# A note on Bondal-Orlov's reconstruction theorem

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ABSTRACT. Let $X$ be a smooth projective irreducible finite type scheme defined over a field $k$, which is not necessarily algebraically closed. Denote by $\omega_X$ the dualizing sheaf on $X$. In their famous 2001 paper [BO01], Bondal and Orlov showed that $X$ can be reconstructed from its bounded derived category $D^b(X)$ of coherent sheaves on it whenever either $\omega_X$ or its dual is ample. In this expository article, we explain the proof of the reconstruction theorem due to Bondal and Orlov. We follow [Huy06].

## 1. INTRODUCTION

A famous theorem of Gabriel says that two smooth projective $k$-varieties $X$ and $Y$ are isomorphic if and only if there is an equivalence of categories $\mathfrak{Coh}(X)$ with $\mathfrak{Coh}(Y)$. Given two abelian categories $\mathcal{A}$ and $\mathcal{B}$, one can have an equivalence of derived categories $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$ even if $\mathcal{A}$ and $\mathcal{B}$ are not equivalent. Therefore, one can ask when can we expect an isomorphism of varieties from an equivalence of their bounded derived categories of coherent sheaves? In [Muk81], Mukai established an equivalence of categories $D^b(A) \simeq D^b(\check{A})$, where $A$ is an abelian variety and $\check{A}$ its dual abelian variety. Therefore, equivalence between bounded derived category of coherent sheaves fails to ensure isomorphism of varieties, in general. It should be noted that the canonical line bundle on $A$ is trivial.

In their famous paper [BO01], Bondal and Orlov shows how to reconstruct a smooth projective variety $X$ from $D^b(X)$ when $\omega_X$ or its dual is ample. More precisely,

**Theorem 1.0.1** (Bondal–Orlov)**.** *Let $X$ be a smooth projective variety over $k$ with canonical line bundle $\omega_X$. Assume that $\omega_X$ (resp., $\omega_X^\vee$) is ample. Let $Y$ be any smooth projective variety over $k$. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as $k$-varieties. In particular, $\omega_Y$ (resp., $\omega_Y^\vee$) is ample.*

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The main idea behind the proof is to "cohomologically" characterize closed points, invertible sheaves and Zariski topology of a smooth projective $k$-variety, and for this, the main technical tool for us is the Serre functor $(- \otimes \omega_X)[\dim_k(X)]$ on $D^b(X)$ (c.f. Definition 2.2.1). Then we show that, $F$ defines a bijection between the set of closed points of $X$ with that of $Y$, and sends $\omega_X$ to $\omega_Y$. Then we recover the Zariski topology on $Y$ from that of $X$ using amplensee of $\omega_X$ or its dual, and finally establish an isomorphism of $k$-schemes from $X$ onto $Y$.

## 2. Preliminaries from derived category

2.1. **Derived category.** Let $\mathcal{A}$ be an abelian category. For example, $\mathcal{A} = \mathfrak{Coh}(X)$, the category of coherent sheaves on $X$. Denote by $\mathcal{K}om(\mathcal{A})$ the category of complexes of $\mathcal{A}$. its objects are of the form

$$E^\bullet : \quad \cdots \to E^{i-1} \xrightarrow{d_E^{i-1}} E^i \xrightarrow{d_E^i} E^{i+1} \to \cdots ,$$

where $E^i \in \mathcal{A}$ and $d_E^i \circ d_E^{i-1} = 0$, for all $i \in \mathbb{Z}$. Such a complex $E^\bullet$ is said to be *bounded* if $E^i = 0$, for $|i| \gg 0$. Denote by $\mathcal{K}om^b(\mathcal{A})$ the full subcategory of $\mathcal{K}om(\mathcal{A})$, whose objects are bounded complexes. Note that, both $\mathcal{K}om(\mathcal{A})$ and $\mathcal{K}om^b(\mathcal{A})$ are abelian. The *$i$-th cohomology* of $E^\bullet \in \mathcal{K}om(\mathcal{A})$ is the object defined by

$$\mathcal{H}^i(E^\bullet) := \frac{\mathrm{Ker}(d_E^i)}{\mathrm{image}(d_E^{i-1})} \in \mathcal{A}.$$

Given two such objects $E^\bullet, F^\bullet \in \mathcal{K}om(\mathcal{A})$, a morphism $f : E^\bullet \to F^\bullet$ in $\mathcal{K}om(\mathcal{A})$ is given by the following commutative diagram.

$$\begin{tikzcd}
E^\bullet \arrow[d, "f"] & : & \cdots \arrow[r] & E^{i-1} \arrow[r, "d_E^{i-1}"] \arrow[d, "f^{i-1}"] & E^i \arrow[r, "d_E^i"] \arrow[d, "f^i"] & E^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
F^\bullet & : & \cdots \arrow[r] & F^{i-1} \arrow[r, "d_F^{i-1}"] & F^i \arrow[r, "d_F^i"] & F^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

A morphism of complexes $f : E^\bullet \to F^\bullet$ is said to be a *quasi-isomorphism* if the induced morphism of cohomologies

$$\mathcal{H}^i(f) : \mathcal{H}^i(E^\bullet) \longrightarrow \mathcal{H}^i(F^\bullet) \tag{2.1.1}$$

is an isomorphism in $\mathcal{A}$, for all $i$. Denote by $D(\mathcal{A})$ the derived category of $\mathcal{A}$; its objects are the same as the objects of $\mathcal{K}om(\mathcal{A})$, but the morphisms in $D(\mathcal{A})$ are obtained by inverting all quasi-isomorphisms. To illustrate it little more, given any two objects $E^\bullet, F^\bullet \in D(\mathcal{A})$, a morphism $f : E^\bullet \to F^\bullet$ in $D(\mathcal{A})$ is given by a diagram (also called a *roof*) of the form

$$
\begin{tikzcd}
 & G^\bullet \arrow[dl, "qis"', "\varphi"] \arrow[dr, "\psi"] & \\
E^\bullet & & F^\bullet,
\end{tikzcd}
\tag{2.1.2}
$$

where $G^\bullet$ is an object of $D(\mathcal{A})$ and $\varphi$ is a quasi-isomorphism of complexes. In the derived category $D(\mathcal{A})$, we convert all quasi-isomorphisms to isomorphisms. Thus, in $D^b(\mathcal{A})$, we may think of the above roof as $\psi \circ \varphi^{-1} : E^\bullet \to F^\bullet$. Similarly, considering only bounded complexes in $\mathcal{A}$, we get a full subcategory $D^b(\mathcal{A})$ of $D(\mathcal{A})$, called the bounded derived category of $\mathcal{A}$. We refer the reader to [Huy06] for more details.

Both the categories $D(\mathcal{A})$ and $D^b(\mathcal{A})$ admits a natural shift functor given by sending $E^\bullet$ to the complex $E^\bullet[1]$, whose $i$-th term is $E^{i+1}$, for all $i$. The categories $D(\mathcal{A})$ and $D^b(\mathcal{A})$ are triangulated, and the shift functor is an exact equivalence of categories.

2.2. **Serre functor.** Let $k$ be a field. Let $\mathcal{A}$ be a $k$-linear additive category.

**Definition 2.2.1.** A *Serre functor* on $\mathcal{A}$ is a $k$-linear equivalence of categories

$$S : \mathcal{A} \longrightarrow \mathcal{A}$$

such that for any two objects $A, B \in \mathcal{A}$, there is a natural $k$-linear isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(B, S(A))^*,$$

which is functorial in both $A$ and $B$. We write the induced $k$-bilinear pairing as

$$\mathrm{Hom}(B, S(A)) \times \mathrm{Hom}(A, B) \longrightarrow k\,, \quad (f, g) \longmapsto \langle f | g \rangle\,.$$

**Lemma 2.2.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $k$-linear additive categories with finite dimensional Hom's. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively, then any $k$-linear equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ commutes with Serre functors (i.e., there is an isomorphism of functors $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$).*

**Proposition 2.2.3.** *Let $\mathcal{A}$ be a $k$-linear additive category. Then any two Serre functors on $\mathcal{A}$ are isomorphic.*

Let $X$ be a smooth projective $k$-variety of dimension $n \geq 1$. Note that, for any locally free coherent sheaf $E$ on $X$, the functor

$$- \otimes E : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(X)\,, \quad F \longmapsto F \otimes E$$

is exact. Let $\omega_X$ be the dualizing sheaf on $X$. Let $D^*(X) = D^*(\mathfrak{Coh}(X))$, where $* \in \{\emptyset, b\}$. Consider the composite functor

$$S_X : D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X), \tag{2.2.4}$$

where $[n] : D^*(X) \to D^*(X)$ is the $n$-th shift functor given by sending a complex $E^\bullet$ to $E^\bullet[n]$. Since both the functors $\omega_X \otimes -$ and $[n]$ are exact, their composite functor $S_X := \omega_X \otimes (-)[n]$ is exact.

