MA5114: Riemannian Geometry

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If you find any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\bigsqcup$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
$A :=$ ...	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Riemannian Geometry

MA5114 Syllabus

Metric: Definition of Riemannian metric and Riemannian manifolds.

Connections: Definition, Levi-Civita connection, covariant derivatives, parallel transport.

Geodesics: The concepts of geodesics, geodesics in the upper half plane, first variational formula, local existence and uniqueness of geodesics, the exponential map, Hopf-Rinow theorem.

Curvature: Curvature tensor and fundamental form, computation of curvature with examples, Ricci, sectional and scalar curvature.

References:

1. Loring W. Tu, *Differential geometry: Connections, curvature, and characteristic classes*, Graduate Texts in Mathematics, 275. Springer, Cham, 2017. xvi+346 pp.

2. John M. Lee, Introduction to Riemannian Manifolds, *Graduate Texts in Mathematics*, Springer Cham. doi:10.1007/978-3-319-91755-9.

1.1 Review of Manifold Theory

1.1.1 Real manifold

A topological space X is said to be *locally Euclidean* if every point $x \in X$ has an open neighbourhood U_x such that there is a homeomorphism φ_{U_x} from U_x onto an open subset of $\mathbb{R}^{n_x}$, for some $n_x \in \mathbb{N}$. The pair (U_x, φ_{U_x}) is called a coordinate chart of X at x. If $\varphi_{U_x}(x) = 0$, we say that (U_x, φ_{U_x}) is a coordinate chart on X *centered at* x. If $n_x = n$, for all $x \in X$, then X is said to have *dimension* n. A *topological manifold* (of dimension n) is a Hausdorff second countable locally Euclidean space (of dimension n).

Let M be a C^∞ manifold over $\mathbb{R}$.

1.1.2 Complex manifold

1.2 Riemannian Manifold

Let V be a vector space over $\mathbb{R}$. Recall that an *inner product* on V is a $\mathbb{R}$–bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

which is

(i) symmetric, i.e., $\langle v, w \rangle = \langle w, v \rangle$, for all $v, w \in V$, and

(ii) positive definite, i.e., $\langle v, v \rangle \geq 0$, for all $v \in V$, with equality holds if and only if $v = 0$.

For example, the Euclidean inner product on the $\mathbb{R}$–vector space $\mathbb{R}^n$ is given by the formula (dot product)

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle := \sum_{j=1}^{n} x_j y_j,$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. We generally use this to the *length* of a vector $v \in \mathbb{R}^n$ to be the real number

$$\|v\| := \sqrt{\langle v, v \rangle},$$

and the angle between two non-zero vectors u and v in $\mathbb{R}^n$ to be the real number $\theta \in [0, \pi] \subset \mathbb{R}$ such that

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|\|v\|}.$$

The *arc length* of a parametrized curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined to be the real number

$$s := \int_a^b \|\gamma'(t)\| dt,$$

where $\gamma'(t)$ is the derivative of γ with respect to t.

Definition 1.2.1. A *Riemannian metric* on M is a C^∞ section h of the vector bundle $TX \otimes_{\mathbb{C}} \overline{TX}$ such that for each $x \in M$ we have $h_x(\xi_x \otimes \overline{\xi}_x) > 0$, for all C^∞ vector field ξ defined on an open neighbourhood of x.

1.3 Vector bundles

1.3.1 Real and complex manifolds

1.4 Vector bundles

1.4.1 Tangent space

Let M be a C^∞ manifold. Let τ_M be the set of all open subsets of M. For a non-empty open subset U of M, we denote by $C^\infty_M(U)$ the set of all real valued C^∞ functions $U \to \mathbb{R}$ on U. Note that, $C^\infty_M(U)$ is an $\mathbb{R}$–algebra with respect to the point-wise addition and multiplication of real valued functions on U. For $U = \emptyset$, we set $C^\infty_M(\emptyset) = 0$, the zero $\mathbb{R}$–algebra. Given two open subsets $U, V \subseteq M$, the restriction of functions defines an $\mathbb{R}$–algebra homomorphism

$$res_{U,V} : C^\infty_M(U) \to C^\infty_M(V),\ f \mapsto f\big|_V.$$

that satisfies the following properties:

(i) $res_{U,U} = \mathrm{Id}_{C^\infty_{M,p}(U)}$, for all open subset U of M.

(ii) $res_{V,W} \circ res_{U,V} = res_{U,W}$, for all open subsets U, V, W of M with $W \subseteq V \subseteq U$.

(iii) Let U be an open subset of M and let $\{V_i : i \in I\}$ be an open cover of U. If $f \in C^\infty_{M,p}(U)$ satisfies $res_{U,V_i}(f) = 0$, for all $i \in I$, then $f = 0$.

