MA3201: Topology

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Disclaimer: This note will be updated from time to time.
If you find any potential mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\coprod$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
$A := \ldots$	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

MA3201 Syllabus

MA3201 (Topology)

- **Metric Spaces:** Metric space topology, equivalent metrics, sequences, complete metric spaces, limits and continuity, uniform continuity, extension of uniformly continuous functions. [1 week]
- **Topological Spaces:** Definition, examples, bases, sub-bases, product topology, subspace topology, metric topology, second countability and separability. [2 weeks]
- **Continuity:** Continuous functions on topological spaces, homeomorphisms, quotient topology. [1 week]
- **Connectedness:** Definition, example, path connectedness and local connectedness. [2 weeks]
- **Compactness:** Definition, limit point compactness, sequential compactness, net and directed set, local compactness, Tychonoff theorem, Stone-Weierstrass theorem, ArzelaAscoli theorem. [3 weeks]
- **Separation Axioms:** Hausdorff, regular and normal spaces; Urysohn lemma and Tietze extension theorem; compactification. [2 weeks]
- **Metrizability:** Urysohn metrization theorem. [1 week]

Chapter 1

Metric Space

1.1 Definition and Examples

A *metric* on a set X is a map

$$d : X \times X \to [0, \infty) := \{t \in \mathbb{R} : t \geq 0\}$$

such that

(i) $d(x_1, x_2) \geq 0$, $\forall\, x_1, x_2 \in X$, with equality holds if and only if $x_1 = x_2$;

(ii) $d(x_1, x_2) = d(x_2, x_1)$, for all $x_1, x_2 \in X$, and

(iii) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$, for all $x_1, x_2, x_3 \in X$.

A *metric space* is a pair (X, d) consisting of a set X and a metric d on it.

Example 1.1.1. The *absolute value* of a real number $x \in \mathbb{R}$ is a non-negative real number $|x|$, defined by

$$|x| := \begin{cases} x, & \text{if} \quad x \geq 0, \\ -x, & \text{if} \quad x < 0. \end{cases}$$

Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$d(x, y) = |x - y|, \ \forall\, x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, d)$ is a metric space.

Example 1.1.2 (Euclidean metric on $\mathbb{R}^n$)**.** Fix an integer $n \geq 1$, and let

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

be the map defined by

$$d\left((x_1, \ldots, x_n),\, (y_1, \ldots, y_n)\right) := \left(\sum_{j=1}^{n} |x_j - y_j|^2\right)^{1/2},$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{R}^n$, called the *Euclidean metric* on $\mathbb{R}^n$.

Example 1.1.3 (Euclidean metric on $\mathbb{C}^n$)**.** Fix an integer $n \geq 1$, and let

$$d : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$$

be the map defined by

$$d\left((z_1, \ldots, z_n), (w_1, \ldots, w_n)\right) := \left(\sum_{j=1}^{n} |z_j - w_j|^2\right)^{1/2},$$

for all $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$. It is straight-forward to check that d is a metric on $\mathbb{C}^n$, called the *Euclidean metric* on $\mathbb{C}^n$.

Example 1.1.4 (Taxicab/rectilinear metric)**.** Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{j=1}^{n} |x_j - y_j|,$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. Verify that d is a metric on $\mathbb{R}^n$.

Example 1.1.5. Given a non-empty set X, let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) := \begin{cases} 1, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Then d is a metric on X, called the *discrete metric* on X.

Definition 1.1.6. Let (X, d) be a metric space. A non-empty subset $A \subseteq X$ is said to be ***bounded*** if there exists a real number M such that

$$d(x, y) \leq M, \ \forall\, x, y \in A.$$

If $A \subseteq X$ is a bounded subset of X, then the number

$$\mathrm{diam}(A) := \sup\{d(a, b) : a, b \in A\}$$

is called the ***diameter*** of A in (X, d).

Exercise 1.1.7 (Standard bounded metric)**.** Let (X, d) be a metric space. Show that

$$d'(x, y) := \min\{1, d(x, y)\}, \ \forall\, x, y \in X$$

defines a metric on X.

Exercise 1.1.8. Let (X, d) be a metric space. Show that the map $d' : X \times X \to \mathbb{R}$ defined by

$$d'(x_1, x_2) := \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}, \ \ \forall\, x_1, x_2 \in X,$$

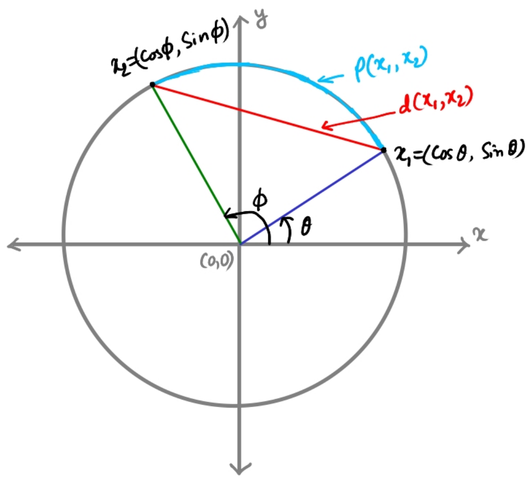
is a metric on X.

Exercise 1.1.9 (Subspace)**.** Let (X, d) be a metric space. For any non-empty subset Y of X, show that the restriction map
$$d_Y : Y \times Y \to \mathbb{R}, \ \ (y_1, y_2) \mapsto d(y_1, y_2),$$
is a metric on Y, called the *induced metric* on Y from (X, d). Then the pair (Y, d_Y) is called the *subspace* of the metric space (X, d).

Example 1.1.10. Let d be the Euclidean metric on $\mathbb{R}$. Then

- $([0, 1], d)$ is a subspace of $(\mathbb{R}, d)$.
- $(\mathbb{Q}, d)$ is a subspace of $(\mathbb{R}, d)$.

Exercise 1.1.11. Consider the unit circle
$$S^1 := \{(\cos t, \sin t) \in \mathbb{R}^2 : 0 \leq t < 2\pi\}$$
in $\mathbb{R}^2$. Given two points $\mathbf{x}_1 := (\cos\theta, \sin\theta), \mathbf{x}_2 := (\cos\phi, \sin\phi) \in S^1$, where $0 \leq \theta, \phi < 2\pi$, define
$$\rho(\mathbf{x}_1, \mathbf{x}_2) := \min\{|\theta - \phi|, 2\pi - |\theta - \phi|\}.$$
Show that ρ is a metric on S^1 that is not induced from the Euclidean metric d on $\mathbb{R}^2$.



Exercise 1.1.12. Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that the rule
$$d((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, \ \forall\, (x_1, y_1), (x_2, y_2) \in X \times Y$$
defines a metric on $X \times Y$, called the *product metric* on $X \times Y$. (*Caution: This is a non-standard terminology, and has nothing to do with product in general sense*).

Definition 1.1.13. Let $\Bbbk$ be the field of real numbers or the field of complex numbers with the Euclidean metric on it. A *norm* on a $\Bbbk$-vector space X is a map $\|\cdot\| : X \to \mathbb{R}$ satisfying the following properties:

(i) $\|x\| \geq 0$, $\forall\, x \in V$, and $\|x\| = 0$ if and only if $x = 0$ in X.

(ii) $\|\alpha x\| = |\alpha|\|x\|$, $\forall\, \alpha \in \mathbb{k}$, $x \in X$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall\, x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *normed linear space*.

Example 1.1.14. Let $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ be defined by

(i) $\|(x_1, \ldots, x_n)\|_1 := \sum\limits_{j=1}^{n} |x_j|$, $\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$,

(ii) $\|(x_1, \ldots, x_n)\|_2 := \left(\sum\limits_{j=1}^{n} x_j^2\right)^{1/2}$, $\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$, and

(iii) $\|(x_1, \ldots, x_n)\|_\infty := \max\limits_{1 \leq j \leq n} |x_j|$, $\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Then $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^n$.

Example 1.1.15. Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ together with the Euclidean metric on it. Let X be a non-empty set and let $\mathcal{B}(X)$ be the set of all $\mathbb{k}$-valued bounded functions defined on X. Define

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\},\ \forall\, f \in \mathcal{B}(X).$$

Then $\|\cdot\|_\infty$ is a norm on $\mathcal{B}(X)$.

Proposition 1.1.16. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the map $d : X \times X \to \mathbb{R}$ defined by*

$$d(x, y) := \|x - y\|,\ \ \forall\, x, y \in X,$$

is a metric on X, called the norm-induced metric on $(X, \|\cdot\|)$.

Proof. Let $x, y \in X$ be arbitrary. Then by definition of norm, we have $d(x, y) = \|x - y\| \geq 0$, for all $x, y \in X$, with equality holds if and only if $x - y = 0$, i.e., $x = y$. Note that, $d(y, x) = \|y - x\| = |-1|\|x - y\| = d(x, y)$. Moreover, given any $z \in X$, we have

$$\begin{aligned} d(x, y) &= \|x - y\| = \|(x - z) + (z - y)\| \\ &\leq \|x - z\| + \|z - y\| \\ &= d(x, z) + d(z, y) \\ &= d(x, z) + d(y, z). \end{aligned}$$

Therefore, d is a metric on X. $\square$

Example 1.1.17. Given any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, the following formulae

- $d_1((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := |x_1 - y_1| + \cdots + |x_n - y_n|$,
- $d_2((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$, and
- $d_\infty((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$,

define metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ induced by the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively.

Exercise 1.1.18. Let $\Bbbk$ be the field $\mathbb{R}$ or $\mathbb{C}$ together with the standard Euclidean norm on it. Fix an integer $n \geq 1$. For any real number $p \geq 1$, show that the map $\|\cdot\|_p : \Bbbk^n \to \mathbb{R}$ defined by

$$\|(x_1, \ldots, x_n)\|_p := (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \ \forall\, (x_1, \ldots, x_n) \in \Bbbk^n$$

is a norm on $\Bbbk^n$, for all $n \geq 1$. The normed linear space $(\Bbbk^n, \|\cdot\|_p)$ is denoted by $\ell_p^n(\Bbbk)$.

Exercise 1.1.19. Fix a real number p with $0 < p < 1$, and an integer $n \geq 2$.

(i) Show that the the map $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := (|x_1 - y_1|^p + \cdots + |x_n - y_n|^p)^{\frac{1}{p}},$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, is not a metric on $\mathbb{R}^n$. (*Hint:* Show that the triangle inequality fails for $x = (1, 1, 0, \ldots, 0), y = (0, 1, 0, \ldots, 0)$ and $z = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$).

(ii) Verify if the map $d'_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d'_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := |x_1 - y_1|^p + \cdots + |x_n - y_n|^p,$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$.

Exercise 1.1.20 (ℓ_p space)**.** Let $\Bbbk$ be the field of real numbers or the field of complex numbers together with the Euclidean metric on it. A *sequence* in $\Bbbk$ is a map $f : \mathbb{N} \to \Bbbk$; we generally denote it by $(a_n)_{n=1}^\infty$, where $a_n := f(n)$, $\forall\, n \in \mathbb{N}$. Fix a natural number $p \geq 1$. Let

$$\ell_p(\Bbbk) := \left\{ (a_n)_{n=1}^\infty : a_n \in \Bbbk,\ \forall\, n \in \mathbb{N}, \text{ and } \sum_{n=1}^\infty |a_n|^p < \infty \right\}.$$

Given $a = (a_n)_{n=1}^\infty$ let

$$\|a\|_p := \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}.$$

Show that $\|\cdot\|_p$ is a norm on $\ell_p(\Bbbk)$, and hence $\ell_p(\Bbbk)$ is a metric space.

Exercise 1.1.21. Fix real numbers a and b with $a < b$, and let

$$C[a, b] := \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$$

be the set of all real-valued continuous maps defined on $[a, b]$. Show that the map $\|\cdot\| : C[a, b] \to \mathbb{R}$ defined by

$$\|f\| := \int_a^b |f(t)| dt, \ \forall\, f \in C[a, b],$$

is a norm on the $\mathbb{R}$-vector space $C[a, b]$, which makes $C[a, b]$ a metric space.

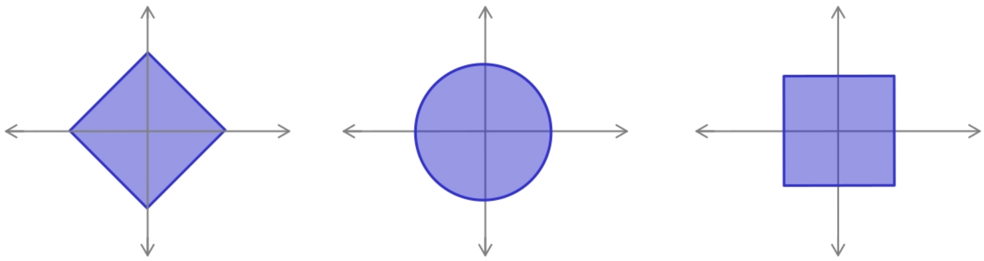
1.2 Topological properties

Let (X, d) be a metric space. Given a point $x_0 \in X$ and a real number $\delta > 0$, the *open ball in (X, d) with center at x_0 and radius δ* is the subset

$$B_d(x_0, \delta) := \{x \in X : d(x, x_0) < \delta\}.$$

Example 1.2.1. (i) In the real line $\mathbb{R}$ with the Euclidean metric d, the open ball with center at $0 \in \mathbb{R}$ and radius $r > 0$ is the open interval $(-r, r)$.

(ii) On $\mathbb{R}^2$, the open balls with center at the origin $(0, 0) \in \mathbb{R}^2$ and radius 1 with respect to the metrics d_1, d_2 and d_∞ (see Example 1.1.17) are given as follow:



$(\mathbb{R}^2, d_1)$ $(\mathbb{R}^2, d_2)$ $(\mathbb{R}^2, d_\infty)$

Let U be a non-empty subset of X. A point $x \in U$ is said to be an *interior point* of U if there exists a real number $\delta_x > 0$ such that $B(x, \delta_x) \subseteq U$. A subset $U \subseteq X$ is said to be *open* in (X, d) if either $U = \emptyset$ or each point of U is an interior point of U. A subset $Z \subseteq X$ is said to be *closed* if its complement $X \setminus Z$ is open in (X, d).

Example 1.2.2. Given $a, b \in \mathbb{R}$ with $a \leq b$, show that each of the intervals listed below are open with respect to the Euclidean metric on $\mathbb{R}$.

- $(a, b) := \{t \in \mathbb{R} : a < t < b\}$,
- $(-\infty, a) := \{t \in \mathbb{R} : t < a\}$,
- $(a, \infty) := \{t \in \mathbb{R} : a < t\}$, and
- $(-\infty, \infty) := \mathbb{R}$.

Lemma 1.2.3. *Let (X, d) be a metric space.*

(i) X and $\emptyset$ are both open and closed in (X, d).

(ii) Arbitrary union of open subsets of X is open.

(iii) Finite intersection of open subsets of X is open.

Proof. (i) Clear.

(ii) Let $\{U_\alpha : \alpha \in I\}$ be an indexed family of open subsets of (X,d). Let $x \in \bigcup_{\alpha\in I} U_\alpha$. Then there exists $\alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Then there exists a real number $\delta > 0$ such that $B(x,\delta) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in I} U_\alpha$. Thus $\bigcup_{\alpha\in I} U_\alpha$ is open in (X,d).

(iii) Let $U_1,\ldots,U_n$ be a finite collection of open subsets of (X,d). Let $x_0 \in \bigcap_{j=1}^{n} U_j$. Since $x_0 \in U_j$ and U_j is open in (X,d), there exists a $\delta_j > 0$ such that $B(x_0,\delta_j) \subseteq U_j$, for each $j = 1,\ldots,n$. Let $\delta := \min\{\delta_1,\ldots,\delta_n\} > 0$. Then $B(x_0,\delta) \subseteq B(x_0,\delta_j) \subseteq U_j$, for all $j = 1,\ldots,n$, and hence $B(x_0,\delta) \subseteq \bigcap_{j=1}^{n} U_j$.

□

Corollary 1.2.4. *Let (X,d) be a metric space. Then arbitrary intersections of closed subsets are closed, and a finite unions of closed subsets are closed.*

Example 1.2.5. Given $a,b \in \mathbb{R}$ with $a \leq b$, let

$$[a,b] := \{t \in \mathbb{R} : a \leq t \leq b\}.$$

Since $[a,b] = \mathbb{R} \setminus ((-\infty,a) \bigcup (b,\infty))$, it is closed in $\mathbb{R}$.

Definition 1.2.6. A point $x_0 \in X$ is said to be a *limit point* of a subset $A \subseteq X$ if for each real number $\delta > 0$ we have

$$(B(x_0,\delta) \setminus \{x_0\}) \bigcap A \neq \emptyset.$$

Example 1.2.7. (i) Consider the Euclidean space $\mathbb{R}$. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then $0 \in \mathbb{R}$ is a limit point of A.

(ii) Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Equip X with the discrete metric d. Then 0 is not a limit point of (X,d).

(iii) Let $X = [0,1] \cup \{2\}$ equipped with the metric d' induced from the Euclidean space $\mathbb{R}$. Then (X,d) is a metric subspace of $\mathbb{R}$. Let $A = B_{d'}(1,1) = \{x \in X : d'(x,1) < 1\}$. Then A is an open ball in X with center 1 and radius 1. However, $2 \in X$ is not a limit point of A.

Proposition 1.2.8. *Let (X,d) be a metric space. A subset Z of X is closed in (X,d) if and only if Z contains all of its limit points.*

Proof. Suppose that Z is closed in (X,d). If $x_0 \in U := X \setminus Z$, then U being open in (X,d), there exists a $\delta > 0$ such that $B(x_0,\delta) \subseteq U$, and so $B(x_0,\delta) \cap Z = \emptyset$. Therefore, x_0 cannot be a limit point of Z.

Conversely, suppose that Z contains all of its limit points in (X,d). Let $U := X \setminus Z$, and $x_0 \in U$. Since $x_0 \notin Z$ and x_0 is not a limit point of Z, there exists a $\delta > 0$ such that $B(x_0,\delta) \bigcap Z = \emptyset$. Therefore, $B(x_0,\delta) \subseteq U$. Since $x_0 \in U$ is arbitrary, U is open in (X,d) and hence Z is closed. □

Lemma 1.2.9. *Let (X,d) be a metric space and let $A \subseteq X$. Let $\mathscr{C}_A$ be the collection of all closed subsets of (X,d) containing A. Then $\bigcap_{Z\in\mathscr{C}_A} Z$ is the smallest closed subset of X containing A, called the* ***closure of A in (X,d)****.*

Proof. It follows from Lemma 1.2.3 that $\overline{A} := \bigcap_{Z \in \mathscr{C}_A} Z$ is a closed subset of X. Clearly $A \subseteq \overline{A}$. Let W be any closed subset of X containing A. Then $W \in \mathscr{C}_A$, and hence $\bigcap_{Z \in \mathscr{C}_A} \subseteq W$. □

Proposition 1.2.10. *Let (X,d) be a metric space. A subset $A \subseteq X$ is closed if and only if $\overline{A} = A$.*

Proof. Suppose that A is closed. Let $\mathscr{C}_A$ be the collection of all closed subsets of (X,d) containing A. If A is closed, then $A \in \mathscr{C}_A$, and hence $A \subseteq \overline{A} = \bigcap_{Z \in \mathscr{C}_A} Z \subseteq A$ shows that $A = \overline{A}$. Converse is obvious since $\overline{A} = \bigcap_{Z \in \mathscr{C}_A} Z$ is closed. □

Proposition 1.2.11. *Let (X,d) be a metric space. Given any two distinct points $x,y \in X$ there exist positive real numbers r_x and r_y such that $B(x,r_x) \cap B(y,r_y) = \emptyset$.*

Proof. Let $x,y \in X$ with $x \neq y$. Then $r := d(x,y) > 0$. Then the open balls $B(x,r/2)$ and $B(y,r/2)$ do not intersect each others. Indeed, if there were $z \in B(x,r/2) \cap B(y,r/2)$, then $d(x,z) < r/2$ and $d(z,y) < r/2$ gives $r = d(x,y) \leq d(x,z) + d(z,y) < r/2 + r/2 = r$, which is not possible. □

Definition 1.2.12 (Equivalent Metrics)**.** Two metrics d_1 and d_2 on a non-empty set X are said to be *topologically equivalent* if for any subset $U \subseteq X$, U is open in (X,d_1) if and only if U is open in (X,d_2).

Proposition 1.2.13. *Let d_1 and d_2 be two metrics on a non-empty set X. Then the following are equivalent.*

(i) d_1 and d_2 are topologically equivalent.

(ii) given any point $x \in X$ and a real number $r > 0$, there exists real numbers $r', r'' > 0$ such that

$$B_{d_2}(x,r'') \subseteq B_{d_1}(x,r) \quad \textit{and} \quad B_{d_1}(x,r') \subseteq B_{d_2}(x,r).$$

Proof. Suppose that d_1 and d_2 are topologically equivalent metrics on X. Let $x \in X$ and $r > 0$ be given. Since $B_{d_1}(x,r)$ is open in (X,d_2), there exists $r'' > 0$ such that $B_{d_2}(x,r'') \subseteq B_{d_1}(x,r)$. Similarly, since $B_{d_2}(x,r)$ is open in (X,d_1), there exists $r' > 0$ such that $B_{d_1}(x,r') \subseteq B_{d_2}(x,r)$.

Conversely, suppose that given any point $x \in X$ and a real number $r > 0$, there exists real numbers $r', r'' > 0$ such that

$$B_{d_2}(x,r'') \subseteq B_{d_1}(x,r) \quad \text{and} \quad B_{d_1}(x,r') \subseteq B_{d_2}(x,r).$$

Let $U \subseteq X$. Suppose that U is open in (X,d_1). Then for given $x \in U$, there exists $r_x > 0$ such that $B_{d_1}(x,r_x) \subseteq U$. Then by assumption, there exists $r''_x > 0$ such that $B_{d_2}(x,r''_x) \subseteq B_{d_1}(x,r_x) \subseteq U$, and hence x is an interior point of U with respect to the d_2-metric on X. Therefore, U is open in (X,d_2). Similarly, if U is open in (X,d_2), then for each $x \in U$ there exists $s_x > 0$ such that $B_{d_2}(x,s_x) \subseteq U$. But then by our assumption, there exists $s'_x > 0$ such that $B_{d_1}(x,s'_x) \subseteq B_{d_2}(x,s_x) \subseteq U$, and hence x is an interior point of U with respect to the d_1 metric on X. Therefore, U is open in (X,d_1). □

Lemma 1.2.14. *Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. Let d_1 and d_2 be the metrics on X induced by the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then d_1 is topologically equivalent to d_2 if and only if there exist positive real numbers $\alpha, \beta > 0$ such that*

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1, \ \ \forall\, x \in X.$$

Proof. Suppose that there exist positive real numbers $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1, \ \ \forall\, x, y \in X.$$

Let $U \subseteq X$ be open in (X, d_1). Then for given any $x \in U$, there exists a real number $r_x > 0$ such that

$$B_{d_1}(x, r_x) = \{y \in X : \|x - y\|_1 < r_x\} \subseteq U.$$

Since

$$B_{d_2}(x, \alpha r_x) = \{y \in X : \|x - y\|_2 < \alpha r_x\} \subseteq B_{d_1}(x, r_x) \subseteq U,$$

we see that U is open in (X, d_2). Now suppose that U is open in (X, d_2). Then given $x \in U$, there exists a real number $s_x > 0$ such that

$$B_{d_2}(x, s_x) = \{y \in X : \|x - y\|_2 < s_x\} \subseteq U.$$

Since

$$B_{d_1}(x, s_x/\beta) = \{y \in X : \|x - y\|_1 < s_x/\beta\} \subseteq \{y \in X : \|x - y\|_2 < \beta\} = B_{d_2}(x, s_x) \subseteq U,$$

we see that U is open in (X, d_1). Therefore, d_1 and d_2 are equivalent metrics on X.

Conversely, suppose that d_1 and d_2 are topologically equivalent metrics on X. Since $B_{d_1}(0, 1) = \{x \in X : \|x\|_1 < 1\}$ is open in (X, d_2), there exists a real number $r > 0$ such that $B_{d_2}(0, r) \subseteq B_{d_1}(0, 1)$. In other words,

$$\|x\|_1 < 1 \quad \text{whenever} \quad \|x\|_2 < r.$$

Now given any $x \in X$ with $x \neq 0$, let $y = (r/\|x\|_1)x \in X$ so that $\|y\|_1 = r$. Then we have $\|(r/\|x\|_1)x\|_2 < 1$, i.e., $\|x\|_2 < \frac{1}{r}\|x\|_1$. So we set $\beta = 1/r > 0$ to get $\|x\|_2 \le \beta\|x\|_1$, $\forall\, x \in X$. Similarly, since $B_{d_2}(0, 1) = \{x \in X : \|x\|_2 < 1\}$ is open in (X, d_1), we can find a positive real number $\alpha > 0$ such that $\alpha\|x\|_1 \le \|x\|_2$, $\forall\, x \in X$. This completes the proof. □

Exercise 1.2.15. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are said to be *equivalent* if there exist positive real numbers $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1, \ \forall\, x \in X.$$

Show that norm equivalence on X is an equivalence relation.

Example 1.2.16. For any real number $p \geq 1$, we show that the ℓ_p-metric on $\mathbb{R}^n$ is equivalent to the ℓ_∞-metric on it. Indeed, given any point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ note that

$$\begin{aligned}
\|(x_1, \ldots, x_n)\|_\infty &= \max\{|x_1|, \ldots, |x_n|\} \\
&\leq (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \\
&= \|(x_1, \ldots, x_n)\|_p \\
&\leq n^{\frac{1}{p}} \max\{|x_1|, \ldots, |x_n|\} \\
&= n^{\frac{1}{p}} \|(x_1, \ldots, x_n)\|_\infty.
\end{aligned}$$

Therefore, ℓ_p-norm on $\mathbb{R}^n$ is equivalent to the ℓ_∞-norm on it, and hence the metrics induced by them on $\mathbb{R}^n$ are topologically equivalent. As a result, for any real numbers $p, q \geq 1$, the metrics on $\mathbb{R}^n$ induced by the ℓ_p-norm and the ℓ_q-norms on it are topologically equivalent.

The following results shows that any two norm induced topologies on a finite dimensional vector space are the same. We need notion of compact set, continuous maps and some related results to prove the following lemma. So you may skip it for the first reading.

Lemma 1.2.17. *Any two norm-induced metrics on a finite dimensional vector space are equivalent.*

Proof. Let X be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Without loss of generality we may assume that $X = \mathbb{C}^n$, for some $n \in \mathbb{N}$. Let $\|\cdot\|$ be a norm on X. It suffices to show that $\|\cdot\|$ is equivalent to the ℓ_2-norm $\|\cdot\|_2$ on $\mathbb{C}^n$. Let $\mathbf{z} = (z_1, \ldots, z_n) = z_1 e_1 + \cdots + z_n e_n \in \mathbb{C}^n$ be given. Then

$$\|\mathbf{z}\| \leq \sum_{j=1}^{n} |z_j| \|e_j\| \leq \left(\sum_{j=1}^{n} \|e_j\| \right) \|x\|_2.$$

Setting $M = \sum\limits_{j=1}^{n} \|e_j\| > 0$, we have

$$\|\mathbf{z}\| \leq M \|\mathbf{z}\|_2, \ \ \forall\, \mathbf{z} \in \mathbb{C}^n.$$

Since

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq M\|x - y\|_2, \ \forall\, x, y \in \mathbb{X},$$

the map $x \mapsto \|x\|$ from the $\mathbb{C}^n$ equipped with the ℓ_2-norm into $\mathbb{R}$ is continuous.

Let

$$\mathbb{S}^1 = \{x \in X : \|x\|_2 = 1\}$$

be the unit sphere in X with respect to the ℓ_2-norm. Note that $\mathbb{S}^1$ is closed and bounded, and so it is compact. Then the continuous map $x \mapsto \|x\|$ attains a minimum value at some point, say $x_0 \in \mathbb{S}^1$. Then $\|x\| \geq \|x_0\|, \forall\, x \in \mathbb{S}^1$. Let $K := \|x_0\| > 0$. Since $\|x_0\|_2 = 1$, it follows that $x_0 \neq 0$ and that $K > 0$. Now for given $x \in X \setminus \{0\}$, we have $\|x\|/\|x\|_2 = \|x/\|x\|_2\| \geq K$, and so

$$K\|x\|_2 \leq \|x\|, \ \forall\, x \in X.$$

This completes the proof. $\square$

Proposition 1.2.18. *Let (X,d) be a metric space. Define a map $\overline{d} : X \times X \to \mathbb{R}$ by*

$$\overline{d}(x,y) := \min\{d(x,y),1\},\ \forall\, x,y \in X.$$

*Then $\overline{d}$ is a metric on X topologically equivalent to d. The metric $\overline{d}$ is called the **standard bounded metric** on X corresponding to d.*

Proof. Clearly $\overline{d}(x,y) \geq 0$, $\forall\, x,y \in X$, with equality holds if and only if $x = y$. Also $\overline{d}(x,y) = \overline{d}(y,x)$, $\forall\, x,y \in X$. To check the triangle inequality:

$$\overline{d}(x,y) \leq \overline{d}(x,z) + \overline{d}(z,y),\ \forall\, x,y,z \in X,$$

note that if $d(x,z) + d(y,z) \geq 1$, then the inequality follows. Assume that $d(x,z) + d(y,z) < 1$. Then from the triangle inequality for d, we have $d(x,y) < d(x,z) + d(y,z) < 1$, and hence $\overline{d}(x,y) = d(x,y)$. Since $\overline{d}(x,z) = d(x,z)$ and $\overline{d}(y,z) = d(y,z)$ in this case, the triangle inequality for $\overline{d}$ follows.

To show that $\overline{d}$ is topologically equivalent to d, let U be any non-empty open subset of (X,d). Let $a \in U$ be given. Then there exists $r > 0$ such that $B_d(a,r) \subseteq U$. Let $\delta = \min\{r,1\}$. Since

$$B_{\overline{d}}(a,\delta) = B_d(a,\delta) \subseteq B_d(a,r) \subseteq U,$$

we see that U is open in $(X,\overline{d})$. Conversely, if U is open in $(X,\overline{d})$, then given a point $a \in U$, there exists $r > 0$ such that $B_{\overline{d}}(a,r) \subseteq U$. Then choosing $\delta = \min\{r,1\}$, we see that

$$B_d(a,\delta) = B_{\overline{d}}(a,\delta) \subseteq B_{\overline{d}}(a,r) \subseteq U,$$

and hence U is open in (X,d). This completes the proof. □

Corollary 1.2.19. *Boundedness of a subset in a metric space is not a topological property.*

Given an index set J and a non-empty set X, let $X^J := Map(J,X)$ be the set of all set maps from J to X. If $x \in X^J$, for each $\alpha \in J$ we denote by x_α the element $x(\alpha) \in X$.

Exercise 1.2.20 (Uniform metric)**.** Let (X,d) be a metric space. Given a non-empty index set J and points $x,y \in X^J := Map(J,X)$, we define

$$\overline{d}_u(x,y) := \sup\{\overline{d}(x_\alpha,y_\alpha) : \alpha \in J\},$$

where $\overline{d}$ is the standard bounded metric on X corresponding to the metric d. Show that $\overline{d}_u$ is a metric on X^J, called the ***uniform metric*** on X^J induced by d, and the topology on X^J induced by the uniform metric $\overline{d}_u$ is called the ***uniform topology***.

Answer: Let $x,y \in X^J$ be given. Since $d(x_\alpha,y_\alpha) \geq 0$, $\forall\, \alpha \in J$, we have

$$0 \leq \overline{d}(x_\alpha,y_\alpha) = \min\{d(x_\alpha,y_\alpha),1\} \leq 1,\ \forall\, \alpha \in J,$$

and hence $0 \leq \overline{d}_u(x,y) := \sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in J\} \leq 1$, $\forall\, x, y \in X^J$. Clearly $\overline{d}_u(x,y) = 0$ if and only if $x_\alpha = y_\alpha$, $\forall\, \alpha \in J$, if and only if $x = y$ in X^J. To check triangle inequality, let $x, y, z \in X^J$ be arbitrary. Since $\overline{d}$ is a metric on X, we have

$$\begin{aligned}\overline{d}(x_\alpha, y_\alpha) &\leq \overline{d}(x_\alpha, z_\alpha) + \overline{d}(y_\alpha, z_\alpha),\ \forall\, \alpha \in J.\\ &\leq \overline{d}_u(x,z) + \overline{d}_u(y,z),\ \forall\, \alpha \in J.\end{aligned}$$

Taking supremum over $\alpha \in J$, we have $\overline{d}_u(x,y) \leq \overline{d}_u(x,z) + \overline{d}_u(y,z)$, $\forall\, x, y, z \in X^J$. Therefore, $\overline{d}_u$ is a metric on X^J. □

Proposition 1.2.21. *Consider the real line $\mathbb{R}$ with the standard Euclidean metric on it. Fix a non-empty set J, and consider the set $\mathbb{R}^J := Map(J, \mathbb{R})$. Then the uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is an infinite set.*

Proof. Let τ_u, τ_b and τ_p be the uniform topology, box topology and the product topology on X^J, respectively. We show that $\tau_p \subseteq \tau_u \subseteq \tau_b$; and all such inclusions are strict if J is infinite.

Let $x \in X^J$ be given. Let $\prod\limits_{\alpha \in J} V_\alpha$ be a basic open subset of (X^J, τ_p) containing x. Then V_α is an open subset of (X, d) containing x_α, for all $\alpha \in J$, and that $V_\alpha \neq X$, for all $\alpha \in \{\alpha_1, \ldots, \alpha_n\} \subseteq J$. For each $i \in \{1, \ldots, n\}$, there exists $\delta_i > 0$ such that

$$B_{\overline{d}}(x_{\alpha_i}, \delta_i) \subseteq V_{\alpha_i},\ \ \forall\, i = 1, \ldots, n.$$

□

Chapter 2

Point Set Topology

2.1 Topological space

A topology on a set X is given by specifying which subsets of X are 'open'. Naturally those subsets should satisfies certain properties as we are familiar from basic analysis and metric space courses.

Definition 2.1.1. A *topology* is on a set X is a collection τ of subsets of X satisfying the following properties:

(i) $\emptyset$ and X are in τ,

(ii) for any collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of objects of τ, their union $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \tau$,

(iii) for a finite collection of objects $U_1, \ldots, U_n \in \tau$, their intersection $\bigcap\limits_{i=1}^{n} U_i \in \tau$.

The pair (X, τ) is called a *topological space*, and the objects of τ are called *open subsets* of (X, τ). For notational simplicity, we suppress τ and denote a topological space (X, τ) simply by X.

Joke: An empty set may contain some air since it is open!

Remark 2.1.2. One can also define a topology on a set X by considering a collection τ_c of subsets of X such that

(i) both $\emptyset$ and X are in τ_c,

(ii) τ_c is closed under arbitrary intersections, and

(iii) τ_c is closed under finite unions.

This is known as the *closed set axioms for a topology*. In this settings, objects of τ_c are called *closed subsets* of X. It is easy to switch between these two definitions by taking complements of objects of τ and τ_c in X. However, unless explicitly mentioned, we usually work with open set axioms for topology.

Example 2.1.3. (i) If $X = \emptyset$ then $\tau = \{\emptyset\}$ is the only topology on $\emptyset$.

(ii) For any set X, $\tau_{\text{disc}} := \mathcal{P}(X)$ and $\tau_{\text{triv}} := \{\emptyset, X\}$ are topologies on X, called the *discrete topology* and the *indiscrete topology* on X, respectively. Note that, $\tau_{\text{disc}}(X)$ and $\tau_{\text{triv}}(X)$ are different if X has at least two elements.

(iii) Let $X \neq \emptyset$, and let $\tau = \{A \in \mathcal{P}(X) : X \setminus A \text{ is finite}\}$. Then (X, τ) is a topological space; such a topology is called the *cofinite topology* on X.

(iv) Consider the set $\mathbb{R}^n$. Let $\tau_E(\mathbb{R}^n)$ be the set of all subsets $U \subseteq \mathbb{R}^n$ such that given any $x \in U$ there exists a real number $r > 0$ such that

$$B(x, r) := \{y \in \mathbb{R}^n : \|x - y\| < r\} \subseteq U.$$

Then the set $\tau_E(\mathbb{R}^n)$ is a topology on $\mathbb{R}^n$ (verify!), called the *standard topology* or the *Euclidean topology* on $\mathbb{R}^n$.

(v) Any metric space (X, d) is a topological space where the topology on X is given by the collection of all open subsets of (X, d).

Let (X, τ) be a topological space. Given a subset Y of X, let

$$\tau_Y := \{U \cap Y : U \in \tau\}.$$

Clearly $\emptyset, Y \in \tau_Y$. Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of elements of τ_Y. Then for each $\alpha \in \Lambda$, we have $A_\alpha = U_\alpha \cap Y$, for some $U_\alpha \in \tau$. Then $\bigcup\limits_{\alpha \in \Lambda} A_\alpha = \bigcup\limits_{\alpha \in \Lambda} (U_\alpha \cap Y) = \left(\bigcup\limits_{\alpha \in \Lambda} U_\alpha\right) \cap Y \in \tau_Y$. If $V_1, V_2 \in \tau_Y$, then $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$, for some $U_1, U_2 \in \tau$. Then $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y \in \tau_Y$. Therefore, τ_Y is a topology on Y, called the ***subspace topology*** on Y induced from (X, τ).

Example 2.1.4. (i) Consider the real line $\mathbb{R}$ with the Equipped with the Euclidean topology on it. Then the set $\mathbb{Q}$ inherits a subspace topology where a subset $U \subseteq Q$ is open if and only if $U = V \cap \mathbb{Q}$, for some open subset V of $\mathbb{R}$.

(ii) The subspace topology on $\mathbb{Z}$ induced from the Euclidean topology on $\mathbb{R}$ is discrete topology on $\mathbb{Z}$.

(iii) Consider $Y = [0, 1) \subset \mathbb{R}$. Note that open subsets of Y in the subspace topology induced from $\mathbb{R}$ are of the form $U \cap [0, 1)$, for some open subset U of $\mathbb{R}$. Note that $[0, 1/2)$ is open in Y, but not in $\mathbb{R}$.

Exercise 2.1.5. Are the subspace topology on the unit circle S^1 in the Euclidean plane $\mathbb{R}^2$ and the metric subspace topology on S^1 induced from the Euclidean metric on $\mathbb{R}^2$ the same?

Proposition 2.1.6. *Let Y be a subspace of a topological space X. If $U \subseteq Y$ is open in Y, and Y is open in X, then U is open in X.*

Proof. Since U is open in Y, $U = Y \cap V$, for some open subset V of X. Since Y is open in X, $U = Y \cap V$ is open in X. □

Proposition 2.1.7. *Let Y be a closed subspace of X. If $Z \subseteq Y$ is closed in Y, then Z is closed in X.*

Proof. Note that, $X \setminus Z = (X \setminus Y) \cup (Y \setminus Z)$. Since Y is closed in X, $X \setminus Y$ is open in X. Since Z is closed in Y, $Y \setminus Z$ is open in Y, and hence $Y \setminus Z = U \cap Y$, for some open subset U of X. We claim that $X \setminus Z = (X \setminus Y) \cup U$. Since $Y \setminus Z = U \cap Y \subseteq U$, we have $X \setminus Z \subseteq (X \setminus Y) \cup U$. Again, since $Z \subseteq Y$ and $Y \setminus Z = U \cap Y$, we must have $U \subseteq X \setminus Z$. Therefore, $X \setminus Z = (X \setminus Y) \cup U$, and hence $X \setminus Z$ is open in X, which in turn gives that Z is closed in X. □

Exercise* **2.1.8** (Zariski topology)**.** This exercise is for readers familiar with basic theory of commutative rings, and is not required for this course. Let A be a commutative ring with identity. Let $\operatorname{Spec}(A)$ be the set of all prime ideals of A, known as the *spectrum of* A. For each subset $E \subseteq A$, let

$$V(E) := \{\mathfrak{p} \in \operatorname{Spec}(A) : E \subseteq \mathfrak{p}\}.$$

Prove the following.

(i) $V(A) = \emptyset$ and $V(0) = \operatorname{Spec}(A)$.

(ii) $V(E) = V(\mathfrak{a})$, where $\mathfrak{a} \subseteq A$ is the ideal generated by $E \subseteq A$.

(iii) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V\big(\mathfrak{a} \cap \mathfrak{b}\big) = V(\mathfrak{a}\mathfrak{b})$, for all ideals $\mathfrak{a}, \mathfrak{b}$ of A.

(iv) $\bigcap\limits_{i \in I} V(\mathfrak{a}_i) = V(\bigcup\limits_{i \in I} \mathfrak{a}_i) = V(\sum\limits_{i \in I} \mathfrak{a}_i)$, for any collection of ideals $\{\mathfrak{a}_i : i \in I\}$ of A.

(v) Conclude that the collection $\{V(\mathfrak{a}) : \mathfrak{a} \text{ is an ideal of } A\}$ satisfies axioms for closed subsets of a topological space. The resulting topology on $\operatorname{Spec}(A)$ is called the *Zariski topology* on $\operatorname{Spec}(A)$.

(vi) For any ideal $\mathfrak{a}$ of A, show that $\operatorname{Spec}(A/\mathfrak{a})$ is homeomorphic to the closed subspace $V(\mathfrak{a})$ of $\operatorname{Spec}(A)$.

(vii) Let X be a topological space. A point $\xi \in X$ is said to be a

(a) *closed point* of X if $\overline{\{\xi\}} = \{\xi\}$, and

(b) *generic point* of X if $\overline{\{\xi\}} = X$.

If A is an integral domain, show that $\operatorname{Spec}(A)$ contains a unique generic point, which is precisely the zero ideal of A.

(viii) Show that the Zariski topology on $\operatorname{Spec}(A)$ is not even T1 let alone be it Hausdorff.

(ix) Show that, a point $\mathfrak{m} \in \operatorname{Spec}(A)$ is closed if and only if $\mathfrak{m}$ is a maximal ideal of A.

(x) Let k be an algebraically closed field, e.g., $k = \mathbb{C}$, and let $A = k[x_1, \ldots, x_n]$, the polynomial ring over k with variables $x_1, \ldots, x_n$. Use Hilbert's Nullstellensatz to show that the set of all closed points of $\operatorname{Spec}(A)$ is in bijection with the set $k^n := \big\{(a_1, \ldots, a_n) : a_i \in k, \ \forall\, i \in \{1, \ldots, n\}\big\}$.

(xi) Let k be an algebraically closed field; for example, $k = \mathbb{C}$. Fix a subset $S \subseteq A := k[x_1, \ldots, x_n]$, and let $\mathfrak{a}_S \subseteq A$ be the ideal generated by S. Show that the set

$$Z(S) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0, \ \forall\, f \in S\}$$

is in bijection with the set of all closed points of $V(\mathfrak{a}_S) \subseteq \operatorname{Spec}(A)$.

The space Spec(A) carries rich algebro-geometric structure. They are called *affine schemes*, and are building blocks of all schemes in the sense that any scheme is build up suitably gluing affine schemes.

Definition 2.1.9 (Basis)**.** Let (X, τ) be a topological space. A subset $\mathcal{B} \subseteq \tau$ is said to be a *basis* for the topology τ on X if given any $U \in \tau$ and any $x \in U$, there exists an element $V \in \mathcal{B}$ such that $x \in V$ and $V \subseteq U$. The elements of $\mathcal{B}$ are called *basic open subsets* of X.

Remark 2.1.10. For some technical reason we include the empty subset $\emptyset$ of X in a basis for X.

Example 2.1.11. (i) If τ is the discrete topology on X, then $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for (X, τ).

(ii) Let $\mathcal{B}$ be the set of all open intervals $(a, b) \subset \mathbb{R}$, where $a < b$. Then $\mathcal{B}$ is a basis for the Euclidean topology on $\mathbb{R}$.

(iii) Let (X, d) be a metric space. Then $\mathcal{B} = \{B(x, r) : x \in X,\ r > 0\}$ is a basis for the metric topology on (X, d).

Lemma 2.1.12. *Let $\mathcal{B}$ be a basis for a topological space (X, τ). Then*

(i) $\bigcup_{V \in \mathcal{B}} V = X$, *and*

(ii) any non-empty open subset of X is a unions of members from $\mathcal{B}$.

Proof. (i) Follows from the Definition 2.1.9 by taking $X = U \in \tau$.
(ii) Since $\mathcal{B} \subseteq \tau$ and τ is closed under arbitrary union, it remains to show that any $U \in \tau$ can be written as a union of members of $\mathcal{B}$. Let $U \in \tau$ be arbitrary. Since $\mathcal{B}$ is a basis for the topology τ on X, for each $x \in U$ there is a basic open subset $V_x \in \mathcal{B}$ with $x \in V_x$ such that $V_x \subseteq U$. Then $U = \bigcup_{x \in U} V_x$. This completes the proof. □

Proposition 2.1.13. *Let $\mathcal{B}$ be a basis for a topological space (X, τ). Let $Y \subseteq X$. Then $\mathcal{B}_Y := \{V \cap Y : V \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof. Since $\mathcal{B}$ is a basis for (X, τ), we have $\bigcup_{V \in \mathcal{B}} V = X$. Then $\bigcup_{V \in \mathcal{B}} (V \cap Y) = Y$. Let $U \cap Y \in \tau_Y$ and $y \in U \cap Y$. Then there exists $V \in \mathcal{B}$ such that $y \in V \subseteq U$. Then $y \in V \cap Y \subseteq U \cap Y$. Therefore, $\mathcal{B}_Y$ is a basis for the subspace topology τ_Y on Y. □

Proposition 2.1.14 (Topology generated by a basis)**.** *Let X be a set. Let $\mathcal{B}$ be a collection of subsets of X satisfying the following properties.*

(i) $\bigcup_{V \in \mathcal{B}} V = X$, *and*

(ii) given any $V_1, V_2 \in \mathcal{B}$ and a point $x \in V_1 \cap V_2$, there exists a $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq V_1 \cap V_2$.

Then there is a unique topology $\tau_{\mathcal{B}}$ on X such that $\mathcal{B}$ is a basis for $\tau_{\mathcal{B}}$. Such a topology $\tau_{\mathcal{B}}$ on X is called the topology generated by $\mathcal{B}$.

Proof. Take

$$\tau_{\mathcal{B}} := \{U \in \mathcal{P}(X) : \text{for each } x \in U, \exists\, V_x \in \mathcal{B} \text{ such that } x \in V_x \subseteq U\}.$$

Clearly $\emptyset$ and X are in $\tau_{\mathcal{B}}$. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any collection of objects from $\tau_{\mathcal{B}}$, and let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then $x \in U_\lambda$, for some $\lambda \in \Lambda$. Then by construction of $\tau_{\mathcal{B}}$ there exists a $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U_\lambda$, and hence $x \in V_x \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$. Thus $\bigcup_{\alpha\in\Lambda} U_\alpha \in \tau_{\mathcal{B}}$. Let $U_1, U_2 \in \tau_{\mathcal{B}}$. Let $x \in U_1 \cap U_2$ be arbitrary. Then there exist $V_1, V_2 \in \mathcal{B}$ such that $x \in V_1 \subseteq U_1$ and $x \in V_2 \subseteq U_2$. Then by property (ii) of $\mathcal{B}$, there exists a $W \in \mathcal{B}$ such that $x \in W \subseteq V_1 \cap V_2 \subseteq U_1 \cap U_2$. Therefore, $U_1 \cap U_2 \in \tau_{\mathcal{B}}$. Thus, $\tau_{\mathcal{B}}$ is topology on X. It follows from the definition of $\tau_{\mathcal{B}}$ that $\mathcal{B}$ is a basis for $(X, \tau_{\mathcal{B}})$.

Let τ be a topology on X such that $\mathcal{B}$ is a basis for τ. Let $U \in \tau$ be arbitrary. Then given any $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V \subseteq U$. Then $U \in \tau_{\mathcal{B}}$ by construction of $\tau_{\mathcal{B}}$. Therefore, $\tau \subseteq \tau_{\mathcal{B}}$. Conversely, let $U \in \tau_{\mathcal{B}}$ be arbitrary. Then $U = \bigcup_{x\in U} V_x$, where $V_x \in \mathcal{B}$ with $x \in V_x \subseteq U$, by Lemma 2.1.12. Since $\mathcal{B} \subseteq \tau$ and τ is closed under arbitrary union of its elements, $U \in \tau$. Therefore, $\tau_{\mathcal{B}} \subseteq \tau$, and hence $\tau_{\mathcal{B}} = \tau$. This proves uniqueness part. $\square$

Proposition 2.1.15. *Let $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies τ and τ' on X. Then the following are equivalent.*

(i) $\tau \subseteq \tau'$,

(ii) given $V \in B$ and $x \in V$, there exists $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\tau \subseteq \tau'$. Let $V \in \mathcal{B}$ and $x \in V$ be given. Since $\mathcal{B} \subseteq \tau \subseteq \tau'$ and $\mathcal{B}'$ is a basis for τ', there exists a $V' \in \mathcal{B}'$ such that $x \in V' \subseteq V$.

(ii) $\Rightarrow$ (i): Let $U \in \tau$ be arbitrary. Since $\mathcal{B}$ is a basis for τ, by Lemma 2.1.12 we have $U = \bigcup_{x\in U} V_x$, where $V_x \in \mathcal{B}$ with $x \in V_x \subseteq U$. Then by (ii) there exists $W_x \in \mathcal{B}'$ such that $x \in W_x \subseteq V_x$, for all $x \in U$. Then $U = \bigcup_{x\in U} W_x \in \tau'$. $\square$

Example 2.1.16. Given $a, b \in \mathbb{R}$ with $a < b$, let $[a, b) := \{t \in \mathbb{R} : a \leq t < b\}$. Let $\mathcal{B}_\ell = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R} \text{ with } a < b\}$. Note that $\mathcal{B}_\ell$ is a basis for a topology τ_ℓ on $\mathbb{R}$, called the *lower limit topology* on $\mathbb{R}$. We denote by $\mathbb{R}_\ell$ the topological space $(\mathbb{R}, \tau_\ell)$. Let $a, b \in \mathbb{R}$ with $a < b$ and let $c \in (a, b)$. Then $c \in [c, b) \subset (a, b)$. Then it follows from Proposition 2.1.15 that $\tau_E \subseteq \tau_\ell$, where τ_E is the Euclidean topology on $\mathbb{R}$. Fix $a, b \in \mathbb{R}$ with $a < b$. Then $a \in [a, b)$, but there is no open interval (c, d) in $\mathbb{R}$ such that $a \in (c, d) \subseteq [a, b)$. Therefore, τ_ℓ is strictly finer than the Euclidean topology on $\mathbb{R}$.

Example 2.1.17 (K-topology)**.** Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $\mathcal{B}_K$ be the set of all open intervals in $\mathbb{R}$ along with the subsets of the form $(a, b) \setminus K$, where $a, b \in \mathbb{R}$ with $a < b$. It is easy to check that $\mathcal{B}_K$ is a basis for a topology τ_K on $\mathbb{R}$ which is strictly finer than the Euclidean topology on $\mathbb{R}$. The topology τ_K on $\mathbb{R}$ generated by $\mathcal{B}_K$ is called the *K-topology* on $\mathbb{R}$. Note that $(-1, 1) \setminus K$ is open in the K-topology on $\mathbb{R}$, but not in the Euclidean topology on $\mathbb{R}$ because there is no open interval (a, b) containing 0 and contained in $(-1, 1) \setminus K$.

Exercise 2.1.18. Show that the lower limit topology and the K-topology on $\mathbb{R}$ are not comparable in the sense that neither $\tau_\ell \subseteq \tau_K$ nor $\tau_K \subseteq \tau_\ell$.

Let X be a non-empty set and let $\mathcal{P}(X)$ be the power set of X. Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be such that $\bigcup_{V \in \mathcal{S}} V = X$. Then we can use $\mathcal{S}$ to construct a topology on X as follow: let $\mathcal{B} \subseteq \mathcal{P}(X)$ be the set of all finite intersections of elements from $\mathcal{S}$. Note that $\mathcal{S} \subseteq \mathcal{B}$ and so $\bigcup_{V \in \mathcal{B}} V = X$. Let $V, W \in \mathcal{B}$ be arbitrary. Then $V = \bigcap_{j=1}^{m} V_j$ and $W = \bigcap_{k=1}^{n} W_k$, for some $V_1, \ldots, V_m, W_1, \ldots, W_n \in \mathcal{S}$. Then their intersection $V \cap W = \left(\bigcap_{j=1}^{m} V_j\right) \cap \left(\bigcap_{k=1}^{n} W_k\right)$ is again a finite intersection of elements from $\mathcal{S}$, and hence is an element of $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis for a topology $\tau_{\mathcal{B}}$ on X, called the ***topology generated by the subbasis $\mathcal{S}$***. This motivates us to define the notion of subbasis for a topological space as follow.

Definition 2.1.19. Let X be a topological space. A set $\mathcal{S} \subseteq \mathcal{P}(X)$ of subsets of X is said to be a ***subbasis for the topology on X*** if $\bigcup_{V \in \mathcal{S}} V = X$ and the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ forms a basis for the topology on X.

Example 2.1.20. (i) The collection $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ is a subbasis for the Euclidean topology on $\mathbb{R}$.

(ii) The collection $\mathcal{S}_\ell := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{[b, \infty) : b \in \mathbb{R}\}$ is a subbasis for the lower limit topology on $\mathbb{R}$.

(iii) The collection $\mathcal{S} := \{[a, b] : a, b \in \mathbb{R},\ a < b\}$ is a subbasis for a topology τ on $\mathbb{R}$, where τ is strictly finer than the Euclidean topology on $\mathbb{R}$. Is τ discrete?

2.1.1 Order topology

Let $(X, \leq)$ be a simply ordered (i.e., totally ordered) set. Given $a, b \in X$ with $a < b$, there are four types of subset of X that are called intervals determined by a and b, namely

$$\begin{aligned}
(a, b) &:= \{x \in X : a < x < b\}, \\
[a, b] &:= \{x \in X : a \leq x \leq b\}, \\
[a, b) &:= \{x \in X : a \leq x < b\}, \\
(a, b] &:= \{x \in X : a < x \leq b\}.
\end{aligned}$$

Let $\mathcal{B}$ be the set of all subsets of X of the following forms

(i) all open intervals (a, b) in X,

(ii) all intervals of the form $[a_0, b)$, where a_0 is the smallest element, if exists, of X,

(iii) all intervals of the form $(a, b_0]$, where b_0 is the largest element, if exists, of X.

Then the collection $\mathcal{B}$ is a basis for a topology on X, called the *order topology* on $(X, \leq)$.

Example 2.1.21. The Euclidean topology on $\mathbb{R}$ is the order topology on it.

Exercise 2.1.22. Consider the set $\mathbb{R} \times \mathbb{R}$ and give a partial order relation on it by setting $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. Draw intervals $((a, b), (c, d))$ in $\mathbb{R} \times \mathbb{R}$, for the case $a < c$, and the case $a = c$ with $b < d$.

2.2 Interior point and limit point

Let X be a topological space, and let $A \subseteq X$.

Definition 2.2.1. A point $x \in A$ is said to be an interior point of A if there exists an open subset U of X such that $x \in U$ and $U \subseteq A$.

Proposition 2.2.2. *A subset $A \subseteq X$ is open if and only if either $A = \emptyset$ or each of the points of A are interior points.*

Proof. Since empty subset is open by definition, we may assume that $A \neq \emptyset$. If A is open in X, given any $x \in A$, we can take $U = A$ so that $x \in U \subseteq A$ holds, so that x is an interior point of A. Conversely, suppose that each point $x \in A$ is an interior point. Then given $a \in A$, there exists an open subset V_a of X such that $a \in V_a \subseteq A$. Then $A = \bigcup_{a \in A} V_a$. Since arbitrary union of open subsets of X is open in X, that A is open in X. □

Example 2.2.3. The subset $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ is open in the Euclidean space $\mathbb{R}$, and hence all of its points are interior points.

