Fundamental Group Schemes

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- Nori's Fundamental group schemes
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- Topological fundamental group of a space X, together with a marked point $x_0 \in X$, is the group $\pi_1^{\text{top}}(X, x_0)$ of homotopy classes of loops in X based at x_0 .
- The notion of homotopy classes of loops is difficult to adopt when X is an algebraic variety or a scheme.
- However, there is a different way of looking at $\pi_1^{\text{top}}(X, x_0)$, namely the group of automorphisms of the universal cover of X.
- This approach gives a better way to extend the notion of fundamental group to the case of algebraic varieties or schemes.
- In [SGA1], Alexander Grothendieck introduced étale fundamental group $\pi_1^{\text{\'et}}(X, x_0)$ of a scheme X with a geometric point x_0 of X as a replacement of the notion of topological fundamental group.

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Definition of $\pi_1^{\text{\'et}}(X, x)$

- A local homomorphism of local rings $\phi: A \to B$ is said to be unramified if $B/\phi(m_A)B$ is a finite separable field extension of A/m_A , where m_A is the maximal ideal of A. A morphism of schemes $f: Y \to X$ is said to be étale if the homomorphism $\mathcal{O}_{X,f(Y)} \to \mathcal{O}_{Y,Y}$ is flat and unramified.
- The definition of $\pi_1^{\text{\'et}}(X,x_0)$ involves a category F'et(X) of finite étale covers of X, together with a fiber functor $F_{x_0}: \text{F\'et}(X) \longrightarrow (\text{Set})$ given by sending a finite étale cover $f: Y \to X$ to the underlined set of points of the fiber $Y_{x_0}:=Y\times_X\{x_0\}$; and then $\pi_1^{\text{\'et}}(X,x_0):=\text{Aut}(F_{x_0})$.
- When X is a smooth projective variety over \mathbb{C} , it turns out that

$$\pi_1^{\text{\'et}}(X, x_0) \cong \widehat{\pi_1^{\text{top}}(X_{\text{an}}, x_0)},$$

the profinite completion of the topological fundamental group of the underlined complex manifold $X_{\rm an}$ of X.

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- Since the definition of $\pi_1^{\text{\'et}}(X, x_0)$ involves only finite étale covers of X, it does not take care of inseparable covers of X in positive characteristics.
- To remedy the situation, for a connected proper reduced scheme X defined over a
 perfect field k, in [Nor76] Madhav Nori considered more general kind of covers of
 X, namely "essentially finite vector bundles" over X.
- Let EF(X) be the category of essentially finite vector bundles on X. Fix a point $x_0 \in X$, and let $T_{x_0} : EF(X) \longrightarrow \mathcal{V}ect_k$ be the functor which sends a vector bundle $E \in EF(X)$ to its fiber $E_{x_0} \in \mathcal{V}ect_k$ at x_0 .
- The the quadruple $(EF(X), \mathcal{O}_X, \otimes, \mathcal{T}_{X_0})$ forms a "neutral Tannakian category".
- The affine k-group scheme $\pi_1^N(X, x_0)$ representing the functor $\underline{\operatorname{Aut}}^{\otimes}(T_{x_0})$ is called Nori's fundamental group scheme of X with base point at $x_0 \in X$.
- We have an equivalence of categories $\mathcal{R}ep_k^{\mathrm{fd}}(\pi_1^N(X,x_0))\cong \mathrm{EF}(X).$

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- The definition of $\pi_1^S(C, x_0)$ is extended for higher dimensional connected proper k-varieties X independently by V. Mehta and A. Langer [Lan11, Lan12].
- We have an exact k-linear tensor equivalence of categories $\mathcal{R}ep_k^{\mathrm{fd}}(\pi_1^S(X,x_0))\cong\mathscr{C}_X^{\mathrm{nf}}$, where $\mathscr{C}_X^{\mathrm{nf}}$ is the category of numerically flat vector bundles on X.
- In general $\pi_1^S(X, x_0)$ carries more geometric information about X than that of $\pi_1^N(X, x_0)$ and $\pi_1^{\text{\'et}}(X, x_0)$. There are faithfully flat homomorphisms

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- Let k be an algebraically closed field of characteristic $p \ge 0$.
- Let *X* be an irreducible smooth projective *k*-variety of dimension $d \in \{1, 2\}$.
- Fix an integer $n \ge 2$, and let $\mathcal{H}ilb_X^n$ be the Hilbert scheme parametrizing 0-cycles of length n on X.
- $\mathcal{H}ilb_X^n$ is a smooth projective k-variety of dimension nd, for d=1,2.
- $\mathcal{H}ilb_X^n$ is very important and studied by many authors; see e.g., [FGA05]
- It is an interesting problem to find the S-fundamental group scheme and Nori's fundamental group schemes of $\mathcal{H}ilb_X^n$.
- Let $S^n(X) := X^n/S_n$ be the *n*-fold symmetric product of X. This is a normal projective k-variety of dimension nd.

