation.
- (ii) Homotopically trivial morphisms of complexes form an ‘ideal’ in the morphisms of $\text{Kom}(\mathcal{A})$.
- (iii) If f^\bullet and g^\bullet are two homotopically equivalent morphisms of complexes in $\text{Kom}(\mathcal{A})$, then the induced morphisms $\mathcal{H}^i(f^\bullet)$ and $\mathcal{H}^i(g^\bullet)$ from $\mathcal{H}^i(A^\bullet)$ to $\mathcal{H}^i(B^\bullet)$ coincide.
- (iv) Let $f^\bullet : A^\bullet \rightarrow B^\bullet$ and $g^\bullet : B^\bullet \rightarrow A^\bullet$ be two morphisms of complexes. If $f^\bullet \circ g^\bullet \sim \text{Id}_{B^\bullet}$ and $g^\bullet \circ f^\bullet \sim \text{Id}_{A^\bullet}$, then f^\bullet and g^\bullet are quasi-isomorphisms, and $\mathcal{H}^i(f^\bullet)^{-1} = \mathcal{H}^i(g^\bullet)$, for all $i \in \mathbb{Z}$.

Now we complete the construction of derived category $D(\mathcal{A})$. Take $\text{Ob}(D(\mathcal{A})) := \text{Ob}(\text{Kom}(\mathcal{A}))$. As discussed before, a morphism $f : A^\bullet \rightarrow B^\bullet$ in $D(\mathcal{A})$ is given by equivalence class of roofs of the form

$$\begin{array}{ccc} & C^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array}$$

where $C^\bullet \xrightarrow{\text{qis}} A^\bullet$ is a quasi-isomorphism of complexes in $\text{Kom}(\mathcal{A})$, and two such roofs

$$\begin{array}{ccc} & C_1^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array} \quad \begin{array}{ccc} & C_2^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array}$$

are considered to be equivalent if they are dominated by a third one of the same type

(1.5.23)

$$\begin{array}{ccccc} & & C_0^\bullet & & \\ & \swarrow \text{qis} & & \searrow \text{qis} & \\ & C_1^\bullet & & C_2^\bullet & \\ & \swarrow \text{qis} & \searrow \text{qis} & \swarrow \text{qis} & \searrow \\ A^\bullet & & & & B^\bullet \end{array}$$

such that the above diagram commutes in the homotopy category $K(\mathcal{A})$. In particular, the compositions $C_0^\bullet \rightarrow C_1^\bullet \rightarrow A^\bullet$ and $C_0^\bullet \rightarrow C_2^\bullet \rightarrow A^\bullet$ are homotopically equivalent. To define composition of morphisms in $D(\mathcal{A})$, consider two roofs

$$\begin{array}{ccc} & C_1^\bullet & \\ \swarrow \text{qis} & & \searrow \\ A^\bullet & & B^\bullet \end{array} \quad \begin{array}{ccc} & C_2^\bullet & \\ \swarrow \text{qis} & & \searrow \\ B^\bullet & & C^\bullet \end{array}$$

representing two morphisms in $D(\mathcal{A})$. It is natural to guess that, one should be able to define their composition to be a morphism represented by 'the' following roof

(1.5.24)

$$\begin{array}{ccccc} & & C_0^\bullet & & \\ & \swarrow \text{qis} & & \searrow & \\ & C_1^\bullet & & C_2^\bullet & \\ & \swarrow \text{qis} & \searrow \text{qis} & \swarrow \text{qis} & \searrow \\ A^\bullet & & B^\bullet & & C^\bullet, \end{array}$$

which commutes in the homotopy category $K(\mathcal{A})$. Now we need to ensure that such a diagram exists uniquely, up to *equivalence of roofs* as defined in (1.5.23) (c.f., Proposition 1.5.29). For this, we need the concept of “mapping cone”.

Definition 1.5.25. The *mapping cone* of a morphism $f^\bullet : A^\bullet \rightarrow B^\bullet$ in $Kom(\mathcal{A})$ is a complex $C(f^\bullet)$ defined as follow:

$$C(f^\bullet)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d_{C(f^\bullet)}^i = \begin{pmatrix} -d_{A^\bullet}^{i+1} & 0 \\ f^{i+1} & d_{B^\bullet}^i \end{pmatrix}, \quad \forall i \in \mathbb{Z}.$$

Note that $C(f^\bullet)$ is a complex. Moreover, there are natural morphisms of complexes

$$(1.5.26) \quad \tau : B^\bullet \rightarrow C(f^\bullet) \quad \text{and} \quad \pi : C(f^\bullet) \rightarrow A^\bullet[1]$$

given by natural injection $B^i \rightarrow A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \rightarrow A^\bullet[1]^i = A^{i+1}$, respectively, for all i . Then we have the following.

- (i) The composition $B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$ is trivial. In fact,

$$0 \longrightarrow B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1] \longrightarrow 0$$

is a short exact sequence in $Kom(\mathcal{A})$, and gives us a long exact sequence of cohomologies

$$\cdots \rightarrow \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet) \rightarrow \mathcal{H}^i(C(f^\bullet)) \rightarrow \mathcal{H}^{i+1}(A^\bullet) \rightarrow \cdots.$$

- (ii) The composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet)$ is homotopic to the trivial morphism. Indeed, take $h^\bullet : A^\bullet \rightarrow C(f^\bullet)$ to be morphism of complexes defined by the natural injective morphism $h^i : A^i \rightarrow C(f^\bullet)^{i-1} = A^i \oplus B^{i-1}$, for all i . Then we have

$$h^{i+1} \circ d_{A^\bullet}^i + d_{C(f^\bullet)}^{i-1} \circ h^i = \tau^i \circ f^i, \quad \forall i \in \mathbb{Z}.$$

Remark 1.5.27. It follows from the above construction that any commutative diagram of complexes

$$\begin{array}{ccccccc} A_1^\bullet & \xrightarrow{f_1^\bullet} & B_1^\bullet & \xrightarrow{\tau_1} & C(f_1^\bullet) & \xrightarrow{\pi_1} & A_1^\bullet[1] \\ \downarrow \phi & & \downarrow \psi & & \downarrow \exists! \phi[1] \oplus \psi & & \downarrow \phi[1] \\ A_2^\bullet & \xrightarrow{f_2^\bullet} & B_2^\bullet & \xrightarrow{\tau_2} & C(f_2^\bullet) & \xrightarrow{\pi_2} & A_2^\bullet[1] \end{array}$$

can be completed by a **dashed arrow** as above (c.f., axiom (TR3) in Definition 1.2.4).

Proposition 1.5.28. Let $f : A^\bullet \rightarrow B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone. Let $\tau : B^\bullet \rightarrow C(f)$ and $\pi : C(f) \rightarrow A^\bullet[1]$ be the natural morphisms as in

(1.5.26). Then there is a morphism of complexes $g : A^\bullet[1] \rightarrow C(\tau)$ which is an isomorphism in $K(\mathcal{A})$ such that the following diagram commutes in the homotopy category $K(\mathcal{A})$.

$$\begin{array}{ccccc} B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A^\bullet[1] & \xrightarrow{-f} & B^\bullet[1] \\ & & \searrow \tau_\tau & & \downarrow g & & \nearrow \pi_\tau \\ & & & & C(\tau) & & \end{array}$$

Proof. Define $g : A^\bullet[1] \rightarrow C(\tau)$ by setting

$$g^i : A^\bullet[1]^i = A^{i+1} \rightarrow C(\tau)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$$

to be the morphism $g^i = (-f^{i+1}, \text{Id}_{A^{i+1}}, 0)$, for all i . Clearly, g is a morphism of complexes, and its inverse (in $K(\mathcal{A})$) is given by the morphism of complexes $g^{-1} : C(\tau) \rightarrow A^\bullet[1]$ defined by projection onto the middle factor. Clearly, $\pi_\tau \circ g = -f$ in $\text{Kom}(\mathcal{A})$. However, the diagram

$$\begin{array}{ccc} C(f) & \xrightarrow{\pi} & A^\bullet[1] \\ & \searrow \tau_\tau & \downarrow g \\ & & C(\tau) \end{array}$$

does not commute in $\text{Kom}(\mathcal{A})$. We show that, $\pi \circ g \sim \tau_\tau$. For this, note that $g^{-1} \circ \tau_\tau = \pi$ in $\text{Kom}(\mathcal{A})$. Since $g \circ g^{-1} \sim \text{Id}_{C(\tau)}$, we have $\tau_\tau \sim g \circ \pi$. This completes the proof. \square

Now we use the above proposition to complete the proof of existence and uniqueness of composition of morphisms in $D(\mathcal{A})$.

Proposition 1.5.29. *Let $f : A^\bullet \rightarrow B^\bullet$ and $g : C^\bullet \rightarrow B^\bullet$ be morphism of complexes with f a quasi-isomorphism. Then there is a complex C_0^\bullet together with a quasi-isomorphism $C_0^\bullet \rightarrow C^\bullet$ and a morphism $C_0^\bullet \rightarrow A^\bullet$ such that the following diagram commutes in the homotopy category $K(\mathcal{A})$.*

$$\begin{array}{ccc} C_0^\bullet & \xrightarrow{qis} & C^\bullet \\ \downarrow & & \downarrow g \\ A^\bullet & \xrightarrow[f]{qis} & B^\bullet \end{array}$$

Proof. Note that, there is a natural morphism of complexes $\phi^i : C(\tau \circ g) \rightarrow A^\bullet[1]$ given by the natural projection

$$\phi^i : C(\tau \circ g)^i = C^{i+1} \oplus C(f)^i = C^{i+1} \oplus A^{i+1} \oplus B^i \xrightarrow{pr_2} A^{i+1} = A^\bullet[1]^i$$

onto the middle factor, for each i . By Proposition 1.5.28, there is morphism of complexes $\psi : C(\tau) \xrightarrow{\sim} A^\bullet[1]$ which is an isomorphism in $K(\mathcal{A})$. Then the following

diagram is commutative in $K(\mathcal{A})$.

$$\begin{array}{ccccccc}
 C_0^\bullet := C(\tau \circ g)[-1] & \xrightarrow{\quad} & C^\bullet & \xrightarrow{\tau \circ g} & C(f) & \xrightarrow{\tau \circ g} & C(\tau \circ g) \\
 \downarrow \phi[-1] & & \downarrow g & & \parallel & & \downarrow \phi \\
 A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & C(\tau) \xrightarrow{\sim} A^\bullet[1]
 \end{array}$$

Define $C_0^\bullet := C(\tau \circ g)[-1]$. Since f is a quasi-isomorphism, it follows from the commutativity of the above diagram that $C_0^\bullet \rightarrow C^\bullet$ is a quasi-isomorphism. \square

Remark 1.5.30. Existence and uniqueness of composition of morphisms in $D(\mathcal{A})$ follows from the above Proposition 1.5.29 (c.f., diagram (1.5.24)). This completes the construction of the derived category $D(\mathcal{A})$.

Proposition 1.5.31. *The categories $K(\mathcal{A})$ and $D(\mathcal{A})$ are additive.*

Proof. Let $A^\bullet, B^\bullet \in \text{Kom}(\mathcal{A})$. Since $\text{Kom}(\mathcal{A})$ is an abelian category, it follows from Proposition 1.5.22 that the quotient

$$\text{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Kom}(\mathcal{A})}(A^\bullet, B^\bullet) / \sim,$$

is an abelian group. Thus $K(\mathcal{A})$ is an additive category.

To see $D(\mathcal{A})$ is an additive category, let $f_1, f_2 \in \text{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ be two morphisms in $D(\mathcal{A})$ represented by following equivalence classes of roofs

$$\begin{array}{ccc}
 & C_1^\bullet & \\
 \phi_1 \swarrow & & \searrow \psi_1 \\
 A^\bullet & & B^\bullet
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & C_2^\bullet & \\
 \phi_2 \swarrow & & \searrow \psi_2 \\
 A^\bullet & & B^\bullet
 \end{array},$$

respectively. It follows from Proposition 1.5.29 that there is an object $C^\bullet \in D(\mathcal{A})$ and quasi-morphisms $\delta_i : C^\bullet \rightarrow C_i^\bullet$, for $i = 1, 2$, such that the following diagram commutes in the homotopy category.

$$(1.5.32) \quad
 \begin{array}{ccc}
 C^\bullet & \xrightarrow{\delta_2} & C_2^\bullet \\
 \text{qis} \downarrow \delta_1 & & \downarrow \phi_2 \\
 C_1^\bullet & \xrightarrow[\text{qis}]{\phi_1} & A^\bullet
 \end{array}$$

Note that, both $\phi_1 \circ \delta_1$ and $\phi_2 \circ \delta_2$, are quasi-isomorphisms, and are equal in $K(\mathcal{A})$. Let $\delta = \phi_1 \circ \delta_1 = \phi_2 \circ \delta_2$ in $K(\mathcal{A})$. Then in $D(\mathcal{A})$, we can write

$$f_1 + f_2 = \psi_1 \circ \phi_1^{-1} + \psi_2 \circ \phi_2^{-1} = (\psi_1 \circ \delta_1 + \psi_2 \circ \delta_2) \circ \delta^{-1}.$$

This defines $f_1 + g_1$ in $D(\mathcal{A})$. One can check that, $f + g$ as defined above, is well-defined in $D(\mathcal{A})$. Note that, the roof $A^\bullet \xleftarrow[\text{qis}]{-\phi_1} C_1^\bullet \xrightarrow{\psi_1} B^\bullet$ is the additive inverse of f_1 in $D(\mathcal{A})$. Now one can check that $\text{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ is an abelian group. \square

Definition 1.5.33. A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $K(\mathcal{A})$ (resp., in $D(\mathcal{A})$) is said to be a *distinguished triangle* if it is isomorphic in $K(\mathcal{A})$ (resp., $D(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1],$$

where f is a morphism of complexes with mapping cone $C(f)$, and τ and π are natural morphisms as defined in (1.5.26).

Proposition 1.5.34. *The categories $D(\mathcal{A})$ and $K(\mathcal{A})$ together with the shift functor are triangulated. Moreover, the natural functor $Q : K(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is an exact functor of triangulated categories.*

Proof. Triangulated structure on both $K(\mathcal{A})$ and $D(\mathcal{A})$ are given by shift functor $A^\bullet \mapsto A^\bullet[1]$ together with the collection of ‘distinguished triangles’ as defined above. Verification of axioms (TR1) – (TR4) requires crucial use of mapping cone. \square

Example 1.5.35. Let $\mathcal{A} = \text{Vect}_{fd}(k)$ be the category, whose objects are finite dimensional k -vector spaces, and morphisms between objects are k -linear homomorphisms. Then $D(\mathcal{A})$ is equivalent to the category $\prod_{i \in \mathbb{Z}} \mathcal{A}$ of graded k -vector spaces. Note that, any complex of k -vector spaces $A^\bullet \in D(\text{Vect}_{fd}(k))$ is isomorphic to its cohomology complex $\bigoplus_{i \in \mathbb{Z}} H^i(A^\bullet)[-i]$ with trivial differentials. More generally, this holds for any *semisimple* abelian category \mathcal{A} (i.e., when \mathcal{A} is abelian and any short exact sequence in \mathcal{A} splits).

Remark 1.5.36. Contrary to the category $\text{Kom}(\mathcal{A})$ of complexes in \mathcal{A} , the derived category $D(\mathcal{A})$ is not abelian, in general. However, $D(\mathcal{A})$ is always triangulated. $D^b(\mathcal{A})$ is abelian if and only if \mathcal{A} is semisimple (see <https://math.stackexchange.com/questions/189769>).

Corollary 1.5.37. (a) *The functor $Q : \text{Kom}(\mathcal{A}) \longrightarrow D(\mathcal{A})$ identifies set underlying set of objects of both categories (Apply property (ii) to the identity functor $\text{Kom}(\mathcal{A}) \rightarrow \text{Kom}(\mathcal{A})$).*

(b) *For any complex $A^\bullet \in D(\mathcal{A})$, its cohomology objects $\mathcal{H}^i(A^\bullet)$ are well-defined objects in \mathcal{A} . (This is because, quasi-isomorphisms of $\text{Kom}(\mathcal{A})$ turns into isomorphisms in $D(\mathcal{A})$.)*

(c) *Considering $A \in \mathcal{A}$ as a complex $A^\bullet \in D(\mathcal{A})$ concentrated at degree zero only, gives an equivalence between \mathcal{A} and the full subcategory of objects of $D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$.*

Proposition 1.5.38. *Let \mathcal{A} be an abelian category and $K(\mathcal{A})$ its homotopy category. Let \mathcal{C} be any additive category.*

- (1) An additive functor $F : K(\mathcal{A}) \rightarrow \mathcal{C}$ factors through an additive functor $\tilde{F} : D(\mathcal{A}) \rightarrow \mathcal{C}$ if and only if F send quasi-isomorphisms to isomorphisms.
- (2) Let \mathcal{B} be an abelian category, and $G : K(\mathcal{A}) \rightarrow K(\mathcal{B})$ an additive functor which maps quasi-isomorphism to quasi-isomorphism. Then G induces an additive functor $\tilde{G} : D(\mathcal{A}) \rightarrow D(\mathcal{B})$ such that the following diagram commutes.

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{G} & K(\mathcal{B}) \\ Q_{\mathcal{A}} \downarrow & & \downarrow Q_{\mathcal{B}} \\ D(\mathcal{A}) & \xrightarrow{\tilde{G}} & D(\mathcal{B}) \end{array}$$

Proof. If $F : K(\mathcal{A}) \rightarrow \mathcal{C}$ sends quasi-isomorphisms to quasi-isomorphism, we define $\tilde{F} : D(\mathcal{A}) \rightarrow \mathcal{C}$ by sending an object $E^\bullet \in D(\mathcal{A})$ to $F(E^\bullet) \in \mathcal{C}$, and any morphism $f/\phi : A^\bullet \xleftarrow[\sim]{\phi} C^\bullet \xrightarrow{f} B^\bullet$ in $D(\mathcal{A})$ to the morphism

$$F(f) \circ F(\phi)^{-1} : F(A^\bullet) \xrightarrow[\simeq]{F(\phi)^{-1}} F(C^\bullet) \xrightarrow{F(f)} F(B^\bullet).$$

in \mathcal{C} . Converse follows from the construction of $D(\mathcal{A})$ and Theorem 1.5.18.