(iv) If for each $i \in I$ we are given $f_i \in C^\infty_{M,p}(V_i)$ such that $f_i\big|_{V_i \cap V_j} = f_j\big|_{V_i \cap V_j}$, for all $i, j \in I$, then there exists a (unique) $f \in C^\infty_M(U)$ such that $f\big|_{V_i} = f_i$, for all $i \in I$.

In other words,

$$C^\infty_M : \tau_M \to Alg_{\mathbb{R}}$$

is a sheaf of $\mathbb{R}$–algebras on M.

Consider the set of all pairs (U, f), where U is an open neighbourhood of p in M and $f : U \to \mathbb{R}$ is a C^∞ function. Given any two such pairs (U, f) and (V, g), we define $(U, f) \sim (V, g)$, if there exists an open neighbourhood W of p in M such that $W \subseteq U \cap V$ and $f\big|_W = g\big|_W$. Note that $\sim$ is an equivalence relation on the set of all such pairs (U, f). The $\sim$–equivalence class of (U, f) is denote by $\langle (U, f) \rangle$, and is called the *germ of f* at p. Let $C^\infty_{M,p}$ be the set of all germs of C^∞ functions defined on some open neighbourhood of p in M. Note that $C^\infty_{M,p}$ is an $\mathbb{R}$–algebra with respect to the point-wise addition and multiplication of $\mathbb{R}$–valued functions. Moreover, for each open neighbourhood U of p in M, we have a natural $\mathbb{R}$–algebra homomorphism

$$\varphi_{U,p} : C^\infty_M(U) \longrightarrow C^\infty_{M,p}$$

given by sending $f \in C^\infty_M(U)$ to its equivalence class $\langle (U, f) \rangle \in C^\infty_{M,p}$. Then for given open neighbourhoods U, V of p in M with $V \subseteq U$, we have the following commutative diagram of

$\mathbb{R}$–algebra homomorphisms

$$\begin{tikzcd} C^\infty_M(U) \arrow[rr, "res_{U,V}"] \arrow[dr, "\varphi_{U,p}"'] & & C^\infty_M(V) \arrow[dl, "\varphi_{V,p}"] \\ & C^\infty_{M,p} & \end{tikzcd}$$

Let $\tau_{M,p}$ be the set of all open neighbourhoods of p in M. Given $U, V \in \tau_{M,p}$, we write $U \leq V$ if $V \subseteq U$. Given any $\mathbb{R}$–algebra A and family of $\mathbb{R}$–algebra homomorphisms $\{\psi_{U,p} : C^\infty_M(U) \to A \mid U \in \tau_{M,p}\}$ satisfying the condition

$$\psi_{V,p} \circ res_{U,V} = \psi_{U,p}, \tag{1.4.1}$$

for each pair of open neighbourhoods $V \subseteq U$ of p in M, the map

$$\psi : C^\infty_{M,p} \to A,$$

which sends $\langle(U, f)\rangle \in C^\infty_{M,p}$ to $\psi_{U,p}(f) \in A$, is a well-defined (c.f (1.4.1)) $\mathbb{R}$–algebra homomorphism satisfying

$$\psi \circ \varphi_{U,p} = \psi_{U,p}, \;\; \forall\, U \in \tau_{M,p}.$$

In other words,

$$C^\infty_{M,p} = \varinjlim_{U \in \tau_{M,p}} C^\infty_M(U),$$

the direct limit of the directed system of $\mathbb{R}$–algebras $\Big(\{C^\infty_M(U)\}_{U\in\tau_{M,p}}, \{res_{U,V}\}_{V\subseteq U\in\tau_{M,p}}\Big)$.

For notational simplicity, sometimes we express the germ of (U, f) by its representing C^∞ function f only. Evaluation of functions at p gives a surjective $\mathbb{R}$–algebra homomorphism

$$ev_p : C^\infty_{M,p} \to \mathbb{R}, \;\; f \mapsto f(p),$$

with kernel

$$\mathfrak{m}_p := \{f \in C^\infty_{M,p} : f(p) = 0\}.$$

Note that $\mathfrak{m}_p$ is a maximal ideal of $C^\infty_{M,p}$ because $\mathbb{R}$ is a field. Since any element $f \in C^\infty_{M,p} \setminus \mathfrak{m}_p$ satisfies $f(p) \neq 0$, we can find a small enough open neighbourhood, say V, of p in M such that $f\big|_V$ takes non-zero values on V. Therefore, f is a unit in $C^\infty_{M,p}$. This shows that, $\mathfrak{m}_p$ is the unique maximal ideal of $C^\infty_{M,p}$. Thus, $(C^\infty_{M,p}, \mathfrak{m}_p, \mathbb{R})$ is a local $\mathbb{R}$–algebra with the maximal ideal $\mathfrak{m}_p$ and the residue field $\mathbb{R}$.