**Theorem 2.2.5** (Grothendieck-Serre duality)**.** *Let $X$ be a smooth projective variety over a field $k$. Then the functor $S_X : D^b(X) \longrightarrow D^b(X)$ as defined in (2.2.4) is a Serre functor in the sense of Definition 2.2.1.*

*Proof.* Given any two objects $E^\bullet, F^\bullet \in D^b(X)$, we need to give an isomorphism of $k$-vector spaces

$$\eta_{E^\bullet,F^\bullet} : \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet))^* \tag{2.2.6}$$

which is functorial in both $E^\bullet$ and $F^\bullet$. Recall that, for $E^\bullet, F^\bullet \in D^b(X)$ we define

$$\mathrm{Ext}^i_{D^b(X)}(E^\bullet, F^\bullet) := \mathcal{H}^i\big(R\,\mathrm{Hom}^\bullet(E^\bullet, F^\bullet)\big)\,, \quad \forall\, i\,,$$

and we have a natural isomorphism

$$\mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i]) = \mathrm{Ext}^i(E^\bullet, F^\bullet), \quad \forall\, i\,. \tag{2.2.7}$$

Since $X$ is smooth and projective, choosing a resolution by complex of locally free sheaves on $X$, we may assume that $E^i$ is locally free, for all $i$. Then we have functorial isomorphisms

$$\begin{aligned}
\mathrm{Hom}^i(E^\bullet, F^\bullet) &= \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(E^j, F^{i+j}) = \bigoplus_{j\in\mathbb{Z}} H^0(X, \mathcal{H}om(E^j, F^{i+j}))\\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^n(F^{i+j}, E^j \otimes \omega_X)^*, \quad \text{by classical Serre duality theorem.}\\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(F^{i+j}, E^j \otimes \omega_X[n])^*, \quad \text{by (2.2.7).}\\
&\cong \mathrm{Hom}^{n-i}(F^\bullet, E^\bullet \otimes \omega_X)^*.
\end{aligned}$$

Hence the theorem follows. □

## 3. Bondal–Orlov's reconstruction theorem

Let $k$ be a field, not necessarily algebraically closed. By a *$k$-variety* we mean an integral separated finite type $k$-scheme. Let $X$ be a smooth projective $k$-variety. Denote by $\mathfrak{Coh}(X)$ the category of coherent sheaves of $\mathcal{O}_X$-modules on $X$. This is an abelian category. Denote by $D^b(X)$ the bounded derived category of $\mathfrak{Coh}(X)$.

3.1. **Equality of dimensions.** A rank one invertible sheaf $L$ on $X$ is said to have *finite order* if $L^r \cong \mathcal{O}_X$ for some integer $r > 0$. The smallest positive integer $r$ such that $L^r \cong \mathcal{O}_X$ is called the *order* of $L$. If $L^r \ncong \mathcal{O}_X$ , $\forall\ r > 0$, we say that $L$ has *infinite order*. For any $x \in X$, let $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at $x$. For any closed point $x \in X$, we can consider $k(x)$ as a coherent sheaf on $X$ supported at $x$ by taking its push-forward along the closed embedding $\iota_x : \mathrm{Spec}(k(x)) \hookrightarrow X$. This is the skyscraper sheaf supported at $x$ given by

$$k(x)(U) = \begin{cases} k(x), & \text{if } x \in U, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 3.1.1.** *Let $X$ and $Y$ be smooth projective varieties over $k$. If there is an exact equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ of bounded derived categories, then $\dim_k(X) = \dim_k(Y)$. In this case, both $\omega_X$ and $\omega_Y$ have the same order (can be infinity too).*

*Proof.* Since both $X$ and $Y$ are smooth projective $k$-varieties, by Theorem 2.2.5, they admit natural Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim_k(Y)]$, respectively. By Lemma 2.2.2, any $k$-linear equivalence $F : D^b(X) \longrightarrow D^b(Y)$ commutes with Serre functors $S_X$ and $S_Y$ (i.e., there is a natural isomorphism of functors $F \circ S_X \cong S_Y \circ F$).

For a closed point $x \in X$, we have $k(x) \cong k(x) \otimes \omega_X \cong S_X(k(x))[-\dim_k(X)]$. So,

$$\begin{aligned} F(k(x)) &\cong F(k(x) \otimes \omega_X) = F(S_X(k(x))[-\dim_k(X)]) \\ &\cong F(S_X(k(x)))[-\dim_k(X)], \quad \text{since } F \text{ is exact.} \\ &\cong S_Y(F(k(x)))[-\dim_k(X)], \quad \text{since } F \circ S_X \cong S_Y \circ F. \\ &\cong F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]\,. \end{aligned} \tag{3.1.2}$$

Since $F$ is an equivalence of categories, $F(k(x))$ is a non-trivial bounded complex. Let $i$ be the maximal (resp., minimal) integer such that $\mathcal{H}^i(F(k(x))) \neq 0$. Now from (3.1.2) we have

$$\begin{aligned} 0 \neq \mathcal{H}^i(F(k(x))) &\cong \mathcal{H}^i(F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x)) \otimes \omega_Y) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x))) \otimes \omega_Y\,. \end{aligned} \tag{3.1.3}$$

Since $\omega_Y$ is a line bundle, (3.1.3) contradicts maximality (resp., minimality) of $i$ whenever $\dim_k(X) < \dim_k(Y)$ (resp., $\dim_k(X) > \dim_k(Y)$). Therefore, $\dim_k(X) = \dim_k(Y)$.

To see that both $\omega_X$ and $\omega_Y$ have the same order, assume that $\omega_X^k \cong \mathcal{O}_X$. Let $n = \dim_k(X) = \dim_k(Y)$. Note that, $S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)}$. Since $F \circ S_X \cong S_Y \circ F$, choosing a quasi-inverse of the equivalence $F$, we have

$$\begin{aligned} & F^{-1} \circ S_Y^k[-kn] \circ F \cong S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)} \\ \Rightarrow & S_Y^k[-kn] \cong \mathrm{Id}_{D^b(Y)} \ . \end{aligned}$$

Applying $\mathcal{O}_Y$ to the above isomorphism of functors, we get $\omega_Y^k \cong \mathcal{O}_Y$. □

**Remark 3.1.4.** In the proof of above Proposition, to show both $\omega_X$ and $\omega_Y$ have the same order, under the assumption that $\dim(X) = \dim(Y)$, we don't need $F$ to be exact.

3.2. **Point like objects.**

**Definition 3.2.1.** A *graded category* is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ and an equivalence functor $T_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, known as *shift functor*. A functor $F : \mathcal{D} \to \mathcal{D}'$ between graded categories is called *graded* if there is an isomorphism of functors $F \circ T_{\mathcal{D}} \xrightarrow{\simeq} T_{\mathcal{D}'} \circ F$.

**Example 3.2.2.** Any triangulated category is a graded category, and any morphism between two triangulated categories is a graded morphism.

**Definition 3.2.3.** Let $\mathcal{D}$ be a $k$-linear triangulated category with Serre functor $S$. An object $P \in \mathcal{D}$ is said to be *point like* of codimension $s$ if

(i) $S(P) \cong P[s]$,
(ii) $\mathrm{Hom}(P, P[i]) = 0$, for $i < 0$, and
(iii) $k(P) := \mathrm{Hom}(P, P)$ is a field.