The second assertion follows by applying the first one to the composition $F : K(\mathcal{A}) \xrightarrow{G} K(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} D(\mathcal{B})$. \square

Definition 1.5.39. Let $Kom^*(\mathcal{A})$, with $*$ = +, −, or b , be the full subcategory of $Kom(\mathcal{A})$, whose objects are complexes $A^\bullet \in Kom(\mathcal{A})$ with $A^i = 0$ for all $i \ll 0$, $i \gg 0$, or $|i| \ll 0$, respectively.

Note that, $Kom^*(\mathcal{A})$ is again abelian, for $*$ \in {+, −, b }. So following similar construction (i.e., by dividing out first by homotopy equivalence, and then by localizing with respect to quasi-isomorphisms), we can construct a category, denoted by $D^*(\mathcal{A})$. There is a natural forgetful functor

$$\mathcal{F}^* : D^*(\mathcal{A}) \longrightarrow D(\mathcal{A}),$$

which just forgets boundedness condition.

Proposition 1.5.40. The natural forgetful functor $\mathcal{F}^* : D^*(\mathcal{A}) \longrightarrow D(\mathcal{A})$, where $*$ = +, −, or b , gives an equivalence of $D^*(\mathcal{A})$ with the full triangulated subcategories of all complexes $A^\bullet \in D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for $i \ll 0$, for $i \gg 0$, or for $|i| \ll 0$, respectively.

To give an idea how this works, let $A^\bullet \in \text{Kom}(\mathcal{A})$ be such that $\mathcal{H}^i(A^\bullet) = 0$, for $i > n$. Then the commutative diagram

$$(1.5.41) \quad \begin{array}{ccccccc} B^\bullet : & \cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow \text{Ker}(d_A^n) \longrightarrow 0 \longrightarrow \cdots \\ & & & \parallel & & \parallel & \downarrow \text{ } \downarrow \\ A^\bullet : & \cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow A^n \longrightarrow A^{n+1} \longrightarrow \cdots \end{array}$$

defines a quasi-isomorphism between a complex $B^\bullet \in \text{Kom}^-(\mathcal{A})$ and A^\bullet . Similarly, if $\mathcal{H}^i(A^\bullet) = 0$ for $i < m$, then the commutative diagram

$$(1.5.42) \quad \begin{array}{ccccccc} A^\bullet : & \cdots & \longrightarrow & A^{m-1} & \longrightarrow & A^m & \longrightarrow A^{m+1} \longrightarrow \cdots \\ & & & \downarrow & & \downarrow & \parallel \\ C^\bullet : & \cdots & \longrightarrow & 0 & \longrightarrow & \text{Coker}(d_A^{m-1}) & \longrightarrow A^{m+1} \longrightarrow \cdots \end{array}$$

defines a quasi-isomorphism of a complex $C^\bullet \in \text{Kom}^+(\mathcal{A})$ and A^\bullet . Similar idea works for $\text{Kom}^b(\mathcal{A})$. However, one need to pass from $\text{Kom}^*(\mathcal{A})$ to the derived category $D^*(\mathcal{A})$ by inverting quasi-isomorphisms. This needs some technical care.

Let $\mathcal{A} \subset \mathcal{B}$ be full abelian subcategory of an abelian category \mathcal{B} . Then there is an obvious functor $\iota : D(\mathcal{A}) \rightarrow D(\mathcal{B})$. One might expect that this is an equivalence of $D(\mathcal{A})$ with the full subcategory $D_{\mathcal{A}}(\mathcal{B})$ of $D(\mathcal{B})$ consisting of objects $E^\bullet \in D(\mathcal{B})$ with $\mathcal{H}^i(E^\bullet) \in \mathcal{A}$, for all $i \in \mathbb{Z}$. However, this is not true, in general. There are several issues.

- $D_{\mathcal{A}}(\mathcal{B})$ need not be triangulated!
- The functor ι is neither faithful nor full, in general.

However, the next proposition answers when the above expectation holds true. First, we need a definition.

Definition 1.5.43. Let \mathcal{A} be an abelian category. A *thick subcategory* of \mathcal{A} is a full abelian subcategory $\mathcal{B} \subset \mathcal{A}$ of \mathcal{A} such that for any short exact sequence (in \mathcal{A})

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

with $A, C \in \mathcal{B}$, we have $B \in \mathcal{B}$.

Let $E, F \in \mathcal{A}$. We say that F is *embedded* in E (or, F is a *subobject* of E) if there is a monomorphism $F \rightarrow E$ in \mathcal{A} . An object $I \in \mathcal{A}$ is called *injective* if the functor

$$\text{Hom}_{\mathcal{A}}(-, I) : \mathcal{A} \rightarrow \text{Ab}$$

is exact.

Proposition 1.5.44. *Let $\mathcal{A} \subset \mathcal{B}$ be a thick abelian subcategory of an abelian category \mathcal{B} . Assume that any object $A \in \mathcal{A}$ is a subobject of an object $I_A \in \mathcal{A}$, which is *injective as an object of \mathcal{B}* . Then the natural functor $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ induces an equivalence*

$$D^*(\mathcal{A}) \rightarrow D^*(\mathcal{B}), \quad \text{where } * = + \text{ or } b,$$

of $D^(\mathcal{A})$ and the full triangulated subcategory $D^*(\mathcal{A}) \subset D^*(\mathcal{B})$ of complexes with cohomologies in \mathcal{A} .*

P.S.: I have not seen similar statement for $* = \emptyset$ or $-$.

Next, we want to get a computable description of Hom's in the derived category. In the next section, using derived functor, we show that

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]) = \mathrm{Ext}^i(A^\bullet, B^\bullet), \quad \forall i.$$

1.6. Derived functors. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. We want to know when such a functor give rise to a natural functor between derived categories. Note that, if F is not exact, then image of an acyclic complex $(\mathcal{H}^i(A^\bullet) = 0, \forall i)$ may not be acyclic. So to get a induced functor at the level of derived categories, F should be exact.

Lemma 1.6.1. *Let $F : K^*(\mathcal{A}) \rightarrow K^*(\mathcal{B})$, where $* \in \{\emptyset, -, +, b\}$, be an exact functor of triangulated categories. Then F induces a functor $\tilde{F} : D^*(\mathcal{A}) \rightarrow D^*(\mathcal{B})$ making the following diagram commutative*

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{F} & K^*(\mathcal{B}) \\ \downarrow & & \downarrow \\ D^*(\mathcal{A}) & \xrightarrow{\tilde{F}} & D^*(\mathcal{B}) \end{array}$$

if and only if one of the following (equivalent) conditions holds:

- (i) *F sends a quasi-isomorphism to a quasi-isomorphism.*
- (ii) *F sends any acyclic complex to an acyclic complex.*

However, if the functor F is not exact or F does not satisfies one of the equivalent conditions in (i) and (ii) above, still there is a bit complicated way to induce a natural functor between derived categories. This new functor is called the derived functor of F , but they will not produce a commutative diagram as in the above lemma. However, derived functor encodes more information about objects of the abelian categories.

To ensure existence of a derived functors, we need to assume some kind of exactness of F . If F is left exact (resp., right exact), we generally get a right derived

functor (resp., let derived functor)

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \quad (\text{resp., } LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})).$$

Both constructions are similar, and we only discuss the case of left exact functor.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor of abelian categories. Assume that \mathcal{A} has *enough injective* (meaning that, for any $A \in \mathcal{A}$, there is an injective object I of \mathcal{A} together with a monomorphism $\iota : A \hookrightarrow I$ in \mathcal{A}). Let $\mathcal{I}_{\mathcal{A}} \subset \mathcal{A}$ be the full subcategory of \mathcal{A} consisting of injective objects of \mathcal{A} . Note that, $\mathcal{I}_{\mathcal{A}}$ is additive, but not necessarily abelian. However, the construction of homotopy category works for any additive category. Therefore, $K^*(\mathcal{I}_{\mathcal{A}})$ is defined, and is a triangulated category. Now the inclusion functor $\mathcal{I}_{\mathcal{A}} \hookrightarrow \mathcal{A}$ induces a natural exact functor $K^*(\mathcal{I}_{\mathcal{A}}) \rightarrow K^*(\mathcal{A})$, and composing it with the exact functor $Q_{\mathcal{A}} : K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$, we get a natural exact functor $K^+(\mathcal{I}_{\mathcal{A}}) \rightarrow D^+(\mathcal{A})$.

Proposition 1.6.2. *The functor $Q_{\mathcal{A}} : K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ induces a natural equivalence of categories $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \rightarrow D^+(\mathcal{A})$.*

Then we have the following diagram

$$\begin{array}{ccccc} K^+(\mathcal{I}_{\mathcal{A}}) & \hookrightarrow & K^+(\mathcal{A}) & \xrightarrow{K(F)} & K^+(\mathcal{B}) \\ & \searrow \iota & \downarrow Q_{\mathcal{A}} & & \downarrow Q_{\mathcal{B}} \\ & & D^+(\mathcal{A}) & & D^+(\mathcal{B}) \end{array}$$

ι^{-1} (red curved arrow from $D^+(\mathcal{A})$ to $K^+(\mathcal{I}_{\mathcal{A}})$)

where ι^{-1} is the quasi-inverse functor. Such a quasi-isomorphism (ι^{-1}) is obtained by choosing a complex of injective objects quasi-isomorphic to a given bounded below complex in $D^+(\mathcal{A})$. Note that, the functor $K(F)$ is well-defined at the level of homotopy category, because F is left exact and we are working with bounded below complexes.

Definition 1.6.3. The *right derived functor* of a left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is the functor

$$RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1} : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}).$$

Proposition 1.6.4. (i) *There is a natural morphism of functors*

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}.$$

(ii) *The right derived functor $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is an exact functor of triangulated categories.*

(iii) *Let $G : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ be an exact functor of triangulated categories. Then any morphism of functors*

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}},$$

factorize through a unique morphism of functors $RF \rightarrow G$.

Proof. (i) Let $A^\bullet \in K^+(\mathcal{A})$. Note that, $Q_{\mathcal{A}}(A^\bullet) = A^\bullet$. Let $I^\bullet := \iota^{-1}(A^\bullet) \in K^+(\mathcal{I}_{\mathcal{A}})$. Then the natural isomorphism of functors $\text{Id}_{D^+(\mathcal{A})} \xrightarrow{\cong} \iota \circ \iota^{-1}$ gives rise to a functorial isomorphism $A^\bullet \xrightarrow{\cong} \iota(I^\bullet) \cong I^\bullet$. Since I^\bullet is a complex of injective objects, the above isomorphism in $D^+(\mathcal{A})$ gives rise to a unique morphism $A^\bullet \rightarrow I^\bullet$ in the homotopy category $K(\mathcal{A})$ by Proposition 1.6.13 (see below). Therefore, we have a functorial morphism

$$K(F)(A^\bullet) \rightarrow K(F)(I^\bullet) = RF(A^\bullet),$$

which gives the required morphism of functors.

- (ii) Note that, $K^+(\mathcal{I}_{\mathcal{A}})$ is a triangulated category and $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \rightarrow D^+(\mathcal{A})$ is an exact equivalence of categories. Then ι^{-1} being the adjoint of ι , it is also exact (c.f. Proposition 1.2.13). Now $RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1}$ being a composition of exact functors, is exact.
- (iii) See [GM03, III.6.11].

□

One can rephrase the Proposition 1.6.4 as a universal property to define derived functor of a left exact functor as follow.

Definition 1.6.5 (Universal property of derived functor). Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then the right derived functor of F , if it exists, is an exact functor $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ of triangulated categories such that

- (i) there is a natural morphism of functors $Q_{\mathcal{B}} \circ K(F) \rightarrow RF \circ Q_{\mathcal{A}}$, and
(ii) for any exact functor $G : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, there is a natural bijection

$$\text{Hom}(RF, G) \xrightarrow{\cong} \text{Hom}(Q_{\mathcal{B}} \circ F, G \circ Q_{\mathcal{A}}).$$

Definition 1.6.6. Let $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ be a right derived functor of a left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$, we define

$$R^i F(A^\bullet) := \mathcal{H}^i(RF(A^\bullet)) \in \mathcal{B}, \quad \forall i \in \mathbb{Z}.$$

The induced functors $R^i F : \mathcal{A} \rightarrow \mathcal{B}$ given by composition

$$\mathcal{A} \hookrightarrow D^+(\mathcal{A}) \xrightarrow{R^i F} \mathcal{B}$$

are known as *higher derived functors* of F .

Given any $A \in \mathcal{A}$, choosing an injective resolution

$$A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

we see that, $R^i F(A) = \mathcal{H}^i(\dots \rightarrow F(I^0) \rightarrow F(I^1) \rightarrow \dots)$. In particular,

$$R^0 F(A) = \text{Ker}(F(I^0) \rightarrow F(I^1)) = F(A),$$

since F is left exact.

Definition 1.6.7. An object $A \in \mathcal{A}$ is called *F-acyclic* if $R^i F(A) \cong 0$, for all $i \neq 0$.

Corollary 1.6.8. *With the above assumptions, any short exact sequence*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{A} give rise to a long exact sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow R^1 F(A) \rightarrow R^1 F(B) \rightarrow R^1 F(C) \rightarrow R^2(A) \rightarrow \dots$$

To see how it works, note that any short exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ in \mathcal{A} gives rise to a distinguished triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ in $D(\mathcal{A})$. Again any distinguished triangle $A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$ in $D(\mathcal{A})$ give rise to a long exact sequence of cohomologies

$$\dots \rightarrow \mathcal{H}^{i-1}(C^\bullet) \rightarrow \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet) \rightarrow \mathcal{H}^i(C^\bullet) \rightarrow \mathcal{H}^{i+1}(A^\bullet) \rightarrow \dots$$

Now the above corollary follows by considering the distinguished triangle $RF(A) \rightarrow RF(B) \rightarrow RF(C) \rightarrow RF(A)[1]$ in $D(\mathcal{B})$.

Example 1.6.9. Let \mathcal{A} be an abelian category, and let \mathbf{Ab} be the category of abelian groups. Consider the covariant functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \longrightarrow \mathbf{Ab}.$$

Clearly, $\mathrm{Hom}(A, -)$ is left exact. If \mathcal{A} contains enough injectives (for example, if \mathcal{A} is $\mathfrak{Mod}(\mathcal{O}_X)$ or $\mathfrak{Coh}(X)$ for X a noetherian scheme; c.f., [Har77, Exercise III. 3.6]), then we define

$$(1.6.10) \quad \mathrm{Ext}^i(A, -) := H^i(R\mathrm{Hom}_{\mathcal{A}}(A, -)).$$

Proposition 1.6.11. *Let \mathcal{A} be an abelian category with enough injectives, and let $A, B \in \mathcal{A}$. Then there are natural isomorphisms*

$$\mathrm{Ext}_{\mathcal{A}}^i(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]), \quad \forall i,$$

where A and B are considered as complexes in $D(\mathcal{A})$ concentrated at degree 0 place.

Proof. Let

$$B \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

be an injective resolution of B in \mathcal{A} . It follows from the construction of right derived functor that $R\mathrm{Hom}(A, B)$, as an object of $D^+(\mathbf{Ab})$, is isomorphic to the complex

$$\mathrm{Hom}(A, I^\bullet) : \mathrm{Hom}(A, I^0) \rightarrow \mathrm{Hom}(A, I^1) \rightarrow \mathrm{Hom}(A, I^2) \rightarrow \dots$$

Therefore, $\mathrm{Ext}^i(A, B) = H^i(\mathrm{Hom}(A, I^\bullet))$.

Note that, a morphism $f \in \mathrm{Hom}(A, I^i)$ is a *cycle* (i.e., $f \in \mathrm{Ker}(\mathrm{Hom}(A, I^i) \rightarrow \mathrm{Hom}(A, I^{i+1}))$) if and only if f defines a morphism of complexes

$$f : A \longrightarrow I^\bullet[i].$$

$$\mathrm{Ext}^i(A, B) \cong H^i(\mathrm{Hom}(A, I^\bullet)) \cong \mathrm{Hom}_{K(A)}(A, I^\bullet[i]).$$
$$\mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, I^\bullet[i]).$$

Lemma 1.6.12. *Let \mathcal{A} be an abelian category with enough injectives. Let $A^\bullet, I^\bullet \in \text{Kom}^+(\mathcal{A})$ be two bounded below complexes such that I^i is injective, for all i . Then*

$$\mathrm{Hom}_{K(A)}(A^\bullet, I^\bullet) = \mathrm{Hom}_{D(A)}(A^\bullet, I^\bullet).$$

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$
$$\begin{array}{ccc}
 & C^\bullet & \\
 qis \swarrow & & \searrow \psi \\
 A^\bullet & \xrightarrow[\psi']{\phi} & I^\bullet
 \end{array}$$

Proposition 1.6.13. *Let \mathcal{A} be an abelian category with enough injectives. Then for any quasi-isomorphism $\phi : B^\bullet \rightarrow A^\bullet$ in $\text{Kom}^+(\mathcal{A})$, the induced map*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, I^\bullet),$$

Sketch of a proof. Complete the morphism $\phi : B^\bullet \rightarrow A^\bullet$ to a distinguished triangle in the triangulated category $K^+(\mathcal{A})$. Applying the functor $\text{Hom}(-, I^\bullet)$ and then taking the associated long exact $\text{Hom}(-, I^\bullet)$ -sequence, we see that it is enough to show that $\text{Hom}_{K(\mathcal{A})}(E^\bullet, I^\bullet) = 0$, for any **acyclic complex** E^\bullet .

$$f^i - d_{I^\bullet}^{i-1} \circ h^i : E^i \longrightarrow I^i$$

factors through $E^i/E^{i-1} \rightarrow I^i$. Since I^i is injective, this lifts to a morphism $h^{i+1} : E^{i+1} \rightarrow I^i$ so that $f^i - d_{I^\bullet}^{i-1} \circ h^i = h^{i+1} \circ d_{E^\bullet}^i$. Thus the induction works! \square

Remark 1.6.14. In practice, we need to deal with many important abelian categories without enough injective objects, or sometimes the functor F is defined at the level of homotopy categories only. However, one can still construct derived functors in that setup under certain assumption. Let us explain briefly.