Definition 1.4.2. A $\mathbb{R}$–*derivation* at a point $p \in M$ is a $\mathbb{R}$–linear map $D : C^\infty_{M,p} \to \mathbb{R}$ that satisfies the *Leibniz rule*:

$$D(f \cdot g) = (Df)g(p) + f(p)Dg,$$

for all $f, g \in C^\infty_{M,p}$. A *tangent vector* on M at $p \in M$ is a derivation on M at p. The set of all tangent vectors on M at p is denoted by T_pM.

Exercise 1.4.3. Let $D : C^\infty_{M,p} \to \mathbb{R}$ be a $\mathbb{R}$–derivation. Think of a real number $c \in \mathbb{R}$ as an element of $C^\infty_{M,p}$ by considering the germ at p of the constant C^∞ function $c : M \to \mathbb{R}$ that sends all points of M to the real number c. Show that $D(c) = 0$.

Moreover, for any $f \in C^\infty_{M,p}$, we have $ev_p(f - f(p)) = 0$ so that $f - f(p) \in \mathfrak{m}_p$. Since the composite map

$$\mathbb{R} \xrightarrow{\alpha \mapsto c_\alpha} C^\infty_{M,p} \xrightarrow{f \mapsto f(p)} \mathbb{R}$$

is the identity map on $\mathbb{R}$, it follows that the natural map

$$C^\infty_{M,p} \longrightarrow \mathbb{R} \oplus \mathfrak{m}_p, \;\; f \longmapsto (f(p), f - f(p)),$$

is an isomorphism of $\mathbb{R}$–vector spaces.

The restriction of D on $\mathfrak{m}_p \subset C^\infty_{M,p}$ is a $\mathbb{R}$–linear map, also denoted by the same symbol,

$$D : \mathfrak{m}_p \to \mathbb{R}.$$

If $f, g \in \mathfrak{m}_p$, then by Leibniz rule we see that

$$D(fg) = D(f)g(p) + f(p)D(g) = 0.$$

Therefore, $D(\mathfrak{m}_p^2) = \{0\}$, and so D gives rise to a $\mathbb{R}$–linear map

$$v_D : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}.$$

Thus we obtain a map

$$v : T_pM \to \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}), \; D \mapsto v_D,$$

which is clearly $\mathbb{R}$–linear and injective. To show that v is surjective, note that for given an $\mathbb{R}$–linear map $\varphi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$, the composite map

$$D_\varphi : C^\infty_{M,p} \xrightarrow{f \mapsto f(p)} \mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\varphi} \mathbb{R}$$

is an $\mathbb{R}$–linear derivation satisfying $v_{D_\varphi} = \varphi$. Thus we get an isomorphism of $\mathbb{R}$–vector spaces

$$T_pM \longrightarrow \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R}),$$

which gives a purely algebraic description of the tangent space of M at p. Note that, looking at the Taylor series expansion of $f \in C^\infty_{M,p}$ about p, the above isomorphism says that T_pM is the *linear (first order) approximation of M at p.*

Let $\mathbb{R}[\epsilon] := \mathbb{R}[t]/(t^2)$ be the $\mathbb{R}$–algebra of *dual numbers*. Note that $\mathbb{R}[\epsilon] = \{a + b\epsilon : a, b \in \mathbb{R} \text{ and } \epsilon^2 = 0\}$. Clearly, $\mathbb{R}[\epsilon]$ is a local $\mathbb{R}$–algebra with the maximal ideal

$$\mathfrak{m} = \{b\epsilon : b \in \mathbb{R}\} \subset \mathbb{R}[\epsilon].$$

Definition 1.4.4. Let k be a field, and let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local k–algebras. A k–algebra homomorphism $\varphi : A \to B$ is said to be a *local k–algebra homomorphism* if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Exercise 1.4.5. Given local k–algebras $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$, let $\mathrm{Hom}_{\mathbf{Alg}_k^{\mathrm{loc}}}(A, B)$ be the set of all local k–algebra homomorphisms from $(A, \mathfrak{m}_A)$ into $(B, \mathfrak{m}_B)$. Show that $\mathrm{Hom}_{\mathbf{Alg}_k^{\mathrm{loc}}}(A, B)$ is a k–vector space. We denote by $\mathbf{Alg}_k^{\mathrm{loc}}$ the category whose objects are local k–algebras and morphisms are local k–algebra homomorphisms.