Example 2.2.4. Consider the subset $A = [0, 1)$ of $\mathbb{R}$. Note that any point of A other than 0 is an interior point of it when the real line is equipped with the Euclidean topology or the lower limit topology. However, $0 \in A$ is not an interior point of A if $\mathbb{R}$ is equipped with the Euclidean topology, but if we equip $\mathbb{R}$ with the lower limit topology, then 0 is an interior point of A in $\mathbb{R}_\ell$.

Definition 2.2.5. A point $x \in X$ is said to be a *limit point* of $A \subseteq X$ if given any open subset U of X containing x, there exists an element $a \in A$ such that $a \neq x$ and $a \in U$; in other words, $(U \setminus \{x\}) \cap A \neq \emptyset$.

Proposition 2.2.6. *Let X be a topological space, and let Z be a non-empty subset of X. Then Z is closed in X if and only if it contains all of its limit points.*

Proof. Suppose that Z is closed in X. If $x \notin Z$, then x is in the open subset $U = X \setminus Z$, and that $(U \setminus \{x\}) \cap Z = \emptyset$. Then x cannot be a limit point of Z. Conversely, suppose that Z contain all of its limit points in X. Let $U = X \setminus Z$. If $U = \emptyset$, then $Z = X$, and hence it is closed in X. Assume that $U \neq \emptyset$. Then given any $x \in U$, x is not a limit point of Z. Then there exists an open subset V_x of X such that

$$(V_x \setminus \{x\}) \cap Z = \emptyset.$$

Since $x \notin Z$, we have $V_x \cap Z = \emptyset$. Then $V_x \subseteq U = X \setminus Z$. Therefore, x is an interior point of U. Thus, U is open in X, and hence Z is closed in X. □

Proposition 2.2.7. *Let X be a topological space. Given a subset A of X, let*

$$\mathscr{C}_A := \{Z \subseteq X : A \subseteq Z \text{ and } Z \text{ is closed in } X\}$$

be the set of all closed subsets of X containing A. Then $\bigcap_{Z \in \mathscr{C}_A} Z$ is the smallest closed subset of X containing A. We call $\bigcap_{Z \in \mathscr{C}_A} Z$ the closure of A in X, and denote it by $\overline{A}$.

Proof. Clearly $A \subseteq \bigcap_{Z \in \mathscr{C}_A} Z$. Since $X \setminus \left(\bigcap_{Z \in \mathscr{C}_A} Z \right) = \bigcup_{Z \in \mathscr{C}_A} (X \setminus Z)$ is open in X by definition of a topological space, the subset $\bigcap_{Z \in \mathscr{C}_A} Z$ is closed in X. If W is any closed subset of X with $A \subseteq W$, then $W \in \mathscr{C}_A$, and hence $\bigcap_{Z \in \mathscr{C}_A} Z \subseteq W$. This completes the proof. $\square$

Lemma 2.2.8. *Let X be a topological space and let $A \subseteq X$. Then $\overline{A} = A \cup A'$, where A' is the set of all limit points of A in X. In particular, $x \in \overline{A}$ if and only if given any open subset V of X containing x, we have $V \cap A \neq \emptyset$.*

Proof. Let $\mathscr{C}_A$ be the set of all closed subsets of X containing A. Since $\overline{A} = \bigcap_{Z \in \mathscr{C}_A} Z$, we have $A \subseteq \overline{A}$. Let $x \in A'$ be arbitrary. Let $Z \in \mathscr{C}_A$. If $x \notin Z$, then $V := X \setminus Z$ is an open neighbourhood of x in X, and so $V \cap A \neq \emptyset$. But this is not possible since $A \subseteq Z$ by assumption. Therefore, $x \in Z$, $\forall\, Z \in \mathscr{C}_A$, and hence $A \cup A' \subseteq Z$, $\forall\, Z \in \mathscr{C}_A$. Therefore, $A \cup A' \subseteq \overline{A}$. Conversely, let $x \in \overline{A}$ be arbitrary. Suppose that $x \notin A$. Let U be an open neighbourhood of x in X. $\square$

2.3 Continuity

Let (X, d_1) and (Y, d_2) be metric spaces. Recall that a map $f : (X, d_1) \to (Y, d_2)$ is said to be *continuous at* $x_0 \in X$ if for given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_2(f(x_0), f(x)) < \epsilon, \quad \text{whenever } d_1(x_0, x) < \delta.$$

If f is continuous at every point of X, then we call f a continuous map. This motivates us to extend the notion of continuity of maps between arbitrary topological spaces by replacing open balls with open subsets. Here is the formal definition.

Definition 2.3.1 (Continuous map)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous at* $x_0 \in X$ if for each open subset V of Y containing $f(x_0)$, there exists an open subset U of X such that $x_0 \in U$ and $f(U) \subseteq V$. We say that f is continuous if it is continuous at every point of X.

Lemma 2.3.2. *Let $f : X \to Y$ be a map of topological spaces. Then f is continuous if and only if for given any open subset V of Y, the subset $f^{-1}(V)$ is open in X.*

Proof. Suppose that f is continuous. Let V be any open subset of Y. If $f^{-1}(V) = \emptyset$, we have nothing to check. Assume that $f^{-1}(V) \neq \emptyset$. Let $x_0 \in f^{-1}(V)$ be given. Then $f(x_0) \in V$. Since

f is continuous at x_0, there exists an open subset U of X with $x_0 \in U$ such that $f(U) \subseteq V$. Then $U \subseteq f^{-1}(V)$ with $x_0 \in U$, and so x_0 is an interior point of $f^{-1}(V)$. Since $x_0 \in f^{-1}(V)$ is chosen arbitrarily, $f^{-1}(V)$ is open in X by Proposition 2.2.2. Converse is obvious. □

The following result shows that to check continuity of a map it suffices to show that inverse image of every basic open subset is open.

Corollary 2.3.3. *Let X and Y be topological spaces. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Then a map $f : X \to Y$ is continuous if and only if given any basic open subset $V \in \mathcal{B}_Y$ of Y, its inverse image $f^{-1}(V)$ is open in X.*

Proof. If f is continuous, then $f^{-1}(B)$ is open in X, for any basic open subset B of Y. To show the converse, let U be any open subset of Y. Then by Lemma 2.1.12, $U = \bigcup\limits_{\alpha \in \Lambda} B_\alpha$, for some collection $\{B_\alpha : \alpha \in \Lambda\}$ of basic open subsets of Y. Since $f^{-1}(B_\alpha)$ is open in X by assumption, the subset $f^{-1}(U) = \bigcup\limits_{\alpha \in \Lambda} f^{-1}(B_\alpha)$ is open in X. Therefore, f is continuous. □

As an immediate consequence of Lemma 2.3.2, we have the following.

Corollary 2.3.4. *Let X be a non-empty set together with two topologies τ_1 and τ_2. For each $j = 1, 2$, let $X_j = (X, \tau_j)$ be the topological space whose underlying set is X and the topology is τ_j. Then τ_2 is finer than τ_1 (i.e., $\tau_1 \subseteq \tau_2$) if and only if the identity map $\mathrm{Id}_X : X_2 \to X_1$ is continuous.*

Example 2.3.5. Let $\mathbb{R}$ be the real line with the Euclidean topology on it, and let $\mathbb{R}_\ell$ be the real line with the lower limit topology on it. Since any open interval (a, b) in $\mathbb{R}$ can be written as

$$(a, b) = \bigcup_{n \in \mathbb{N}} [a - \frac{1}{n}, b),$$

and each of $[a - \frac{1}{n}, b)$ are open in $\mathbb{R}_\ell$, it follows that (a, b) is open in $\mathbb{R}_\ell$. However, $[a, b)$ is open in $\mathbb{R}_\ell$ but not in $\mathbb{R}$. Therefore, the lower limit topology on the real line is strictly finer than the Euclidean topology on it. Then by Corollary 2.3.4, the identity map $\mathbb{R}_\ell \to \mathbb{R}$ is continuous, while the identity map $\mathbb{R} \to \mathbb{R}_\ell$ is not continuous.

Lemma 2.3.6. *Let X and Y be topological spaces and let $f : X \to Y$ be a map. Then the following are equivalent.*

(i) f is continuous.

(ii) for every closed subset Z of Y, $f^{-1}(Z)$ is closed in X.

(iii) $f(\overline{A}) \subseteq \overline{f(A)}$, for all $A \subseteq X$.

Proof. (i) ⇒ (ii): Let Z be a closed subset of Y. Then $Y \setminus Z$ is open in Y. Since f is continuous by assumption (i), $f^{-1}(Y \setminus Z)$ is open in X. Since $f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z)$ (verify!), we conclude that $f^{-1}(Z)$ is closed in X.

(ii) $\Rightarrow$ (iii): Let $A \subseteq X$. Since $\overline{f(A)}$ is closed in Y by Proposition 2.2.7, $f^{-1}(\overline{f(A)})$ is closed in X by assumption (ii). Since

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}),$$

and $\overline{A}$ is the smallest closed subset of X containing A, we have $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Therefore, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

(iii) $\Rightarrow$ (i): Let V be a non-empty open subset of Y. Since $Z := Y \setminus V$ is closed in Y, we have $\overline{Z} = Z$ by Proposition 2.2.7. Apply assumption (iii) to the subset $A := f^{-1}(Z) \subseteq X$ to get $f(\overline{A}) \subseteq \overline{f(A)}$. Since $f(A) = f(f^{-1}(Z)) \subseteq Z$, we have

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{Z} = Z.$$

Then $\overline{A} \subseteq f^{-1}(Z) = A$, and hence $\overline{A} = A$. Then A is closed in X by Proposition 2.2.7. Since

$$A = f^{-1}(Z) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),$$

we see that $f^{-1}(V)$ is open in X. Therefore, f is continuous. $\square$

Definition 2.3.7. Let X be a topological space.

(i) A collection $\{V_i : i \in I\}$ of open neighbourhoods of $a \in X$ is said to be a *local basis for X at $a \in X$* if for given any open neighbourhood U of a in X, there exists $i \in I$ such that $V_i \subseteq U$.

(ii) A topological space X is said to be *first countable* if there exists a countable local basis for X at each points of X.

(iii) A topological space X is said to be *second countable* if there exists a countable basis for the topology on X.

Example 2.3.8. (i) Any second countable space is clearly first countable.

(ii) The real line $\mathbb{R}$ with the standard topology is second countable. Indeed, the collection $\{(a, b) : a, b \in \mathbb{Q} \text{ with } a < b\}$ is a countable basis for the topology on $\mathbb{R}$.

(iii) Any metric space (X, d) is first countable. Indeed, for each $a \in X$, the collection $\{B_d(a, 1/n) : n \in \mathbb{N}\}$ is a countable local basis for the metric topology on X at a. However, it need not be second countable. For example, one can take the discrete metric on an uncountable set.

Example 2.3.9. The lower limit topological space $\mathbb{R}_\ell$ is first countable, but not second countable. Indeed, for each $a \in \mathbb{R}$, the collection $\{[a, x) : x \in \mathbb{Q} \text{ with } a < x\}$ is a countable local basis for $\mathbb{R}_\ell$ at a. Suppose that $\mathcal{B}$ be a basis for the topology on $\mathbb{R}_\ell$. For each $x \in \mathbb{R}$, the subset $V_x := [x, \infty)$ is an open neighbourhood of x in $\mathbb{R}_\ell$, and hence it contains an element, say $V_x \in \mathcal{B}$. Since $\inf(V_x) = x$ by construction, we see that $x \neq y$ in $\mathbb{R}$ implies that $V_x \neq V_y$. Therefore, the collection $\mathcal{B}$ must be uncountable.

Remark 2.3.10. Let X be a topological space. Let $\mathcal{B} := \{U_n : n \in \mathbb{N}\}$ be a countable local basis for X at $a \in X$. For each $n \in \mathbb{N}$, let $V_n := \bigcap_{i=1}^{n} U_i$. Then the collection $\{V_n : n \in \mathbb{N}\}$ is a countable local basis for X at a satisfying $V_{n+1} \subseteq V_n$, $\forall\, n \in \mathbb{N}$.

Lemma 2.3.11 (Sequence Lemma)**.** *Let X be a topological space and let A be a non-empty subset of X. If there is a sequence $(a_n)_{n\in\mathbb{N}}$ in A converging to a point $x \in X$, then $x \in \overline{A}$. The converse holds if X has a countable local basis at x.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of points in A converging to $x \in X$. Then for any open neighbourhood V of x in X, $\exists\, n_V \in \mathbb{N}$ such that $a_n \in V, \ \forall\, n \geq n_V$. Therefore, $x \in \overline{A}$.

Conversely suppose that $x \in \overline{A}$. If $x \in A$, we may take the constant sequence given by $a_n = x, \forall\, n \in \mathbb{N}$. Suppose that $x \notin A$. Then x is a limit point of A. Let $\{V_n : n \in \mathbb{N}\}$ be a countable local basis for X at x. Note that, for each $n \in \mathbb{N}$, the subset $U_n := \bigcap\limits_{i=1}^{n} V_i$ is an open neighbourhood of a in X, and satisfies $U_{n+1} \subseteq U_n$, for all $n \in \mathbb{N}$. Clearly $\{U_n : n \in \mathbb{N}\}$ is a countable local basis for X at a. Choose $a_n \in U_n$, for all $n \in \mathbb{N}$. Let $\mathcal{O}$ be any open neighbourhood of x in X. Then $V_n \subseteq \mathcal{O}$, for some n, and hence $U_m \subseteq \mathcal{O}$, for all $m \geq n$. Therefore, $a_m \in \mathcal{O}$, for all $m \geq n$. Therefore, $(a_n)_{n\in\mathbb{N}}$ converges to x in X. □

Theorem 2.3.12 (Sequential Criterion for Continuity)**.** *Let $f : X \to Y$ be a map of topological spaces. If f is continuous at $a \in X$, then for any sequence $(x_n)_{n\in\mathbb{N}}$ converging to a, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(a) \in Y$. Converse holds if X has a countable local basis at a.*

Proof. Suppose that f is continuous at $a \in X$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to a. Let V be an open neighbourhood of $f(a)$ in Y. Since f is continuous at a, there exists an open neighbourhood U of a in X such that $f(U) \subseteq V$. Since $(x_n)_{n\in\mathbb{N}}$ converges to a in X, there exists $n_U \in \mathbb{N}$ such that $x_n \in U, \forall\, n \geq n_U$. Then $f(x_n) \in f(U) \subseteq V, \forall\, n \geq n_U$. Therefore, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(a)$ in Y.

For the converse part, assume that X has a countable local basis $\{U_n : n \in \mathbb{N}\}$ at a. In view of Remark 2.3.10, replacing U_n with $\bigcap\limits_{i=1}^{n} U_i$, if required, we may assume that $U_{n+1} \subseteq U_n, \forall\, n \in \mathbb{N}$. Suppose on the contrary that there is an open neighbourhood $V \subseteq Y$ of $f(a)$ for which there is no open neighbourhood $U \subseteq X$ of a satisfying $f(U) \subseteq V$. Then for each $n \in \mathbb{N}$, $f(U_n) \nsubseteq V$, and so we can choose $x_n \in U_n$ such that $f(x_n) \notin V$. Then $(x_n)_{n\in\mathbb{N}}$ converges to $a \in X$ while its image $(f(x_n))_{n\in\mathbb{N}}$ does not converges to $f(a)$ by construction. This contradicts our assumption, and completes the proof. □

Exercise 2.3.13. Let A be a non-empty subset of a topological space. Show that the subspace topology on A induced from X is the smallest topology on A such that the inclusion map

$$\iota_A : A \hookrightarrow X, \ \ a \mapsto a,$$

is continuous.

Answer: Since the subspace topology τ_A on A induced from X is given by $\tau_A = \{U \cap A : U \text{ is open in } X\}$, given an open subset U of X, the subset $\iota_A^{-1}(U) = U \cap A$ is in τ_A. If τ' is any topology on A such that the inclusion map $\iota_A : A \hookrightarrow X$ is continuous, then $U \cap A \in \tau'$, for all open subset U of X, and hence $\tau_A \subseteq \tau'$. □

Exercise 2.3.14. Show that the direct image of open (resp., closed) subset under a continuous map of topological spaces need not be open (resp., closed). (*Hint:* Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, $\forall\, x \in \mathbb{R}$, and note that $f((-1,1)) = [0,1)$. Take $g : [1,\infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$, $\forall\, x$. Note that the image of the closed subset $[1,\infty)$ under g is not closed in $\mathbb{R}$.)

Let X and Y be topological spaces. A map $f : X \to Y$ is said to be a *constant map* if there exists $y_0 \in Y$ such that $f(x) = y_0$, $\forall\, x \in X$.

Corollary 2.3.15. *Let X and Y be two topological spaces. Then any constant map $f : X \to Y$ is continuous.*

Proof. Let $f(x) = y_0$, for all $x \in X$. Let V be any open subset of Y. If $y_0 \in Y$, then $f^{-1}(V) = X$, and if $y_0 \notin V$, then $f^{-1}(V) = \emptyset$. Therefore, $f^{-1}(V)$ is open in X, and hence f is continuous. $\square$

Proposition 2.3.16. *Let $\Bbbk$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the Euclidean topology on it. Let X be a topological space and $f, g : X \to \Bbbk$ be two continuous maps. Then the maps $f+g, fg : X \to \Bbbk$ defined by point-wise addition and multiplication of real numbers,*

$$(f+g)(x) := f(x) + g(x),\ \forall\, x \in X,$$
$$(fg)(x) := f(x)g(x),\ \forall\, x \in X,$$

are continuous. Moreover, if $f(x) \neq 0$, $\forall\, x \in X$, then the map

$$g : X \to \mathbb{R},\ \ x \mapsto \frac{1}{f(x)},$$

is continuous. The set $\mathcal{C}(X,\Bbbk)$ of all $\Bbbk$-valued continuous functions on X forms an $\Bbbk$-algebra.

Proof. Suppose that $f, g \in \mathcal{C}(X,\Bbbk)$ be given. Let $x_0 \in X$ be arbitrary. Since f and g are continuous at $x_0 \in X$, given a real number $r > 0$ there exists open subsets U and V of X containing x_0 such that

$$|f(x) - f(x_0)| < r/2,\ \forall\, x \in U,\ \text{ and}$$
$$|g(x) - g(x_0)| < r/2,\ \forall\, x \in V.$$

Then for any $x \in U \cap V$, we have

$$\begin{aligned} |(f+g)(x) - (f+g)(x_0)| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Therefore, $f+g$ is continuous at x_0. Since $x_0 \in X$ is chosen arbitrarily, $f+g$ is continuous on X. Similarly, one can show that $f \cdot g$ is continuous on X (verify!). $\square$

Corollary 2.3.17. *Let X be a topological space and $f, g : X \to \mathbb{R}$ be two continuous maps. Then*

(i) $A := \{x \in X : f(x) < g(x)\}$ *is open in X, and*

(ii) $B := \{x \in X : f(x) \leq g(x)\}$ *is closed in* X.

Proof. Since $g - f : X \to \mathbb{R}$ is continuous by Proposition 2.3.16 and since $\mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ is open in $\mathbb{R}$ and $\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : t \geq 0\}$ is closed in $\mathbb{R}$, the subset $A = (g - f)^{-1}(\mathbb{R}^+)$ is open in X, and the subset $B = (g - f)^{-1}(\mathbb{R}_{\geq 0})$ is closed in X. □

Lemma 2.3.18. *Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f$.*

Proof. Let V be an open subset of Z. Since g is continuous, $g^{-1}(V)$ is open in Y, and since f is continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. Therefore, $g \circ f$ is continuous. □

Lemma 2.3.19. *Let $f : X \to Y$ be continuous map. Given any non-empty subset $A \subseteq X$, equip A with the subspace topology induced from X. Then the restriction map $f\big|_A : A \to Y$ is continuous.*

Proof. Since inclusion map $\iota_A : A \hookrightarrow X$ and $f : X \to Y$ are continuous, so is their composition map $f\big|_A : A \to Y$. □

Exercise 2.3.20. Give example of maps $f : X \to Y$ and $g : Y \to Z$ of topological spaces such that both g and $g \circ f$ are continuous, but f is not continuous. (*Hint:* Consider a constant function).

Definition 2.3.21. Let (Y, d) be a metric spaces, and let $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ be a sequence of maps from a non-empty set X into Y. We say that $(f_n)_{n \in \mathbb{N}}$ *converges uniformly* to the map $f : X \to Y$ if for given any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \epsilon, \; \forall\, n \geq n_\epsilon \text{ and } x \in X.$$

Theorem 2.3.22 (Uniform Limit Theorem)**.** *Let $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ be a sequence of continuous maps from a topological space X into a metric space (Y, d). If $(f_n)_{n \in \mathbb{N}}$ converges uniformly to a map $f : X \to Y$, then f is continuous.*

Proof. □

Definition 2.3.23. Let X and Y be topological spaces. A map $f : X \to Y$ is said to be

(i) a ***homeomorphism*** if f is continuous and there exists a continuous map $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

(ii) an ***embedding*** if f is homeomorphism onto its image $f(X) \subseteq Y$, where $f(X)$ is equipped with the subspace topology induced from Y.

Note that, a homeomorphism is a continuous bijective map of topological spaces whose inverse is also continuous. If there is a homeomorphism $f : X \to Y$ then we say that X is homeomorphic to Y, and express it as $X \cong Y$.

Example 2.3.24. (i) For any topological space X, the identity map $\mathrm{Id}_X : X \to X$ is a homeomorphism.

(ii) Let X be the real line $\mathbb{R}$ with the usual topology on it. Then for any $a \in \mathbb{R}$, the translation by a map
$$t_a : \mathbb{R} \to \mathbb{R}, \;\; x \mapsto x + a,$$
is a homeomorphism. Indeed, t_a is a continuous bijective map with the continuous inverse t_{-a} (verify!).

(iii) Equip $\mathbb{C}$ and $\mathbb{R}^2$ with the usual Euclidean topologies. Then the map $f : \mathbb{C} \to \mathbb{R}^2$ given by
$$f(a + ib) = (a, b), \; \forall\, a + ib \in \mathbb{C},$$
is a homeomorphism.

(iv) The map $f : \mathbb{R} \to \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ defined by $f(x) = e^x$, $\forall\, x \in \mathbb{R}$, is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$, where $\mathbb{R}^+$ is equipped with the subspace topology induced from the usual topology on $\mathbb{R}$.

(v) The map $f : (-1, 1) \to \mathbb{R}$ defined by
$$f(t) = \frac{t}{t^2 - 1}, \; \forall\, t \in (-1, 1),$$
is a homeomorphism (verify!), while the inclusion map $\iota : (-1, 1) \hookrightarrow \mathbb{R}$ is an embedding.

Exercise 2.3.25. Equip $\mathbb{R}^n$ with the standard Euclidean metric
$$\|(x_1, \ldots, x_n)\|_2 := \sqrt{x_1^2 + \cdots + x_n^2},$$
and let $B(0, r) := \{x \in \mathbb{R}^n : \|x\|_2 < 1\}$ be the open unit ball in $\mathbb{R}^n$ centered at the origin. Show that the map $f : B(0, 1) \to \mathbb{R}^n$ given by
$$f(x) = \frac{x}{1 - \|x\|_2}, \; \forall\, x \in B(0, 1),$$
is a homeomorphism.

Remark 2.3.26. Note that "being homeomorphic topological spaces" is an equivalence relation on the collection of all topological spaces. Indeed, any topological space X is homeomorphic to itself via the identity map $\mathrm{Id}_X : X \to X$. If $f : X \to Y$ is a homeomorphism of topological spaces, then $f^{-1} : Y \to X$ is a homeomorphism from Y into X. If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms of topological spaces, then $g \circ f : X \to Z$ is a homeomorphism.

Lemma 2.3.27 (Pasting lemma)**.** *Let X and Y be topological spaces. Let A and B be closed subsets of X such that $X = A \cup B$. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps. If $f(x) = g(x)$, $\forall\, x \in A \cap B$, then there is a unique continuous map $h : X \to Y$ such that $h\big|_A = f$ and $h\big|_B = g$.*

Proof. Uniqueness of h is obvious. To show existence, define $h : X \to Y$ by setting
$$h(x) = \begin{cases} f(x), & \text{if} \quad x \in A, \\ g(x), & \text{if} \quad x \in B. \end{cases}$$

Since $f(x) = g(x)$, $\forall\, x \in A \cap B$, the map h is well-defined. To check continuity of h, note that given any open subset V of Y, we have

$$h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V).$$

Since f and g are continuous, $h^{-1}(V)$ is open in X. This completes the proof. $\square$

Exercise 2.3.28 (Sheaf of continuous maps)**.** Let X and Y be topological spaces. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of an open subset U of X.

(i) If V is an open subset of U, for any continuous map $f : U \to V$, show that $\left(f\big|_U\right)\big|_V = f\big|_V$.

(ii) Let $f, g : U \to Y$ be continuous maps such that $f\big|_{U_\alpha} = g\big|_{U_\alpha}$, $\forall\, \alpha \in \Lambda$. Show that $f(x) = g(x)$, $\forall\, x \in U$.

(iii) Let $\{f_\alpha : U_\alpha \to Y\}_{\alpha \in \Lambda}$ be a family of continuous maps such that

$$f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta},\ \forall\, \alpha, \beta \in \Lambda.$$

Show that there exists a unique continuous map $f : U \to Y$ such that $f\big|_{U_\alpha} = f_\alpha$, $\forall\, \alpha \in \Lambda$.

Exercise 2.3.29. Let $(Y, \leq)$ be a totally ordered set. Given $y_1, y_2 \in Y$, define

$$\min\{y_1, y_2\} := \begin{cases} y_1, & \text{if } \quad y_1 \leq y_2, \\ y_2, & \text{if } \quad y_2 \leq y_1. \end{cases}$$

Equip Y with the order topology induced by the total ordering $\leq$ on Y. Given a topological space X and continuous maps $f, g : X \to Y$, show that

(i) $V := \{x \in X : f(x) < g(x)\}$ is open in X,

(ii) $Z := \{x \in X : f(x) \leq g(x)\}$ is closed in X, and

(iii) the map $h : X \to Y$ defined by

$$h(x) = \min\{f(x), g(x)\},\ \forall\, x \in X,$$

is continuous.

Solution: (i) Let $a \in V$ be given. Then $f(a) < g(a)$. If there is an element $y \in Y$ with $f(a) < y < g(a)$, then

$$U_a := f^{-1}((-\infty, y)) \cap g^{-1}((y, \infty))$$

is an open subset of X with $a \in U_a \subseteq V$. If there is no $y \in Y$ with $f(a) < y < g(a)$, then

$$U_a' := f^{-1}((-\infty, g(a))) \cap g^{-1}((f(a), \infty))$$

is an open neighbourhood of a in X. Let $b \in U_a'$ be arbitrary. Then

$$f(b) < g(a) \quad \text{and} \quad f(a) < g(b).$$

Since there exists no $y \in Y$ with $f(a) < y < g(a)$ and $(Y, \leq)$ is totally ordered by assumption, we must have $f(a) < f(b)$. Therefore, $U_a' \subseteq V$. Thus, a is an interior point of V, and hence V is open in X.

(ii) Since $V := \{x \in X : g(x) < f(x)\}$ is open by part (i), we have $Z = X \setminus V$ because $(Y, \leq)$ is totally ordered set. Therefore, Z is closed in X.

(iii) Since $f, g : X \to Y$ are continuous, the subsets

$$\begin{aligned} A &:= \{x \in X : f(x) \leq g(x)\} \\ \text{and} \quad B &:= \{x \in X : g(x) \leq f(x)\}. \end{aligned}$$

are closed in X by part (ii). Since $(Y, \leq)$ is a totally ordered set, we have $Y = A \cup B$. Since $f, g : X \to Y$ are continuous, so are their restriction maps

$$f' := f\big|_A : A \to Y \quad \text{and} \quad g' := g\big|_B : B \to Y$$

by Lemma 2.3.19. Since $h : X \to Y$ is given by

$$h(x) = \min\{f(x), g(x)\} = \begin{cases} f'(x), & \text{if} \quad x \in A, \\ g'(x), & \text{if} \quad x \in B, \end{cases}$$

and $f'\big|_{A \cap B} = g'\big|_{A \cap B}$, we conclude by pasting lemma 2.3.27 that h is continuous. $\square$

Exercise 2.3.30. Let X and Y be topological spaces. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an indexed family of subsets of X such that $X = \bigcup\limits_{\alpha \in \Lambda} A_\alpha$. Let $f : X \to Y$ be a map such that $f\big|_{A_\alpha} : A_\alpha \to Y$ is continuous, $\forall\, \alpha \in \Lambda$.

(i) If $\{A_\alpha : \alpha \in \Lambda\}$ is a finite collection and if each A_α is closed in X, show that f is continuous.

(ii) Show by an example that if $\{A_\alpha : \alpha \in \Lambda\}$ is at least countably infinite collection, f need not be continuous even if all A_α are closed in X.

(iii) An indexed family of subsets $\{A_\alpha : \alpha \in \Lambda\}$ is said to be *locally finite* if each point $x \in X$ has an open neighbourhood in X that intersects A_α for finitely many $\alpha \in \Lambda$. If $\{A_\alpha : \alpha \in \Lambda\}$ is a locally finite family of closed subsets of X, show that f is continuous.

Exercise 2.3.31. Let X, Y and Z be topological spaces. A map $F : X \times Y \to Z$ is said to be ***continuous in each variable separately*** if the maps

$$\begin{aligned} &F_{y_0} : X \to Z, \ \ x \mapsto F(x, y_0), \\ \text{and} \quad &F_{x_0} : Y \to Z, \ \ y \mapsto F(x_0, y), \end{aligned}$$

are continuous, for all $x_0 \in X$ and $y_0 \in Y$. If F is continuous, show that it is continuous in each variable separately.

Exercise 2.3.32. Consider the map $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x,y) := \begin{cases} \frac{xy}{x^2+y^2}, & \text{if} \quad (x,y) \neq (0,0), \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

(i) Show that F is continuous in each variable separately.

(ii) Compute the map $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x,x), \ \forall\, x \in \mathbb{R}$.

(iii) Conclude that F is not continuous.

Exercise 2.3.33. Let X be a topological space. Let $A \subseteq X$ and let $\overline{A}$ be the closure of A in X. Let Y be a Hausdorff space and let $f : A \to Y$ be a continuous map. Show that there can be at most one continuous function $g : A \to Y$ such that $g\big|_A = f$.

2.4 Product topology

Before proceeding further, let us introduce a terminology, namely *category*, that is a systematic common framework to study for various mathematical objects.

Definition 2.4.1. A *category* $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X,Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X,Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X,Y,Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X,Y) \times \mathrm{Mor}_{\mathscr{C}}(Y,Z) \to \mathrm{Mor}_{\mathscr{C}}(X,Z), \ \ (f,g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X,Y,Z,W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y,Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z,W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X,X)$ such that given any objects $Y,Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

Example 2.4.2. (i) In the category of sets (Set), we consider the collection of all sets as $\mathrm{ob}(Set)$ and given any two objects (i.e., sets) X and Y, we have a collection $\mathrm{Mor}_{(Set)}(X,Y)$ which consists of all set theoretic maps from X into Y.

(ii) In the category of groups $(\mathcal{G}rp)$, we take $\mathrm{ob}(\mathcal{G}rp)$ to be the collection of all groups and given any two objects (groups) G and H, we take $\mathrm{Mor}_{(\mathcal{G}rp)}(G,H) = \mathrm{Hom}(G,H)$, the set of all group homomorphisms from G into H.

(iii) Let k be a field. In the category of k-vector spaces $(\mathcal{V}ect_k)$, we take $\mathrm{ob}(\mathcal{V}ect_k)$ to be the collection of all k-vector spaces, and given any two objects (k-vector spaces) V and W, we take $\mathrm{Mor}_{(\mathcal{V}ect_k)}(V, W) = \mathrm{Hom}_k(V, W)$, the set of all k-linear maps from V into W.

(iv) In the category of topological spaces $\mathcal{T}op$, we take $\mathrm{ob}(\mathcal{T}op)$ to be the collection of all topological spaces, and given any two objects (topological spaces) X and Y, we take $\mathrm{Mor}_{\mathcal{T}op}(X, Y)$ to be the set of all continuous maps from X into Y.

One can easily verify that all the axioms of the above Definition 2.4.3 are satisfied for each of the above mentioned examples.

Let $\mathscr{C}$ be a category (think of any one from the above examples).

Definition 2.4.3. The *product* of an indexed family of objects $\{X_\alpha : \alpha \in \Lambda\}$ in $\mathscr{C}$ is a pair $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha\in\Lambda})$, consisting of an object P in $\mathscr{C}$ and a family of morphisms $\{\pi_\alpha : P \to X_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}$, satisfying the following universal property: given any object T of $\mathscr{C}$ and a family of morphisms $\{f_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}$, there exists a unique morphism $f : T \to P$ in $\mathscr{C}$ such that $\pi_\alpha \circ f = f_\alpha$, for all $\alpha \in \Lambda$.

$$\begin{tikzcd} T \arrow[r, dashed, "f"] \arrow[dr, "f_\alpha"'] & P \arrow[d, "\pi_\alpha"] \\ & X_\alpha \end{tikzcd}$$

It follows from the universal property of product that, if it exists, then it is unique upto a unique isomorphism making the above diagram commutative. Indeed, if $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha\in\Lambda})$ and $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha\in\Lambda})$ are two products of an indexed family of objects $\{X_\alpha : \alpha \in \Lambda\}$ in $\mathscr{C}$, then applying universal property of $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha\in\Lambda})$ for the test object $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha\in\Lambda})$, we get a unique morphism $f : P' \to P$ such that $\pi_\alpha \circ f = \pi'_\alpha$, $\forall\, \alpha \in \Lambda$. Similarly, applying universal property of $(P', (\pi'_\alpha : P' \to X_\alpha)_{\alpha\in\Lambda})$ for the test object $(P, (\pi_\alpha : P \to X_\alpha)_{\alpha\in\Lambda})$ we have a unique morphism $g : P \to P'$ such that $\pi'_\alpha \circ g = \pi_\alpha$, $\forall\, \alpha \in \Lambda$.

$$\begin{tikzcd} P' \arrow[r, dashed, "f"] \arrow[rr, bend left, "\mathrm{Id}_{P'}"] \arrow[dr, "\pi'_\alpha"'] & P \arrow[r, dashed, "g"] \arrow[d, "\pi_\alpha"] & P' \arrow[dl, "\pi'_\alpha"] \\ & X_\alpha & \end{tikzcd}$$

Since both the morphisms $g \circ f, \mathrm{Id}_{P'} \in \mathrm{Mor}_{\mathscr{C}}(P', P')$ make the following diagram commutative,

$$\begin{tikzcd} P' \arrow[rr, dashed, "g\circ f", "\mathrm{Id}_{P'}"'] \arrow[dr, "\pi'_\alpha"'] & & P' \arrow[dl, "\pi'_\alpha"] \\ & X_\alpha & \end{tikzcd}$$

for all $\alpha \in \Lambda$, it follows from the uniqueness assertation in Definition 2.4.3 that $g \circ f = \mathrm{Id}_{P'}$. Similarly one can show that $f \circ g = \mathrm{Id}_P$. Therefore, the product of $\{X_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}$, if it exists, is unique upto a unique isomorphism making the diagram in Definition 2.4.3 commutative. We denote the product of $\{X_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}$ by $\prod_{\alpha\in\Lambda} X_\alpha$.

Lemma 2.4.4. *The product $\prod\limits_{\alpha\in\Lambda} X_\alpha$ of a family of sets $\{X_\alpha : \alpha \in \Lambda\}$ exists in the category of sets.*

Proof. Let

$$\prod_{\alpha\in\Lambda} X_\alpha := \left\{ f : \Lambda \to \bigsqcup_{\alpha\in\Lambda} X_\alpha \mid f(\alpha) \in X_\alpha,\ \forall\, \alpha \in \Lambda \right\}.$$

Define a map $\pi_\beta : \prod\limits_{\alpha\in\Lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f_\beta := f(\beta)$, $\forall\, \beta \in \Lambda$. Suppose that we are given a set T together with maps $f_\alpha : T \to X_\alpha$, for each $\alpha \in \Lambda$. Define a map $F : T \to \prod\limits_{\alpha\in\Lambda} X_\alpha$ which sends $t \in T$ to $F(t) \in \prod\limits_{\alpha\in\Lambda} X_\alpha$ defined by $F(t)(\alpha) = f_\alpha(t)$, $\forall\, \alpha \in \Lambda$. Then $(\pi_\alpha \circ F)(t)(\alpha) = \pi_\alpha(F(t)) = F(t)(\alpha) = f_\alpha(t)$, $\forall\, t \in T$. Therefore, $\pi_\alpha \circ F = f_\alpha$, $\forall\, \alpha \in \Lambda$. To show uniqueness of F, suppose that $G : T \to \prod\limits_{\alpha\in\Lambda} X_\alpha$ be any map such that $\pi_\alpha \circ G = f_\alpha$, $\forall\, \alpha \in \Lambda$. Then for any $t \in T$, we have

$$G(t)(\alpha) = \pi_\alpha(G(t)) = f_\alpha(t) = F(t)(\alpha),\ \ \forall\, \alpha \in \Lambda,$$

and hence $G(t) = F(t)$, $\forall\, t \in T$. This proves that $G = F$. $\square$

Theorem 2.4.5. *Let $\mathcal{T}op$ be the category of topological spaces. The categorical product of a family of topological spaces $\{X_\alpha : \alpha \in \Lambda\}$ exists in $\mathcal{T}op$, and is* ***unique upto a unique homeomorphism*** *in the sense that if $(P, \{\pi_\alpha : P \to X_\alpha\}_{\alpha\in\Lambda})$ and $(P', \{\pi'_\alpha : P' \to X_\alpha\}_{\alpha\in\Lambda})$ are products of the family of topological spaces $\{X_\alpha : \alpha \in \Lambda\}$, then there exists a unique homeomorphism $\Phi : P' \to P$ such that $\pi_\alpha \circ \Phi = \pi'_\alpha$, for all $\alpha \in \Lambda$.*

Proof. Uniqueness follows from the universal property of product in a category. We only prove existence. Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Let τ_α be the topology on X_α, for all $\alpha \in \Lambda$. If the product $\prod\limits_{\alpha\in\Lambda} X_\alpha$ exists in $\mathcal{T}op$, its underlying set of points may be described as in Lemma 2.4.4. We just need to give a suitable topology on the set $\prod\limits_{\alpha\in\Lambda} X_\alpha$ that makes it the product in $\mathcal{T}op$. First of all, we need the projection maps

$$\pi_\alpha : \prod_{\beta\in\Lambda} X_\beta \longrightarrow X_\alpha$$

to be continuous, for all $\alpha \in \Lambda$. Let

$$\mathcal{S} := \bigcup_{\alpha\in\Lambda} \{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \tau_\alpha\}.$$

Note that, $\pi_\alpha^{-1}(U_\alpha) = \prod\limits_{\beta\in\Lambda} U_\beta$, where $U_\beta = X_\beta$, for all $\beta \neq \alpha$ in Λ (here product is taken in the category of sets). Clearly the union of all elements of $\mathcal{S}$ is the set $X := \prod\limits_{\alpha\in\Lambda} X_\alpha$, and hence $\mathcal{S}$ is a subbasis for a topology on X. Let $\mathcal{B}$ be the set of all finite intersections of elements from $\mathcal{S}$. Then $\mathcal{B}$ is a basis for the topology on X generated by the subbasis $\mathcal{S}$. A typical element of $\mathcal{B}$ is of the form $\bigcap\limits_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j}) = \prod\limits_{\beta\in\Lambda} U_\beta$, where $U_\beta = X_\beta$, for all $\beta \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$, for some

$n \in \mathbb{N}$. Clearly the maps $\pi_\alpha : \prod\limits_{\beta\in\Lambda} X_\beta \to X_\alpha$ are continuous by construction of topology on $\prod\limits_{\beta\in\Lambda} X_\beta$.

Let T be a topological space, and consider a family $\mathcal{F} := \{f_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ of continuous maps from X into X_α, for all $\alpha \in \Lambda$. By universal property of product set $X := \prod\limits_{\beta\in\Lambda} X_\beta$, there exists a unique set map $f : T \to \prod\limits_{\beta\in\Lambda} X_\beta$ such that $\pi_\alpha \circ f = f_\alpha$, $\forall\, \alpha \in \Lambda$. To check continuity of f, it suffices to check that $f^{-1}(B)$ is open in T, for all $B \in \mathcal{B}$. Now a basic open subset $B \in \mathcal{B}$ is of the form $B = \bigcap\limits_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j})$, for some $n \in \mathbb{N}$. Then

$$f^{-1}(B) = \bigcap_{j=1}^{n} f^{-1}\left(\pi_{\alpha_j}^{-1}(U_{\alpha_j})\right) = \bigcap_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j})$$

is open in T. Therefore, f is continuous. This completes the proof. $\square$

Let τ_α be the topology on X_α, for all $\alpha \in \Lambda$. It is clear from the above construction of product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$ that the collection

$$\mathcal{B} := \left\{ \prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \in \tau_{X_\alpha} \text{ and } U_\alpha = X_\alpha, \text{ for all but finitely many } \alpha \in \Lambda \right\}$$

is a basis for the product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$. However, we can further cut down $\mathcal{B}$ to construct another basis for the product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$ by looking at basic open subsets of X_α's. Indeed, fixing a basis $\mathcal{B}_\alpha$ for each X_α, one can easily verify that the collection

$$\mathcal{B}' := \left\{ \prod_{\alpha\in\Lambda} V_\alpha : V_\alpha \in \mathcal{B}_\alpha \cup \{X_\alpha\} \text{ and } V_\alpha = X_\alpha, \text{ for all but finitely many } \alpha \in \Lambda \right\}$$

forms a basis for the product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$.

Proposition 2.4.6. *Let X and Y be two topological spaces. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for X and Y, respectively. Then the collection*

$$\mathcal{B} := \{U \times V : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$$

is a basis for the product topological space $X \times Y$.

Proof. We first show that $\mathcal{B}_X \times \mathcal{B}_Y$ forms a basis for some topology on $X \times Y$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, respectively, given any $(x, y) \in X \times Y$, there exist $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ such that $x \in U$ and $y \in V$, so that $(x, y) \in U \times V$. Let $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ be given. Since $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, there exist $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ such that $x \in U \subseteq U_1 \cap U_2$ and $y \in V \subseteq V_1 \cap V_2$. Then $(x, y) \in U \times V \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$. Therefore, $\mathcal{B}$ is a basis for a topology on $X \times Y$. Let $A \subseteq X \times Y$ be any non-empty open subset in the product topological space $X \times Y$. Let $(x, y) \in A$ be given. Then there exist $U \in \tau_X$ and $V \in \tau_Y$ such that $(x, y) \in U \times V \subseteq A$. Since

$\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for (X, τ_X) and (Y, τ_Y), respectively, there exist $U_1 \in \mathcal{B}_X$ and $V_1 \in \mathcal{B}_Y$ such that $x \in U_1 \subseteq U$ and $y \in V_1 \subseteq V$. Then $(x, y) \in U_1 \times V_1 \subseteq \mathcal{A}$. Therefore, $\mathcal{B}$ is a basis for the product topology on $X \times Y$. □

Exercise 2.4.7. Show that the product topology on $\mathbb{R}^n$ coincides with the Euclidean topology on $\mathbb{R}^n$.

Let X and Y be two sets. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the maps defined by

$$\pi_X(x, y) = x, \ \forall \ (x, y) \in X \times Y,$$
$$\text{and } \pi_Y(x, y) = y, \ \forall \ (x, y) \in X \times Y.$$

The maps π_X and π_Y are called the projection maps onto X and Y, respectively. Note that, $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, for any subsets $U \subseteq X$ and $V \subseteq Y$, respectively.

Theorem 2.4.8. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then the collection*

$$\mathcal{S} := \{\pi_X^{-1}(U) : U \in \tau_X\} \cup \{\pi_Y^{-1}(V) : V \in \tau_Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Since $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, for any subsets $U \subseteq X$ and $V \subseteq Y$, respectively, we see that $\mathcal{S} \subseteq \tau_{X \times Y}$, where $\tau_{X \times Y}$ is the product topology on $X \times Y$. Since the topology $\tau_{\mathcal{S}}$ generated by $\mathcal{S}$ consists of arbitrary unions of finite intersections of elements from $\mathcal{S}$, we have $\tau_{\mathcal{S}} \subseteq \tau_{X \times Y}$. On the other hand, every basic open subset $U \times V$ for the product topology $\tau_{X \times Y}$ can be written as finite intersection

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V).$$

Therefore, these two topologies coincides. □

Theorem 2.4.9. *Let A be a subspace of X and B be a subspace of Y. Then the product topology on $A \times B$ coincides with the subspace topology on $A \times B$ induced from $X \times Y$.*

Proof. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for the topologies on X and Y, respectively. Then $\mathcal{B}_A = \{U \cap A : U \in \mathcal{B}_X\}$ and $\mathcal{B}_B = \{U \cap A : U \in \mathcal{B}_Y\}$ are bases for the subspace topologies on A and B, respectively. Then $\mathcal{B} = \{(U \cap A) \times (V \cap B) : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the product topology on $A \times B$. Note that $\mathcal{B}_{X \times Y} = \{U \times V : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$. Then $\mathcal{B}' = \{(U \times V) \cap (A \times B) : U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$ is a basis for the subspace topology on $A \times B$ induced from $X \times Y$. Since $(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B)$, for all $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$, we have $\mathcal{B} = \mathcal{B}'$. Hence the result follows. □

Lemma 2.4.10. *Let X, Y and Z be topological spaces. Equip $Y \times Z$ with the product topology, and let $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ be the projection maps onto the first and the second factors, respectively. Then a map $f : X \to Y \times Z$ is continuous if and only if both $\pi_Y \circ f : X \to Y$ and $\pi_Z \circ f : X \to Z$ are continuous.*

Proof. If f is continuous, then $\pi_Y \circ f$ and $\pi_Z \circ f$ are continuous by Lemma 2.3.18. Conversely, assume that both $\pi_Y \circ f : X \to Y$ and $\pi_Z \circ f : X \to Z$ are continuous. Then for given any open subsets $U \subseteq Y$ and $V \subseteq Z$, the subsets $(\pi_Y \circ f)^{-1}(U)$ and $(\pi_Z \circ f)^{-1}(V)$ are open in X. Then the subset

$$\begin{aligned} f^{-1}(U \times V) &= \{x \in X : f(x) \in U \times V\} \\ &= \{x \in X : (\pi_Y \circ f)(x) \in U \text{ and } (\pi_Z \circ f)(x) \in V\} \\ &= (\pi_Y \circ f)^{-1}(U) \cap (\pi_Z \circ f)^{-1}(V) \end{aligned}$$

is open in X. Since a basic open subset of $Y \times Z$ is of the form $U \times V$, where U and V are open subsets of Y and Z, respectively, it follows from Corollary 2.3.3 that f is continuous. $\square$

Corollary 2.4.11. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be maps of topological spaces. Equip $X_1 \times X_2$ and $Y_1 \times Y_2$ with the product topologies. Then the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by*

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)), \ \forall\, (x_1, x_2) \in X_1 \times X_2,$$

is continuous if and only if both f_1 and f_2 are continuous.

Proof. Let $\pi_1 : Y_1 \times Y_2 \to Y_1$ and $\pi_2 : Y_1 \times Y_2 \to Y_2$ be the projection maps onto the first and the second factors, respectively. Then in view of Lemma 2.4.10, it suffices to show that

$$\begin{aligned} &\pi_1 \circ (f_1 \times f_2) : X_1 \times X_2 \to Y_1, \\ \text{and } \ &\pi_2 \circ (f_1 \times f_2) : X_1 \times X_2 \to Y_2 \end{aligned}$$

are continuous. Let V_1 and V_2 be open subsets of Y_1 and Y_2, respectively. Then

$$\begin{aligned} (\pi_1 \circ (f_1 \times f_2))^{-1}(V_1) &= \{(x_1, x_2) \in X_1 \times X_2 : \pi_1(f_1(x_1), f_2(x_2)) \in V_1\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) \in V_1\} \\ &= f_1^{-1}(V_1) \times X_2 \end{aligned}$$

is open in $X_1 \times X_2$. Similarly, $(\pi_2 \circ (f_1 \times f_2))^{-1}(V_2) = X_1 \times f_2^{-1}(V_2)$ is open in $X_1 \times X_2$. This completes the proof. $\square$

2.5 Hausdorff space

Definition 2.5.1. A topological space X is said to be *Hausdorff* or, *T2* or, *separated** if each pair of distinct points of X can be separated by a pair of disjoint open neighbourhoods of them. In other words, give $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open subsets V_1, V_2 of X with $x_1 \in V$, $x_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$.

Lemma 2.5.2. *A topological space is Hausdorff if and only if the image of the diagonal map*

$$\Delta_X : X \to X \times X, \ \ x \mapsto (x, x)$$

*Not to be confused with the notion of a *separable* topological space.

is closed in $X \times X$.

Proof. Suppose that X is Hausdorff. It is enough to show that $U := (X \times X) \setminus \Delta_X(X)$ is open in $X \times X$. Since any point of U is of the form $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$, there are open neighbourhoods $x_j \in V_j \subset X$, $j = 1, 2$, such that $V_1 \cap V_2 = \emptyset$. Then $(x_1, x_2) \in V_1 \times V_2 \subseteq U$, and hence U is open.

Conversely suppose that X is separated. If $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $(x_1, x_2) \in U := (X \times X) \setminus \Delta_X(X)$. Since U is open in $X \times X$, there exist open subsets $V_1, V_2 \subset X$ with $x_j \in V_j$, $j = 1, 2$, such that $(x_1, x_2) \in V_1 \times V_2 \subseteq U = (X \times X) \setminus \Delta_X(X)$. Then $(V_1 \times V_2) \cap \Delta_X(X) = \emptyset$, and hence $V_1 \cap V_2 = \emptyset$. □

Exercise 2.5.3. Let $f, g : X \to Y$ be continuous maps of topological spaces. If Y is Hausdorff, show that the subset $\{x \in X : f(x) = g(x)\}$ is closed in X. (*Hint:* Look at the inverse image of $\Delta_Y(Y) \subset Y \times Y$ under the map $(f, g) : X \to Y \times Y$ given by $x \mapsto (f(x), g(x))$.)

Exercise 2.5.4. Let X and Y be topological spaces with Y Hausdorff. Let $A \subset X$ be such that $\overline{A} = X$. If $f, g : X \to Y$ are continuous maps satisfying $f\big|_A = g\big|_A$, show that $f = g$.

Definition 2.5.5. Let X be a topological space. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to converge to a point $x \in X$ if for given any open neighbourhood U of x in X, there exists $n_U \in \mathbb{N}$ such that $x_n \in U$, $\forall\, n \geq n_U$.

Proposition 2.5.6. *Let X be a Hausdorff topological space. Then a sequence in X can converge to at most one point in X.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a Hausdorff space X. Suppose on the contrary that $(x_n)_{n \in \mathbb{N}}$ converge to two distinct points, say $x, y \in X$. Since X is Hausdorff, there exist open neighbourhoods U and V of x and y, respectively, in X such that $U \cap V = \emptyset$. Then there exist $n_U, n_V \in \mathbb{N}$ such that $x_n \in U$, $\forall\, n \geq n_U$ and $x_n \in V$, $\forall\, n \geq n_V$. Then for $n_0 := \max\{n_U, n_V\}$, we have $x_n \in U \cap V, \forall\, n \geq n_0$, which contradicts our assumption that $U \cap V = \emptyset$. Therefore, $(x_n)_{n \in \mathbb{N}}$ can converge to at most one point in X. □

Definition 2.5.7. Fix a topological space Z. A *Z-topological space* (or, a *topological space over Z*) is a pair (X, f), where X is a topological space and $f : X \to Z$ is a continuous maps. Given two Z-topological spaces (X, f) and (Y, g), a *morphism from (X, f) to (Y, g)* is given by a continuous map $\varphi : X \to Y$ such that $g \circ \varphi = f$. In other words, the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Definition 2.5.8. The *fiber product* of a family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha \in \Lambda}$ is a pair $(F, \{\pi_\alpha : F \to X_\alpha\}_{\alpha \in \Lambda})$, where F is a topological space and $\{\pi_\alpha : F \to X_\alpha\}_{\alpha \in \Lambda}$ is a family of continuous maps indexed by Λ such that

(FP1) $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta, \ \ \forall\, \alpha, \beta \in \Lambda$, and

(FP2) *Universal property:* given any topological space T and a family of continuous maps $\{g_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ satisfying $f_\alpha \circ g_\alpha = f_\beta \circ g$, there exists a unique continuous map $g : T \to F$ such that $\pi_\alpha \circ g = g_\alpha$, for all $\alpha \in \Lambda$.

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "g_\alpha"] \arrow[ddr, bend right, "g_\beta"'] & & \\
& F \arrow[r, "\pi_\alpha"] \arrow[d, "\pi_\beta"'] & X_\alpha \arrow[d, "f_\alpha"] \\
& X_\beta \arrow[r, "f_\beta"] & Z
\end{tikzcd}$$

Proposition 2.5.9. *Fiber product of a family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha\in\Lambda}$ exists, and is unique up to a unique homeomorphism in the sense that if $(F, \{\pi_\alpha : F \to X_\alpha\}_{\alpha\in\Lambda})$ and $(F', \{\pi'_\alpha : F' \to X_\alpha\}_{\alpha\in\Lambda})$ are fiber products of the family of Z-topological spaces $\{f_\alpha : X_\alpha \to Z\}_{\alpha\in\Lambda}$, then there exists a unique homeomorphism $\Phi : F' \to F$ such that the following diagram commutes, for all $\alpha, \beta \in \Lambda$.*

$$\begin{tikzcd}
F' \arrow[dr, dashed, "\exists ! \Phi"] \arrow[drr, bend left, "\pi'_\alpha"] \arrow[ddr, bend right, "\pi'_\beta"'] & & \\
& F \arrow[r, "\pi_\alpha"] \arrow[d, "\pi_\beta"'] & X_\alpha \arrow[d, "f_\alpha"] \\
& X_\beta \arrow[r, "f_\beta"] & Z
\end{tikzcd}$$

Proof. Uniqueness follows from the universal property of fiber product. We only show its existence. Consider the subset

$$F = \left\{ (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} : f_\alpha(x_\alpha) = f_\beta(x_\beta),\ \forall\, \alpha, \beta \in \Lambda \right\}.$$

of the product $\prod\limits_{\alpha\in\Lambda} X_\alpha$, and equip it with the subspace topology induced from the product topology on $\prod\limits_{\alpha\in\Lambda} X_\alpha$. For each $\alpha \in \Lambda$, let $\pi_\alpha : F \to X_\alpha$ be the restriction of the projection map onto the α'th factor; clearly this is continuous. By construction of F, we have $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta$, for all $\alpha, \beta \in \Lambda$. Let T be a topological space and $\{g_\alpha : T \to X_\alpha\}_{\alpha\in\Lambda}$ be a family of continuous maps such that $f_\alpha \circ g_\alpha = f_\beta \circ g_\beta$, for all $\alpha, \beta \in \Lambda$. Define a map $g : T \to F$ by

$$g(t) = (g_\alpha(t))_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} X_\alpha.$$

Since $f_\alpha \circ g_\alpha = f_\beta \circ g_\beta$, for all $\alpha, \beta \in \Lambda$, it follows that $g(t) \in F$, $\forall\, t \in T$, and that $\pi_\alpha \circ g = g_\alpha$, $\forall\, \alpha \in \Lambda$. Uniqueness of g is clear due to the condition that $\pi_\alpha \circ g = g_\alpha$, $\forall\, \alpha \in \Lambda$. It remains to show that g is continuous. For that, we take a basic open subset of F of the form

$$V := F \cap \prod_{\alpha\in\Lambda} U_\alpha,$$

where U_α is an open subset of X_α, and that $U_\alpha = X_\alpha$, for all but finitely many $\alpha \in \Lambda$, say

$U_{\alpha_1}, \ldots, U_{\alpha_n}$ are only proper open subsets. Then $g^{-1}(V) = \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$, and hence is open in T. This completes the proof. $\square$

Exercise 2.5.10. Given a topological space Z and non-empty subsets X and Y of Z, show that the fiber product of the inclusion maps $\iota_X : X \hookrightarrow Z$ and $\iota_Y : Y \hookrightarrow Z$ is $X \cap Y$ in Z.