An object $E$ of an additive category is called *simple* if $\mathrm{Hom}(E, E)$ is a field.

**Example 3.2.4.** Let $X$ be a smooth projective $k$-variety of dimension $n$.

(i) For any closed point $x \in X$, we have $S_X(k(x)) = (k(x) \otimes \omega_X)[n] \cong k(x)[n]$. Therefore, $k(x) \in D^b(X)$ is a point like object of codimension $d$.
(ii) Let $\omega_X \cong \mathcal{O}_X$ (for example when $X$ is an abelian variety or a K3 surface). Then any simple object $E \in \mathfrak{Coh}(X)$ defines a point like object of codimension $n$ in $D^b(X)$.

**Proposition 3.2.5.** *Let $\mathcal{A}$ be an abelian category, and $A^\bullet \in D^b(\mathcal{A})$. Let*

$$i^+ := \max\{i : \mathcal{H}^i(A^\bullet) \neq 0\} \quad \textit{and} \quad i^- := \min\{i : \mathcal{H}^i(A^\bullet) \neq 0\}.$$

*Then in $D^b(\mathcal{A})$, there are morphisms $\phi : A^\bullet \to \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ and $\psi : \mathcal{H}^{i^-}(A^\bullet)[-i^-] \to A^\bullet$ such that $\mathcal{H}^{i^+}(\phi) = \mathrm{Id}_{\mathcal{H}^{i^+}(A^\bullet)}$ and $\mathcal{H}^{i^-}(\psi) = \mathrm{Id}_{\mathcal{H}^{i^-}(A^\bullet)}$.*

*Proof.* There is a natural quasi-isomorphism of complexes

$$\begin{tikzcd} A^\bullet_- : \arrow[d, "qis"'] & \cdots \arrow[r] & A^{i^+-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d^{i^+}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\ A^\bullet : & \cdots \arrow[r] & A^{i^+-1} \arrow[r] & A^{i^+} \arrow[r, "d^{i^+}"] & A^{i^++1} \arrow[r] & \cdots . \end{tikzcd}$$

Since the natural morphism of complexes $A^\bullet_- \longrightarrow \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ induces identity morphism at $i^+$-th cohomology, the first part follows. The second part is similar. □

**Corollary 3.2.6.** *With the above notations, for any $B \in \mathcal{A}$, we have the following natural isomorphisms*

*(i)* $\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[-i^+])$, *and*
*(ii)* $\mathrm{Hom}_{D^b(\mathcal{A})}(B, \mathcal{H}^{i^-}(A^\bullet)) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(B[-i^-], A^\bullet)$.

*Proof.* Send $f \in \mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B)$ to $f[-i^+]$ and use above Proposition 3.2.5. To get the inverse map, send any $\phi \in \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[i^+])$ to $\mathcal{H}^{i^+}(\phi)[-i^+]$. The second part is similar. □

**Remark 3.2.7.** Let $A^\bullet \in D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for all $i < m$. Then there is a distinguished triangle

$$\mathcal{H}^m(A^\bullet)[-m] \longrightarrow A^\bullet \xrightarrow{\varphi} B^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[1-m]$$

in the derived category $D(\mathcal{A})$ such that

$$\mathcal{H}^i(B^\bullet) \cong \begin{cases} \mathcal{H}^i(A^\bullet) & \text{if } i \le m, \text{ and} \\ 0, & \text{if } i > m. \end{cases}$$

**Remark 3.2.8.** Let $X$ be a smooth projective $k$-variety of dimension $d$. Then any point like object $P \in D^b(X)$ has codimension $d$. This follows from assumption (i) in the Definition 3.2.3, because looking at minimal $i$ with non-zero cohomologies, the isomorphism $P \otimes \omega_X[d] \cong P[s]$ implies

$$\mathcal{H}^i(P) \otimes \omega_X[d] \cong \mathcal{H}^i(P)[s]. \tag{3.2.9}$$

This forces $d = s$.

**Lemma 3.2.10.** *Let $M$ be a finitely generated non-zero module over a noetherian ring $A$. Then there is a finite chain of $A$-submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

*such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ (as $A$-modules), for some $\mathfrak{p}_i \in \mathrm{Supp}(M)$.*

*Proof.* Denote by $\mathrm{Ass}(M)$ the set of all associated primes of $M$. Recall that, $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ for any finitely generated $A$-module $M$. Since $M \neq 0$, we can choose a $\mathfrak{p}_1 \in \mathrm{Ass}(M)$ to get an $A$-submodule

$$M_1 := \mathrm{image}(A/\mathfrak{p}_1 \hookrightarrow M) \subset M.$$

If $M_1 \neq M$, we do the same for $M/M_1$ to choose a $\mathfrak{p}_2 \in \mathrm{Ass}(M/M_1)$ and apply the same to obtain a sequence $M_1 \subsetneq M_2 \subseteq M$ with $M_2/M_1 \cong A/\mathfrak{p}_2$. Since $(M/M_1)_{\mathfrak{p}_2} \neq 0$, we see that $\mathfrak{p}_2 \in \mathrm{Supp}(M)$. Since $M$ is finitely generated, the result follows by induction. $\square$

**Corollary 3.2.11.** *With the above notation, if* $\mathrm{Supp}(M) = \{\mathfrak{m}\}$, *for some maximal ideal* $\mathfrak{m}$ *of* $A$, *there is a surjective (resp., injective)* $A$*-module homomorphism* $M \twoheadrightarrow A/\mathfrak{m}$ *(resp.,* $A/\mathfrak{m} \hookrightarrow M$*).*

*Proof.* Since $\mathrm{Ass}(M) = \{\mathfrak{m}\}$, the result follows from the above Lemma 3.2.10. $\square$

**Definition 3.2.12.** Support of a complex $E^\bullet \in D^b(X)$ is the union of the supports of its cohomologies. In other words, $\mathrm{Supp}(E^\bullet)$ is the closed subset of $X$ defined by

$$\mathrm{Supp}(E^\bullet) := \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(\mathcal{H}^i(E^\bullet)) .$$

**Lemma 3.2.13.** *Let* $E^\bullet \in D^b(X)$ *with* $\mathrm{Supp}(E^\bullet) = Z_1 \cup Z_2$, *for some disjoint closed subsets* $Z_1$ *and* $Z_2$ *in* $X$. *Then* $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, *for some non-zero objects* $E_j^\bullet \in D^b(X)$ *with* $\mathrm{Supp}(E_j^\bullet) \subseteq Z_j$, *for all* $j = 1, 2$.

*Proof.* This is clear for any $E \in \mathfrak{Coh}(X)$, and hence the result follows for $E^\bullet \cong E[n] \in D^b(X)$, for $E \in \mathfrak{Coh}(X)$ and $n \in \mathbb{Z}$. Let

$$i^+_{E^\bullet} := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\} \quad \text{and} \quad i^-_{E^\bullet} := \min\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\};$$

and we drop the subscript $E^\bullet$ when there is no confusion likely to arise. The *length* of an object $E^\bullet \in D^b(X)$ is the difference $i^+ - i^-$. For general case, we use induction on the length of a complex.