Let $\alpha : C^\infty_{M,\,p} \to \mathbb{R}[\epsilon]$ be a *local* $\mathbb{R}$*–algebra homomorphism*. For given an $f \in C^\infty_{M,\,p}$, there exist unique $f_0, D_\alpha(f) \in \mathbb{R}$ such that

$$\alpha(f) = f_0 + D_\alpha(f)\epsilon.$$

Given $f \in C^\infty_{M,\,p}$, note that $g := f - f(p) \in \mathfrak{m}_p$. Since $\alpha(\mathfrak{m}_p) \subseteq \mathfrak{m}$, we have $g_0 = 0$. Since α is an $\mathbb{R}$–algebra homomorphism we have

$$D_\alpha(f - f(p))\epsilon = \alpha(g) = \alpha(f) - \alpha(f(p)) = [f_0 - f(p)] + D_\alpha(f)\epsilon,$$

From this we conclude that $f_0 = f(p)$. Moreover, for given $f, g \in C^\infty_{M,\,p}$ we have

$$\begin{aligned} \alpha(fg) &= (fg)_0 + D_\alpha(fg)\epsilon, \\ \text{and } \alpha(f)\alpha(g) &= (f_0 + D_\alpha(f)\epsilon)\,(g_0 + D_\alpha(g)\epsilon) = f_0 g_0 + (D_\alpha(f)g_0 + f_0 D_\alpha(g))\,\epsilon. \end{aligned}$$

Comparing the above two expression, we see that D_α satisfies the Leibniz rule:

$$D_\alpha(fg) = D_\alpha(f)g(p) + f(p)D_\alpha(g). \tag{1.4.6}$$

Thus $\alpha \mapsto D_\alpha$ defines a map

$$D : \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{R}}^{\mathrm{loc}}}(C^\infty_{M,\,p}, \mathbb{R}[\epsilon]) \longrightarrow T_pM := Der_{\mathbb{R}}(C^\infty_{M,\,p}, \mathbb{R}), \tag{1.4.7}$$

where $\mathrm{Alg}_{\mathbb{R}}^{\mathrm{loc}}$ is the category whose objects are local $\mathbb{R}$–algebras and morphisms are local homomorphisms of local $\mathbb{R}$–algebras. Given $\alpha, \beta \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{R}}^{\mathrm{loc}}}(C^\infty_{M,\,p}, \mathbb{R}[\epsilon])$ and real numbers $c, d \in \mathbb{R}$, we have

$$D_{c\alpha + d\beta}(f) = cD_\alpha(f) + dD_\beta(f) = (cD_\alpha + dD_\beta)(f),\ \forall\, f \in C^\infty_{M,\,p}. \tag{1.4.8}$$

Thus D is an $\mathbb{R}$–linear homomorphism. To show that D is surjective, for given an $\mathbb{R}$–linear derivation $\xi : C^\infty_{M,\,p} \to \mathbb{R}$, note that the map $\widetilde{\xi} : C^\infty_{M,\,p} \to \mathbb{R}[\epsilon]$ defined by

$$\widetilde{\xi}(f) = f(p) + \xi(f)\epsilon,\ \forall\, f \in C^\infty_{M,\,p},$$

is a local $\mathbb{R}$–algebra homomorphism such that $D_{\widetilde{\xi}} = \xi$. Thus, D is an $\mathbb{R}$–linear isomorphism.

1.4.2 Tangent bundle

1.4.3 Operations on vector bundles

1.5 Connection and curvature

1.5.1 Directional derivative in Euclidean space

Let $f \in C^\infty_{\mathbb{R}^n}(U)$ be a C^∞ function defined on an open neighbourhood, say U, of p in $\mathbb{R}^n$. Fix a tangent vector (need not be of unit length), say

$$X_p = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_p$$

at $p = (p_1, \ldots, p_n)$ in $\mathbb{R}^n$. To compute the *directional derivative* of f at p in the direction X_p, we consider a straight-line passing through p in the direction X_p given parametrically by the map $t \mapsto (x_1(t), x_2(t), x_3(t))$, for $t \in (-\epsilon, \epsilon) \subset \mathbb{R}$, where

$$x_i(t) := p_i + ta_i, \;\; i \in \{1, \ldots, n\}.$$

Set $a := (a_1, \ldots, a_n)$. Then the directional derivative $D_{X_p} f$ is given by

$$\begin{aligned} D_{X_p} f = \lim_{t \to 0} \frac{f(p + ta) - f(p)}{t} &= \frac{d}{dt}\Big|_{t=0} f(p + ta) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p \cdot \frac{dx_i}{dt} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p \cdot a_i \\ &= \left(\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_p \right) f \\ &= X_p(f). \end{aligned}$$

1.5.2 Flat connection and monodromy

1.6 Affine connection