

ON HIGGS BUNDLES AND HIGGS FUNDAMENTAL GROUP SCHEMES

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ABSTRACT. Let X be a connected reduced proper scheme defined over an algebraically closed field k . We discuss a natural extension of Nori's theory of principal G -bundle as a functor to the case of principal G -Higgs bundles, for G an affine k -group scheme. Then we use this to show invariance of base points for Higgs fundamental group schemes of smooth projective k -varieties.

1. INTRODUCTION

Let X be a connected reduced proper scheme defined over an algebraically closed field k . Let G be an affine group scheme over k , and denote by $\mathcal{R}ep_k(G)$ the category of all k -linear representations of G . In [Nor76], Nori established a one-to-one correspondence between the principal G -bundles on X and the functors $\mathcal{R}ep_k(G) \rightarrow \mathcal{Q}coh(X)$ satisfying certain axioms. In this note, we show that this correspondence can be generalized to the case of principal G -Higgs bundles on X .

Let $\mathcal{H}iggs_G(X)$ the category of all principal G -Higgs bundles on X . Let $\mathcal{H}iggs(X)$ be the category of all quasi-coherent Higgs sheaves on X , and let

$$\mathcal{F}un_{\text{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$$

be the full subcategory of the functor category $\mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$ whose objects satisfies axioms (HF1) – (HF6) as stated in Proposition 2.5.2; see also (2.5.4). Then we have the following.

Theorem 1.0.1. *There is an equivalence of categories*

$$\Phi : \mathcal{H}iggs_G(X) \longrightarrow \mathcal{F}un_{\text{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)).$$

Let X be a connected smooth projective k -variety. Let $\mathcal{H}iggs_0^{\text{nf}}(X)$ be the full subcategory of $\mathcal{H}iggs(X)$, whose objects are Higgs numerically flat (in short, *H-nflat*) Higgs bundles on X (see Definition 3.1.7). This is a k -linear symmetric monoidal

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category, and fixing a closed point $x \in X(k)$, we get a k -linear exact faithful tensor functor $\mathcal{F}_x^H : \text{Higgs}_0^{\text{nf}}(X) \rightarrow \text{Vect}(k)$ defined by sending a H-nflat Higgs bundle (E, θ) to its fiber $E_x \in \text{Vect}(k)$ at x . This gives us a neutral Tannakian category $(\text{Higgs}_0^{\text{nf}}(X), \otimes, \mathcal{O}_X, \mathcal{F}_x^H)$, and the affine k -group scheme $\pi_1^H(X, x)$ Tannakian dual to this is called the *Higgs fundamental group scheme* of X with base point at x . Then using Theorem 1.0.1, we prove the following.

Theorem 1.0.2. *Let X be a connected smooth projective k -variety. Fix a closed point $x \in X(k)$, and let $G := \pi_1^H(X, x)$. Then there is a principal G -Higgs bundle $\mathfrak{P} := (\mathcal{P}, \theta_{\mathcal{P}})$ on X such that given any object $\mathfrak{E} := (E, \theta) \in \text{Higgs}_0^{\text{nf}}(X)$, there is an object $\rho : G \rightarrow \text{GL}(V)$ in $\text{Rep}_k(G)$ such that $\mathfrak{E} = \mathfrak{P} \times^{\rho} V$.*

As an immediate corollary to this, we obtain the following.

Corollary 1.0.3. *Let X be a connected smooth projective k -variety. For any two points $x_1, x_2 \in X(k)$, the affine k -group schemes $\pi_1^H(X, x_1)$ and $\pi_1^H(X, x_2)$ are isomorphic.*

2. HIGGS BUNDLES

2.1. Notations. A k -scheme X is said to be connected if $H^0(X, \mathcal{O}_X) \cong k$. For a k -scheme X , denote by $\mathfrak{Q}\mathfrak{Coh}(X)$ the category of coherent sheaves on X , and let $\mathfrak{Coh}(X)$ (resp., $\text{Vect}(X)$) be the full subcategory of $\mathfrak{Q}\mathfrak{Coh}(X)$, whose objects are coherent sheaves (resp., locally free coherent sheaves) on X . There are natural fully faithful embeddings $\text{Vect}(X) \subset \mathfrak{Coh}(X) \subset \mathfrak{Q}\mathfrak{Coh}(X)$. The objects of $\text{Vect}(X)$ are also referred to as vector bundles on X . When $X = \text{Spec}(k)$, the category $\text{Vect}(\text{Spec}(k))$ coincides with the category of all finite dimensional k -vector spaces $\text{Vect}(k)$, and hence we simply denote it by $\text{Vect}(k)$. For a locally free coherent sheaf (vector bundle) E on X and a point $x \in X$, on contrary to the usual notation of stalk, we denote by E_x the fiber of E at x ; whereas the notation $\mathcal{O}_{X,x}$ is preserved to denote the stalk at x of the structure sheaf \mathcal{O}_X . For any group scheme G over k , denote by $\text{Lie}(G)$ the Lie algebra of G .