Let \mathcal{A} and \mathcal{B} be abelian categories.

Case I. F is defined only at the level of homotopy category: let

$$(1.6.15) \quad F : K^+(\mathcal{A}) \rightarrow K(\mathcal{B})$$

be an exact functor of triangulated categories. Then the right derived functor

$$(1.6.16) \quad RF : D^+(\mathcal{A}) \rightarrow D(\mathcal{B})$$

of F satisfying the properties (i)–(iii) of Proposition 1.6.4 exists if there is a triangulated subcategory $K_F \subset K^+(\mathcal{A})$ adapted to F , meaning that K_F satisfies the following conditions:

- (i) if $A^\bullet \in K_F$ is acyclic, then so is $F(A^\bullet)$, and
- (ii) any $A^\bullet \in K^+(\mathcal{A})$ is quasi-isomorphic to a complex in K_F .

Roughly, with the above hypotheses (i)–(iii), we may localize the subcategory $K^+(K_F)$ with respect to the quasi-isomorphisms of objects from K_F to produce an equivalence of categories $K^+(K_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A})$. Moreover, the functor $K(F) : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$ give rise to a functor $K^+(K_F)_{qis} \rightarrow K^+(\mathcal{B})$. Then by choosing a quasi-inverse of the above equivalence, we get the required derived functor $D^+(\mathcal{A}) \rightarrow D(\mathcal{B})$.

Case II. $F : \mathcal{A} \rightarrow \mathcal{B}$ a left exact functor, but \mathcal{A} has not enough injectives. In this situation, we may construct the right derived functor of F by looking at the F -adapted class of objects $\mathcal{I}_F \subset \mathcal{A}$, which is defined by the following properties.

- (a) \mathcal{I}_F is stable under finite sums,
- (b) if $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^i \in \mathcal{I}_F$, for all i , then $F(A^\bullet)$ is acyclic, and
- (c) any object of \mathcal{A} can be embedded inside an object of \mathcal{I}_F .

Let $K^+(\mathcal{I}_F)_{qis}$ be the localization of $K^+(\mathcal{I}_F)$ by quasi-isomorphism of complexes with objects from \mathcal{I}_F . Then the above hypotheses (a)–(c) gives rise to an equivalence of categories

$$(1.6.17) \quad \iota_q : K^+(\mathcal{I}_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A}).$$

Then $K(F) : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$ induces a functor $K(F)_{qis} : K^+(\mathcal{I}_F)_{qis} \rightarrow K^+(\mathcal{B})$. Now choosing a quasi-inverse ι_q^{-1} of (1.6.17) and composing with $Q_{\mathcal{B}} \circ K(F)_{qis}$, we get the required right derived functor $RF : D^+(\mathcal{A}) \rightarrow D(\mathcal{B})$ of F as discussed before.

The definition of Ext group as given in Example (1.6.9) can be generalized for complexes as follow. Given $A^\bullet, B^\bullet \in Kom(\mathcal{A})$, let $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ be the complex defined by

$$(1.6.18) \quad \text{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_{j \in \mathbb{Z}} \text{Hom}(A^j, B^{i+j})$$

with the differential

$$d(f) := d_B \circ f - (-1)^i f \circ d_A, \quad \forall f \in \text{Hom}(A^*, B^{i+*}).$$

$$\begin{array}{ccc} A^j & \xrightarrow{f_i^j} & B^{i+j} \\ d_A^j \downarrow & & \downarrow d_B^{i+j} \\ A^{j+1} & \xrightarrow{f_i^{j+1}} & B^{i+j+1} \end{array}$$

The complex $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ is known as the *internal hom of A^\bullet into B^\bullet* .

Note that, any $A^\bullet \in Kom(\mathcal{A})$ gives rise to an exact functor

$$(1.6.19) \quad \text{Hom}^\bullet(A^\bullet, -) : K^+(\mathcal{A}) \longrightarrow K(\mathbf{Ab}), \quad B^\bullet \longmapsto \text{Hom}^\bullet(A^\bullet, B^\bullet).$$

Let $\mathcal{I} \subset K^+(\mathcal{A})$ be the full triangulated subcategory, whose objects are complexes I^\bullet with I^i injective object of \mathcal{A} , for all i . Then \mathcal{I} is F -adapted, where $F = \text{Hom}^\bullet(A^\bullet, -)$, as defined in Remark 1.6.14. Then the right derived functor

$$(1.6.20) \quad R\text{Hom}^\bullet(A^\bullet, -) : D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab}).$$

of $\text{Hom}^\bullet(A^\bullet, -)$ exists. Then we define

$$(1.6.21) \quad \text{Ext}_{\mathcal{A}}^i(A^\bullet, B^\bullet) := H^i(R\text{Hom}^\bullet(A^\bullet, B^\bullet)), \quad \forall i.$$

Now the proof of Proposition 1.6.11 can be modified to prove the following.

Theorem 1.6.22. *Let \mathcal{A} be an abelian category with enough injectives, and let $A^\bullet, B^\bullet \in Kom(\mathcal{A})$ be two bounded (or bounded below) complexes. Then there are natural isomorphisms of abelian groups*

$$\text{Ext}_{\mathcal{A}}^i(A^\bullet, B^\bullet) \cong \text{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]), \quad \forall i.$$

It follows from Theorem 1.6.22 that the abelian group $\text{Ext}_{\mathcal{A}}^i(A^\bullet, B^\bullet)$ depends on the “isomorphism classes” of A^\bullet and B^\bullet in the derived category, not on the complexes. If $A_1^\bullet \xrightarrow{\sim} A_2^\bullet$ is a quasi-isomorphism of complexes, then the induced morphism

$$R\text{Hom}^\bullet(A_1^\bullet, B^\bullet) \longrightarrow R\text{Hom}^\bullet(A_2^\bullet, B^\bullet)$$

is an isomorphism in $D(\mathbf{Ab})$, because their cohomologies are isomorphic. Therefore, the functor $\mathrm{Hom}^\bullet(-, B^\bullet)$ descends to the derived category to give a bifunctor

$$(1.6.23) \quad D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab}),$$

which is exact in each variable.

Definition 1.6.24. An abelian category \mathcal{A} is said to have *enough projectives* if for each object $A \in \mathcal{A}$ there is a projective object P in \mathcal{A} together with an epimorphism $A \rightarrow P$ in \mathcal{A} .

If the abelian category \mathcal{A} has enough projectives, then for any complex $B^\bullet \in \mathrm{Kom}(\mathcal{A})$, the left exact functor

$$\mathrm{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathbf{Ab})$$

admits a right derived functor

$$R\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab}).$$

One can check that, this depends only on B^\bullet as an object of derived category. Therefore, it defines a bifunctor

$$(1.6.25) \quad D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \longrightarrow D(\mathbf{Ab}).$$

If \mathcal{A} has enough injectives and enough projectives, both bifunctors in (1.6.23) and (1.6.25) give rise to the same bifunctor

$$(1.6.26) \quad R\mathrm{Hom}^\bullet(-, -) : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab}).$$

Remark 1.6.27. If \mathcal{A} has enough injectives, but not necessarily have enough projectives, using (1.6.23) we can get the derived functor

$$(1.6.28) \quad R\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab}).$$

Note that, thanks to Theorem 1.6.22, composition of morphisms in the derived category can be used to define composition for Ext groups:

$$(1.6.29) \quad \mathrm{Ext}_{\mathcal{A}}^i(A^\bullet, B^\bullet) \times \mathrm{Ext}_{\mathcal{A}}^j(B^\bullet, C^\bullet) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{i+j}(A^\bullet, C^\bullet),$$

for all $A^\bullet, B^\bullet, C^\bullet \in D^+(\mathcal{A})$. This follows because

$$\mathrm{Ext}_{\mathcal{A}}^j(B^\bullet, C^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(B^\bullet, C^\bullet[j]) \cong \mathrm{Hom}_{D(\mathcal{A})}(B^\bullet[i], C^\bullet[i+j]).$$

Proposition 1.6.30 (Grothendieck's composite functor theorem). *Let $F_1 : \mathcal{A} \rightarrow \mathcal{B}$ and $F_2 : \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors of abelian categories. Suppose that there are adapted classes $\mathcal{I}_{F_1} \subset \mathcal{A}$ and $\mathcal{I}_{F_2} \subset \mathcal{B}$ for F_1 and F_2 , respectively, such that $F_1(\mathcal{I}_{F_1}) \subseteq \mathcal{I}_{F_2}$. Then the derived functors $RF_1 : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, $RF_2 : D^+(\mathcal{B}) \rightarrow D^+(\mathcal{C})$ and $R(F_2 \circ F_1) : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{C})$ exists, and there is a natural isomorphism of functors*

$$R(F_2 \circ F_1) \xrightarrow{\cong} RF_2 \circ RF_1.$$

Proof. Clearly RF_1 and RF_2 exists by given assumptions (c.f., Remark 1.6.14). Since $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, we see that \mathcal{I}_{F_1} is $(F_2 \circ F_1)$ -adapted. Therefore, $R(F_2 \circ F_1)$ exists. Then the natural morphism of functors

$$(1.6.31) \quad R(F_2 \circ F_1) \longrightarrow RF_2 \circ RF_1$$

follows from the universal property of derived functor $R(F_2 \circ F_1)$ (c.f., Definition 1.6.5). To see (1.6.31) is an isomorphism, given any complex $A^\bullet \in D^+(\mathcal{A})$ we choose a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$ quasi-isomorphic to A^\bullet . Then we have

$$(1.6.32) \quad R(F_2 \circ F_1)(A^\bullet) \cong (K(F_2) \circ K(F_1))(I^\bullet)$$

and

$$(1.6.33) \quad RF_2(RF_1(A^\bullet)) \cong RF_2(K(F_1)(I^\bullet)) \cong K(F_2)(K(F_1)(I^\bullet)).$$

Now it follows from the natural isomorphism between (1.6.32) and (1.6.33) that the morphism of functor in (1.6.31) is an isomorphism. \square

Remark 1.6.34. If both \mathcal{A} and \mathcal{B} have enough injectives, then the hypotheses of the above Proposition 1.6.30 are satisfied if $F_1(\mathcal{I}_{\mathcal{A}}) \subset \mathcal{I}_{\mathcal{B}}$.

1.7. Serre functor. Let k be a field. Let \mathcal{A} be a k -linear additive category.

Definition 1.7.1. A *Serre functor* on \mathcal{A} is a k -linear equivalence of categories

$$S : \mathcal{A} \longrightarrow \mathcal{A}$$

such that for any two objects $A, B \in \mathcal{A}$, there is a natural k -linear isomorphism

$$\eta_{A,B} : \text{Hom}(A, B) \longrightarrow \text{Hom}(B, S(A))^*,$$

which is functorial in both A and B . We write the induced k -bilinear pairing as

$$\text{Hom}(B, S(A)) \times \text{Hom}(A, B) \longrightarrow k, \quad (f, g) \longmapsto \langle f|g \rangle.$$

Proposition 1.7.2. Let \mathcal{A} be a k -linear additive category together with a Serre functor $S : \mathcal{A} \rightarrow \mathcal{A}$. Then for any $A, B \in \mathcal{A}$, the following diagram commutes.

$$(1.7.3) \quad \begin{array}{ccc} \text{Hom}(A, B) & \xrightarrow{\eta_{A,B}} & \text{Hom}(B, S(A))^* \\ S_{A,B} \downarrow & \searrow \text{red dashed } \exists \eta_{B,S(A)}^* & \uparrow S_{B,S(A)}^* \\ \text{Hom}(S(A), S(B)) & \xrightarrow{\eta_{S(A),S(B)}} & \text{Hom}(S(B), S^2(A))^* \end{array}$$

Proof. By abuse of notation, we denote by $\eta_{B,S(A)}^*$ the composite k -linear homomorphism

$$\eta_{B,S(A)}^* : \text{Hom}(S(A), S(B)) \hookrightarrow \text{Hom}(S(A), S(B))^{**} \xrightarrow{\eta_{B,S(A)}^*} \text{Hom}(B, S(A))^*.$$

Therefore, it suffices to show that both upper and lower triangles in (1.7.3) commutes. Note that, commutativity of upper triangle is equivalent to

$$\langle f|g \rangle = \langle S_{A,B}(g)|f \rangle, \quad \forall f \in \text{Hom}(B, S(A)), g \in \text{Hom}(A, B).$$

Applying functoriality of η in the second variable, we have the following commutative diagram.

$$(1.7.4) \quad \begin{array}{ccc} \text{Hom}(A, B) & \xrightarrow{\eta_{A,B}} & \text{Hom}(B, S(A))^* \\ \uparrow - \circ g & & \uparrow (S(g) \circ -)^* \\ \text{Hom}(B, B) & \xrightarrow{\eta_{B,B}} & \text{Hom}(B, S(B))^* \end{array}$$

Applying commutativity of (1.7.4) to $\text{Id}_B \in \text{Hom}(B, B)$ we have $\langle f|g \rangle = \langle S(g) \circ f | \text{Id} \rangle$. Applying functoriality of η in the first variable, we have the following commutative diagram

$$(1.7.5) \quad \begin{array}{ccc} \text{Hom}(B, B) & \xrightarrow{\eta_{B,B}} & \text{Hom}(B, S(B))^* \\ f \circ - \downarrow & & \downarrow (- \circ f)^* \\ \text{Hom}(B, S(A)) & \xrightarrow{\eta_{B,S(A)}} & \text{Hom}(S(A), S(B))^*, \end{array}$$

which gives $\langle (S(g) \circ f) | \text{Id}_B \rangle = \langle S(g) | f \rangle$. This completes the proof. \square

Remark 1.7.6. In order to avoid trouble with identifying $\text{Hom}(A, B)$ with its double dual $\text{Hom}(A, B)^{**}$, we always assume that a k -linear additive category \mathcal{A} has finite dimensional Hom's (i.e., $\dim_k \text{Hom}_{\mathcal{A}}(A, B) < \infty$, for all $A, B \in \text{Ob}(\mathcal{A})$).

Lemma 1.7.7. *Let \mathcal{A} and \mathcal{B} be k -linear additive categories with finite dimensional Hom's. If \mathcal{A} and \mathcal{B} are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively, then any k -linear equivalence $F : \mathcal{A} \rightarrow \mathcal{B}$ commutes with Serre functors (i.e., there is an isomorphism of functors $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$).*

Proposition 1.7.8. *Let \mathcal{A} be a k -linear additive category. Then any two Serre functors on \mathcal{A} are isomorphic.*

Let X be a smooth projective variety of dimension $n \geq 1$ defined over a field k . Note that, for any locally free coherent sheaf E on X , the functor

$$- \otimes E : \mathfrak{Coh}(X) \rightarrow \mathfrak{Coh}(X), \quad F \mapsto F \otimes E$$

is exact. Let ω_X be the dualizing sheaf on X . Let $D^*(X) = D^*(\mathfrak{Coh}(X))$, where $*$ $\in \{\emptyset, b, -, +\}$. Consider the composite functor

$$(1.7.9) \quad S_X : D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X),$$

where $[n] : D^*(X) \rightarrow D^*(X)$ is the n -th shift functor given by sending a complex E^\bullet to $E^\bullet[n]$. Since both the functors $\omega_X \otimes -$ and $[n]$ are exact, their composite functor $S_X := \omega_X \otimes (-)[n]$ is exact.

Theorem 1.7.10 (Grothendieck-Serre duality). *Let X be a smooth projective variety over a field k . Then the functor $S_X : D^b(X) \rightarrow D^b(X)$ as defined in (1.7.9) is a Serre functor in the sense of Definition 1.7.1.*

Proof. Given any two objects $E^\bullet, F^\bullet \in D^b(X)$, we need to give an isomorphism of k -vector spaces

$$(1.7.11) \quad \eta_{E^\bullet, F^\bullet} : \operatorname{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \xrightarrow{\cong} \operatorname{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet))^*$$

which is functorial in both E^\bullet and F^\bullet . Thanks to Theorem 1.6.22, we have

$$\operatorname{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i]) = \operatorname{Ext}^i(E^\bullet, F^\bullet), \quad \forall i.$$

Since X is smooth and projective, choosing a resolution by complex of locally free sheaves on X , we may assume that E^i is locally free, for all i . Then we have functorial isomorphisms

$$\begin{aligned} \operatorname{Hom}^i(E^\bullet, F^\bullet) &= \bigoplus_{j \in \mathbb{Z}} \operatorname{Hom}(E^j, F^{i+j}) = \bigoplus_{j \in \mathbb{Z}} H^0(X, \mathcal{H}om(E^j, F^{i+j})) \\ &\cong \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}^n(F^{i+j}, E^j \otimes \omega_X)^*, \quad \text{by classical Serre duality theorem.} \\ &\cong \bigoplus_{j \in \mathbb{Z}} \operatorname{Hom}_{D^b(X)}(F^{i+j}, E^j \otimes \omega_X[n])^*, \quad \text{by Proposition 1.6.11.} \\ &\cong \operatorname{Hom}^{n-i}(F^\bullet, E^\bullet \otimes \omega_X)^*. \end{aligned}$$

Since for any two complexes A^\bullet, B^\bullet , we have

$$\operatorname{Ext}_{\mathcal{A}}^i(A^\bullet, B^\bullet) := H^i(R\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)), \quad \forall i,$$

the theorem follows. \square

Theorem 1.7.10 is a special case of Grothendieck-Verdier duality.