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- Let *X* be an irreducible smooth projective surface over $k = \overline{k}$.
- Let $n \ge 2$. Then there is a morphism of k-schemes $\varphi : \mathcal{H}ilb_X^n \longrightarrow S^n(X)$, known as the Hilbert-Chow morphism, which sends

$$Z \in \mathcal{H}ilb_X^n \longmapsto \sum_{x \in \text{Supp}(Z)} \ell(\mathcal{O}_{Z,x})[x] \in S^n(X),$$

- It is known that φ is a proper morphism.
- Fix a closed point $x_0 \in X$, and let $\widetilde{nx_0} \in \mathcal{H}ilb_X^n$ be such that $\varphi(\widetilde{nx_0}) = nx_0 \in S^n(X)$, where $\varphi : \mathcal{H}ilb_X^n \to S^n(X)$ is the Hilbert-Chow morphism.

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Fundamental group schemes of $\mathcal{H}ilb_X^n$

Notation: For an affine k-group scheme G, denote by G_{ab} the abelianization of G.
 This is the largest abelian affine quotient k-group scheme of G.

Theorem (Paul, Sebastian [PS20])

Assume that char(k) = p > 3. Let X be an irreducible smooth projective surface over k. Then there is an isomorphism of affine k-group schemes

$$f^?: \pi_1^?(X, x_0)_{ab} \xrightarrow{\simeq} \pi_1^?(\mathcal{H}ilb_X^n, \widetilde{nx_0}),$$

where ? = S, N, ét. In particular, $\pi_1^?(\mathcal{H}ilb_x^n, \widetilde{nx_0})$ is abelian.

- In [PS21], we have proved similar results for the case $\dim_k(X) = 1$ and $\operatorname{char}(k) = p > 0$.
- Remark: The case of étale fundamental groups was known from the work of Biswas and Hogadi in [BH15]; however, our approach is different.

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Notation: For an affine k-group scheme G, denote by G_{ab} the abelianization of G.
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$$f^{?}: \pi_{1}^{?}(X, x_{0})_{ab} \xrightarrow{\simeq} \pi_{1}^{?}(\mathcal{H}ilb_{X}^{n}, \widetilde{nx_{0}}),$$

where ? = S, N, ét. In particular, $\pi_1^?(\mathcal{H}ilb_X^n, \widetilde{nx_0})$ is abelian.

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- A point of $S^n(X)$ is of the form $\overline{x} = r_1 x_1 + \dots + r_p x_p$, with $x_j \in X$ and $r_1 \ge \dots \ge r_p$ positive integers with $\sum_{j=1}^p r_j = n$. We call $\langle r_1, \dots, r_p \rangle$ the type of \overline{x} .
- Let $W \subset S^n(X)$ be the open subscheme consisting of points of types $\langle 1, 1, \ldots, 1 \rangle$ and $\langle 2, 1, 1, \ldots, 1 \rangle$, and let $V = \varphi^{-1}(W) \subset \mathcal{H}ilb_X^n$.
- Consider the diagram



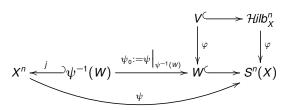
• Then the functor $\mathscr{G}:\mathscr{C}^{\mathrm{nf}}_{\mathcal{H}lib_X^n}\longrightarrow\mathscr{C}^{\mathrm{nf}}_{X^n},\ E\longmapsto \left(j_*\psi_0^*\varphi_*(E|_V)\right)^{\vee\vee}$ defines a homomorphism $f:\pi_1^S(X^n,(x_0,\ldots,x_0))\longrightarrow\pi_1^S(\mathcal{H}ibb_X^n,\widetilde{nx_0}).$

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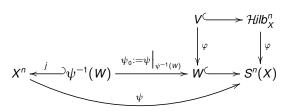
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Outline of proof (cont.)

• The homomorphism $\prod_{i=1}^n \pi_1^S(X, x_0) \cong \pi_1^S(X^n, (x_0, \dots, x_0)) \stackrel{f}{\longrightarrow} \pi_1^S(\mathcal{H}ilb_X^n, \widetilde{nx_0})$ is S_n -invariant, and gives rise to a homomorphism of affine k-group schemes

$$\widetilde{f}^S: \pi_1^S(X, x_0)_{ab} \longrightarrow \pi_1^S(\mathcal{H}ilb_X^n, \widetilde{nx_0}).$$

- Then we use the following result of Deligne and Milne [DM82] to show that \tilde{f}^S is a faithfully flat closed immersion, and hence is an isomorphism of k-group schemes
- Let $\theta: G \to G'$ be a homomorphism of affine algebraic groups over k. Then
 - ⊕ is faithfully flat if and only if the functor $\tilde{\theta}$: $\mathcal{R}ep_k(G') \to \mathcal{R}ep_k(G)$ is fully faithful and given any subobject $W \subset \tilde{\theta}(V')$, with $V' \in \mathcal{R}ep_k(G')$, there is a subobject $W' \subset V'$ in $\mathcal{R}ep_k(G')$ such that $\tilde{\theta}(W') \cong W$ in $\mathcal{R}ep_k(G)$.
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Thank you!