2.2. The category of Higgs sheaves. Let X be a connected reduced proper k -scheme.

Definition 2.2.1. A *Higgs sheaf* on X is a pair (E, θ) , where E is a quasi-coherent sheaf of \mathcal{O}_X -modules on X and $\theta : E \rightarrow E \otimes \Omega_X^1$ is an \mathcal{O}_X -module homomorphism such that $\theta \wedge \theta = 0$ in $H^0(X, \text{End}(E) \otimes \Omega_X^2)$. When E is coherent we call (E, θ) a *coherent Higgs sheaf* on X . Similarly, for E a locally free coherent sheaf on X , we call (E, θ) a *Higgs bundle* on X .

Given two Higgs sheaves $\mathfrak{E} = (E, \theta)$ and $\mathfrak{E}' = (E', \theta')$ on X , a morphism from \mathfrak{E} to \mathfrak{E}' is given by an \mathcal{O}_X -module homomorphism $f : E \rightarrow E'$ such that the following diagram commutes

$$(2.2.2) \quad \begin{array}{ccc} E & \xrightarrow{\theta} & E \otimes \Omega_X^1 \\ \downarrow f & & \downarrow f \otimes \text{Id}_{\Omega_X^1} \\ E' & \xrightarrow{\theta'} & E' \otimes \Omega_X^1. \end{array}$$

Moreover, the direct sum and tensor product of two Higgs sheaves \mathfrak{E} and \mathfrak{E}' are again Higgs sheaves, given by

$$\begin{aligned} \mathfrak{E} \oplus \mathfrak{E}' &:= (E \oplus E', \theta \oplus \theta'), \quad \text{and} \\ \mathfrak{E} \otimes \mathfrak{E}' &:= (E \otimes E', \theta \otimes \text{Id}_{E'} + \text{Id}_E \otimes \theta'). \end{aligned}$$

Let $\mathcal{Higgs}(X)$ be the category whose objects are Higgs sheaves on X and morphisms are defined by commutative diagrams as in (2.2.2). Then $\mathcal{Higgs}(X)$ is an abelian category. In fact, $\mathcal{QCoh}(X)$ admits a natural fully faithful embedding inside $\mathcal{Higgs}(X)$ by considering zero Higgs field. We denote by $\mathcal{Higgs}_{\mathcal{Coh}}(X)$ the full subcategory of $\mathcal{Higgs}(X)$ whose objects are coherent Higgs sheaves on X . Denote by $\mathcal{Higgs}_0(X)$ the full subcategory of $\mathcal{Higgs}(X)$ whose objects are locally free coherent Higgs sheaves on X . Thus, we have fully faithful embeddings

$$\begin{array}{ccccc} \text{Vect}(X) & \hookrightarrow & \mathcal{Coh}(X) & \hookrightarrow & \mathcal{QCoh}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Higgs}_0(X) & \hookrightarrow & \mathcal{Higgs}_{\mathcal{Coh}}(X) & \hookrightarrow & \mathcal{Higgs}(X). \end{array}$$

Proposition 2.2.3. *Direct limit of a direct system of coherent Higgs sheaves exists in the category of quasi-coherent Higgs sheaves.*

Proof. Obvious. □

2.3. Principal G -Higgs bundles. Let G be a k -group scheme.

Definition 2.3.1. A *principal G -bundle* on X is a k -variety P together with a G -action $\sigma : P \times G \rightarrow P$ on P , and a G -invariant morphism of k -schemes $\pi : P \rightarrow X$ such that the morphism $(\text{pr}_1, \sigma) : P \times_k G \rightarrow P \times_X P$ induced by σ and the projection map $\text{pr}_1 : P \times G \rightarrow P$, is an isomorphism.

Let P be a principal G -bundle on X . Let $\rho : G \rightarrow \text{GL}(V)$ be a finite dimensional k -linear representation of G . Then G -acts on $P \times V$ by $(z, v) \cdot g := (z \cdot g, \rho(g)^{-1}(v))$, for all $z \in P$, $v \in V$ and $g \in G$. The associated quotient $P \times^\rho V := (P \times V)/G$ is

a vector bundle of rank $r = \dim_k(V)$ on X , denoted by ρ_*P . Using Grothendieck's theory of flat descent [Gro71], the vector bundle ρ_*P can be constructed as a locally free coherent sheaf on X by taking G -invariants of $\mathcal{O}_P \otimes_k V$. The vector bundle $\text{ad}(P) := P \times^{\text{ad}} \mathfrak{g}$ associated to the adjoint representation

$$(2.3.2) \quad \text{ad} : G \longrightarrow \text{GL}(\mathfrak{g})$$

of G on its Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is called the *adjoint vector bundle* of P . Note that, $\text{ad}(P)$ is a Lie algebra bundle on X .

Definition 2.3.3. A *principal G -Higgs bundle* on X is a pair $\mathfrak{P} := (P, \theta)$, where P is a principal G -bundle on X and $\theta \in H^0(X, \text{ad}(P) \otimes \Omega_X^1)$ such that $\theta \wedge \theta = 0$ in $H^0(X, \text{ad}(P) \otimes \Omega_X^2)$.