Theorem 1.7.12 (Grothendieck-Verdier duality). *Let $f : X \rightarrow Y$ be a morphism of smooth schemes over a field k of relative dimension $\dim(f) := \dim(X) - \dim(Y)$. Let*

$$(1.7.13) \quad \omega_f := \omega_X \otimes f^* \omega_Y$$

be the relative dualizing sheaf of f . Then for any $F^\bullet \in D^b(X)$ and $E^\bullet \in D^b(Y)$, there is a a functorial isomorphism

$$(1.7.14) \quad Rf_* R\mathcal{H}om(F^\bullet, Lf^*(E^\bullet) \otimes_{\omega_f} [\dim(f)]) \xrightarrow{\cong} R\mathcal{H}om(Rf_* F^\bullet, E^\bullet).$$

We shall see some interesting applications of Serre functors in $D^b(X)$. For this, we need concept of local Hom complex, and spectral sequences.

1.8. Local $\mathcal{H}om^\bullet$ complex. Let X be a noetherian scheme. For $E \in \mathfrak{QCo}h(X)$, the functor

$$(1.8.1) \quad \mathcal{H}om(E, -) : \mathfrak{QCo}h(X) \longrightarrow \mathfrak{QCo}h(X), \quad F \mapsto \mathcal{H}om(E, F),$$

is left exact. Moreover, $\mathcal{H}om(E, F) \in \mathfrak{Co}h(X)$ if both E, F are coherent. Since $\mathfrak{QCo}h(X)$ has enough injectives (c.f., [Har77, Chapter III, Exercise 3.6]), its right derived functor

$$(1.8.2) \quad R\mathcal{H}om(E, -) : D^+(\mathfrak{QCo}h(X)) \longrightarrow D^+(\mathfrak{QCo}h(X))$$

exists. Then for any $E, F \in \mathfrak{QCo}h(X)$ and any integer i , we define

$$(1.8.3) \quad \mathcal{E}xt^i(E, F) := R^i\mathcal{H}om(E, F) := \mathcal{H}^i(R\mathcal{H}om(E, F)) \in \mathfrak{QCo}h(X).$$

If $E \in \mathfrak{Co}h(X)$, the above definition is local in the sense that its stalk at $x \in X$ can be computed as

$$(1.8.4) \quad \mathcal{E}xt^i(E, F)_x = \text{Ext}_{\mathcal{O}_{X,x}}^i(E_x, F_x),$$

which follows from commutativity of the following diagram.

$$(1.8.5) \quad \begin{array}{ccc} \mathfrak{QCo}h(X) & \xrightarrow{\mathcal{H}om(E, -)} & \mathfrak{QCo}h(X) \\ \downarrow & & \downarrow \\ \text{Mod}(\mathcal{O}_{X,x}) & \xrightarrow{\text{Hom}(E_x, -)} & \text{Mod}(\mathcal{O}_{X,x}). \end{array}$$

Note that, $\mathcal{E}xt^i(E, F) \in \mathfrak{Co}h(X)$ whenever both $E, F \in \mathfrak{Co}h(X)$.

When $E \in \mathfrak{Co}h(X)$, the functor (1.8.2) restricts to the bounded below derived category of coherent sheaves

$$(1.8.6) \quad R\mathcal{H}om(E, -) : D^+(X) \longrightarrow D^+(X).$$

Since for a non-regular local ring A , the groups $\text{Ext}_A^i(M, -)$ can be non-trivial even for $i \gg 0$, only for non-singular schemes X , the above functor $R\mathcal{H}om(E, -)$ restricts to $D^b(X)$, the bounded derived category of coherent sheaves on X .

As discussed before, the above construction easily generalizes for bounded above complexes of coherent sheaves $E^\bullet \in D^-(X)$. For this, we note that the following functor is exact.

$$\mathcal{H}om^\bullet(E^\bullet, -) : K^+(\mathfrak{QCo}h(X)) \longrightarrow K^+(\mathfrak{QCo}h(X))$$

given by sending a complex $F^\bullet \in K^+(\mathfrak{QCo}h(X))$ to the complex $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$, where

$$\mathcal{H}om^i(E^\bullet, F^\bullet) := \prod_{p \in \mathbb{Z}} \mathcal{H}om(E^p, F^{i+p})$$

and the differentials are given by $d^i = d_{E^\bullet} - (-1)^i d_{F^\bullet}$, for all $i \in \mathbb{Z}$. The following lemma follows from corresponding local statement for modules over a ring.

Lemma 1.8.7. *Let $F^\bullet \in D^-(X)$ be a complex of injective sheaves. If F^\bullet or $E^\bullet \in K^+(\mathfrak{Q}\mathfrak{Coh}(X))$ is acyclic, then $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$ is acyclic.*

The above Lemma 1.8.7 applied to the class

$$\mathcal{I} := \{I^\bullet \in K^+(\mathfrak{Q}\mathfrak{Coh}(X)) : I^i \text{ is injective } \mathcal{O}_X\text{-module}\}$$

shows that \mathcal{I} is adapted to the functor $\mathcal{H}om^\bullet(E^\bullet, -)$ (see Remark 1.6.14), and hence, the right derived functor

$$(1.8.8) \quad R\mathcal{H}om^\bullet(E^\bullet, -) : D^+(\mathfrak{Q}\mathfrak{Coh}(X)) \longrightarrow D^+(\mathfrak{Q}\mathfrak{Coh}(X))$$

exists. Note that, we are working with $\mathfrak{Q}\mathfrak{Coh}(X)$, because $\mathfrak{Coh}(X)$ has not enough injectives. Similarly, to see that the functor

$$\mathcal{H}om^\bullet(-, F^\bullet) : D^-(\mathfrak{Q}\mathfrak{Coh}(X))^{\text{op}} \longrightarrow D^+(\mathfrak{Q}\mathfrak{Coh}(X))$$

descends to the derived category for any $F^\bullet \in D^+(\mathfrak{Q}\mathfrak{Coh}(X))$. Therefore, we get a bifunctor

$$(1.8.9) \quad R\mathcal{H}om^\bullet(-, -) : D^-(\mathfrak{Q}\mathfrak{Coh}(X))^{\text{op}} \times D^+(\mathfrak{Q}\mathfrak{Coh}(X)) \longrightarrow D^+(\mathfrak{Q}\mathfrak{Coh}(X)).$$

This enables us to define

$$(1.8.10) \quad \mathcal{E}xt^i(E^\bullet, F^\bullet) := R^i\mathcal{H}om^\bullet(E^\bullet, F^\bullet) := \mathcal{H}^i(R\mathcal{H}om^\bullet(E^\bullet, F^\bullet)) \in \mathfrak{Q}\mathfrak{Coh}(X), \quad \forall i.$$

Proposition 1.8.11. *Let X be a non-singular noetherian k -scheme. Then any bounded complex $E^\bullet \in D^b(X)$ is isomorphic to a bounded complex $\mathcal{E}^\bullet \in D^b(X)$ of locally free coherent sheaves of \mathcal{O}_X -modules on X . In fact, one can choose \mathcal{E}^\bullet to have length at most $\dim_k(X)$.*

Proof. Since X is a noetherian non-singular k -scheme, $\mathfrak{Coh}(X)$ has enough projectives, So any $E^i \in \mathfrak{Coh}(X)$ admits a finite resolution

$$0 \rightarrow F_i^\ell \rightarrow F_i^{\ell-1} \rightarrow \cdots \rightarrow F_i^0 \rightarrow E \rightarrow 0,$$

with F_i^j a locally free coherent sheaves of \mathcal{O}_X -modules on X , and $\ell \leq \dim_k(X)$. □

1.9. Spectral sequence. In this subsection, we explain how spectral sequence occur when we compose two derived functors. Let \mathcal{A} be an abelian category.

Definition 1.9.1. A spectral sequence in \mathcal{A} is given by a collection of objects

$$(E_r^{p,q}, E^n), \quad n, p, q, r \in \mathbb{Z}, \quad r \geq 1$$

and morphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$$

satisfying that the following conditions.

- (i) $d_r^{p+r, q-r+1} \circ d_r^{p,q} = 0$, for all p, q, r . This yields a complex $E_r^{p+\bullet, q-\bullet, r+\bullet}$, for all $r \geq 1$.

(ii) There are isomorphisms

$$E_{r+1}^{p,q} \cong H^0(E_r^{p+\bullet, q-\bullet, r+\bullet}),$$

which are part of the data.

(iii) For any (p, q) , there is an r_0 such that $d_r^{p,q} = d_r^{p-r, q+r-1} = 0$, for all $r \geq r_0$. In particular, $E_r^{p,q} \cong E_{r_0}^{p,q}$, for all $r \geq r_0$. This object is denoted by $E_\infty^{p,q}$.

(iv) There is a decreasing filtration

$$\dots \subset F^{p+1}E^n \subset F^pE^n \subset F^{p-1}E^n \subset \dots \subset F^0E^n := E^n$$

such that $\bigcap_{p \in \mathbb{Z}} F^pE^n = 0$ and $\bigcup_{p \in \mathbb{Z}} F^pE^n = E^n$, and isomorphisms

$$E_\infty^{p,q} \cong F^pE^{p+q} / F^{p+1}E^{p+q}.$$

Let us try to visualize a spectral sequence. In page E_1 , we have the following data.

$$\begin{array}{ccccccc} E_1^{p-2, q+1} & \longrightarrow & E_1^{p-1, q+1} & \longrightarrow & E_1^{p, q+1} & \longrightarrow & E_1^{p+1, q+1} & \longrightarrow & \dots \\ E_1^{p-2, q} & \longrightarrow & E_1^{p-1, q} & \longrightarrow & E_1^{p, q} & \longrightarrow & E_1^{p+1, q} & \longrightarrow & \dots \\ E_1^{p-2, q-1} & \longrightarrow & E_1^{p-1, q-1} & \longrightarrow & E_1^{p, q-1} & \longrightarrow & E_1^{p+1, q-1} & \longrightarrow & \dots \end{array}$$

In page E_2 , we have the following data.

$$\begin{array}{ccccccc} E_2^{p-2, q+1} & & E_2^{p-1, q+1} & & E_2^{p, q+1} & & E_2^{p+1, q+1} & & \dots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ E_2^{p-2, q} & & E_2^{p-1, q} & & E_2^{p, q} & & E_2^{p+1, q} & & \dots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ E_2^{p-2, q-1} & & E_2^{p-1, q-1} & & E_2^{p, q-1} & & E_2^{p+1, q-1} & & \dots \end{array}$$

In some sense, the condition (iv) says that the objects $E_r^{p,q}$ converges towards a subquotient of certain filtration of E^n . Usually objects of one layer, say $E_r^{p,q}$ with r fixed, are given, and objects of the next layer can be obtained using (ii). It is enough to give objects $E_r^{p,q}$ with $r \geq m$, for some m ; the information is just the same. We express the spectral sequence by writing

$$E_r^{p,q} \implies E^{p+q}.$$

In most of the applications, only $E_r^{p,q}$ are given for $r \geq 2$, and in most of the cases, we don't need to go beyond page E_2 or E_3 .

Definition 1.9.2. A double complex $K^{\bullet,\bullet}$ is given by the following data: for each pair of integers (i, j) , an object $K^{i,j} \in \mathcal{A}$ and morphisms

$$d_I^{i,j} : K^{i,j} \longrightarrow K^{i+1,j} \quad \text{and} \quad d_{II}^{i,j} : K^{i,j} \longrightarrow K^{i,j+1}$$

such that

$$d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0.$$

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & K^{i-1,j-1} & \xrightarrow{d_I^{i-1,j-1}} & K^{i,j-1} & \xrightarrow{d_I^{i,j-1}} & K^{i+1,j-1} \longrightarrow \cdots \\
 & & \downarrow d_{II}^{i-1,j-1} & & \downarrow d_{II}^{i,j-1} & & \downarrow d_{II}^{i+1,j-1} \\
 \cdots & \longrightarrow & K^{i-1,j} & \xrightarrow{d_I^{i-1,j}} & K^{i,j} & \xrightarrow{d_I^{i,j}} & K^{i+1,j} \longrightarrow \cdots \\
 & & \downarrow d_{II}^{i-1,j} & & \downarrow d_{II}^{i,j} & & \downarrow d_{II}^{i+1,j} \\
 \cdots & \longrightarrow & K^{i-1,j+1} & \xrightarrow{d_I^{i-1,j+1}} & K^{i,j+1} & \xrightarrow{d_I^{i,j+1}} & K^{i+1,j+1} \longrightarrow \cdots \\
 & & \downarrow d_{II}^{i-1,j+1} & & \downarrow d_{II}^{i,j+1} & & \downarrow d_{II}^{i+1,j+1} \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

The associated *total complex* $K^\bullet := \text{tot}(K^{\bullet,\bullet})$ is defined by $K^n := \bigoplus_{i+j=n} K^{i,j}$ with differentials $d = d_I + d_{II}$.

The total complex $K^\bullet = \text{tot}(K^{\bullet,\bullet})$ admits a natural decreasing filtration $\{F^\ell K^n\}_\ell$ given by

$$(1.9.3) \quad F^\ell K^n := \bigoplus_{j \geq \ell} K^{n-j,j},$$

which satisfies $d_I(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n . Due to symmetry of the situation, there is another such natural filtration.

Example 1.9.4. The complex $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ is an example of a total complex of the double complex $K^{i,j} := \text{Hom}(A^{-i}, B^j)$ together with the differentials $d_I = (-1)^{j-i+1} d_A$ and $d_{II} = d_B$ (there are different sign conventions in the literature; however one can choose one sign convention, and final conclusion would be the same).

Definition 1.9.5. A *filtered complex* is a complex K^\bullet together with a decreasing filtration

$$\cdots \subset F^\ell K^n \subset F^{\ell-1} K^n \subset \cdots \subset F^0 K^n := K^n, \quad \forall n,$$

such that $d^n(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n .

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} \longrightarrow \cdots \\
& & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & F^\ell(K^{n-1}) & \longrightarrow & F^\ell(K^n) & \longrightarrow & F^\ell(K^{n+1}) \longrightarrow \cdots \\
& & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & F^{\ell+1}(K^{n-1}) & \longrightarrow & F^{\ell+1}(K^n) & \longrightarrow & F^{\ell+1}(K^{n+1}) \longrightarrow \cdots
\end{array}$$

Consider the filtrations $\{F^\ell K^n\}_\ell$ of the total complex $K^\bullet = \text{tot}(K^{\bullet,\bullet})$ in (1.9.3). The associated graded objects

$$\text{gr}^\ell(K^n) := F^\ell(K^n)/F^{\ell+1}(K^n) = K^{n-\ell,\ell}$$

forms a complex $K^{\bullet,\ell}[-\ell]$ (up to a global sign $(-1)^\ell$). Hence $\mathcal{H}^k(\text{gr}^\ell(K^\bullet)) = \mathcal{H}^{k-\ell}(K^{\bullet,\ell})$, for all ℓ , and the cohomology of the complex $\mathcal{H}_I^n(K^{\bullet,\bullet}) := (\mathcal{H}^n(K^{\bullet,j}))_{j \in \mathbb{Z}}$, with respect to d_{II} , gives $\mathcal{H}_{II}^\ell(\mathcal{H}_I^{k-\ell}(K^{\bullet,\bullet}))$.

Assuming the following finiteness condition: for each n , there is $\ell_+(n)$ and $\ell_-(n)$ such that $F^\ell K^n = 0$, for all $\ell \geq \ell_+(n)$ and $F^\ell K^n = K^n$, for all $\ell \leq \ell_-(n)$, one can show that any filtered complex gives rise to a spectral sequence. In case of double complex, we have the following.

Proposition 1.9.6. *Let $K^{\bullet,\bullet}$ be a double complex such that for any n , $K^{n-\ell,\ell} = 0$, for $|\ell| \gg 0$. Then there is a spectral sequence*

$$E_2^{p,q} := \mathcal{H}_{II}^p \mathcal{H}_I^q(K^{\bullet,\bullet}) \implies \mathcal{H}^{p+q}(K^\bullet).$$

Definition 1.9.7. Let $A^\bullet \in K^+(\mathcal{A})$. A Cartan-Eilenberg resolution of A^\bullet is a double complex $C^{\bullet,\bullet}$ together with a morphism of complexes $A^\bullet \rightarrow C^{\bullet,0}$ such that

- (i) $C^{i,j} = 0$, for $j < 0$,
- (ii) the sequences $A^n \rightarrow C^{n,0} \rightarrow C^{n,1} \rightarrow \cdots$ are injective resolutions of A^n , and the induced sequences

$$\begin{aligned}
\text{Ker}(d_A^n) &\rightarrow \text{Ker}(d_I^{n,0}) \rightarrow \text{Ker}(d_I^{n,1}) \rightarrow \cdots \\
\text{Im}(d_A^n) &\rightarrow \text{Im}(d_I^{n,0}) \rightarrow \text{Im}(d_I^{n,1}) \rightarrow \cdots \\
\mathcal{H}^n(A^\bullet) &\rightarrow \mathcal{H}_I^n(C^{\bullet,0}) \rightarrow \mathcal{H}_I^n(C^{\bullet,1}) \rightarrow \cdots
\end{aligned}$$

are injective resolutions of $\text{Ker}(d_A^n)$, $\text{Im}(d_A^n)$ and $\mathcal{H}^n(A^\bullet)$, respectively, and

- (iii) any short exact sequences of the form

$$0 \rightarrow \text{Ker}(d_I^{i,j}) \rightarrow C^{i,j} \rightarrow \text{Im}(d_I^{i,j}) \rightarrow 0$$

split.