Exercise 2.5.11. Let $f : X \to Z$ and $g : Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, show that the subset

$$FP(f,g) := \{(x,y) \in X \times Y : f(x) = g(y)\}$$

is closed in the product topological space $X \times Y$. (*Hint:* Note that $FP(f,g) = (f \times g)^{-1}(\Delta_Z(Z))$, where $f \times g : X \times Y \to Z \times Z$ is the continuous map defined by $(f \times g)(x,y) = (f(x), g(y))$, for all $(x,y) \in X \times Y$; see Corollary 2.4.11).

Proposition 2.5.12. *A topological space X is Hausdorff if and only if given any two X-topological spaces $f : U \to X$ and $g : V \to X$, the subset*

$$FP(f,g) := \{(u,v) \in U \times V : f(u) = g(v)\}$$

is closed in $U \times V$.

Proof. If X is Hausdorff, the assertion follows by Exercise 2.5.11. For the converse part, take $U = V = X$ and $f = g = \mathrm{Id}_X$ so that $FP(f,g) = \Delta_X(X)$, where $\Delta_X : X \to X \times X$ is the diagonal map. $\square$

Exercise 2.5.13. Let Z be a Hausdorff topological space. Given a finite family of Z-topological spaces $\{f_k : X_k \to Z \mid k = 1, \ldots, n\}$, show that the subset

$$FP(f_1, \ldots, f_n) := \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n : f_1(x_1) = \cdots = f_n(x_n)\}$$

is closed in the product topological space $X_1 \times \cdots \times X_n$. (*Hint:* We only prove for the case $n = 3$; general case is similar. For $n = 3$, we have $FP(f_1, f_2, f_3) = (FP(f_1, f_2) \times X_3) \cap (X_1 \times FP(f_2, f_3))$. Since both $FP(f_1, f_2)$ and $FP(f_2, f_3)$ are closed in $X_1 \times X_2$ and $X_2 \times X_3$, respectively, by previous Exercise 2.5.11, the result follows.)

Exercise 2.5.14. Given a Hausdorff space Z, can you generalize Exercise 2.5.13 to arbitrary family of Z-topological spaces?

2.5.1 Exercises

Ex.1 Let X be a non-empty set. Let $\{\tau_\alpha : \alpha \in \Lambda\}$ be a family of topologies on X.

(i) Show that $\bigcap_{\alpha \in \Lambda} \tau_\alpha$ is a topology on X.

(ii) Is $\bigcup_{\alpha \in \Lambda} \tau_\alpha$ a topology on X? Justify your answer.

(iii) Show that there is a unique largest topology on X contained in all τ_α, $\forall\, \alpha \in \Lambda$.

(iv) Show that there is a unique smallest topology on X containing all τ_α, $\forall\, \alpha \in \Lambda$.

Ex.2 Let $\mathscr{B}$ be a basis for a topology on X. Show the topology $\tau_{\mathscr{B}}$ on X generated by $\mathscr{B}$ is the intersection of all topologies on X that contain $\mathscr{B}$. Prove the same statement if $\mathscr{B}$ is a subbasis for some topology on X.

Ex.3 Given $a, b \in \mathbb{Q}$, consider the subsets

$$\begin{aligned} (a,b) &= \{t \in \mathbb{R} : a < t < b\} \subset \mathbb{R}, \\ \text{and } \ [a,b) &= \{t \in \mathbb{R} : a \leq t < b\} \subset \mathbb{R}. \end{aligned}$$

(i) Show that the collection $\mathcal{B} = \{(a,b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for the standard (Euclidean) topology on $\mathbb{R}$.

(ii) Show that the collection $\mathcal{B}'_\ell = \{[a,b) : a, b \in \mathbb{Q}, a < b\}$ is a basis for some topology τ'_ℓ on $\mathbb{R}$ that is different from the lower limit topology on $\mathbb{R}$. Compare τ'_ℓ with the lower limit topology on $\mathbb{R}$.

Ex.4 Show that the countable collection

$$\{(a,b) \times (c,d) : a, b, c, d \in \mathbb{Q} \ \text{ with } a < b \text{ and } c < d\}$$

is a basis for the standard Euclidean topology on $\mathbb{R}^2$.

Ex.5 Let $(X, \leq)$ be a totally ordered set. Equip X with the order topology induced by $\leq$. Let Y be a non-empty subset of X. Then the partial order relation $\leq$ on X induces a partial order relation $\leq_Y$ on Y defined by

$$y_1 \leq_Y y_2 \ \text{ in } \ (Y, \leq_Y), \ \text{ if } \ y_1 \leq y_2 \ \text{ in } \ (X, \leq).$$

(i) Show that $(Y, \leq_Y)$ is a totally ordered set.

(ii) Show that the order topology on $\mathbb{R}$ induced by the standard partial order relation on it is the standard Euclidean topology on $\mathbb{R}$.

(iii) Show by an example that the subspace topology on Y induced from the order topology on X could be different from the order topology on Y induced by $\leq_Y$.

(*Hint:* Consider the subset $Y = [0,1) \cup \{2\}$ of $X := \mathbb{R}$. Then Y admits two topologies, namely the subspace topology τ_Y induced from the order topology on $X = \mathbb{R}$, and the order topology $\tau_{(Y,\leq)}$ on Y induced by the partial order relation $\leq$ on Y induced from that of X. Note that, $\{2\} = (3/2, 5/2) \cap Y$ is open in the subspace topology on Y induced from $X = \mathbb{R}$, while any basic open subset of $(Y, \tau_{(Y,\leq)})$ containing 2 is of the form

$$V_a := \{t \in Y : a < t \leq 2\},$$

for some $a \in Y$; such a subset V_a necessarily contains a point of Y less than 2.)

Ex.6 Let X be a totally ordered set. A subset $Y \subseteq X$ is said to be *convex* if for given any $a, b \in Y$ with $a < b$, the interval

$$(a,b) := \{x \in X : a < x < b\}$$

is contained in Y. Let Y be a convex proper subset of X. Does it follow that Y is an interval in X?

Ex.7 A map $f : X \to Y$ of topological spaces is said to be an *open map* if for given any open subset U of X, its image $f(U)$ is open in Y. Show that the projection maps

$$\pi_X : X \times Y \to X,\ (x, y) \mapsto x,$$
$$\text{and}\ \ \pi_Y : X \times Y \to Y,\ (x, y) \mapsto y$$

are open maps.

Ex.8 Let X be a topological space. Let $A \subseteq Y \subseteq X$. Equip Y with the subspace topology induced from X. If A is closed in Y and Y is closed in X, show that A is closed in X.

Ex.9 Let X and Y be topological spaces. Let A and B be closed subsets of X and Y, respectively. Show that $A \times B$ is closed in the product topological space $X \times Y$.

Ex.10 Let X be a topological space. If $U \subseteq X$ is open and $A \subseteq X$ is closed, show that $U \setminus A$ is open in X, and $A \setminus U$ is closed in X.

Ex.11 Let $(X, \leq)$ be a totally ordered set. Equip X with the order topology. Given any $a, b \in X$, show that $\overline{(a, b)} \subseteq [a, b]$. Under what conditions does equality hold?

Ex.12 Let A, B, and A_α be subsets of a topological space X. Show that

(i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iii) $\overline{\bigcup_{\alpha \in \Lambda} A_\alpha} \supseteq \bigcup_{\alpha \in \Lambda} \overline{A_\alpha}$. Give an example where equality fails to hold.

Ex.13 Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Show that

$$\overline{A \times B} = \overline{A} \times \overline{B}$$

holds in the product space $X \times Y$.

Ex.14 Show that every order topology is Hausdorff.

Ex.15 Let X and Y be topological spaces. Show that the product topological space $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Ex.16 Show that a subspace of a Hausdorff topological space is Hausdorff.

Ex.17 In cofinite topology on $\mathbb{R}$, to what point or points does the sequence $(\frac{1}{n})_{n \in \mathbb{N}}$ converge?

Ex.18 Determine the closure of (a, b) in $\mathbb{R}_\ell$ and $\mathbb{R}_K$, where $K = \{1/n : n \in \mathbb{N}\}$.

Ex.19 Let X be a topological space. Given a subset $A \subseteq X$, we define the ***interior of** A* to be the subset $Int(A)$ consisting of all interior points of A in X, and define the ***boundary of** A* to be the subset

$$Bd(A) := \overline{A} \cap \overline{(X \setminus A)}.$$

(i) Show that $Int(A) \cap Bd(A) = \emptyset$ and $\overline{A} = Int(A) \cup Bd(A)$.

(ii) Show that $Bd(A) = \emptyset$ if and only if A is both open and closed in X.

(iii) Show that A is open in X if and only if $Bd(A) = \overline{A} \setminus A$.

(iv) If A is open, is it true that $A = Int(\overline{A})$? Justify your answer.

Ex.20 Find the boundary and the interior of the following subsets of $\mathbb{R}^2$.

(i) $A = \{(x, 0) : x \in \mathbb{R}\}$.

(ii) $B = \{(x, y) : x > 0, y \neq 0\}$.

(iii) $C = A \cup B$.

(iv) $D = \{(x, y) : x \in \mathbb{Q}\}$.

(v) $E = \{(x, y) : 0 < x^2 - y^2 \leq 1\}$.

(vi) $F = \{(x, y) : x \neq 0, y \leq 1/x\}$.

Exercise 2.5.15. Fix a pint $p \in \mathbb{R}$. Show that the map $f_p : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, p),$$

is an embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ (c.f. Definition 2.3.23).

Exercise 2.5.16. Embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ via the map

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0).$$

Let $N = (0, \ldots, 0, 1)$ be the north pole of the unit n-sphere $S^n \subset \mathbb{R}^{n+1}$. Equip $S^n \setminus \{N\}$ with the subspace topology induced from S^n.

(i) For each $v \in S^n$, show that the straight-line joining N to v in $\mathbb{R}^{n+1}$ intersects $\mathbb{R}^n$ at a unique point $x_v \in \mathbb{R}^n$.

(ii) Show that the map $f : S^n \setminus \{N\} \to \mathbb{R}^n$ given by

$$f(v) = x_v, \ \forall\, v \in S^n,$$

is a homeomorphism.

2.6 Quotient space

Let's recall the notion of quotients from algebra course. Let G be a group and H a normal subgroup of G. Then we have a relation $\sim$ on G defined by

$$g_1 \sim g_2 \quad \text{if} \quad g_1^{-1} g_2 \in H.$$

Clearly this is an equivalence relation on G, and we have a partition of G into a disjoint union of its subsets (equivalence classes)

$$G = \bigcup_{g \in G} gH,$$

where $gH = \{g' \in G : g \sim g'\}$ is the equivalence class of g in G, for all $g \in G$.

Now question is does there exists a pair (Q, q) consisting of a group Q and a map $q : G \to Q$ such that

(QG1) $q : G \to Q$ is a surjective group homomorphism satisfying $q(g) = q(g')$ whenever $g \sim g'$, and

(QG2) given any group G' and a group homomorphism $f : G \to G'$ with $H \subseteq \mathrm{Ker}(f)$, there should exists a unique group homomorphism $\widetilde{f} : Q \to G'$ such that $\widetilde{f} \circ q = f$?

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "q"'] & G' \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd} \tag{2.6.1}$$

Interesting point is that, without knowing existence of such a pair (Q, q), it follows immediately from the properties (QG1) and (QG2) that such a pair (Q, q), if it exists, must be unique up to a unique isomorphism of groups in the sense that, given another such pair (Q', q') satisfying the above two conditions, there is a unique group isomorphism $\phi : Q \to Q'$ such that $\phi \circ q = q'$).

Exercise 2.6.2. Prove the above mentioned uniqueness statement.

Now question is about its existence. The condition (QG1) suggests that the elements of Q should be the fibers of the map q, which are nothing but the $\sim$-equivalence classes

$$[g]_\sim = \{g' \in G : g' \sim g\} = gH, \ \forall\, g \in G.$$

This suggests us to consider $\{gH : g \in G\}$ as a possible candidate for the set Q. Now question is what should be the appropriate group structure on it? Take any group homomorphism $f : G \to G'$ such that $H \subseteq \mathrm{Ker}(f)$. This says that $f(g_1) = f(g_2)$ if $g_1 \sim g_2$ (equivalently, $g_1^{-1}g_2 \in H$). The commutativity of the diagram (2.6.1) tells us to send $gH \in Q$ to $f(g) \in G'$ to define the map $\widetilde{f} : Q \to G'$ (note that this is well-defined!), and since we want $\widetilde{f} : Q \to G'$ to be a group homomorphism, we should define a binary operation on $Q = \{gH : g \in G\}$ in such a way that $(g_1H) * (g_2H) \xmapsto{\widetilde{f}} f(g_1)f(g_2) = f(g_1g_2)$, for all $g_1, g_2 \in G$. So the obvious choice is to define

$$(g_1H) * (g_2H) := (g_1g_2)H, \ \forall\, g_1, g_2 \in G. \tag{2.6.3}$$

Clearly this is a well-defined binary operation on $Q = \{gH : g \in G\}$, since H is normal.

Exercise 2.6.4. Verify that (2.6.3) makes Q a group such that the pair (Q, q) satisfies the condition (QG1) and (QG2).

Exercise 2.6.5. Verify analogous stories for the cases rings and vector spaces.

We are going to witness the same phenomenon in topology! Let X be a topological space.

Definition 2.6.6. Given an equivalence relation $\sim$ on X, the associated *quotient topological space* (or, *identification space*) $X/\sim$ is a pair (Q, q) consisting of a topological space Q and a continuous map $q : X \to Q$ such that

(QT1) q is surjective and satisfies $q(x) = q(x')$ whenever $x \sim x'$ in $(X, \sim)$; and

(QT2) given any topological space Y and a continuous map $f : X \to Y$ satisfying $f(x) = f(x')$ whenever $x \sim x'$ in $(X, \sim)$, there is a unique continuous map $\widetilde{f} : Q \to Y$ such that $\widetilde{f} \circ q = f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

The map q is called the *quotient map* (or, *identification map*) for $(X, \sim)$.

As an immediate corollary to the Definition 2.6.6, we have the following.

Corollary 2.6.7. *If (Q, q) is a quotient space for $(X, \sim)$, then the topology on Q is the largest topology on the set Q such that the map $q : X \to Q$ is continuous.*

Proof. By a topology on a set S we mean a collection τ of subsets of S that satisfies axioms for open subsets in S. Suppose on the contrary that the statement in Corollary 2.6.7 is false. Then there is a topology τ' on the set Q finer than the quotient topology on Q such that the set map $q : X \to Q' := (Q, \tau')$ is continuous. Then by property (QT2) of (Q, q) in Definition 2.6.6, there is a unique (continuous) map $f : Q \to Q'$ such that $f \circ q = q$. Since $q : X \to Q$ is surjective, it admits a right inverse (set theoretically). This forces $f : Q \to Q'$ to be the identity map. This is not possible because f is continuous and the topology on Q' is finer than that of Q by our assumption (see Corollary 2.3.4). Hence the result follows. □

Remark 2.6.8. In Definition 2.6.6, the first condition suggests what should be the underlying set of points of Q and the map $q : X \to Q$, and the second condition suggests what should be the topology on the set Q.

Theorem 2.6.9. *Given a topological space X and an equivalence relation $\sim$ on X, the associated quotient space (Q, q) for $(X, \sim)$ exists, and is unique up to a unique homeomorphism (i.e., if (Q, q) and (Q', q') are two quotient spaces for $(X, \sim)$, then there is a unique homeomorphism $\varphi : Q \to Q'$ such that $\varphi \circ q = q'$).*

Proof. We first prove uniqueness of the pair (Q, q), up to a unique homeomorphism. Let (Q', q') be another quotient space for the pair $(X, \sim)$. Since q' is continuous and $q'(x) = q'(y)$ whenever $x \sim y$ in $(X, \sim)$, we have a unique continuous map $\widetilde{q} : Q' \to Q$ such that $\widetilde{q} \circ q' = q$.

$$\begin{tikzcd} & X \arrow[dl, "q'"'] \arrow[d, "q"] \arrow[dr, "q'"] & \\ Q' \arrow[r, dashed, "\widetilde{q}"'] & Q \arrow[r, dashed, "\widetilde{q'}"'] & Q' \end{tikzcd} \tag{2.6.10}$$

Similarly, interchanging the role of (Q, q) and (Q', q') we get a unique continuous map $\widetilde{q'} : Q \to Q'$ such that $\widetilde{q'} \circ q = q'$. Then we have $(\widetilde{q'} \circ \widetilde{q}) \circ q' = q'$. Since the identity map $\mathrm{Id}_{Q'} : Q' \to Q'$ is continuous and satisfies $\mathrm{Id}_{Q'} \circ q' = q'$, we must have $\widetilde{q'} \circ \widetilde{q} = \mathrm{Id}_{Q'}$. Similarly, we have

$\widetilde{q} \circ \widetilde{q}' = \mathrm{Id}_Q$. Therefore, both $\widetilde{q}$ and $\widetilde{q}'$ are homeomorphisms. Thus the pair (Q, q) is unique, up to a unique homeomorphism.

Now (following Remark 2.6.8) we give an explicit construction of $(Q, \sim)$. For each $x \in X$, the *equivalence class of* x in $(X, \sim)$ is the subset

$$[x] := \{x' \in X : x \sim x'\} \subseteq X.$$

Let Q be the set of all distinct equivalence classes of elements of X. Consider the map $q : X \to Q$ defined by sending each point $x \in X$ to its equivalence class $[x] \in Q$. Note that, the map q is surjective. As suggested in Corollary 2.6.7, we define a topology on Q by declaring a subset $U \subseteq Q$ to be *open* if its inverse image $q^{-1}(U) \subseteq X$ is open in X. Clearly this makes $q : X \to Q$ continuous. It remains to check property (QT2) as in Definition 2.6.6. Let Y be any topological space and $f : X \to Y$ any continuous map satisfying $f(x) = f(x')$ for $x \sim x'$ in $(X, \sim)$. Define a map $\widetilde{f} : Q \to Y$ by $\widetilde{f}([x]) = f(x)$, for all $[x] \in Q$. Clearly $\widetilde{f}$ is well-defined, and by its construction it satisfies

$$\widetilde{f} \circ q = f. \tag{2.6.11}$$

Since f is continuous, for any open subset $V \subseteq Y$, the subset

$$q^{-1}(\widetilde{f}^{-1}(V)) = (\widetilde{f} \circ q)^{-1}(V) = f^{-1}(V)$$

is open in X, and hence $\widetilde{f}^{-1}(V)$ is open in Q by definition of the topology on Q. Therefore, $\widetilde{f}$ is continuous. If $g : Q \to Y$ is any continuous map satisfying $g \circ q = f$, then $g([x]) = (g \circ q)(x) = f(x)$, for all $[x] \in Q$, and hence $g = \widetilde{f}$. □

Theorem 2.6.12. *Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective continuous map. Then the following are equivalent.*

(i) The pair (Y, p) is a quotient space for some equivalence relation $\sim$ on X.

(ii) A subset $U \subseteq Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

(iii) A subset $Z \subseteq Y$ is closed in Y if and only if $p^{-1}(Z)$ is closed in X.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): Follow from Corollary 2.6.7.

We show (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) together. Consider the relation $\sim$ on X defined by

$$x_1 \sim x_2 \text{ in } X, \text{ if } p(x_1) = p(x_2).$$

It is easy to see that $\sim$ is an equivalence relation on X. Clearly $p(x) = p(x')$ whenever $x \sim x'$ in X. Let T be a topological space and let $f : X \to T$ be a continuous map such that $f(x) = f(x')$ if $x \sim x'$ in X. Since $p : X \to Y$ is surjective, for each $y \in Y$ we can choose a point $x_y \in p^{-1}(y) \subseteq X$ by axiom of choice. Since $f(x) = f(x')$ for all $x, x' \in p^{-1}(y)$, we get a well-defined map $\widetilde{f} : Y \to T$ defined by

$$\widetilde{f}(y) = f(x_y), \ \forall\, y \in Y.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & T \\ Y \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

Clearly $\widetilde{f} \circ p = f$. Let $V \subseteq T$ be an open (resp., closed) subset of T. Since f is continuous, $p^{-1}(\widetilde{f}^{-1}(V)) = f^{-1}(V)$ is open (resp., closed) in X. Then it follows from the assumption (ii) (resp., (iii)) that $\widetilde{f}^{-1}(V)$ is open (resp., closed) in Y. Therefore, $\widetilde{f} : Y \to T$ is continuous. If $g : Y \to T$ is any continuous map such that $g \circ p = f$, then $g(p(x)) = f(x)$, $\forall\, x \in X$ gives $g(y) = \widetilde{f}(y)$, for all $y \in Y$. Therefore, $\widetilde{f}$ is the unique continuous map such that $\widetilde{f} \circ p = f$. Thus (Y, p) is the quotient space of X by the equivalence relation $\sim$ on X (see Definition 2.6.6). $\square$

Remark 2.6.13. It follows from construction of quotient space in Theorem 2.6.9, and the proof of Theorem 2.6.12 that if $f : X \to Y$ is a quotient map then the set of all fibers $\{f^{-1}(y) : y \in Y\}$ of f gives a partition of X, and hence defines an equivalence relation $\sim$ on X such that the associated quotient space $X/\sim$ is homeomorphic to (Y, f).

Exercise 2.6.14. If $f : X \to Y$ and $g : Y \to Z$ are quotient maps, show that $g \circ f : X \to Z$ is a quotient map.

Exercise 2.6.15. Give an example of a quotient map $p : X \to Y$ and a topological space Z such that $p \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is not a quotient map.

Definition 2.6.16. A map $f : X \to Y$ is said to be *open* (resp., *closed*) if $f(U)$ is open (resp., closed) in Y for any open (resp., closed) subset U of X.

Exercise 2.6.17. Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two open maps of topological spaces. Show that the product map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)(x_1, x_2) = (f(x_1), f_2(x_2)),\ \forall\, (x_1, x_2) \in X_1 \times X_2,$$

is an open map.

Corollary 2.6.18. *A surjective continuous open (or, closed) map is a quotient map.*

Proof. Let $f : X \to Y$ be a surjective map. Then for any $V \subseteq Y$ we have $f(f^{-1}(V)) = V$. Suppose that f is also continuous and open (resp., closed). Then for any $V \subseteq Y$ with $f^{-1}(V)$ open (resp., closed) in X, $V = f(f^{-1}(V))$ is open (resp., closed) in Y. Hence the result follows from Theorem 2.6.12. $\square$

Remark 2.6.19. Corollary 2.6.18 fails without continuity assumption on f. For example, take a set X with at least two elements. Let τ_0 and τ_1 be the trivial topology and the discrete topology on X, respectively. Then the identity map $\mathrm{Id}_X : (X, \tau_0) \to (X, \tau_1)$ is a surjective open map, which is not continuous let alone be a quotient map.

Exercise 2.6.20. Let $q : X \to Q$ be a quotient map, and let $Z \subseteq Q$. Show by an example that the restriction map $q\big|_{q^{-1}(Z)} : q^{-1}(Z) \to Z$ need not be a quotient map, in general. If Z is open or q is an open map, show that the above restriction map become a quotient map.

Corollary 2.6.21. *Let $f : X \to Y$ be a continuous surjective map. If X is compact (see Definition 2.11.1) and Y is Hausdorff, then f is a quotient map.*

Proof. Let Z be a closed subset of X. Since X is compact, Z is compact by Proposition 2.11.8. Since f is continuous, $f(Z)$ is a compact subset of Y by Proposition 2.11.11. Since Y is Hausdorff, $f(Z)$ is closed in Y by Corollary 2.11.10. Hence the result follows from Corollary 2.6.18. □

Proposition 2.6.22. *Let $\sim$ be an equivalence relation on a topological space X, and let (Q, q) be the associated quotient space. Given a topological space Y, a map $\phi : Q \to Y$ is continuous if and only if the composite map $\phi \circ q : X \to Y$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\phi\circ q"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof. Since the quotient map q is continuous, the composite map $\phi \circ q$ is continuous whenever ϕ is continuous. Conversely, let $\phi \circ q$ be continuous. Since for any open subset $V \subseteq Y$, we have $q^{-1}(\phi^{-1}(V)) = (\phi \circ q)^{-1}(V)$ is open in X, by construction of topology of Q, the subset $\phi^{-1}(V)$ is open in Q. Thus ϕ is continuous. □

Example 2.6.23. (i) **Circle**: Let $I = [0, 1] \subset \mathbb{R}$ be the unit closed interval in $\mathbb{R}$. Define a map

$$f : [0, 1] \to S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

by

$$f(t) = (\cos 2\pi i t, \sin 2\pi i t), \;\; \forall\, t \in [0, 1].$$

Clearly f is a surjective continuous map. Since $[0, 1]$ is compact and S^1 is Hausdorff, it follows from Corollary 2.6.21 that $f : I \to S^1$ is a quotient map. Note that $f^{-1}(1, 0) = \{0, 1\}$ and $f^{-1}(x, y)$ is singleton for $(x, y) \in S^1 \setminus \{(1, 0)\}$. Therefore, S^1 is the quotient space of $[0, 1]$ for the equivalence relation on $[0, 1]$ which only identify the end points of $[0, 1]$ to a single point.

(ii) **Cylinder**: Let $I = [0,\, 1] \subset \mathbb{R}$. Consider the unit square

$$I \times I = \{(x,\, y) \in \mathbb{R}^2 \,:\, 0 \leq \, x,\, y \, \leq 1\}$$

in $\mathbb{R}^2$. Define an equivalence relation $\sim_1$ on $I \times I$ by setting

$$(x, y) \sim_1 (x', y'), \;\text{ if }\;\; x' = x + 1 = 1 \;\text{ and }\; y = y'.$$

This identifies points of two vertical sides of $I \times I$ (see Figure 2.1 below), and the associated quotient space $(I \times I)/{\sim_1}$ is homeomorphic to the cylinder

$$S^1 \times [0,\, 1] = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\; 0 \leq z \leq 1\}.$$

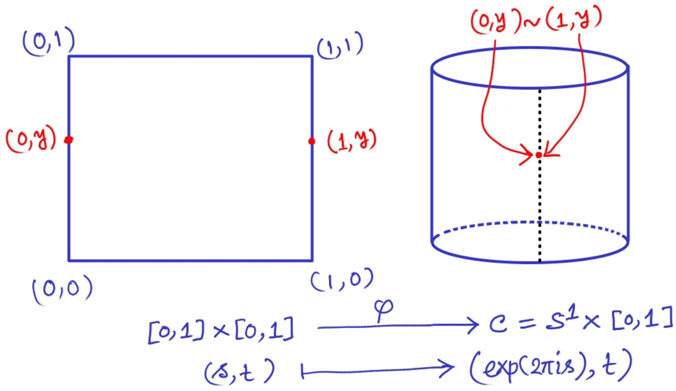


FIGURE 2.1

Indeed, we can define a (continuous) map

$$\phi : I \times I \to S^1 \times [0, 1]$$

by $\phi(s, t) = (\exp(2\pi i s), t)$, for all $(s, t) \in I \times I$. Then the set $\{\phi^{-1}(z, t) : (z, t) \in S^1 \times [0, 1]\}$ of all fibers of ϕ is precisely the partition of $I \times I$ given by the equivalence relation $\sim_1$ on $I \times I$. It follows from Corollary 2.6.21 that ϕ is a quotient map, and by Remark 2.6.13 the associated quotient space $(I \times I)/\sim$ is homeomorphic to $S^1 \times I$.

(iii) **Torus**: Consider an equivalence relation $\sim_2$ on the cylinder $S^1 \times I$ defined by

$$(z, t) \sim_2 (z', t') \text{ if } z = z', \text{ and } t' = t + 1 = 1.$$

This identifies each point of the bottom circle of $S^1 \times I$ with the corresponding point of the top circle on $S^1 \times I$ (see Figure 2.2 below).

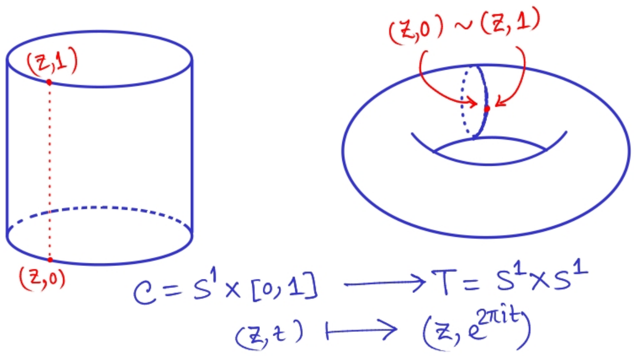


FIGURE 2.2

Then the associated quotient space $(S^1 \times I)/\sim_2$ is homeomorphic to the torus $T := S^1 \times S^1$ in $\mathbb{R}^4$ (see the remark below). Indeed, we can define a (continuous) map

$$\psi : S^1 \times I \to S^1 \times S^1$$

by $\psi(z,t) = (z, \exp(2\pi i t))$, for all $(z, t) \in S^1 \times I$. As before, it is easy to see that the set of all fibers of the map ψ is precisely the partition of $S^1 \times I$ defined by the equivalence relation $\sim_2$ on the cylinder $S^1 \times I$. As before, it follows from Corollary 2.6.21 that ψ is a quotient map, and by Remark 2.6.13 the associated quotient space $(S^1 \times I)/\sim$ is homeomorphic to $S^1 \times S^1$.

Remark 2.6.24. To see the quotient space $(S^1 \times I)/\sim_2$ is homeomorphic a torus inside $\mathbb{R}^3$, we need to use the parametric equation of a torus T in $\mathbb{R}^3$; this is given by the map

$$I \times I \to T \subset \mathbb{R}^3, \ \ (s,t) \mapsto (x(s,t), y(s,t), z(s,t)),$$

where

$$\begin{aligned} x(s,t) &= (d + r\cos 2\pi t)\cos 2\pi s, \\ y(s,t) &= (d + r\cos 2\pi t)\sin 2\pi s, \\ z(s,t) &= r \sin 2\pi t, \end{aligned}$$

where $d, r \in \mathbb{R}$ with $0 < r < d$ (here d is the radius of the circle passing through the center of the torus tube, and r is the radius of the circular section of the torus). Then we consider the map

$$\psi : S^1 \times I \to T$$

defined by

$$\psi(e^{2\pi i s}, t) = (x(s,t), y(s,t), z(s,t)), \ \forall\, s, t \in [0,1].$$

(iv) Define a relation $\rho \subset \mathbb{R} \times \mathbb{R}$ on $\mathbb{R}$ by $(x, y) \in \rho$ if $x - y \in \mathbb{Z}$. Note that this is an equivalence realtion on $\mathbb{R}$. Let $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Define a map $f : \mathbb{R} \to S^1$ by

$$f(t) = (\cos 2\pi i t, \sin 2\pi i t), \ \ \forall\, t \in \mathbb{R}.$$

Clearly f is a surjective continuous map. We show that f is an open map. For this, it suffices to show that image of an open interval $(a, b) \subset \mathbb{R}$ is open in S^1. Let $a, b \in \mathbb{R}$ with $a < b$. Since $f(t + 1) = f(t)$, $\forall\, t \in \mathbb{R}$, it follows that $f((a, b)) = S^1$ if $b - a \geq 1$. Suppose that $b - a < 1$. Let $p := (x, y) \in f((a, b))$ be arbitrary. Then $(x, y) = (\cos 2\pi i t, \sin 2\pi i t)$, for some $t \in (a, b)$. Taking $r = \min\{t - a, b - t\} > 0$, we see that $B(p, r) \cap S^1 \subseteq f((a, b))$ (verify). Thus, $f((a, b))$ is open in S^1, and hence f is an open map. Then by Corollary 2.6.18 f is a quotient map. Since the fibers of f can be identified with $\mathbb{Z}$ (verify!), we may denote the associated quotient space by $\mathbb{R}/\mathbb{Z}$. Then f induces a homeomorphism of $\mathbb{R}/\mathbb{Z}$ onto S^1.

(v) **Cone**: Let $I = [0, 1] \subseteq \mathbb{R}$. The *cone* of a topological space X is the quotient space $CX := (X \times I)/\sim$ of $X \times I$ for the equivalence relation $\sim$ on $X \times I$ defined by

$$(x, t) \sim (x', t'), \ \text{ if } \ t = t' = 1. \tag{2.6.25}$$

The associated set of all partitions of $X \times I$ is the set

$$\big\{X \times \{1\}, \{(x,t)\} : x \in X, 0 \leq t < 1\big\}.$$

Thus we identify all points of $X \times \{1\} \subseteq X \times I$ into a single point, called the *vertex* of the

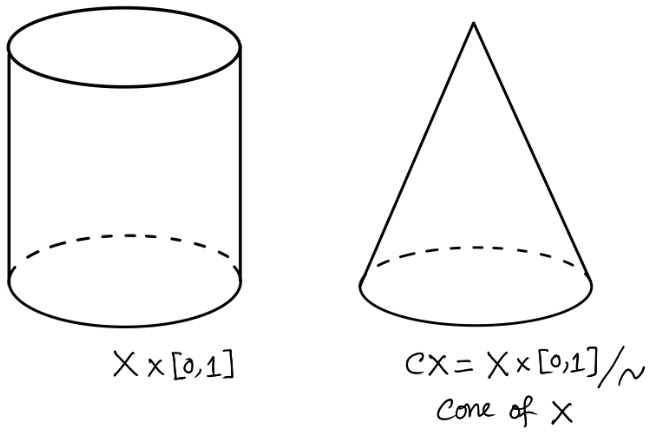


FIGURE 2.3

cone CX, and the remaining points of $X \times [0,1)$ remains as they are.

If X is a compact subset of an Euclidean space $\mathbb{R}^n$, then we can construct CX more geometrically as follow. Embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$, and fix a point $v \in \mathbb{R}^{n+1}$ which lies outside the image of this embedding; for example take $v = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Note that $\ell_{[v,x]} := \{tv + (1-t)x : 0 \leq t \leq 1\} \subseteq \mathbb{R}^{n+1}$ is the straight line segment in $\mathbb{R}^{n+1}$ joining v and $x \in X$. The subset

$$\bigcup_{x \in X} \ell_{[v,x]} \subseteq \mathbb{R}^{n+1}$$

with the subspace topology induced from $\mathbb{R}^{n+1}$ is called the *geometric cone* of X. We show that the geometric cone of X is homeomorphic to the cone of X, i.e.,

$$CX \cong \bigcup_{x \in X} \ell_{[v,x]}.$$

Define a map

$$f : X \times I \to \bigcup_{x \in X} \ell_{[v,x]}$$

by $f(x,t) = tv + (1-t)x$, for all $(x,t) \in X \times I$. Clearly f is a surjective continuous map, and $f(x,t) = f(x,t')$ if and only if either $x = x'$ and $t = t'$, or $t = t' = 1$. Since X is compact and its image is Hausdorff (being a subspace of $\mathbb{R}^{n+1}$), it follows from Corollary 2.6.21 that f is a quotient map. Since the fibers of the map f are precisely the equivalence classes for the equivalence relation on $X \times I$ defined in (2.6.25), it follows from Remark 2.6.13 that $CX = (X \times I)/\sim$ is homeomorphic to the geometric cone of X.

(vi) **The space** X/A: Let A be a subset of a topological space X. Define an equivalence relation $\sim$ on X by

$$x \sim x' \text{ if both } x \text{ and } x' \text{ are in } A.$$

We denote by X/A the associated quotient space $X/\sim$. Here we collapse the subspace A into a single point, and the remaining points of $X \setminus A$ remains as they were. For example, $CX = (X \times I)/(X \times \{1\})$.

(vii) **The space B^n/S^{n-1}**: Consider the closed unit ball

$$B^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1\}$$

in $\mathbb{R}^n$, and its boundary

$$\partial B^n = \{(x_1, \ldots, x_n) \in \mathbb{B}^n : \sum_{j=1}^{n} x_j^2 = 1\} = S^{n-1}.$$

Then the associated quotient space is denoted by B^n/S^{n-1} is homeomorphic to S^n. This is quite easy to visualize for $n = 1$ and 2. For $n = 1$, $B^1 = [-1, 1] \subseteq \mathbb{R}$, and $S^0 = \{-1, 1\}$ is its boundary. If we identify all points of $S^0 = \{-1, 1\}$ into a single point and keep all other points of B^1 as they were, we get a circle S^1 in $\mathbb{R}^2$; see Figure 2.4 below.

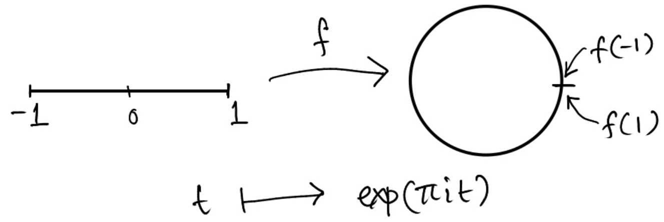


FIGURE 2.4

The case $n = 2$ is explained in the Figure 2.5 below.

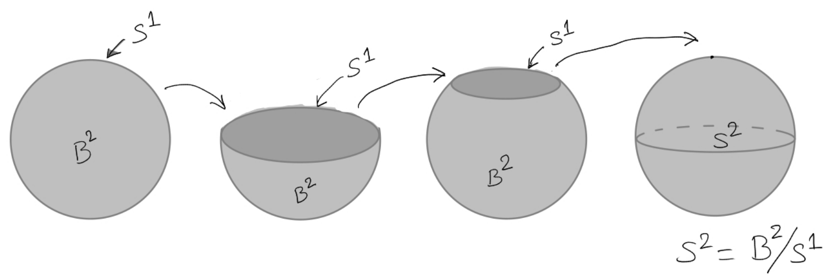


FIGURE 2.5

In general, it suffices to construct a surjective continuous map

$$f : B^n \to S^n$$

such that $f\big|_{B^n \setminus S^{n-1}}$ is injective and $f(S^{n-1})$ is a singleton subset of S^n. Then by Corollary 2.6.21, f become a quotient map producing a homeomorphism of B^n/S^{n-1} onto S^n. To construct such a map f, note that $\mathbb{R}^n$ is homeomorphic to $B^n \setminus S^{n-1}$ and $S^n \setminus \{p\}$, for any $p \in S^n$ (see Exercise 2.5.15 and Exercise 2.5.16). Fix two homeomorphisms $h_1 : B^n \setminus$

$S^{n-1} \to \mathbb{R}^n$ and $h_2 : \mathbb{R}^n \to S^n \setminus \{p\}$, and define

$$f(x) := \begin{cases} h_2(h_1(x)), & \text{if} \quad x \in B^n \setminus S^{n-1}, \\ p, & \text{if} \quad x \in S^{n-1}. \end{cases} \tag{2.6.26}$$

It is easy to check that f has desired properties (verify).

Exercise 2.6.27. Consider the Euclidean plane $X = \mathbb{R}^2$. Define a relation $\sim$ on $\mathbb{R}^2$ by

$$(x_1, y_1) \sim (x_2, y_2), \text{ if } (x_1 - x_2, y_1 - y_2) \in \mathbb{Z} \times \mathbb{Z}.$$

Example 2.6.28 (Attaching spaces along a map)**.** Let X and Y be two topological spaces. Suppose we wish to attach X by identifying points of a subspace $A \subseteq X$ with points of Y in a continuous way. This can be done by using a continuous map $f : A \to Y$. Indeed, we identify $x \in A$ with its image $f(x) \in Y$. This defines an equivalence relation on $X \sqcup Y$, and we denote the associated quotient space by $X \sqcup_f Y$, and call it the *space Y with X attached along A via f*. Let us discuss some examples.

(i) Let $I = [0, 1] \subset \mathbb{R}$. Given a space X, we call $X \times I$ the *cylinder over* X. Let $f : X \to Y$ be a continuous map of topological spaces. Then f induces a continuous map $\widetilde{f} : X \times \{0\} \to Y$ given by

$$\widetilde{f}(x, 0) = f(x), \ \forall \ (x, 0) \in X \times \{0\}.$$

If we attach Y with the *cylinder* $X \times I$ of X along its base $X \times \{0\} \subset X \times I$ via the map $\widetilde{f}$, by identifying $(x, 0) \sim f(x)$, then the associated quotient space $M_f = (X \times I) \sqcup_{\widetilde{f}} Y$ is called the *mapping cylinder of f* (see Figure 2.6).

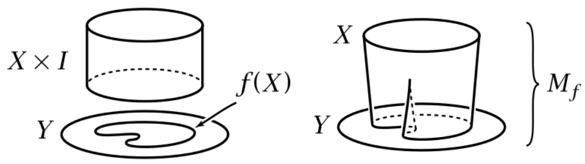


FIGURE 2.6: Mapping Cylinder

(ii) Let $CX = (X \times I)/(X \times \{1\})$ be the *cone over* X obtained by collapsing the subspace $X \times \{1\}$ of the cylinder $X \times I$ over X to a single point. Let $f : X \to Y$ be a continuous map. If we attach this cone CX with Y along its base $X \times \{0\} \subset CX$ by identifying $(x, 0) \in X \times \{0\}$ with $f(x) \in Y$, then the resulting quotient space $C_f = Y \sqcup_f CX$ is called the *mapping cone of f* (see Figure 2.7). Note that, the mapping cone C_f can also be obtained as a quotient space of the mapping cylinder M_f by collapsing $X \times \{1\} \subset M_f$ to a point.

(iii) Let X be a topological space. The *suspension* SX of X is the quotient space of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

For example, if we take X to be the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then $X \times I$ is a cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \leq z \leq 1\}$, and then we collapse two circular edges of C to two points to get SX, which is homeomorphic to the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

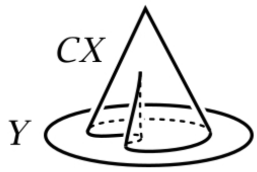


FIGURE 2.7: Mapping Cone

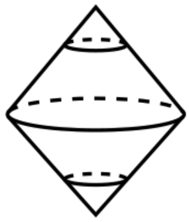


FIGURE 2.8: Suspension

We can think of SX as a *double cone on* X: take disjoint union of two cones $C_1 := (X \times I)/(X \times \{1\})$ and $C_2 := (X \times I)/(X \times \{0\})$, and then attach C_1 with C_2 via the continuous map $f : X \times \{0\} \to C_2$ given by $f(x, 0) = (x, 0) \in C_2$, $\forall\, (x, 0) \in X \times \{0\} \subset C_1$.

The following exercise shows that a quotient of a Hausdorff space need not be Hausdorff in general.

Exercise 2.6.29. Consider the two disjoint copies of real line $X = \mathbb{R} \times \{0, 1\} \subset \mathbb{R}^2$, and the equivalence relation $\sim$ on X defined by $(t, 0) \sim (t, 1)$, for all $t \in \mathbb{R} \setminus \{0\}$. The associated quotient space $X/\sim$ is called the *real line with double origin*. Show that $X/\sim$ is not Hausdorff.

Definition 2.6.30 (Local homeomorphism)**.** Let X and Y be topological spaces. We say that X is locally homeomorphic to Y if for each $x \in X$ there exists an open neighbourhood $U_x \subseteq X$ of x and a continuous map $f_x : U_x \to Y$ such that $f_x(U_x)$ is open in Y and $f_x : U_x \to f_x(U_x)$ is a homeomorphism.

Remark 2.6.31. Note that if X is locally homeomorphic to Y, then Y need not be locally homeomorphic to X.

Exercise 2.6.32 (Hausdorffness is not a local property)**.** Define an equivalence relation $\sim$ on the Euclidean line $\mathbb{R}$ by $x \sim y$ if either $x = y$ or $|x| = |y| > 1$. Let $Q := X/\sim$ be the associated quotient space. Show that every point $z \in Q$ has an open neighbourhood homeomorphic to $(-1, 1)$, but Q is not Hausdorff.

Proof. Let $z \in \mathbb{Q}$ be given. Let $x \in \pi^{-1}(z)$, where $\pi : \mathbb{R} \to Q$ is the quotient map. If $|x| \leq 1$, then $\pi^{-1}(z) = \{x\}$ is singleton. $\square$

Proposition 2.6.33. *Let $\rho \subseteq X \times X$ be an equivalence relation on a topological space X, and let $q : X \to Q := X/\rho$ be the quotient map. Then we have the following.*

(i) Q is a T1 space if and only if every ρ-equivalence class is closed in X.

(ii) If Q is Hausdorff then ρ is a closed subspace of the product space $X \times X$. The converse holds if $q : X \to Q$ is an open map.

Proof. (i) Let Q be T1. Let $x \in X$. Choose a $y \in X \setminus [x]$. Then $[x] \neq [y]$ in Q. Since Q is T1, there is an open subset $V_y \subseteq Q$ such that $[y] \in V_y$ and $[x] \notin V_y$. Then $q^{-1}(V_y)$ is an open neighbourhood of y with $q^{-1}(V_y) \cap [x] = \emptyset$. Therefore, y is an interior point of $X \setminus [x]$. Therefore, $X \setminus [x]$ is open, and hence $[x] \subseteq X$ is closed.

Conversely, suppose that $[x] \subseteq X$ is closed, for all $x \in X$. To show Q is T1, we need to show that $\{[x]\}$ is closed in Q, for all $x \in X$. Since $q^{-1}(Q \setminus \{[x]\}) = \{y \in X : q(y) \neq [x]\} = X \setminus [x]$ is open, $Q \setminus \{[x]\}$ is open in Q, for all $[x] \in Q$. Therefore, Q is a T1 space.

(ii) Consider the commutative diagram of continuous maps

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\Delta_X"'] & Q \arrow[d, "\Delta_Q"] \\ X \times X \arrow[r, "q \times q"] & Q \times Q, \end{tikzcd}$$

where $q \times q : X \times X \to Q \times Q$ is the product map given by

$$(q \times q)(x, y) = (q(x), q(y)), \ \forall \, (x, y) \in X \times X.$$

Note that, $(q \times q)^{-1}(\Delta_Q(Q)) = \{(x, y) \in X \times X : q(x) = q(y)\} = \rho$. If Q is Hausdorff, then $\Delta_Q(Q)$ is closed by Lemma 2.5.2. Since $q \times q$ is continuous, ρ is closed in $X \times X$.

Now we assume that q is an open map, and that ρ is closed in $X \times X$. Since $q \times q$ is a continuous surjective open map (verify!), it is a quotient map by Corollary 2.6.18. Since $(q \times q)^{-1}(\Delta_Q(Q)) = \rho$ is closed in $X \times X$, the diagonal $\Delta_Q(Q)$ is closed in $Q \times Q$ by Theorem 2.6.12 (iii). Therefore, Q is Hausdorff by Lemma 2.5.2.

□

Now we give an example to show that even if X is Hausdorff and the equivalence relation ρ is closed in $X \times X$, the associated quotient space $Q = X/\rho$ need not be Hausdorff without the assumption that q is an open map. For this, we first recall the following.

Definition 2.6.34. A topological space X is said to be *normal* if any two disjoint closed subsets can be separated by a pair of disjoint open subsets containing them. In other words, given two closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there are open subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Most of the familiar examples of topological spaces are generally normal (e.g., $\mathbb{R}^n$), and a closed subspace of a normal space is normal. The following example shows that a Hausdorff space need not be normal.

Example 2.6.35. Let $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Consider the topology τ_K on $\mathbb{R}$ whose basis for open subsets is given by the collection

$$\mathscr{B} = \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R} \text{ with } a < b\}.$$

Clearly this topology on $\mathbb{R}$ is strictly finer than the Euclidean topology on $\mathbb{R}$, and hence $(\mathbb{R}, \tau_K)$ is a Hausdorff space. Note that in this topology, K and $\{0\}$ are disjoint closed subsets that cannot be separated by a pair of disjoint open subsets containing them. Therefore, $(\mathbb{R}, \tau_K)$ is not normal.

Exercise 2.6.36. Start with a Hausdorff space X that is not normal. Choose two disjoint closed subsets $A, B \subset X$ that cannot be separated by two disjoint open subsets containing them. Take $\rho = \Delta_X(X) \cup (A \times A) \cup (B \times B)$. Note that ρ is an equivalence relation on X, and is closed in $X \times X$ (why?). Show that the associated quotient space X/ρ is T1 but not Hausdorff.

Exercise 2.6.37. (i) Let $p : X \to Y$ be a continuous map. If there is a continuous map $f : Y \to X$ such that $p \circ f = \mathrm{Id}_Y$, then show that p is a quotient map.

(ii) Let $A \subset X$. A *retraction of* X *onto* A is a continuous map $r : X \to A$ such that $r(a) = a,\ \forall\, a \in A$. Show that a retraction is a quotient map.

Proof. (i) Let $V \subseteq Y$ be such that $p^{-1}(V)$ is open in X. Since f is continuous and $p \circ f = \mathrm{Id}_Y$, we have $V = (p \circ f)^{-1}(V) = f^{-1}(p^{-1}(V))$ is open in Y. Therefore, p is a quotient map.

(ii) Let $\iota_A : A \hookrightarrow X$ be the inclusion map of A into X. Since $\iota_A : A \to X$ is continuous with $r \circ \iota_A = \mathrm{Id}_A$, that r is a quotient map by part (i). □

Exercise 2.6.38. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first factor. Let $A := \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0\} \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}$. Let $q : A \to \mathbb{R}$ be the map $\pi_1\big|_A$. Show that q is a quotient map that is neither open nor closed.

Proof. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection map onto the first factor. Note that $A = (\overline{\mathbb{R}^+} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$, and the restriction map $\pi_1\big|_{\mathbb{R} \times \{0\}} : \mathbb{R} \times \{0\} \to \mathbb{R} \times \{0\}$ is the identity map of $\mathbb{R} \times \{0\}$. Therefore,

□

2.7 Projective space and Grassmannian†

In this section we discuss two special examples of quotient spaces, namely *projective space* and *Grassmannian*† that naturally occurs in algebraic topology and geometry.

2.7.1 Real and complex projective spaces

Fix an integer $n \geq 0$. Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$v \sim v' \text{ if } v' = \lambda \cdot v, \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}. \tag{2.7.1}$$

†This section § 2.7 is not in the syllabus and may be skipped for examination.

In other words, identify all points lying on the same straight-line in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. Then the associated quotient space

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

is called the *real projective n-space*. As a set, $\mathbb{RP}^n$ consists of all straight-lines in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. So an element of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \{\lambda \cdot (a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R} \setminus \{0\}\}. \tag{2.7.2}$$

Let $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map for the projective n-space. Note that the unit n-sphere $S^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : \sum\limits_{j=0}^{n} a_j^2 = 1\}$ is a compact connected subspace of $\mathbb{R}^{n+1} \setminus \{0\}$. Since the restriction map $q\big|_{S^n} : S^n \longrightarrow \mathbb{RP}^n$ is continuous and surjective, $\mathbb{RP}^n$ is compact and connected.

Exercise 2.7.3. Prove the following.

(i) Show that the map $f := q\big|_{S^n} : S^n \to \mathbb{RP}^n$ is a quotient map.

(ii) For each $\ell \in \mathbb{RP}^n$, show that $f^{-1}(\ell) = \{v, -v\}$, for some $v \in S^n$.

(iii) Show that $\mathbb{RP}^n$ is Hausdorff.

Outline of solution. Note that, the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is given by sending $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ to the straight line

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) : \lambda \in \mathbb{R}\} \in \mathbb{RP}^n.$$

Since $\mathbb{RP}^n$ consists of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin, given a straight-line $\ell \in \mathbb{RP}^n$, choosing any non-zero point $v := (a_0, \ldots, a_n) \in \ell$, we find an element $v/\|v\| \in S^n$ with $f(v/\|v\|) = \ell$, where $\|v\| := \big(\sum\limits_{j=1}^{n} a_j^2\big)^{1/2}$. Thus, f is surjective.

$$f : S^n \overset{\iota}{\hookrightarrow} \mathbb{R}^{n+1} \setminus \{0\} \overset{q}{\longrightarrow} \mathbb{RP}^n.$$

Note that given any subset $V \subseteq \mathbb{RP}^n$, we have $f^{-1}(V) = q^{-1}(V) \cap S^n$. So continuity of f follows from that of q. To see that f is a quotient map, suppose that $f^{-1}(V)$ is open in S^n. To show that V is open in $\mathbb{RP}^n$, fix a point $\ell \in V$. Its fiber $f^{-1}(\ell) = \{v, -v\}$ consists of the two antipodal points of S^n obtained by intersecting the line ℓ with S^n. Since the points v and $-v$ lies on two hemispheres separated by a great circle on S^n, we can find a small enough (connected) open neighbourhood $U \subset f^{-1}(V)$ of v such that $-U := \{-u : u \in U\} \subset S^n$ is an open neighbourhood of $-v$ in S^n, and $U \cap (-U) = \emptyset$. Note that, $-U \subseteq f^{-1}(V)$. Then $f\big|_U$ is a homeomorphism of U onto the open neighbourhood $f(U) \subseteq V$ of ℓ in $\mathbb{RP}^n$. Thus f is a quotient map. $\square$

Next we show that $\mathbb{RP}^n$ can be covered by $n+1$ open subsets each homeomorphic to $\mathbb{R}^n$. Let $p_j : \mathbb{R}^{n+1} \to \mathbb{R}$ be the j-th projection map defined by

$$p_j(x_0, \ldots, x_n) = x_j, \ \ \forall\, (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}.$$

For each $j \in \{0, 1, \ldots, n\}$, consider the *hyperplane*

$$H_j := \{[a_0 : \cdots : a_n] \in \mathbb{RP}^n : a_j = 0\} \subset \mathbb{RP}^n.$$

Since q is a quotient map and

$$\begin{aligned} q^{-1}(H_j) &= \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : a_j = 0\} \\ &= p_j^{-1}(0) \cap (\mathbb{R}^{n+1} \setminus \{0\}), \end{aligned}$$

we conclude that H_j is a closed subset of $\mathbb{RP}^n$. Let $U_j := \mathbb{P}^n \setminus H_j$, $\forall\, j = 0, 1, \ldots, n$. Since any point of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R}\},$$

with $a_j \neq 0$, for some j, we see that $\{U_0, U_1, \ldots, U_n\}$ is an open cover of $\mathbb{RP}^n$.

Proposition 2.7.4. *The open subset $U_j \subset \mathbb{RP}^n$ is homeomorphic to $\mathbb{R}^n$, for all j.*

Proof. Consider the map $\phi_j : U_j \longrightarrow \mathbb{R}^n$ given by

$$[a_0 : \cdots : a_n] \stackrel{\phi_j}{\longmapsto} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right).$$

Note that ϕ_j is a well-defined bijective map with its inverse $\psi_j : \mathbb{R}^n \to U_j$ given by

$$(b_0, \ldots, b_{n-1}) \mapsto [b_0 : \cdots : b_{j-1} : 1 : b_j : \cdots : b_n].$$

Note that $V_j = q^{-1}(U_j) = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : a_j \neq 0\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$, and the map $f_j : V_j \to \mathbb{R}^n$ given by $(a_0, \ldots, a_n) \stackrel{f_j}{\longmapsto} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right)$ is continuous (why?). Since $q^{-1}(\phi_j^{-1}(V)) = f_j^{-1}(V)$, $\forall\, V \subseteq \mathbb{R}^n$, and q is a quotient map, we conclude that ϕ_j is continuous, for all j (c.f. Proposition 2.6.22).

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "q"'] & V_j \arrow[l, hook'] \arrow[r, "f_j"] \arrow[d, "q_j"'] & \mathbb{R}^n \arrow[dl, dashed, bend left, "\psi_j"] \\
\mathbb{RP}^n & U_j \arrow[l, hook'] \arrow[ur, "\phi_j"] &
\end{tikzcd}$$

Since f_j is a quotient map by Corollary 2.6.18, as before we see that $\psi_j = \phi_j^{-1}$ is also continuous. This completes the proof. $\square$

Corollary 2.7.5. $\mathbb{RP}^n$ *is a compact connected Hausdorff space.*

Exercise 2.7.6. Define an equivalence relation $\sim$ on S^n by

$$v \sim v' \text{ if } v' = -v.$$

Show that the associated quotient space $S^n/\sim$ is homeomorphic to $\mathbb{RP}^n$. Conclude that $\mathbb{RP}^n$ is a compact connected Hausdorff space.