Let P and P' be two principal G -bundles on X . Then any homomorphism of principal G -bundles $\varphi : P \rightarrow P'$ induces a homomorphism of their adjoint vector bundles

$$(2.3.4) \quad \text{ad}(\varphi) : \text{ad}(P) \rightarrow \text{ad}(P')$$

Tensoring with Ω_X^1 and taking global section functor, we have a k -linear homomorphism

$$(2.3.5) \quad \tilde{\varphi} : H^0(X, \text{ad}(P) \otimes \Omega_X^1) \longrightarrow H^0(X, \text{ad}(P') \otimes \Omega_X^1).$$

Let $\mathfrak{P} = (P, \theta)$ and $\mathfrak{P}' = (P', \theta')$ be two principal G -Higgs bundles on X .

Definition 2.3.6. A morphism of principal G -Higgs bundles $\mathfrak{P} \rightarrow \mathfrak{P}'$ is given by a morphism of principal G -bundles $\varphi : P \rightarrow P'$ such that the induced homomorphism $\tilde{\varphi}$ in (2.3.5) sends θ to θ' .

2.4. Principal G -Higgs bundle as a functor. Let $\mathcal{Higgs}(X)$ be the category of coherent Higgs sheaves on X (see §2.2). Let G be an affine k -group scheme. Let $\text{Rep}_k^{\text{fd}}(G)$ be the category of finite dimensional k -linear representations of G ; its objects are pair (V, ρ) , where V is a finite dimensional k -vector space and $\rho : G \rightarrow \text{GL}(V)$ is a group homomorphism. A morphism $(V, \rho) \rightarrow (V', \rho')$ in $\text{Rep}_k^{\text{fd}}(G)$ is given by a G -equivariant homomorphism of k -vector spaces $V \rightarrow V'$. The category $\text{Rep}_k^{\text{fd}}(G)$ admits finite direct sum and tensor products.

Let $\mathfrak{P} = (P, \theta)$ be a principal G -Higgs bundle on X . Any finite dimensional k -linear representation $\rho : G \rightarrow \text{GL}(V)$ give rise to a G -module homomorphism

$$(2.4.1) \quad d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) := \text{Lie}(\text{GL}(V)),$$

which in turn give rise to a homomorphism of vector bundles

$$(2.4.2) \quad (d\rho)_P : \text{ad}(P) := P \times^{\text{ad}} \mathfrak{g} \longrightarrow \text{End}(\rho_*P),$$

where $\rho_* P := P \times^\rho V$ is the vector bundle on X associated to P and the representation (V, ρ) . This gives a k -linear homomorphism

$$(2.4.3) \quad \tilde{\rho}_P : H^0(X, \text{ad}(P) \otimes \Omega_X^1) \longrightarrow H^0(X, \text{End}(\rho_* P) \otimes \Omega_X^1).$$

Thus we obtain a Higgs bundle

$$(2.4.4) \quad \rho_* \mathfrak{P} := (\rho_* P, \rho_* \theta)$$

on X , where $\rho_* P := P \times^\rho V$ and $\rho_* \theta = \tilde{\rho}_P(\theta) \in H^0(X, \text{End}(\rho_* P) \otimes \Omega_X^1)$.

A morphism $\varphi : (V, \rho) \longrightarrow (V', \rho')$ in $\mathcal{R}ep_k^{\text{fd}}(G)$ give rise to the following commutative diagram of (Lie algebras) G -module homomorphisms

$$(2.4.5) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\rho} & \mathfrak{gl}(V) \\ \parallel & & \downarrow \tilde{\varphi} \\ \mathfrak{g} & \xrightarrow{d\rho'} & \mathfrak{gl}(V'), \end{array}$$

which makes the following diagram of k -linear maps commutative

$$(2.4.6) \quad \begin{array}{ccc} H^0(X, \text{ad}(P) \otimes \Omega_X^1) & \xrightarrow{\tilde{\rho}_P} & H^0(X, \text{End}(\rho_*(P)) \otimes \Omega_X^1) \\ \parallel & & \downarrow \tilde{\varphi} \\ H^0(X, \text{ad}(P) \otimes \Omega_X^1) & \xrightarrow{\tilde{\rho}'_P} & H^0(X, \text{End}(\rho'_*(P)) \otimes \Omega_X^1). \end{array}$$

Thus we get a homomorphism of Higgs bundles

$$(2.4.7) \quad \varphi_P : \rho_* \mathfrak{P} \longrightarrow \rho'_* \mathfrak{P};$$

(see (2.4.4)). The above construction is functorial, and hence give rise to a covariant functor

$$(2.4.8) \quad \Phi_{\mathfrak{P}} : \mathcal{R}ep_k^{\text{fd}}(G) \longrightarrow \mathcal{H}iggs_0(X),$$

which sends an object $(V, \rho) \in \mathcal{R}ep_k^{\text{fd}}(G)$ to the Higgs bundle $\rho_* \mathfrak{P} := (\rho_* P, \rho_* \theta)$ as defined in (2.4.4), and a morphism $\varphi : (V, \rho) \rightarrow (V', \rho')$ to φ_P as defined in (2.4.7).

Proposition 2.4.9. *The functor $\Phi_{\mathfrak{P}}$ defined in (2.4.8) preserve finite direct sums and tensor products.*

Proof. It is well-known that $(V, \rho) \mapsto \rho_* P = P \times^\rho V$ is a covariant additive tensor functor of tensor abelian categories $\mathcal{R}ep_k^{\text{fd}}(G) \rightarrow \mathcal{V}ect(X)$. Therefore, it is enough to check what happens to the Higgs fields.