Let $E^\bullet \in D^b(X)$ be a complex of length at least 2. Let $m = i^-_{E^\bullet}$, and write $\mathcal{H} := \mathcal{H}^m(E^\bullet)$. The sheaf $\mathcal{H}$ can be decomposed as $\mathcal{H} \cong \mathcal{H}_1 \bigoplus \mathcal{H}_2$, with $\mathrm{Supp}(\mathcal{H}_j) \subset Z_j$, for $j = 1, 2$. By Proposition 3.2.5, we have a natural morphism $\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet$ inducing identity morphism on the $m$-th cohomology; complete it to a distinguished triangle

$$\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet \longrightarrow F^\bullet := C(\varphi) \longrightarrow \mathcal{H}[1-m] .$$

Then from long exact sequence of cohomologies we have

$$\mathcal{H}^i(F^\bullet) = \begin{cases} \mathcal{H}^i(E^\bullet), & \text{if } i > m, \text{ and} \\ 0, & \text{if } i \leq m; \end{cases}$$

(c.f. Remark 3.2.7). Since the length of $F^\bullet$ is less than the length of $E^\bullet$, induction hypothesis applied to $F^\bullet$ gives a decomposition $F^\bullet \cong F_1^\bullet \bigoplus F_2^\bullet$ with $\mathrm{Supp}(\mathcal{H}^i(F_j^\bullet)) \subset Z_j$, for all $j = 1, 2$, and $i \in \mathbb{Z}$. Since $\mathcal{H}^{-q}(F_1^\bullet)$ and $\mathcal{H}_2$ are coherent sheaves of $\mathcal{O}_X$-modules with disjoint supports, we have

$$\mathrm{Hom}_{D^b(X)}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2) = 0, \ \ \forall\, p \in \mathbb{Z},$$

which can be verified locally. Then $\mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[1-m]) = 0$ follows from the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) \Longrightarrow E^{p+q} := \mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[p+q]).$$

Similarly, we have $\mathrm{Hom}(F_2^\bullet, \mathcal{H}_1[1-m]) = 0$. Choose a complex $E_j^\bullet$ to complete a distinguished triangle

$$E_j^\bullet \longrightarrow F_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow E_j^\bullet[1], \ \ \forall\, j = 1, 2,$$

we have a decomposition $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$. Since $\mathrm{Supp}(F_j^\bullet) \subset Z_j$, it follows that $\mathrm{Supp}(E_j^\bullet) \subset Z_j$, for all $j = 1, 2$. □

**Lemma 3.2.14.** *Let $E^\bullet$ be a simple object in $D^b(X)$ with zero dimensional support. If $\mathrm{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, then $E^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer $m$.*

*Proof.* Since $E^\bullet$ is supported in dimension zero, $\mathrm{Supp}(E)$ is a finite subset of closed points in $X$. If $\mathrm{Supp}(E)$ is not a singleton set, then it has disjoint components. Then in $D^b(X)$, we have an isomorphism $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, with $E_j^\bullet \not\cong 0$, $\forall\, i = 1, 2$, which contradicts simplicity of $E^\bullet$. Therefore, $\mathrm{Supp}(E^\bullet)$ is a closed point, say $x \in X$. Let $i^+ := \max\{i : \mathcal{H}^i(E^\bullet) \neq 0\}$ and $i^- := \min\{j : \mathcal{H}^j(E^\bullet) \neq 0\}$. Since both $\mathcal{H}^{i^+}(E^\bullet)$ and $\mathcal{H}^{i^-}(E^\bullet)$ have support $\{x\}$, they are given by finite modules over the noetherian local ring $\mathcal{O}_{X,x}$ supported at $\mathfrak{m}_x$. Then applying Corollary 3.2.11, we get a non-trivial $\mathcal{O}_{X,x}$-module homomorphism $\phi : \mathcal{H}^{i^+}(E^\bullet) \longrightarrow \mathcal{H}^{i^-}(E^\bullet)$ given by the composition

$$\mathcal{H}^{i^+}(E^\bullet) \twoheadrightarrow k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \hookrightarrow \mathcal{H}^{i^-}(E^\bullet).$$

Now it follows from Proposition 3.2.5 that the following composite morphism is non-trivial.

$$E^\bullet[i^+] \longrightarrow \mathcal{H}^{i^+}(E^\bullet) \xrightarrow{\phi} \mathcal{H}^{i^-}(E^\bullet) \longrightarrow E^\bullet[i^-]\,.$$

Since $\mathrm{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, we must have $i^- - i^+ \geq 0$. Hence, $i^- = i^+ =: m$ (say). Therefore, $E^\bullet \cong E[m]$, for some $E \in \mathfrak{Coh}(X)$ with $\mathrm{Supp}(E) = \{x\}$. Since $\mathrm{Hom}(E[m], E[m]) \cong \mathrm{Hom}(E, E)$, so $E$ is simple. Then the natural surjective homomorphism $E \to k(x)$ must be isomorphism. Therefore, $E^\bullet \cong k(x)[m]$. □

**Proposition 3.2.15** (Bondal–Orlov)**.** *Let $X$ be a smooth projective $k$-variety with $\omega_X$ or $\omega_X^\vee$ ample. Then any point like object in $D^b(X)$ is isomorphic to an object of the form $k(x)[m]$, for some closed point $x \in X$ and some integer $m$.*

**Remark 3.2.16.** Above result fails if neither $\omega_X$ nor $\omega_X^\vee$ is ample; c.f. Example 3.2.4.

*Proof.* Note that $X$ is projective because there is an ample line bundle on $X$. Clearly for any closed point $x \in X$ and any integer $m$, the shifted skyscraper sheaf $k(x)[m] \in D^b(X)$ is a point like object of codimension $d = \dim(X)$ (c.f., Example 3.2.4).

To see the converse, let $P \in D^b(X)$ be a point like object of codimension $n$. It follows from $P \otimes \omega_X[d] \cong P[n]$ that $n = d$ (c.f., Remark 3.2.8). Then we have,

$$\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P), \quad \forall\, i \in \mathbb{Z}\,. \tag{3.2.17}$$

Suppose that $\omega_X$ is ample. Let

$$m \mapsto P_E(m) := \chi(E \otimes \omega_X^m)$$

be the Hilbert polynomial of $E \in \mathfrak{Coh}(X)$. Since $\deg(P_E(m)) = \dim(\mathrm{Supp}(E))$, taking tensor product with $\omega_X$ makes difference only if $\dim(\mathrm{Supp}(E)) > 0$. Therefore, form (3.2.17) we conclude that $\mathcal{H}^i(P)$ is supported in dimension zero. Since $P$ is simple, the result follows from Lemma 3.2.14. The same argument applies for $\omega_X^\vee$ ample. $\square$

3.3. **Invertible objects.** Now we realize line bundles on $X$ as objects of $D^b(X)$.

**Definition 3.3.1.** Let $\mathcal{D}$ be a triangulated category together with a Serre functor $T_\mathcal{D} : \mathcal{D} \to \mathcal{D}$. An object $L \in \mathcal{D}$ is said to be *invertible* if for each point like object $P \in \mathcal{D}$, there is an integer $n_P$ (which also depends on $L$) such that

$$\mathrm{Hom}_\mathcal{D}(L, P[i]) = \begin{cases} k(P), & \text{if } i = n_P, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we characterize invertible objects in $D^b(X)$. For this, we need the following well-known result form commutative algebra.

**Lemma 3.3.2.** *Let $M$ be a finitely generated module over a noetherian local ring $(A, \mathfrak{m})$. If $\mathrm{Ext}^1(M, A/\mathfrak{m}) = 0$, then $M$ is free.*

*Proof.* Let $k = A/\mathfrak{m}$. Then any $k$-basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for the $A$-module $M$ by Nakayama lemma. Thus we get a short exact sequence of $A$-modules

$$0 \longrightarrow N \xrightarrow{\iota} A^n \xrightarrow{\phi} M \longrightarrow 0.$$

Note that, $N = \mathrm{Ker}(\phi)$ is finitely generated, and $\iota$ induces a trivial homomorphism $\widetilde{\iota} : N/\mathfrak{m}N \longrightarrow k^n$. Since $\mathrm{Ext}^1(M, k) = 0$, the induced homomorphism

$$\mathrm{Hom}(A^n, k) \longrightarrow \mathrm{Hom}(N, k)$$

is surjective. Since $\mathrm{Hom}_A(A^n, k) \cong \mathrm{Hom}_k(k^n, k)$ and $\mathrm{Hom}_A(N, k) \cong \mathrm{Hom}_k(N/\mathfrak{m}N, k)$, the homomorphism $\mathrm{Hom}_k(k^n, k) \longrightarrow \mathrm{Hom}_k(N/\mathfrak{m}N, k)$ induced by $\widetilde{\iota}$ is surjective.