The *complex projective n-space* $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim$ defined by

$$v \sim v' \text{ if } v' = \lambda v, \text{ for some } \lambda \in \mathbb{C}.$$

So the points of $\mathbb{CP}^n$ are precisely one dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^{n+1}$ (i.e., complex lines in $\mathbb{C}^{n+1}$ passing through the origin $0 \in \mathbb{C}^{n+1}$).

Exercise 2.7.7. Show that $\mathbb{CP}^n$ is a compact connected Hausdorff space.

Remark on notations: The real projective n-space $\mathbb{RP}^n$ is also denoted by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n(\mathbb{R})$. Similar notations $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^n(\mathbb{C})$ are also used for complex projective n-space $\mathbb{CP}^n$.

2.7.2 Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$

Fix two positive integers k and n, with $k < n$. Let

$$(\mathbb{R}^n)^k := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}}$$

be k-fold product of $\mathbb{R}^n$ together with the product topology. A typical element of $(\mathbb{R}^n)^k$ is of the form $(v_1, \ldots, v_k)$, where $v_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$, for all $j = 1, \ldots, k$. Note that, we can identify $(\mathbb{R}^n)^k$ with $M_{k,n}(\mathbb{R})$ using the bijective map

$$(v_1, \ldots, v_k) \longmapsto \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix}.$$

Consider the subset

$$X := \{(v_1, \ldots, v_k) \in (\mathbb{R}^n)^k \mid \{v_1, \ldots, v_k\} \text{ is } \mathbb{R}\text{-linearly independent}\}$$

with the subspace topology induced from $(\mathbb{R}^n)^k$. Given $A := (v_1, \ldots, v_k)$ and $A' := (v'_1, \ldots, v'_k)$ in X, we define $A \sim A'$ if

$$\mathrm{Span}_{\mathbb{R}}\{v_1, \ldots, v_k\} = \mathrm{Span}_{\mathbb{R}}\{v'_1, \ldots, v'_k\}.$$

Clearly $\sim$ is an equivalence relation on X. The associated quotient topological space $X/\sim$ is known as the *Grassmannian of k-dimensional* $\mathbb{R}$*-linear subspaces of* $\mathbb{R}^n$, and is denoted by $\mathrm{Gr}(k, \mathbb{R}^n)$. As a set, $\mathrm{Gr}(k, \mathbb{R}^n)$ consists of all $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$ of dimension k.

Corollary 2.7.8. $\mathrm{Gr}(1, \mathbb{R}^n)$ *is homeomorphic to* $\mathbb{RP}^{n-1}$.

Remark 2.7.9 (Plücker embedding)**.** Given a n-dimensional $\mathbb{R}$-vector space V, its k-th exterior power $\bigwedge^k V$ is a $\mathbb{R}$-vector space of dimension $\binom{n}{k}$. Sending $W \in \mathrm{Gr}(k, \mathbb{R}^n)$ to its k-th exterior power $\bigwedge^k W \subset \bigwedge^k \mathbb{R}^n$, we get a continuous map

$$\Phi : \mathrm{Gr}(k, \mathbb{R}^n) \longrightarrow \mathbb{RP}^N,$$

where $N = \binom{n}{k} - 1$. It turns out that Φ is a closed embedding (homeomorphism onto a closed subspace of $\mathbb{RP}^N$). From this, one can conclude that $\mathrm{Gr}(k, \mathbb{R}^n)$ is a compact Hausdorff space. We shall not go into detailed proofs of the above statements.

2.8 Topological group*

Definition 2.8.1. A *topological group** is a topological space G which is also a group G such that the binary map (group operation)

$$m : G \times G \to G, \quad (x, y) \mapsto xy,$$

and the inversion map

$$\mathrm{inv} : G \to G, \quad x \mapsto x^{-1},$$

involved in its group structure, are continuous. Here we consider $G \times G$ as the product topological space.

We recast the above definition of topological group in more formal language, without using points of G. This formalism, with appropriate type of spaces and maps between them, defines *Lie group*, *algebraic group*, *group-scheme* and more generally, a *group object* in a category (for curious readers!). Denote by $*$ the topological space whose underlying set is singleton. This space is unique up to a unique homeomorphism. Given any topological space X, any map $* \to X$ is continuous, and they are in bijection with the underlying set of points of X. On the other hand, the space $*$ is the *final object* in the category of topological spaces in the sense that, given any topological space X, there is a unique continuous map $X \to *$. Clearly the product space $X \times *$ is homeomorphic to X, and the set of all such homeomorphisms are in bijection with the set of all *automorphisms of* X (i.e., homeomorphisms of X onto itself). Unless explicitly specified, we consider the homeomorphism $X \times * \to X$ given by the identity map $\mathrm{Id}_X : X \to X$ of X.

Now the above Definition 2.8.1 essentially says that, a topological group is a pair (G, m), where G is a topological space and $m : G \times G \to G$ is a continuous map such that the following axioms holds.

*Additional materials; may be skipped for exam.

(TG1) *Associativity:* The following diagram is commutative.

$$\begin{tikzcd} G \times G \times G \arrow[r, "m\times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

(TG2) *Existence of neutral element:* There is a continuous map $e : * \to G$ such that the following diagram is commutative.

$$\begin{tikzcd} * \times G \arrow[r, "e\times \mathrm{Id}_G"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{Id}_G \times e"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

(TG3) *Existence of inverse:* There is a continuous map $\mathrm{inv} : G \to G$ such that the following diagram is commutative.

$$\begin{tikzcd} G \arrow[r, "(\mathrm{Id}_G, \mathrm{inv})"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{inv}, \mathrm{Id}_G)"'] \arrow[d] \\ * \arrow[r, "e"] & G & * \arrow[l, "e"'] \end{tikzcd}$$

Example 2.8.2. (i) Any abstract group is a topological group with respect to the discrete topology on it.

(ii) $(\mathbb{R}, +)$, the real line with usual addition of real numbers, is a topological group.

(iii) $(\mathbb{R}^*, \cdot)$, the subspace of non-zero real numbers with usual multiplication is a topological group.

(iv) $(\mathbb{Z}, +)$ is a topological group, where the topology on $\mathbb{Z}$ is discrete.

(v) For any integer $n \geq 1$, the Euclidean space $\mathbb{R}^n$ with the component wise addition, i.e.,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n), \;\; \forall\, a_i, b_i \in \mathbb{R},$$

is a topological group.

(vi) Given integers $m, n \geq 1$, the set of all $(m \times n)$-matrices with real entries $\mathrm{M}_{m,n}(\mathbb{R})$, considered as the Euclidean topological space $\mathbb{R}^{mn}$, is a topological group with respect to the usual matrix addition.

(vii) $\mathrm{GL}_n(\mathbb{R})$, the subspace of all invertible $(n \times n)$-matrices with real entries, is a topological group with respect to multiplication of matrices.

(viii) Circle group: The space $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$, together with the multiplication of complex numbers, is a topological group.

(ix) Any abstract subgroup of a topological group is a topological group with respect to the subspace topology.

(x) Product of two topological groups is a topological group.

Exercise 2.8.3. Let G be a topological group. If $U \subseteq G$ is an open neighbourhood of identity $e \in G$, show that there is an open neighbourhood $V \subset G$ of identity such that $V^2 := \{ab : a, b \in V\} \subseteq U$. (Hint: Use continuity of the multiplication map m.)

Exercise 2.8.4. Show that for any $a \in G$, the *right translation by a map*

$$R_a : G \to G, \;\; g \mapsto ga,$$

is a homeomorphism. Prove the same statement for the *left translation by a map* given by $L_a(g) = ga$, for all $g \in G$. (*Hint:* Note that R_a is the composite map $g \mapsto (g, a) \overset{m}{\mapsto} ga$ with inverse $R_{a^{-1}}$.)

Exercise 2.8.5. Show that a topological group G is Hausdorff if and only if it is a T1 space. (*Hint:* $\Delta_G(G)$ is precisely the inverse image of $\{e\} \subseteq G$ under the map $(x, y) \mapsto x^{-1}y$.)

Lemma 2.8.6. *Let G be a topological group. Let H be the connected component of G containing the neutral element $e \in G$. Then H is a closed normal subgroup of G.*

Proof. Since connected components are closed, H is closed. Since for any $a \in H$, the set $Ha^{-1} = \{ha^{-1} : h \in H\} = R_{a^{-1}}(H)$ contains e, and is homeomorphic to H, we must have $Ha^{-1} \subseteq H$. Since this holds for all $a \in H$, we see that H is a subgroup of G. To see that H is normal, note that, for any $g \in G$, the set $gHg^{-1} = L_g(R_{g^{-1}}(H))$ is a connected subset of G containing e, and hence $gHg^{-1} \subseteq H$. This completes the proof. $\square$

Definition 2.8.7. A *right action* of a topological group G on a topological space X is a continuous map $\sigma : X \times G \to X$ such that $\sigma(x, e) = x$, and $\sigma(\sigma(x, g_1), g_2) = \sigma(x, m(g_1, g_2))$, for all $x \in X$ and $g_1, g_2 \in G$, where $m : G \times G \to G$ is the product operation (multiplication map) on G. Similarly, one can left action of G on X.

Without using points, a right G-action σ on X can be defined by commutativity of the following diagrams.

(i)

$$\begin{tikzcd} X \times * \arrow[r, "\mathrm{Id}_X \times e"] \arrow[dr, "\cong"'] & X \times G \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

(ii)

$$\begin{tikzcd} X \times G \times G \arrow[r, "\sigma \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_X \times m"'] & X \times G \arrow[d, "\sigma"] \\ X \times G \arrow[r, "\sigma"] & X \end{tikzcd}$$

A right G-action σ on X induces an equivalence relation on X, which gives a partition of X as a disjoint union of equivalence classes. A typical equivalence class is of the form

$$\mathrm{orb}_G(x) := \{x' \in X : x' = xg, \;\text{ for some }\; g \in G\} = xG,$$

and is called the *G-orbit* of $x \in X$. The associated quotient space, denoted by X/σ or X/G, consists of all G-orbits of elements of X as its points. For this reason, X/G is also called *orbit space*. If the G-action on X is *transitive* (i.e., given any $x, x' \in X$, there exists $g \in G$ such that $x' = xg$), then X is called a *homogeneous space*. In this case, the associated quotient space X/G is singleton.

Exercise 2.8.8. Given a subgroup H of a topological group G, the *H-action on G* defined by

$$G \times H \mapsto G, \ \ (g, h) \mapsto gh$$

gives a partition of G into all right cosets of H in G. Show that the orbit space G/H is a homogeneous space.

Exercise 2.8.9. Let H be a subgroup of a topological group G, and let $G/H = \{gH : g \in G\}$ be the associated quotient space of G by the natural H-action on it. Let $q : G \to G/H$ be the associated quotient map. Show that q is continuous surjective and open. (*Hint:* Since $q^{-1}(\{aH\}) = aH$, $\forall\, aH \in G/H$, given an open subset $U \subseteq G$, we see that

$$q^{-1}(q(U)) = q^{-1}\Big(\bigcup_{g\in U} \{gH\}\Big) = \bigcup_{g\in U} gH = \bigcup_{h\in H} L_h(U)$$

is open in G, and hence $q(U)$ is open in G/H.)

Exercise 2.8.10. Let $I = [0, 1] \subset \mathbb{R}$. Define the $\mathbb{Z}_2$-action on $I \times I$ which gives identifications $(0, t) \sim (1, 1 - t)$, for each $t \in I$. Convince yourself that the associated quotient space is homeomorphic to the *Möbius strip* (see Figure 2.9). Note that, Möbius strip has only one side!

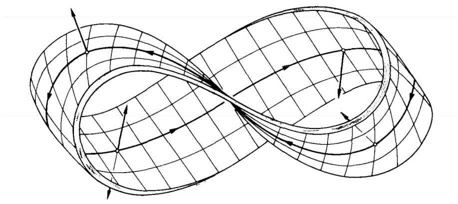


FIGURE 2.9: Möbius strip

Exercise 2.8.11. Define a $\mathbb{Z}_2$-action on the n-sphere $S^n \subset \mathbb{R}^{n+1}$ which identifies $v \in S^n$ with its *antipodal point* $-v \in S^n$. Show that the associated quotient space $S^n/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^n$.

Exercise 2.8.12. Show that $\mathbb{R}/\mathbb{Q}$ is a non-Hausdorff topological group. Hint: Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, and right cosets are just translates of $\mathbb{Q}$.

Exercise 2.8.13. Let $\sigma : X \times G \to X$ be a right action of a topological group on a space X. For each $g \in G$, show that the induced map

$$\sigma_g : X \to X, \ \ x \mapsto xg := \sigma(x, g)$$

is a homeomorphism.

Exercise 2.8.14. Let X be a topological space together with an action of a topological group G. Show that the quotient map $q : X \to X/G$ is open. (Hint: For $V \subseteq X$ open, show that $q^{-1}(q(V)) = \bigcup_{g \in G} Vg$ is open by Exercise 2.8.13, where $Vg = \{vg : v \in V\}, \forall g \in G$.)

Proposition 2.8.15. *Let H be a subgroup of a topological group G. Then the orbit space G/H is Hausdorff if and only if H is closed in G. (Here G/H is not necessarily a group because H need not be a normal subgroup of G.)*

Proof. If G/H is Hausdorff, then it is a T1 space so that $H = \mathrm{orb}_H(e) \in G/H$ is a closed point. Since H is the inverse image of this point under the quotient map $q : G \to G/H$ (continuous), H is closed in G. Conversely, suppose that H is closed in G. Since the equivalence relation given by the H-action on G is precisely the inverse image of H under the continuous map

$$G \times G \longrightarrow G, \ \ (g_1, g_2) \mapsto g_1^{-1} g_2,$$

and the quotient map $q : G \to G/H$ is open by Exercise 2.8.14, the converse part follows from Proposition 2.6.33 because H is closed in G. □

Corollary 2.8.16. *The topological group $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.*

Definition 2.8.17. A *homomorphism* of topological groups is a continuous group homomorphism. An *isomorphism* of topological groups is a bijective bi-continuous homomorphism of topological groups.

Exercise 2.8.18. If $f : G \to H$ is a homomorphism of topological groups with H Hausdorff, show that $\mathrm{Ker}(f) := \{g \in G : f(g) = e_H\}$ is a closed normal subgroup of G.

Exercise 2.8.19. If $f : G \to H$ is a homomorphism of topological groups, show that the induced map $G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism of topological groups.

Exercise 2.8.20. Show that $f : \mathbb{R} \to S^1$ defined by $f(t) = e^{2\pi i t}$, for all $t \in \mathbb{R}$, is a surjective homomorphism of topological groups. Use Exercise 2.8.19 to show that $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

Exercise 2.8.21. Let $f : G \to H$ be a continuous bijective homomorphism of topological groups. Show that $f^{-1} : H \to G$ is continuous (Hint: Use Exercise 2.8.14).

Corollary 2.8.22. *A bijective homomorphism $f : G \to H$ of topological groups is an isomorphism.*

Exercise 2.8.23. Consider the $\mathbb{Z}$-action on $\mathbb{R}$ given by $\sigma(t, n) = t + n$, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Show that the associated quotient space $\mathbb{R}/\sigma$ is homeomorphic to S^1.

Exercise 2.8.24. Show that $\mathrm{GL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$ as topological groups.

Exercise 2.8.25. Show that $\mathrm{GL}_n(\mathbb{R})$ is disconnected, and has precisely two connected components, whereas $\mathrm{GL}_n(\mathbb{C})$ is path-connected. (*Hint:* For the first part, use determinant map. For the second part, given $A \in \mathrm{GL}_n(\mathbb{C})$ use left and right translation homeomorphisms to move it to an upper triangular matrix, and then use convex combination map for its entries to move it to the identity matrix in $\mathrm{GL}_n(\mathbb{C})$.)

Exercise* 2.8.26. Show that the group

$$SO_n = \{A \in \mathrm{GL}_n(\mathbb{R}) : AA^t = A^tA = I_n \text{ and } \det(A) = 1\}$$

is compact and connected.

Exercise* 2.8.27 (Universal property of product)**.** Let G_1 and G_2 be two topological groups. Let P be a topological group together with homomorphisms of topological groups $p_1 : P \to G_1$ and $p_2 : P \to G_2$ such that given given any topological group H and homomorphisms of topological groups $f_1 : H \to G_1$ and $f_2 : H \to G_2$, there is a unique homomorphism of topological groups $f : H \to P$ such that the following diagram commutes.

$$\begin{tikzcd}
 & H \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
G_1 & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_2.
\end{tikzcd}$$

Prove that there is a unique isomorphism of topological groups $\phi : P \to G_1 \times G_2$.

2.9 Connectedness

Let X be a topological space. A *separation* of X is a pair of open subsets $U, V \subseteq X$ such that $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$.

Definition 2.9.1. A topological space X is said to be *connected* if there is no separation of X by non-empty pair of disjoint open subsets of X that covers X. If X is not connected, it is called disconnected. A subset $A \subseteq X$ is said to be connected if the topological space A, with the subspace topology induced from X, is connected.

Example 2.9.2. (i) The empty subset $\emptyset \subseteq X$ is always connected because there is no separation of it.

(ii) The punctured real line $\mathbb{R} \setminus \{0\}$ is disconnected in $\mathbb{R}$ since it has a separation given by the open subsets $(-\infty, 0)$ and $(0, \infty)$.

(iii) The subset $[0, 1] \setminus \{1/2\} \subset \mathbb{R}$ is disconnected, since it has a separation given by the subsets $[0, 1/2), (1/2, 1]$ open in $[0, 1] \setminus \{1/2\}$.

(iv) Let $L_{m,c} := \{(x, mx + c) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be a straight-line in the Euclidean plane $\mathbb{R}^2$. Then the subset $\mathbb{R}^2 \setminus L_{m,c}$ is disconnected, since it has a separation given by the subsets $U = \{(x, y) : y < mx + c\}$ and $V = \{(x, y) : y > mx + c\}$ open in $\mathbb{R}^2 \setminus L_{m,c}$.

(v) Since $\mathbb{R}_\ell = (-\infty, 0) \cup [0, \infty)$ and both $(-\infty, 0)$ and $[0, \infty)$ are open in $\mathbb{R}_\ell$, the space $\mathbb{R}_\ell$ is disconnected.

Proposition 2.9.3. *A topological space X is connected if and only if the only subsets of X that are both open and closed are the empty subset of X and X itself.*

Proof. Suppose that X is connected. Suppose on the contrary that there is a non-empty proper subset $U \subset X$ of X that is both open and closed in X. Then $V := X \setminus U$ is a non-empty proper open subset of X such that $U \cap V = \emptyset$ and $U \cup V = X$. This is not possible since X is connected.

Conversely, suppose that the only non-empty subset of X that is both open and closed in X is X itself. Suppose that $X = U \cup V$ for some open subsets U and V of X with $U \cap V = \emptyset$. If $U \neq \emptyset$, then $V = X \setminus U$ is both closed and also open in X, and hence it must be empty set. Thus X has no separation in X, and hence is connected. □

Exercise 2.9.4. Let τ and τ' be two topologies on a non-empty set X. If $\tau \subseteq \tau'$ and (X, τ') is connected, show that (X, τ') is connected. Give an example to show that the converse does not hold, in general.

Exercise 2.9.5. Let τ_c be the cofinite topology on an infinite set X. Show that (X, τ) is connected.

Definition 2.9.6. A topological space is said to be *totally disconnected* if its only connected non-empty subsets are singleton subsets.

Example 2.9.7. (i) Let X be a discrete topological space. Let A be a connected subspace of X. Suppose on the contrary that A contains at least two distinct points, say $a, b \in A$. Let $U = A \setminus \{a\}$ and $V = \{a\}$. Clearly $U \cup V = A$ and $U \cap V = \emptyset$. Since both U and V are open in X, and hence in A, we get a separation of A, which contradicts our assumption that A is connected. Therefore, A must be a one-point space. Therefore, X is totally disconnected.

(ii) The set $\mathbb{Q}$ equipped with the subspace topology induced from $\mathbb{R}$ is totally disconnected. Indeed, if $A \subseteq \mathbb{Q}$ is non-empty with at least two points, say $a, b \in A$ with $a < b$, then choosing an irrational number $c \in \mathbb{R}$ with $a < c < b$, we see that $U = (-\infty, c) \cap A$ and $V = (c, \infty) \cap A$ are non-empty open subsets of $A \subseteq \mathbb{Q}$ with $U \cap V = \emptyset$ and $U \cup V = A$ making A disconnected. Note that singleton subsets of $\mathbb{Q}$ are not open in $\mathbb{Q}$, and hence the subspace topology on $\mathbb{Q}$ induced from $\mathbb{R}$ is not discrete.

(iii) $\mathbb{R}_\ell$ is totally disconnected. Indeed, if $A \subseteq \mathbb{R}_\ell$ contains at least two distinct points, say $a, b \in A$ with $a < b$, then both $U := (-\infty, b) \cap A$ and $V := [a, \infty) \cap A$ are non-empty open subsets of $A \subseteq \mathbb{R}_\ell$ with $U \cap V = \emptyset$ and $U \cup V = A$ making A disconnected.

Lemma 2.9.8. *Let X be a topological space. Let $U, V \subset X$ be two non-empty disjoint open subsets of X such that $X = U \cup V$. If A is a connected subset of X, then either $A \subseteq U$ or $A \subseteq V$.*

Proof. Since $A \subseteq X = U \cup V$, A intersects at least one of U and V. Suppose that $A \cap U \neq \emptyset$. Then $A \cap U$ and $A \cap V$ are open subsets of A with $(A \cap U) \cup (A \cap V) = A$. Since A is connected and $A \cap U \neq \emptyset$, we must have $A \cap V = \emptyset$, and hence $A \subseteq U$. □

Exercise 2.9.9. Let $p : X \to Y$ be a quotient map of topological spaces. Suppose that Y is connected and each fiber $p^{-1}(y)$ is connected, for all $y \in Y$. Show that X is connected.

Answer: Suppose on the contrary that X is not connected. Then there exists a pair of non-empty open subsets U_1 and U_2 of X such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2 = \emptyset$. Fix a $j \in \{1, 2\}$. Let $x \in U_j$ and $y = f(x) \in p(U_j) \subseteq Y$. Since $p^{-1}(y)$ is connected and $p^{-1}(y) \cap U_j \neq \emptyset$, we have $p^{-1}(y) \subseteq U_j$. Therefore, $p^{-1}(p(U_j)) \subseteq U_j$, and hence $p^{-1}(p(U_j)) = U_j$. Since U_1 and U_2 are open in X and p is a quotient map, both $p(U_1)$ and $p(U_2)$ are open in Y. Since U_1 and U_2 are non-empty and pair-wise disjoint, so are $p(U_1)$ and $p(U_2)$. Since p is surjective and $U_1 \cup U_2 = X$, we have $p(U_1) \cup p(U_2) = Y$. Thus, we have a separation of Y, which contradicts the fact that Y is connected. Therefore, X must be connected. □

Lemma 2.9.10. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of connected subsets of X having a common point, say $x_0 \in \bigcap_{\alpha\in\Lambda} A_\alpha$. Suppose on the contrary that $A := \bigcup_{\alpha\in\Lambda} A_\alpha$ is not connected. Then there exists a pair of disjoint non-empty subsets $U, V \subseteq A$ open in A such that $A = U \cup V$. Since A_α is connected, A_α is entirely contained in exactly one of U and V. Suppose that $A_{\alpha_0} \subseteq U$. If $A_\beta \subseteq V$, for some $\beta \in \Lambda$, then $x_0 \in \bigcap_{\alpha\in\Lambda} A_\alpha \subseteq A_{\alpha_0} \cap A_\beta \subseteq V$, contradicting our assumption that $A_{\alpha_0} \subseteq U$. Therefore, $A_\alpha \subseteq U$, for all $\alpha \in \Lambda$, and hence $A = \bigcup_{\alpha\in\Lambda} A_\alpha \subseteq U$. This forces V to be an empty set, which is a contradiction. Therefore, A must be connected. □

Exercise 2.9.11. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of connected subsets of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset$, for all $n \in \mathbb{N}$. Show that $\bigcup_{n\in\mathbb{N}} A_n$ is connected.

Answer: Since $A_n \cap A_{n+1} \neq \emptyset$, $\forall\, n \in \mathbb{N}$, it follows that $A_n \neq \emptyset$, $\forall\, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $B_n = \bigcup_{k=1}^{n} A_k$. For $n = 1$, $B_1 = A_1$ is connected. Suppose that $n > 1$, and assume inductively that B_{n-1} is connected. Since both A_n and B_{n-1} is connected, it follows that $B_n = B_{n-1} \cup A_n$ is connected because $\emptyset \neq A_{n-1} \cap A_n \subseteq B_{n-1} \cap A_n$. Then by induction B_n is connected, $\forall\, n \in \mathbb{N}$. Since B_n is connected, $\forall\, n \in \mathbb{N}$, and since $\bigcap_{n\in\mathbb{N}} B_n = A_1 \neq \emptyset$, it follows that $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$ is connected. □

Exercise 2.9.12. Let $\{A_\alpha : \alpha \in \Lambda\}$ be a collection of connected subsets of X. Let B be a connected subset of X such that $A_\alpha \cap B \neq \emptyset$, for all $\alpha \in \Lambda$. Show that $B \cup \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)$ is connected.

Exercise 2.9.13. Let $A \subset X$. If C is a connected subspace of X such that $C \cap A \neq \emptyset$ and $C \cap (X \setminus A) \neq \emptyset$, then $C \cap \mathrm{Bd}(A) \neq \emptyset$, where $\mathrm{Bd}(A) = \overline{A} \cap \overline{(X \setminus A)}$ is the set of all boundary points of A.

Answer: Let $U = C \cap A$ and $V = C \cap (X \setminus A)$. Then both U and V are non-empty disjoint subsets of C with $U \cup V = C$. Suppose on the contrary that $C \cap \mathrm{Bd}(A) = \emptyset$. We claim that both U and V are closed in C. If V contains a limit point x of U, then $x \in V \cap U' \subseteq C \cap \overline{(X \setminus A)} \cap \overline{A} = \emptyset$, which is a contradiction. Similarly, U does not contain any limit point of V. Since $U \cup V = C$, both U and V are closed in C. Since $U \cap V = \emptyset$, both U and V are open in C. Thus we get a separation of C, which is not possible since C is connected. Therefore, we must have $C \cap \mathrm{Bd}(A) = \emptyset$. □

Lemma 2.9.14. *Let $f : X \to Y$ be a continuous map of topological spaces, and let A be a connected subset of X. Then $f(A)$ is a connected subset of Y.*

Proof. Suppose on the contrary that $f(A)$ is disconnected. Then $f(A) = U \cup V$, for a pair of non-empty disjoint subsets $U, V \subset f(A)$ open in $f(A)$. Since $g := f|_A : A \to f(A)$ is continuous by Lemma 2.3.19, both $g^{-1}(U)$ and $g^{-1}(V)$ are open subsets of A. Since both U and V are non-empty and g is surjective, both $g^{-1}(U)$ and $g^{-1}(V)$ are non-empty. Since $f(A) = U \cup V$, it follows that $A = f^{-1}(U) \cup f^{-1}(V)$. But this is not possible since A is connected. Therefore, $f(A)$ must be connected. □

Corollary 2.9.15. *Let X and Y be homeomorphic topological spaces. Then X is connected if and only if Y is connected.*

Theorem 2.9.16 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is a totally ordered set together with the order topology. Let $a, b \in X$ and $y \in Y$ be such that $f(a) < y < f(b)$ in Y. Then there exists $c \in X$ such that $f(c) = y$.*

Proof. Note that $U := f(X) \cap (-\infty, y)$ and $V := f(X) \cap (y, \infty)$ are disjoint open subsets of $f(X)$ containing $f(a)$ and $f(b)$, respectively. If $f^{-1}(y) = \emptyset$, then $f(X) = U \cup V$. Which is not possible since $f(X)$ is connected by Lemma 2.9.14. Therefore, $f^{-1}(y) \neq \emptyset$. □

Theorem 2.9.17. *If X and Y are connected topological spaces, so is their product space $X \times Y$.*

Proof. Let X and Y be connected topological spaces, and let $X \times Y$ be their product space. Fix a point $b \in Y$. Since $X \times \{b\}$ is homeomorphic to X, it follows from Lemma 2.9.14 that $X \times \{b\}$ is connected. Similarly, since Y is connected, $\{x\} \times Y$ is connected, for all $x \in X$. Since $(X \times \{b\}) \cap (\{x\} \times Y) = \{(x, b)\} \neq \emptyset$, it follows from Lemma 2.9.10 that

$$T_{x,b} := (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected. Since $\bigcup\limits_{x \in X} T_{x,b} = X \times Y$ and $\bigcap\limits_{x \in X} T_{x,b} = X \times \{b\} \neq \emptyset$, it follows from Lemma 2.9.10 that $X \times Y$ is connected. □

Corollary 2.9.18. *Let X and Y be two topological spaces. Then $X \times Y$ is connected if and only if both X and Y are connected.*

Proof. One direction is already proved in Theorem 2.9.17. Since both of the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous and surjective, the converse part follows from Lemma 2.9.14. □

Corollary 2.9.19. *A finite Cartesian product of connected spaces is connected.*

Proof. Let $X_1, \ldots, X_n$ be connected topological spaces. For $n = 1$, the result holds trivially. Assume that $n > 1$, and the result holds for any $n - 1$ number of connected topological spaces. Then $Y := X_1 \times \cdots \times X_{n-1}$ is connected by induction hypothesis. Since $X_1 \times \cdots \times X_n$ is homeomorphic to $Y \times X_n$, and that $Y \times X_n$ is connected by Corollary 2.9.17, it follows from Lemma 2.9.14 that $X_1 \times \cdots \times X_n$ is connected. □

Lemma 2.9.20. *Let X be a topological space. Let A be a connected subset of X. If $B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$, then B is connected in X.*

Proof. If possible suppose that B is not connected in X. Then there exists a pair of non-empty subsets $V_1, V_2 \subseteq B$ open in B such that $V_1 \cup V_2 = B$ and $V_1 \cap V_2 = \emptyset$. Now $V_1 = U_1 \cap B$ and $V_2 = U_2 \cap B$, for some open subsets U_1, U_2 of X. Since $A \subseteq B$, $W_1 := U_1 \cap A = V_1 \cap A$ and $W_2 := U_2 \cap A = V_2 \cap A$ are open subsets of A with $W_1 \cup W_2 = V_1 \cap V_2 \cap A = \emptyset$ and $W_1 \cup W_2 = (V_1 \cap A) \cup (V_2 \cap A) = (V_1 \cup V_2) \cap A = B \cap A = A$. We claim that both W_1 and W_2 are non-empty. Indeed, since $U_1 \cap B = V_1 \neq \emptyset$, there exists a point $b \in B \subseteq \overline{A}$ such that $b \in U_1$. Since $b \in \overline{A}$, we must have $W_1 = U_1 \cap A \neq \emptyset$. Similarly, we have $W_2 \neq \emptyset$. Thus we get a separation of A, which contradicts our assumption that A is connected in X. Therefore, B must be connected. □

Definition 2.9.21. A subset $I \subseteq \mathbb{R}$ is said to be an *interval* in $\mathbb{R}$ if given any two points $a, b \in I$ with $a < b$, we have $(a, b) := \{x \in \mathbb{R} : a < x < b\} \subseteq I$.

Proposition 2.9.22. *A connected subset of $\mathbb{R}$ is an interval.*

Proof. Let $I \subseteq \mathbb{R}$ be a connected subset of $\mathbb{R}$. If I is an empty set or a singleton subset of $\mathbb{R}$, the result holds trivially. Assume that I contains at least two distinct points. Let $a, b \in I$ be arbitrary. Let $x \in \mathbb{R}$ be such that $a < x < b$. Let $U_x = (-\infty, x) \cap I$ and $V_x = (x, \infty) \cap I$. Note that $a \in U_x$ and $b \in V_x$. Then U_x and V_x are non-empty open subsets of I. If $x \notin I$, then $I = U_x \cup V_x$. This is not possible since I is connected. Therefore, I must be an interval in $\mathbb{R}$. □

Theorem 2.9.23. *Any non-empty interval in $\mathbb{R}$ is connected.*

Proof. In view of Lemma 2.9.20, it suffices to show that any open interval in $\mathbb{R}$ is connected. Let I be an open interval in $\mathbb{R}$. If possible suppose that I is not connected. Then there exists a pair of non-empty subsets $U, V \subseteq I$ open in I (and hence in $\mathbb{R}$) such that $U \cup V = I$ and $U \cap V = \emptyset$. Fix two points $a \in U$ and $b \in V$. Without loss of generality, we may assume that $a < b$. Let

$$A := \{x \in \mathbb{R} : [a, x) \subseteq U\}.$$

Since $b \in V$ and $V \subseteq I$ is open, there exists a $\delta > 0$ such that $(b - \delta, b + \delta) \subseteq V$. Since $[a, b) \cap (b - \delta, b + \delta) \neq \emptyset$ and $U \cap V = \emptyset$, we must have $b \notin A$. Then $x < b$, $\forall\, x \in A$. Therefore, A is a bounded above subset of $I \subseteq \mathbb{R}$, and so it has a least upper bound, say $\ell := \sup(A) \in \mathbb{R}$. Clearly $a \leq \ell \leq b$. Since I is an interval, $\ell \in I$. Since $I = U \cup V$, either $\ell \in U$ or $\ell \in V$.

Case 1: Suppose that $\ell \in U$. Then U being an open set, $(\ell - \epsilon, \ell + \epsilon) \subseteq U$, for some $\epsilon > 0$. On the other hand, since $\ell = \sup(A)$ and $\epsilon/2 > 0$, there exists $x_0 \in A$ such that $\ell - \frac{\epsilon}{2} < x_0$. Then $[a, \ell + \epsilon) = [a, x_0) \cup (\ell - \epsilon, \ell + \epsilon) \subseteq U$, and so $\ell + \epsilon \in A$, which is not possible since $\ell = \sup(A)$.

Case 2: Suppose that $\ell \in V$. Since V is open, there exists $\epsilon' > 0$ such that $(\ell - \epsilon', \ell + \epsilon') \subseteq V$. Since $\ell = \sup(A)$, there exists $x_1 \in A$ such that $\ell - \frac{\epsilon'}{2} < x_1$. Then $\ell - \frac{\epsilon'}{2} \in [a, x_1) \cap (\ell - \epsilon', \ell + \epsilon') \subseteq U \cap V$, which is not possible since $U \cap V = \emptyset$.

Since we are getting contradictions in both cases, I must be connected. □

Corollary 2.9.24. *The Euclidean space $\mathbb{R}^n$ is connected, for all $n \geq 1$.*

Proof. For $n = 1$, since $\mathbb{R}$ is an interval in itself, it is connected by Theorem 2.9.23. Suppose that $n > 1$. Since $\mathbb{R}^n$ is homeomorphic to the product topological space $X_1 \times \cdots \times X_n$, where $X_j = \mathbb{R}$, $\forall\, j = 1, \ldots, n$, the result follows from Corollary 2.9.19. $\square$

Exercise 2.9.25. Show that the unit circle $S^1 \subseteq \mathbb{R}^2$ is connected. (*Hint:* Since $f : [0,1] \to S^1$ defined by $f(t) = e^{2\pi i t}$, $\forall\, t \in [0,1]$ is continuous surjective and $[0,1]$ is connected by Theorem 2.9.23, the result follows from Lemma 2.9.14.)

Exercise 2.9.26. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point $x \in S^1$ such that $f(x) = f(-x)$.

Answer: Suppose on the contrary that $f(x) \neq f(-x)$, for all $x \in S^1$. Consider the map $g : S^1 \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(-x), \ \ \forall\, x \in S^1.$$

Then g is a continuous map with $g(S^1) \subseteq \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$. Since S^1 is connected by Exercise 2.9.25, either $g(S^1) \subseteq (0, \infty)$ or $g(S^1) \subseteq (-\infty, 0)$ by Lemma 2.9.8. But this is not possible since if $g(x) > 0$, for some $x \in S^1$, then $-x \in S^1$ and that $g(-x) = -g(x) < 0$. Therefore, there must be a point $x_0 \in S^1$ such that $g(x_0) = 0$, which gives $f(x_0) = f(-x_0)$. $\square$

Exercise 2.9.27. Let $f : [0,1] \to [0,1]$ be a continuous map. Show that f has a fixed point (i.e., there exists a point $a \in [0,1]$ such that $f(a) = a$.).

Answer: Suppose on the contrary that f has no fixed point. Then for each $x \in [0,1]$, either $f(x) < x$ or $f(x) > x$ holds. Since f is continuous, both $U := \{x \in [0,1] : f(x) < x\}$ and $V := \{x \in [0,1] : f(x) > x\}$ are open subsets of $[0,1]$. Clearly $U \cap V = \emptyset$ and $U \cup V = [0,1]$. Since $[0,1]$ is connected, we must have either $U = \emptyset$ or $V = \emptyset$. Suppose that $U = \emptyset$. Then 1 must lie in $[0,1] = V$, which is not possible since $f(1) \in [0,1]$. Similarly, if $V = \emptyset$, then $[0,1] = U$ forces that $f(0) < 1$, which is not possible. Therefore, f must have a fixed point. $\square$

Exercise 2.9.28. Show that no two of the subspaces $(0,1)$, $(0,1]$, and $[0,1]$ of $\mathbb{R}$ are homeomorphic.

Answer: Suppose on the contrary that $f : (0,1] \to (0,1)$ is a homeomorphism. Then $f|_{(0,1)} : (0,1) \to (0,1) \setminus \{f(1)\}$ is a homeomorphism. Since $(0,1) \setminus \{f(1)\} = (0, f(1)) \cup (f(1), 1)$ gives a separation of $(0,1) \setminus \{f(1)\}$, and continuous image of a connected space is connected, we get a contradiction. Therefore, there is no homeomorphism of $(0,1]$ with $(0,1)$.

Suppose on the contrary that there is a homoemorphism $g : (0,1] \to [0,1]$. Let $g(1) = a \in [0,1]$. If $0 < a < 1$, then $g|_{(0,1)} : (0,1) \to [0,a) \cup (a,1]$ is a homoemorphism, which is not possible since continuous image of a connected space is connected and $[0,a) \cup (a,1]$ is disconnected. Therefore, no such homoemorphism could exist. If $f(1) = a \in \{0,1\}$, by removing 1 from $(0,1]$ and its image a from $[0,1]$ we reduce the problem to the first case, and get a contradiction.

Suppose on the contrary that there is a homeomorphism $h : [0,1] \to (0,1)$. Since $b := h(0) \in (0,1)$, by removing 0 from $[0,1]$ and its image b from $(0,1)$, we get a contradiction as before. $\square$

Exercise 2.9.29. If $n > 1$, show that the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic to each other.

Answer: Suppose on the contrary that there is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$, where $n > 1$. Let $p \in \mathbb{R}^n$ be such that $f(p) = 0$ in $\mathbb{R}$. Let $X = \mathbb{R}^n \setminus \{p\}$. Then $f\big|_X : X \to \mathbb{R} \setminus \{0\}$ is a homeomorphism. But this is not possible since $X = \mathbb{R}^n \setminus \{p\}$ is connected and $\mathbb{R} \setminus \{0\}$ is disconnected. Therefore, $\mathbb{R}^n$ cannot be homeomorphic to $\mathbb{R}$, for $n > 1$. $\square$

Theorem 2.9.30. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of connected topological spaces. Then the product space $\prod_{\alpha \in \Lambda} X_\alpha$ is connected.*

Proof. Thanks to Theorem 2.9.17 we may assume that Λ is infinite and that X_α contains at least two points, for infinitely many $\alpha \in \Lambda$. Let $X := \prod_{\alpha \in \Lambda} X_\alpha$ be equipped with the product topology. Fix a point $\mathbf{a} = (a_\alpha)_{\alpha \in \Lambda} \in X$. Let $\mathscr{F}(\lambda)$ be the collection of all finite subsets of Λ. Given a finite subset $K \in \mathscr{F}(\Lambda)$ of Λ, let

$$X_K := \{\mathbf{x} = (x_\alpha)_{\alpha \in \Lambda} \in X : x_\alpha = a_\alpha, \ \forall \alpha \in \Lambda \setminus K\}.$$

Then the subset $X_K \subseteq X$, with the subspace topology induced from X, is homeomorphic to $X_{\alpha_1} \times \cdots \times X_{\alpha_n}$, where $K = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Lambda$. Since X_α is connected, for each $\alpha \in \Lambda$, it follows from Corollary 2.9.19 that X_K is connected. Let $Y = \bigcup_{F \in \mathscr{F}(\Lambda)} X_F$. Since X_F is connected, for all $F \in \mathscr{F}(\Lambda)$, and since $\mathbf{a} \in \bigcap_{F \in \mathscr{F}(\Lambda)} X_F$, it follows from Lemma 2.9.10 that Y is connected.

Note that if X_α is not singleton for infinitely many $\alpha \in \Lambda$, then choosing a point $b = (b_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} X_\alpha$ with $b_\alpha \neq a_\alpha$, $\forall \alpha \in \Lambda$, we see that $b \notin Y$. Therefore, $Y \neq X$. We now show that the closure of Y in X is X itself. Let $\mathbf{b} = (b_\alpha)_{\alpha \in \Lambda} \in X$ be arbitrary. Let $U = \prod_{\alpha \in \Lambda} U_\alpha$ be a non-empty basic open subset of X containing $\mathbf{b}$. Then $U_\alpha \subseteq X_\alpha$ is an open neighbourhood of b_α in X_α, $\forall \alpha \in \Lambda$, and there is a finite subset $G \in \mathscr{F}(\Lambda)$ such that $U_\alpha = X_\alpha, \ \forall \alpha \in \Lambda \setminus G$. Consider the point $\mathbf{c} = (c_\alpha)_{\alpha \in \Lambda} \in X$ defined by

$$c_\alpha = \begin{cases} b_\alpha, & \text{if} \quad \alpha \in G, \\ a_\alpha, & \text{if} \quad \alpha \in \Lambda \setminus G. \end{cases}$$

Then $\mathbf{c} \in U \cap X_G \subseteq U \cap Y$. Therefore, $\mathbf{c} \in \overline{Y}$, and hence $\overline{Y} = X$. Since closure of a connected set is connected (see Lemma 2.9.14), the result follows. $\square$

However, the following example shows that the conclusion of the Theorem 2.9.30 fails if we equip $X = \prod_{\alpha \in \Lambda} X_\alpha$ with the box topology instead of the product topology.

Example 2.9.31. For each $n \in \mathbb{N}$, let X_n be the Euclidean space $\mathbb{R}$. Let $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ be the set $\prod_{n \in \mathbb{N}} X_n$ of all sequences of real numbers. Equip the set $\prod_{n \in \mathbb{N}} X_n$ with the box topology. Consider the subsets

$$U = \{(a_n)_{n\in\mathbb{N}} : (a_n)_{n\in\mathbb{N}} \text{ is a bounded sequence}\},$$
$$\text{and} \quad V = \{(a_n)_{n\in\mathbb{N}} : (a_n)_{n\in\mathbb{N}} \text{ is an unbounded sequence}\}.$$

of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Clearly both U and V are non-empty subsets of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ with $U \cap V = \emptyset$. Given a point $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{\text{box}}$, the subset $W := \prod_{n\in\mathbb{N}} U_n$, where $U_n = (a_n - 1, a_n + 1) \subset \mathbb{R}$, $\forall\, n \in \mathbb{N}$, is an open neighbourhood of $\mathbf{a}$ in the box topological space $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Note that all sequences in W are bounded (resp., unbounded) if $\mathbf{a}$ is bounded (resp., unbounded). Therefore, both U and V are open in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$. Thus we get a separation of $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, and hence the space $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is disconnected.

Let X be a topological space. Let

$$\rho := \{(a, b) \in X \times X : \text{there is a connected subspace of } X \text{ containing } a \text{ and } b\} \subseteq X \times X.$$

Clearly ρ is reflexive and symmetric. If $(a, b), (b, c) \in \rho$, then there are connected subspaces A and B of X with $a, b \in A$ and $b, c \in B$. Then $A \cap B \neq \emptyset$, and hence $A \cup B$ is connected by Lemma 2.9.10. Therefore, $(a, c) \in \rho$. Thus, ρ is an equivalence relation on X. The ρ-equivalence classes in X are called the *connected components* of X. Clearly, connected components of X are precisely maximal connected subsets of X, and gives a partition of X. Moreover, any non-empty connected subspace of X is contained in exactly one of the connected components of X. Since closure of a connected subspace is connected, it follows that connected components of X are closed in X. Therefore, if X has only finitely many connected components, then the connected components of X are both open and closed in X.

Example 2.9.32. All connected components of $\mathbb{Q}$ are one-point space. Indeed, if A is a subspace of $\mathbb{Q}$ containing at least two points, say $a, b \in A$, then choosing an irrational number $\alpha \in \mathbb{R}$ with $a < \alpha < b$ we get a separation of A by two non-empty disjoint open subsets $(-\infty, \alpha) \cap A$ and $(\alpha, \infty) \cap A$ of A. Moreover, any one-point subspace of $\mathbb{Q}$ does not admit any separation, and hence is connected. Therefore, the only connected subspaces of $\mathbb{Q}$ are one-point subspaces.

Lemma 2.9.33. *Let X be a topological space. Let $A \subseteq X$ be a non-empty connected subset of A that is both open and closed in X. Then A is a connected component of X.*

Proof. Since A is a non-empty connected subspace of A, there is a unique connected component of X, say C such that $A \subseteq C$. Since A is closed in X, its complement $U := X \setminus A$ is open in X. Since $C = A \cup (C \cap U)$ and C is connected, $C \cap U$ must be an empty set. Since $A \subseteq C$, this forces $A = C$. □

2.10 Path-connectedness

Definition 2.10.1. A *path* in X from $x_0 \in X$ to $x_1 \in X$ is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A topological space X is said to be *path-connected* if given any two points x_0 and x_1 of X, there is a path in X from x_0 to x_1.

Remark 2.10.2. A path in X joining $x_0 \in X$ to $x_1 \in X$ can equivalently be defined to be a continuous map $\gamma : [a,b] \to X$ with $\gamma(a) = x_0$ and $\gamma(b) = x_1$, where $a, b \in \mathbb{R}$ with $a < b$. To see this, note that given $a, b \in \mathbb{R}$ with $a < b$, we have continuous maps

$$\sigma : [0,1] \to [a,b] \text{ and } \sigma' : [a,b] \to [0,1]$$

defined by

$$\sigma(t) = (1-t)a + tb, \ \forall\, t \in [0,1],$$
$$\text{and } \sigma'(s) = \frac{s-a}{b-a}, \ \forall\, s \in [a,b].$$

Note that both σ and σ' are continuous with $\sigma \circ \sigma' = \mathrm{Id}_{[a,b]}$ and $\sigma' \circ \sigma = \mathrm{Id}_{[0,1]}$.

Then given a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, the composite map $\sigma' \circ \gamma : [a,b] \to X$ is continuous and $(\sigma' \circ \gamma)(a) = x_0$ and $(\sigma' \circ \gamma)(b) = x_1$. Conversely, given a continuous map $\delta : [a,b] \to X$ with $\delta(a) = x_0$ and $\delta(b) = x_1$, the composite map $\delta \circ \sigma$ is continuous and satisfies $(\delta \circ \sigma)(0) = x_0$ and $(\delta \circ \sigma)(1) = x_1$.

Example 2.10.3. (i) Any interval I in $\mathbb{R}$ is path-connected. Indeed, given any two points $a, b \in I$, the map $\gamma : [0,1] \to \mathbb{R}$ defined by

$$\gamma(t) = (1-t)a + tb, \ \forall\, t \in [0,1],$$

is continuous with $\gamma(0) = a$ and $\gamma(1) = b$. Note that, $\gamma(t) = a + t(b-a) \in I$, $\forall\, t \in [0,1]$. Therefore, γ is a path in I from a to b.

(ii) Consider the subspace $X_n = \mathbb{R}^n \setminus \{0\}$ of the Euclidean space $\mathbb{R}^n$. For $n = 1$, it follows from the intermediate value theorem that there is no path in $\mathbb{R} \setminus \{0\}$ joining a negative real number to a positive real number. Therefore, $\mathbb{R} \setminus \{0\}$ is not path-connected. In fact, it is not connected.

(iii) Given a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ be the Euclidean norm of x in $\mathbb{R}^n$. The subspace $B(0,r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ of $\mathbb{R}^n$ is path-connected. Indeed, let $a, b \in B(0,r)$ be arbitrary. Consider the convex combination

$$\gamma_{a,b}(t) := (1-t)a + tb, \ \forall\, t \in [0,1].$$

Since $\|\gamma_{a,b}(t)\| \leq (1-t)\|a\| + t\|b\| < (1-t)r + tr = r$, the map $t \mapsto \gamma_{a,b}(t)$ gives a path in $B(0,r)$ joining a to b. Therefore, $B(0,r)$ is path-connected.

Proposition 2.10.4. *Continuous image of a path-connected space is path-connected.*

Proof. Let $f : X \to Y$ be a surjective continuous map of topological spaces. Assume that X is path-connected. Let $y_0, y_1 \in Y$ be given. Since f is surjective, there exists $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Since X is path-connected, there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $f \circ \gamma : [0,1] \to Y$ is a continuous map with $(f \circ \gamma)(0) = y_0$ and $(f \circ \gamma)(1) = y_1$. Thus, Y is path-connected. □

Example 2.10.5. Assume that $n \geq 2$. We show that $\mathbb{R}^n \setminus \{0\}$ is path-connected. Let $a, b \in \mathbb{R}^n \setminus \{0\}$ be arbitrary. Consider the convex combination of a and b, namely

$$\gamma(t) := (1-t)a + tb, \;\; t \in [0,1].$$

If $\gamma(t) \neq 0$, $\forall\, t \in [0,1]$, then $t \mapsto \gamma(t)$ gives a path in $\mathbb{R}^n \setminus \{0\}$ from a to b, and we are done. If $\gamma(t) = 0$, for some $t \in [0,1]$, then choose a point $c \in \mathbb{R}^n \setminus \{0\}$ that does not lie on the straight-line passing through a and b in $\mathbb{R}^n$. Then the straight-lines in $\mathbb{R}^n$ joining a to c and c to b do not pass through the origin, and thus we get a path in $\mathbb{R}^n \setminus \{0\}$ from a to b.

Example 2.10.6. For each integer $n \geq 2$, the subspace $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ of the Euclidean space $\mathbb{R}^n$ is path-connected. To see this, note that $\mathbb{R}^n \setminus \{0\}$ is path-connected and the map $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by

$$f(x) = \frac{x}{\|x\|}, \;\; \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

is continuous and surjective. Therefore, it follows from Proposition 2.10.4 that S^{n-1} is path-connected, for all $n \geq 2$.

Definition 2.10.7. Given two paths $\gamma : [0,1] \to X$ and $\delta : [0,1] \to X$ with $\alpha(1) = \beta(0)$, we denote by $\gamma \star \delta : [0,1] \to X$ the *composite path* defined by

$$(\gamma \star \delta)(t) := \begin{cases} \gamma(2t), & \text{if} \quad 0 \leq t \leq 1/2, \\ \delta(2t-1), & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Note that $\gamma \star \delta$ is a continuous map, and hence is a path in X from $\gamma(0)$ to $\delta(1)$.

Proposition 2.10.8. *Let X be a topological space. The relation "being path-connected" is an equivalence relation on X.*

Proof. Let

$$\rho := \{(x,y) \in X \times X : \;\text{there is a path in } X \text{ from } x \text{ to } y\}.$$

For each $x \in X$, the constant map $c_x : [0,1] \to X$ sending all points of $[0,1]$ to x is a path from x to itself in X. Therefore, $(x,x) \in \rho$, for all $x \in X$. Thus ρ is reflexive. Let $(x,y) \in \rho$. Then there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then the map $\overline{\gamma} : [0,1] \to X$ defined by $\overline{\gamma}(t) = \gamma(1-t)$, $\forall\, t \in [0,1]$, is a path in X from y to x, and hence $(y,x) \in \rho$. Thus ρ is symmetric. Let $(a,b), (b,c) \in \rho$. Let $\gamma, \delta : [0,1] \to X$ be two continuous maps with $\gamma(0) = a$, $\gamma(1) = b = \delta(0)$ and $\delta(1) = c$. Then the map $\gamma \star \delta : [0,1] \to X$ as defined in Definition 2.10.7 is a path in X joining a to c. Thus $(a,c) \in \rho$, and hence ρ is transitive. Therefore, ρ is an equivalence relation on X. The ρ-equivalence classes in X are called *path-components* of X, and X can be written as a disjoint union of its path-componenets. $\square$

Proposition 2.10.9. *Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of path-connected subspaces of a topological space X. If $\bigcap\limits_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, then $\bigcup\limits_{\alpha \in \Lambda} A_\alpha$ is path-connected.*

Proof. Choose a point $p \in \bigcap_{\alpha \in \Lambda} A_\alpha$. Let $a, b \in \bigcup_{\alpha \in \Lambda} A_\alpha$ be arbitrary. Then $a \in A_\alpha$ and $b \in A_\beta$, for some $\alpha, \beta \in \Lambda$. Since A_α and A_β are path-connected, there exist continuous maps (paths)

$$\gamma_{a,p} : [0,1] \to A_\alpha \quad \text{and} \quad \gamma_{p,b} : [0,1] \to A_\beta$$

joining a to p in A_α and p to b in A_β, respectively. Then the map

$$\gamma_{a,p} \star \gamma_{p,b} : [0,1] \to A_\alpha \cup A_\beta \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$$

as defined in Definition 2.10.7 is a path in $\bigcup_{\alpha \in \Lambda} A_\alpha$ joining a to b. This completes the proof. $\square$

Proposition 2.10.10. *A path-connected space X is connected.*

Proof. Fix a point $a \in X$. Since X is path-connected, given any $x \in X$, there is a continuous map $\gamma_x : [0,1] \to X$ with $\gamma_x(0) = a$ and $\gamma_x(1) = x$. For each $x \in X$, the subset $\gamma_x([0,1]) \subseteq X$ is connected by Lemma 2.9.14, and $a \in \bigcap_{x \in X} \gamma_x([0,1])$. Then $X = \bigcup_{x \in X} \gamma_x([0,1])$ is connected by Lemma 2.9.10. $\square$

However, the following example shows that a connected space need not be path-connected.

Example 2.10.11 (Topologist's sine curve). Consider the subspace $S = \{(x, \sin \frac{1}{x}) : x \in (0, \infty)\}$ of the Euclidean space $\mathbb{R}^2$. Since $(0, \infty) \subset \mathbb{R}$ is path-connected and the map

$$f : (0, \infty) \to S$$

defined by

$$f(x) = \left(x, \sin \frac{1}{x}\right), \quad \forall\, x \in (0, \infty),$$

is continuous and surjective, it follows from Proposition 2.10.4 that S is path-connected, and hence S is connected. It follows from Lemma 2.9.20 that the subspace $\overline{S}$ is connected. The subspace $\overline{S}$ is called *topologist's sine curve*. Note that, $\overline{S} = S \cup A$, where $A = \{0\} \times [-1, 1] \subset \mathbb{R}^2$. To see this, fix a point $(0, t) \in A$ and a real number $r > 0$. Consider the open neighbourhood $U = (-r, r) \times (t - r, t + r) \subset \mathbb{R}^2$ of $(0, t)$. By Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{4n\pi} < \frac{1}{2n\pi} < r$. Then it follows from the intermediate value theorem (Theorem 2.9.16) that, there exists a real number $x \in (\frac{1}{4n\pi}, \frac{1}{2n\pi}) \subseteq (0, r)$ such that $\sin \frac{1}{x} = t$. Therefore, $(0, t) \in \overline{S}$, and hence $\overline{S} = S \cup A$.

We claim that there is no path in $\overline{S}$ joining $(0, 0)$ to $(x, \sin \frac{1}{x}) \in S$, where $x > 0$, and hence $\overline{S}$ is not path-connected. Suppose on the contrary that there is a path $\gamma : [0, 1] \to \overline{S}$ in $\overline{S}$ joining the origin $(0, 0)$ to $(x, \sin \frac{1}{x}) \in S$, for some $x > 0$. Since $\{0\} \times [-1, 1]$ is closed, its inverse image $\gamma^{-1}(\{0\} \times [-1, 1]) \subseteq [0, 1]$ is closed and bounded. Then there exists $a \in [0, 1]$ such that $a = \sup\{t : t \in \gamma^{-1}(\{0\} \times [-1, 1])\}$. Replacing $[0, 1]$ by $[a, 1]$, if required, we may assume that

$$\gamma(0) \in \{0\} \times [-1, 1] \ \text{ and } \ \gamma(t) \in S, \ \forall\, t > 0.$$

Let $\gamma(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are continuous maps such that

- $x(0) = 0$,
- $x(t) > 0$, $\forall\, t > 0$, and
- $y(t) = \sin \frac{1}{x(t)}$, $\forall\, t > 0$.