Let $(V_1, \rho_1), (V_2, \rho_2) \in \mathcal{R}ep_k^{\text{fd}}(G)$. It follows from the commutative diagram of G -module homomorphisms

$$(2.4.10) \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\rho_1} & \mathfrak{gl}(V_1) \\ \downarrow d\rho_2 & & \downarrow \\ \mathfrak{gl}(V_2) & \longrightarrow & \mathfrak{gl}(V_1 \oplus V_2) \end{array}$$

and the corresponding induced homomorphisms of vector bundles induced by P that $(\rho_1 \oplus \rho_2)_* \theta = (\rho_{1*} \theta) \oplus (\rho_{2*} \theta)$. Similarly, for the case of tensor product representation $\rho_1 \otimes \rho_2 : G \rightarrow \text{GL}(V_1 \otimes V_2)$, we have $(\rho_1 \otimes \rho_2)_* \theta = (\rho_{1*} \theta \otimes \text{Id}) + (\text{Id} \otimes \rho_{2*} \theta)$. Hence the result follows. \square

2.5. Recovering G -Higgs bundle from the associated functor. Let $\mathcal{H}iggs(G, X)$ be the category whose objects are principal G -Higgs bundles on X , and morphisms are morphisms of principal G -Higgs bundles (see Definition 2.3.6). Given any two categories \mathcal{C} and \mathcal{D} , we denote by $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$ and morphisms are natural transformations of those functors.

Following [Nor76], let

$$(2.5.1) \quad \mathcal{F}un_{\text{HF}}(\mathcal{R}ep_k^{\text{fd}}(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k^{\text{fd}}(G), \mathcal{H}iggs(X))$$

be the full subcategory of $\mathcal{F}un(\mathcal{R}ep_k^{\text{fd}}(G), \mathcal{H}iggs(X))$ whose objects are functors

$$\mathcal{F} : \mathcal{R}ep_k^{\text{fd}}(G) \longrightarrow \mathcal{H}iggs(X)$$

satisfying the following axioms (HF1) – (HF6):

- (HF1) \mathcal{F} is a faithful k -linear exact functor,
- (HF2) \mathcal{F} sends trivial G -module to $(\mathcal{O}_X, 0)$ in $\mathcal{H}iggs(X)$,
- (HF3) $\mathcal{F} \circ \otimes = \otimes \circ (\mathcal{F} \times \mathcal{F})$,
- (HF4) \otimes in $\mathcal{R}ep_k^{\text{fd}}(G)$ is associative and compatible with \mathcal{F} ,
- (HF5) \otimes in $\mathcal{R}ep_k^{\text{fd}}(G)$ is commutative and compatible with \mathcal{F} , and
- (HF6) if $V \in \mathcal{R}ep_k^{\text{fd}}(G)$ is of rank n , then $\mathcal{F}(V)$ is a Higgs bundle of rank n over X .

Let $\mathcal{R}ep_k(G)$ be the category of all (including infinite dimensional) k -linear representations of G . Note that, $\mathcal{R}ep_k^{\text{fd}}(G)$ is a full subcategory of $\mathcal{R}ep_k(G)$, and [Nor76, Lemma 2.1] generalizes to the following.

Proposition 2.5.2. *Any functor $\mathcal{F} : \mathcal{R}ep_k^{\text{fd}}(G) \longrightarrow \mathcal{H}iggs(X)$ satisfying axioms (HF1) – (HF6) extends uniquely to a functor $\widehat{\mathcal{F}} : \mathcal{R}ep_k(G) \longrightarrow \mathcal{H}iggs(X)$ such that*

- (i) the axioms (HF1) – (HF5) holds for $\widehat{\mathcal{F}}$,
- (ii) $\widehat{\mathcal{F}}$ restricts to \mathcal{F} on $\mathcal{R}ep_k^{\text{fd}}(G)$,

- (iii) the underlined \mathcal{O}_X -module of $\widehat{\mathcal{F}}(V)$ is flat, for all $V \in \mathcal{R}ep_k(G)$, and is faithfully flat if $V = 0$, and
- (iv) $\widehat{\mathcal{F}}$ preserves direct limits.

Proof. In view of Proposition 2.2.3, given any object $V \in \mathcal{R}ep_k(G)$, we define $\widehat{\mathcal{F}}(V) := \varinjlim \mathcal{F}(W)$, where W runs through the directed system of all finite dimensional G -invariant k -linear subspaces of V . Then the result follows. \square

Henceforth, we use the same notation \mathcal{F} to denote the extended functor $\widehat{\mathcal{F}}$ as in Proposition 2.5.2. The category under consideration would be clear from the context.

It follows from the construction discussed in the subsection §2.4 that given any principal G -Higgs bundle $\mathfrak{P} = (P, \theta)$ on X , the associated covariant functor

$$\Phi_{\mathfrak{P}} : \mathcal{R}ep_k^{\text{fd}}(G) \longrightarrow \mathcal{H}iggs_0(X)$$

defined in (2.4.8) satisfies the axioms (HF1) – (HF6), and hence extends uniquely to a covariant functor, also denoted by

$$(2.5.3) \quad \Phi_{\mathfrak{P}} : \mathcal{R}ep_k(G) \longrightarrow \mathcal{H}iggs(X)$$

satisfying the conditions (i) – (iv) as in Proposition 2.5.2. We want to show that the converse also holds. More precisely, we construct a natural equivalence between the category of principal G -Higgs bundles on X and the full subcategory

$$(2.5.4) \quad \mathcal{F}un_{\text{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$$

of functors $\mathcal{F} : \mathcal{R}ep_k(G) \rightarrow \mathcal{H}iggs(X)$ as described in Proposition 2.5.2.