Since $\widetilde{\iota} = 0$, this forces $N/\mathfrak{m}N = 0$. Then $N = 0$ by Nakayama lemma, and hence $M$ is a free $A$-module. □

**Proposition 3.3.3** (Bondal–Orlov). *Let $X$ be a smooth projective $k$-variety. Any invertible object in $D^b(X)$ is of the form $L[m]$, for some line bundle $L$ on $X$ and some integer $m$. Conversely, if any point like object of $D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and some integer $\ell$, then for any line bundle $L$ on $X$ and any integer $m$, $L[m] \in D^b(X)$ is invertible.*

**Remark 3.3.4.** Note that, by Proposition 3.2.15 the condition in the converse part of the above Proposition is satisfied when $\omega_X$ or $\omega_X^\vee$ is ample.

*Proof of Proposition 3.3.3.* **Step 1.** Let $E^\bullet \in D^b(X)$ be an invertible object. Let $m = \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then by Proposition 3.2.5, there is a morphism

$$E^\bullet \longrightarrow \mathcal{H}^m(E^\bullet)[-m]$$

in $D^b(X)$ inducing identity morphism at $m$-th cohomology $\mathcal{H}^m(E^\bullet)$. This gives

$$\mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,, \tag{3.3.5}$$

(c.f., Corollary 3.2.6). Fix a closed point $x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$. Then by Lemma 3.2.10, there is an associated prime ideal $\mathfrak{p} \subseteq \mathfrak{m}_{x_0}$ and a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow \mathcal{O}_{X,x_0}/\mathfrak{p}$, which gives a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x_0)$. Therefore, by (3.3.5), we have

$$0 \neq \mathrm{Hom}_{D^b(X)}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,.$$

This forces $n_{k(x_0)} = -m$ (c.f., Definition 3.3.1).

**Step 2.** *We show that,* $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $n_{k(x_0)} = -m$, it follows from the definition of invertible object $E^\bullet \in D^b(X)$ that

$$\mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = \mathrm{Hom}(E^\bullet, k(x_0)[1 + n_{k(x_0)}]) = 0. \tag{3.3.6}$$

Consider the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(E^\bullet), k(x_0)[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), k(x_0)) \Longrightarrow E^{p+q} := \mathrm{Hom}(E^\bullet, k(x_0)[p+q]). \tag{3.3.7}$$

Since $\mathcal{H}^{m+1}(E^\bullet) = 0$, we have

$$E_2^{3,-m-1} = \mathrm{Hom}(\mathcal{H}^{m+1}(E^\bullet), k(x_0)[3]) = 0. \tag{3.3.8}$$

Also

$$E_2^{-1,-m+1} = \mathrm{Hom}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)[-1]) = \mathrm{Ext}^{-1}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)) = 0. \tag{3.3.9}$$

Now using (3.3.8) and (3.3.9), and taking $H^0$ of the complex

$$\cdots \longrightarrow 0 = E_2^{-1,-m+1} \xrightarrow{d} E_2^{1,-m} \xrightarrow{d} E_2^{3,-m-1} = 0 \longrightarrow \cdots,$$

we see that $E_3^{1,-m} = E_2^{1,-m}$; similarly, $E_r^{1,-m} = E_2^{1,-m}$, for all $r \geq 2$. The following picture of page $E_2$ could be useful to understand the situation.

$$\begin{tikzcd}
0 \arrow[drr] & E_2^{0,-m+1} \arrow[drr] & E_2^{1,-m+1} \arrow[drr] & E_2^{2,-m+1} & E_2^{3,-m+1} \\
0 & E_2^{0,-m} \arrow[drr] & E_2^{1,-m} \arrow[drr] & E_2^{2,-m} & E_2^{3,-m} \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

This shows that,

$$E_2^{1,-m} = E_\infty^{1,-m}. \tag{3.3.10}$$

Since $E_\infty^{1,-m}$ is isomorphic to a subquotient of

$$E^{1-m} = \mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = 0 \tag{3.3.11}$$

(see, (3.3.6) and (3.3.7)), using (3.3.10) we conclude that $E_2^{1,-m} = 0$. Therefore,

$$\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0, \ \ \forall\, x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))\,. \tag{3.3.12}$$

**Step 3.** *We show that $\mathcal{H}^m(E^\bullet)$ is a locally free $\mathcal{O}_X$-module.*

For this, we consider the *local-to-global* spectral sequence

$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q(\mathcal{H}^m(E^\bullet), k(x_0))) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m(E^\bullet), k(x_0))\,, \tag{3.3.13}$$

which allow us to pass from the global vanishing $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ to the local one $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $\mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))$ is a skyscraper sheaf supported at $x_0$, it is flasque, and hence is $\Gamma$-acyclic. Then form (3.3.13), we have

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{3.3.14}$$

Again,

$$E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{3.3.15}$$

Since at page $E_2$, we have morphisms

$$0 = E_2^{-2,2} \xrightarrow{d} E_2^{0,1} \xrightarrow{d} E_2^{2,0} = 0,$$

we have $E_3^{0,1} = \mathcal{H}^0(\cdots \to 0 \to E_2^{0,1} \to 0 \to \cdots) = E_2^{0,1}$. Similar computations shows that $E_r^{0,1} = E_2^{0,1}$, for all $r \geq 2$. Hence we conclude that,

$$E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))) = E_\infty^{0,1}\,. \tag{3.3.16}$$

Since $E^1 = \operatorname{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ by Step 2, we have $E_2^{0,1} = E_\infty^{0,1} = 0$. Since $k(x_0)$ is a skyscraper sheaf supported at $x_0$, we see that $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))$ is supported over $\{x_0\}$, and hence is globally generated. Since

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = E_2^{0,1} = 0,$$

we have $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$. Since $\mathcal{H}^m(E^\bullet) \in \mathfrak{Coh}(X)$, we have

$$\operatorname{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))_{x_0} = 0. \tag{3.3.17}$$

The by Lemma 3.3.2, $\mathcal{H}^m(E^\bullet)_{x_0}$ is free $\mathcal{O}_{X,x_0}$-module. Since freeness is an open property, there is a non-empty open (dense) subset $U$ of $X$ containing $x_0$ such that $U \subseteq \operatorname{Supp}(\mathcal{H}^m(E^\bullet))$ and $\mathcal{H}^m(E^\bullet)\big|_U$ is a free $\mathcal{O}_U$-module. Since $X$ is irreducible, $\mathcal{H}^m(E^\bullet)$ is locally free on $X$.

**Step 4.** *We show that, $\mathcal{H}^m(E^\bullet)$ is a line bundle on $X$.*

Since $\operatorname{Supp}(\mathcal{H}^m(E^\bullet)) = X$, there is a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x)$, for each $x \in X$. Then following argument of Step 1, we have

$$\operatorname{Hom}(E^\bullet, k(x)[-m]) = \operatorname{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \neq 0, \;\; \forall\, x \in X. \tag{3.3.18}$$

Now it follows from Definition 3.3.1 of invertible objects that

$$n_{k(x)} = -m, \;\forall\; x \in X. \tag{3.3.19}$$

If $r$ is the rank of $\mathcal{H}^m(E^\bullet)$, we have

$$\begin{aligned} k(x) = \operatorname{Hom}(E^\bullet, k(x)[-m]) &= \operatorname{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \\ &= \operatorname{Hom}(\mathcal{O}_{X,x}^{\oplus r}, k(x)) \cong k(x)^{\oplus r}\,. \end{aligned} \tag{3.3.20}$$

Therefore, $r = 1$, and hence $\mathcal{H}^m(E^\bullet)$ is a line bundle on $X$.

**Step 5.** *We show that, $\mathcal{H}^i(E^\bullet) = 0$, for all $i < m$.*

From the spectral sequence in (3.3.7), we have

$$\begin{aligned} E_2^{q,-m} &= \operatorname{Hom}(\mathcal{H}^m(E^\bullet), k(x)[q]) \\ &= \operatorname{Ext}^q(\mathcal{H}^m(E^\bullet), k(x)) \\ &\cong H^q(X, \mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))) = 0, \quad \forall\, q > 0, \end{aligned} \tag{3.3.21}$$

because $\mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))$ is a skyscraper sheaf supported on $\{x\}$, and hence is $\Gamma$-acyclic.