Proposition 1.9.8. *If \mathcal{A} has enough injectives, then any $A^\bullet \in K^+(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

Theorem 1.9.9 (Grothendieck spectral sequence). *Let \mathcal{A}, \mathcal{B} and \mathcal{C} be abelian categories. Let $F_1 : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$ and $F_2 : K^+(\mathcal{B}) \rightarrow K(\mathcal{C})$ be exact functors. Suppose that \mathcal{A} and \mathcal{B} contains enough injectives, and for any complex $I^\bullet \in K^+(\mathcal{A})$ of injective objects from \mathcal{A} , its image $F_1(I^\bullet)$ is inside an F_2 -adapted triangulated subcategory \mathcal{K}_{F_2} . Then for any complex $A^\bullet \in D^+(\mathcal{A})$, there is a spectral sequence*

$$(1.9.10) \quad E_2^{p,q} := R^p F_2(R^q F_1(A^\bullet)) \implies E^{p+q} := R^{p+q}(F_2 \circ F_1)(A^\bullet).$$

Proof. Note that, if $F_1 = \text{Id}$, then for a left exact functor $F : K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$, the above spectral sequence reads

$$(1.9.11) \quad E_2^{p,q} := R^p F(\mathcal{H}^q(A^\bullet)) \implies E^{p+q} := R^{p+q} F(A^\bullet).$$

It follows from construction of derived functors that, given $A^\bullet \in D^+(\mathcal{A})$ isomorphic to a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$, we have $RF_1(A^\bullet) \cong F_1(I^\bullet)$ and

$$(1.9.12) \quad R^p F_2(R^q F_1(A^\bullet)) = R^p F_2(\mathcal{H}^q(F_1(I^\bullet))).$$

Since

$$(1.9.13) \quad \begin{aligned} R^n(F_2 \circ F_1)(A^\bullet) &= \mathcal{H}^n(R(F_2 \circ F_1)(I^\bullet)) \cong \mathcal{H}^n(RF_2(RF_1(A^\bullet))) \\ &\cong \mathcal{H}^n(RF_2(F_1(I^\bullet))) \cong R^n F_2(F_1(I^\bullet)), \end{aligned}$$

using (1.9.12), the general case (1.9.10) follows from the special case (1.9.11) above.

Therefore, it suffices to prove the result with $F_1 = \text{Id}$ and $F := F_2$. For this we need an appropriate double complex, which is provided by a Cartan-Eilenberg resolution of A^\bullet . Let $C^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of A^\bullet , and set $K^{\bullet,\bullet} := F(C^{\bullet,\bullet})$. Since F is additive, it preserve direct sums, and since $C^{i,j} \cong \text{Ker}(d_I^{i,j}) \oplus \text{Im}(d_I^{i,j})$, we have $\mathcal{H}_I^q(K^{\bullet,p}) = F\mathcal{H}_I^q(C^{\bullet,p})$. Fixing q , and allowing p to vary, we see that $\mathcal{H}_I^q(C^{\bullet,p})$ defines an injective resolution of $\mathcal{H}^q(K^{\bullet,\bullet}) = \mathcal{H}^q(A^\bullet)$. Then we have

$$\mathcal{H}_{II}^p \mathcal{H}_I^q(K^{\bullet,\bullet}) = R^p F(\mathcal{H}^q(A^\bullet)).$$

Now applying spectral sequence in Proposition 1.9.6 and using the fact that the natural morphism $A^\bullet \rightarrow \text{tot}(C^{\bullet,\bullet})$ is a quasi-isomorphism, we see that

$$\begin{aligned} \mathcal{H}^{p+q}(\text{tot}(K^{\bullet,\bullet})) &= \mathcal{H}^{p+q}(F(\text{tot}(C^{\bullet,\bullet}))) \\ &= \mathcal{H}^{p+q}(RF(A^\bullet)) \\ &= R^{p+q} F(A^\bullet). \end{aligned}$$

This completes the proof. □

Next we give some useful examples of Grothendieck spectral sequences, which will appear in next sections.

Example 1.9.14. Let \mathcal{A} be an abelian category with enough injectives. Let $A^\bullet \in D(\mathcal{A})$ and $B^\bullet \in D^+(\mathcal{A})$. Take $F_1 = \text{Id}$ so that $R^q F_1(B^\bullet) = \mathcal{H}^q(B^\bullet)$, and take $F_2 = \text{Hom}^\bullet(A^\bullet, -)$. Then we have,

$$R^p F_2(R^q F_1(B^\bullet)) = R^p \text{Hom}^\bullet(A^\bullet, \mathcal{H}^q(B^\bullet)) = \text{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)),$$

and

$$R^{p+q}(F_2 \circ F_1)(B^\bullet) = R^{p+q} \text{Hom}^\bullet(A^\bullet, B^\bullet) = \text{Ext}^{p+q}(A^\bullet, B^\bullet).$$

Since $\text{Ext}^{p+q}(A^\bullet, B^\bullet) = \text{Hom}(A^\bullet, B^\bullet[p+q])$ and $\text{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)) \cong \text{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p])$, by Theorem 1.9.9, we have a spectral sequence

$$(1.9.15) \quad E_2^{p,q} := \text{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p]) \implies \text{Hom}(A^\bullet, B^\bullet[p+q]).$$

Similarly, if \mathcal{A} has enough projectives so that we can compute $R^p \text{Hom}^\bullet(A^\bullet, B^\bullet)$ for $A^\bullet \in D^-(\mathcal{A})$ as the right derived functor of the contravariant functor $\text{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\text{op}} \rightarrow K(\mathbf{Ab})$, we have the spectral sequence

$$(1.9.16) \quad E_2^{p,q} := \text{Hom}(\mathcal{H}^{-q}(A^\bullet), B^\bullet[p]) \implies \text{Hom}(A^\bullet, B^\bullet[p+q]).$$

If $B^\bullet \in D^+(\mathcal{A})$ is bounded below, and \mathcal{A} has enough injectives, but may not have enough projectives, then also we have this spectral sequence.

Remark 1.9.17. It should be noted that, a spectral sequence $E_2^{p,q} \Rightarrow E^{p+q}$ given at page E_2 **does not imply** that the term $E_\infty^{p,q}$ lies in page E_2 .

Example 1.9.18. Let X be a noetherian scheme so that $\mathfrak{QCo}\mathfrak{h}(X)$ has enough injectives. Then for any $E^\bullet, F^\bullet \in D^b(X)$, we have the following spectral sequences

$$(1.9.19) \quad E_2^{p,q} := \text{Ext}^p(E^\bullet, \mathcal{H}^q(F^\bullet)) \implies \text{Ext}^{p+q}(E^\bullet, F^\bullet),$$

and

$$(1.9.20) \quad E_2^{p,q} := \text{Ext}^p(\mathcal{H}^{-q}(E^\bullet), F^\bullet) \implies \text{Ext}^{p+q}(E^\bullet, F^\bullet),$$

Remark 1.9.21. If X is a projective k -variety, $\dim_k H^i(X, E) < \infty$ for any coherent sheaf E on X . Using this, one can deduce that $\dim_k \text{Ext}^i(E, F) < \infty$, for any $E, F \in \mathfrak{Co}\mathfrak{h}(X)$. Then using the spectral sequences (1.9.19) and (1.9.20) one can show that $\text{Ext}^i(E^\bullet, F^\bullet)$ has finite dimension, for all $E^\bullet, F^\bullet \in D^b(X)$.

Example 1.9.22. Let $E^\bullet \in D^-(X)$. Then by definition of local Hom complex, we have $\Gamma \circ \mathcal{H}om^\bullet(E^\bullet, -) = \text{Hom}^\bullet(E^\bullet, -)$. Since for a complex I^\bullet of injective sheaves of \mathcal{O}_X -modules, the complex $\mathcal{H}om^\bullet(E^\bullet, I^\bullet)$ is Γ -acyclic (meaning that, $\text{Ext}^i(E^\bullet, I^\bullet) = R^i \Gamma(\mathcal{H}om^\bullet(E^\bullet, I^\bullet)) = 0$ for all $i \neq 0$, which indeed holds), we have

$$R\Gamma \circ R\mathcal{H}om^\bullet(E^\bullet, -) = R\text{Hom}^\bullet(E^\bullet, -).$$

Therefore, applying Theorem 1.9.9 we have the following spectral sequence relating local and global Ext:

$$(1.9.23) \quad E_2^{p,q} := H^p(X, \mathcal{E}xt^q(E^\bullet, F^\bullet)) \implies \text{Ext}^{p+q}(E^\bullet, F^\bullet).$$

1.10. Theorem of Bondal and Orlov. A famous theorem of Gabriel says that two k -varieties X and Y are isomorphic if and only if there is an equivalence of categories $\mathcal{Coh}(X)$ with $\mathcal{Coh}(Y)$. In [Muk81], Mukai established an equivalence $D^b(A) \simeq D^b(\check{A})$, where A is an abelian variety and \check{A} its dual abelian variety. Therefore, equivalence between bounded derived category of coherent sheaves fails to ensure isomorphism of varieties, in general. In their famous paper [BO01], Bondal and Orlov shows how to reconstruct a smooth projective variety X from $D^b(X)$ when ω_X or its dual is ample (see Theorem 1.10.5).

Let k be a field. A k -variety is an integral separated finite type k -scheme. For any smooth projective k -variety X , we define $D^b(X) := D^b(\mathcal{Coh}(X))$. A rank one invertible sheaf L on X is said to have *finite order* if $L^r \cong \mathcal{O}_X$ for some integer $r > 0$. The smallest positive integer r such that $L^r \cong \mathcal{O}_X$ is called the *order* of L . If $L^r \not\cong \mathcal{O}_X$, $\forall r > 0$, we say that L has *infinite order*. For any $x \in X$, let $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at x . For any closed point $x \in X$, we can consider $k(x)$ as a coherent sheaf on X supported at x by taking its push-forward along the closed embedding $\iota_x : \text{Spec}(k(x)) \hookrightarrow X$. This is the skyscraper sheaf supported at x given by

$$k(x)(U) = \begin{cases} k(x), & \text{if } x \in U, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 1.10.1. *Let X and Y be smooth projective varieties over k . If there is an exact equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ of bounded derived categories, then $\dim_k(X) = \dim_k(Y)$. In this case, both ω_X and ω_Y have the same order (can be infinity too).*

Proof. Since both X and Y are smooth projective k -varieties, by Theorem 1.7.10, they admit natural Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim_k(Y)]$, respectively. By Lemma 1.7.7, any k -linear equivalence $F : D^b(X) \rightarrow D^b(Y)$ commutes with Serre functors S_X and S_Y (i.e., there is a natural isomorphism of functors $F \circ S_X \cong S_Y \circ F$).

For a closed point $x \in X$, we have $k(x) \cong k(x) \otimes \omega_X \cong S_X(k(x))[-\dim_k(X)]$. So,

$$\begin{aligned} F(k(x)) &\cong F(k(x) \otimes \omega_X) = F(S_X(k(x))[-\dim_k(X)]) \\ &\cong F(S_X(k(x))[-\dim_k(X)]), \quad \text{since } F \text{ is exact.} \\ (1.10.2) \quad &\cong S_Y(F(k(x))[-\dim_k(X)]), \quad \text{since } F \circ S_X \cong S_Y \circ F. \\ &\cong F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]. \end{aligned}$$

Since F is an equivalence of categories, $F(k(x))$ is a non-trivial bounded complex. Let i be the maximal (resp., minimal) integer such that $\mathcal{H}^i(F(k(x))) \neq 0$. Now from

(1.10.2) we have

$$\begin{aligned}
 0 \neq \mathcal{H}^i(F(k(x))) &\cong \mathcal{H}^i(F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]) \\
 &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x)) \otimes \omega_Y) \\
 (1.10.3) \quad &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x))) \otimes \omega_Y.
 \end{aligned}$$

Since ω_Y is a line bundle, (1.10.3) contradicts maximality (resp., minimality) of i whenever $\dim_k(X) < \dim_k(Y)$ (resp., $\dim_k(X) > \dim_k(Y)$). Therefore, $\dim_k(X) = \dim_k(Y)$.

To see that both ω_X and ω_Y have the same order, assume that $\omega_X^k \cong \mathcal{O}_X$. Let $n = \dim_k(X) = \dim_k(Y)$. Note that, $S_X^k[-kn] \cong \text{Id}_{D^b(X)}$. Since $F \circ S_X \cong S_Y \circ F$, choosing a quasi-inverse of the equivalence F , we have

$$\begin{aligned}
 F^{-1} \circ S_Y^k[-kn] \circ F &\cong S_X^k[-kn] \cong \text{Id}_{D^b(X)} \\
 \Rightarrow S_Y^k[-kn] &\cong \text{Id}_{D^b(Y)}.
 \end{aligned}$$

Applying \mathcal{O}_Y to the above isomorphism of functors, we get $\omega_Y^k \cong \mathcal{O}_Y$. \square

Remark 1.10.4. In the proof of above Proposition, to show both ω_X and ω_Y have the same order, under the assumption that $\dim(X) = \dim(Y)$, we don't need F to be exact.

Theorem 1.10.5 (Bondal–Orlov). *Let X be a smooth projective variety over k with canonical line bundle ω_X . Assume that ω_X (resp., ω_X^\vee) is ample. Let Y be any smooth projective variety over k . If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k -varieties. In particular, ω_Y (resp., ω_Y^\vee) is ample.*

The main idea behind the proof is to “cohomologically” characterize closed points, invertible sheaves and Zariski topology of a smooth projective k -variety. For this we need “Serre functor” as defined in Definition 1.7.1. Note that, both $D^b(X)$ and $D^b(Y)$ admits Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim(Y)]$, respectively.

Definition 1.10.6. A *graded category* is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category \mathcal{D} and an equivalence functor $T_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$, known as *shift functor*. A functor $F : \mathcal{D} \rightarrow \mathcal{D}'$ between graded categories is called *graded* if there is an isomorphism of functors $F \circ T_{\mathcal{D}} \xrightarrow{\cong} T_{\mathcal{D}'} \circ F$.

Example 1.10.7. Any triangulated category is a graded category, and any morphism between two triangulated categories is a graded morphism.

Definition 1.10.8. Let \mathcal{D} be a k -linear triangulated category with Serre functor S . An object $P \in \mathcal{D}$ is said to be *point like* of codimension s if

- (i) $S(P) \cong P[s]$,
- (ii) $\text{Hom}(P, P[i]) = 0$, for $i < 0$, and
- (iii) $k(P) := \text{Hom}(P, P)$ is a field.

An object E of an additive category is called *simple* if $\text{Hom}(E, E)$ is a field.

Example 1.10.9. Let X be a smooth projective k -variety of dimension n .

- (i) For any closed point $x \in X$, we have $S_X(k(x)) = (k(x) \otimes \omega_X)[n] \cong k(x)[n]$. Therefore, $k(x) \in D^b(X)$ is a point like object of codimension d .
- (ii) Let $\omega_X \cong \mathcal{O}_X$ (for example when X is an abelian variety or a K3 surface). Then any simple object $E \in \mathcal{Coh}(X)$ defines a point like object of codimension n in $D^b(X)$.

Proposition 1.10.10. Let \mathcal{A} be an abelian category, and $A^\bullet \in D^b(\mathcal{A})$. Let

$$i^+ := \max\{i : \mathcal{H}^i(A^\bullet) \neq 0\} \quad \text{and} \quad i^- := \min\{i : \mathcal{H}^i(A^\bullet) \neq 0\}.$$

Then in $D^b(\mathcal{A})$, there are morphisms $\phi : A^\bullet \rightarrow \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ and $\psi : \mathcal{H}^{i^-}(A^\bullet)[-i^-] \rightarrow A^\bullet$ such that $\mathcal{H}^{i^+}(\phi) = \text{Id}_{\mathcal{H}^{i^+}(A^\bullet)}$ and $\mathcal{H}^{i^-}(\psi) = \text{Id}_{\mathcal{H}^{i^-}(A^\bullet)}$.

Proof. There is a natural quasi-isomorphism of complexes

$$\begin{array}{ccccccc} A_-^\bullet : & \cdots & \longrightarrow & A^{i^+-1} & \longrightarrow & \text{Ker}(d^{i^+}) & \longrightarrow 0 \longrightarrow \cdots \\ \text{qis} \downarrow & & & \parallel & & \downarrow & \downarrow \\ A^\bullet : & \cdots & \longrightarrow & A^{i^+-1} & \longrightarrow & A^{i^+} & \xrightarrow{d^{i^+}} A^{i^++1} \longrightarrow \cdots \end{array}$$

Since the natural morphism of complexes $A_-^\bullet \rightarrow \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ induces identity morphism at i^+ -th cohomologies, the first part follows. Second part is similar. \square

Corollary 1.10.11. With the above notations, for any $B \in \mathcal{A}$, we have the following natural isomorphisms

- (i) $\text{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B) \cong \text{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[-i^+])$, and
- (ii) $\text{Hom}_{D^b(\mathcal{A})}(B, \mathcal{H}^{i^-}(A^\bullet)) \cong \text{Hom}_{D^b(\mathcal{A})}(B[-i^-], A^\bullet)$.

Proof. Send $f \in \text{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B)$ to $f[-i^+]$ and use above Proposition. To get the inverse map, send any $\phi \in \text{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[i^+])$ to $\mathcal{H}^{i^+}(\phi)[-i^+]$. Second part is similar. \square

Exercise 1.10.12. Let $A^\bullet \in D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for all $i < m$. Then there is a distinguished triangle

$$\mathcal{H}^m(A^\bullet)[-m] \longrightarrow A^\bullet \xrightarrow{\varphi} B^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[1-m]$$

in the derived category $D(\mathcal{A})$ such that

$$\mathcal{H}^i(B^\bullet) \cong \begin{cases} \mathcal{H}^i(A^\bullet) & \text{if } i \leq m, \text{ and} \\ 0, & \text{if } i > m. \end{cases}$$

Remark 1.10.13. Let X be a smooth projective k -variety of dimension d . Then any point like object $P \in D^b(X)$ has codimension d . This follows from assumption (i) in the Definition 1.10.8, because looking at minimal i with non-zero cohomologies, the isomorphism $P \otimes \omega_X[d] \cong P[s]$ implies

$$(1.10.14) \quad \mathcal{H}^i(P) \otimes \omega_X[d] \cong \mathcal{H}^i(P)[s].$$

This forces $d = s$.

Lemma 1.10.15. *Let M be a finitely generated non-zero module over a noetherian ring A . Then there is a finite chain of A -submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ (as A -modules), for some $\mathfrak{p}_i \in \text{Supp}(M)$.