Let $\mathfrak{P} = (P, \theta)$ and $\mathfrak{P}' = (P', \theta')$ be two principal G -Higgs bundles on X . Let $\varphi : \mathfrak{P} \rightarrow \mathfrak{P}'$ be a morphism of principal G -Higgs bundles (see Definition 2.3.6). Then for an object $(V, \rho) \in \mathcal{R}ep_k(G)$, we have a homomorphism of vector bundles

$$(2.5.5) \quad \varphi_{\rho} : \rho_* P \longrightarrow \rho_* P'.$$

In particular, for the adjoint representation $\text{ad} : G \rightarrow \text{GL}(\mathfrak{g})$, we have a homomorphism of adjoint vector bundles $\text{ad}(P) \rightarrow \text{ad}(P')$. Since the induced homomorphism of Lie algebras $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a G -module homomorphism, we have a commutative diagram of vector bundle homomorphisms

$$(2.5.6) \quad \begin{array}{ccc} \text{ad}(P) & \longrightarrow & \text{End}(\rho_* P) \\ \downarrow & & \downarrow \\ \text{ad}(P') & \longrightarrow & \text{End}(\rho_* P'), \end{array}$$

where the horizontal homomorphisms are induced by the G -module homomorphism $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, and the left and the right vertical homomorphisms are induced

by the adjoint action of G on \mathfrak{g} and the induced G -action on $\mathfrak{gl}(V)$, respectively. Now tensoring the commutative diagram (2.5.6) with Ω_X^1 , it follows that φ_ρ sends the Higgs field $\rho_*\theta \in H^0(X, \text{End}(\rho_*P) \otimes \Omega_X^1)$ to $\rho_*\theta' \in H^0(X, \text{End}(\rho_*P') \otimes \Omega_X^1)$. Thus we have a homomorphism of Higgs bundles

$$(2.5.7) \quad \Phi_\varphi(\rho) : \Phi_{\mathfrak{P}}(\rho) \longrightarrow \Phi_{\mathfrak{P}'}(\rho)$$

where $\Phi_{\mathfrak{P}}(\rho) := \rho_*\mathfrak{P} = (\rho_*P, \rho_*\theta)$ and $\Phi_{\mathfrak{P}'}(\rho) := \rho_*\mathfrak{P}' = (\rho_*P', \rho_*\theta')$.

Given a morphism

$$(2.5.8) \quad \eta : (V_1, \rho_1) \longrightarrow (V_2, \rho_2)$$

in $\text{Rep}_k(G)$ and any principal G -Higgs bundle $\mathfrak{P} = (P, \theta)$ on X , the construction just before the Proposition 2.4.9 give rise to a homomorphism of flat Higgs sheaves

$$(2.5.9) \quad \Phi_{\mathfrak{P}}(\eta) : \Phi_{\mathfrak{P}}(\rho_1) \longrightarrow \Phi_{\mathfrak{P}}(\rho_2).$$

Now it follows from the construction in the preceding paragraph that, given any morphism of principal G -Higgs bundles $\varphi : \mathfrak{P} \longrightarrow \mathfrak{P}'$ on X , the following diagram is commutative.

$$(2.5.10) \quad \begin{array}{ccc} \rho_{1*}\mathfrak{P} = (\rho_{1*}P, \rho_{1*}\theta) & \xrightarrow{\Phi_\varphi(\rho_1)} & \rho_{1*}\mathfrak{P}' = (\rho_{1*}P', \rho_{1*}\theta') \\ \downarrow \Phi_{\mathfrak{P}}(\eta) & & \downarrow \Phi_{\mathfrak{P}'}(\eta) \\ \rho_{2*}\mathfrak{P} = (\rho_{2*}P, \rho_{2*}\theta) & \xrightarrow{\Phi_\varphi(\rho_2)} & \rho_{2*}\mathfrak{P}' = (\rho_{2*}P', \rho_{2*}\theta'). \end{array}$$

In other words, $\varphi : \mathfrak{P} \longrightarrow \mathfrak{P}'$ induces a morphism of functors $\Phi_\varphi : \Phi_{\mathfrak{P}} \longrightarrow \Phi_{\mathfrak{P}'}$.

$$(2.5.11) \quad \begin{array}{ccc} & \Phi_{\mathfrak{P}} & \\ & \curvearrowright & \\ \text{Rep}_k(G) & \xrightarrow{\Phi_\varphi} & \text{Higgs}(X) \\ & \curvearrowleft & \\ & \Phi_{\mathfrak{P}'} & \end{array}$$

Thus the above construction give rise to a functor

$$(2.5.12) \quad \Phi : \text{Higgs}_G(X) \longrightarrow \text{Fun}_{\text{HF}}(\text{Rep}_k(G), \text{Higgs}(X)).$$

defined by sending a principal G -Higgs bundle $\mathfrak{P} \in \text{Higgs}_G(X)$ on X to the functor $\Phi_{\mathfrak{P}}$ as defined in (2.5.3), and a morphism of principal G -Higgs bundles $\varphi : \mathfrak{P} \rightarrow \mathfrak{P}'$ on X to the morphism of functors Φ_φ defined in (2.5.11).