Suppose that $i < m$. Then it follows from Definition 3.3.1 and (3.3.19) that

$$E^{-i} = \operatorname{Hom}(E^\bullet, k(x)[-i]) = 0, \;\; \forall\, x \in X. \tag{3.3.22}$$

Now to show $\mathcal{H}^i(E^\bullet) = 0$, it is enough to show that

$$E_2^{0,-i} = \operatorname{Hom}(\mathcal{H}^i(E^\bullet), k(x)) = 0, \forall\, x \in X. \tag{3.3.23}$$

Since $E^{-i} = 0$, if we can show that

$$E_2^{0,-i} = E_\infty^{0,-i}, \tag{3.3.24}$$

then from the spectral sequence (3.3.7) we would get $E_2^{0,-i} = 0$. We prove this by induction on $i$.

If $i = m-1$, then $E_2^{2,-i-1} = E_2^{2,-m} = 0$ by (3.3.21). Since negative indexed Ext groups between two coherent sheaves are zero, we have $E_2^{-2,-(m-2)} = 0$. Then (3.3.24), for the case $i = m-1$, follows from the complex

$$\cdots \to 0 = E_2^{-2,-(m-2)} \xrightarrow{d} E_2^{0,1-m} \xrightarrow{d} E_2^{2,-m} = 0 \to \cdots .$$

Therefore, $\mathcal{H}^{m-1}(E^\bullet) = 0$. Assume inductively that $\mathcal{H}^i(E^\bullet) = 0$, for all $i \in \mathbb{Z}$, with $i_0 < i \leq m-1$. Then putting $m = i_0 + 1$ in (3.3.21) and using $\mathcal{H}^{i_0+1}(E^\bullet) = 0$, we have $E_2^{2,-i_0-1} = 0$. Then (3.3.24) follows from the complex

$$\cdots \to 0 = E_2^{-2,1-i_0} \xrightarrow{d} E_2^{0,-i_0} \xrightarrow{d} E_2^{2,-i_0-1} = 0 \to \cdots .$$

This completes induction. Therefore, $\mathcal{H}^i(E^\bullet) = 0,\ \forall\, i < m$, and hence for all $i \neq m$.

**Step 6.** Now we prove converse part of the Proposition 3.3.3. Suppose that any point like object $P \in D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and $\ell \in \mathbb{Z}$. Let $L$ be a line bundle on $X$, and $m \in \mathbb{Z}$. Then from Definition 3.3.1 we get

$$\begin{aligned} \mathrm{Hom}(L[m], P[i]) &\cong \mathrm{Hom}(L, k(x)[\ell + i - m]) \\ &= \mathrm{Ext}^{\ell+i-m}(\mathcal{O}_X, L^\vee \otimes k(x)) \\ &\cong H^{\ell+i-m}(X, L^\vee \otimes k(x)), \end{aligned} \tag{3.3.25}$$

which vanishes except for $i = m - \ell$. Then we set $n_P := m - \ell$. This completes the proof. □

**Remark 3.3.26.** Let $\mathcal{D}$ be a (tensor) triangulated category admitting a Serre functor $S$. If we naively define *Picard group* of $\mathcal{D}$ to be the set $\mathrm{Pic}(\mathcal{D})$ of all invertible objects in $\mathcal{D}$, then for a smooth projective $k$-variety $X$ with $\omega_X$ or $\omega_X^\vee$ ample, we have $\mathrm{Pic}(D^b(X)) = \mathrm{Pic}(X) \times \mathbb{Z}$.

### 3.4. Spanning class of $D^b(X)$.

**Definition 3.4.1.** A collection $\Omega$ of objects in a triangulated category $\mathcal{D}$ is called a *spanning class of $\mathcal{D}$* (or *spans $\mathcal{D}$*) if for all $B \in \mathcal{D}$ the following conditions hold.

(i) If $\mathrm{Hom}(A, B[i]) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
(ii) If $\mathrm{Hom}(B[i], A) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

**Remark 3.4.2.** If a triangulated category $\mathcal{D}$ admits a Serre functor, then the conditions (i) and (ii) in the above Definition 3.4.1 are equivalent.

**Proposition 3.4.3.** *Let $X$ be a smooth projective $k$-variety. Then the objects of the form $k(x)$, with $x \in X$ a closed point, spans $D^b(X)$.*

*Proof.* It is enough to show that, for any non-zero object $E^\bullet \in D^b(X)$ there exists closed points $x_1, x_2 \in X$ and integers $i_1, i_2$ such that

$$\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0 \quad \text{and} \quad \mathrm{Hom}(k(x_2), E^\bullet[i_2]) \neq 0\,.$$

Since $\mathrm{Hom}(k(x_2), E^\bullet[i_2]) \cong \mathrm{Hom}(E^\bullet, k(x_2)[\dim(X) - i_2])^*$ by Serre duality, it is enough to show that $\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0$, for some closed point $x \in X$ and some $i \in \mathbb{Z}$. Let $m := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then $\mathrm{Hom}(E^\bullet, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x))$ by Corollary 3.2.6. Now choosing a closed point $x$ in the support of $\mathcal{H}^m(E^\bullet)$, we see that $\mathrm{Hom}(E^\bullet, k(x)[-m]) \neq 0$. This completes the proof. $\square$

**Remark 3.4.4.** Spanning class in $D^b(X)$ is not unique. For a smooth projective $k$-variety $X$, for a choice of an ample line bundle $L$ on $X$, we shall see later that, $\{L^{\otimes i} : i \in \mathbb{Z}\}$ forms a spanning class in $D^b(X)$.

3.5. **Proof of the reconstruction theorem.** Now we are in a position to prove the reconstruction theorem of Bondal and Orlov in the light of the following well-known results.

**Proposition 3.5.1.** [Sta20, Tag01PR] *Let $X$ be a quasi-compact scheme. Let $L$ be an invertible sheaf of $\mathcal{O}_X$-modules on $X$. Consider the graded algebra $S := \bigoplus_{i \geq 0} H^0(X, L^i)$, and its ideal $S_+ = \bigoplus_{i > 0} H^0(X, L^i)$. For each homogeneous element $s \in H^0(X, L^i)$, for $i > 0$, let $X_s := \{x \in X : s_x \notin \mathfrak{m}_x L^i_x\}$. Then the following are equivalent.*

*(i) $L$ is ample.*
*(ii) The collection of open sets $X_s$, with $s \in S_+$ homogeneous, covers $X$, and the natural morphism $X \longrightarrow \mathbf{Proj}(S)$ is an open immersion.*
*(iii) The collection of open sets $X_s$, with $s \in S_+$ homogeneous, forms a basis for the Zariski topology on $X$.*

**Proposition 3.5.2.** *Let $X$ be a smooth projective $k$-variety. Let $L$ be a line bundle on $X$. If $L$ or $L^\vee$ is ample, then the natural morphism of $k$-schemes*

$$X \longrightarrow \mathbf{Proj}\left(\bigoplus_n H^0(X, L^n)\right)$$

*is an isomorphism.*

**Theorem 1.0.1** (Bondal–Orlov)**.** *Let $X$ be a smooth projective variety over $k$ with canonical line bundle $\omega_X$. Assume that $\omega_X$ (resp., $\omega_X^\vee$) is ample. Let $Y$ be any smooth projective variety* *over $k$. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as $k$-varieties. In particular, $\omega_Y$ (resp., $\omega_Y^\vee$) is ample.*

*Proof.* **Step 1.** If $F(\mathcal{O}_X) = \mathcal{O}_Y$, and $\omega_Y$ or $\omega_Y^\vee$ is ample, the theorem follows.