Proof. Denote by $\text{Ass}(M)$ the set of all associated primes of M . Recall that, $\text{Ass}(M) \subseteq \text{Supp}(M)$ for any finitely generated A -module M . Since $M \neq 0$, we can choose a $\mathfrak{p}_1 \in \text{Ass}(M)$ to get an A -submodule

$$M_1 := \text{image}(A/\mathfrak{p}_1 \hookrightarrow M) \subset M.$$

If $M_1 \neq M$, we do the same for M/M_1 to choose a $\mathfrak{p}_2 \in \text{Ass}(M/M_1)$ and apply the same to obtain a sequence $M_1 \subsetneq M_2 \subseteq M$ with $M_2/M_1 \cong A/\mathfrak{p}_2$. Since $(M/M_1)_{\mathfrak{p}_2} \neq 0$, we see that $\mathfrak{p}_2 \in \text{Supp}(M)$. Since M is finitely generated, the result follows by induction. \square

Corollary 1.10.16. *With the above notation, if $\text{Supp}(M) = \{\mathfrak{m}\}$, for some maximal ideal \mathfrak{m} of A , there is a surjective (resp., injective) A -module homomorphism $M \twoheadrightarrow A/\mathfrak{m}$ (resp., $A/\mathfrak{m} \hookrightarrow M$).*

Proof. Since $\text{Ass}(M) = \{\mathfrak{m}\}$, the result follows from the above Lemma 1.10.15. \square

Definition 1.10.17. Support of a complex $E^\bullet \in D^b(X)$ is the union of the supports of its cohomologies. In other words, $\text{Supp}(E^\bullet)$ is the closed subset of X defined by

$$\text{Supp}(E^\bullet) := \bigcup_{i \in \mathbb{Z}} \text{Supp}(\mathcal{H}^i(E^\bullet)).$$

Lemma 1.10.18. *Let $E^\bullet \in D^b(X)$ with $\text{Supp}(E^\bullet) = Z_1 \cup Z_2$, for some disjoint closed subsets Z_1 and Z_2 in X . Then $E^\bullet \cong E_1^\bullet \oplus E_2^\bullet$, for some non-zero objects $E_j^\bullet \in D^b(X)$ with $\text{Supp}(E_j^\bullet) \subseteq Z_j$, for all $j = 1, 2$.*

Proof. This is clear for any $E \in \mathfrak{Coh}(X)$, and hence the result follows for $E^\bullet \cong E[n] \in D^b(X)$, for $E \in \mathfrak{Coh}(X)$ and $n \in \mathbb{Z}$. Let

$$i_{E^\bullet}^+ := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\} \quad \text{and} \quad i_{E^\bullet}^- := \min\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\};$$

and we drop the subscript E^\bullet when there is no confusion likely to arise. The *length* of an object $E^\bullet \in D^b(X)$ is the difference $i^+ - i^-$. For general case, we use induction on the length of a complex.

Let $E^\bullet \in D^b(X)$ be a complex of length at least 2. Let $m = i_{E^\bullet}^-$, and write $\mathcal{H} := \mathcal{H}^m(E^\bullet)$. The sheaf \mathcal{H} can be decomposed as $\mathcal{H} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\text{Supp}(\mathcal{H}_j) \subset Z_j$, for $j = 1, 2$. By Proposition 1.10.10, we have a natural morphism $\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet$ inducing identity morphism on the m -th cohomology; complete it to a distinguished triangle

$$\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet \longrightarrow F^\bullet := C(\varphi) \longrightarrow \mathcal{H}[1-m].$$

Then from long exact sequence of cohomologies we have

$$\mathcal{H}^i(F^\bullet) = \begin{cases} \mathcal{H}^i(E^\bullet), & \text{if } i > m, \text{ and} \\ 0, & \text{if } i \leq m; \end{cases}$$

(c.f. Exercise 1.10.12). Since the length of F^\bullet is less than the length of E^\bullet , induction hypothesis applied to F^\bullet gives a decomposition $F^\bullet \cong F_1^\bullet \oplus F_2^\bullet$ with $\text{Supp}(\mathcal{H}^i(F_j^\bullet)) \subset Z_j$, for all $j = 1, 2$, and $i \in \mathbb{Z}$. Since $\mathcal{H}^{-q}(F_1^\bullet)$ and \mathcal{H}_2 are coherent sheaves of \mathcal{O}_X -modules with disjoint supports, we have

$$\text{Hom}_{D^b(X)}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) = \text{Ext}^p(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2) = 0, \quad \forall p \in \mathbb{Z},$$

which can be verified locally. Then $\text{Hom}(F_1^\bullet, \mathcal{H}_2[1-m]) = 0$ follows from the spectral sequence

$$E_2^{p,q} := \text{Hom}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) \implies E^{p+q} := \text{Hom}(F_1^\bullet, \mathcal{H}_2[p+q]);$$

c.f., Example 1.9.14. Similarly, we have $\text{Hom}(F_2^\bullet, \mathcal{H}_1[1-m]) = 0$. Choose a complex E_j^\bullet to complete a distinguished triangle

$$E_j^\bullet \longrightarrow F_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow E_j^\bullet[1], \quad \forall j = 1, 2,$$

we have a decomposition $E^\bullet \cong E_1^\bullet \oplus E_2^\bullet$. Since $\text{Supp}(F_j^\bullet) \subset Z_j$, it follows that $\text{Supp}(E_j^\bullet) \subset Z_j$, for all $j = 1, 2$. \square

Lemma 1.10.19. *Let E^\bullet be a simple object in $D^b(X)$ with **zero dimensional support**. If $\text{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, then $E^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer m .*

Proof. Since E^\bullet is supported in dimension zero, $\text{Supp}(E)$ is a finite subset of closed points in X . If $\text{Supp}(E)$ is not a singleton set, then it has disjoint components. Then in $D^b(X)$, we have an isomorphism $E^\bullet \cong E_1^\bullet \oplus E_2^\bullet$, with $E_j^\bullet \neq 0$, $\forall i = 1, 2$, which contradicts simplicity of E^\bullet . Therefore, $\text{Supp}(E^\bullet)$ is a closed point, say $x \in X$. Let

$i^+ := \max\{i : \mathcal{H}^i(E^\bullet) \neq 0\}$ and $i^- := \min\{j : \mathcal{H}^j(E^\bullet) \neq 0\}$. Since both $\mathcal{H}^{i^+}(E^\bullet)$ and $\mathcal{H}^{i^-}(E^\bullet)$ have support $\{x\}$, they are given by finite modules over the noetherian local ring $\mathcal{O}_{X,x}$ supported at \mathfrak{m}_x . Then applying Corollary 1.10.16, we get a non-trivial $\mathcal{O}_{X,x}$ -module homomorphism $\phi : \mathcal{H}^{i^+}(E^\bullet) \longrightarrow \mathcal{H}^{i^-}(E^\bullet)$ given by the composition

$$\mathcal{H}^{i^+}(E^\bullet) \longrightarrow k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \hookrightarrow \mathcal{H}^{i^-}(E^\bullet).$$

Now it follows from Proposition 1.10.10 that the following composite morphism is non-trivial.

$$E^\bullet[i^+] \longrightarrow \mathcal{H}^{i^+}(E^\bullet) \xrightarrow{\phi} \mathcal{H}^{i^-}(E^\bullet) \longrightarrow E^\bullet[i^-].$$

Since $\text{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, we must have $i^- - i^+ \geq 0$. Hence, $i^- = i^+ =: m$ (say). Therefore, $E^\bullet \cong E[m]$, for some $E \in \mathfrak{Coh}(X)$ with $\text{Supp}(E) = \{x\}$. Since $\text{Hom}(E[m], E[m]) \cong \text{Hom}(E, E)$, so E is simple. Then the natural surjective homomorphism $E \rightarrow k(x)$ must be isomorphism. Therefore, $E^\bullet \cong k(x)[m]$. \square

Proposition 1.10.20 (Bondal–Orlov). *Let X be a smooth projective k -variety with ω_X or ω_X^\vee ample. Then any point like object in $D^b(X)$ is isomorphic to an object of the form $k(x)[m]$, for some closed point $x \in X$ and some integer m .*

Remark 1.10.21. Above result fails if neither ω_X nor ω_X^\vee is ample; c.f. Example 1.10.9.

Proof. Note that X is projective because there is an ample line bundle on X . Clearly for any closed point $x \in X$ and any integer m , the shifted skyscraper sheaf $k(x)[m] \in D^b(X)$ is a point like object of codimension $d = \dim(X)$ (c.f., Example 1.10.9).

To see the converse, let $P \in D^b(X)$ be a point like object of codimension n . It follows from $P \otimes \omega_X[d] \cong P[n]$ that $n = d$ (c.f., Remark 1.10.13). Then we have,

$$(1.10.22) \quad \mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P), \quad \forall i \in \mathbb{Z}.$$

Suppose that ω_X is ample. Let

$$m \mapsto P_E(m) := \chi(E \otimes \omega_X^m)$$

be the Hilbert polynomial of $E \in \mathfrak{Coh}(X)$. Since $\deg(P_E(m)) = \dim(\text{Supp}(E))$, taking tensor product with ω_X makes difference only if $\dim(\text{Supp}(E)) > 0$. Therefore, from (1.10.22) we conclude that $\mathcal{H}^i(P)$ is supported in dimension zero. Since P is simple, the result follows from Lemma 1.10.19. The same argument applies for ω_X^\vee ample. \square

Next we characterize line bundles as objects of $D^b(X)$.

Definition 1.10.23. Let \mathcal{D} be a triangulated category together with a Serre functor $T_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$. An object $L \in \mathcal{D}$ is said to be *invertible* if for each point like object

$P \in \mathcal{D}$, there is an integer n_P (which also depends on L) such that

$$\mathrm{Hom}_{\mathcal{D}}(L, P[i]) = \begin{cases} k(P), & \text{if } i = n_P, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we characterize invertible objects in $D^b(X)$. For this, we need the following well-known result from commutative algebra.

Lemma 1.10.24. *Let M be a finitely generated module over a noetherian local ring (A, \mathfrak{m}) . If $\mathrm{Ext}^1(M, A/\mathfrak{m}) = 0$, then M is free.*

Proof. Let $k = A/\mathfrak{m}$. Then any k -basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for the A -module M by Nakayama lemma. Thus we get a short exact sequence of A -modules

$$0 \longrightarrow N \xrightarrow{\iota} A^n \xrightarrow{\phi} M \longrightarrow 0.$$

Note that, $N = \mathrm{Ker}(\phi)$ is finitely generated, and ι induces a trivial homomorphism $\tilde{\iota}: N/\mathfrak{m}N \longrightarrow k^n$. Since $\mathrm{Ext}^1(M, k) = 0$, the induced homomorphism

$$\mathrm{Hom}(A^n, k) \longrightarrow \mathrm{Hom}(N, k)$$

is surjective. Since $\mathrm{Hom}_A(A^n, k) \cong \mathrm{Hom}_k(k^n, k)$ and $\mathrm{Hom}_A(N, k) \cong \mathrm{Hom}_k(N/\mathfrak{m}N, k)$, the homomorphism $\mathrm{Hom}_k(k^n, k) \longrightarrow \mathrm{Hom}_k(N/\mathfrak{m}N, k)$ induced by $\tilde{\iota}$ is surjective. Since $\tilde{\iota} = 0$, this forces $N/\mathfrak{m}N = 0$. Then $N = 0$ by Nakayama lemma, and hence M is a free A -module. \square

Proposition 1.10.25 (Bondal–Orlov). *Let X be a smooth projective k -variety. Any invertible object in $D^b(X)$ is of the form $L[m]$, for some line bundle L on X and some integer m . Conversely, if any point like object of $D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and some integer ℓ , then for any line bundle L on X and any integer m , $L[m] \in D^b(X)$ is invertible.*

Remark 1.10.26. Note that, by Proposition 1.10.20 the condition in the converse part of the above Proposition is satisfied when ω_X or ω_X^\vee is ample.

Proof of Proposition 1.10.25. Step 1. Let $E^\bullet \in D^b(X)$ be an invertible object. Let $m = \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then by Proposition 1.10.10, there is a morphism

$$E^\bullet \longrightarrow \mathcal{H}^m(E^\bullet)[-m]$$

in $D^b(X)$ inducing identity morphism at m -th cohomology $\mathcal{H}^m(E^\bullet)$. This gives

$$(1.10.27) \quad \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m]),$$

(c.f., Corollary 1.10.11). Fix a closed point $x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$. Then by Lemma 1.10.15, there is an associated prime ideal $\mathfrak{p} \subseteq \mathfrak{m}_{x_0}$ and a surjective homomorphism

$\mathcal{H}^m(E^\bullet) \twoheadrightarrow \mathcal{O}_{X,x_0}/\mathfrak{p}$, which gives a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x_0)$. Therefore, by (1.10.27), we have

$$0 \neq \mathrm{Hom}_{D^b(X)}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m]).$$

This forces $n_{k(x_0)} = -m$ (c.f., Definition 1.10.23).

Step 2. *We show that, $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.*

Since $n_{k(x_0)} = -m$, it follows from the definition of invertible object $E^\bullet \in D^b(X)$ that

$$(1.10.28) \quad \mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = \mathrm{Hom}(E^\bullet, k(x_0)[1+n_{k(x_0)}]) = 0.$$

Consider the spectral sequence (see Example 1.9.18)

$$(1.10.29) \quad \begin{aligned} E_2^{p,q} &:= \mathrm{Hom}(\mathcal{H}^{-q}(E^\bullet), k(x_0)[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), k(x_0)) \\ &\implies E^{p+q} := \mathrm{Hom}(E^\bullet, k(x_0)[p+q]). \end{aligned}$$

Since $\mathcal{H}^{m+1}(E^\bullet) = 0$, we have

$$(1.10.30) \quad E_2^{3,-m-1} = \mathrm{Hom}(\mathcal{H}^{m+1}(E^\bullet), k(x_0)[3]) = 0.$$

Also

$$(1.10.31) \quad E_2^{-1,-m+1} = \mathrm{Hom}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)[-1]) = \mathrm{Ext}^{-1}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)) = 0.$$

Now using (1.10.30) and (1.10.31), and taking H^0 of the complex

$$\cdots \longrightarrow 0 = E_2^{-1,-m+1} \xrightarrow{d} E_2^{1,-m} \xrightarrow{d} E_2^{3,-m-1} = 0 \longrightarrow \cdots,$$

we see that $E_3^{1,-m} = E_2^{1,-m}$; similarly, $E_r^{1,-m} = E_2^{1,-m}$, for all $r \geq 2$. The following picture of page E_2 could be useful to understand the situation.

$$\begin{array}{ccccccc} 0 & & E_2^{0,-m+1} & & E_2^{1,-m+1} & & E_2^{2,-m+1} & & E_2^{3,-m+1} \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ 0 & & E_2^{0,-m} & & E_2^{1,-m} & & E_2^{2,-m} & & E_2^{3,-m} \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

This shows that,

$$(1.10.32) \quad E_2^{1,-m} = E_\infty^{1,-m}.$$

Since $E_\infty^{1,-m}$ is isomorphic to a subquotient of

$$(1.10.33) \quad E^{1-m} = \mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = 0$$

(see, (1.10.28) and (1.10.29)), using (1.10.32) we conclude that $E_2^{1,-m} = 0$. Therefore,

$$(1.10.34) \quad \mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0, \quad \forall x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet)).$$

Step 3. *We show that $\mathcal{H}^m(E^\bullet)$ is a locally free \mathcal{O}_X -module.*

For this, we consider the *local-to-global* spectral sequence (see Example 1.9.22)

$$(1.10.35) \quad E_2^{p,q} := H^p(X, \mathcal{E}xt^q(\mathcal{H}^m(E^\bullet), k(x_0))) \implies \text{Ext}^{p+q}(\mathcal{H}^m(E^\bullet), k(x_0)),$$

which allow us to pass from the global vanishing $\text{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ to the local one $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $\mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))$ is a skyscraper sheaf supported at x_0 , it is flasque, and hence is Γ -acyclic. Then from (1.10.35), we have

$$(1.10.36) \quad E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))) = 0.$$

Again,

$$(1.10.37) \quad E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m(E^\bullet), k(x_0))) = 0.$$

Since at page E_2 , we have morphisms

$$0 = E_2^{-2,2} \xrightarrow{d} E_2^{0,1} \xrightarrow{d} E_2^{2,0} = 0,$$

we have $E_3^{0,1} = \mathcal{H}^0(\cdots \rightarrow 0 \rightarrow E_2^{0,1} \rightarrow 0 \rightarrow \cdots) = E_2^{0,1}$. Similar computations shows that $E_r^{0,1} = E_2^{0,1}$, for all $r \geq 2$. Hence we conclude that,

$$(1.10.38) \quad E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))) = E_\infty^{0,1}.$$

Since $E^1 = \text{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ by Step 2, we have $E_2^{0,1} = E_\infty^{0,1} = 0$. Since $k(x_0)$ is a skyscraper sheaf supported at x_0 , we see that $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))$ is supported over $\{x_0\}$, and hence is globally generated. Since

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))) = E_2^{0,1} = 0,$$

we have $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$. Since $\mathcal{H}^m(E^\bullet) \in \mathfrak{Coh}(X)$, we have

$$(1.10.39) \quad \text{Ext}_{\mathcal{O}_{X,x_0}}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))_{x_0} = 0.$$

The by Lemma 1.10.24, $\mathcal{H}^m(E^\bullet)_{x_0}$ is free \mathcal{O}_{X,x_0} -module. Since freeness is an open property, there is a non-empty open (dense) subset U of X containing x_0 such that $U \subseteq \text{Supp}(\mathcal{H}^m(E^\bullet))$ and $\mathcal{H}^m(E^\bullet)|_U$ is a free \mathcal{O}_U -module. Since X is irreducible, $\mathcal{H}^m(E^\bullet)$ is locally free on X .

Step 4. *We show that, $\mathcal{H}^m(E^\bullet)$ is a line bundle on X .*

Since $\text{Supp}(\mathcal{H}^m(E^\bullet)) = X$, there is a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x)$, for each $x \in X$. Then following argument of Step 1, we have

$$(1.10.40) \quad \text{Hom}(E^\bullet, k(x)[-m]) = \text{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \neq 0, \quad \forall x \in X.$$

Now it follows from Definition 1.10.23 of invertible objects that

$$(1.10.41) \quad n_{k(x)} = -m, \quad \forall x \in X.$$

If r is the rank of $\mathcal{H}^m(E^\bullet)$, we have

$$\begin{aligned} k(x) &= \text{Hom}(E^\bullet, k(x)[-m]) = \text{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \\ (1.10.42) \quad &= \text{Hom}(\mathcal{O}_{X,x}^{\oplus r}, k(x)) \cong k(x)^{\oplus r}. \end{aligned}$$

Therefore, $r = 1$, and hence $\mathcal{H}^m(E^\bullet)$ is a line bundle on X .