Theorem 2.5.13. *The functor Φ defined in (2.5.12) is an equivalence of categories.*

Proof. We first show that Φ is essentially surjective. Let $\mathcal{F} : \text{Rep}_k(G) \rightarrow \text{Higgs}(X)$ be a functor satisfying axioms (HF1) – (HF6). We need to show that there is a (unique) principal G -Higgs bundle $\mathfrak{P} = (P, \theta)$ on X such that $\Phi_{\mathfrak{P}} \cong \mathcal{F}$. Let $k[G]$ be the

function k -algebra of the affine k -group scheme G . There is a natural regular G -action on $k[G]$ given by

$$(2.5.14) \quad (g \cdot f)(a) := f(ga), \quad \forall \, g, a \in G \text{ and } f \in k[G].$$

Let E be the underlined \mathcal{O}_X -module of the Higgs sheaf $\mathcal{F}(k[G])$. Then the relative spectrum $\mathcal{P} := \text{Spec}_{\mathcal{O}_X}(E)$ together with the natural projection $\mathcal{P} \rightarrow X$ (affine morphism) is a principal G -bundle on X (see proof of [Nor76, Lemma 2.3, p. 32]). Since the associated locally free adjoint vector bundle (sheaf) $\text{ad}(\mathcal{P}) = (\mathcal{O}_{\mathcal{P}} \otimes \mathfrak{g})^G$ is naturally isomorphic to the locally free coherent sheaf $\text{End}(E)$, the Higgs field

$$\theta \in H^0(X, \text{End}(\mathcal{E}) \otimes \Omega_X^1) = H^0(X, \text{ad}(\mathcal{P}) \otimes \Omega_X^1)$$

of $\mathcal{F}(k[G])$ can be considered as a Higgs field on \mathcal{P} . Now with this $\mathfrak{P} := (\mathcal{P}, \theta)$, we have $\Phi_{\mathfrak{P}} \cong \mathcal{F}$. Thus Φ is essentially surjective.

To see Φ is faithful, note that if $\Phi_{\varphi} = \Phi_{\psi}$, for some morphisms of principal G -Higgs bundles $\varphi, \psi : \mathfrak{P}_1 \rightarrow \mathfrak{P}_2$, then for any k -linear representation $\rho : G \rightarrow \text{GL}(V)$ we have $\Phi_{\varphi}(\rho) = \Phi_{\psi}(\rho)$; see (2.5.7). In particular, taking $V = k[G]$ together with the natural regular G -action described in (2.5.14), we see that $\varphi = \psi$. To see Φ is full, given morphism of functors $\mathcal{F} : \Phi_{\mathfrak{P}_1} \rightarrow \Phi_{\mathfrak{P}_2}$ in $\text{Fun}_{\text{HF}}(\text{Rep}_k(G), \text{Higgs}(X))$, we can use the G -module $k[G]$ as above to get a morphism of G -Higgs bundles $\psi : \mathfrak{P}_1 \rightarrow \mathfrak{P}_2$ on X such that $\Phi_{\psi} = \mathcal{F}$. Thus Φ is an equivalence of categories. \square

Let $\mathfrak{P} := (P, \theta)$ be a principal G -Higgs bundle on X . Given any morphism of k -schemes $f : Y \rightarrow X$, we can pullback P along f to get a principal G -bundle $f^*P := P \times_X Y$ on Y . Then the image of θ under the induced natural k -linear homomorphism

$$(2.5.15) \quad H^0(X, \text{ad}(P) \otimes \Omega_X^1) \longrightarrow H^0(Y, \text{ad}(f^*P) \otimes \Omega_Y^1)$$

gives a Higgs field $f^*\theta$ on f^*P . Thus we obtain a principal G -Higgs bundle $f^*\mathfrak{P} := (f^*P, f^*\theta)$ on Y .

Let $\sigma : G \rightarrow H$ is a homomorphism of affine k -group schemes. Given a principal G -bundle P on X , we can extend the structure group of P by σ to get a principal H -bundle on X as follow: take quotient of $P \times H$ by the equivalence relation

$$(z, h) \cdot g \sim (z \cdot g, \sigma(g)^{-1}h), \quad \forall \, z \in P, g \in G, \text{ and } h \in H,$$

induced by the twisted G -action on $P \times H$ to obtain a principal H -bundle

$$\sigma_*P := (P \times H) / \sim$$

on X . Let $\mathcal{R}_{\sigma} : \text{Rep}_k(H) \longrightarrow \text{Rep}_k(G)$ be the functor obtained by sending an object $\rho : H \rightarrow \text{GL}(V)$ of $\text{Rep}_k(H)$ to the object $\rho \circ \sigma : G \rightarrow \text{GL}(V)$ of $\text{Rep}_k(G)$. Considering the adjoint representations of both G and H to their Lie algebras \mathfrak{g} and \mathfrak{h} , respectively,

and the Lie algebra homomorphism $d\sigma : \mathfrak{g} \rightarrow \mathfrak{h}$ induced by σ , we get a k -linear homomorphism (denoted by the same symbol)

$$(2.5.16) \quad \sigma_* : H^0(X, \mathrm{ad}(P) \otimes \Omega_X^1) \rightarrow H^0(X, \mathrm{ad}(\sigma_* P) \otimes \Omega_X^1).$$

Thus, given a principal G -Higgs bundle $\mathfrak{P} := (P, \theta)$ on X , we obtain a principal H -Higgs bundle $\sigma_* \mathfrak{P} := (\sigma_* P, \sigma_* \theta)$ on X . Then we have the following.