Indeed, assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Since $F$ is an exact equivalence of categories, $F \circ S_X \cong S_Y \circ F$ and $\dim(X) = \dim(Y) = n$ (say), (see Proposition 3.1.1). Then we have

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X))[-kn] = S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k, \ \forall\, k\,. \tag{3.5.3}$$

Since $F$ is fully faithful, we have

$$H^0(X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k), \ \forall\, k\,. \tag{3.5.4}$$

The product structure on the graded $k$-algebra $\bigoplus\limits_k H^0(X, \omega_X^k)$ can be expressed in terms of following composition: for $s_i \in H^0(X, \omega_X^{k_i})$, $i = 1, 2$, we have

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1\,.$$

Note that, $s_1 \cdot s_2 = s_2 \cdot s_1$ follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "s_1"] \arrow[d, "s_2"'] & \omega_X^{k_1} \arrow[d, "S_X^{k_1}(s_2)[-k_1 n]"] \\ \omega_X^{k_2} \arrow[r, "S_X^{k_2}(s_1)[-k_2 n]"] & \omega_X^{k_1+k_2} \end{tikzcd} \tag{3.5.5}$$

Similarly, we have product structure on $\bigoplus\limits_k H^0(Y, \omega_Y^k)$. Therefore, $F$ naturally induces an isomorphism of graded $k$-algebras

$$\widetilde{F} : \bigoplus_k H^0(X, \omega_X^k) \longrightarrow \bigoplus_k H^0(Y, \omega_Y^k)\,, \tag{3.5.6}$$

which induces isomorphism of $k$-schemes

$$X \xrightarrow{\cong} \mathbf{Proj}\Big(\bigoplus_k H^0(X, \omega_X^k)\Big) \xrightarrow{\cong} \mathbf{Proj}\Big(\bigoplus_k H^0(Y, \omega_Y^k)\Big) \xrightarrow{\cong} Y, \tag{3.5.7}$$

whenever $\omega_Y$ or its dual $\omega_Y^\vee$ is ample (c.f., Proposition 3.5.2). Therefore, it is enough to show that $F(\mathcal{O}_X) = \mathcal{O}_Y$, and $\omega_Y$ or $\omega_Y^\vee$ is ample whenever $\omega_X$ or $\omega_X^\vee$ is ample.

**Step 2.** We can assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Indeed, it follows from Definition 3.2.3 and Definition 3.3.1 that an exact equivalence $F : D^b(X) \to D^b(X)$ induce bijections

$$\begin{tikzcd} \{\text{point like objects of } D^b(X)\} \arrow[r, "F", "\simeq"'] \arrow[d, equal] & \{\text{point like objects of } D^b(Y)\} \\ \{k(x)[m] : x \in X_{closed} \text{ and } m \in \mathbb{Z}\} & \{k(y)[m] : y \in Y_{closed} \text{ and } m \in \mathbb{Z}\} \arrow[u, hook, "(*)"'] \end{tikzcd} \tag{3.5.8}$$

and

$$
\begin{tikzcd}
\{\text{invertible objects of } D^b(X)\} \arrow[r, "F"', "\simeq"] \arrow[d, equal] & \{\text{invertible objects of } D^b(Y)\} \arrow[d, hook, "(**)"] \\
\{L[m] : L \in \mathrm{Pic}(X) \text{ and } m \in \mathbb{Z}\} & \{M[m] : M \in \mathrm{Pic}(Y) \text{ and } m \in \mathbb{Z}\},
\end{tikzcd}
\tag{3.5.9}
$$

where $X_{closed}$ (resp., $Y_{closed}$) is the set of all closed points of $X$ (resp., $Y$), and the vertical inclusions and equalities are given by Proposition 3.2.15 and Proposition 3.3.3. Therefore, $F(\mathcal{O}_X) = M[m]$, for some $M \in \mathrm{Pic}(Y)$ and some $m \in \mathbb{Z}$.

If $F(\mathcal{O}_X) \neq \mathcal{O}_Y$, replacing $F$ with the following composite functor

$$
D^b(X) \xrightarrow{F} D^b(Y) \xrightarrow{(M^\vee \otimes -)[-m]} D^b(Y), \tag{3.5.10}
$$

which is an exact equivalence sending $\mathcal{O}_X$ to $\mathcal{O}_Y$, we may assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Therefore, it remains to show is that $\omega_Y$ or its dual is ample.

**Step 3.** We establish bijections $X_{closed} \xleftrightarrow{F} Y_{closed}$ and $\mathrm{Pic}(X) \xleftrightarrow{F} \mathrm{Pic}(Y)$.

Using the equivalence $F$, we first show that the vertical inclusion $(*)$ in the diagram (3.5.8) is a bijection. This immediately imply that the vertical inclusion $(**)$ in the diagram (3.5.9) is bijective by Proposition 3.3.3. Then Step 3 will follow.

By horizontal bijection in the diagram (3.5.8), for any closed point $y \in Y$ there is a closed point $x_y \in X$ and $m_y \in \mathbb{Z}$ such that $F(k(x_y)[m_y]) \cong k(y)$. Suppose on the contrary that there is a point like object $P \in D^b(Y)$, which is not of the form $k(y)[m]$, for any closed point $y \in Y$ and integer $m$. Because of bijection in (3.5.8), there is a unique closed point $x_P \in X$ and integer $m_P$ such that $F(k(x_P)[m_P]) \cong P$. Then $x_P \neq x_y$, for all closed point $y \in Y$. Hence, for any closed point $y \in Y$ and any integer $m$, we have

$$
\begin{aligned}
\mathrm{Hom}(P, k(y)[m]) &= \mathrm{Hom}(F(k(x_P)[m_P]), k(y)[m]) \\
&= \mathrm{Hom}(k(x_P)[m_P], k(x_y)[m_y + m]) \\
&= \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) = 0,
\end{aligned}
\tag{3.5.11}
$$

because $k(x_P)$ and $k(x_y)$ being skyscraper sheaves supported at different points, $\mathrm{Ext}^i(k(x_P), k(x_y)) = 0$, for all $i$. Since the objects $k(y)$, with $y \in Y$ a closed point, form a spanning class of $D^b(X)$ (c.f. Definition 3.4.1), $P \cong 0$ by Proposition 3.4.3, which contradicts our assumption that $P$ is a point like object in $D^b(Y)$. Therefore, point like objects of $D^b(Y)$ are exactly of the form $k(y)[m]$, for $y \in Y$ a closed point and $m \in \mathbb{Z}$.

Note that, for any closed point $x \in X$, there is a closed point $y_x \in Y$ such that $F(k(x)) \cong k(y_x)[m_x]$, for some $m_x \in \mathbb{Z}$. Since $F$ is fully faithful and $F(\mathcal{O}_X) = \mathcal{O}_Y$, we have $\mathrm{Hom}(\mathcal{O}_X, k(x)) = \mathrm{Hom}(\mathcal{O}_Y, k(y_x)[m_x]) = \mathrm{Ext}^{m_x}(\mathcal{O}_Y, k(y_x)) \neq 0$. This forces

$m_x = 0$, and hence $F(k(x)) \cong k(y_x)$ (no shift!). This immediately imply that, for any $L \in \text{Pic}(X)$, $F(L) \cong M$, for some $M \in \text{Pic}(Y)$. Indeed, from bijections in the diagram (3.5.9), we find unique $M \in \text{Pic}(Y)$ and $m_L \in \mathbb{Z}$ such that $F(L) \cong M[m_L]$. Take closed points $x \in X$ and $y_x \in Y$ such that $F(k(x)) \cong k(y_x)$. Then

$$\begin{aligned} \text{Ext}^{-m_L}(M, k(y_x)) &= \text{Hom}(M, k(y_x)[-m_L]) = \text{Hom}(M[m_L], k(y_x)) \\ &= \text{Hom}(F(L), F(k(x))) = \text{Hom}(L, k(x)) \neq 0. \end{aligned}$$

This forces $m_L = 0$.

**Step 4.** Recovering Zariski topology from derived category to conclude ampleness.

Let $Z$ be a quasi-compact $k$-scheme. Denote by $Z_0$ the subset of all closed points of $Z$. Take line bundles $L_1$ and $L_2$ on $Z$, and take $\alpha \in \text{Hom}(L_1, L_2) = H^0(X, L_1^\vee \otimes L_2)$. For each closed point $z \in Z$, let

$$\alpha_z^* : \text{Hom}(L_2, k(z)) \longrightarrow \text{Hom}(L_1, k(z)) \tag{3.5.12}$$

be the homomorphism induced by $\alpha$. Then $U_\alpha := \{z \in Z : \alpha_z^* \neq 0\}$ is a Zariski open subset of $Z$, and hence $U_\alpha \cap Z$ is open in $Z_0$.