Now we construct a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that $t_n \to 0$ as $n \to \infty$, while $y(t_n) = (-1)^n$, for all $n \in \mathbb{N}$, which would contradict continuity of $y = \pi_2 \circ \gamma$, and hence of γ.

To construct such a sequence, we proceed as follow: given $n \in \mathbb{N}$, choose a real number u_n with $0 < u_n < x(1/n)$ such that $\sin \frac{1}{u_n} = (-1)^n$. This is always possible. Then use intermediate value theorem to find $t_n \in (0, 1/n)$ such that $x(t_n) = u_n$. This sequence $(t_n)_{n\in\mathbb{N}}$ do the job. Therefore, there is no path in $\overline{S}$ joining $(0,0)$ to a point of S.

Exercise 2.10.12. A *hyperplane* in the Euclidean space $\mathbb{R}^n$ is the zero locus of a non-constant linear polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$. In other words, a hyperplane in $\mathbb{R}^n$ is a subspace of the form

$$H = \{(a_1, \ldots, a_n) : f(a_1, \ldots, a_n) = 0\},$$

where f is a non-constant linear polynomial over $\mathbb{R}$, i.e., f is of the form $f = c_0 + c_1x_1 + \cdots + c_nx_n$ with coefficients $c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that not all of $c_1, \ldots, c_n$ are simultaneously zero. Show that a hyperplane in $\mathbb{R}^n$ is path-connected.

Proposition 2.10.13. *Assume that $n \geq 2$. For any countable subset A of $\mathbb{R}^n$, the subspace $\mathbb{R}^n \setminus A$ of $\mathbb{R}^n$ is path-connected.*

Proof. Given a point $p \in \mathbb{R}^n$, let $\mathscr{L}_p$ be the set of all straight-lines in $\mathbb{R}^n$ passing through p. Since for each $q \in \mathbb{R}^n \setminus \{p\}$ there is a unique straight-line in $\mathbb{R}^n$ passing through p and q, the set $\mathscr{L}_p$ is uncountable. To show $\mathbb{R}^n \setminus A$ is path-connected, we fix any two points $a, b \in \mathbb{R}^n$, and join them by a path in $\mathbb{R}^n \setminus A$.

Let $a, b \in \mathbb{R}^n \setminus A$ be arbitrary. Since A is countable, there exists a straight-line $L \in \mathscr{L}_a$ passing through a that does not intersect A. For each point $p \in L$, we have a unique straight-line $L_{p,b}$ in $\mathbb{R}^n$ passing through p and b. Since there are uncountably many points on L and A is a countable set, we can choose a straight-line $L_{p,b}$ in $\mathbb{R}^n$ that passes through b and $p \in L$ and does not intersect A. Since L and $L_{p,b}$ are path-connected (as being continuous image of the Euclidean line $\mathbb{R}$) and $L \cap L_{p,b} \neq \emptyset$, it follows from Prposition 2.10.9 that $L \cup L_{p,b}$ is path-connected. Since $a, b \in Y_{a,b} := L \cup L_{p,b}$, we have a path in $Y_{a,b} \subseteq \mathbb{R}^n \setminus A$ joining a to b. This completes the proof. $\square$

Exercise 2.10.14. Assume that $n \geq 2$. Let W be a $\mathbb{R}$-linear subspace of the Euclidean space $\mathbb{R}^n$. If $\dim_{\mathbb{R}}(W) \leq n - 2$, show that $\mathbb{R}^n \setminus W$ is path-connected.

Proof. If $n = 2$, then $\dim_{\mathbb{R}}(W) = 0$. Then $W = \{(0,0)\} \subset \mathbb{R}^2$, and hence $\mathbb{R}^2 \setminus W$ is path-connected. Assume that $n \geq 3$. Choose an ordered basis, say $\{v_1, \ldots, v_{n-2}\}$ for W and extend it to an ordered basis $\{v_1, \ldots, v_{n-2}, v_{n-1}, v_n\}$ for $V = \mathbb{R}^n$. Fix two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus W$. Write $\mathbf{a} = \sum\limits_{i=1}^{n} a_iv_i$ and $\mathbf{b} = \sum\limits_{i=1}^{n} b_iv_i$, where $a_i, b_i \in \mathbb{R}$, $\forall\, i = 1, \ldots, n$. Since $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus W$, we have $(a_{n-1}, a_n), (b_{n-1}, b_n) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since $\mathbb{R}^2 \setminus \{(0,0)\}$ is path-connected, there is a path, say

$\delta : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ from (a_{n-1}, a_n) to (b_{n-1}, b_n). Since $(a_1, \ldots, a_{n-2}), (b_1, \ldots, b_{n-2}) \in \mathbb{R}^{n-2}$ and $n - 2 \geq 1$, there is a path, say $\gamma : [0,1] \to \mathbb{R}^{n-2}$ joining $(a_1, \ldots, a_{n-2})$ to $(b_1, \ldots, b_{n-2})$. Let us denote by $p_j^k : \mathbb{R}^k \to \mathbb{R}$, the projection map onto the j-th coordinate of $\mathbb{R}^k$. More precisely,

$$p_j^k(x_1, \ldots, x_n) = x_j, \ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^k.$$

Note that, p_j^k is continuous, for all $k \in \mathbb{N}$ and $j \leq k$. Let $\gamma_i = p_i^{n-2} \circ \gamma$ and $\delta_j = p_j^2 \circ \delta$, for all $i \in \{1, \ldots, n-2\}$ and $j \in \{1, 2\}$. Then the map $\alpha : [0,1] \to \mathbb{R}^n$ defined by

$$\alpha(t) = \sum_{j=1}^{n-2} \gamma_j(t) v_j + \delta_1(t) v_{n-1} + \delta_2(t) v_n, \ \forall\, t \in [0,1],$$

is continuous with $\alpha(0) = \mathbf{a}$ and $\alpha(1) = \mathbf{b}$. Since δ is a path in $\mathbb{R}^2 \setminus \{(0,0)\}$, the image of α lands in $\mathbb{R}^n \setminus W$. Therefore, $\mathbb{R}^n \setminus W$ is path-connected. $\square$

Exercise 2.10.15. Let $\{W_j : j \in \mathbb{N}\}$ be a countable family of $\mathbb{R}$-linear subspaces of $\mathbb{R}^3$ such that $\dim_{\mathbb{R}}(W_j) \leq 1$, $\forall\, j \in \mathbb{N}$. Is $\mathbb{R}^3 \setminus \bigcup\limits_{j \in \mathbb{N}} W_j$ connected? Is it path-connected?

Hint: Note that, W_j is either a point or a straight-line in $\mathbb{R}^3$ passing through the origin. Let $a, b \in \mathbb{R}^3 \setminus \bigcup\limits_{j \in \mathbb{N}} W_j$ be given. Since there are uncountably many planes in $\mathbb{R}^3$ passing through a and b, there is at least one such plane P whose intersection with $\bigcup\limits_{j \in \mathbb{N}} W_j$ is at most countable. Since a plane is homeomorphic to $\mathbb{R}^2$, we see that $P \setminus \big(\bigcup\limits_{j \in \mathbb{N}} W_j\big)$ is path-connected. Thus $\mathbb{R}^3 \setminus \bigcup\limits_{j \in \mathbb{N}} W_j$ is path-connected. $\square$

Proposition 2.10.16. *A connected open subspace of $\mathbb{R}^n$ is path-connected.*

Proof. Note that, given $a \in \mathbb{R}^n$ and any real number $r > 0$, the Euclidean open ball

$$B(a, r) := \{x \in \mathbb{R}^n : \|a - x\| < r\}$$

in $\mathbb{R}^n$ is path-connected. This follows by observing that given any point $x \in B(a, r)$ there is a path

$$t \mapsto \gamma_x(t) := (1-t)a + tx, \ \forall\, t \in [0,1],$$

in $B(a, r)$ joining a to x.

Let A be a non-empty connected open subset of $\mathbb{R}^n$. Fix a point $a \in A$, and let

$$U_a = \{x \in A : \text{there is a path in } A \text{ joining } a \text{ to } x\}.$$

Note that $U_a \neq \emptyset$, since $a \in U_a$. Clearly U_a is path-connected. Let $x \in U_a$ be arbitrary. Let $\gamma_{a,x} : [0,1] \to A$ be a path in A joining a to x. Since $x \in A$ and A is open in $\mathbb{R}^n$, there exists a real number $r > 0$ such that

$$B(x, r) := \{y \in \mathbb{R}^n : d(x, y) < r\} \subseteq A,$$

where $d(x, y)$ is the Euclidean distance between x and y in $\mathbb{R}^n$. Since $B(x, r)$ is path-connected, for each $y \in B(x, r)$ there is a path, say

$$\gamma_{x,y} : [0, 1] \to B(x, r)$$

in $B(x, r)$ joining x to y. Then the map $\gamma_{a,x} \star \gamma_{x,y} : [0, 1] \to A$ as defined in Definition 2.10.7 is a path in A joining a to y. Therefore, $B(x, r) \subseteq U_a$, and hence U_a is open in A.

If $A \setminus U_a \neq \emptyset$, choose a point $y \in A \setminus U_a$. Then there is no path in A joining a to y. Since A is open in $\mathbb{R}^n$, there is a real number $r_0 > 0$ such that $B(y, r_0) \subseteq A$. We show that $B(y, r_0) \cap U_a = \emptyset$. For otherwise, if $x \in B(y, r_0) \cap U_a$, then there is a path, say $\gamma_x : [0, 1] \to A$, in A joining a to x. Since $B(y, r_0)$ is also path-connected, there is a path, say $\gamma_{x,y} : [0, 1] \to B(y, r_0) \subseteq A$ in $B(y, r_0)$ joining x to y. Then $(\gamma_{a,x} \star \gamma_{x,y}) : [0, 1] \to A$ is a path in A joining a to y. This contradicts the fact that $y \in A \setminus U_a$. Therefore, $A \setminus U_a$ is also open in A. Thus, U_a is both open and closed in the connected space A, and hence is equal to A by Lemma 2.9.33. □

Exercise 2.10.17. Show that any non-empty open subset of $\mathbb{R}^n$ contains a basis for the $\mathbb{R}$-vector space $\mathbb{R}^n$.

Proof. We denote by $\|\bullet\|$ the standard Euclidean norm on $\mathbb{R}^n$. Fix an ordered basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$. Replacing e_i with $e_i/\|e_i\|$, if required, we may assume that $\|e_i\| = 1$, for all $i = 1, \ldots, n$. (for simplicity, you may think of the standard ordered basis for $\mathbb{R}^n$). Note that, for any real number $t > 0$, the subset $\{te_1, \ldots, te_n\}$ is a basis for $\mathbb{R}^n$, and is contained in the open ball $B(0, t)$. Then for given any $a \in \mathbb{R}^n$, consider the subset

$$B(0, t) + a := \{x + a : x \in B(0, t)\} \subseteq B(a, t).$$

Note that

$$f(t) := \det(x + te_1, \ldots, x + te_n) \in \mathbb{R}[t]$$

is a non-zero polynomial in one variable t, and hence it has only finitely many zeros in $\mathbb{R}$. Therefore, the subset $\{x + te_1, \ldots, x + te_n\}$ is $\mathbb{R}$-linearly independent except for finitely many $t > 0$. Now for an arbitrary non-empty open subset U of $\mathbb{R}^n$, fixing a point $a \in U$, we can find a real number $\delta > 0$ such that $B(a, \delta) \subseteq U$ and it contains a basis for the $\mathbb{R}$-vector space $\mathbb{R}^n$. □

Exercise 2.10.18. Let U be a non-empty open subset of $\mathbb{R}^n$. Let W be an $\mathbb{R}$-linear subspace of $\mathbb{R}^n$. If $U \cap W \neq \emptyset$, then $U \cap W$ contains a basis for the $\mathbb{R}$-vector space W that can be extended to a basis for $\mathbb{R}^n$ contained in U.

Proof. Let $d = \dim_{\mathbb{R}}(W)$. Choosing a basis, say $\{w_1, \ldots, w_d\}$ for W, we have an $\mathbb{R}$-linear isomorphism

$$f : \mathbb{R}^d \to W, (x_1, \ldots, x_n) \mapsto x_1w_1 + \ldots + x_nw_n,$$

which is, in fact, a homeomorphism. Then $f^{-1}(U \cap W)$ is a non-empty open subset of $\mathbb{R}^d$, and hence it contains a basis, say $\{v_1, \ldots, v_d\}$ for $\mathbb{R}^n$ by Exercise 2.10.17. Since f is an $\mathbb{R}$-linear isomorphism, $\{f(v_1), \ldots, f(v_d)\}$ is a basis for W which is contained in $U \cap W$. Replacing each $f(v_i)$ with $\lambda_i f(v_i)$, if required, we may assume that $\{f(v_1), \ldots, f(v_d)\} \subseteq B_{\mathbb{R}^n}(a, \delta) \cap W$, for some $a \in U \cap W$ and $\delta > 0$ such that $B_{\mathbb{R}^n}(a, \delta) \subseteq U$. Then applying a linear translation

$$T_{-a} : \mathbb{R}^n \to \mathbb{R}^n, \;\; x \mapsto x - a,$$

we can get an $\mathbb{R}$-linearly independent subset

$$\{f(v_1) - a, \ldots, f(v_d) - a\} \subseteq B_{\mathbb{R}^n}(0, \delta)$$

of $\mathbb{R}^n$. Extend it to a $\mathbb{R}$-linear basis, say

$$\{f(v_1), \ldots, f(v_d), u_{d+1}, \ldots, u_n\} \subseteq B_{\mathbb{R}^n}(0, \delta)$$

for $\mathbb{R}^n$. Then apply the opposite linear translation map

$$T_a : \mathbb{R}^n \to \mathbb{R}^n, \;\; y \mapsto y + a,$$

to get a basis $\{f(v_1), \ldots, f(v_d), u_{d+1} + a, \ldots, u_n + a\}$ for $\mathbb{R}^n$. Since $\|u_i\| < \delta$, for all $i = d + 1, \ldots, n$, it follows that $u_{d+1} + a, \ldots, u_n + a \in B_{\mathbb{R}^n}(a, \delta) \subseteq U$. This completes the proof. $\square$

Exercise 2.10.19. Fix an integer $n \geq 3$. Let U be a non-empty connected open subset of $\mathbb{R}^n$. If W is a $\mathbb{R}$-linear subspace of $\mathbb{R}^n$ with $\dim_{\mathbb{R}}(W) \leq n - 2$, show that $U \setminus W$ is connected.

Proof. Recall that a non-empty connected open subset of $\mathbb{R}^n$ is path-connected. We show that $U \setminus W$ is path-connected. If $U \cap W = \emptyset$, there is nothing to prove. We assume that $U \cap W \neq \emptyset$. Fix two points $a, b \in U \setminus W$. Using Exercise 2.10.18 we can find an ordered basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ contained in U such that $\{v_1, \ldots, v_d\} \subseteq W$, where $d = \dim_{\mathbb{R}}(W) \leq n - 2$. Then each element $x \in U$ can be uniquely expressed as

$$u = u_1 v_1 + \cdots + u_n v_n,$$

for some $(u_1, \ldots, u_n) \in \mathbb{R}^n$ such that $u \in W$ if and only if $(u_{d+1}, \ldots, u_n) = 0$ in $\mathbb{R}^{n-d}$.

Now the maps $\pi_U : U \to \mathbb{R}^{n-d}$ and $\pi_W : U \to \mathbb{R}^d$ defined by

$$\pi_U(x_1 v_1 + \cdots + x_n v_n) = (x_{d+1}, \cdots, x_n).$$
$$\pi_W(x_1 v_1 + \cdots + x_n v_n) = (x_1, \cdots, x_d),$$

are continuous. Note that $\pi_U(U) \subseteq \mathbb{R}^{n-d} \setminus \{0\}$. Since U and W are path-connected, so are $\pi_U(U)$ and $\pi_W(U)$.

We can write $a = a_1 v_1 + \ldots + a_n v_n$ and $b = b_1 v_1 + \ldots + b_n v_n$, for some $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Since $a, b \in U \setminus W$, we have

$$(a_{d+1}, \ldots, a_n), (b_{d+1}, \ldots, b_n) \in \pi_U(U) \subseteq \mathbb{R}^{n-d} \setminus \{0\}.$$

Since $\pi_U(U) \subseteq \mathbb{R}^{n-d} \setminus \{0\}$ is path-connected, there is a path $\gamma : [0, 1] \to \pi_U(U)$ joining $(a_{d+1}, \ldots, a_n)$ to $(b_{d+1}, \ldots, b_n)$. Since $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in \pi_W(U) \subseteq \mathbb{R}^d$ and $\pi_W(U)$ is path-connected, there is a path $\delta : [0, 1] \to \pi_W(U)$ joining $(a_1, \ldots, a_d)$ to $(b_1, \ldots, b_d)$.

We denote by $\pi_i^m : \mathbb{R}^m \to \mathbb{R}$ the projection map onto the i-th factor of $\mathbb{R}^m$. Then the map $\alpha : [0,1] \to \mathbb{R}^n$ defined by

$$\alpha(t) = \sum_{i=1}^{d} \pi_i^d \gamma(t) v_i + \sum_{j=1}^{n} \pi_j^{n-d} \delta(t) v_j,\ \forall\, t \in [0,1],$$

is continuous. Note that $\alpha(0) = a$ and $\alpha(1) = b$. It remains to show that $\alpha([0,1]) \subseteq U \setminus W$. Left as an exercise! $\square$

Exercise 2.10.20. Let $n \geq 3$. Let $W_1, \ldots, W_k$ be finitely many $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$ with $\dim_{\mathbb{R}}(W_i) \leq n-2$, for all i. Show that $\mathbb{R}^n \setminus \bigcup\limits_{i=1}^{k} W_i$ is path-connected. If we remove countably infinitely many $\mathbb{R}$-linear subspaces $\{W_k : k \in \mathbb{N}\}$ with $\dim_{\mathbb{R}}(W_k) \leq n-2$, $\forall\, k \in \mathbb{N}$, from $\mathbb{R}^n$, what can we say about connectedness of $\mathbb{R}^n \setminus \left(\bigcup\limits_{k \in \mathbb{N}} W_k\right)$?

Definition 2.10.21. A topological space X is said to be *locally connected at* $x \in X$ if each open neighbourhood of x contains a connected open neighbourhood of x. If X is locally connected at each point of it, then X is said to be locally connected. Similarly, X is said to be *locally path-connected at* $x \in X$ if each open neighbourhood of x contains a path-connected open neighbourhood of x; and X is said to be *locally path-connected* if it is locally path-connected at each of its points.

Example 2.10.22. The Euclidean line $\mathbb{R}$ is both locally connected and locally path-connected. The subspace $[0,1) \cup (1,2] \subset \mathbb{R}$ is locally connected and locally path-connected, but neither connected nor path-connected. The topologist's sine curve is connected but not locally connected.

Proposition 2.10.23. *Let X be a topological space. Then X is locally connected (resp., locally path-connected) if and only if for each open subset U of X, any connected component (resp., path-component) of U is open in X.*

Proof. Suppose that X is locally connected (resp., locally path-connected). Fix an open subset U of X. Let $C \subseteq U$ be a connected component (resp., path-component) of U. Let $x \in C$ be given. Since X is locally connected (resp., locally path-connected), there is a connected (resp., path-connected) open subset $V_x \subseteq X$ such that $x \in V_x$ and $V_x \subseteq U$. Since C is a connected component (resp., path-component) of U containing x and $V_x \cap C \neq \emptyset$, we have $V_x \subseteq C$. Therefore, C is open in X.

Conversely, suppose that any connected component (resp., path-component) of an open subset of X is open in X. Let $x_0 \in X$ and let U be an open neighbourhood of x_0 in X. Let $C \subseteq U$ be a connected component (resp., path-component) of U containing x_0. Then C is open in X. Thus, X is locally connected (resp., locally path-connected). $\square$

Theorem 2.10.24. *Let X be a topological space. Each path-component of X lies in a connected component of X. If X is locally path-connected, then the path-components and the connected components of X are the same.*

Proof. Let C be a path-component of X. Then C is path-connected, and hence is connected. Then C is contained in a connected component of X.

Assume that X is locally path-connected. It suffices to show that connected components of X are path-connected. Let C be a connected component of X. Fix a point $x_0 \in C$, and consider the subset

$$U := \{x \in C : \text{there is a path in } C \text{ joining } x_0 \text{ to } x\}.$$

Clearly U is non-empty subset of C since $x_0 \in U$. Note that U is path-connected. Indeed, given any two points $x, y \in U$, we have paths γ and δ in C joining x_0 to x and y, respectively. Let

$$\overline{\gamma} : [0,1] \to C, \quad t \mapsto \gamma(1-t),$$

be the inverse path in C joining x to x_0. Then $\delta \star \overline{\gamma}$ is a path in C from x to y.

We show that U is both open and closed in X, and hence coincides with C. Let $x \in U$ be given. Since X is locally path-connected, there is a path-connected open neighbourhood $V_x \subseteq X$ of x. Since V_x is a connected subspace of X containing x, it must be contained in the connected component C of X containing x. Since V_x is path-connected, given a point of $y \in V_x$, there is a path, say $\delta : [0,1] \to V_x$ in $V_x \subseteq C$ from x to y. Since $x \in U$, there is a path, say $\gamma : [0,1] \to C$ from x_0 to x. Then $\gamma \star \delta$ is a path in C joining x_0 to y. Thus, $V_x \subseteq U$, and hence U is open in X.

We claim that U is closed in X. Let $y \in X \setminus U$ be arbitrary. Since X is locally path-connected, there is a path-connected open neighbourhood, say $V_y \subseteq X$ of y. We claim that $V_y \cap U = \emptyset$. Suppose on the contrary that there is a point $z \in V_y \cap U$. Since $z \in U \subseteq C$ and V_y is connected, we must have $V_y \subseteq C$. Choose a paths γ in C from x_0 to z, and a path δ in $V_y \subseteq C$ from z to y. Then $\gamma \star \delta$ is a path in C joining x_0 to y. Then $y \in U$, which contradicts our choice of y as a point of $X \setminus U$. Therefore, we must have $V_y \cap U = \emptyset$. Thus, no point of $X \setminus U$ can be a limit point of U. Therefore, U is closed in X. Since $U \subseteq C$ is both open and closed in X, it must be a connected component of X by Lemma 2.9.33, and hence $U = C$. □

2.11 Compactness

Let X be a topological space. Let $A \subseteq X$. A collection $\mathcal{F} = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a *cover* of A if $A \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. If all members V_α of $\mathcal{F}$ are open subsets of X, then $\mathcal{F}$ is called an *open cover* of A in X. A *subcover* of $\mathcal{F}$ is a subcollection of $\mathcal{F}$ such that union of all its members cover A.

Definition 2.11.1. A topological space X is said to be *compact* if every open cover of X has a finite subcover. In other words, given a family of open subsets $\mathcal{F} = \{U_\alpha : \alpha \in \Lambda\}$ such that $X = \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\mathcal{F}$ such that $X = \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. A subset K of a topological space X is said to be *compact* if it is compact with respect to the subspace topology on K induced from X.

Example 2.11.2. The Euclidean space $\mathbb{R}$ is not compact, as it has an open cover $\{(n, n+2) : n \in \mathbb{N}\}$ which has no finite subcover.

Example 2.11.3. The subspace $K := \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact. To see this, for each $n \in \mathbb{N}$ we consider the open interval $U_n := (\frac{1}{n} - r_n, \frac{1}{n} + r_n)$, where $r_n := \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+1}\right)$. Then U_n is an open neighbourhood of $\frac{1}{n}$ and that $U_n \cap U_m = \emptyset$, for all $n \neq m$ in $\mathbb{N}$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of K, which has no finite subcover. However, $K \cup \{0\}$ is compact. To see this, consider an open cover $\mathcal{F} = \{U_\alpha : \alpha \in \Lambda\}$ of $K \cup \{0\}$. Then $0 \in U_{\alpha_0}$, for some $\alpha_0 \in \Lambda$. Since U_{α_0} is an open subset of $\mathbb{R}$, there exists a real number $r > 0$ such that $(-r, r) \subseteq U_{\alpha_0}$. Then by Archimedean property of $\mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < r$, $\forall\, n \geq n_0$. For each $j \in \{1, \ldots, n_0\}$, we can choose an open subset $U_{\alpha_j} \in \mathcal{F}$ such that $\frac{1}{j} \in U_{\alpha_j}$, $\forall\, j = 1, \ldots, n_0$. Then $K \cup \{0\} \subseteq U_{\alpha_0} \bigcup \big(\bigcup\limits_{j=1}^{n_0} U_{\alpha_j}\big)$. Therefore, $K \cup \{0\}$ is compact.

Exercise 2.11.4. Show that any subset of $\mathbb{R}$ is compact in the cofinite topology on $\mathbb{R}$.

Exercise 2.11.5. Show that a finite union of compact topological spaces is compact.

Exercise 2.11.6. If K is a compact subset of a metric space (X, d), show that K is closed and bounded in (X, d). Show by an example that a closed and bounded subset of a metric space need not be compact.

Lemma 2.11.7. *Let X be a topological space. A subspace $K \subseteq X$ is compact if and only if every open cover of K in X has a finite subcover.*

Proof. Let $K \subseteq X$. Assume that K is compact. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets U_α of X such that $K \subseteq \bigcup\limits_{\alpha \in \Lambda} U_\alpha$. Then $\mathscr{F}_K := \{U_\alpha \cap K : \alpha \in \Lambda\}$ is a collection of open subsets of K such that $K = \bigcup\limits_{\alpha \in \Lambda} (U_\alpha \cap K)$. Since K is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $K = \bigcup\limits_{j=1}^{n} (U_{\alpha_j} \cap K)$. Then $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a required finite subcover for K.

Conversely, suppose that every open cover of K has a finite subcover. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of K such that $K = \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. Note that, for each $\alpha \in \Lambda$, we have $V_\alpha = U_\alpha \cap K$, for some open subset U_α of X. Since $K \subseteq \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, the collection $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ is an open cover of K in X. Then there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $K \subseteq \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. Then $K = \bigcup\limits_{j=1}^{n} V_{\alpha_j}$. This completes the proof. □

Proposition 2.11.8. *A closed subspace of a compact space is compact.*

Proof. Let K be a closed subspace of a compact space X. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of K in X. Then $\mathscr{F} \cup \{X \setminus K\}$ is an open cover of X. Since X is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = (X \setminus K) \bigcup \big(\bigcup\limits_{j=1}^{n} U_{\alpha_j}\big)$. Since K does not intersect $X \setminus K$, we have $K \subseteq \bigcup\limits_{j=1}^{n} U_{\alpha_j}$. Thus, K is compact. □

Proposition 2.11.9. *Let K be a compact subspace of a Hausdorff space X. Assume that $X \setminus K \neq \emptyset$. Then given $x \in X \setminus K$, there exists a pair of open subsets U and V of X such that $x \in U$, $K \subseteq V$ and $U \cap V = \emptyset$.*

Proof. Since X is Hausdorff, for each $y \in K$ there exists a pair of open subsets U_y and V_y of X containing x and y, respectively, such that $U_y \cap V_y = \emptyset$. Then $\{V_y : y \in K\}$ is an open cover of K. Since K is compact, there exists finite number of points $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} V_{y_j}$. Let $U := \bigcap\limits_{j=1}^{n} U_{y_j}$ and $V := \bigcup\limits_{j=1}^{n} V_{y_j}$. Then U and V are open subsets of X containing x and K, respectively. Since $U_{y_j} \cap V_{y_j} = \emptyset$, for all $j = 1, \ldots, n$, it follows that $U \cap V = \emptyset$. □

Corollary 2.11.10. *A compact subspace of a Hausdorff space is closed.*

Proof. Let K be a compact subspace of a Hausdorff space X. It follows from Proposition 2.11.9 that a point of $X \setminus K$ cannot be a limit point of K. Therefore, K is closed in X. □

Proposition 2.11.11. *Continuous image of a compact space is compact.*

Proof. Let $f : X \to Y$ be a continuous map of topological spaces with X compact. We show that $f(X)$ is a compact subspace of Y. Let $\mathcal{F} = \{V_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of Y such that $f(X) \subseteq \bigcup\limits_{\alpha \in \Lambda} V_\alpha$. Since f is continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = \bigcup\limits_{j=1}^{n} f^{-1}(V_{\alpha_j})$. Then $f(X) \subseteq \bigcup\limits_{j=1}^{n} V_{\alpha_j}$. Therefore, $f(X)$ is compact. □

Corollary 2.11.12. *Let $f : X \to Y$ be a continuous bijective map. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. To show f is a homeomorphism, we show that $f^{-1} : Y \to X$ is continuous. Let $g = f^{-1}$. Let $Z \subseteq X$ be a closed subset of X. Since X is compact, Z is compact by Proposition 2.11.8. Then $g^{-1}(Z) = f(Z)$ is compact by Proposition 2.11.11. Since Y is Hausdorff, $g^{-1}(Z) = f(Z)$ is closed by Proposition 2.11.8. Therefore, f^{-1} is continuous, and hence f is a homeomorphism. □

Exercise 2.11.13. Let τ_1 and τ_2 be two topologies on a non-empty set X. Assume that both (X, τ_1) and (X, τ_2) are compact and Hausdorff. Show that either $\tau_1 = \tau_2$ or they are not comparable. (*Hint:* Use Proposition 2.11.8 and Corollary 2.11.10).

Lemma 2.11.14 (Tube lemma)**.** *Let X and Y be topological spaces. If B is a compact subspace of Y, given a point $x \in X$ and an open subset $W \subseteq X \times Y$ containing the slice $\{x\} \times B$, there exists an open neighbourhood $U \subseteq X$ of x and an open subset $V \subseteq Y$ containing B such that $U \times V \subseteq W$.*

Proof. Note that the product topology on $X \times Y$ has a basis consisting of subsets of the form $U \times V$, where U and V are open subsets of X and Y, respectively. Since $W \subseteq X \times Y$ is an open subset containing the slice $\{x\} \times B$, for each $y \in B$ we can choose open neighbourhoods $U_y \subseteq X$ and $V_y \subseteq Y$ of x and y, respectively, such that $(x, y) \in U_y \times V_y \subseteq W$. Then $\{V_y : y \in B\}$ is an open cover of B. Since B is compact, there are finite number of open subsets $V_{y_1}, \dots, V_{y_n} \subseteq Y$ such that $B \subset \bigcup_{j=1}^{n} V_{y_j}$. Let $U = \bigcap_{j=1}^{n} U_{y_j}$ and $V = \bigcup_{j=1}^{n} V_{y_j}$. Then U and V are open subsets of X and Y, respectively, and that $x \in U$ and $B \subseteq V$. Clearly, $\{x\} \times B \subseteq U \times V \subseteq W$, as required. □

Example 2.11.15. Tube Lemma 2.11.14 fails if B is not compact. For example, let $X = Y = B = \mathbb{R}$ with the Euclidean topology on them. Then the slice $\{0\} \times \mathbb{R}$ (i.e., the y-axis in $\mathbb{R}^2$) has an open neighbourhood

$$W = \left\{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{y^2 + 1}\right\}$$

in $\mathbb{R}^2$ which contains no open neighbourhood of $\{0\} \times \mathbb{R}$ of the form $U \times \mathbb{R}$, where U is an open neighbourhood of 0 in $\mathbb{R}$ (verify!).

Corollary 2.11.16 (Generalized tube lemma)**.** *Let A and B be compact subspaces of X and Y, respectively. Let $N \subseteq X \times Y$ be an open subset containing $A \times B$. Then there exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq N$.*

Proof. For each $a \in A$, the slice $\{a\} \times B$ is contained in N. Then by Tube lemma 2.11.14 there is an open neighbourhood $U_a \subseteq X$ of a and open subset $V_a \subseteq Y$ containing B such that $\{a\} \times B \subseteq U_a \times V_a \subseteq N$. Since $\{U_a : a \in A\}$ is an open cover of the compact space A, there is a finite subcover, say $\{U_{a_1}, \dots, U_{a_n}\}$ for A. Let $U = \bigcup_{j=1}^{n} U_{a_j}$ and $V = \bigcap_{j=1}^{n} V_{a_j}$. Then U and V are open subsets of X and Y, respectively. Clearly, $A \subseteq U$ and $B \subseteq V$. Then $A \times B \subseteq U \times V \subseteq N$. □

Theorem 2.11.17. *Finite product of compact spaces is compact.*

Proof. It suffices to show that product of two compact spaces is compact. Then the general case follows by induction on the number of compact spaces. Let X and Y be compact topological spaces. Let $\mathcal{F} = \{W_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of $X \times Y$ such that $\bigcup_{\alpha \in \Lambda} W_\alpha = X \times Y$. For each $x \in X$, since the slice $\{x\} \times Y$ is a compact subspace of $X \times Y$, there exists $\alpha(x, 1), \dots, \alpha(x, n(x)) \in \Lambda$ such that $\{x\} \times Y \subseteq \bigcup_{j=1}^{n(x)} W_{\alpha(x,j)}$. Then by tube lemma (Lemma 2.11.14) there exists an open neighbourhood $U_x \subseteq X$ of x such that

$$\{x\} \times Y \subseteq U_x \times Y \subseteq \bigcup_{j=1}^{n(x)} W_{\alpha(x,j)}.$$

Since $\{U_x : x \in X\}$ is an open cover of X and X is compact, there exists $x_1, \dots, x_m \in X$ such that $X = \bigcup_{i=1}^{m} U_{x_i}$. Then

$$X \times Y \subseteq \bigcup_{i=1}^{m} U_{x_i} \times Y \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n(x_i)} W_{\alpha(x_i,j)} \subseteq X \times Y.$$

Therefore, $\{W_{\alpha(x_i,j)} : 1 \le j \le n(x_i),\ 1 \le i \le m\} \subseteq \mathcal{F}$ is a required finite subcover for $X \times Y$, and hence $X \times Y$ is compact. □

Definition 2.11.18. A collection $\mathscr{C}$ of subsets of X is said to have *finite intersection property* if for any finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathscr{C}$, we have $\bigcap_{j=1}^{n} C_j \neq \emptyset$.

Theorem 2.11.19. *A topological space X is compact if and only if given any collection $\mathscr{C}$ of closed subsets of X having finite intersection property, the intersection $\bigcap_{C \in \mathscr{C}} C$ is non-empty.*

Proof. Suppose that X is compact. Let $\mathscr{C}$ be a collection of closed subsets of X having finite intersection property. Suppose on the contrary that $\bigcap_{Z \in \mathscr{C}} Z = \emptyset$. Then taking complement in X, we have $\bigcup_{Z \in \mathscr{C}} (X \setminus Z) = X$. Then the collection

$$\mathscr{U} := \{X \setminus Z : Z \in \mathscr{C}\}$$

is an open cover of X. Since X is compact, there exists a finitely many elements $Z_1, \ldots, Z_n \in \mathscr{C}$ such that $\bigcup_{j=1}^{n} (X \setminus Z_j) = X$. Taking complement in X, we have $\bigcap_{j=1}^{n} Z_j = \emptyset$, which contradicts the fact that $\mathscr{C}$ has finite intersection property. Therefore, $\bigcap_{Z \in \mathscr{C}} Z \neq \emptyset$.

Conversely, suppose that given any collection $\mathscr{C}$ of closed subsets of X having finite intersection property, the intersection $\bigcap_{Z \in \mathscr{C}} Z$ is non-empty. Suppose on the contrary that X is not compact. Then there exists an open cover, say $\mathscr{U}$ of X that has no finite subcover. Then $\mathscr{C} := \{X \setminus U : U \in \mathscr{U}\}$ is a collection of closed subsets of X. Since $\mathscr{U}$ has no finite subcover, given any finite collection $\{X \setminus U_j : j = 1, \ldots, n\}$ of elements of $\mathscr{C}$, we have

$$\bigcap_{j=1}^{n} (X \setminus U_j) = X \setminus \left(\bigcup_{j=1}^{n} U_j \right) \neq \emptyset.$$

In other words, $\mathscr{C}$ has finite intersection property. Then by assumption, we have $\bigcap_{Z \in \mathscr{C}} Z \neq \emptyset$. Taking complement in X, we see that $\mathscr{U}$ is not an open cover of X, which is a contradiction. Therefore, X must be compact. $\square$

Remark 2.11.20. Let X be a topological space. A sequence of subsets $\{Z_n : n \in \mathbb{N}\}$ of X is said to be *nested* if $Z_{n+1} \subseteq Z_n$, $\forall\, n \in \mathbb{N}$. Let $\{Z_n : n \in \mathbb{N}\}$ be a nested sequence of closed subsets of a compact topological space X. If $Z_n \neq \emptyset$, $\forall\, n \in \mathbb{N}$, then the collection $\{Z_n : n \in \mathbb{N}\}$ satisfies finite intersection property, and hence $\bigcap_{n \in \mathbb{N}} Z_n \neq \emptyset$ whenever X is compact.

Exercise 2.11.21. Let $f : X \to Y$ be a map of topological spaces, and let

$$G_f := \{(x, y) \in X \times Y : y = f(x)\}$$

be the *graph of f*.

(i) If f is continuous and Y is Hausdorff, then G_f is closed in $X \times Y$.

(ii) ***Kuratowski's theorem:*** If Y is compact, show that the projection map $\pi_1 : X \times Y \to X$ is a closed map.

(iii) If Y is compact and G_f is closed, then f is continuous.

Proof. (i) The map $f : X \to Y$ induces a map $(f \times \mathrm{Id}_Y) : X \times Y \to Y \times Y$ defined by

$$(f \times \mathrm{Id}_Y)(x, y) = (f(x), y), \ \forall\, (x, y) \in X \times Y,$$

where $\mathrm{Id}_Y : Y \to Y$ is defined by $\mathrm{Id}_Y(y) = y$, $\forall\, y \in Y$. Since both f and Id_Y are continuous, $(f \times \mathrm{Id}_Y)$ is continuous. Since Y is Hausdorff, $\Delta_Y(Y) = \{(y, y) : y \in Y\}$ is closed in $Y \times Y$. Since $G_f = (f \times \mathrm{Id}_Y)^{-1}(\Delta_Y(Y))$, we see that G_f is closed in $X \times Y$.

(ii) Let C be a closed subset of $X \times Y$. Let $x_0 \in X \setminus \pi_1(C)$ be arbitrary. Then the slice $\{x_0\} \times Y$ is contained in the open subset $(X \times Y) \setminus C$ of $X \times Y$. Since Y is compact, by tube lemma 2.11.14 there is an open neighbourhood W of x_0 in X such that

$$\{x_0\} \times Y \subset W \times Y \subseteq (X \times Y) \setminus C.$$

Then $x_0 \in W$ and $W \subseteq X \setminus \pi_1(C)$. Thus $X \setminus \pi_1(C)$ is open in X, and hence $\pi_1(C)$ is closed in X. Thus, π_1 is a closed map.

(iii) Let $Z \subset Y$ be any closed subset of Y. To show f is continuous, we show that $f^{-1}(Z)$ is closed in X. Since $X \times Z$ is closed in $X \times Y$, and since G_f is closed in $X \times Y$ by assumption, the subset

$$G_f \cap (X \times Z) = \{(x, y) \in X \times Y : f(x) \in Z\}$$

is closed in $X \times Y$. Note that $\pi_1(G_f \cap (X \times Z)) = f^{-1}(Z)$, where $\pi_1 : X \times Y \to X$ is the projection map onto the first component. Since Y is compact, the projection map $\pi_1 : X \times Y \to X$ is closed by part (ii). Therefore, $f^{-1}(Z)$ is closed. This completes the proof. $\square$

Definition 2.11.22. Let (X, d) be a metric space. A subset A of X is said to be *bounded* if there exists a real number $M > 0$ such that $d(a, b) < M, \ \forall\, a, b \in A$. If A is a bounded subset of X, then

$$\mathrm{diam}(A) := \sup_{a, b \in X} d(a, b)$$

exists in $\mathbb{R}$, and is called the *diameter* of A.

Lemma 2.11.23. *Any closed and bounded interval in $\mathbb{R}$ is compact.*

Proof. Let $a, b \in \mathbb{R}$ with $a \leq b$, and consider the closed interval

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} \subset \mathbb{R}.$$

Let $\mathscr{F} := \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $[a, b]$ in $\mathbb{R}$. Let

$$K := \{x \in (a, b] : [a, x] \text{ can be covered by finitely many members of } \mathscr{F}\}.$$

Since $[a, b] \subseteq \bigcup_{\alpha \in \Lambda} V_\alpha$, there exists $\alpha_0 \in \lambda$ such that $a \in V_{\alpha_0}$. Since V_{α_0} is open in $\mathbb{R}$, there exists a real number $r > 0$ such that $(a - r, a + r) \subseteq V_{\alpha_0}$. Then for any $x \in (a, a + r)$, we have $[a, x] \subseteq V_{\alpha_0}$, and hence $x \in K$. Therefore, K is non-empty. Clearly K is bounded above by

b. Then by the *least upper bound property* of $\mathbb{R}$, the least upper bound $c := \sup K$ exists in $\mathbb{R}$. Clearly $a < c \le b$. We claim that $c \in F$. Since $c \in [a,b]$, there exists $\beta \in \Lambda$ such that $c \in V_\beta$. Since V_β is open in $\mathbb{R}$, there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq V_\beta$. Since $c = \sup K$, there exists an element $x \in K$ such that $c - \delta < x < c$. Then $[a,x]$ can be covered by finitely many objects, say $V_{\alpha_1}, \ldots, V_{\alpha_n} \in \mathscr{F}$, and hence $[a,c]$ can be covered by $\{V_{\alpha_1}, \ldots, V_{\alpha_n}, V_\beta\} \subseteq \mathscr{F}$. Then $c \in K$. Now we show that $c = b$. If not, then $c < b$. Since $c \in V_\gamma$, for some $\gamma \in \Lambda$, there exists $t \in (c,b)$ such that $[c,t] \subseteq V_\gamma$. Then $[a,t] = [a,c] \cup [c,t]$ can be covered by finitely many elements from $\mathscr{F}$, and hence $t \in K$. Since $t > c = \sup K$, we get a contradiction. Therefore, $c = b$. This completes the proof. □

Theorem 2.11.24. *A subspace K of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose that K is a compact subspace of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, K is closed in $\mathbb{R}^n$ by Corollary 2.11.10. Let $B_d(0,n) = \{x \in \mathbb{R}^n : \|x\| < n\}$, where $\|x\| \in \mathbb{R}$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. Since $\{B_d(0,n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$, and hence of A, and since $B(0,n) \subseteq B(0,n+1)$, $\forall\, n \in \mathbb{N}$, by compactness of K we can find $n_0 \in \mathbb{N}$ such that $A \subseteq B(0,n)$. Therefore, K is bounded.

Conversely, suppose that K is closed and bounded in $\mathbb{R}^n$. Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Euclidean metric on $\mathbb{R}^n$ defined by

$$d(x,y) := \|x-y\| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Since K is bounded, there exists a real number $M > 0$ such that $d(x,y) < M$, $\forall\, x,y \in K$. Fix a point $x_0 \in K$, and let $\ell := d(x_0, \mathbf{0})$ be the Euclidean distance of x_0 from the origin $\mathbf{0}$ of $\mathbb{R}^n$. Then by triangle inequality, we have

$$d(x,0) \le d(x,x_0) + d(x_0,0) \le M + \ell.$$

Let $r := M + \ell > 0$. Since the closed interval $[-r,r] \subset \mathbb{R}$ is compact by Lemma 2.11.23, its n-fold product $[-r,r]^n \subset \mathbb{R}^n$ is compact by Theorem 2.9.17. Note that, $K \subseteq [-r,r]^n$. Since K is closed in $\mathbb{R}^n$, it is closed in the compact space $[-r,r]^n$. Therefore, K is compact by Proposition 2.11.8. □

Theorem 2.11.25 (Extreme value theorem)**.** *Let X be a compact topological space and Y an ordered set together with the order topology on it. Let $f : X \to Y$ be a continuous map. Then there exist $a, b \in X$ such that $f(a) \le f(x) \le f(b)$, $\forall\, x \in X$.*

Proof. Since X is compact and f is continuous, $A := f(X)$ is a compact subspace of Y. We claim that A has a largest element and a smallest element (i.e., there exist $M, m \in A$ such that $m \le a \le M$, $\forall\, a \in A$). Suppose on the contrary that A has no largest element. Then for each $a \in A$ there exists $a' \in A$ such that $a < a'$ so that $a \in (-\infty, a') := \{y \in Y : y < a'\} \subseteq Y$. Then the collection $\mathscr{F} = \{(-\infty, a) : a \in A\}$ is an open cover of A. Since A is compact, $\mathscr{F}$ has a finite subcollection $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$, which covers A. Let $a_m = \max\{a_1, \ldots, a_n\} \in A$. Then $a_m \notin \bigcup\limits_{j=1}^{n} (-\infty, a_j) = A$, which contradicts the fact that $a_m \in A$. Therefore, A must have a largest element, say $M \in A$. A similar argument shows that A has a smallest element, say $m \in A$. Then $m = f(a)$ and $M = f(b)$, for some $a, b \in X$, and that $f(a) \leq f(x) \leq f(b),\ \forall\, x \in X$. $\square$

Let (X, d) be a metric space. Let A be a non-empty subset of X. The *distance from $x \in X$ to A* is the real number

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

Proposition 2.11.26. *Let A be a non-empty subset of a metric space (X, d). Then the map $\phi_A : X \to \mathbb{R}$ defined by*

$$\phi_A(x) = d(x, A),\ \forall\, x \in X,$$

is continuous.

Proof. Given any $x, y \in X$, we have

$$\begin{aligned}
\phi_A(x) = d(x, A) &\leq d(x, a),\ \forall\, a \in A, \\
&\leq d(x, y) + d(y, a),\ \forall\, a \in A, \\
\Rightarrow\ d(x, A) - d(x, y) &\leq d(y, a),\ \forall\, a \in A, \\
\Rightarrow\ d(x, A) - d(x, y) &\leq d(y, A) \\
\Rightarrow\ d(x, A) - d(y, A) &\leq d(x, y).
\end{aligned}$$

Interchanging x and y, we have $d(y, A) - d(x, A) \leq d(x, y)$. Thus, ϕ_A is continuous. $\square$

A non-empty subset A of X is said to be *bounded* if there exists a real number $M > 0$ such that $d(a_1, a_2) \leq M,\ \forall\, a_1, a_2 \in A$. The *diameter* of a non-empty bounded subset A of X is the real number

$$\text{diam}(A) := \sup\{d(a_1, a_2) : a_1, a_2 \in A\}.$$

Definition 2.11.27. A *Lebesgue number* of an open cover $\mathscr{F}$ of a metric space (X, d) is a real number $\delta > 0$ such that given any non-empty subset A of X of diameter $\text{diam}(A) < \delta$, there exists an element $U \in \mathscr{F}$ such that $A \subseteq U$.

Lemma 2.11.28 (Lebesgue number lemma)**.** *If (X, d) is a compact metric space, every open cover of X has a Lebesgue number.*

Proof. Let $\mathscr{F} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. If $X \in \mathscr{F}$, then every positive real number is a Lebesgue number for $\mathscr{F}$. Assume that $X \notin \mathscr{F}$. Since X is compact, there is a finite subfamily $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\} \subseteq \mathscr{F}$ that covers X. Consider the map $f : X \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{n} \sum_{j=1}^{n} d(x, Z_j),\ \forall\, x \in X,$$

where $Z_j := X \setminus U_{\alpha_j},\ \forall\, j \in \{1, \ldots, n\}$. Since each of the maps $x \mapsto d(x, Z_j)$ is continuous by Proposition 2.11.26, it follows that f is continuous. We claim that $f(x) > 0$, for all $x \in X$. Let $x \in X$ be arbitrary. Since $X = \bigcup\limits_{j=1}^{n} U_{\alpha_j}$, we have $x \in U_{\alpha_i}$, for some $i \in \{1, \ldots, n\}$. Since U_{α_i} is open in (X, d), there exists a real number $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U_{\alpha_i}$. Then $d(x, Z_i) \geq \epsilon$, and hence $f(x) = \frac{1}{n}\sum_{j=1}^{n} d(x, Z_j) \geq \epsilon/n > 0$. Since f is continuous and X is compact, there exists $x_0 \in X$ such that $f(x_0) \leq f(x)$, $\forall\, x \in X$. We claim that $\delta := f(x_0) > 0$ is a required Lebesgue number of $\mathscr{F}$. Let $A \subseteq X$ be such that $\mathrm{diam}(A) < \delta$. Fix a point $a \in A$. Then $A \subseteq B_d(a, \delta)$. Clearly $\delta \leq f(a)$. Let $\ell \in \{1, \ldots, n\}$ be such that $d(a, Z_\ell) = \max\{d(a, Z_j) : 1 \leq j \leq n\}$. Then

$$\delta \leq f(a) = \frac{1}{n}\sum_{j=1}^{n} d(a, Z_j) \leq d(a, Z_\ell),$$

and hence $B_d(a, \delta) \subseteq X \setminus Z_\ell = U_{\alpha_\ell}$. This completes the proof. □

Definition 2.11.29. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be *uniformly continuous* if given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$d_Y(f(x_1), f(x_2)) < \epsilon, \ \text{ whenever } \ d_X(x_1, x_2) < \delta.$$

Note that a uniformly continuous map is continuous, but converse need not be true.

Theorem 2.11.30 (Uniform continuity theorem)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. If X is compact, then any continuous map $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous.*

Proof. Let $\epsilon > 0$ be given. Consider the open cover $\{B_{d_Y}(y, \epsilon/2) : y \in f(X)\}$ of $f(X) \subseteq Y$. Since f is continuous, $\mathscr{U} := \{f^{-1}(B_{d_Y}(y, \epsilon/2)) : y \in f(X)\}$ is an open cover of X. Since X is compact, there exists $y_1, \ldots, y_n \in f(X)$ such that $X = \bigcup_{j=1}^{n} f^{-1}(B_{d_Y}(y_j, \epsilon/2))$. Since X is compact, by Lebesgue number lemma 2.11.28 there exists a real number $\delta > 0$ such that given any subset A of X with $\mathrm{diam}(A) < \delta$, we have $A \subseteq f^{-1}(B_{d_Y}(y_{j_A}, \epsilon/2))$, for some $j_A \in \{1, \ldots, n\}$. Then given any $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, we can have $f(x_1), f(x_2) \in B_{d_Y}(y_{j_A}, \epsilon/2)$. Therefore, by triangle inequality we have

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq d_Y(f(x_1), y_{j_A}) + d_Y(f(x_2), y_{j_A}) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore, f is uniformly continuous. □

Definition 2.11.31. Let X be a topological space. A point $x \in X$ is said to be an *isolated point* of X if the singleton subset $\{x\}$ is an open in X.

Proposition 2.11.32. *Let X be a Hausdorff topological space. If $x \in X$ is not an isolated point of X, then for given any non-empty open subset U of X, there exists a non-empty open subset V of X such that $V \subseteq U$ and $x \notin \overline{V}$.*

Proof. If $x \notin U$, then U being non-empty, there exists a point $y \in U$ with $y \neq x$. If $x \in U$, then since X has no isolated points, there exists a point $y \in U$ with $y \neq x$. Since X is Hausdorff, there exists a pair of open neighbourhoods V_1 and V_2 of x and y, respectively, such that $V_1 \cap V_2 = \emptyset$. Note that, $W_1 := U \cap V_1$ and $W_2 := U \cap V_2$ are open neighbourhoods of x and y, respectively, in U such that $W_1 \cap W_2 = \emptyset$. Then $V := W_2$ is the required non-empty open subset of X such that $x \notin \overline{W_2}$. This completes the proof. □

Theorem 2.11.33. *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. We now show that there is no surjective map $f : \mathbb{N} \to X$, which would imply that X is uncountable. To see this, let $f : \mathbb{N} \to X$ be any map. Let $x_n = f(n)$, $\forall\, n \in \mathbb{N}$. Applying Proposition 2.11.32 to the non-empty open subset $U_1 = X$ and the point $x_1 = f(1)$, we can find a non-empty open subset $U_2 \subseteq U_1 = X$ such that $x_1 \notin \overline{U_2}$. Suppose that $n \geq 2$, and we have constructed non-empty open subsets $U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1$ such that $x_{n-1} \notin \overline{U_n}$. Again by Proposition 2.11.32 we can find a non-empty open subset $U_{n+1} \subseteq U_n$ such that $x_n \notin \overline{U_{n+1}}$. Thus we have a nested sequence of non-empty closed subsets

$$\overline{U_1} \supseteq \overline{U_2} \supseteq \overline{U_3} \supseteq \cdots$$

of X; clearly this has finite intersection property. Since X is compact, it follows from Theorem 2.11.19 that there is a point, say $x \in \bigcap\limits_{n \in \mathbb{N}} \overline{U_n}$. Since $x_n \notin \overline{U_{n+1}}$, for all $n \in \mathbb{N}$, we see that $x \neq x_n$, for all $n \in \mathbb{N}$. Therefore, the map $f : \mathbb{N} \to X$ cannot be surjective. □

Proposition 2.11.34. *Any interval in $\mathbb{R}$ having more than one points has no isolated points.*

Proof. Let I be an interval in $\mathbb{R}$ having at least two points. Let $a \in I$ be arbitrary. If possible suppose that the singleton subset $\{a\} \subseteq I$ is open in I. Then there exists an open subset V of $\mathbb{R}$ such that $\{a\} = V \cap I$. Then there exists a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq V$. Since I has at least two points, there exists $b \in I$ such that $b \neq a$. Then either $a < b$ or $b < a$. Assume that $a < b$. Then either $b < a + \delta$ or $a + \delta \leq b$. If $b < a + \delta$, then $b \in V \cap I = \{a\}$, which is not possible since $a \neq b$. Then $a + \delta \leq b$. Since $a < a + \frac{\delta}{2} < a + \delta \leq b$ and I is an interval, $a + \frac{\delta}{2} \in I$. Then $a + \frac{\delta}{2} \in (a - \delta, a + \delta) \cap I \subseteq V \cap I = \{a\}$, implies that $a + \frac{\delta}{2} = a$, which is not possible since $\delta > 0$. Similarly, if $b < a$ we get contradiction. Therefore, $\{a\}$ cannot be open in I, and hence I has no isolated points. □

Corollary 2.11.35. *Any interval in $\mathbb{R}$ having more than one point is uncountable. In particular, the real line $\mathbb{R}$ is uncountable.*

Proof. Let I be an interval in $\mathbb{R}$ having at least two points, say $a, b \in I$. Without loss of generality, we may assume that $a < b$. Then $[a, b] \subseteq I$. Since $[a, b] \subset \mathbb{R}$ is compact and Hausdorff and has no isolated points by Proposition 2.11.34, it follows from Theorem 2.11.33 that $[a, b]$ is uncountable. Then I is uncountable, and hence $\mathbb{R}$ is uncountable. □

Exercise 2.11.36. Show that a connected metric space having more than one point is uncountable.

Proof. Let (X, d) be a connected metric space with at least two points. Fix a point $a \in X$, and consider the map $f : X \to \mathbb{R}$ defined by

$$f(x) = d(a, x),\ \forall\, x \in X.$$

Note that f is continuous by Proposition 2.11.26. Since X is connected, $f(X)$ is a connected subspace of $\mathbb{R}$ by Lemma 2.9.14. Then $f(X)$ is an interval in $\mathbb{R}$ by Proposition 2.9.22. Let $b \in X$ be such that $a \neq b$. Since $f(a) = d(a,a) = 0$ and $f(b) = d(a,b) > 0$, the image set $f(X) \subset \mathbb{R}$ is an interval in $\mathbb{R}$ having more than one points, and hence is uncountable by Corollary 2.11.35. Then it follows that X is uncountable. □

Exercise 2.11.37. Let $f : X \to Y$ be a *closed map* (i.e., image of closed subsets are closed) such that $f^{-1}(y)$ is compact, for all $y \in Y$. If K is a compact subset of Y, show that $f^{-1}(K)$ is compact.