Proposition 2.5.17. *With the above notations, if \mathfrak{P} is a principal G -Higgs bundle on X , then the following hold.*

- (i) *For any morphism $f : Y \rightarrow X$ of k -schemes, pulled-back of $\Phi_{\mathfrak{P}}$ along f is the functor $f^* \circ \Phi_{\mathfrak{P}} = \Phi_{f^* \mathfrak{P}}$, and hence $f^* \circ \Phi_{\mathfrak{P}} \in \mathrm{Fun}_{\mathrm{HF}}(\mathrm{Rep}_k(G), \mathrm{Higgs}(Y))$.*
- (ii) *For any homomorphism $\sigma : G \rightarrow H$ of affine k -group schemes, $\Phi_{\mathfrak{P}} \circ \mathcal{R}_{\sigma} = \Phi_{\sigma_* \mathfrak{P}}$.*

Proof. Follows by chasing construction of the functor Φ in (2.5.12). \square

3. HIGGS FUNDAMENTAL GROUP SCHEMES

3.1. Numerically flat Higgs bundles. Let us first recall some definitions from [BBG19] that we need. Let X be a connected smooth projective k -variety. Let E be a locally free coherent sheaf of rank r (≥ 2) on X . Fix a positive integer s with $s < r$, and consider the functor:

$$(3.1.1) \quad \mathrm{Gr}(E, s) : (\mathrm{Sch}/X)^{\mathrm{op}} \longrightarrow (\mathrm{Set})$$

given by sending $g : T \rightarrow X \in (\mathrm{Sch}/X)$ to the set

$$\mathrm{Gr}(E, s)(T) := \{q : g^* E \twoheadrightarrow F \mid q \text{ is surjective and } F \text{ is a locally free coherent sheaf of rank } s \text{ on } T\} / \sim,$$

where two such quotients $q : g^* E \twoheadrightarrow F$ and $q' : g^* E \twoheadrightarrow F'$ are said to be equivalent, denoted $q \sim q'$, if $\mathrm{Ker}(q) = \mathrm{Ker}(q')$. There is a projective X -scheme

$$(3.1.2) \quad p : \mathrm{Gr}(E, s) \longrightarrow X$$

which represents the functor $\mathrm{Gr}(E, s)$, meaning that there is a natural isomorphism of functors $\mathrm{Gr}(E, s) \xrightarrow{\sim} \mathrm{Mor}_{(\mathrm{Sch}/X)}(-, \mathrm{Gr}(E, s))$. In particular, the identity morphism of $\mathrm{Gr}(E, s)$ corresponds to an exact sequence of locally free coherent sheaves

$$(3.1.3) \quad 0 \longrightarrow \mathcal{S}(E, s) \xrightarrow{\Psi} p^* E \xrightarrow{\mathcal{F}} \mathcal{Q}(E, s) \longrightarrow 0,$$

known as the universal exact sequence over $\mathrm{Gr}(E, s)$.

If $\mathfrak{E} := (E, \theta)$ is a Higgs bundle on X , then its pullback $p^* \mathfrak{E} := (p^* E, p^* \theta)$ is a Higgs bundle on $\mathrm{Gr}(E, s)$, where $p : \mathrm{Gr}(E, s) \rightarrow X$ is the Grassmannian as described in (3.1.2). The Higgs field $p^* \theta$ naturally induces a Higgs field on the universal quotient

$\mathcal{Q}(E, s)$ making it a quotient Higgs bundle of $(p^*E, p^*\theta)$ if and only if the universal kernel bundle $\mathcal{S}(E, s)$ is preserved under $p^*\theta$ in the sense that

$$p^*\theta(\mathcal{S}(E, s)) \subseteq \mathcal{S}(E, s) \otimes \Omega_{\mathrm{Gr}(E, s)}^1.$$

Let $\mathfrak{Gr}(E, s) \subseteq \mathrm{Gr}(E, s)$ be the subscheme defined by the vanishing locus of the following composite homomorphism

$$(3.1.4) \quad \mathcal{S}(E, s) \xrightarrow{\Psi} p^*E \xrightarrow{p^*\theta} p^*E \otimes p^*\Omega_X^1 \xrightarrow{\mathcal{F} \otimes \mathrm{Id}} \mathcal{Q}(E, s) \otimes p^*\Omega_X^1,$$

(see (3.1.3)). It follows that $\mathfrak{Gr}(E, s)$ is the closed subscheme of $\mathrm{Gr}(E, s)$ parametrizing the quotient Higgs bundles of (E, θ) , and we call it the *Higgs Grassmannian* of (E, θ) . This closed embedding of $\mathfrak{Gr}(E, s)$ into $\mathrm{Gr}(E, s)$ gives rise to an exact sequence on $\mathfrak{Gr}(E, s)$

$$(3.1.5) \quad 0 \longrightarrow \mathcal{S}(\mathfrak{E}, s) \xrightarrow{\Psi} p^*\mathfrak{E} \xrightarrow{\mathcal{F}} \mathcal{Q}(\mathfrak{E}, s) \longrightarrow 0,$$

where $\mathcal{Q}(\mathfrak{E}, s)$ may be called the universal Higgs quotient for \mathfrak{E} (c.f., (3.1.3)).