Fix a line bundle $L_0 \in \text{Pic}(X)$. Then it follows from Proposition 3.5.1 that the collection of all such $U_\alpha$, where $\alpha \in H^0(X, L_0^n)$ and $n \in \mathbb{Z}$, forms a basis for the Zariski topology on $Z$ if and only if either $L_0$ or $L_0^\vee$ is ample.

By Step 3, the exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$ sends closed points of $X$ to closed points of $Y$ bijectively, and sends line bundles on $X$ to line bundles on $Y$ bijectively. In particular, $F(\omega_X^i) \cong \omega_Y^i$, for all $i \in \mathbb{Z}$. Then the natural isomorphisms $H^0(X, \omega_X^i) \cong H^0(Y, \omega_Y^i)$, $\forall\, i \in \mathbb{Z}$, give rise to a bijection between the collection of open subsets

$$\begin{aligned} \mathcal{B}_X &:= \{U_\alpha : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}, \text{ and} \\ \mathcal{B}_Y &:= \{V_\alpha : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}. \end{aligned}$$

Since $\omega_X$ (resp., $\omega_X^\vee$) is ample, $\mathcal{B}_X$ is a basis for the Zariski topology on $X$, and hence $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}$ is a basis for the Zariski topology on $X_0$. Therefore, $\mathcal{B}_{Y_0} := \{V_\alpha \cap Y_0 : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}$ is a basis for the Zariski topology on $Y_0$, and hence $\mathcal{B}_Y$ is a basis for the Zariski topology on $Y$ (see Lemma 3.5.13 below). Therefore, $\omega_Y$ (resp., $\omega_Y^\vee$) is ample. This completes the proof. □

I thank Arideep Saha for some useful discussion leading to the following Lemma.

**Lemma 3.5.13.** *Let $X$ be a scheme locally of finite type over* $\text{Spec}(\Bbbk)$, *where $\Bbbk$ is a field or $\mathbb{Z}$. Let $X_0$ be a subset of $X$ containing all closed points of $X$. Let $\mathcal{B}_X := \{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of $X$ such that $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in \Lambda\}$ is a basis for the subspace Zariski topology on $X_0$. Then $\mathcal{B}$ is a basis for the Zariski topology on $X$.*

*Proof.* **Step 1.** First we show that, *if an open set $U \subset X$ contains a closed point $x_0$, then for any point $x \in X$ which contains $x_0$ in its closure (i.e., $x_0 \in \overline{\{x\}}$), we have $x \in U$.* Since $\mathcal{B}_{X_0}$ is a basis, there is $\alpha \in \Lambda$ such that $x_0 \in U_\alpha \cap X_0 \subseteq U \cap X_0$. If $x \notin U_\alpha$, then $x$ belongs to the closed set $X \setminus U_\alpha$, and hence $\overline{\{x\}} \subseteq X \setminus U_\alpha$, which contradicts the assumption that $x_0 \in \overline{\{x\}}$. Therefore, $x \in U_\alpha$. Since closure of any point in $X$ contains a closed point, it follows that $\mathcal{B}_X$ is an open cover for $X$.

It remains to show that for $x \in U_\alpha \cap U_\beta$, there is $\gamma \in \Lambda$ such that $x \in U_\gamma \subseteq U_\alpha \cap U_\beta$.

**Step 2.** Assume that, *for any open subset $U$ of $X$ with $x \in U$, there is a closed point $x_0 \in \overline{\{x\}} \cap U$.* For then, taking $U = U_\alpha \cap U_\beta$, we can find a $\gamma \in \Lambda$ such that

$$x_0 \in U_\gamma \cap X_0 \subseteq U_\alpha \cap U_\beta \cap X_0.$$

Then we will have $U_\gamma \subseteq U_\alpha \cap U_\beta$. Indeed, for each $z \in U_\gamma$, by *above assumption* there is a closed point $z_0 \in \overline{\{z\}} \cap U_\alpha \cap U_\beta$. Then by Step 1, we have $z \in U_\alpha \cap U_\beta$.

**Step 3.** We now prove the *assumption* of Step 2. Since the statement is local, we may assume that $X = \text{Spec}(A)$, for some finitely generated $\Bbbk$-algebra $A$. For each $f \in A$, let $D_f := \{\mathfrak{q} \in \text{Spec}(A) : f \notin \mathfrak{q}\}$. Since $\{D_f : f \in A\}$ forms a basis for the Zariski topology on Spec$(A)$, any point $\mathfrak{p} \in \text{Spec}(A)$ is contained in $D_f$, for some $f \in A \setminus \{0\}$. We claim that, there is a closed point (maximal ideal) $\mathfrak{m} \in D_f$ with $\mathfrak{p} \subset \mathfrak{m}$. If not, then all closed points (maximal ideal) $\mathfrak{m} \in \text{Max}(A/\mathfrak{p}) \subset \text{Spec}(A/\mathfrak{p})$ lies outside $D_f$. Since $A/\mathfrak{p}$ is a finitely generated $\Bbbk$-algebra, we have

$$\text{Jac}(A/\mathfrak{p}) = \bigcap_{\mathfrak{m} \in \text{Max}(A/\mathfrak{p})} \mathfrak{m} = \bigcap_{\mathfrak{q} \in \text{Spec}(A/\mathfrak{p})} \mathfrak{q} = \text{Nil}(A/\mathfrak{p}),$$

which is zero because $A/\mathfrak{p}$ is an integral domain. This contradicts the fact that $f \neq 0$ in $A/\mathfrak{p}$. This completes the proof. $\square$

Although we don't need full strength of the following Lemma 3.5.14 here, let me mention it here since it can be useful in may purpose. I thank Saurav Bhaumik for explaining it to me.

**Lemma 3.5.14.** *Any polarized reduced projective scheme locally of finite type over a field can be reconstructed from its set of closed points.*

*Proof.* Let $X$ be a reduced projective $k$-scheme, which is locally of finite type over Spec$(k)$. If $\mathscr{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is the ideal sheaf of a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, for some integer $n \geq 1$, then $X \cong \text{Proj}(S/I)$, where $I := \bigoplus_{i \geq 0} H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ is the homogeneous ideal of the graded $k$-algebra $S := k[x_0, \ldots, x_n]$. Therefore, it suffices to show that, $I$ coincides with the ideal of homogeneous polynomials in $S$ vanishing at each closed point of $X$. It follows from the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n_k, \mathscr{I}_X(i)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_X(i)) \longrightarrow H^0(X, \mathcal{O}_X(i))$$

that $H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ can be identified with the set of all homogeneous polynomials of degree $i$ in $S$ that vanishes at each point of $X$. Therefore, it suffices to show that, if $X$ is a finite type reduced $k$-subscheme of a $k$-scheme $\widetilde{X}$, a section $s \in H^0(\widetilde{X}, L)$ of a line bundle $L$ on $\widetilde{X}$ vanishes at every closed points of $X$ if and only if $s|_X = 0$. This can be checked locally. Take an affine open subset $U = \text{Spec}(A)$ of $X$ such that $L\big|_U$ is trivial. Then $s\big|_U$ is given by an element $f \in A$. Since $s$ vanishes at every closed points of $X$, $f \in \text{Jac}(A)$. Since $X$ is locally of finite type over Spec$(k)$, $\text{Jac}(A) = \text{Nil}(A)$, which is zero because $X$ is reduced. Therefore, $f = 0$, and hence $s\big|_X = 0$. Hence the result follows. □

**Remark 3.5.15.** When $k$ is an algebraically closed, there is a more geometric proof of ampleness of $\omega_Y$ or its dual. The idea is to use the fact that line bundle is very ample if and only if it separates points and tangent vectors; see [Huy06].

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