Proof. Let U be an open subset of X such that $f^{-1}(y) \subseteq U$, for some $y \in Y$. Since f is a closed map, $f(X \setminus U)$ is closed in Y. Since $f^{-1}(y) \subseteq U$, we see that $V_y := Y \setminus f(X \setminus U)$ is an open neighbourhood of y in Y. Since $(X \setminus U) \cap f^{-1}(V_y) = \emptyset$, we conclude that

$$f^{-1}(y) \subseteq f^{-1}(V_y) \subseteq U.$$

Let K be a compact subset of Y. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $f^{-1}(K)$ in X. For each $y \in K$, the fiber $f^{-1}(y)$ being compact, we can find a finite subset $\Lambda_y \subseteq \Lambda$ such that

$$f^{-1}(y) \subseteq U_y := \bigcup_{\alpha \in \Lambda_y} U_\alpha.$$

Then there exists an open neighbourhood $V_y \subseteq Y$ of y such that

$$f^{-1}(y) \subseteq f^{-1}(V_y) \subseteq U_y.$$

Then the collection $\{V_y : y \in K\}$ being an open cover of the compact subset $K \subseteq Y$, we can find $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup\limits_{i=1}^{n} V_{y_i}$. Then we have

$$f^{-1}(K) \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{y_i}) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda_{y_i}} U_\alpha$$

Since $\bigcup\limits_{i=1}^{n} \Lambda_{y_i}$ is a finite subset of Λ, we conclude that $f^{-1}(K)$ is compact. □

Exercise 2.11.38. Let $\mathbb{R}_K = (\mathbb{R}, \tau_K)$, the real line equipped with the K-topology, where $K = \{1/n : n \in \mathbb{N}\}$.

(i) Show that $[0,1]$ is not compact in $\mathbb{R}_K$.

(ii) Show that $\mathbb{R}_K$ is connected.

(iii) Show that $\mathbb{R}_K$ is not path-connected.

Proof. (i) For each $n \in \mathbb{N}$, fix a real number δ_n such that $\frac{1}{n+1} < \delta_n < \frac{1}{n}$. Set $\delta_0 = 1$. Then the collection

$$\mathcal{U} := \{(\delta_{n+1}, \delta_n) : n \in \mathbb{N} \cup \{0\}\} \cup ((-1,2) \setminus K)$$

is an open cover of $[0,1]$ in $\mathbb{R}_K$. Note that $\mathcal{U}$ has no finite subcover.

(ii) $\square$

2.11.1 Limit point compactness

Definition 2.11.39. A topological space X is said to be *limit point compact* if every infinite subset of X has a limit point in X.

Proposition 2.11.40. *A compact space is limit point compact.*

Proof. Let X be a compact topological space. Let K be an infinite subset of X. Suppose on the contrary that K has no limit points in X. Then K is closed in X. Since X is compact, K is also compact. Since K has no limit points in X, for each $x \in K$, there exists an open neighbourhood V_x of x in X such that $K \cap (V_x \setminus \{x\}) = \emptyset$. Then $V_x \cap K = \{x\}$. Since $\{V_x : x \in K\}$ is an open cover of K in X and K is compact, there are finitely many points $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} V_{x_j}$. Since $K \cap V_{x_j} = \{x_j\}$, $\forall\, j = 1, \ldots, n$, it follows that $K = \{x_1, \ldots, x_n\}$ is a finite set, which is a contradiction to our assumption that K is infinite. Therefore, K must have a limit point in X. $\square$

Example 2.11.41. Let Y be the two points space with only open subsets $\emptyset$ and Y itself, and let $X = \mathbb{N} \times Y$, where $\mathbb{N}$ is equipped with the subspace topology induced from $\mathbb{R}$. Let K be any infinite subset of $X = \mathbb{N} \times Y$. Given $(n, y) \in K$, note that any open neighbourhood U of (n, y) in X contains an open neighbourhood $V_n := \{n\} \times Y$, which intersects K. Therefore, if $(n, y_1) \in K$ and $y_2 \neq y_1$ in Y, then (n, y_2) is a limit points of K. Thus, X is limit point compact. However, the collection $\mathscr{C} = \{\{n\} \times Y : n \in \mathbb{N}\}$ is an open cover of X which has no finite subcover. Therefore, X is not compact.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. A *subsequence* of $(x_n)_{n \in \mathbb{N}}$ is a sequence $(x_{n_k})_{k \in \mathbb{N}}$ in X, where $(n_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers.

Definition 2.11.42. A topological space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

Example 2.11.43. A compact topological space need not be sequentially compact. For example, for each $\alpha \in [0, 1] \subset \mathbb{R}$ let $X_\alpha = [0, 1] \subset \mathbb{R}$, and consider the product space $X := \prod\limits_{\alpha \in [0,1]} X_\alpha$. Then X is compact by Tychonoff's theorem. As a set, X consists of all functions $f : [0, 1] \to [0, 1]$. For each $n \in \mathbb{N}$, consider the map $f_n : [0, 1] \to [0, 1]$ defined by sending $x \in [0, 1]$ to the n-th place digit x_n of the binary representation of x. Then $f_n \in X$, $\forall\, n \in \mathbb{N}$, and thus we have a sequence $(f_n)_{n \in \mathbb{N}}$ in X. Note that (f_n) has no convergent subsequence. Indeed, given any subsequence $(f_{n_k})_{k \in \mathbb{N}}$, let $x \in [0, 1]$ be the real number such that $x_k = 0$ if and only if k is even. Then $(f_{n_k}(x))_{k \in \mathbb{N}}$ is the sequence whose all even terms are 0 and odd terms are 1, and hence is not convergent. Therefore, the subsequence $(f_{n_k})_{k \in \mathbb{N}}$ cannot converges to a function $f : [0, 1] \to [0, 1]$ in the product space X. Therefore, X is not sequentially compact.

Exercise 2.11.44. Show that the set $[0, 1]^{\mathbb{N}}$ equipped with the box topology is not limit point compact, and hence is not compact.

Answer: Let $X = [0,1]^{\mathbb{N}}$ be equipped with the box topology. For each $n \in \mathbb{N}$, let $f_n \in [0,1]^{\mathbb{N}}$ be defined by

$$f_n(m) = \begin{cases} 1, & \text{if} \quad m = n, \\ 0, & \text{if} \quad m \neq n. \end{cases}$$

Note that $f_n \neq f_m$, for $m \neq n$. Therefore, $A = \{f_n : n \in \mathbb{N}\} \subset X$ is an infinite subset of X. We claim that A has no limit point in X. Indeed, for each $n \in \mathbb{N}$, let $U_n := \prod\limits_{k \in \mathbb{N}} V_k$, where

$$V_k = \begin{cases} [0,1/2), & \text{if} \quad k \neq n, \\ (1/2,1], & \text{if} \quad k = n. \end{cases}$$

Then U_n is an open neighbourhood of f_n in the box topology on X. Clearly, $f_m \notin U_n$ for $m \neq n$. Therefore, A is a discrete subspace of X. If $f \in X \setminus A$, then we have the following cases:

Case 1: $f(n) \in (0,1)$, for all $n \in \mathbb{N}$. Then take $U = \prod\limits_{k \in \mathbb{N}} V_k$, where $V_1 = (0,1)$ and $V_k = [0,1]$, for all $k \geq 2$. Then U is an open neighbourhood of f (even in the product topology) in X, but $U \cap A = \emptyset$.

Case 2: There exists $m, n \in \mathbb{N}$ with $m \neq n$ such that $f(m) = f(n) = 1$. Take $V_m = V_n = (1/2, 1]$ and $V_k = [0,1]$, for all $k \in \mathbb{N} \setminus \{m,n\}$. Then $U = \prod\limits_{k \in \mathbb{N}} V_k$ is an open neighbourhood of f (even in the product topology) in X, but $U \cap A = \emptyset$.

Case 3: $f(n) = 0$, for all $n \in \mathbb{N}$. Take $V_k = [0,1/2)$, $\forall\, k \in \mathbb{N}$. Then $U = \prod\limits_{k \in \mathbb{N}} V_k$ is an open neighbourhood of f (in the box topology) in X, but $U \cap A = \emptyset$.

Therefore, no points of $X \setminus A$ can be a limit point of A, and hence A is closed in X. Therefore, A has no limit point in X, since A is discrete. Therefore, X is not limit point compact, and hence is not compact. □

Theorem 2.11.45. *Let X be a topological space. Consider the following statements:*

(i) X is compact.

(ii) X is limit point compact.

(iii) X is sequentially compact.

Then (i) $\Rightarrow$ (ii) always holds. If X is first countable, then (ii) $\Rightarrow$ (iii) holds. If X is metrizable, then (iii) $\Rightarrow$ (i) holds. In particular, all these statements are equivalent for metrizable topological spaces.

Proof. (i) $\Rightarrow$ (ii): Proved in Proposition 2.11.40.

(ii) $\Rightarrow$ (iii): Let X be limit point compact and first countable. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. Consider the set $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then there exists $x \in A$ such that $x_n = x$, for infinitely many $n \in \mathbb{N}$. This gives a constant subsequence of $(x_n)_{n \in \mathbb{N}}$ which is clearly convergent. If A is infinite, then X being limit point compact, there is a limit point, say $a \in X$ of A. Since X is first countable by assumption, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ that converges to a (see Sequence Lemma 2.3.11).

(iii) $\Rightarrow$ (i): Suppose that X is sequentially compact and metrizable. We use the following two results to show that X is compact.

Lemma 2.11.46. *If (X, d) is a sequentially compact metric space, then every open cover of X has a positive Lebesgue number.*

Proof. Suppose on the contrary that there exists an open cover $\mathscr{F} = \{V_\alpha : \alpha \in \Lambda\}$ of X that has no positive Lebesgue number. Then for each $n \in \mathbb{N}$ there exists a subset $A_n \subseteq X$ of diameter $\text{diam}(A_n) < 1/n$ such that A_n is not contained in any of the member of $\mathscr{F}$. Choose an element $x_n \in A_n$, for each $n \in \mathbb{N}$. Since X is sequentially compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$, converging to $a \in X$. Then $a \in V_\alpha$, for some $\alpha \in \Lambda$. Since V_α is open in X, there exists a real number $\epsilon > 0$ such that $B_d(a, \epsilon) \subseteq V_\alpha$. Then there exists $k \in \mathbb{N}$ such that $\frac{1}{n_k} < \epsilon/2$ and that $d(a, x_{n_k}) < \epsilon/2$. Then we have

$$A_{n_k} \subseteq B_d(x_{n_k}, \epsilon/2) \subseteq V_\alpha.$$

This contradicts our assumption that $\mathscr{F}$ has no positive Lebesgue number. This completes the proof. $\square$

Lemma 2.11.47. *If (X, d) is a sequentially compact metric space, given a real number $\epsilon > 0$ there exists finitely many points $x_1, \ldots, x_n \in X$ such that $X = \bigcup\limits_{j=1}^{n} B_d(x_j, \epsilon)$.*

Proof. Suppose on the contrary that X cannot be covered by finitely many ϵ-balls. Start with a point $x_1 \in X$, and choose $x_2 \in X \setminus B_d(x_1, \epsilon)$ noting that $B_d(x_1, \epsilon)$ cannot cover X. Assume that $n \geq 2$, and $x_n \in X$ is chosen from $X \setminus \big(\bigcup\limits_{j=1}^{n-1} B_d(x_j, \epsilon)\big)$. Then choose $x_{n+1} \in X \setminus \big(\bigcup\limits_{j=1}^{n} B_d(x_j, \epsilon)\big)$. Thus we get a sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d). Since $d(x_i, x_n) \geq \epsilon$, for all $i \neq n$, we see that $(x_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence, which contradicts our assumption that (X, d) is sequentially compact. Therefore, X can be covered by finitely many ϵ-balls. $\square$

To complete the proof, we start with an open cover $\mathscr{F} = \{V_\alpha : \alpha \in \Lambda\}$ of X, and use Lemma 2.11.46 to find a positive Lebesgue number $\delta > 0$ for $\mathscr{F}$. Set $\epsilon = \frac{\delta}{3} > 0$, and use Lemma 2.11.47 to find a finite open cover $\{B_d(x_j, \epsilon) : x_1, \ldots, x_n \in X\}$ of X by ϵ-balls. Since $\text{diam}(B_d(x_j, \epsilon)) = 2\epsilon = 2\delta/3 < \delta$, by definition of Lebesgue number, each of $B_d(x_j, \epsilon)$ is contained in $V_{\alpha_j} \in \mathscr{F}$, for some $\alpha_j \in \Lambda$. Then $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\} \subseteq \mathscr{F}$ is a required finite subcover for X. Therefore, X is compact. $\square$

2.11.2 Local compactness

Definition 2.11.48. A topological space X is said to be *locally compact at $x \in X$* if there exists a compact subspace $K_x \subseteq X$ containing an open neighbourhood $V_x \subseteq X$ of x. If X is locally compact at each $x \in X$ then we say that X is locally compact.

Example 2.11.49. (i) Any compact topological space is locally compact.

(ii) $\mathbb{R}$ equipped with the discrete topology is locally compact, but not compact.

(iii) The real line $\mathbb{R}$ is locally compact, but not compact. Indeed, given a point $x \in \mathbb{R}$, we have a compact subspace $[x-1, x+1] \subset \mathbb{R}$ that contains an open neighbourhood $(x-1, x+1) \subset \mathbb{R}$ of x. Therefore, $\mathbb{R}$ is locally compact. Since the open cover $\{(n, n+2) : n \in \mathbb{Z}\}$ of $\mathbb{R}$ has no finite subcover for $\mathbb{R}$, it follows that $\mathbb{R}$ is not compact.

(iv) The Euclidean space $\mathbb{R}^n$ is locally compact. Indeed, given a point $a \in \mathbb{R}^n$, take $K = \overline{B(a,1)}$, the closed ball in $\mathbb{R}^n$ with center a and radius 1. This is compact and contains the open ball $B(a,1)$.

(v) Any non-empty open interval in $\mathbb{R}$ is locally compact.

Exercise 2.11.50. Is $\mathbb{R}_\ell$ locally compact?

Exercise 2.11.51. Let $f : X \to Y$ be a continuous surjective map of topological spaces.

(i) If X is locally compact, is Y necessarily locally compact?

(ii) What if f is open?

Answer: (i) No! Take $X = \mathbb{R}$ equipped with the discrete topology so that it is locally compact. Take $Y = \mathbb{R}_\ell$, which is not locally compact. Then the identity map $\mathrm{Id} : \mathbb{R}_{disc} \to \mathbb{R}_\ell$ is continuous and surjective.

(ii) Yes! Because, in this case given $f(x) \in f(X) = Y$, by local compactness of X we can find a compact subset $K \subseteq X$ containing an open neighbourhood $V \subseteq X$ of x, and then f being continuous and open, $f(K)$ is a compact subspace of Y containing the open neighbourhood $f(V)$ of $f(x) \in Y$. □

Exercise 2.11.52. Let τ_1 and τ_2 be the subspace topologies on

$$\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : t \geq 0\} \subset \mathbb{R}$$

induced from the Euclidean topology and the lower limit topology on $\mathbb{R}$, respectively. Which of the spaces $(\mathbb{R}_{\geq 0}, \tau_1)$ and $(\mathbb{R}_{\geq 0}, \tau_2)$ are locally compact?

Exercise 2.11.53. Let τ_1 and τ_2 be two topologies on a non-empty set X. Suppose that $\tau_1 \subseteq \tau_2$. Prove or give counter examples of the following.

(i) If (X, τ_2) is compact, so is (X, τ_1).

(ii) If (X, τ_2) is locally compact, so is (X, τ_1).

Proposition 2.11.54. *If X and Y are locally compact topological spaces, so is their product $X \times Y$.*

Proof. Let $(x, y) \in X \times Y$ be given. Since X and Y are locally compact, there exist compact subsets $K \subseteq X$ and $L \subseteq Y$ and open neighbourhoods $U \subseteq X$ and $V \subseteq Y$ of x and y, respectively, such that

$$(x, y) \in U \times V \subseteq K \times L.$$

Since $U \times V$ is open in $X \times Y$ and $K \times L$ is compact in $X \times Y$ (see Theorem 2.11.17), we conclude that $X \times Y$ is locally compact. □

Exercise 2.11.55. Show that any non-empty open subset of $\mathbb{R}^n$ is locally compact.

Proposition 2.11.56. *Let X be a Hausdorff topological space. Then X is locally compact if and only if for given any point $x \in X$, there exists an open neighbourhood V of x in X whose closure $\overline{V}$ in X is compact.*

Proof. Suppose that X is locally compact. Let $x \in X$ be given. Then there exists a compact subset K of X that contains an open neighbourhood, say V, of x. Since X is Hausdorff, K is closed in X by Corollary 2.11.10. Since $V \subseteq K$ and K is closed in X, we have $\overline{V} \subseteq K$. Then $\overline{V}$ is compact by Lemma 2.11.8. Converse part is trivial. □

Exercise 2.11.57. Show that $\mathbb{Q}$ is not locally compact. (*Hint:* Use Proposition 2.11.56).

Exercise 2.11.58. Show that $\mathbb{R}^{\mathbb{N}}$ is not locally compact in the product topology on it.

Exercise 2.11.59. Let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that the map $f : (0, 1) \to S^1$ defined by

$$f(t) = (\cos 2\pi t,\ \sin 2\pi t),\ \forall\, t \in \mathbb{R},$$

is a homeomorphism of $(0, 1)$ onto $S^1 \setminus \{(0, 0)\}$. Construct a homeomorphism $g : \mathbb{R} \to (0, 1)$, and precompose it with f to get a homeomorphism of $\mathbb{R}$ onto the subspace $S^1 \setminus \{(0, 0)\}$.

Example 2.11.60. Let X be the Euclidean space $\mathbb{R}^n$, where $n \geq 2$. Given a point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, for each $j \in \{1, \ldots, n\}$ there exists a compact subspace $[x_j - 1, x_j + 1] \subset \mathbb{R}$ that contains an open neighbourhood $(x_j - 1, x_j + 1)$ of x_j. Then $K = [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1] \subset \mathbb{R}^n$ is a compact subspace of $\mathbb{R}^n$ containing an open neighbourhood $(x_1 - 1, x_1 + 1) \times \cdot \times (x_n - 1, x_n + 1)$ of $\mathbf{x}$. Therefore, $\mathbb{R}^n$ is locally compact. However, $\mathbb{R}^n$ is not compact, because the open cover $\{B(0, n) : n \in \mathbb{N}\}$ of $\mathbb{R}^n$ has no finite subcover.

Exercise 2.11.61. Fix an integer $n \geq 2$. Consider the *unit n-sphere*

$$S^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^{n+1},$$

in $\mathbb{R}^{n+1}$. Show that S^n is compact and Hausdorff. Use stereographic projection map to construct a homeomorphism of $\mathbb{R}^n$ onto the punctured n-sphere $S^n \setminus \{*\}$, where $* \in S^n$ is a chosen point.

Theorem 2.11.62. *A topological space X is locally compact and Hausdorff if and only if there exists a topological space Y satisfying the following properties:*

(i) Y contains X as a subspace,

(ii) $Y \setminus X$ is singleton, and

(iii) Y is compact and Hausdorff.

If Y and Y' are two topological spaces satisfying the above three properties, then there exists a unique homeomorphism $f : Y \to Y'$ such that $f\big|_X = \mathrm{Id}_X$.

Proof. **Step 1 (Uniqueness):** Suppose that Y and Y' are two topological spaces satisfying the above three properties. Suppose that $Y \setminus X = \{*\}$ and $Y' \setminus X = \{*'\}$. Define a map $f : Y \to Y'$ by

$$f(y) = \begin{cases} y, & \text{if} \quad y \in X, \\ *', & \text{if} \quad y = *. \end{cases}$$

Clearly f is a bijective map such that $f\big|_X = \mathrm{Id}_X$. Since Y and Y' are T1 spaces by assumption, that X is open in both Y and Y'. Let $U \subseteq Y$ be an open subset. If $* \notin U$, then $U \subseteq X$, and hence $f(U) = U$ is open in Y'. If $* \in U$, then $Z := Y \setminus U \subseteq X$, and hence $f(Z) = Z$. Since Z is closed in the compact space Y, it is compact. But $Z \subseteq X$ and $X \subseteq Y'$. Therefore, Z is a compact subspace of the Hausdorff space Y', and hence is closed in Y'. Then $f(U) = f(Y \setminus Z) = Y \setminus f(Z) = Y' \setminus Z$ is open in Y'. Thus, f is an open map. Interchanging the roles of Y and Y' we see that f^{-1} is also open. Therefore, f is a homeomorphism.

Step 2 (Construction of Y): Let X be locally compact and Hausdorff. Let $*$ be any object that is not an element of X, and let $Y := X \cup \{*\}$. Let τ_X be the collection of all open subsets of X. Let

$$\tau_Y := \tau_X \cup \{Y \setminus K : K \text{ is a compact subspace of } X\}.$$

We show that τ_Y gives a topology on Y.

Let $U, V \in \tau_Y$ be arbitrary. We show that $U \cap V \in \tau_Y$.
Case 1: If $U, V \in \tau_X$, then $U \cap V \in \tau_X \subseteq \tau_Y$.
Case 2: If $U \in \tau_X$ and $V = Y \setminus K$, for some compact subspace K of X, then $U \cap V = U \cap (Y \setminus K) = U \cap (X \setminus K)$. Since K is a compact subspace of X and X is **Hausdorff**, K is closed in X, and hence $X \setminus K$ is open in X. Then $U \cap V = U \cap (X \setminus K) \in \tau_X \subseteq \tau_Y$.
Case 3: If $U = X \setminus K_1$ and $V = Y \setminus K_2$, for some compact subspaces K_1 and K_2 of X, then $U \cap V = Y \setminus (K_1 \cup K_2) \in \tau_Y$, because $K_1 \cup K_2$ is a compact subspace of X. Thus, τ_Y is closed under finite intersection.

Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an indexed family of objects from τ_Y. We show that $\bigcup\limits_{\alpha \in \Lambda} V_\alpha \in \tau_Y$.
Case 1: If $V_\alpha \in \tau_X$, $\forall\, \alpha \in \Lambda$, then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha \in \tau_X \subseteq \tau_Y$.
Case 2: If each V_α is of the form $V_\alpha = Y \setminus K_\alpha$, for some compact subspace K_α of X, then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = Y \setminus \left(\bigcap\limits_{\alpha \in \Lambda} K_\alpha \right)$. Since K_α is a compact subspace of the **Hausdorff** space X, it is closed in X, and hence $\bigcap\limits_{\alpha \in \Lambda} K_\alpha$ is closed in X. Since $\bigcap\limits_{\alpha \in \Lambda} K_\alpha$ is a closed subset of a compact space K_β, where $\beta \in \Lambda$, it is compact. Therefore, $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = Y \setminus \left(\bigcap\limits_{\alpha \in \Lambda} K_\alpha \right) \in \tau_Y$.
Case 3: Let $\Lambda = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$, and suppose that $V_\alpha \in \tau_X$, $\forall\, \alpha \in \Lambda_1$ and $V_\beta = Y \setminus K_\beta$, for all $\beta \in \Lambda_2$. Then $U := \bigcup\limits_{\alpha \in \Lambda_1} V_\alpha \in \tau_X$ by case 1, and $V := \bigcup\limits_{\beta \in \Lambda_2} V_\beta = Y \setminus K$, for some compact subspace K of X, as discussed in case 2. Then $\bigcup\limits_{\alpha \in \Lambda} V_\alpha = U \cup V \in \tau_Y$ by case 2. Therefore, τ_Y is closed under arbitrary union. Therefore, τ_Y is a topology on Y.

Step 3 (X is a subspace of Y): Let $V \in \tau_Y$ be arbitrary. If $V \in \tau_X$, then $V \cap X = V \in \tau_X$. If $V \notin \tau_X$, then $V = Y \setminus K$, for some compact subspace K of X. Then $V \cap X = (Y \setminus K) \cap X = (X \setminus K) \cap X \in \tau_X$, since K being a compact subspace of the Hausdorff space, K is closed in X, and hence $X \setminus K$ is open in X. Therefore, the subspace topology on X induced from Y coincides with the topology on X. Thus, X is a subspace of Y.

Step 4 (Y is compact): Let $\mathscr{U} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then there exists $\alpha_0 \in \Lambda$ such that $* \in V_{\alpha_0}$. Clearly $V_{\alpha_0} = Y \setminus K$, for some compact subspace K of X. Then the collection of all V_α's that does not contain $*$ forms an open cover of X, and hence of the compact subspace K of X. Then we can choose finitely many such objects, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$ from $\mathscr{U}$ that covers K. Then $\{V_{\alpha_0}, V_{\alpha_1}, \ldots, V_{\alpha_n}\} \subseteq \mathscr{U}$ is a finite subcover of Y. Therefore, Y is compact.

Step 4 (Y is Hausdorff): Let $x, y \in Y$ be two distinct points. If $x, y \in X$, then X being **Hausdorff** we can separate them by a pair of disjoint open neighbourhoods of them in X, and hence in Y. If $y = *$, then $x \in X$. Since X is **locally compact** we can find a compact subspace K of X containing an open neighbourhood $V \subseteq X$ of x. Then $U := Y \setminus K$ is an open neighbourhood of $*$ in Y disjoint from the open neighbourhood V of x. Therefore, Y is Hausdorff.

Step 5 (Converse part): Suppose that there is a compact Hausdorff topological space Y that contains X as a subspace of it and $Y \setminus X = \{*\}$ is singleton. Since Y is Hausdorff, it is a T1 space, and hence $Y \setminus X$ is closed in Y. Then X is open in Y. Clearly X is Hausdorff. Let $x \in X$ be arbitrary. Then Y being **Hausdorff**, there is a pair of disjoint open neighbourhoods U and V of x and $*$, respectively, in Y. Since $K := Y \setminus V$ is a closed subspace of the **compact** space Y, it is compact. Since $K \subseteq X$, K is a compact subspace of X. Clearly $x \in U \subseteq K$. Therefore, X is locally compact. This completes the proof. $\square$

Definition 2.11.63. Let X be a topological space that is not compact. A *compactification* of X is a compact topological space Y containing X as its subspace such that the closure of X in Y is Y itself. If Y is a compactification of X such that $Y \setminus X$ is a singleton subset of Y, then Y is called an *one-point compactification* of X.

Exercise 2.11.64. Find the one-point compactification of the closed interval $[a, b] \subseteq \mathbb{R}$, when it is equipped with the subspace topology induced from $\mathbb{R}$.

Exercise 2.11.65. Let $f : X_1 \to X_2$ be a homeomorphism of locally compact Hausdorff spaces. Let Y_1 and Y_2 be one-point compactifications of X_1 and X_2, respectively. Show that there is a unique homeomorphism $\widetilde{f} : Y_1 \to Y_2$ such that $\widetilde{f}|_{X_1} = f$.

Proof. Let $Y_1 \setminus X_1 = \{*_1\}$ and $Y_2 \setminus X_2 = \{*_2\}$. Define a map $\widetilde{f} : Y_1 \to Y_2$ by setting

$$\widetilde{f}(y) = \begin{cases} f(y), & \text{if} \quad y \in X_1, \\ *_2, & \text{if} \quad y = *_1. \end{cases}$$

Since f is bijective, so is $\widetilde{f}$. Clearly $\widetilde{f}$ is the unique bijective map from Y_1 onto Y_2 such that $\widetilde{f}|_{X_1} = f$. Let V be an open subset of Y_2. If $*_2 \notin V$, then $V \subseteq X_2$, and hence $\widetilde{f}^{-1}(V) = f^{-1}(V) \subseteq X_1$ is open in X_1, and hence is open in Y_1, since X_1 is open in Y_1. If $*_2 \in V$, then $V = Y_2 \setminus K$, for some compact subspace K of X_2. Then $\widetilde{f}^{-1}(V) = \widetilde{f}^{-1}(Y_2) \setminus f^{-1}(K) = Y_1 \setminus f^{-1}(K)$. Since f is a homeomorphism and K is compact in X_2, its inverse image $f^{-1}(K)$ is compact in X_1, and hence in Y_1. Therefore, $\widetilde{f}^{-1}(V)$ is open in Y_1. Therefore, $\widetilde{f}$ is continuous. By symmetry, the same argument applied to $\widetilde{f}^{-1}$ shows that $\widetilde{f}^{-1}$ is continuous. Therefore, $\widetilde{f}$ is a homeomorphism. $\square$

Exercise 2.11.66. Let X be a locally compact Hausdorff space with the one-point compactification $\widehat{X}$. Let Y be a locally compact Hausdorff space. Fix a point $y_0 \in Y$, and let $f : X \to Y$ be a continuous map such that $f^{-1}(K)$ is compact for all closed subset K of Y not containing y_0. Show that there is a unique continuous map $\widetilde{f} : \widehat{X} \to Y$ such that $\widetilde{f}\big|_X = f$.

Exercise 2.11.67. Show that one-point compactification of $\mathbb{R}$ is homeomorphic to the unit circle S^1 in $\mathbb{R}^2$.

Proof. Note that the maps $f : \mathbb{R} \to (-1,1)$ and $g : (-1,1) \to (0,1)$ defined by

$$f(s) = \frac{s}{1+|s|}, \ \forall\, s \in \mathbb{R},$$
$$\text{and } \ g(t) := \frac{1+t}{2}, \ \ \forall\, t \in (-1,1),$$

are homeomorphisms (verify!). Note that the map $f : (0,1) \to S^1$ by

$$h(u) = e^{2\pi i u}, \ \forall\, u \in (0,1),$$

is a homeomorphism onto its image in S^1. Therefore, the composite map

$$h \circ g \circ f : \mathbb{R} \to S^1$$

is a homeomorphism of $\mathbb{R}$ onto its image $S^1 \setminus \{(1,0)\} \subset S^1$. Moreover, S^1 is compact and Hausdorff. Therefore, S^1 is homeomorphic to the one-point compactification of $\mathbb{R}$. $\square$

Exercise 2.11.68. Find the one-point compactification of $\mathbb{R}^n \setminus \{0\}$. Show that it is path-connected, for all $n \geq 1$.

Exercise 2.11.69. Equip $\mathbb{N}$ with the subspace topology induced from $\mathbb{R}$. Show that one-point compactification of $\mathbb{N}$ is homeomorphic to $\{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$.

Proof. Define a map $f : \mathbb{N} \to X := \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ by sending $n \in \mathbb{N}$ to $f(n) := 1/n$. Clearly f is an injective map with $X \setminus f(\mathbb{N}) = \{0\}$. Since the topology on $\mathbb{N}$ is discrete, f is continuous. Since X is compact and Hausdorff, it follows that X is homeomorphic to one-point compactification of $\mathbb{N}$. $\square$

Exercise 2.11.70. Let H be a subgroup of a locally compact topological group G. Show that the quotient space $G/H = \{gH : g \in G\}$ (which need not be a group) is locally compact.

Proof. Let $\pi : G \to G/H$ be the quotient map. Since π is continuous, surjective and open by Exercise 2.8.9, it follows from Exercise 2.11.51 that G/H is locally compact. $\square$

We say that a topological space X has property P locally if for each $x \in X$, every open neighbourhood $U_x \subseteq X$ of x contains a neighbourhood V_x of x that has property P. For example, local connectedness, local path-connectedness etc. are local property. However, the definition of local compactness given in Definition 2.11.48 is not “local” in nature, in general. Nevertheless, the following proposition says that local compactness is indeed a local property for a Hausdorff space.

Proposition 2.11.71. *Let X be a Hausdorff topological space. Then X is locally compact at $x \in X$ if and only if given any open subset $U \subseteq X$ containing x, there exists an open neighbourhood $V \subseteq X$ of x such that $\overline{V}$ is a compact subspace of X and that $\overline{V} \subseteq U$.*

Proof. Let X be a Hausdorff space. Suppose that X is locally compact. Let Y be the one-point compactification of X, and let $Y \setminus X = \{*\}$. Let $x \in X$ be arbitrary. Let U be an open neighbourhood of x in X. Since Y is Hausdorff, $X = Y \setminus \{*\}$ is open in Y, and therefore, U is open in Y. Then $K := Y \setminus U$ is a closed subspace of the compact space Y, and hence is compact. Since Y is a Hausdorff space, there exist open subsets V and W of Y such that $x \in V$, $K \subseteq W$ and $V \cap W = \emptyset$. Let $\overline{V}$ be the closure of V in Y. Then $\overline{V} \cap K = \emptyset$, and hence $\overline{V} \subseteq U$. Since Y is compact and $\overline{V}$ is closed in Y, $\overline{V}$ is compact.

Conversely, suppose that for each $x \in X$ and an open neighbourhood U of x in X there exists an open neighbourhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq U$. Then $K = \overline{V}$ is a compact subspace of X containing the open neighbourhood V of x in X. Therefore, X is locally compact. □

Corollary 2.11.72. *An open or a closed subspace of a locally compact Hausdorff space is locally compact and Hausdorff.*

Proof. Let X be a locally compact space. Let A be a closed subspace of X. Then for given $a \in A$, there exists a compact subspace K of X containing an open neighbourhood $V \subseteq X$ of x. Then $A \cap V$ is an open neighbourhood of a in A contained in $A \cap K$. Since A is closed in X, $A \cap K$ is a closed subspace of the compact space K, and hence $A \cap K \subseteq A$ is compact. Thus, $A \cap V$ is an open neighbourhood of a in A contained in the compact subspace $A \cap K$ of A. Thus, A is locally compact. Note that, here we have not used Hausdorff property of X.

Assume that X is a locally compact Hausdorff space, and A is open in X. Let $a \in A$. Since A is an open neighbourhood of a in X, by Proposition 2.11.71 there exists an open neighbourhood V of a in X such that $\overline{V}$ is a compact subspace of X and $\overline{V} \subseteq A$. Then $K = \overline{V}$ is a required compact subspace of A containing an open neighbourhood V of a in A. Thus, A is locally compact. □

Corollary 2.11.73. *A topological space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.*

Proof. If X is locally compact and Hausdorff then we can take Y to be the one-point compactification of X. Then Y is a compact Hausdorff space that contains X as its open subspace.

Conversely, suppose that X is homeomorphic to an open subspace of a compact Hausdorff space Y. Then X is locally compact and Hausdorff by Corollary 2.11.72. □

Exercise 2.11.74. (i) Let $p : X \to Y$ be a quotient map of topological spaces. If Z is a locally compact Hausdorff space, show that the map

$$\pi := p \times \mathrm{Id}_Z : X \times Z \longrightarrow Y \times Z$$

is a quotient map (c.f. Exercise 2.6.15).

(ii) Let $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ be quotient maps of topological spaces. If Y_1 and X_2 are locally compact and Hausdorff, show that the product map $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient map.

Proof. (i) Since p is continuous and surjective, it follows from Corollary 2.4.11 that $\pi := p \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is continuous and surjective. Let $A \subseteq Y \times Z$ be such that $\pi^{-1}(A)$ is open in $X \times Z$. We show that A is open in $Y \times Z$. Assume that $A \neq \emptyset$. Let $(y, z) \in A$ be given. Since p is surjective, there exists $(x, z) \in X \times Z$ such that $\pi(x, z) = (p(x), z) = (y, z)$. Since $\pi^{-1}(A)$ is open in $X \times Z$, there exists open neighbourhoods $U_1 \subseteq X$ and $V_1 \subseteq Z$ of x and z, respectively, such that $(x, z) \in U_1 \times V_1 \subseteq \pi^{-1}(A)$. Now by Proposition 2.11.71 we can find an open neighbourhood V of z such that $\overline{V}$ is compact and $\overline{V} \subseteq V_1$. Since $\pi(U_1 \times V) = p(U_1) \times V \subseteq A$, we have $p^{-1}(p(U_1)) \times V \subseteq \pi^{-1}(A)$. For each $x \in p^{-1}(p(U_1))$, applying Tube Lemma 2.11.14 to the slice $\{x\} \times \overline{V}$ we can find an open neighbourhood $U_x \subseteq X$ of x and an open subset $V_x \subseteq Z$ such that

$$\{x\} \times \overline{V} \subseteq U_x \times V_x \subseteq \pi^{-1}(A).$$

Then $U_2 := \bigcup_{x \in p^{-1}(p(U_1))} U_x$ is an open subset of X containing $p^{-1}(p(U_1))$ such that

$$p^{-1}(p(U_1)) \times \overline{V} \subseteq U_2 \times \overline{V} \subseteq \pi^{-1}(A).$$

Iterating this step we can a sequence of open subsets $\{U_n\}_{n \in \mathbb{N}}$ of X such that

$$p^{-1}(p(U_{n-1})) \times \overline{V} \subseteq U_n \times \overline{V} \subseteq \pi^{-1}(A).$$

Then $U := \bigcup_{n \in \mathbb{N}} U_n$ is an open subset of X such that

$$(x, z) \in U \times \overline{V} \subseteq \pi^{-1}(A).$$

Since $\pi^{-1}(\pi(U \times V)) = p^{-1}(p(U)) \times V = U \times V$, it follows that $\pi(U \times V)$ is an open neighbourhood of (y, z) contained in A. Therefore, A is open in $Y \times Z$.

(ii) Since the following diagram commutes,

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r, "p_1 \times \mathrm{Id}_{X_2}"] \arrow[dr, "p_1 \times p_2"'] & Y_1 \times X_2 \arrow[d, "\mathrm{Id}_{Y_1} \times p_2"] \\
& X_2 \times Y_2,
\end{tikzcd}$$

the result follows from part (i) and Exercise 2.6.14. □

2.11.3 Net & Tychonoff's Theorem

A *directed set* is a partially ordered set $(I, \leq)$ such that given any two elements $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.11.75. A *net* in X is a map $f : I \to X$, where $(I, \leq)$ is a directed set. Like a sequence, we usually denote by x_α the image $f(\alpha)$ of $\alpha \in I$, and express a net as $(x_\alpha)_{\alpha \in I}$.

A net $(x_\alpha)_{\alpha \in I}$ in X is said to converge to a point $x \in X$, written as $(x_\alpha)_{\alpha \in I} \longrightarrow x$, if for given an open neighbourhood $U \subseteq X$ of x there exists an element $\alpha_U \in I$ such that $x_\beta \in U$, for all $\beta \in I$ satisfying $\alpha_U \leq \beta$.

Remark 2.11.76. Clearly, any sequence is a net but not the other way around. If $(I, \leq)$ is equal to $(\mathbb{N}, \leq)$, then the notion of net and its convergence coincides with the notion of a sequence and its convergence.

Exercise 2.11.77. Let X and Y be two topological spaces. Let $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ be two nets in X and Y, respectively, indexed by the same directed set $(I, \leq)$. If $(x_\alpha)_{\alpha \in I}$ converges to x in X and $(y_\alpha)_{\alpha \in I}$ converges to y in Y, show that the net $((x_\alpha, y_\alpha))_{\alpha \in I}$ converges to (x, y) in the product space $X \times Y$.

Proposition 2.11.78. *In a Hausdorff space X a net $(x_\alpha)_{\alpha \in I}$ can converge to at most one point of X.*

Proof. Suppose on the contrary that $(x_\alpha)_{\alpha \in I}$ converge to x and y in X, where $x \neq y$. Since X is Hausdorff, there exists a pair of open neighbourhoods V_x and V_y of x and y, respectively, such that $V_x \cap V_y = \emptyset$. Then by definition of convergence of a net in X, there exist $\alpha_0, \beta_0 \in I$ such that

$$\alpha_0 \leq \alpha \implies x_\alpha \in V_x, \quad \text{and} \quad \beta_0 \leq \beta \implies x_\beta \in V_y.$$

Since $(I, \leq)$ is a directed set, there exists $\gamma \in I$ such that $\alpha_0 \leq \gamma$ and $\beta_0 \leq \gamma$. Then $x_\gamma \in V_x \cap V_y$. But this is not possible, since $V_x \cap V_y = \emptyset$. $\square$

Theorem 2.11.79. *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if there exists a net $(x_\alpha)_{\alpha \in I}$ of points of A converging to x.*

Proof. Suppose that there is a net $(x_\alpha)_{\alpha \in I}$ of points of A that converges to $x \in X$. Let $U \subseteq X$ be an open neighbourhood of x. Since $(x_\alpha)_{\alpha \in I} \longrightarrow x$ in X, there exists $\alpha \in I$ such that $x_\alpha \in U$, and hence $A \cap U \neq \emptyset$. Thus, $x \in \overline{A}$.

Conversely, suppose that $x \in \overline{A}$. If $x \in A$, then we can take the constant sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n = x$, $\forall\, n \in \mathbb{N}$, that clearly converges to x in X. Suppose that $x \in \overline{A} \setminus A$. Let

$$I_x := \{V : V \ \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Given $V_1, V_2 \in I_x$, we define $V_1 \leq V_2$ if $V_2 \subseteq V_1$. Then $(I_x, \leq)$ is a partially ordered set. Given $V_1, V_2 \in I_x$, $V := V_1 \cap V_2 \in I_x$ and it satisfies $V_1 \leq V$ and $V_2 \leq V$. Therefore, $(I_x, \leq)$ is a directed set. Since $x \in \overline{A}$, for each open neighbourhood $V \in I_x$ of x in X, we can choose an element $x_V \in A \cap V$. Thus we have a net $(x_V)_{V \in I_x}$ of points of A. Given any open neighbourhood U of x in X, $U \in I_x$. Then if $V \in I_x$ with $U \leq V$, then $V \subseteq U$ and hence $x_V \in V$ implies $x_V \in U$. Thus the net $(x_V)_{V \in I_x}$ converges to x in X. $\square$

Corollary 2.11.80. *Let $A \subseteq X$. Then A is closed in X if and only if limit point of every convergent net of points of A is in A.*

Definition 2.11.81. Let $(I, \leq)$ be a directed set. A subset $J \subseteq I$ is said to be *cofinal* in $(I, \leq)$ if for each $i \in I$, there exists $j \in J$ such that $i \leq j$.

Proposition 2.11.82. *If J is a cofinal subset of $(I, \leq)$, then the partial order relation induced from $(I, \leq)$ makes $(J, \leq)$ a directed set.*

Proof. Clearly $(J, \leq)$ is a partially ordered set. Given $j_1, j_2 \in J$, there exists $i \in I$ such that $j_1 \leq i$ and $j_2 \leq i$. Since J is cofinal in $(I, \leq)$, there exists $j \in J$ such that $i \leq j$. Then by transitivity of the partial order relation we have $j_1 \leq j$ and $j_2 \leq j$. $\square$

Definition 2.11.83. Let $(I, \leq)$ be a directed set, and let $f : I \to X$ be a net in X. A *subnet* of points of X is a composite map $f \circ g : J \to X$, where $(J, \leq)$ is a directed set and $g : J \to I$ is a map satisfying the following properties:

(i) $\alpha \leq \beta$ in $(J, \leq)$ implies that $g(\alpha) \leq g(\beta)$ in $(I, \leq)$, and

(ii) the subset $g(J) = \{g(\alpha) : \alpha \in J\} \subseteq I$ is cofinal in $(I, \leq)$.

Proposition 2.11.84. *If a net $(x_\alpha)_{\alpha \in I}$ in X converges to $x \in X$, so is any of its subnet.*

Proof. Let $f : I \to X$ be a net in X indexed by a directed set $(I, \leq)$. Let $f \circ g$ be a subnet of f, where $g : J \to I$ is a map of directed sets satisfying the conditions (i) and (ii) as in Definition 2.11.83. Let U be an open neighbourhood of x in X. Since f converges to x, there exists $\alpha \in I$ such that $f(\beta) \in U$, for all $\beta \in I$ satisfying $\alpha \leq \beta$. Since $g(J)$ is cofinal in $(I, \leq)$, there exists $j \in J$ such that $\alpha \leq g(j)$. Then for any $k \in J$ with $j \leq k$ we have $\alpha \leq g(j) \leq g(k)$. Then we have $(f \circ g)(k)) \in U$, for all $k \in J$ satisfying $j \leq k$. Therefore, the subnet $f \circ g$ converges to x. $\square$

Definition 2.11.85. Let $(x_\alpha)_{\alpha \in I}$ be a net in X. A point $x \in X$ is said to be an *accumulation point* (or, *cluster point*) of $(x_\alpha)_{\alpha \in I}$ if for given any open neighbourhood $U \subseteq X$ of x the subset $I_U := \{\alpha \in I : x_\alpha \in U\}$ is cofinal in $(I, \leq)$.

Lemma 2.11.86. *A point $x \in X$ is an accumulation point of a net $(x_\alpha)_{\alpha \in I}$ in X if and only if there exists a subnet of $(x_\alpha)_{\alpha \in I}$ converging to x.*

Proof. Let $x \in X$ be an accumulation point of a net $f : (I, \leq) \to X$ in X. For notational simplicity, we write f as $(x_\alpha)_{\alpha \in I}$, where $x_\alpha := f(\alpha)$, $\forall\, \alpha \in I$. Then for given any open neighbourhood U of x in X, the subset

$$I_U := \{\alpha \in I : x_\alpha \in U\}$$

is cofinal in $(I, \leq)$. Let

$$K := \{(\alpha, U) : U \text{ is an open neighbourhood of } x \text{ and } \alpha \in I_U\}.$$

Given $(\alpha, U), (\beta, V) \in K$, we define

$$(\alpha, U) \leq (\beta, V) \ \text{ if } \alpha \leq \beta \text{ in } (I, \leq) \text{ and } V \subseteq U.$$

Clearly $(K, \leq)$ is a partially ordered set. Since $(I, \leq)$ is directed,for given $(\alpha, U), (\beta, V) \in K$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Let $W := U \cap V$. Then W is an open neighbourhood of x in X with $W \subseteq U$ and $W \subseteq V$. Since $I_W := \{\delta \in I : x_\delta \in W\}$ is cofinal in $(I, \leq)$, there exists $\delta \in I_W$ such that $\gamma \leq \delta$. Then $(\delta, W) \in K$ and it satisfies

$$(\alpha, U) \leq (\delta, W) \quad \text{and} \quad (\beta, V) \leq (\delta, W).$$

Therefore, K is a directed set. Define a map $g : K \to I$ by

$$g((\alpha, U)) = \alpha, \;\; \forall\, (\alpha, U) \in K.$$

Then we have

(i) $g((\alpha, U)) \leq g((\beta, V))$ if $(\alpha, U) \leq (\beta, V)$ in $(K, \leq)$, and

(ii) $g(K) = I$ is cofinal in $(I, \leq)$.

Therefore, $f \circ g : K \to X$ is a subnet of the net $f : I \to X$. We claim that $f \circ g$ converges to x in X. Given any open neighbourhood U of x in X, there exists $\alpha \in I_U := \{\alpha \in I : x_\alpha \in U\}$ such that $(\alpha, U) \in K$. If $(\beta, V) \in K$ satisfies $(\alpha, U) \leq (a \in I : x_\alpha \in U\}$ such that $(\alpha, U) \in K$. If $(\beta, V) \in K$ satisfies $(\alpha, U) \leq (\beta, V)$ in K, then $g(\beta, V) = f(\beta) \in V \subseteq U$. Therefore, the subnet $\left(x_{g(\alpha,U)}\right)_{(\alpha,U)\in K}$ converges to x in X.

Conversely, suppose that $(x_\alpha)_{\alpha \in I}$ has a subnet $(x_{g(\beta)})_{\beta \in K}$ that converges to x in X. Let U be any open neighbourhood of x in X. Then there exists $\beta_U \in K$ such that $x_{g(\beta)} \in U$, $\forall\, \beta \geq \beta_U$ in K. Since $g(K) \subseteq I$ is cofinal in $(I, \leq)$, for given any $\alpha \in I$ there exists $\beta_\alpha \in K$ such that $g(\beta_\alpha) \geq \alpha$. Since $(K, \leq)$ is directed, there exists $\gamma \in K$ such that $\beta_\alpha \leq \gamma$ and $\beta_U \leq \gamma$. Then $g(\gamma) \geq g(\beta_\alpha) \geq \alpha$ in $(I, \leq)$ and that $x_{g(\gamma)} \in U$. Therefore, the subset $I_U := \{\alpha \in I : x_\alpha \in U\}$ is cofinal in $(I, \leq)$, and hence x is an accumulation point of $(x_\alpha)_{\alpha \in I}$ in X. □

Theorem 2.11.87. *A topological space X is compact if and only if every net in X has a convergent subnet.*

Proof. Let X be a compact topological space. Let $(I, \leq)$ be a directed set and let $f : I \to X$ be a net in X, which we write as $(x_\alpha)_{\alpha \in I}$, for notational simplicity. For each $\alpha \in I$, consider the subset

$$F_\alpha := \{x_\beta : \beta \in I \text{ with } \alpha \leq \beta\} \subseteq X.$$

Since $(I, \leq)$ is a directed set, given a finite number of points $\alpha_1, \ldots, \alpha_n \in I$, there exists $\alpha \in I$ such that $\alpha_i \leq \alpha$, $\forall\, i \in \{1, \ldots, n\}$. Then $x_\alpha \in \bigcap\limits_{j=1}^{n} F_{\alpha_j}$, and hence $\bigcap\limits_{j=1}^{n} \overline{F_{\alpha_j}} \neq \emptyset$, where $\overline{F_\alpha}$ denotes the closure of F_α in X. Thus the collection $\mathscr{F} := \{\overline{F_\alpha} : \alpha \in I\}$ of closed subsets of X satisfies finite intersection property. Since X is compact, there exists a point $x \in \bigcap\limits_{\alpha \in J} \overline{F_\alpha}$. In view of Lemma 2.11.86 it suffices to show that x is an accumulation point of the net $(x_\alpha)_{\alpha \in I}$. Let U be an open neighbourhood of x in X. We need to show that the subset

$$I_U := \{\gamma \in I : x_\gamma \in U\}$$

is cofinal in $(I, \leq)$. For this let $\alpha \in I$ be given. Since $x \in \overline{F_\alpha}$, we can choose $x_\beta \in U \cap F_\alpha$, for some $\beta \geq \alpha$ in $(I, \leq)$. Then $\beta \in I_U := \{\gamma \in I : x_\gamma \in U\}$, and hence the subset I_U is cofinal in $(I, \leq)$.

Conversely, suppose that every net in X has a convergent subnet. Let $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ be an indexed family of non-empty closed subsets of X satisfying finite intersection property. Let $\mathscr{G}$ be the set of all possible finite intersections of members from $\mathscr{F}$. Then all members of $\mathscr{G}$ are non-empty by assumption on $\mathscr{F}$. Therefore, given $Z \in \mathscr{G}$ we can choose an element $x_V \in Z$. This defines a map $f : \mathscr{G} \to X$. Given $Z_1, Z_2 \in \mathscr{G}$, we say that

$$Z_1 \leq Z_2 \text{ if } Z_2 \subseteq Z_1.$$

Clearly $(\mathscr{G}, \leq)$ is a partially ordered set. Given $Z_1, Z_2 \in \mathscr{G}$, the subset $Z_3 := Z_1 \cap Z_2 \in \mathscr{G}$ and satisfies $Z_1 \leq Z_3$ and $Z_2 \leq Z_3$. Therefore, $(\mathscr{G}, \leq)$ is a directed set, and hence $(x_Z)_{Z \in \mathscr{G}}$ is a net in X. Then by assumption, it has a convergent subnet, which produces an accumulation point, say $x \in X$, of the net $(x_Z)_{Z \in \mathscr{G}}$. We show that $x \in \bigcap\limits_{\alpha \in \Lambda} F_\alpha$. Let U be an open neighbourhood of x in X. Since x is an accumulation point of the net $(x_Z)_{Z \in \mathscr{G}}$, the subset

$$\mathscr{G}_U := \{Z \in \mathscr{G} : x_Z \in U\}$$

is cofinal in $(\mathscr{G}, \leq)$. Since $\mathscr{F} \subseteq \mathscr{G}$, for given $\alpha \in \Lambda$ there exists $Z \in \mathscr{G}$ such that $F_\alpha \leq Z$ in $(\mathscr{G}, \leq)$ and that $x_Z \in U$. Since $x_Z \in Z$ and $Z \subseteq F_\alpha$ by construction, we have $x_Z \in U \cap F_\alpha$. Therefore, $x \in \overline{F_\alpha} = F_\alpha$. Since this holds for all $\alpha \in \Lambda$, we see that $x \in \bigcap\limits_{\alpha \in \Lambda} F_\alpha$. In view of Theorem 2.11.19 this completes the proof. □

Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and let $X := \prod\limits_{\alpha \in \Lambda} X_\alpha$ be the associated product topological space. Given a subset $\Gamma \subseteq \Lambda$, it follows from the universal property of product that the indexed family of continuous maps $\{\pi_\gamma : X \to X_\gamma\}_{\gamma \in \Gamma}$ gives rise to a unique continuous map

$$\pi_\Gamma : X \to \prod_{\gamma \in \Gamma} X_\gamma$$

such that $p_\gamma \circ \pi_\Gamma = \pi_\gamma$, where $p_\gamma : \prod\limits_{\delta \in \Gamma} X_\delta \to X_\gamma$ is the projection map onto the γ-th factor, for all $\gamma \in \Gamma$.

By a *partially defined element* f of the product space $X = \prod\limits_{\alpha \in \Lambda} X_\alpha$ we mean an element $f \in \prod\limits_{\gamma \in \Gamma} X_\gamma$, where $\Gamma \subseteq \Lambda$; in this case, we say Γ the *domain of definition* of f, and we express it symbolically by $\mathscr{D}(f) = \Gamma$. Note that, given a subset $\Gamma \subseteq \Lambda$, an element $f \in X$ defines a partially defined element

$$f\big|_\Gamma := \pi_\Gamma \circ f,$$

of X with domain Γ, which we may call the *restriction of f on* Γ (note the abuse of notation and terminology). However, given a partially defined element of X, in general, there is no unique choice of $g \in X$ whose restriction over Γ is f.

Let $(f_i)_{i \in I}$ be a net in $X = \prod\limits_{\alpha \in \Lambda} X_\alpha$. A partially defined element f of X with domain $\Gamma \subseteq \Lambda$ is said to be a *partial accumulation point* of the net $(f_i)_{i\in I}$ in X if f is an accumulation point of the *restricted net* $(f_i|_\Gamma)_{i\in I}$ in $\prod_{\gamma\in\Gamma} X_\gamma$.

Corollary 2.11.88 (Tychonoff's Theorem)**.** *[Chernoff] Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of compact topological spaces. Then the product $X := \prod_{\alpha\in\Lambda} X_\alpha$ is compact in the product topology.*

Proof. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be equipped with the product topology. Let $(I, \leq)$ be a directed set and

$$\phi : (I, \leq) \longrightarrow X$$

a net in X. For notational simplicity, we sometimes write it as $(f_i)_{i\in I}$, where $f_i := \phi(i)$, $\forall\, i \in I$. We use Zorn's lemma to show that $(f_i)_{i\in I}$ has a convergent subnet.

Let $\mathscr{P}$ be the set of all partial accumulation points of the net $(f_i)_{i\in I}$. Since each X_α is compact, it follows from Theorem 2.11.87 that the restricted net $\left(f_i|_{\{\alpha\}}\right)_{i\in I}$ on $\{\alpha\} \subseteq \Lambda$ has an accumulation point in X_α, and hence the set $\mathscr{P}$ is non-empty. Given $f, g \in \mathscr{P}$, we define

$$f \leq g, \text{ if } \mathscr{D}(f) \subseteq \mathscr{D}(g) \text{ and } g|_{\mathscr{D}(f)} = f. \tag{2.11.89}$$

Clearly this is a partial order relation on $\mathscr{P}$.