Step 5. *We show that, $\mathcal{H}^i(E^\bullet) = 0$, for all $i < m$.*

From the spectral sequence in (1.10.29), we have

$$\begin{aligned} E_2^{q,-m} &= \text{Hom}(\mathcal{H}^m(E^\bullet), k(x)[q]) \\ &= \text{Ext}^q(\mathcal{H}^m(E^\bullet), k(x)) \\ (1.10.43) \quad &\cong H^q(X, \mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))) = 0, \quad \forall q > 0, \end{aligned}$$

because $\mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))$ is a skyscraper sheaf supported on $\{x\}$, and hence is Γ -acyclic.

Suppose that $i < m$. Then it follows from Definition 1.10.23 and (1.10.41) that

$$(1.10.44) \quad E^{-i} = \text{Hom}(E^\bullet, k(x)[-i]) = 0, \quad \forall x \in X.$$

Now to show $\mathcal{H}^i(E^\bullet) = 0$, it is enough to show that

$$(1.10.45) \quad E_2^{0,-i} = \text{Hom}(\mathcal{H}^i(E^\bullet), k(x)) = 0, \quad \forall x \in X.$$

Since $E^{-i} = 0$, if we can show that

$$(1.10.46) \quad E_2^{0,-i} = E_\infty^{0,-i},$$

then from the spectral sequence (1.10.29) we would get $E_2^{0,-i} = 0$. We prove this by induction on i .

If $i = m - 1$, then $E_2^{2,-i-1} = E_2^{2,-m} = 0$ by (1.10.43). Since negative indexed Ext groups between two coherent sheaves are zero, we have $E_2^{-2,-(m-2)} = 0$. Then (1.10.46), for the case $i = m - 1$, follows from the complex

$$\cdots \rightarrow 0 = E_2^{-2,-(m-2)} \xrightarrow{d} E_2^{0,1-m} \xrightarrow{d} E_2^{2,-m} = 0 \rightarrow \cdots.$$

Therefore, $\mathcal{H}^{m-1}(E^\bullet) = 0$. Assume inductively that $\mathcal{H}^i(E^\bullet) = 0$, for all $i \in \mathbb{Z}$, with $i_0 < i \leq m - 1$. Then putting $m = i_0 + 1$ in (1.10.43) and using $\mathcal{H}^{i_0+1}(E^\bullet) = 0$, we have $E_2^{2,-i_0-1} = 0$. Then (1.10.46) follows from the complex

$$\cdots \rightarrow 0 = E_2^{-2,1-i_0} \xrightarrow{d} E_2^{0,-i_0} \xrightarrow{d} E_2^{2,-i_0-1} = 0 \rightarrow \cdots.$$

This completes induction. Therefore, $\mathcal{H}^i(E^\bullet) = 0$, $\forall i < m$, and hence for all $i \neq m$.

Step 6. Now we prove converse part of the Proposition 1.10.25. Suppose that any point like object $P \in D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and

$\ell \in \mathbb{Z}$. Let L be a line bundle on X , and $m \in \mathbb{Z}$. Then from Definition 1.10.23 we get

$$\begin{aligned} \text{Hom}(L[m], P[i]) &\cong \text{Hom}(L, k(x)[\ell + i - m]) \\ &= \text{Ext}^{\ell+i-m}(\mathcal{O}_X, L^\vee \otimes k(x)) \\ (1.10.47) \quad &\cong H^{\ell+i-m}(X, L^\vee \otimes k(x)), \end{aligned}$$

which vanishes except for $i = m - \ell$. Then we set $n_P := m - \ell$. This completes the proof. \square

Remark 1.10.48. Let \mathcal{D} be a (tensor) triangulated category admitting a Serre functor S . If we naively define *Picard group* of \mathcal{D} to be the set $\text{Pic}(\mathcal{D})$ of all invertible objects in \mathcal{D} , then for a smooth projective k -variety X with ω_X or ω_X^\vee ample, we have $\text{Pic}(D^b(X)) = \text{Pic}(X) \times \mathbb{Z}$.

Definition 1.10.49. A collection Ω of objects in a triangulated category \mathcal{D} is called a *spanning class of \mathcal{D}* (or *spans \mathcal{D}*) if for all $B \in \mathcal{D}$ the following conditions hold.

- (i) If $\text{Hom}(A, B[i]) = 0$, $\forall A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
- (ii) If $\text{Hom}(B[i], A) = 0$, $\forall A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Remark 1.10.50. If a triangulated category \mathcal{D} admits a Serre functor, then the conditions (i) and (ii) in the above Definition 1.10.49 are equivalent.

Proposition 1.10.51. Let X be a smooth projective k -variety. Then the objects of the form $k(x)$, with $x \in X$ a closed point, spans $D^b(X)$.

Proof. It is enough to show that, for any non-zero object $E^\bullet \in D^b(X)$ there exists closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$\text{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0 \quad \text{and} \quad \text{Hom}(k(x_2), E^\bullet[i_2]) \neq 0.$$

Since $\text{Hom}(k(x_2), E^\bullet[i_2]) \cong \text{Hom}(E^\bullet, k(x_2)[\dim(X) - i_2])^*$ by Serre duality, it is enough to show that $\text{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0$, for some closed point $x \in X$ and some $i \in \mathbb{Z}$. Consider the spectral sequence

$$(1.10.52) \quad E_2^{p,q} := \text{Hom}(\mathcal{H}^{-q}(E^\bullet), k(x)[p]) \implies \text{Hom}(E^\bullet, k(x)[p+q]).$$

Let $m := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then the differential $d_r^{0,-m} : E_r^{0,-m} \rightarrow E_r^{r,1-m-r}$ is 0, for all $r \geq 2$. Since $\mathcal{H}^{-q}(E^\bullet), k(x) \in \mathfrak{Coh}(X)$, we have

$$E_2^{p,q} = \text{Ext}^p(\mathcal{H}^{-q}(E^\bullet), k(x)) = 0, \quad \forall p < 0.$$

Therefore, all differentials with target $E_r^{0,-m}$ are also 0. Thus, $E_\infty^{0,-m} = E_2^{0,-m}$.

If we choose a closed point $x \in \text{Supp}(\mathcal{H}^m(E^\bullet))$, then

$$E_\infty^{0,-m} = E_2^{0,-m} = \text{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \neq 0,$$

and hence $\text{Hom}(E^\bullet, k(x)[-m]) \neq 0$ by (1.10.52). \square

Remark 1.10.53. Spanning class in $D^b(X)$ is not unique. For a smooth projective k -variety X , for a choice of an ample line bundle L on X , we shall see later that, $\{L^{\otimes i} : i \in \mathbb{Z}\}$ forms a spanning class in $D^b(X)$.

Now we are in a position to prove the reconstruction theorem of Bondal and Orlov in the light of the following well-known results.

Proposition 1.10.54. [Sta20, Tag01PR] *Let X be a quasi-compact scheme. Let L be an invertible sheaf of \mathcal{O}_X -modules on X . Consider the graded algebra $S := \bigoplus_{i \geq 0} H^0(X, L^i)$, and its ideal $S_+ = \bigoplus_{i > 0} H^0(X, L^i)$. For each homogeneous element $s \in H^0(X, L^i)$, for $i > 0$, let $X_s := \{x \in X : s_x \notin \mathfrak{m}_x L_x^i\}$. Then the following are equivalent.*

- (i) L is ample.
- (ii) For each $x \in X$, there is $i \geq 1$ and a section $s \in H^0(X, L^i)$ such that X_s is an affine open neighbourhood of x .
- (iii) The collection of open sets X_s , with $s \in S_+$ homogeneous, covers X , and the natural morphism $X \rightarrow \mathbf{Proj}(S)$ is an open immersion.
- (iv) $\{X_s : s \text{ is a homogeneous element of } S_+\}$ forms a basis for Zariski topology of X .

Proposition 1.10.55. *Let X be a smooth projective k -variety. Let L be a line bundle on X . If L or L^\vee is ample, then the natural morphism of k -schemes*

$$X \longrightarrow \mathbf{Proj} \left(\bigoplus_n H^0(X, L^n) \right)$$

is an isomorphism.

Theorem 1.10.5 (Bondal–Orlov). *Let X be a smooth projective variety over k with canonical line bundle ω_X . Assume that ω_X (resp., ω_X^\vee) is ample. Let Y be any smooth projective variety over k . If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k -varieties. In particular, ω_Y (resp., ω_Y^\vee) is ample.*

Proof. Step 1. If $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or ω_Y^\vee is ample, the theorem follows.

Indeed, assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Since F is an exact equivalence of categories, $F \circ S_X \cong S_Y \circ F$ and $\dim(X) = \dim(Y) = n$ (say), (see Proposition 1.10.1). Then we have

$$(1.10.56) \quad F(\omega_X^k) = F(S_X^k(\mathcal{O}_X))[-kn] = S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k, \quad \forall k.$$

Since F is fully faithful, we have

$$(1.10.57) \quad H^0(X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k), \quad \forall k.$$

The product structure on the graded k -algebra $\bigoplus_k H^0(X, \omega_X^k)$ can be expressed in terms of following composition: for $s_i \in H^0(X, \omega_X^{k_i})$, $i = 1, 2$, we have

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1.$$

Note that, $s_1 \cdot s_2 = s_2 \cdot s_1$ follows from the commutativity of the following diagram.

$$(1.10.58) \quad \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{s_1} & \omega_X^{k_1} \\ s_2 \downarrow & & \downarrow S_X^{k_1}(s_2)[-k_1 n] \\ \omega_X^{k_2} & \xrightarrow{S_X^{k_2}(s_1)[-k_2 n]} & \omega_X^{k_1+k_2} \end{array}$$

Similarly, we have product structure on $\bigoplus_k H^0(Y, \omega_Y^k)$. Therefore, F naturally induces an isomorphism of graded k -algebras

$$(1.10.59) \quad \tilde{F} : \bigoplus_k H^0(X, \omega_X^k) \longrightarrow \bigoplus_k H^0(Y, \omega_Y^k),$$

which induces isomorphism of k -schemes

$$(1.10.60) \quad X \xrightarrow{\cong} \mathbf{Proj}\left(\bigoplus_k H^0(X, \omega_X^k)\right) \xrightarrow{\cong} \mathbf{Proj}\left(\bigoplus_k H^0(Y, \omega_Y^k)\right) \xrightarrow{\cong} Y,$$

whenever ω_Y or its dual ω_Y^\vee is ample (c.f., Proposition 1.10.55). Therefore, it is enough to show that $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or ω_Y^\vee is ample whenever ω_X or ω_X^\vee is ample.

Step 2. We can assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Indeed, it follows from Definition 1.10.8 and Definition 1.10.23 that an exact equivalence $F : D^b(X) \rightarrow D^b(Y)$ induce bijections

$$(1.10.61) \quad \begin{array}{ccc} \{\text{point like objects of } D^b(X)\} & \xrightarrow[\cong]{} & \{\text{point like objects of } D^b(Y)\} \\ \parallel & & \uparrow (*) \\ \{k(x)[m] : x \in X_{\text{closed}} \text{ and } m \in \mathbb{Z}\} & & \{k(y)[m] : y \in Y_{\text{closed}} \text{ and } m \in \mathbb{Z}\} \end{array}$$

and

$$(1.10.62) \quad \begin{array}{ccc} \{\text{invertible objects of } D^b(X)\} & \xrightarrow[\cong]{} & \{\text{invertible objects of } D^b(Y)\} \\ \parallel & & \downarrow (**) \\ \{L[m] : L \in \text{Pic}(X) \text{ and } m \in \mathbb{Z}\} & & \{M[m] : M \in \text{Pic}(Y) \text{ and } m \in \mathbb{Z}\}, \end{array}$$

where X_{closed} (resp., Y_{closed}) is the set of all closed points of X (resp., Y), and the vertical inclusions and equalities are given by Proposition 1.10.20 and Proposition 1.10.25. Therefore, $F(\mathcal{O}_X) = M[m]$, for some $M \in \text{Pic}(Y)$ and some $m \in \mathbb{Z}$.

If $F(\mathcal{O}_X) \neq \mathcal{O}_Y$, replacing F with the following composite functor

$$(1.10.63) \quad D^b(X) \xrightarrow{F} D^b(Y) \xrightarrow{(M^\vee \otimes -)[-m]} D^b(Y),$$

which is an exact equivalence sending \mathcal{O}_X to \mathcal{O}_Y , we may assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Therefore, it remains to show is that ω_Y or its dual is ample.

Step 3. We establish bijections $X_{\text{closed}} \xleftarrow{F} Y_{\text{closed}}$ and $\text{Pic}(X) \xleftarrow{F} \text{Pic}(Y)$.

Using the equivalence F , we first show that the vertical inclusion $(*)$ in the diagram (1.10.61) is a bijection. This immediately imply that the vertical inclusion $(**)$ in the diagram (1.10.62) is bijective by Proposition 1.10.25. Then Step 3 will follow.

By horizontal bijection in the diagram (1.10.61), for any closed point $y \in Y$ there is a closed point $x_y \in X$ and $m_y \in \mathbb{Z}$ such that $F(k(x_y)[m_y]) \cong k(y)$. Suppose on the contrary that there is a point like object $P \in D^b(Y)$, which is not of the form $k(y)[m]$, for any closed point $y \in Y$ and integer m . Because of bijection in (1.10.61), there is a unique closed point $x_P \in X$ and integer m_P such that $F(k(x_P)[m_P]) \cong P$. Then $x_P \neq x_y$, for all closed point $y \in Y$. Hence, for any closed point $y \in Y$ and any integer m , we have

$$\begin{aligned} \text{Hom}(P, k(y)[m]) &= \text{Hom}(F(k(x_P)[m_P]), k(y)[m]) \\ (1.10.64) \quad &= \text{Hom}(k(x_P)[m_P], k(x_y)[m_y + m]) \\ &= \text{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) = 0, \end{aligned}$$

because $k(x_P)$ and $k(x_y)$ being skyscraper sheaves supported at different points, $\text{Ext}^i(k(x_P), k(x_y)) = 0$, for all i . Since the objects $k(y)$, with $y \in Y$ a closed point, form a spanning class of $D^b(X)$ (c.f. Definition 1.10.49), $P \cong 0$ by Proposition 1.10.51, which contradicts our assumption that P is a point like object in $D^b(Y)$. Therefore, point like objects of $D^b(Y)$ are exactly of the form $k(y)[m]$, for $y \in Y$ a closed point and $m \in \mathbb{Z}$.

Note that, for any closed point $x \in X$, there is a closed point $y_x \in Y$ such that $F(k(x)) \cong k(y_x)[m_x]$, for some $m_x \in \mathbb{Z}$. Since F is fully faithful and $F(\mathcal{O}_X) = \mathcal{O}_Y$, we have $\text{Hom}(\mathcal{O}_X, k(x)) = \text{Hom}(\mathcal{O}_Y, k(y_x)[m_x]) = \text{Ext}^{m_x}(\mathcal{O}_Y, k(y_x)) \neq 0$. This forces $m_x = 0$, and hence $F(k(x)) \cong k(y_x)$ (no shift!). This immediately imply that, for any $L \in \text{Pic}(X)$, $F(L) \cong M$, for some $M \in \text{Pic}(Y)$. Indeed, from bijections in the diagram (1.10.62), we find unique $M \in \text{Pic}(Y)$ and $m_L \in \mathbb{Z}$ such that $F(L) \cong M[m_L]$. Take closed points $x \in X$ and $y_x \in Y$ such that $F(k(x)) \cong k(y_x)$. Then

$$\begin{aligned} \text{Ext}^{-m_L}(M, k(y_x)) &= \text{Hom}(M, k(y_x)[-m_L]) = \text{Hom}(M[m_L], k(y_x)) \\ &= \text{Hom}(F(L), F(k(x))) = \text{Hom}(L, k(x)) \neq 0. \end{aligned}$$

This forces $m_L = 0$.

Step 4. Recovering Zariski topology from derived category to conclude ampleness.

Let Z be a quasi-compact k -scheme. Denote by Z_0 the subset of all closed points of Z . Take line bundles L_1 and L_2 on Z , and take $\alpha \in \text{Hom}(L_1, L_2) = H^0(X, L_1^\vee \otimes L_2)$.

For each closed point $z \in Z$, let

$$(1.10.65) \quad \alpha_z^* : \text{Hom}(L_2, k(z)) \longrightarrow \text{Hom}(L_1, k(z))$$

be the homomorphism induced by α . Then $U_\alpha := \{z \in Z : \alpha_z^* \neq 0\}$ is a Zariski open subset of Z , and hence $U_\alpha \cap Z$ is open in Z_0 .

Fix a line bundle $L_0 \in \text{Pic}(X)$. Then it follows from Proposition 1.10.54 that the collection of all such U_α , where $\alpha \in H^0(X, L_0^n)$ and $n \in \mathbb{Z}$, forms a basis for the Zariski topology on Z if and only if either L_0 or L_0^\vee is ample.