Definition 3.1.6. Let $\mathfrak{E} = (E, \theta)$ be a Higgs bundle on X . If $\mathrm{rk}(E) = 1$ and E is numerically effective, we say that (E, θ) is *Higgs numerically effective* (in short, *H-nef*). When $r := \mathrm{rk}(E) > 1$, we define *H-nefness* inductively by requiring that

- (1) the universal Higgs quotient $\mathcal{Q}(\mathfrak{E}, s)$ is *H-nef*, for all $s = 1, \dots, r-1$, and
- (2) $\det(E) := \bigwedge^r E$ is *nef*.

Definition 3.1.7. A Higgs bundle $\mathfrak{E} := (E, \theta)$ on X is said to be *Higgs numerically flat* (in short, *H-nflat*) if both \mathfrak{E} and its dual Higgs bundle \mathfrak{E}^\vee are *H-nef*.

Let $\mathrm{Higgs}_0(X) \subset \mathrm{Higgs}(X)$ be the full subcategory of Higgs bundles (locally free) on X . Let $\mathrm{Higgs}_0^{\mathrm{nf}}(X)$ be the full subcategory of $\mathrm{Higgs}_0(X)$ whose objects are Higgs numerically flat in the sense of Definition 3.1.7. It is known that $\mathrm{Higgs}_0^{\mathrm{nf}}(X)$ is an abelian category closed under tensor product, and has a structure of a k -linear symmetric monoidal category. As before, fixing a closed point $x \in X(k)$, we have a faithful exact k -linear tensor functor

$$(3.1.8) \quad \mathcal{F}_x^H : \mathrm{Higgs}_0^{\mathrm{nf}}(X) \longrightarrow \mathrm{Vect}(k)$$

given by sending $(E, \theta) \in \mathrm{Higgs}_0^{\mathrm{nf}}(X)$ to its fiber $E_x \in \mathrm{Vect}(k)$ at x . It turns out that the quadruple $(\mathrm{Higgs}_0^{\mathrm{nf}}(X), \otimes, \mathcal{O}_X, \mathcal{F}_x^H)$ is a neutral Tannakian category, and the associated affine k -group scheme $\pi_1^H(X, x)$ representing the functor of k -algebras $\underline{\mathrm{Aut}}^\otimes(\mathcal{F}_x^H)$ is called the *Higgs fundamental group scheme of X with base point at x* .

Remark 3.1.9. The notion of Higgs fundamental group scheme of a connected reduced proper k -scheme (not necessarily smooth) should make sense following the same line of arguments given in [BBG19].

Theorem 3.1.10. *Let X be a connected smooth projective k -variety. Fix a closed point $x \in X(k)$, and let $G := \pi_1^H(X, x)$. Then there is a principal G -Higgs bundle $\mathfrak{P} := (\mathcal{P}, \theta_{\mathcal{P}})$ on X such that given any object $\mathfrak{E} := (E, \theta) \in \text{Higgs}_0^{\text{nf}}(X)$, there is an object $\rho : G \rightarrow \text{GL}(V)$ in $\text{Rep}_k(G)$ such that $\mathfrak{E} = \mathfrak{P} \times^{\rho} V$.*

Proof. It follows from [DM82, Theorem 2.11] that the fiber functor \mathcal{F}_x^H in (3.1.8) defines an equivalence of k -linear tensor abelian categories

$$(3.1.11) \quad \widehat{\mathcal{F}_x^H} : \text{Higgs}_0^{\text{nf}}(X) \longrightarrow \text{Rep}_k^{\text{fd}}(G),$$

whose composition with the forgetful functor $\text{Rep}_k^{\text{fd}}(G) \rightarrow \text{Vect}(k)$ gives the fiber functor \mathcal{F}_x^H . Now one can check that the inverse of the equivalence $\widehat{\mathcal{F}_x^H}$ in (3.1.11) give rise to an object of $\mathcal{F}_{\text{unHF}}(\text{Rep}_k(G), \text{Higgs}(X))$ (see (2.5.4) for the definition of this category), and hence by Theorem 2.5.13, it is isomorphic to a functor $\Phi_{\mathfrak{P}}$ for some unique principal G -Higgs bundle \mathfrak{P} on X . From this the result follows. \square

Corollary 3.1.12. *Let X be a connected smooth projective k -variety. For any two points $x_1, x_2 \in X(k)$, the affine k -group schemes $\pi_1^H(X, x_1)$ and $\pi_1^H(X, x_2)$ are isomorphic.*

Proof. Using Theorem 3.1.10 above, the result follows from the proof of [PS20, Lemma 2.2.2], mutatis mutandis. \square

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