Let $\mathscr{C} = \{f_t : t \in T\}$ be a totally ordered subset of $(\mathscr{P}, \leq)$. Then any two members of $\mathscr{C}$ agree on the common parts of their domains. Let $\mathcal{D} := \bigcup_{t\in T} \mathscr{D}(f_t) \subseteq \Lambda$. Define a map $f : \mathcal{D} \to \coprod_{\alpha\in\mathcal{D}} X_\alpha$ by sending $\alpha \in \mathcal{D}$ to $f_t(\alpha)$ if $\alpha \in \mathscr{D}(f_t)$. If $\alpha \in \mathscr{D}(f_t) \cap \mathscr{D}(f_{t'})$, for some $t, t' \in T$, then the set $\mathscr{C}$ being totally ordered, either $f_t \leq f_{t'}$ or $f_{t'} \leq f_t$, and hence $f_t(\alpha) = f_{t'}(\alpha)$ by definition of the partial order relation on $\mathscr{P}$ given in (2.11.89). Therefore, f is a well-defined element of $\prod_{\alpha\in\mathcal{D}} X_\alpha$, which we write symbolically as

$$f := \bigcup_{t\in T} f_t.$$

Clearly f is a partially defined element of X. We show that $f \in \mathscr{P}$, i.e., a partial accumulation point of X. To see this, note that any basic open neighbourhood U of f in the product topological space $\prod_{\alpha\in\mathcal{D}} X_\alpha$ is ***finitely supported*** (i.e., $U = \prod_{\alpha\in\mathcal{D}} U_\alpha$, where U_α is an open subset of X_α and $U_\alpha \neq X_\alpha$, for all α in a finite subset $Supp(U) \subseteq \mathcal{D}$). Since $\mathscr{C} = \{f_t : t \in T\}$ is a totally ordered subset of $\mathscr{P}$ and $f = \bigcup_{t\in T} f_t$, it follows that $Supp(U) \subseteq \mathscr{D}(f_{t_U})$, for some $t_U \in T$. Since f_{t_U} is a partial accumulation point of the net $(f_i)_{i\in I}$ and that $f_{t_U}|_{Supp(U)} = f|_{Supp(U)}$, it follows that f is a partial accumulation point of the net $(f_i)_{i\in I}$, i.e., $f \in \mathscr{P}$. Clearly f is an upper bound of $\mathscr{C}$ in $\mathscr{P}$. Therefore, by Zorn's lemma $\mathscr{P}$ contains a maximal element, say g.

To complete the proof, it suffices to show that $\mathscr{D}(g) = \Lambda$. Suppose on the contrary that $\mathscr{D}(g) \neq \Lambda$. Choose an index $\alpha \in \Lambda \setminus \mathscr{D}(g)$. Since g is a cluster point of the restricted net

$\left(f_i|_{\mathscr{D}(g)}\right)_{i\in I}$ in $\prod\limits_{\alpha\in\mathscr{D}(g)} X_\alpha$, that g is a limit point of a subnet, say

$$\phi'\circ\psi : (J,\leq) \xrightarrow{\quad\psi\quad} (I,\leq) \xrightarrow{\phi':=\phi|_{\mathscr{D}(g)}} \prod_{\alpha\in\mathscr{D}(g)} X_\alpha,$$

of the restricted net $(f_i|_{\mathscr{D}(g)})_{i\in I}$. Since X_α is a non-empty and compact, the net

$$\phi\circ\psi|_{\{\alpha\}} := \pi_\alpha(\phi\circ\psi)$$

in X_α (note the abuse of notation, which possibly reduces confusion a bit!) has a cluster point, say $x_\alpha\in X_\alpha$. Define a map

$$h:\mathscr{D}(g)\cup\{\alpha\} \longrightarrow \coprod_{\beta\in\mathscr{D}(g)\cup\{\alpha\}} X_\beta$$

by setting

$$h(\beta) = \begin{cases} g(\beta), & \text{if } \beta\in\mathscr{D}(g), \\ x_\alpha, & \text{if } \beta=\alpha. \end{cases}$$

Clearly $h\in\prod\limits_{\beta\in\mathscr{D}(g)\cup\{\alpha\}} X_\beta$ and it is a limit point of the net $(\phi\circ\psi)|_{\mathscr{D}(g)\cup\{\alpha\}}$, which is a subnet of

$$\phi|_{\mathscr{D}(g)\cup\{\alpha\}} = \left(f_i|_{\mathscr{D}(g)\cup\{\alpha\}}\right)_{i\in I}$$

in $\prod\limits_{\beta\in\mathscr{D}(g)\cup\{\alpha\}} X_\beta$. Therefore, $h\in\mathscr{P}$ and $g<h$, which contradicts maximality of g in $\mathscr{P}$. Therefore, we must have $\mathscr{D}(g)=\Lambda$, and hence g is a cluster point of the net $(f_i)_{i\in I}$ in the product space $X=\prod\limits_{\alpha\in\Lambda} X_\alpha$. This completes the proof. $\square$

2.12 Second countability and separability

Recall that a topological space X is first countable if for each $x\in X$ there is a countable collection $\mathscr{B}_x$ of open neighbourhoods of x in X such that given any open neighbourhood U of x in X, there exists $V\in\mathscr{B}_x$ such that $V\subseteq U$. A topological space X is said to be second countable if it has a countable basis for its topology.

Example 2.12.1. (i) A second countable space is first countable.

(ii) Any metric space is first countable.

(iii) The set $\mathbb{R}$, equipped with the discrete metric d, is not second countable.

(iv) The Euclidean space $\mathbb{R}^n$ is second countable, for all $n\in\mathbb{N}$.

(v) The space $\mathbb{R}_\ell$ is first countable, but not second countable.

Lemma 2.12.2. *(i) Subspace of a first countable space is first countable.*

(ii) Countable product of first countable spaces is first countable.

(iii) Subspace of a second countable space is first countable.

(iv) Countable product of second countable spaces is first countable.

Proof. (i) Let X be a first countable space, and let Y be a subspace of X. Given $y \in Y$, we have a countable local basis, say $\mathcal{B}_y$, of X at y. Then $\mathcal{B}'_y := \{V \cap Y : V \in \mathcal{B}_y\}$ is a countable collection of open neighbourhoods of y in Y. Given an open neighbourhood, say U of y in Y, we have $U = W \cap Y$, for some open neighbourhood W of y in X. Then there exists $V \in \mathcal{B}_y$ such that $V \subseteq W$. Then $V \cap Y \subseteq U$, and hence $\mathcal{B}'_y$ is a countable local basis for Y at y. Therefore, Y is first countable.

(ii) Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of first countable spaces, and let $X := \prod_{n\in\mathbb{N}} X_n$ be the associated product topological space. Let $x := (x_1, x_2, \ldots) \in X$ be given. For each $n \in \mathbb{N}$, let $\mathcal{B}_{x,n}$ be a countable local basis for X_n at $x_n \in X_n$. Let $\mathcal{B}_x$ be the set of all subsets of X of the form

$$\prod_{n\in\mathbb{N}} V_n,$$

where $V_n \in \mathcal{B}_{x,n}$, for all $n \in F$, for some finite subset F of $\mathbb{N}$, and $V_m = X_m$, for all $m \in \mathbb{N} \setminus F$. Given an open neighbourhood, say $\mathcal{O}$ of x in X, we can find a basic open subset, say $\prod_{n\in\mathbb{N}} U_n$, of x in X such that

$$x \in \prod_{n\in\mathbb{N}} U_n \subseteq \mathcal{O};$$

where U_n is open in X_n, for all $n \in \mathbb{N}$, and there is a finite subset, say $F \subset \mathbb{N}$, such that $U_n \neq X_n$, $\forall\, n \in F$. Since for each $n \in F$, we have $x_n \in U_n$, there exists $V_n \in \mathcal{B}_{x,n}$ such that $x_n \in V_n \subseteq U_n$. Then for each $m \in \mathbb{N} \setminus F$, setting $V_m = X_m$ we see that $V := \prod_{n\in\mathbb{N}} V_n \in \mathcal{B}_x$ and that $x \in V \subseteq \prod_{n\in\mathbb{N}} U_n \subseteq \mathcal{O}$. Therefore, $\mathcal{B}_x$ is a countable local basis for the product space X at x, and hence X is first countable.

Proofs of (iii) and (iv) are similar (in fact, simpler to write) to that of (i) and (ii). □

A subset A of X is said to be *dense* if $\overline{A} = X$. It is straight-forward to see that a subset $A \subseteq X$ is dense in X if and only if given any non-empty open subset U of X, we have $U \cap A \neq \emptyset$.

Definition 2.12.3. A topological space X is said to be

(i) a *Lindelöf space* if every open cover of X has a countable subcover.

(ii) *separable** if it has a countable dense subset.

Remark 2.12.4. Any compact topological space is Lindelöf, however the converse is not true. For example, $\mathbb{R}$ is a Lindelöff space (why?), but not compact.

Example 2.12.5. A separable space need not be Lindelöf. To see this, let X be an uncountable set. Fix a point, say $x_0 \in X$, and define a topology on X by declaring a subset $U \subseteq X$ to be open if and only if either $U = \emptyset$ or $x_0 \in U$. Then $\{x_0\}$ is dense in X, and hence X is separable. Since $\{x_0, x\}$ is open in X, $\forall\, x \in X$, it follows that X is neither first countable nor Lindelöf (verify!).

*Unfortunate choice of terminology!

Example 2.12.6. A Lindelöf space need not be separable. To see this, let X be an uncountable set and let $Y = X \cup \{*\}$, where $* \notin X$. For each $x \in X$, we declare $\{x\}$ to be open in Y. If $U \subseteq Y$ and $* \in U$, we declare U to be open in Y if $Y \setminus U$ is countable. Verify that this gives a topology on Y. Let $\mathcal{U}$ be an open cover of Y. Then $p \in U$, for some $U \in \mathcal{U}$. By construction of the topology on Y, we have $Y \setminus U$ is countable. For each $x \in Y \setminus U$, we may choose an element, say $V_x \in \mathcal{U}$ such that $x \in V_x$. Then the subcollection $\{V_x : x \in Y \setminus U\} \cup \{U\}$ is a countable subcover of $\mathcal{U}$. Therefore, Y is Lindelöf. If A is a countable subset of Y, then X being uncountable there exists $x \in X$ such that $x \notin A$. Then $\{x\}$ is open in Y which does not intersects A, and hence A cannot be dense in Y. Therefore, Y is not separable.

Proposition 2.12.7. *A topological space X is Lindelöf if and only if every basic open cover of X has a countable subcover.*

Proof. Fix a basis $\mathcal{B}$ of open subsets for X. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, choose an element $U_x \in \mathcal{U}$ such that $x \in U_x$. Then we can choose $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U_x$. Then the collection $\mathcal{B}' := \{V_x : x \in X\}$ is an open cover of X consisting of basic open subsets of X, and hence by assumption on X it admits a countable subcover, say $\mathcal{B}_c := \{V_n : n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, we choose one $U_n \in \mathcal{U}$ such that $V_n \subseteq U_n$. Since $\bigcup\limits_{n \in \mathbb{N}} V_n = X$, it follows that $\{U_n : n \in \mathbb{N}\}$ is a countable subcover of $\mathcal{U}$. Therefore, X is Lindelöf. $\square$

Proposition 2.12.8. *Any second countable space is Lindelöf and separable.*

Proof. Let X be a second countable space. Let $\mathcal{B} = \{V_n : n \in \mathbb{N}\}$ be a countable basis for X.

Let $\mathcal{U}$ be an open cover of X. To show that $\mathcal{U}$ has a countable subcover, for each $n \in \mathbb{N}$, we choose an element, say U_n from the subset

$$\mathcal{U}_n := \{U \in \mathcal{U} : V_n \subseteq U\},$$

if $\mathcal{U}_n \neq \emptyset$, to define a map $\Phi : \mathbb{N} \to \mathcal{U} \cup \{\emptyset\}$ by setting

$$\Phi(n) = \begin{cases} U_n, & \text{if} \quad \mathcal{U}_n \neq \emptyset, \\ \emptyset, & \text{if} \quad \mathcal{U}_n = \emptyset. \end{cases}$$

Clearly Φ is a well-defined map. Clearly $\mathcal{U}_c := \Phi(\mathbb{N})$ is a countable subset of $\mathcal{U}$. We show that $\mathcal{U}_c$ is a countable subcover of $\mathcal{U}$. Let $x \in X$ be given. Since $\mathcal{U}$ is an open cover of X, we have $x \in U$, for some $U \in \mathcal{U}$. Since $\mathcal{B}$ is a basis for X, there exists $n \in \mathbb{N}$ such that $x \in V_n \subseteq U$. Then $\mathcal{U}_n$ is non-empty, as it contains U, and so we have $x \in V_n \subseteq U_n$ by construction of Φ. Therefore, $\mathcal{U}_c := \Phi(\mathbb{N})$ is a required countable subcover of $\mathcal{U}$.

To show that X is separable, for each $n \in \mathbb{N}$ we choose one element, say $x_n \in V_n$ to get a countable subset $A := \{x_n : n \in \mathbb{N}\} \subseteq X$. To show that A is dense in X, let U be any non-empty open subset of X. Then U being non-empty, choosing an element $x \in U$ we can find a basic open subset, say $V_n \in \mathcal{B}$, such that $x \in V_n \subseteq U$. Then $x_n \in U$, and hence $A \cap U \neq \emptyset$. $\square$

Corollary 2.12.9. *For any countable index set I, the product space $\mathbb{R}^I$ is Lindelöf.*

Proof. Follows from Example 2.3.8, Lemma 2.12.2 and Proposition 2.12.8. ☐

Proposition 2.12.10. *Any Lindelöf metric space is second countable, and hence is separable.*

Proof. Let (X, d) be a Lindelöf metric space. For each $n \in \mathbb{N}$, the collection

$$\mathcal{U}_n := \{B_d(x, 1/n) : x \in X\}$$

being an open cover of (X, d), admits a countable subcover, say $\mathcal{B}_n \subset \mathcal{U}_n$ by Lindelöf assumption on (X, d). Then $\mathcal{B} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$, being a countable union of countable sets, is a countable open cover for (X, d). We show that $\mathcal{B}$ is a basis for (X, d). Let $x \in X$ and U an open neighbourhood of x in (X, d). Then there exists $n_0 \in \mathbb{N}$ such that $B_d(x, 1/n) \subseteq U$, for all $n \geq n_0$. Since $\mathcal{B}_{2n}$ is an open cover of (X, d), there exists $y \in X$ such that $B_d(y, 1/2n) \in \mathcal{B}_{2n}$ and $x \in B_d(y, 1/2n)$. Then by triangle inequality, we have

$$x \in B_d(y, 1/2n) \subseteq B_d(x, 1/n) \subseteq U.$$

Therefore, $\mathcal{B}$ is a basis for the metric topology on (X, d). Therefore, X is second countable and hence is separable by Proposition 2.12.8. ☐

Theorem 2.12.11. *Let (X, d) be a metric space. Then the following are equivalent.*

(i) X is second countable.

(ii) X is Lindelöf.

(iii) X is separable.

Proof. Note that (i) implies (ii) by Proposition 2.12.8, and (ii) implies (iii) by Proposition 2.12.10. Suppose that (X, d) is separable. Let A be a countable dense subset of X. Then the collection of open balls

$$\mathcal{B} := \{B_d(a, 1/n) : a \in A, n \in \mathbb{N}\}$$

is countable. We show that $\mathcal{B}$ is a basis for the topology on (X, d). Let $x \in X$ and U be an open neighbourhood of x in (X, d). Then there exists $n_0 \in \mathbb{N}$ such that $B_d(x, 1/n) \subseteq U$, for all $n \geq n_0$. Since A is dense in X, for each $n \geq n_0$ we can find an element, say $a_n \in A \cap B_d(x, 1/n)$. Then it follows from the triangle inequality that

$$x \in B_d(a_{2n}, 1/n) \subseteq B_d(x, 1/n) \subseteq U,$$

and hence the collection $\mathcal{B}$ is a basis for (X, d). This completes the proof. ☐

Proposition 2.12.12. *$\mathbb{R}_\ell$ is Lindelöf.*

Proof. To show that the space $\mathbb{R}_\ell$ is Lindelöf, in view of Proposition 2.12.7 it suffices to show that every basic open cover of $\mathbb{R}_\ell$ has a countable subcover. Let $\mathcal{U} = \{[a_\alpha, b_\alpha) : \alpha \in \Lambda\}$ be an open cover of $\mathbb{R}_\ell$ by basic open subsets. Let $C = \bigcup_{\alpha \in \Lambda} (a_\alpha, b_\alpha) \subseteq \mathbb{R}$. We show that $\mathbb{R} \setminus C$ is countable. Let $x \in \mathbb{R} \setminus C$ be given. Since $x \notin \bigcup_{\alpha\in\Lambda} (a_\alpha, b_\alpha)$, so $x = a_\alpha$, for some $\alpha \in \Lambda$; fix such an $\alpha \in \Lambda$. Fix a rational number $r_x \in (a_\alpha, b_\alpha)$. Since $(a_\alpha, b_\alpha) \subseteq C$, we have $(x, r_x) = (a_\alpha, r_x) \subseteq C$. If $x, y \in \mathbb{R} \setminus C$ with $x < y$, then we have $r_x < r_y$ (verify!). Therefore, the map $f : \mathbb{R} \setminus C \to \mathbb{Q}$ defined by

$$x \longmapsto r_x,$$

is injective, and hence $\mathbb{R} \setminus C$ is countable.

For each $x \in \mathbb{R} \setminus C$, choosing a member $U_x \in \mathcal{U}$ we get a countable subcollection

$$\mathcal{U}_1 := \{U_x : x \in \mathbb{R} \setminus C\} \subseteq \mathcal{U}$$

that covers $\mathbb{R} \setminus C$. Since $\mathbb{R}$ is second countable (see Example 2.3.8), so is its subspace $C := \bigcup_{\alpha\in\Lambda} (a_\alpha, b_\alpha)$ by Lemma 2.12.2. Then C is Lindelöf by Proposition 2.12.8, we can find a countable subcollection

$$\{(a_{\alpha_n}, b_{\alpha_n}) : n \in \mathbb{N}\} \subseteq \{(a_\alpha, b_\alpha) : \alpha \in \Lambda\}$$

that covers C. Then the subcollection

$$\mathcal{U}_1 \cup \{[a_{\alpha_n}, b_{\alpha_n}) : n \in \mathbb{N}\} \subseteq \mathcal{U}$$

is countable and covers $\mathbb{R}_\ell$. Thus $\mathbb{R}_\ell$ is Lindelöf. □

Corollary 2.12.13. *The space $\mathbb{R}_\ell$ is not metrizable (i.e., there is no metric on $\mathbb{R}_\ell$ that induces the lower limit topology on it).*

Proof. Since $\mathbb{R}_\ell$ is Lindelöf by Proposition 2.12.12 and not second countable by Example 2.3.9, the result follows from Proposition 2.12.8. □

Example 2.12.14. In Proposition 2.12.8 we have seen that second countable spaces are both Lindelöf and separable. However, the converse need not be true, in general. Let $X := C[0,1]$ be the set of all continuous maps from $[0,1] \subset \mathbb{R}$ into $\mathbb{R}$. Equip X with the subspace topology induced from the product topology on $\mathbb{R}^{[0,1]}$. Then X is Lindelöf and separable, but not second countable. This will be explained in detail later.

Proposition 2.12.15. *Countable product of separable spaces is separable.*

Proof. Let $\{X_n : n \in \mathbb{N}\}$ be a countable collection of separable spaces, and let $X := \prod_{n\in\mathbb{N}} X_n$ be the associated product space. Let A_n be a countable dense subset of X_n, for each $n \in \mathbb{N}$. Note that the Cartesian product $A := \prod_{n\in\mathbb{N}} A_n \subseteq X$ need not be a countable subset of X. We construct a countable subset of A that is dense in X. To do this, for each $n \in \mathbb{N}$, we fix an element $a_n \in A_n$. Then for each $m \in \mathbb{N}$, the subset

$$B_m := \left(\prod_{1\leq n<m} A_n\right) \times \prod_{n\geq m} \{a_n\}$$

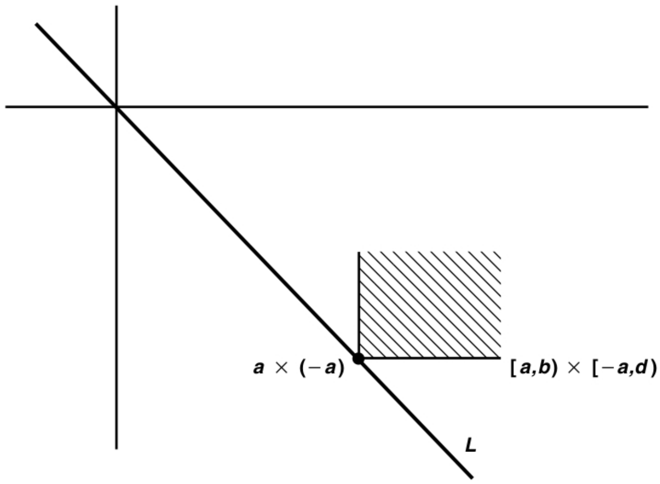
is countable, and hence the subset $B := \bigcup_{m\in\mathbb{N}} B_m$ is countable. Let U be any non-empty open subset of X. Then U contains a non-empty basic open subset of the form

$$V := \prod_{n\in\mathbb{N}} V_n \subseteq U,$$

where V_n is a non-empty open subset of X_n, for all $n \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $V_n = X_n$, for all $n \geq n_0$. Since A_n is dense in X_n, for all $n \in \mathbb{N}$, it follows that $B_{n_0} \cap V \neq \emptyset$. Therefore, $B \cap U \neq \emptyset$, and hence B is a countable dense subset of X. Therefore, X is separable. □

Example 2.12.16. ***Product of Lindelöf spaces need not be Lindelöf***. We have shown in Example 2.12.12 that $\mathbb{R}_\ell$ is Lindelöf. We show that the product space $\mathbb{R}_\ell^2 := \mathbb{R}_\ell \times \mathbb{R}_\ell$ (***Sorgenfrey plane***) is not Lindelöf. Note that $\mathbb{R}_\ell^2$ has basis consisting of open subsets of the form $[a,b) \times [c,d)$, where $a < b$ and $c < d$ in $\mathbb{R}$. Consider the subspace

$$L := \{(x, -x) : x \in \mathbb{R}_\ell\} \subset \mathbb{R}_\ell^2.$$



Clearly L is closed in $\mathbb{R}_\ell^2$. Then the open set $\mathbb{R}_\ell^2 \setminus L$ together with the basic open subsets of the form

$$[a,b) \times [-a,d)$$

where $a < b$ and $-a < d$, forms an open cover $\mathscr{U}$ of $\mathbb{R}_\ell^2$. Since L is uncountable and each of $[a,b) \times [-a,d)$ intersects L at $(a,-a)$ only, it follows that $\mathcal{U}$ has no countable subcover. Therefore, $\mathbb{R}_\ell^2$ is not Lindelöf.

Example 2.12.17. ***Subspace of a Lindelöf space need not be Lindelöf***. To see this, let X be an uncountable set, and let $Y = X \cup \{*\}$, where $*$ is a point outside X. Define a subset U of Y to be open if U is empty, or $U = Y$ or $U \subseteq X$. Clearly this gives a topology on Y. Since Y is the only open subset of Y containing $*$, we see that Y is compact, and hence is Lindelöf. However, X is not Lindelöf in the subspace topology induced from Y.

2.13 Regular and normal spaces

Definition 2.13.1. A topological space X is said to be *regular* if X is a T1 space and given any closed subset A of X and a point $x \in X \setminus A$, there exists a pair of disjoint open subsets U and V of X containing x and A, respectively.

Since every singleton subset of a T1 space are closed, it follows that a regular space is Hausdorff. However, the converse is not true in general.

Example 2.13.2. Let $X = \mathbb{R}_K$, the real line equipped with the K-topology, where $K = \{1/n : n \in \mathbb{N}\}$. Clearly $\mathbb{R}_K$ is Hausdorff. Note that K is closed in $\mathbb{R}_K$ and $0 \notin K$. We show that K and 0 cannot be separated by a pair of disjoint open subsets of $\mathbb{R}_K$ containing them. Suppose on the contrary that there exists a pair of disjoint open subsets U and V of $\mathbb{R}_K$ such that $K \subseteq U$ and $0 \in V$. Since $V \cap K = \emptyset$ by assumption, there is a basic open subset of the form $(a, b) \setminus K$ containing 0 in $\mathbb{R}_K$. Since $0 < b$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < b$. Since $1/n \in K$ and $K \subseteq U$, a basic open neighbourhood of $1/n$ contained in U must be of the form (c, d), with $c < 1/n < d$. Then choosing a point $x \in \mathbb{R}$ with $\max\{c, \frac{1}{n+1}\} < x < \frac{1}{n}$, we see that $z \in ((a, b) \setminus K)\, U \cap V$, a contradiction. Therefore, $\mathbb{R}_K$ is not a regular space.

The next proposition gives an equivalent characterization of regular spaces.

Proposition 2.13.3. *A topological space X is regular if and only if X is a T1 space such that given any $x \in X$ and an open neighbourhood U of x there exists an open neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.*

Proof. Let X be a regular space. Then X is a T1 space. Let $x \in X$ and U be an open neighbourhood of x in X. Then $A := X \setminus U$ is a closed subset of X such that $x \in X \setminus A$. Then by regularity of X, there exists a pair of disjoint open subsets V and W of X containing x and A, respectively. Then $\overline{V} \cap A = \emptyset$, and hence $\overline{V} \subseteq X \setminus A = U$.

For the converse part, let X be a T1 space such that given any $x \in X$ and an open neighbourhood U of x in X there exists an open neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. Let A be a closed subset of X and let $x \in X \setminus A$. Then $U := X \setminus A$ is an open neighbourhood of x in X. Then by assumption on X, there exists an open neighbourhood V of x in X such that $\overline{V} \subseteq U$. Then $W := X \setminus \overline{V}$ is an open subset of X containing $X \setminus U = A$ and $V \cap W = \emptyset$. □

Proposition 2.13.4. *Subspace of a regular space is regular.*

Proof. Let A be a subspace of a regular space X. Since X is Hausdorff, so is its subspace A, and hence A is a T1 space. Let $Z \subseteq A$ be a closed subspace of A and let $a \in Z \setminus A$. Let $\overline{Z}$ be the closure of Z in X. Then $Z = \overline{Z} \cap A$. Since $a \in A \setminus Z$, we have $a \notin \overline{Z}$. Since X is regular and $\overline{Z}$ is a closed subspace of X not containing a, there exists a pair of disjoint open subsets U and V of X containing a and $\overline{Z}$, respectively. Then $A \cap U$ and $A \cap V$ are open neighbourhoods of a and $Z = \overline{Z} \cap A$ in A, respectively. Clearly $(A \cap U) \cap (A \cap V) = \emptyset$. Therefore, A is regular. □

Exercise 2.13.5. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, and let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the associated product space. Let $A_\alpha \subseteq X_\alpha$, for all $\alpha \in \Lambda$, and let $A = \prod_{\alpha\in\Lambda} A_\alpha$. Then $\overline{\prod_{\alpha\in\Lambda} A_\alpha} = \prod_{\alpha\in\Lambda} \overline{A_\alpha}$, where $\overline{A_\alpha}$ is the closure of A_α in X_α, for each $\alpha \in \Lambda$.

Proposition 2.13.6. *Product of regular spaces is regular.*

Proof. Let $\mathscr{F} = \{X_\alpha : \alpha \in \Lambda\}$ be a collection of regular spaces, and let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the associated product space. Let $x = (x_\alpha)_{\alpha\in\Lambda} \in X$ and let U be an open neighbourhood of x in X. Then there is a basic open neighbourhood $\prod_{\alpha\in\Lambda} U_\alpha$ of x contained in U, where $U_\alpha = X_\alpha$, for all $\alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$. Since U_{α_j} is an open neighbourhood of x_{α_j} in the regular space X_{α_j}, there exists an open neighbourhood V_{α_j} of x_{α_j} such that $\overline{V_{\alpha_j}} \subseteq U_{\alpha_j}$, for all $j \in \{1, \ldots, n\}$. Set $V_\alpha = X_\alpha$, for $\alpha \in \Lambda \setminus \{\alpha_1, \ldots, \alpha_n\}$. Then $V = \prod_{\alpha\in\Lambda} V_\alpha$ is an open neighbourhood of x in X such that $\overline{V} = \prod_{\alpha\in\Lambda} \overline{V_\alpha} \subseteq U$. Therefore, X is regular. □

Definition 2.13.7. A topological space X is said to be *completely regular* if X is a T1 space and given a closed subset A and a point $x \in X \setminus A$, there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

Proposition 2.13.8. *A completely regular space is regular.*

Proof. Let X be a completely regular space. Let A be a closed subset of X and let $x \in X \setminus A$. Then there is a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$. Then $U := f^{-1}([0, 1/2))$ and $V := f^{-1}((1/2, 1])$ are pairwise disjoint open subsets of X containing x and A, respectively. Therefore, X is regular. □

Definition 2.13.9. A topological space X is said to be *normal* if X is a T1 space and given any two closed subsets A and B of X with $A \cap B = \emptyset$, there exist open subsets U and V of X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Proposition 2.13.10. *A normal space is regular.*

Proof. Let X be a normal space. Let A be a closed subset of X. Let $x \in X$ be such that $x \notin A$. Since X is a T1 space, $\{x\}$ is a closed subset of X. Then there exist open neighbourhoods U and V of $\{x\}$ and A, respectively, in X such that $U \cap V = \emptyset$. Therefore, X is regular. □

Theorem 2.13.11. *Every metrizable space is normal.*

Proof. Let X be a metrizable space. Fix a metric d on X that induces the topology on X. Let A and B be two non-empty closed subsets of X with $A \cap B = \emptyset$. For each $a \in A$ we can choose a real number $r_a > 0$ such that $B_d(a, r_a) \cap B = \emptyset$. Similarly, for each $b \in B$ we can choose a real number $s_b > 0$ such that $B_d(b, s_b) \cap A = \emptyset$. Then $U := \bigcup_{a\in A} B_d(a, r_a/2)$ and $V := \bigcup_{b\in B} B_d(b, s_b/2)$ are open subsets of X containing A and B, respectively. If there exists a point $x \in U \cap V$, then $x \in B_d(a, r_a/2) \cap B_d(b, s_b/2)$, for some $a \in A$ and $b \in B$. Then $d(a,b) \leq d(a,x) + d(b,x) < r_a/2 + s_b/2$. If $r_a \leq s_b$, then $(r_a + s_b)/2 \leq s_b$, and hence $d(a,b) < s_b$. Then $a \in B_d(b, s_b)$, which is not possible since $A \cap B = \emptyset$. Similarly, if $s_b \leq r_a$, then $d(a,b) < r_a$, and hence $b \in B_d(a, r_a)$, which is not possible. Therefore, we must have $U \cap V = \emptyset$. Therefore, X is normal. □

Lemma 2.13.12. *Let K be a compact subset of a Hausdorff space X. Given any $x \in X \setminus K$ there exists a pair of disjoint open subsets U and V of X containing x and K, respectively.*

Proof. Since X is Hausdorff, for each $y \in K$, there exists a pair of disjoint open subsets U_y and V_y of X containing x and y, respectively. Then $\mathscr{U} = \{V_y : y \in K\}$ is an open cover of K. Since K is compact, there exists finitely many points $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} V_{y_j}$. Then $U := \bigcap\limits_{j=1}^{n} U_{y_j}$ and $V := \bigcup\limits_{j=1}^{n} V_{y_j}$ are open neighbourhoods of x and K, respectively, such that $U \cap V = \emptyset$. This completes the proof. □

Corollary 2.13.13. *A compact Hausdorff space is regular.*

Proof. Let X be a compact Hausdorff space. Let K be a closed subset of X and let $x \in X \setminus K$. Since closed subspace of a compact space is compact, K is compact. Then by Lemma 2.13.12 there exists a pair of disjoint open subsets U and V of X containing x and K, respectively. Thus X is regular. □

Theorem 2.13.14. *A compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. Since X is compact, both A and B are compact. Since X is compact and Hausdorff, it is regular by Corollary 2.13.13. Then for each $a \in A$ there exists a pair of disjoint open subsets U_a and V_a of X containing a and B, respectively. Then $\{U_a : a \in A\}$ is an open cover of A in X. Since A is compact, there exists finitely many points $a_1, \ldots, a_n \in A$ such that $A \subseteq \bigcup\limits_{j=1}^{n} U_{a_j}$. Then $U := \bigcup\limits_{j=1}^{n} U_{a_j}$ and $V := \bigcap\limits_{j=1}^{n} V_{a_j}$ are pairwise disjoint open subsets of X containing A and B, respectively. Therefore, X is normal. □

Theorem 2.13.15. *Every second countable regular space is normal.*

Proof. Let X be a second countable regular space. Let $\mathscr{B}$ be a countable basis for the topology on X. Let A and B be two non-empty closed subsets of X with $A \cap B = \emptyset$. Since X is regular, for each $a \in A$ there exist a pair of disjoint open subsets U_a and W_a containing a and B, respectively. Therefore,

$$\overline{U_a} \cap B = \emptyset, \ \forall\, a \in A. \tag{2.13.16}$$

Since $\mathscr{B}$ is a basis for the topology on X, for each $a \in A$ we can choose a basic open subset $V_a \in \mathscr{B}$ such that $a \in V_a \subseteq U_a$. Since $\{V_a : a \in A\} \subseteq \mathscr{B}$ and $\mathscr{B}$ is countable, we have a countable collection $\mathscr{V} = \{V_n : n \in \mathbb{N}\}$ of open subsets of X such that $A \subseteq V := \bigcup\limits_{n \in \mathbb{N}} V_n$ and $\overline{V_n} \cap B = \emptyset$, for all $n \in \mathbb{N}$. Similarly, we get a countable collection $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ of open subsets of X such that $B \subseteq W := \bigcup\limits_{n \in \mathbb{N}} W_n$ and $\overline{W_n} \cap A = \emptyset$, for all $n \in \mathbb{N}$. However, $V \cap W$ need not be empty. So we perform the following simple trick to modify them to get a pair of disjoint open neighbourhoods of A and B in X. For each $n \in \mathbb{N}$, let

$$V'_n = V_n \setminus \bigcup_{j=1}^{n} \overline{W_j} \text{ and } W'_n = W_n \setminus \bigcup_{j=1}^{n} \overline{V_j}. \tag{2.13.17}$$

Since V_n is open in X and $\bigcup\limits_{j=1}^{n} \overline{W_j}$ is closed in X, the set difference V'_n is open in X. Similarly, W'_n is open in X. Since $A \cap \overline{W_j} = \emptyset$, $\forall\, j$, the collection $\{V'_n : n \in \mathbb{N}\}$ is an open cover of A. Similarly, the collection $\{W'_n : n \in \mathbb{N}\}$ is an open cover of B. Finally, the open subsets $V' := \bigcup\limits_{n\in\mathbb{N}} V'_n$ and $W' := \bigcup\limits_{n\in\mathbb{N}} W'_n$ are disjoint. Indeed, if $x \in V' \cap W'$, then $x \in V'_n \cap W'_m$, for some $n, m \in \mathbb{N}$. Without loss of generality, we many assume that $m \leq n$. Then $x \in W'_n = W_n \setminus \bigcup\limits_{j=1}^{n} \overline{V_j}$ implies that $x \notin \overline{V_m}$, since $m \leq n$, and hence $x \notin V_m$, which contradicts the assumption that $x \in V'_m \subseteq V_m$. This completes the proof. □

Lemma 2.13.18. *A topological space X is normal if and only if X is a T1 space such that given any closed subset A of X and an open neighbourhood U of A in X, there exists an open neighbourhood V of A whose closure $\overline{V}$ in X is contained in U.*

Proof. Suppose that X is a normal space. Then X is a T1 space. Let A be a closed subspace of X and let U be an open neighbourhood of A in X. Then $B := X \setminus U$ is a closed subset of X with $A \cap B = \emptyset$. Then there exist open neighbourhoods V and W of A and B, respectively, in X such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \emptyset$. Then $\overline{V} \cap B = \emptyset$, and hence $\overline{V} \subseteq X \setminus B = U$, as required.

Conversely, suppose that X is a T1 space such that given any closed subset A of X and an open neighbourhood U of A in X, there exists an open neighbourhood V of A in X such that $\overline{V} \subseteq U$. Let A and B be two disjoint closed subsets of X. Then $U := X \setminus B$ is an open neighbourhood of A in X. Then by assumption there exists an open neighbourhood V of A in X such that $\overline{V} \subseteq X \setminus B$. Then $W := X \setminus \overline{V}$ is an open neighbourhood of B such that $V \cap W = \emptyset$. □

Theorem 2.13.19 (Urysohn's lemma)**.** *Given a pairwise disjoint closed subsets A and B of a normal space X, there exists a continuous map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Since $U := X \setminus B$ is an open neighbourhood of A in X and X is a normal space, by Lemma 2.13.18 we can find an open neighbourhood $U_{\frac{1}{2}}$ of A such that

$$A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U. \tag{2.13.20}$$

Since $U_{1/2}$ and U are open neighbourhoods of the closed subsets A and $\overline{U_{1/2}}$, respectively, applying normality we have open subsets $U_{1/4}$ and $U_{3/4}$ of X such that

$$A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U. \tag{2.13.21}$$

Continuing in this way, for each rational number

$$t \in T := \left\{ \frac{m}{2^n} \in \mathbb{Q} \mid m \in \{1, \ldots, 2^n - 1\} \text{ and } n \in \mathbb{N} \right\},$$

we have an open subset U_t containing A such that given $t_1, t_2 \in T$, we have

$$t_1 \leq t_2 \implies A \subseteq U_{t_1} \subseteq \overline{U_{t_1}} \subseteq U_{t_2} \subseteq \overline{U_{t_2}} \subseteq U = X \setminus B. \tag{2.13.22}$$

Define a map $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } \ x \in \bigcap\limits_{t \in T} U_t, \\ \sup\{t \in T : x \notin U_t\}, & \text{if } \ x \notin \bigcap\limits_{t \in T} U_t. \end{cases}$$

Clearly, $f(x) \in [0,1]$, $\forall\, x \in X$, and that $f(A) = \{0\}$ and $f(B) = \{1\}$. All it remains to show that f is continuous. Since the collection of all intervals of the form $[0,a)$ and $(a,1]$, where $a \in (0,1)$, forms a subbasis for the subspace topology on $[0,1]$ induced from $\mathbb{R}$, to show f is continuous, it suffices to show that $f^{-1}([0,a))$ and $f^{-1}((a,1])$ are open in X, for all $a \in (0,1)$. Fix an element $a \in (0,1)$. Note that, *$f(x) < a$ if and only if $x \in U_t$, for some $t < a$*. Indeed, since T is dense in $[0,1]$, it follows from the definition of f that if $f(x) < a$, choosing an element $t_0 \in T$ with $f(x) < t_0 < a$ we have $x \in U_{t_0}$. Conversely, if $x \in U_{t_0}$ for some $t_0 < a$, then $f(x) := \sup\{t \in T : x \notin U_t\} < t_0$. Therefore, $f^{-1}([0,a)) = \{x \in X : f(x) < a\} = \bigcup\limits_{t<a} U_t$ is open in X. Now we show that $f^{-1}((a,1])$ is open in X. For this, note that *$f(x) > a$ if and only if $x \notin \overline{U_t}$, for some $t > a$.* Indeed, if $f(x) > a$, then T being dense in $[0,1]$, choosing $t_0, t_1 \in T$ with $a < t_0 < t_1 < f(x)$ we see that $x \notin U_{t_1}$. Since $\overline{U_{t_0}} \subseteq U_{t_1}$ by construction, it follows that $x \notin \overline{U_{t_0}}$. Conversely, if $x \notin \overline{U_t}$, for some $t > a$, then $x \notin U_t$ where $t > a$, and hence $f(x) = \sup\{s \in T : x \notin U_s\} > a$. Therefore, $f^{-1}((a,1]) = \{x \in X : f(x) > a\} = \bigcup\limits_{t>a} (X \setminus \overline{U_t})$ is open in X. This completes the proof. □

Corollary 2.13.23 (Urysohn's lemma)**.** *Let A and B be two disjoint closed subsets of a normal space X. Then given $a, b \in \mathbb{R}$ with $a < b$, there exists a continuous function $f : X \to [a,b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

Proof. By Urysohn's lemma there exists a continuous map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Let $\varphi : [0,1] \to [a,b]$ be the map defined by

$$\varphi(t) = (1-t)a + tb, \ \ \forall\, t \in [0,1].$$

Then φ is a continuous map such that $\varphi(0) = a$ and $\varphi(1) = b$. Then $\varphi \circ f : X \to [a,b]$ is a continuous map such that $(\varphi \circ f)(A) = \{a\}$ and $(\varphi \circ f)(B) = \{b\}$. □

Corollary 2.13.24. *A normal space is completely regular.*

Proof. Follows from Urysohn's lemma. □

Theorem 2.13.25 (Tietze's extension theorem)**.** *Let X be a normal topological space and let Z be a non-empty closed of X. Then any continuous map $f : Z \to \mathbb{R}$ can be extended to a continuous map $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}\big|_Z = f$. Moreover, if $f(A) \subseteq [a,b]$, for some $a, b \in \mathbb{R}$ with $a \leq b$, then we can find an extension $\widetilde{f}$ of f on X such that $\widetilde{f}(X) \subseteq [a,b]$.*

Proof. The idea is to construct a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ on X that uniformly converges and the restriction of g_n on Z converges to f on Z. Then the limit function of $(g_n)_{n\in\mathbb{N}}$ will be continuous and its restriction on Z will be f. For this we use the following.

Lemma 2.13.26. *With the above notations, for given a continuous map $f : Z \to [-r, r]$, there exists a continuous map $g : X \to [-\frac{r}{3}, \frac{r}{3}]$ such that*

$$|f(z) - g(z)| \leq \frac{2r}{3}, \ \forall\, z \in Z.$$

Proof. Note that

$$A := f^{-1}([-r, -r/3]) \ \text{ and } \ B := f^{-1}([r, r/3])$$

are pairwise disjoint closed subsets of Z. Since Z is closed in X, both A and B are closed in X. Since X is normal, by Uryshon's lemma there exists a continuous map $g : X \to [-r/3, r/3]$ such that

$$g(A) = \{-r/3\} \ \text{ and } \ g(B) = \{r/3\}.$$

Then we have the following three cases.

Case 1: If $z \in A$, then $f(z), g(z) \in [-r, -r/3]$.

Case 2: If $z \in B$, then $f(z), g(z) \in [r/3, r]$.

Case 3: If $z \in Z \setminus (A \cup B)$, then $f(z), g(z) \in (-r/3, r/3)$.

Therefore, in each case we have $|f(z) - g(z)| \leq 2r/3$. This completes the proof. □

Suppose that $f : Z \to [a,b]$ be a continuous map. Without loss of generality, we may replace $[a,b]$ by $[-1,1]$. Therefore, we begin with a continuous map $f : Z \to [-1,1]$. Then by above Lemma 2.13.26 with $r = 1$ we can find a continuous map $g_1 : X \to \mathbb{R}$ such that

$$\begin{aligned} |g_1(x)| &\leq 1/3, \ \forall\, x \in X, \\ \text{and } \ |f(z) - g_1(z)| &\leq 2/3, \ \forall\, z \in Z. \end{aligned}$$

Applying Lemma 2.13.26 again to the continuous map $f - g_1$ with $r = 2/3$, we have a continuous map $g_2 : X \to \mathbb{R}$ such that

$$\begin{aligned} |g_2(x)| &\leq \frac{1}{3}\left(\frac{2}{3}\right), \ \forall\, x \in X, \\ \text{and } \ |f(z) - g_1(z) - g_2(z)| &\leq \left(\frac{2}{3}\right)^2, \ \forall\, z \in Z. \end{aligned}$$

Continuing in this way, by induction, for each $n \in \mathbb{N}$ we have continuous maps $g_1, \ldots, g_n : X \to \mathbb{R}$ such that

$$|g_i(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{i-1}, \forall\, i = 1, \ldots, n;\ x \in X,$$

$$\text{and} \quad \left|f(z) - \sum_{i=1}^{n} g_n(z)\right| \leq \left(\frac{2}{3}\right)^{n}, \forall\, z \in Z.$$

Consider the map $g : X \to \mathbb{R}$ defined by

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \ \forall\, x \in X.$$

Since for each $n \in \mathbb{N}$ we have $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$, $\forall\, x \in X$, and the geometric series $\frac{1}{3}\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$ is convergent to 1, it follows that the series of functions converges $\sum\limits_{n=1}^{\infty} g_n(x)$ is absolutely and uniformly on X with $|g(x)| = |\sum\limits_{n=1}^{\infty} g_n(x)| \leq 1$, for all $x \in X$. Therefore, g is continuous on X with $g(X) \subseteq [-1, 1]$. Since for each $n \in \mathbb{N}$, we have

$$\left|f(z) - \sum_{i=1}^{n} g_n(z)\right| \leq \left(\frac{2}{3}\right)^{n}, \forall\, z \in Z,$$

and since $\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$, taking limit as $n \to \infty$ we see that $|f(z) - g(z)| = 0$, for all $z \in \mathbb{Z}$. Therefore, $g\big|_Z = f$.

For the second part, let $f : Z \to \mathbb{R}$ be a continuous map. Since $\mathbb{R}$ is homeomorphic to the open interval $(-1, 1)$, applying the first part we have a continuous map $g : X \to [-1, 1]$ such that $g\big|_Z = f$. Now we need to replace g with a continuous map $h : X \to (-1, 1)$ such that $h\big|_Z = f$. For this, note that $F := g^{-1}(-1) \cup g^{-1}(1)$ is a closed subset of X disjoint from Z. Since X is normal, by Uryshon's lemma we have a continuous map $\phi : X \to [0, 1]$ such that $\phi(F) = \{0\}$ and $\phi(Z) = \{1\}$. Then the map $h : X \to \mathbb{R}$ defined by

$$h(x) = \phi(x)g(x), \ \forall\, x \in X,$$

is continuous and that $h\big|_Z = g\big|_Z = f$. If $x \in F$, then $h(x) = \phi(x)g(x) = 0$, and if $x \in X \setminus F$, then $|g(x)| < 1$ and $|\phi(x)| \leq 1$ together gives $|h(x)| < 1$. Therefore, the image of h lands inside $(-1, 1)$, as required. This completes the proof. □

Exercise 2.13.27. Prove Uryshon's lemma assuming Tietze extension theorem.

Exercise 2.13.28. Let X be a T1 topological space such that for given a non-empty closed subset Z of X and a continuous map $f : Z \to [a, b] \subseteq \mathbb{R}$, there exists a continuous map $\widetilde{f} : X \to [a, b]$ such that $\widetilde{f}\big|_Z = f$. Then X is normal.

Proof. Let A and B be two pairwise disjoint non-empty closed subsets of X. Define a map $f : A \cup B \to [a, b]$ by

$$f(x) = \begin{cases} a, & \text{if} \quad x \in A, \\ b, & \text{if} \quad x \in B. \end{cases}$$

Since A and B are pairwise disjoint and closed, f is continuous by pasting Lemma 2.3.27. Then by assumption on X, we have a continuous map $\widetilde{f} : X \to [a, b]$ such that $\widetilde{f}\big|_Z = f$. Fixing a point $c \in (a, b)$ we see that $\widetilde{f}^{-1}([a, c))$ and $\widetilde{f}^{-1}((c, b])$ are pairwise disjoint non-empty open subsets of X containing A and B, respectively. Therefore, X is normal. $\square$

Exercise 2.13.29. Let J be any index set, and let $\mathbb{R}^J$ be equipped with the product topology. Let A be a non-empty closed subset of a normal space X. Show that any continuous map $f : A \to \mathbb{R}^J$ can be extended to a continuous map $\widetilde{f} : X \to \mathbb{R}^J$ such that $\widetilde{f}\big|_A = f$.

Lemma 2.13.30. *Let X be a metrizable topological space, and let Y be a subspace of X. Then for any metric d on X inducing its topology, the induced metric d_Y on Y given by*

$$d_Y(y_1, y_2) := d(y_1, y_2),\ \forall\, y_1, y_2 \in Y,$$

induces the subspace topology on Y induced from X. In particular, subspace of a metrizable topological space is metrizable.

Proof. Let d be a metric on X that induced the given topology τ_X on X. Let Y be a subspace of X. Then the subspace topology on Y induced by τ_X is given by $\tau_Y = \{U \cap Y : U \in \tau_X\}$. Let $U \cap Y \in \tau_Y$, where $U \in \tau_X$, be given. Let $y \in U \cap Y$. Since U is open in X, there exists $r > 0$ such that $B_d(y, r) \subseteq U$. Then

$$B_{d_Y}(y, r) := \{y_1 \in Y : d_Y(y, y_1) < r\} = B_d(y, r) \cap Y \subseteq U \cap Y.$$

Therefore, $U \cap Y$ is open with respect to the metric d_Y on Y induced by d. Conversely, suppose that $V \subseteq Y$ be open with respect to the metric d_Y on Y. Then for each $y \in V$, there exists $r_y > 0$ such that $B_{d_Y}(y, r_y) \subseteq V$. Let $U := \bigcup\limits_{y \in V} B_d(y, r_y)$. Then U is open in X. Since $B_d(y, r_y) \cap Y = B_{d_Y}(y, r_y)$, we see that $U \cap Y = V$. Therefore, $V \in \tau_Y$. This completes the proof. $\square$

Proposition 2.13.31. *The space $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology is metrizable.*

Proof. Let $\overline{d}$ be the standard bounded metric on $\mathbb{R}$ defined by

$$\overline{d}(x, y) = \min\{1, |x - y|\},\ \forall\, x, y \in \mathbb{R}.$$

Given $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ and $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$ in $X := \mathbb{R}^{\mathbb{N}}$, we define

$$D(\mathbf{x}, \mathbf{y}) := \sup\left\{\frac{\overline{d}(x_n, y_n)}{n} : n \in \mathbb{N}\right\}.$$

We first show that D is a metric on $\mathbb{R}^{\mathbb{N}}$. Clearly $D(\mathbf{x}, \mathbf{y}) \geq 0$ with equality holds if and only if $x_n = y_n,\ \forall\, n \in \mathbb{N}$, i.e., if and only if $\mathbf{x} = \mathbf{y}$. Also $D(\mathbf{x}, \mathbf{y}) = D(\mathbf{y}, \mathbf{x})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$. To verify triangle inequality, let $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$, $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$ and $\mathbf{z} = (z_n)_{n \in \mathbb{N}}$ be any three points on $\mathbb{R}^{\mathbb{N}}$. Since $\overline{d}$ is a metric on $\mathbb{R}$, we have

$$\frac{\overline{d}(x_n, z_n)}{n} \leq \frac{\overline{d}(x_n, y_n)}{n} + \frac{\overline{d}(y_n, z_n)}{n} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}),\ \ \forall\, n \in \mathbb{N}.$$

Therefore, we have

$$D(\mathbf{x}, \mathbf{z}) = \sup \left\{ \frac{\overline{d}(x_n, y_n)}{n} : n \in \mathbb{N} \right\} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}).$$

Therefore, D is a metric on $X = \mathbb{R}^{\mathbb{N}}$.

We now show that the topology on $X = \mathbb{R}^{\mathbb{N}}$ induced by the metric D coincides with the product topology on it. Let $U \subseteq X$ be a non-empty subset of X. Fix a point $\mathbf{x} \in U$. Suppose that U is open with respect to the metric D on X. Then there exists $\epsilon > 0$ such that $B_D(\mathbf{x}, \epsilon) \subseteq U$. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. Let V be the basic open subset in the product topology on $X = \mathbb{R}^{\mathbb{N}}$ of the form

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots .$$

Clearly $\mathbf{x} \in V$. We show that $V \subseteq B_D(\mathbf{x}, \epsilon)$. □

Corollary 2.13.32. *The product topological space $[0,1]^{\mathbb{N}}$ is metrizable.*

Theorem 2.13.33 (Urysohn's Metrization Theorem)**.** *Every second countable regular space is metrizable.*

Proof. Let X be a second countable regular space. Then X is normal by Theorem 2.13.15. Then X is completely regular by Uryshon's lemma (Theorem 2.13.19). Since the product space $[0,1]^{\mathbb{N}}$ is metrizable by Proposition 2.13.31, it suffices to prove the following.

Lemma 2.13.34. *A second countable completely regular space can be embedded into $[0,1]^{\mathbb{N}}$.*

Proof. □

□

2.14 Complete Metric Spaces

Let (X,d) be a metric space.

Definition 2.14.1. A *sequence* in X is a map $f : \mathbb{N} \to X$. We generally denote a sequence $f : \mathbb{N} \to X$ by its image $\{x_n\}_{n\in\mathbb{N}}$, where $x_n := f(n)$, $\forall\, n \in \mathbb{N}$.

Definition 2.14.2. A sequence $\{x_n\}$ in (X,d) is said to be a *Cauchy sequence* if given any real number $\epsilon > 0$ there exists a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, $\forall\, m,n \geq n_0$.

Definition 2.14.3. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to be *eventually constant* if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n+1}$, $\forall\, n \geq n_0$.

A sequence $\{x_n\}$ in (X,d) is said to be *convergent* if there exists $x_0 \in X$ such that given any $\epsilon > 0$ there exists a natural number n_0 such that $d(x_n, x) < \epsilon$, $\forall\, n \geq n_0$. In this case, we say that x_0 is a *limit point* of the sequence $\{x_n\}$ and we denote this symbolically as $x_0 = \lim\limits_{n\to\infty} x_n$. Clearly an eventually constant sequence is convergent, but the converse need not be true.

Lemma 2.14.4. *Any convergent sequence in a metric space is a Cauchy sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (X,d) converging to a point $x_0 \in X$. Then for given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_0) < \epsilon/2,\ \forall\, n \geq n_0.$$

Then we have

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_m, x_0) < \epsilon/2 + \epsilon/2 = \epsilon,\ \forall\, m,n \geq n_0.$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). □

Definition 2.14.5. A metric space (X,d) is said to be *complete* if every Cauchy sequence in (X,d) is convergent in (X,d).

Example 2.14.6. (i) The open interval $(0,1) \subset \mathbb{R}$ with the Euclidean metric induced from $\mathbb{R}$ is not complete. Indeed, the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, is a Cauchy sequence in $(0,1)$, but it does not converges to a point of $(0,1)$.

(ii) The metric subspace $\mathbb{Q}$ of the Euclidean space $\mathbb{R}$ is not complete. Indeed, given any irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can always find a Cauchy sequence of rational numbers $(x_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} x_n = \alpha$. For example, taking $x_n = \sum\limits_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$, $\forall\, n \in \mathbb{N}$, we see that the sequence (x_n) converges to $e \in \mathbb{R}$, which is not a rational number.

(iii) The real line $\mathbb{R}$ with the Euclidean metric on it is complete. This is a standard result from basic real analysis course.

(iv) Any discrete metric space is complete. Indeed, if d is a discrete metric on a non-empty set X, then any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) is eventually constant.