By Step 3, the exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$ sends closed points of X to closed points of Y bijectively, and sends line bundles on X to line bundles on Y bijectively. In particular, $F(\omega_X^i) \cong \omega_Y^i$, for all $i \in \mathbb{Z}$. Then the natural isomorphisms $H^0(X, \omega_X^i) \cong H^0(Y, \omega_Y^i)$, $\forall i \in \mathbb{Z}$, give rise to a bijection between the collection of open subsets

$$(1.10.66) \quad \mathcal{B}_X := \{U_\alpha : \alpha \in H^0(X, \omega_X^i), i \in \mathbb{Z}\} \text{ and } \mathcal{B}_Y := \{U_\alpha : \alpha \in H^0(Y, \omega_Y^i), i \in \mathbb{Z}\}.$$

Since ω_X^i is ample, for some $i \in \{-1, 1\}$, \mathcal{B}_X is a basis for the Zariski topology on X , and hence $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in H^0(X, \omega_X^i), i \in \mathbb{Z}\}$ is a basis for the Zariski topology on X_0 . Therefore, $\mathcal{B}_{Y_0} := \{U_\alpha \cap Y_0 : \alpha \in H^0(Y, \omega_Y^i), i \in \mathbb{Z}\}$ is a basis for the Zariski topology on Y_0 . Since Y_0 is Zariski dense in Y , the collection \mathcal{B}_Y is a basis for the Zariski topology on Y . Therefore, ω_Y^i is ample, for some $i \in \mathbb{Z}$. This completes the proof. \square

Remark 1.10.67. There is a more geometric proof of Step 4 above for ampleness of ω_Y or ω_Y^\vee in case k is algebraically closed. The idea is to use the fact that line bundle is very ample if and only if it separates points and tangent vectors.

Definition 1.10.68. Let X be a smooth projective k -variety, and $L \in \text{Pic}(X)$. We say that,

- (i) L separates points if for any two points $p, q \in X$, there is a section $s \in H^0(X, L)$ such that $s(p) = 0$ and $s(q) \neq 0$.
- (ii) L separates tangent vectors if for any point $p \in X$ and any tangent vector $v \in T_p X = (\mathfrak{m}_p / \mathfrak{m}_p^2)^*$, there is a non-zero section $s \in H^0(X, L)$ such that $v \notin T_p V$, where V is the divisor of zero locus of s .

1.11. Grothendieck group.

Definition 1.11.1. Let \mathcal{A} be a small abelian category. The *Grothendieck group* of \mathcal{A} , denoted by $K_0(\mathcal{A})$, is the quotient of the free abelian group generated by the set of all isomorphism classes of objects of \mathcal{A} by its normal subgroup generated by all elements $[B] - [A] - [C]$, where $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in \mathcal{A} .

Let \mathcal{A} be an abelian category. For an object $A \in \mathcal{A}$, we denote by $A^\bullet \in \text{Kom}(\mathcal{A})$ the complex defined by

$$A^i = \begin{cases} A, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

We also denote by $A^\bullet \in D^b(\mathcal{A})$, the object in $D^b(\mathcal{A})$ represented by the complex A^\bullet .

Definition 1.11.2. Let \mathcal{T} be a small triangulated category. For example, \mathcal{T} can be the bounded derived category $D^b(\mathcal{A})$ of a small abelian category \mathcal{A} . Let $F(\mathcal{T})$ be the free abelian group generated by the set of all objects of \mathcal{T} . Let $\mathcal{R}(\mathcal{T})$ be the normal subgroup of $F(\mathcal{T})$ generated by the elements $[A] + [C] - [B] \in F(\mathcal{T})$, whenever there is a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in \mathcal{T} . The quotient abelian group

$$(1.11.3) \quad K_0(\mathcal{T}) := F(\mathcal{T})/\mathcal{R}(\mathcal{T})$$

is called the *Grothendieck group* of \mathcal{T} .

Remark 1.11.4. It follows from the axiom (TR3) in Definition 1.2.4 that $A[1] = 0$ in $K_0(\mathcal{T})$.

The following proposition establishes relation between $K_0(\mathcal{A})$ and $K_0(D^b(\mathcal{A}))$.

Proposition 1.11.5. Let \mathcal{A} be a small abelian category, and let $D^b(\mathcal{A})$ be the bounded derived category of \mathcal{A} . Then the natural functor $\iota : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ defined by sending an object $A \in \mathcal{A}$ to $A^\bullet \in D^b(\mathcal{A})$ induces an isomorphism of their Grothendieck groups $K_0(\mathcal{A}) \xrightarrow{\cong} K_0(D^b(\mathcal{A}))$.

1.12. Chern characters for complexes. Let X be a smooth proper \mathbb{C} -variety. For each integer j , with $0 \leq j \leq \dim_{\mathbb{C}}(X)$, let $\text{CH}^j(X)$ be the Chow group of j -cocycles on X . For any bounded complex of coherent sheaf of \mathcal{O}_X -modules $\mathcal{E}^\bullet \in \text{Kom}^b(\mathcal{Coh}(X))$, there is a bounded complex E^\bullet of locally free coherent sheaves of \mathcal{O}_X -modules quasi-isomorphic to \mathcal{E}^\bullet . Then we define the j -th Chern character of \mathcal{E}^\bullet by

$$(1.12.1) \quad \text{ch}_j(\mathcal{E}^\bullet) := \sum_i (-1)^i \text{ch}_j(E^i) \in \text{CH}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proposition 1.12.2. The map $\mathcal{E}^\bullet \longmapsto \text{ch}_j(\mathcal{E}^\bullet)$ defined in (1.12.1) induces a map

$$\text{ch}_j : K_0(D^b(X)) \longrightarrow \text{CH}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 1.12.3. Since X is smooth, for any $\mathcal{E}^\bullet \in \text{Kom}(\mathcal{Mod}(X))$, there is a bounded complex E^\bullet of locally free coherent sheaves of \mathcal{O}_X -modules quasi-isomorphic to \mathcal{E}^\bullet . Then the above definition of Chern character and Chern class map extend to define maps from $D(\mathcal{Mod}(X))$.

Remark 1.12.4. To define Chern class and Chern character of objects of a general triangulated category \mathcal{A} , we need to give compatible symmetric monoidal structure on \mathcal{A} (such a category is called a tensor triangulated category, c.f. Section §2). On a tensor triangulated category, one can define notion of algebraic cycle, Chow group etc., which recovers the classical algebraic cycles and Chow group when we take \mathcal{A} modelled on a scheme X [Kle16, Theorem 3.2.6]. In fact, a tensor triangulated category associated with a scheme completely recovers the scheme [Bal10].

2. TENSOR TRIANGULATED CATEGORY

3. BRIDGELAND STABILITY

The definition of stability condition generalized in different ways from curve to higher dimensional varieties. We follow [MS17, BBHR09, Huy06].

3.1. Stability condition in an abelian category.

Definition 3.1.1. Let \mathcal{C} be an additive category. A *subobject* of an object E in \mathcal{C} is a monomorphism $\iota : F \rightarrow E$ in \mathcal{C} .

Remark 3.1.2. Note that, “being a subobject” is a transitive relation on $\text{Ob}(\mathcal{A})$. (This may fails to hold if we consider equivalence class of monomorphisms instead of just monomorphism).

Let \mathcal{A} be an abelian category. Denote by $K_0(\mathcal{A})$ the Grothendieck group of \mathcal{A} (c.f., Definition 1.11.1). For any complex number z , we denote by $\text{Im}(z)$ (resp., $\text{Re}(z)$) the *imaginary part* (resp., the *real part*) of z .

Definition 3.1.3. A *stability function* on \mathcal{A} is an additive group homomorphism

$$Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

such that for any non-zero object $E \in \mathcal{A}$, we have $\text{Im}(Z(E)) \geq 0$, and if $\text{Im}(Z(E)) = 0$, then $\text{Re}(Z(E)) < 0$.

Note. Here “ $\text{Im}(Z(a)) \geq 0, \forall a \in \mathcal{A}$ ” **does not imply** that “ $\text{Im}(Z(a)) \geq 0, \forall a \in K_0(\mathcal{A})$ ”.

Given a stability function $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$ as in Definition 3.1.3, we may think of

$$\deg_Z(E) := -\text{Re}(Z(E)) \quad \text{and} \quad \text{rk}_Z(E) := \text{Im}(Z(E))$$

to be the *generalized degree* and the *generalized rank* of E with respect to Z , respectively. Define the *generalized slope* of $E \in \mathcal{A} \setminus \{0\}$ with respect to Z by

$$(3.1.4) \quad \mu_Z(E) := \begin{cases} \frac{\deg_Z(E)}{\operatorname{rk}_Z(E)}, & \text{if } \operatorname{rk}_Z(E) \neq 0, \text{ and} \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 3.1.5. Let X be a smooth projective curve defined over an algebraically closed field k . Let $\mathcal{Coh}(X)$ be the category of coherent sheaves of \mathcal{O}_X -modules on X . Let $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$ be the additive group homomorphism defined by sending a non-zero object $E \in \mathcal{A}$ to

$$Z(E) := -\deg(E) + \sqrt{-1} \cdot \operatorname{rk}(E) \in \mathbb{C}.$$

Clearly, Z is a stability function on \mathcal{A} . Note that, $\mu_Z(E)$ coincides with the usual slope $\mu(E) := \deg(E)/\operatorname{rk}(E)$ of E , and hence in this case, Z -(semi)stability coincides with the usual slope (semi)stability of coherent sheaves on X .

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k . Fix an ample class $\omega \in \operatorname{Amp}(X) \subseteq N^1(X) := \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and a divisor class $B \in N^1(X)$.

Definition 3.1.6 (K. Matsuki, R. Wentworth). The *B -twisted Chern character* of $E \in \mathcal{Coh}(X)$ is defined by

$$\operatorname{ch}^B(E) := \operatorname{ch}(E) \cdot e^{-B} = \sum_{i \geq 0} \operatorname{ch}_i(E) \cdot \sum_{j \geq 0} \frac{(-1)^j}{j!} B^j.$$

Thus, for $i \geq 0$, the *B -twisted i -th Chern character of E* , denoted $\operatorname{ch}_i^B(E)$, are given by

$$\begin{aligned} \operatorname{ch}_0^B(E) &= \operatorname{ch}_0(E) = \operatorname{rk}(E), \\ \operatorname{ch}_1^B(E) &= \operatorname{ch}_1(E) - \operatorname{ch}_0(E) \cdot B, \\ \operatorname{ch}_2^B(E) &= \operatorname{ch}_2(E) - \operatorname{ch}_1(E) \cdot B + \frac{1}{2} \operatorname{rk}(E) \cdot B^2, \end{aligned}$$

and so on. Note that, taking $B = 0$, we get back the usual Chern characters.

Define an additive group homomorphism

$$Z_{\omega, B} : K_0(\mathcal{Coh}(X)) \rightarrow \mathbb{C}$$

by sending a non-zero object E of $\mathcal{Coh}(X)$ to the complex number

$$(3.1.7) \quad Z_{\omega, B}(E) := -\omega^{n-1} \cdot \operatorname{ch}_1^B(E) + \sqrt{-1} \cdot \omega^n \cdot \operatorname{ch}_0^B(E).$$

If T is a *torsion* coherent sheaf on X supported in dimension $\leq n-2$, then $\operatorname{rk}(T) = 0$, and the line bundle $\det(E)$ admits a nowhere vanishing global section (c.f., [Kob87,

Proposition 5.6.14]). Then $\det(T) \cong \mathcal{O}_X$, and hence $\mathrm{ch}_1(T) = 0$. Therefore, $Z_{\omega,B}(T) = 0$. This shows that, $Z_{\omega,B}$ is **not a stability function**.

Remark 3.1.8. Let $\mathfrak{Coh}_{\leq n-2}(X)$ be the full subcategory of coherent sheaves on X whose supports have dimension $\leq n-2$, and let \mathcal{A} be the localized category

$$\mathfrak{Coh}_{n,n-2}(X) = \mathfrak{Coh}(X) / \mathfrak{Coh}_{\leq n-2}(X).$$

Then the function

$$Z_{\omega,B} : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

as defined in (3.1.7) above, is a stability function.

Definition 3.1.9. A *stability condition* is a pair (\mathcal{A}, Z) , where \mathcal{A} is an abelian category and $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a stability function such that for any non-zero object E of \mathcal{A} , there is a filtration of E by its subobjects

$$(3.1.10) \quad 0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E,$$

such that all E_i/E_{i-1} are Z -semistable, and their Z -slopes satisfies

$$\mu_Z(E_1) > \mu_Z(E_2/E_1) > \cdots > \mu_Z(E_\ell/E_{\ell-1}).$$

Such a filtration (3.1.10) is known as *Harder-Narasimhan filtration* of E .

Proposition 3.1.11. Let \mathcal{A} be an abelian category. Given a stability function $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$, Harder-Narasimhan filtration of an object $E \in \mathcal{A}$, if it exists, is unique up to isomorphism in \mathcal{A} .

Existence of Harder-Narasimhan filtration requires some additional assumption on the category \mathcal{A} and the stability function Z . For this we need some definitions.

Definition 3.1.12. An additive category \mathcal{A} is said to be *noetherian* if for any object $E \in \mathcal{A}$, and any any ascending chain of subobjects

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E,$$

of E , there is an integer $i_0 \geq 0$ such that $E_i = E_{i+1}$, for all $i \geq i_0$.

Lemma 3.1.13. Let \mathcal{A} be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$ be a stability function. If the image of the imaginary part of Z ,

$$\mathrm{Im}(Z) : K_0(\mathcal{A}) \rightarrow \mathbb{R},$$

is discrete in \mathbb{R} , then for any object $E \in \mathcal{A}$, there is a number $D_E \in \mathbb{R}$ such that for any subobject F of E in \mathcal{A} , we have $D(F) \leq D_E$.

Proof. Suppose that $R(E) = 0$. Then $D(E) > 0$. Given a subobject $F \subset E$, from the exact sequence $0 \rightarrow F \rightarrow E \rightarrow E/F \rightarrow 0$, we have $R(E) = R(F) + R(E/F)$ and $D(E) = D(F) + D(E/F)$, since both $D := -\mathrm{Re} Z$ and $R := \mathrm{Im} Z$ are homomorphisms

of additive groups. Since Z is a stability function, we have $R(F), R(E/F) \geq 0$. Then $R(F) = 0$, and hence $D(F) > 0$. Thus, $0 < D(F) \leq D(E)$ (note that, $D(E/F) > 0$ if $E/F = 0$).

Assume that $R(E) > 0$. Suppose on the contrary that there is an infinite sequence of subobjects $\{F_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} D(F_n) = +\infty$. If for some $n \in \mathbb{N}$, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ implies $D(E/F_n) \geq 0$, and so $D(F_n) \leq D(E)$. So, without loss of generality, we may assume that $R(F_n) < R(E)$, for all $n \in \mathbb{N}$.

□

Theorem 3.1.14. *Let \mathcal{A} be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$ be a stability function. Assume that $R := \text{Im}(Z) : K_0(\mathcal{A}) \rightarrow \mathbb{R}$ has discrete image in \mathbb{R} . Then any non-zero object $E \in \mathcal{A}$ admits a unique Harder-Narasimhan filtration.*

3.2. Stability conditions in a triangulated category. The notion of stability conditions in triangulated category is introduced by Tom Bridgeland in his famous paper [Bri07] inspired by the works of Douglas [Dou01b, Dou02, Dou01a].

Recall that, a *triangulated category* is an additive category \mathcal{A} together with an additive equivalence (shift functor)

$$(3.2.1) \quad T : \mathcal{A} \longrightarrow \mathcal{A},$$

and a set of *distinguished triangles*

$$(3.2.2) \quad A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A),$$

satisfying the following axioms (TR1) – (TR4) (c.f. Definition 1.2.4). We shall always assume that \mathcal{A} is essentially small (i.e., \mathcal{A} is equivalent to a small category).

Notation 3.2.3. *For any additive subcategory \mathcal{B} of a triangulated category \mathcal{A} , we denote by $\mathcal{B}[1]$ the full subcategory of \mathcal{A} , whose objects are of the form $A[1] \in \mathcal{A}$, with $A \in \mathcal{B}$.*

Definition 3.2.4. A *stability condition* on a triangulated category (\mathcal{A}, T) is a pair (Z, \mathcal{P}) , where $Z : K_0(\mathcal{A}) \rightarrow \mathbb{C}$ is an additive group homomorphism (called the *central charge*) and full subcategories $\mathcal{P}(\phi)$ of \mathcal{A} , for each $\phi \in \mathbb{R}$, satisfying the following axioms:

- (i) if $E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \cdot \exp(2\pi i \phi)$, for some $m(E) \in \mathbb{R}_{>0}$.
- (ii) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] := T(\mathcal{P}(\phi))$, for all $\phi \in \mathbb{R}$.
- (iii) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $\text{Hom}(A_1, A_2) = 0$.
- (iv) for each object $E \in \mathcal{A}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_\ell,$$

(depending on E) and a sequence of distinguished triangles

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \longrightarrow E_{\ell-1} \longrightarrow E_{\ell} = E \\
 & \swarrow \text{---} & \searrow & \swarrow \text{---} & \searrow & \swarrow \text{---} & \searrow \\
 & & A_1 & & A_2 & & \cdots & & A_{\ell}
 \end{array}$$

with $A_j \in \mathcal{P}(\phi_j)$, for all $j \in \{1, \dots, \ell\}$.

Let \mathcal{A} be an abelian category and $D^b(\mathcal{A})$ be the bounded derived category of \mathcal{A} .

Proposition 3.2.5. *Let \mathcal{H} be a full additive subcategory of $D^b(\mathcal{A})$. Then \mathcal{H} is a heart of a bounded t -structure on $D^b(\mathcal{A})$ if and only if \mathcal{H} satisfies following properties.*

- (1) *for integers $i > j$ and objects $A, B \in \mathcal{H}$, we have $\text{Hom}(A[i], B[j]) = \text{Hom}(A, B[j-i]) = 0$, and*
- (2) *for $E \in D^b(\mathcal{A})$, there are integers $k_1 > \dots > k_m$, collection of objects $E_i \in D^b(\mathcal{A})$ and $A_i \in \mathcal{H}$, for each $i = 1, \dots, m$, and a sequence of distinguished triangles*

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots \longrightarrow E_{m-1} \longrightarrow E_m = E \\
 & \swarrow \text{---} & \downarrow & \swarrow \text{---} & \downarrow & \swarrow \text{---} & \downarrow & \swarrow \text{---} & \downarrow \\
 & & A_1[k_1] & & A_2[k_2] & & A_{m-1}[k_{m-1}] & & A_m[k_m]
 \end{array}$$

Definition 3.2.6. A Bridgeland stability condition on a triangulated category \mathcal{A} is a pair $\sigma = (\mathcal{P}, Z)$, where

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5. MIRROR SYMMETRY

5.1. Fukaya category.

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