Lemma 2.14.7. *Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in a metric space (X,d). Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. Then $(x_n)_{n\in\mathbb{N}}$ converges to $a \in X$ if and only if $(x_{n_k})_{k\in\mathbb{N}}$ converges to a.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X,d). Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to $a \in X$. Let $\epsilon > 0$ be given. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon/2, \ \forall\, m, n \geq n_1.$$

Since $(x_{n_k})_{k\in\mathbb{N}}$ converges to a, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_{n_k}, a) < \epsilon/2, \ \forall\, k \geq n_2.$$

Let $n_0 := \max\{n_1, n_2\} \in \mathbb{N}$. Since $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, choosing a $k \geq n_0$ we see that

$$d(a, x_n) \leq d(a, x_{n_k}) + d(x_{n_k}, x_n) < \epsilon, \ \forall\, n \geq n_0.$$

Therefore, $(x_n)_{n\in\mathbb{N}}$ converges to a in (X,d). Converse part is obvious. $\square$

Corollary 2.14.8. *A metric space (X,d) is complete if and only if every Cauchy sequence in X has a convergent subsequence.*

Proof. Follows from Lemma 2.14.7. $\square$

Lemma 2.14.9. *Every Cauchy sequence in a metric space (X,d) is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X,d). Then there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq 1, \ \forall\, m, n \geq n_0.$$

Let $d := \max\{d(x_i, x_j) : i, j \leq n_0\}$ and set $M := \max\{d, 1\}$. Then by triangle inequality, we have $d(x_n, x_m) \leq M + 1$, $\forall\, m, n \in \mathbb{N}$. This completes the proof. $\square$

Example 2.14.10. $\mathbb{R}$ is complete with respect to the Euclidean metric on it.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded by Lemma 2.14.9, say $|x_n| \leq M$, for some $M > 0$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in $[-M, M] \subset \mathbb{R}$. Since $[-M, M]$ is compact by Lemma 2.11.23, the sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, say $(x_{n_k})_{k\in\mathbb{N}}$, by Theorem 2.11.45. Then $(x_n)_{n\in\mathbb{N}}$ itself is convergent by Lemma 2.14.7. $\square$

Lemma 2.14.11. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and let $X := \prod\limits_{\alpha\in\Lambda} X_\alpha$ be the associated product topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$ if and only if the sequence $(\pi_\alpha(x_n))_{n\in\mathbb{N}}$ converges to $\pi_\alpha(x)$ in X_α, for all $\alpha \in \Lambda$.*

Proof. Suppose that a sequence $(x_n)_{n\in\mathbb{N}}$ in $X := \prod\limits_{\alpha\in\Lambda} X_\alpha$ converges to $x \in X$. Fix an $\alpha_0 \in \Lambda$, and let $U_{\alpha_0} \subseteq X_{\alpha_0}$ be an open neighbourhood of $\pi_{\alpha_0}(x) \in X_{\alpha_0}$. For $\alpha \in \Lambda$ with $\alpha \neq \alpha_0$, we set $U_\alpha = X_\alpha$. Then $U := \prod\limits_{\alpha\in\Lambda} U_\alpha$ is an open neighbourhood of x in X. Since $(x_n)_{n\in\mathbb{N}}$ converges to

x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$, $\forall\, n \geq n_0$. Then $\pi_{\alpha_0}(x_n) \in U_{\alpha_0}$, $\forall\, n \geq n_0$. Therefore, $(\pi_{\alpha_0}(x_n))_{n\in\mathbb{N}}$ converges to $\pi_{\alpha_0}(x)$.

Conversely, suppose that the sequence $(\pi_\alpha(x_n))_{n\in\mathbb{N}}$ converges to $\pi_\alpha(x)$ in X_α, for all $\alpha \in \Lambda$. Let U be an open neighbourhood of x in X. Then there exists a basic open neighbourhood of x of the form

$$V = \prod_{\alpha\in\Lambda} V_\alpha \subseteq U,$$

where V_α is an open neighbourhood of $\pi_\alpha(x)$ in X_α, for all $\alpha \in \Lambda$, and there is a finite subset, say $F := \{\alpha_1, \ldots, \alpha_m\} \subseteq \Lambda$ such that $V_\alpha = X_\alpha$, for all $\alpha \in \Lambda \setminus F$. Then for each $i \in \{1, \ldots, m\}$, there exists $\ell_i \in \mathbb{N}$ such that

$$\pi_{\alpha_i}(x_n) \in V_{\alpha_i},\ \forall\, n \geq \ell_i.$$

Set $\ell = \max\{\ell_1, \ldots, \ell_m\}$. Then

$$\pi_\alpha(x_n) \in V_\alpha,\ \forall\, n \geq \ell,$$

and hence $x_n \in V \subseteq U$, for all $n \geq \ell$. This completes the proof. □

Definition 2.14.12. A normed linear space $(X, \|\cdot\|)$ is said to be *complete* if $(X, d_{\|\cdot\|})$ is a complete metric space, where $d_{\|\cdot\|}$ is the metric on X induced by the norm $\|\cdot\|$ on it. A complete normed linear space is a called a *Banach space*.

Lemma 2.14.13. *Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the standard Euclidean norm on it. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a $\mathbb{k}$-vector space X. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.*

Sketch of a proof: If d_1 and d_2 are the metrics on X induced by the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, then

$$d_1(x, y) = \|x - y\| \quad \text{and} \quad d_2(x, y) = \|x - y\|_2,\ \forall\, x, y \in X.$$

Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, there exist positive real numbers $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1.$$

Then it follows that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is Cauchy (resp., convergent) with respect to d_1 if and only if it is Cauchy (resp., convergent) with respect to d_2. Hence the result follows. □

Proposition 2.14.14. *For each $k \in \mathbb{N}$, the Euclidean metric space $(\mathbb{R}^k, \|\cdot\|_2)$ is complete.*

Sketch of a proof: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^k$. For each $i \in \{1, \ldots, k\}$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the projection map onto the i-th factor. Since for each $i \in \{1, \ldots, k\}$ we have

$$|\pi_i(x) - \pi_i(y)| \leq \|x - y\|_2,\ \forall\, x, y \in \mathbb{R}^k,$$

it follows that $(\pi_i(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, for all $i \in \{1, \ldots, k\}$. Since $\mathbb{R}$ is complete, the sequence $(\pi_i(x_n))_{n\in\mathbb{N}}$ converges to a point, say $x_i \in \mathbb{R}$, for each $i \in \{1, \ldots, k\}$. Let $\epsilon > 0$ be given. Then for each $i \in \{1, \ldots, k\}$ there exists $n_i \in \mathbb{N}$ such that

$$|x_i - \pi_i(x_n)| < \epsilon/k,\ \forall\, n \geq n_i.$$

Then it follows that

$$\|x - x_n\|_2 \leq \sum_{i=1}^{k} |x_i - \pi_i(x_n)| < k \cdot \frac{\epsilon}{k} = \epsilon, \ \forall\, n \geq n_0 := \max\{n_1, \ldots, n_k\}.$$

This completes the proof. □

Corollary 2.14.15. *$\mathbb{R}^n$ is complete with respect to any norm-induced metric on it, for all $n \in \mathbb{N}$.*

Proof. Since $\mathbb{R}^n$ is complete with respect to the standard Euclidean norm (i.e., the ℓ_2-norm) by Proposition 2.14.14, and since any two norms on a finite dimensional vector space are equivalent by Lemma 1.2.17, the result follows from Lemma 2.14.13. □

Corollary 2.14.16. *Let (X, d) be a complete metric space, and let Z be a non-empty subset of X. Equip Z with the metric $d_Z := d|_{Z \times Z}$. Then Z is closed in X if and only if (Z, d_Z) is complete.*

Proof. Let Z be a closed subset of a complete metric space (X, d). Let $(z_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Z. Since (X, d) is complete, the sequence $(z_n)_{n\in\mathbb{N}}$ converges to a point, say $a \in X$. Then for given any $r > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$z_n \in B_d(a, r), \ \forall\, n \geq n_0.$$

Then $a \in \overline{Z}$. Since Z is closed in X, we have $a \in Z$. Therefore, Z is complete.

Conversely, suppose that (Z, d_Z) is complete. Let $x \in \overline{Z}$ be given. Then by sequence Lemma 2.3.11 there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in Z that converges to x in (X, d). Then $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d), and hence in (Z, d_Y). Since (Z, d_Z) is complete, $(z_n)_{n\in\mathbb{N}}$ converges to a point $z \in Z$. Since (X, d) is Hausdorff, we must have $z = x$ by Proposition 2.5.6. Therefore, $\overline{Z} = Z$. This completes the proof. □

Proposition 2.14.17. *Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the standard Euclidean metric on it. Let*

$$\ell_\infty(\mathbb{k}) = \left\{(x_n) \in \mathbb{k}^{\mathbb{N}} : (x_n) \ \text{ is bounded}\right\}$$

be the set of all bounded sequences in $\mathbb{k}$. Note that $\ell_\infty(\mathbb{k})$ is a $\mathbb{k}$-vector space admitting a norm $\|\cdot\|_\infty$ defined by

$$\|(x_n)\|_\infty := \sup_{n\in\mathbb{N}} |x_n|, \ \forall\, (x_n) \in \ell_\infty(\mathbb{k}).$$

Then the space $\ell_\infty(\mathbb{k})$ is complete.

Proof. Given $(x_n), (y_n) \in \ell_\infty(\mathbb{R})$, we define

$$d_\infty((x_n), (y_n)) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell_\infty(\mathbb{k})$. Let $\epsilon > 0$ be given. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$d_\infty(f_n, f_m) = \sup_{k\in\mathbb{N}} |f_n(k) - f_m(k)| < \epsilon, \ \forall\, n, m \geq n_\epsilon.$$

Then for all $i \in \mathbb{N}$, we have

$$|f_n(i) - f_m(i)| < \epsilon, \ \forall\, n, m \geq n_\epsilon.$$

Therefore, for each $i \in \mathbb{N}$, the sequence $(f_n(i))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{k}$, and hence it converges to a point, say $f(i) \in \mathbb{k}$, by completenss of $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$. This defines a map

$$f : \mathbb{N} \to \mathbb{k}, \ \ i \mapsto f(i).$$

We show that $f \in \ell_\infty(\mathbb{k})$ and that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\ell_\infty(\mathbb{k})$. Left as an exercise! $\square$

Exercise 2.14.18. Let $\mathbb{k}$ be the field $\mathbb{R}$ or $\mathbb{C}$ equipped with the standard Euclidean metric on it. Fix a real number $p \geq 1$, and let

$$\ell_p(\mathbb{k}) = \left\{ (x_n) \in \mathbb{k}^{\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_n|^p \ \text{ is convergent} \right\}.$$

Show that $\ell_p(\mathbb{k})$ is a normed linear space over $\mathbb{k}$ with respect to the norm defined by

$$\|(x_n)\|_p := \left(\sum_{n\in\mathbb{N}} |x_n|^p \right)^{1/p}, \ \forall\ (x_n) \in \ell_p(\mathbb{k}).$$

and the space $\ell_p(\mathbb{k})$ is complete.

Exercise 2.14.19. Let $C[a,b]$ be the set of all continuous real valued functions defined on $[a,b] \subset \mathbb{R}$. Given $f, g \in C[a,b]$, show that

$$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$$

is a metric on $C[a,b]$ that makes it a complete metric space.

Exercise 2.14.20. Let $Z = \{(x_n) \in \ell_\infty(\mathbb{k}) : (x_n) \ \text{ is convergent}\}$ is a closed subset of $\ell_\infty(\mathbb{k})$, and hence is complete with respect to the sup norm induced from $\ell_\infty(\mathbb{k})$.

Lemma 2.14.21. *Let (X,d) and (Y,d) be metric spaces. Let A be a non-empty subset of X, and let $f : A \to Y$ be a uniformly continuous map. If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, then $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y,d).*

Proof. Let $\epsilon > 0$ be given. Then by uniform continuity of f, there exists a $\delta > 0$ such that

$$d_Y(f(a_n), f(a_m)) < \epsilon, \ \text{ whenever } \ d_X(a_n, a_m) < \delta.$$

Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $n_\delta \in \mathbb{N}$ such that $d_X(a_m, a_n) < \delta$, $\forall\ n \geq n_\delta$. Then

$$d_Y(f(a_n), f(a_m)) < \epsilon, \ \forall\ m, n \geq n_\delta.$$

Therefore, $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y,d). $\square$

Exercise 2.14.22. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences in a metric space (X,d). Define a sequence $(c_n)_{n\in\mathbb{N}}$ in X by setting

$$c_n = \begin{cases} a_m, & \text{if} \quad n = 2m, \text{ for some } m \in \mathbb{N}, \text{ and} \\ b_m, & \text{if} \quad n = 2m-1, \text{ for some } m \in \mathbb{N}. \end{cases}$$

(i) Show that both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are subsequences of $(c_n)_{n\in\mathbb{N}}$.

(ii) Show that $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences.

(iii) Conclude that $(c_n)_{n\in\mathbb{N}}$ converges to a point $x\in X$ if and only if both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converges to x.

Lemma 2.14.23. *Let A be a non-empty subset of a metric space (X,d), and let (Y,d) be a complete metric space. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences in A converging to the same point in X. Then for any uniformly continuous map $f : A \to Y$, both the sequences $(f(a_n))_{n\in\mathbb{N}}$ and $(f(b_n))_{n\in\mathbb{N}}$ converges to the same point in (Y,d).*

Proof. Suppose that both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to $x_0 \in X$. For each $n\in\mathbb{N}$, define

$$c_n = \begin{cases} a_m, & \text{if} \quad n = 2m, \text{ for some } m \in \mathbb{N}, \text{ and} \\ b_m, & \text{if} \quad n = 2m-1, \text{ for some } m \in \mathbb{N}. \end{cases}$$

Since both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are subsequences of $(c_n)_{n\in\mathbb{N}}$, the sequence $(c_n)_{n\in\mathbb{N}}$ converges to x_0 by Exercise 2.14.22. Then $(f(c_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y,d) by Lemma 2.14.21, and hence it converges to a point, say $y_0 \in Y$, since (Y,d) is complete. Since both $(f(a_n))_{n\in\mathbb{N}}$ and $(f(b_n))_{n\in\mathbb{N}}$ are subsequences of $(f(c_n))_{n\in\mathbb{N}}$, both $(f(a_n))_{n\in\mathbb{N}}$ and $(f(b_n))_{n\in\mathbb{N}}$ converges to y_0 (c.f. Lemma 2.14.7). □

Theorem 2.14.24 (Uniform Extension Theorem)**.** *Let A be a non-empty subset of a metric space (X,d_X), and let $\overline{A}$ be the closure of A in X. Let (Y,d_Y) be a complete metric space. Then any uniformly continuous map $f : A \to Y$ uniquely extends to a uniformly continuous map $\widetilde{f} : \overline{A} \to Y$ such that $\widetilde{f}\big|_A = f$.*

Proof. Uniqueness of $\widetilde{f}$, if it exists, is clear because Y is Hausdorff (see Exercise 2.5.4). We now show existence of $\widetilde{f}$ and prove its uniform continuity. Let $f : A \to Y$ be a uniformly continuous map. Let $x_0 \in \overline{A}\setminus A$ be given. Choose a sequence $(a_n)_{n\in\mathbb{N}}$ in A that converges to x_0 (c.f. Lemma 2.3.11). Then $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y,d_Y) by Lemma 2.14.21, and hence it converges to a point, say $y_0 \in Y$, since (Y,d_Y) is complete. Then we define $\widetilde{f}(x_0) = y_0$. Since y_0 does not depend on choice of a sequence $(a_n)_{n\in\mathbb{N}}$ in A converging to x_0 by Lemma 2.14.23, the above construction

$$x_0 \longmapsto y_0,$$

gives a well-defined map $\widetilde{f} : \overline{A} \to Y$ such that $\widetilde{f}\big|_A = f$. Note that $\widetilde{f}$ is continuous by Corollary 2.3.12. It remains to show that $\widetilde{f}$ is uniformly continuous.

Let $\epsilon > 0$ be given. Since $f : A \to Y$ is uniformly continuous, there exists $\delta > 0$ such that

$$d_Y(f(a), f(b)) < \epsilon/3, \text{ whenever } d_X(a,b) < \delta. \tag{2.14.25}$$

Let $x, y \in \overline{A}$ be such that

$$d_X(x,y) < \delta/3. \tag{2.14.26}$$

Choose sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in A that converge to x and y, respectively. Since the sequences $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ converge to $\widetilde{f}(x)$ and $\widetilde{f}(y)$, respectively, there exist $n_0 \in \mathbb{N}$ such that

$$d_Y(f(x_n), \widetilde{f}(x)) < \epsilon/3, \text{ and } d_Y(f(y_n), \widetilde{f}(y)) < \epsilon/3, \forall\, n \geq n_0. \tag{2.14.27}$$

Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$, there exists $n_1 \geq n_0$ such that

$$d_X(x_n, x) < \delta/3, \text{ and } d_X(y_n, y) < \delta/3, \forall\, n \geq n_1. \tag{2.14.28}$$

Then for all $n \geq n_1$, by our choice of x and y as in (2.14.26), we have

$$d_X(x_n, y_n) \leq d_X(x_n, x) + d_X(x,y) + d_X(y, y_n) < \delta. \tag{2.14.29}$$

Since $d_X(x,y) < \delta$, it follows from inequalities (2.14.25) and (2.14.27) that

$$\begin{aligned} d_Y(\widetilde{f}(x), \widetilde{f}(y)) &\leq d_Y(\widetilde{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), \widetilde{f}(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

Thus, $\widetilde{f}$ is uniformly continuous. This completes the proof. □

Remark 2.14.30. Note that Theorem 2.14.24 fails if $f : A \to Y$ is just continuous without being uniformly continuous. For example, the map $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = 1/x, \ \forall\, x \in (0,1),$$

is continuous and $\mathbb{R}$ is a complete metric space. However, f cannot be extended to a continuous map $\widetilde{f} : [0,1] \to \mathbb{R}$ satisfying $\widetilde{f}\big|_{(0,1)} = f$.

Proposition 2.14.31. *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ satisfying*

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2), \ \forall\, x_1, x_2 \in X,$$

is an embedding of X into Y, called an isometric embedding of (X, d_X) into (Y, d_Y).

Proof. If $f(x_1) = f(x_2)$, for some $x_1, x_2 \in X$, then $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0$ gives $x_1 = x_2$. Therefore, f is injective. Clearly f is continuous. Fix a point $x_0 \in X$ and a real number $r > 0$. Since

$$f\,(B(x_0, r)) = B_{d_Y}(f(x_0), r) \cap f(X),$$

we conclude that f is an embedding of X into Y. $\square$

Lemma 2.14.32. *Let $f : (X,d) \to (Y,\rho)$ be a continuous map of metric spaces. Let A be a dense subset of (X,d). If*

$$\rho(f(x), f(y)) = d(x,y),\ \forall\, x, y \in A,$$

then f is an isometric embedding.

Proof. Let $x, y \in X$ be given. Choose sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in X that converge to x and y, respectively, in (X,d). Since the composite map

$$\rho \circ (f,f) : X \times X \xrightarrow{(f,f)} Y \times Y \xrightarrow{\rho} \mathbb{R}$$

is continuous, the sequence $(\rho(f(a_n), f(b_n)))_{n\in\mathbb{N}}$ converges to $\rho(f(a), f(b))$. Since

$$\rho(f(a_n), f(b_n)) = d(a_n, b_n),\ \forall\, n \in \mathbb{N},$$

and the map $d : X \times X \to \mathbb{R}$ is continuous, the sequence $(d(a_n, b_n))_{n\in\mathbb{N}}$ converges to $d(a,b)$. Since $\mathbb{R}$ is Hausdorff, by uniqueness of limit of a convergent sequence, we conclude that $\rho(f(a), f(b)) = d(a,b)$. Then the result follows from Proposition 2.14.31. $\square$

Corollary 2.14.33. *Let A be a dense subset of a metric space (X,d) and let (Y,ρ) be a complete metric space. If $f : (A,d) \to (Y,\rho)$ is an isometric embedding, then f uniquely extends to an isometric embedding of (X,d) into (Y,ρ).*

Proof. Let $f : (A,d) \to (Y,\rho)$ be an isometric embedding. Then f is uniformly continuous. Since A is dense in (X,d) and (Y,ρ) is complete, by uniform extension theorem 2.14.24 we have a unique continuous map $\widetilde{f} : (X,d) \to (Y,\rho)$ such that $\widetilde{f}\big|_A = f$. Since

$$\rho(\widetilde{f}(x), \widetilde{f}(y)) = \rho(f(x), f(y)) = d(x,y),\ \forall\, x, y \in A,$$

we see that $\widetilde{f}$ is an isometric embedding by Lemma 2.14.32. $\square$

Lemma 2.14.34. *Let (X,d) be a metric space. Let (Y_1,d_1) and (Y_2,ρ_2) be complete metric spaces, and let $f_1 : (X,d) \to (Y_1,d_1)$ and $f_2 : (X,d) \to (Y_2,\rho_2)$ be two isometric embeddings such that $f_1(X)$ and $f_2(X)$ are dense in (Y_1,d_1) and (Y_2,d_2), respectively. Then there is a unique isometric homeomorphism $\varphi : (Y_1,d_1) \to (Y_2,d_2)$ such that $\varphi \circ f_1 = f_2$.*

$$\begin{tikzcd}
(X,d) \arrow[r, "f_2"] \arrow[d, "f_1"'] & (Y_2,d_2) \\
(Y_1,d_1) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Proof. Consider the composite map $h := f_2 \circ f_1^{-1} : f_1(X) \to f_2(X) \subseteq Y_2$. Let $y_1, z_1 \in f_1(X) \subseteq Y_1$ be given. Since f_1 is an isometric embedding, there exist unique $y, z \in X$ such that $f_1(y) = y_1$, $f_1(z) = z_1$ and $d_1(y_1, z_1) = d(y,z)$. Since f_2 is an isometric embedding, we have $d(y,z) = d_2(f(y), f(z)) = d_2(h(y_1), h(z_1))$. Therefore, $h := f_2 \circ f_1^{-1}$ is an isometric embedding of $f_1(X)$

onto $f_2(X)$. Since (Y_2, d_2) is complete and $f_1(X)$ is dense in (Y_1, d_1), the map h uniquely extends to an isometric embedding $\varphi : (Y_1, d_1) \to (Y_2, d_2)$ such that $\varphi \circ f_1 = f_2$. By symmetry, we have a unique isometric embedding $\psi : (Y_2, d_2) \to (Y_1, d_1)$ such that $\psi \circ f_2 = f_1$. Then by uniqueness part, we must have $\varphi \circ \psi = \mathrm{Id}_{Y_2}$ and $\psi \circ \varphi = \mathrm{Id}_{Y_1}$. Therefore, φ is an isometric homeomorphism. □

Definition 2.14.35 (Pseudo-metric space)**.** Let X be a non-empty set, and let $d : X \times X \to \mathbb{R}$ be a map satisfying the following properties:

(i) $d(x, y) \geq 0$, for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$, and

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$.

Such a map d is called a *pseudo-metric* on X, and the pair (X, d) is called a *pseudo-metric space*. Clearly any metric space is a pseudo-metric space.

Example 2.14.36 (Pseudo metric that is not a metric)**.** Let $X = \mathbb{R}^2$, the Cartesian product of $\mathbb{R}$ with itself. Define $d : X \times X \to \mathbb{R}$ by

$$d\left((x_1, x_2), (y_1, y_2)\right) = |x_1 - y_1|, \;\; \forall\, (x_1, x_2), (y_1, y_2) \in X.$$

Then d is a pseudo metric on X, and (X, d) is a pseudo metric space. Since $d((1, 2), (1, 3)) = 0$ but $(1, 2) \neq (1, 3)$ in $\mathbb{R}^2$, d is not a metric on $\mathbb{R}^2$.

Exercise 2.14.37. Let $C'[0, 1]$ be the set of all real-valued piece-wise continuous functions on $[0, 1] \subset \mathbb{R}$. Given $f, g \in C'[0, 1]$, we define

$$d(f, g) := \int_0^1 |f(t) - g(t)| dt.$$

Show that d is a pseudo metric on $C'[0, 1]$ but not a metric. Let

$$C[0, 1] = \{f \in C'[0, 1] : f \;\text{ is continuous on }\; [0, 1]\}.$$

Show that the restriction of d gives a metric on $C[0, 1]$.

Exercise 2.14.38. Let (X, d) be a pseudo-metric space. Give a point $a \in X$ and a real number $r > 0$, let $B_d(a, r) := \{x \in X : d(a, x) < r\}$.

(i) Show that the set $\mathscr{B}_d := \{B_d(a, r) : a \in X, r \in \mathbb{R}^+\}$ forms a basis for a topology τ_d on X, called the *pseudo-metric topology* on (X, d).

(ii) Show that (X, τ_d) is Hausdorff if and only if d is a metric.

Then next lemma tells us how to construct a metric space out of a pseudo-metric space.

Lemma 2.14.39. *Let (X, d) be a pseudo-metric space. Define a relation $\rho \subseteq X \times X$ on X by setting*

$$(x, y) \in \rho \;\; \text{if} \;\; d(x, y) = 0.$$

Then ρ is an equivalence relation on X. Let $\widetilde{X}$ be the set of all ρ-equivalence classes of elements in X. Then the map $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ defined by

$$\widetilde{d}([x],[y]) = d(x,y),\ \forall\, [x],[y] \in \widetilde{X},$$

is a metric on $\widetilde{X}$. The pair $(\widetilde{X},\widetilde{d})$ is called the metric space associated to the pseudo-metric space (X,d).

Proof. Since $d(x,x) = 0$, $\forall\, x \in X$, ρ is reflexive. Since $d(x,y) = d(y,x)$, $\forall\, x,y \in X$, ρ is symmetric. If $d(x,y) = 0$ and $d(y,z) = 0$, for some $x,y,z \in X$, then by triangular inequality we have $0 \leq d(x,z) \leq d(x,y) + d(y,z) = 0$ and hence $d(x,z) = 0$. Therefore, ρ is transitive. Thus, ρ is an equivalence relation on X. Let $\widetilde{X} = X/\rho$ be the set of all ρ-equivalence classes of elements of X. We denote by

$$\overline{x} := \{y \in X : d(y,x) = 0\} \in \widetilde{X}$$

the ρ-equivalence class of $x \in X$ in X. Define

$$\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}$$

by

$$\widetilde{d}(\overline{x},\overline{y}) := d(x,y),\ \forall\, \overline{x},\overline{y} \in \widetilde{X}.$$

Let $x,x',y,y' \in X$ be such that $\overline{x} = \overline{x'}$ and $\overline{y} = \overline{y'}$. Then $d(x,x') = 0$ and $d(y,y') = 0$. Then

$$\begin{aligned} d(x,y) &\leq x(x,x') + d(x',y) \\ &= d(x',y) \\ &\leq d(x',x) + d(x,y) \\ &= d(x,y) \end{aligned}$$

implies that $d(x,y) = d(x',y)$. Similarly, $d(x',y) = d(x',y')$ since $d(y,y') = 0$. Therefore, $d(x,y) = d(x',y')$, and hence $\widetilde{d}$ is well-defined. Note that $\widetilde{d}(\widetilde{x},\widetilde{y}) = d(x,y) = 0$ if and only if $\widetilde{x} = \widetilde{y}$ in $\widetilde{X}$. Clearly for all $\widetilde{x},\widetilde{y},\widetilde{z} \in \widetilde{X}$, we have

$$\widetilde{d}(\widetilde{x},\widetilde{y}) = d(x,y) = d(y,x) = \widetilde{d}(\widetilde{y},\widetilde{x}),$$

and

$$\widetilde{d}(\widetilde{x},\widetilde{y}) \leq \widetilde{d}(\widetilde{x},\widetilde{z}) + \widetilde{d}(\widetilde{z},\widetilde{y}).$$

Therefore, $(\widetilde{X},\widetilde{d})$ is a metric space. $\square$

Remark 2.14.40. Note that, by choosing one element from each of the ρ-equivalence class in $\widetilde{X}$ (by axiom of choice), we get a subset $Y \subseteq X$. It is clear that $d\big|_{Y\times Y} : Y \times Y \to [0,\infty)$ is a metric on Y.

Lemma 2.14.41. *Let (Y,d) be a metric space. Let A be a dense subset of (Y,d). If every Cauchy sequence in A converges to some point in Y, then (Y,d) is complete.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (Y, d). Since A is dense in Y, for each $n \in \mathbb{N}$, there exists $a_n \in B_d(y_n, 1/n) \cap A$. Let $\epsilon > 0$ be given. Then there exists $n_\epsilon \in \mathbb{N}$ such that $1/n < \epsilon/3$, $\forall\, n \geq n_\epsilon$. Since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $m_\epsilon \in \mathbb{N}$ such that

$$d(y_n, y_m) < \epsilon/3,\ \forall\, m, n \geq m_\epsilon.$$

Set $M = \max\{n_\epsilon, m_\epsilon\}$. Then for all $m, n \geq M$, we have

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, y_n) + d(y_n, y_m) + d(y_m, a_m) \\ &< \frac{1}{n} + \frac{\epsilon}{3} + \frac{1}{m} \\ &< \epsilon. \end{aligned}$$

Then $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, and hence it converges to a point, say $y \in Y$, by assumption. Then there exists $K \in \mathbb{N}$ such that

$$d(a_n, y) < \epsilon/3,\ \forall\, n \geq K.$$

Set $M_0 = \max\{M, K\}$. Then for all $n \geq M_0$, we have

$$\begin{aligned} d(y_n, y) &\leq d(y_n, a_n) + d(a_n, y) \\ &< \frac{1}{n} + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

Therefore, $(y_n)_{n\in\mathbb{N}}$ converges to $y \in Y$, and hence (Y, d) is complete. $\square$

Theorem 2.14.42 (Completion of a metric space)**.** *Given a metric space (X, d), there exists a pair $((\widehat{X}, \widehat{d}), \iota)$ consisting of a complete metric space $(\widehat{X}, \widehat{d})$ and an isometric embedding $\iota : (X, d) \to (\widehat{X}, \widehat{d})$ such that*

(i) $\iota(X)$ is dense in $(\widehat{X}, \widehat{d})$, and

(ii) Universal property: given any complete metric space (Y, ρ) and a uniformly continuous map $f : (X, d) \to (Y, \rho)$, there exists a unique uniformly continuous map $\widetilde{f} : (\widehat{X}, \widehat{d}) \to (Y, \rho)$ such that $\widetilde{f} \circ \iota = f$.

$$\begin{tikzcd} (X,d) \arrow[r, "f"] \arrow[d, "\iota"'] & (Y,\rho) \\ (\widehat{X},\widehat{d}) \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

*The pair $(\widehat{X}, \widehat{d})$ is uniquely determined, up to a unique isometry, by the above two properties (meaning that, if (Y, ρ) is any complete metric space admitting an isometric embedding $f : (X, d) \to (Y, \rho)$ such that the pair $((Y, \rho), f)$ satisfy the above two properties, then there exists a unique isometric homeomorphism $\Phi : (\widehat{X}, \widehat{d}) \to (Y, \rho)$ such that $\Phi \circ \iota = f$). The pair $(\widehat{X}, \widehat{d})$ is called the **completion** of (X, d).*

Proof. Uniqueness of the pair $\big((\widehat{X}, \widehat{d}), \iota\big)$, up to a unique isometric homeomorphism, is already proved in Lemma 2.14.34. It remains to show its existence.

Let $\mathscr{C}(X^{\mathbb{N}})$ be the set of all Cauchy sequences in (X, d). Given $(x_n), (y_n) \in \mathscr{C}(X^{\mathbb{N}})$, we define

$$d((x_n), (y_n)) = \lim_{n \to \infty} d(x_n, y_n).$$

Note that, for all (x_n), (y_n) and (z_n) in $\mathscr{C}(X^{\mathbb{N}})$, we have

(i) $d((x_n), (y_n)) \geq 0$,

(ii) $d((x_n), (y_n)) = d((y_n), (x_n))$, and

(iii) $d((x_n), (y_n)) \leq d((x_n), (z_n)) + d((z_n), (y_n))$.

However, we may have two distinct Cauchy sequences (x_n) and (y_n) in (X, d) with $d((x_n), (y_n)) = 0$. This motivates us to define a relation $\sim$ on the set $\mathscr{C}(X^{\mathbb{N}})$ by setting

$$(x_n) \sim (y_n) \text{ if } \lim_{n \to \infty} d(x_n, y_n) = 0.$$

Clearly this is an equivalence relation on $\mathscr{C}(X^{\mathbb{N}})$. Let

$$\widehat{X} := \mathscr{C}(X^{\mathbb{N}})/\sim$$

be the set of all $\sim$-equivalence classes of Cauchy sequences in (X, d). Define

$$\widehat{d}([(x_n)], [(y_n)]) := \lim_{n \to \infty} d(x_n, y_n).$$

Note that $\widehat{d}$ is a metric on $\widehat{X}$ (see Lemma 2.14.39).

Isometric embedding: Note that the natural map

$$\iota : (X, d) \to (\widehat{X}, \widehat{d})$$

that sends a point $x \in X$ to the constant sequence $(x, x, \cdots) \in \widehat{X}$, is an isometric embedding. Indeed, ι is an injective map, and given $x, y \in X$, we have $\widehat{d}(\iota(x), \iota(y)) = d(x, y)$ (see Proposition 2.14.31).

Image of X under ι is dense in $(\widehat{X}, \widehat{d})$: Let $z \in \widehat{X}$ be given by a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in (X, d). Since the isometric embedding map ι is uniformly continuous, the image sequence $(\iota(x_n))_{n \in \mathbb{N}}$ in $\iota(X) \subseteq \widehat{X}$ is Cauchy, which clearly converge to the point $z = [(x_n)]$ in $(\widehat{X}, \widehat{d})$ (verify!).

Completeness of $(\widehat{X}, \widehat{d})$: Since for any Cauchy sequence $(a_n)_{n \in \mathbb{N}}$ in (X, d), its image $(\iota(x_n))_{n \in \mathbb{N}}$ converges to the point $[(x_n)_{n \in \mathbb{N}}] \in \widehat{X}$ (verify!), completeness of $(\widehat{X}, \widehat{d})$ follows from Lemma 2.14.41.

Universal property of $(\widehat{X}, \widehat{d})$: Let (Y, ρ) be a complete metric space and $f : (X, d) \to (Y, \rho)$ a uniformly continuous map. Since $\iota : (X, d) \to (\widehat{X}, \widehat{d})$ is an isometric embedding with dense image, by Uniform Extension Theorem (Theorem 2.14.24) the map $f \circ \iota^{-1} : \iota(X) \to Y$ uniquely extends to a uniformly continuous map

$$\widetilde{f} : (\widehat{X}, \widehat{d}) \to (Y, \rho)$$

such that $\widetilde{f}|_{\iota(X)} = f \circ \iota^{-1}$. Therefore, $\widetilde{f} \circ \iota = f$. This completes the proof. $\square$

Exercise 2.14.43. Show that the completion of the $(\mathbb{Q}, |\cdot|)$ is the real line $(\mathbb{R}, |\cdot|)$.

Recall that, given a metric space (X, d), the map $\overline{d} : X \times X \to \mathbb{R}$ defined by

$$\overline{d}(x, y) := \min\{1, d(x, y)\}, \ \forall\, x, y \in X,$$

is a metric on X, called the *standard bounded metric on* X.

Lemma 2.14.44. *Let (X, d) be a metric space, and let $\overline{d}$ be the standard bounded metric on X associated to d. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is Cauchy with respect to d if and only if it is Cauchy with respect to $\overline{d}$. Consequently, (X, d) is complete if and only if $(X, \overline{d})$ is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then taking $0 < \epsilon < 1$, we see that $\overline{d}(x_n, x_m) < \epsilon$ if and only if $d(x_n, x_m) < \epsilon$. Therefore, (x_n) is Cauchy in (X, d) if and only if it is Cauchy in $(X, \overline{d})$. Hence the result follows. $\square$

Let I be an index set, and consider the Cartesian product set X^I. Given $f, g \in X^I$, let

$$\rho_d(f, g) := \sup\left\{\overline{d}(f(\alpha), g(\alpha)) : \alpha \in I\right\}. \tag{2.14.45}$$

Then ρ_d is a metric on X^I, called the *uniform metric* on X^I induced from (X, d).

Theorem 2.14.46. *If (X, d) is a complete metric space, then (X^I, ρ_d) is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X^I, ρ_d). Let $\epsilon > 0$ be given. Without loss of generality, we may assume that $\epsilon < 1$. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$\rho_d(f_n, f_m) < \epsilon/2, \ \forall\, m, n \geq n_\epsilon.$$

Since for given any $\alpha \in I$, we have (see (2.14.45))

$$\overline{d}(f_n(\alpha), f_m(\alpha)) \leq \rho_d(f_n, f_m) < \epsilon/2, \ \forall\, n, m \geq n_\epsilon, \tag{2.14.47}$$

it follows that $(f_n(\alpha))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, \overline{d})$, and hence in (X, d) by Lemma 2.14.44, for all $\alpha \in I$. Since (X, d) is complete, it converges to some point, say $f(\alpha) \in X$, for all $\alpha \in I$. Then $\alpha \mapsto f(\alpha)$ defines an element $f \in X^I$. Fix $n \geq n_\epsilon$ and $\alpha \in I$. Then letting $m \to \infty$ in (2.14.47) we see that

$$\overline{d}(f_n(\alpha), f(\alpha)) \leq \epsilon/2 < \epsilon. \tag{2.14.48}$$

Since this inequality holds for all $\alpha \in I$ and $n \geq n_\epsilon$, we conclude that

$$\rho_d(f_n, f) = \sup_{\alpha\in I} \overline{d}(f_n(\alpha), f(\alpha)) \leq \epsilon/2 < \epsilon, \ \forall\, n \geq n_\epsilon.$$

Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in (X^I, ρ_d). This completes the proof. □

Let X be a topological space and let (Y,d) be a metric space. A map $f : X \to Y$ is said to be *bounded* if there exists a constant $M > 0$ such that

$$d(f(x), f(y)) < M, \ \forall\, x, y \in X.$$

A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to converge uniformly to an element $f \in Y^X$ with respect to the uniform metric ρ_d on Y^X induced from (Y,d) if for given any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \epsilon, \ \forall\, n \geq n_\epsilon, \ x \in X.$$

Theorem 2.14.49 (Generalized uniform limit theorem)**.** *Let X be a topological space and let (Y,d) be a metric space. Let $C(X,Y)$ be the set of all continuous maps from X into (Y,d), and let $B(X,Y)$ be the set of all bounded functions from X into (Y,d). Equip Y^X with the uniform metric ρ_d induced from (Y,d). Then*

(i) both $C(X,Y)$ and $B(X,y)$ are closed subsets of (Y^X, ρ_d), and

(ii) if (Y,d) is complete, then $C(X,Y)$ and $B(X,Y)$ are complete.

Proof. We first show that if a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X converges to an element $f \in Y^X$ in the uniform metric ρ_d, then it converges uniformly with respect to the metric $\overline{d}$ on Y. Let $\epsilon > 0$ be given. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$\rho_d(f_n, f) < \epsilon, \ \forall\, n \geq n_\epsilon.$$

Then by definition of uniform metric ρ_d, we see that for any $x \in X$, we have

$$\overline{d}(f_n(x), f(x)) \leq \rho_d(f_n, f) < \epsilon, \ \forall\, n \geq n_\epsilon.$$

Therefore, (f_n) converges uniformly to f with respect to the metric $\overline{d}$ on Y.

To show that $C(X,Y)$ is closed in (Y^X, ρ_d), let $f \in \overline{C(X,Y)}$ be given. Choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X,Y)$ that converges to f with respect to the uniform metric ρ_d. Then by the above argument, the sequence (f_n) converges uniformly to f in $(Y, \overline{d})$, and then f is continuous by uniform limit theorem 2.3.22. Therefore, $f \in C(X,Y)$.

To show that $B(X,Y)$ is closed in (Y^X, ρ_d), let $f \in \overline{B(X,Y)}$ be arbitrary. Choose a sequence (f_n) in $B(X,Y)$ that converges to f with respect to the uniform metric ρ_d. Then for $\epsilon = 1/2$, we can find $N \in \mathbb{N}$ such that

$$\rho_d(f_n, f) < 1/2, \ \forall\, n \geq N.$$

Since f_N is bounded, $\operatorname{diam}(f_N(X)) < M$, for some real number $M > 0$. Then by triangle inequality, we have $\operatorname{diam}(f(X)) \leq M + 1$. Therefore, $f \in B(X,Y)$.

The second part follows from the first part applying Theorem 2.14.46 and Lemma 2.14.16. □

2.15 Compact Metric Spaces

Definition 2.15.1. A metric space (X, d) is said to be totally bounded if for given a real number $\epsilon > 0$, there exists finitely many points $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$.

Theorem 2.15.2. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. Let (X, d) be a compact metric space. Then it is complete by Theorem 2.11.45 and Lemma 2.14.7. For given $\epsilon > 0$, the collection $\{B_d(x, \epsilon) : x \in X\}$ being an open cover of (X, d), by compactness of (X, d), we can find finitely many points $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$. Therefore, (X, d) is totally bounded.

Conversely, assume that (X, d) is complete and totally bounded. We show that (X, d) is sequentially compact, and hence is compact by Theorem 2.11.45. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in (X, d). Since (X, d) is complete, it suffices to show that $(x_n)_{n \in \mathbb{N}}$ has a Cauchy subsequence. Without loss of generality, we may assume that $(x_n)_{n \in \mathbb{N}}$ is not eventually constant (c.f. Definition 2.14.3). Since (X, d) is totally bounded, we can cover X by finitely many open balls of radius 1; at least one of which, say B_1, contains x_n for infinitely many values of $n \in \mathbb{N}$. Let $J_1 = \{n \in \mathbb{N} : x_n \in B_1\}$. Assume that $n \geq 2$, and we have constructed nested sequence of infinite subsets

$$J_1 \supseteq \cdots \supseteq J_{n-1}$$

of $\mathbb{N}$ and open balls B_i of radius $1/i$, for $i = 1, \ldots, n-1$, such that $x_k \in B_i$, $\forall\, k \in J_i; i = 1, \ldots, n-1$. Then covering X by finitely many open balls of radius $1/n$, we can find an open ball, say B_n, of radius $1/n$ that contains x_k for infinitely many values of $k \in J_{n-1}$, and then set $J_n := \{k \in J_{n-1} : x_k \in B_n\}$, and proceed inductively.

Choose $n_1 \in J_1$. Assume that $k \geq 2$, and we have chosen $n_i \in J_i$, for all $i = 1, \ldots, k-1$, in such a way that $n_1 < \cdots < n_{k-1}$. Then we choose $n_k \in J_k$ such that $n_{k-1} < n_k$. We can do this because all J_k' are infinite. Then by induction we can find a sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $n_k < n_{k+1}$, $\forall\, k \in \mathbb{N}$.

Let $\epsilon > 0$ be given. Then there exists $n_\epsilon \in \mathbb{N}$ such that $1/n_\epsilon < \epsilon$. Then for all $i, j \geq n_\epsilon$, we have $n_i, n_j \in J_{n_\epsilon}$, and hence $x_{n_i}, x_{n_j} \in B_{n_\epsilon}$. Therefore, we have

$$d(x_{n_i}, x_{n_j}) < 1/n_\epsilon < \epsilon, \forall\, i, j \geq n_\epsilon,$$

and hence the subsequence $(x_{n_k})_{k \in \mathbb{N}}$ is Cauchy. This completes the proof. $\square$

Definition 2.15.3. Let X be a topological space and (Y, d) a metric space. Let $C(X, Y)$ be the set of all continuous maps from X into (Y, d). A subset $\mathcal{F} \subseteq C(X, Y)$ is said to be *equicontinuous at* $x_0 \in X$ if for given a real number $\epsilon > 0$, there exists an open neighbourhood $U \subseteq X$ of x_0 such that

$$d(f(x), f(y)) < \epsilon, \ \forall\, f \in \mathcal{F},\ x, y \in U.$$

If $\mathcal{F}$ is equicontinuous at every point of X, then $\mathcal{F}$ called *equicontinuous*.

Definition 2.15.4. A topological space X is said to be a *Baire space* if for given any countable collection $\{U_n : n \in \mathbb{N}\}$ of non-empty open dense subsets, the subset $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

Theorem 2.15.5 (Baire Category Theorem)**.**

(i) A compact Hausdorff space is a Baire space.

(ii) A complete metric space is a Baire space.

Proof. □

Lemma 2.15.6. *Any open subspace of a Baire space is a Baire space.*

Proof. □

Theorem 2.15.7. *Let X be a topological space and (Y, d) a metric space. Let $(f_n : X \to Y)_{n\in\mathbb{N}}$ be a sequence of continuous maps that point-wise converges to a map $f : X \to Y$, i.e., for each $x \in X$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ in (Y, d). If X is a Baire space, then the subset*

$$\mathcal{DC}(f) := \{x \in X : f \text{ is continuous at } x\}$$

is dense in X.

Proof. □

2.16 Stone-Weierstrass theorem

Theorem 2.16.1 (Stone–Weierstrass)**.** *Let X be a compact Hausdorff space. Equip $\mathscr{C}(X, \mathbb{R})$ the topology of uniform convergence. Let $\mathcal{A}$ be a $\mathbb{R}$-subalgebra of $\mathscr{C}(X, \mathbb{R})$ which contains a non-zero constant function. Then $\mathcal{A}$ is dense in $\mathscr{C}(X, \mathbb{R})$ if and only if it separates points.*

Proof. □

Theorem 2.16.2 (Stone–Weierstrass: Locally compact version)**.** *Let X be a locally compact Hausdorff space with the one-point compactification $\widehat{X}$. Let $\mathscr{C}_0(X, \mathbb{R})$ be the set of all real valued continuous maps from X into $\mathbb{R}$ that vanishes at $\infty \in \widehat{X}$. Equip $\mathscr{C}_0(X, \mathbb{R})$ with the topology of uniform convergence. Let $\mathcal{A}$ be an $\mathbb{R}$-subalgebra of $\mathscr{C}_0(X, \mathbb{R})$. Then $\mathcal{A}$ is dense in $\mathscr{C}_0(X, \mathbb{R})$ if and only if it separates points and vanishes nowhere.*

Proof. □

Corollary 2.16.3 (Weierstrass Approximation Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous map. Then for given an $\epsilon > 0$, there exists a polynomial $p \in \mathbb{R}[x]$ such that*

$$|f(x) - p(x)| < \epsilon, \; \forall \; x \in [a, b].$$

In other words, $\|f - p\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ denotes the sup norm.

2.17 Ascoli & Arzela's theorem

Theorem 2.17.1 (Ascoli). *Let X be a topological space and let (Y, d) be a metric space. Give $\mathscr{C}(X, Y)$ the topology of compact convergence, and let $\mathcal{F} \subseteq \mathscr{C}(X, y)$.*

(i) If $\mathcal{F}$ is equicontinuous under d and the set

$$\mathcal{F}_a := \{f(a) : f \in \mathcal{F}\}$$

has compact closure in Y for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $\mathscr{C}(X, Y)$.

(ii) The converse holds if X is locally compact and Hausdorff.

Proof. □

Corollary 2.17.2 (Ascoli's theorem: classical version). *Let X be a compact topological space. Fix an integer $k \geq 1$, and equip $\mathbb{R}^k$ with the standard Euclidean metric. Equip $\mathscr{C}(X, \mathbb{R}^k)$ with the uniform topology. Then a subset $\mathcal{F} \subseteq \mathscr{C}(X, \mathbb{R}^k)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and point-wise bounded.*

Proof. □

Theorem 2.17.3 (Arzela). *Let X be a σ-compact Hausdorff space. Fix an integer $k \geq 1$, and equip $\mathbb{R}^k$ with the standard Euclidean metric. Equip $\mathscr{C}(X, \mathbb{R}^k)$ the topology of compact convergence. If a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathscr{C}(X, \mathbb{R}^k)$ is point-wise bounded and equicontinuous, it has a subsequence that converges to a continuous function $f \in \mathscr{C}(X, \mathbb{R}^k)$.*

Proof. □

Chapter 3

Algebraic Topology

MA4104 Syllabus

Homotopy theory: Review of quotient topology, path homotopy, definition of fundamental group, covering spaces, path and homotopy lifting, fundamental group of S^1, deformation retraction, Brouwer's fixed point theorem, Borsuk-Ulam theorem, Van-Kampen's theorem, fundamental group of surfaces, universal covering space, correspondence between covering spaces and subgroups of fundamental group.

Homology Theory: Simplicial complexes and maps, homology groups, computation for surfaces.

3.1 Homotopy of maps

[[Examples and Exercises need to be added.]]

Let I be the closed interval $[0,1] \subset \mathbb{R}$. Let X and Y be topological spaces.

Definition 3.1.1. Let $f_0, f_1 : X \to Y$ be continuous maps. We say that f_0 *is homotopic to* f_1, written as $f_0 \simeq f_1$, if there is a continuous map

$$F : X \times I \longrightarrow Y$$

such that $F(x,0) = f_0(x)$, $\forall\, x \in X$, and $F(x,1) = f_1(x)$, $\forall\, x \in X$. In this case, the continuous map F is called the *homotopy* from f_0 to f_1. A continuous map $f : X \to Y$ is said to be *null homotopic* if f is homotopic to a constant map from X into Y.

Example 3.1.2. 1. Let X be a space. Then any two continuous maps $f, g : X \to \mathbb{R}^2$ are homotopic. To see this, note that the map $F : X \times I \to \mathbb{R}^2$ defined by

$$F(x,t) = (1-t)f(x) + tg(x), \;\; \forall\, (x,t) \in X \times I,$$

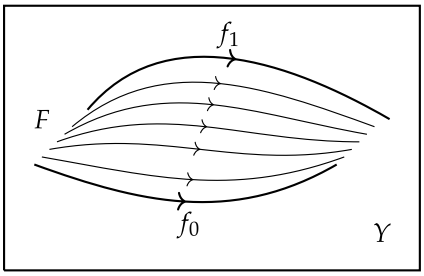


FIGURE 3.1: Homotopy

is continuous and satisfies $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in X$. Thus F is a homotopy from f to g; such a homotopy is called a *straight-line homotopy*, because for each $x \in X$, it movies $f(x)$ to $g(x)$ along the straight-line segment joining them.

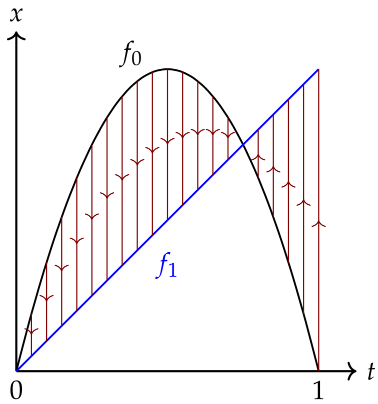


FIGURE 3.2: Example of a straight-line homotopy

Before proceeding further, let us recall the following useful result from basic topology course, that we need frequently in this course.

Lemma 3.1.3 (Joining continuous maps). *Let A and B be two closed subsets of topological space X such that $X = A \cup B$. Let Y be any topological space. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps such that $f(x) = g(x)$, for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & \text{if} \quad x \in A, \\ g(x), & \text{if} \quad x \in B, \end{cases}$$

is continuous.

Proof. Let $Z \subseteq Y$ be a closed subset. It is enough to check that $h^{-1}(Z)$ is closed in X. Note that

$$\begin{aligned} h^{-1}(Z) &= \left(h^{-1}(Z) \cap A\right) \bigcup \left(h^{-1}(Z) \cap B\right) \\ &= f^{-1}(Z) \bigcup g^{-1}(Z). \end{aligned}$$

Since f and g are continuous, $f^{-1}(Z)$ is closed in A and $g^{-1}(Z)$ is closed in B. Since A and B are closed in X, both $f^{-1}(Z)$ and $g^{-1}(Z)$ are closed in X, and so is their union $h^{-1}(Z)$. This completes the proof. □

Lemma 3.1.4. *The relation "being homotopic maps" is an equivalence relation on the set $C(X, Y)$ of all continuous maps from X into Y.*

Proof. For any $f \in C(X, Y)$, taking

$$F : X \times I \to Y, \ \ (x, t) \mapsto f(x)$$

we see that f is homotopic to itself, and hence "being homotopic maps" is a reflexive relation. Let $f_0, f_1 \in C(X, Y)$ be such that f_0 is homotopic to f_1 with homotopy F. Then the continuous map

$$G : X \times I \to Y, \ \ (x, t) \mapsto F(x, 1 - t)$$

is a homotopy from f_1 to f_0. So "being homotopic maps" is a symmetric relation. Let $f_0, f_1, f_2 \in C(X, Y)$ be such that $f_0 \simeq f_1$ with a homotopy F, and $f_1 \simeq f_2$ with a homotopy G. Consider the map $H : X \times I \to Y$ defined by

$$H(x, t) := \begin{cases} F(x, 2t), & \text{if} \quad t \in [0, \frac{1}{2}], \\ G(x, 2t - 1), & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

Since at $t = \frac{1}{2}$, we have $F(x, 2t) = F(x, 1) = f_1(x) = G(x, 0) = G(x, 2t - 1)$, for all $x \in X$, we see that H is a well-defined continuous map (c.f., Lemma 3.1.3). Clearly H satisfies $H(x, 0) = f_0(x)$ and $H(x, 1) = f_2(x)$, for all $x \in X$. Therefore, H is a homotopy from f_0 to f_2. Thus "being homotopic maps" is a transitive relation, and hence is an equivalence relation on $C(X, Y)$. □

Exercise 3.1.5. Let $f, g \in C(X, Y)$, and $F : X \times I \to Y$ be a homotopy from f to g. Use F to construct a homotopy G from f to g with $G \neq F$. Therefore, homotopy between two maps need not be unique. (*Hint: take* $G(x, t) = F(x, t^2)$).

Lemma 3.1.6. *Let $f_0, f_1 : X \to Y$ be two continuous maps such that f_0 is homotopic to f_1. Then for any spaces Z and W, and continuous maps $g : Z \to X$ and $h : Y \to W$, we have $f_0 \circ g \simeq f_1 \circ g$ and $h \circ f_0 \simeq h \circ f_1$.*

$$\begin{tikzcd} Z \arrow[r, "g"] & X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, "h"] & W. \end{tikzcd}$$

Proof. Let $F : X \times I \to Y$ be a continuous map such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, for all $x \in X$. Define $G : Z \times I \to Y$ by setting

$$G(z, t) = F(g(z), t), \ \forall \ (z, t) \in Z \times I.$$

Clearly G is a continuous function with $G(z, 0) = F(g(z), 0) = (f_0 \circ g)(z)$, and $G(z, 1) = F(g(z), 1) = (f_1 \circ g)(z)$, for all $z \in Z$. Therefore, G gives a homotopy $f_0 \circ g \simeq f_1 \circ g$. Similarly, taking

$$H : X \times I \to W, \quad (x,t) \mapsto h(F(x,t)),$$

we see that H is a continuous map satisfying $H(x,0) = h(F(x,0)) = (h \circ f_0)(x)$ and $H(x,1) = h(F(x,0)) = (h \circ f_1)(x)$, for all $x \in X$. Therefore, H gives a homotopy $h \circ f_0 \simeq h \circ f_1$. □

Definition 3.1.7. Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be continuous maps of pointed topological spaces. A *homotopy* from f_0 to f_1 is a continuous map $F : X \times I \to Y$ such that

(i) $F(x,0) = f_0(x)$, $\forall\, x \in X$,

(ii) $F(x,1) = f_1(x)$, $\forall\, x \in X$, and

(iii) $F(x_0,t) = y_0$, $\forall\, t \in [0,1]$.

When we talk about homotopy of continuous maps of pointed topological spaces, we always mean that the homotopy preserve the marked points in the sense of (iii) mentioned above.

Exercise 3.1.8. Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be two continuous maps of pointed topological spaces. If f_0 is homotopic to f_1 in the sense of Definition 3.1.7, show that for any spaces Z and W, and continuous maps $g : (Z, z_0) \to (X, x_0)$ and $h : (Y, y_0) \to (W, w_0)$, we have

(i) $f_0 \circ g$ is homotopic to $f_1 \circ g$ in the sense of Definition 3.1.7, and

(ii) $h \circ f_0$ is homotopic to $h \circ f_1$ in the sense of Definition 3.1.7.

Definition 3.1.9. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there exist a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. In this case, we say that X *is homotopy equivalent to* Y (or, X *and* Y *have the same homotopy type*), and write it as $X \simeq Y$.

Lemma 3.1.10. *Being homotopy equivalent spaces is an equivalence relation.*

Proof. For any space X, we can take $f = g = \mathrm{Id}_X$ to get $f \circ g = \mathrm{Id}_X = g \circ f$ so that X is homotopy equivalent to itself (verify!). It follows from the Definition 3.1.9 that the relation “being homotopy equivalent spaces” is symmetric. Let X, Y and Z be topological spaces such that $X \simeq Y$ and $Y \simeq Z$. Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be homotopy equivalences. Then there are continuous maps $g_1 : Y \to X$ and $g_2 : Z \to Y$ such that $g_1 \circ f_1 \simeq \mathrm{Id}_X$, $f_1 \circ g_1 \simeq \mathrm{Id}_Y$, $g_2 \circ f_2 \simeq \mathrm{Id}_Y$ and $f_2 \circ g_2 \simeq \mathrm{Id}_Z$. Now using Lemma 3.1.6 we have

$$\begin{aligned}(f_2 \circ f_1) \circ (g_1 \circ g_2) &= f_2 \circ (f_1 \circ g_1) \circ g_2 \\ &\simeq f_2 \circ \mathrm{Id}_Y \circ g_2 \\ &= f_2 \circ g_2 \simeq \mathrm{Id}_Z\,.\end{aligned}$$

Similarly, we have $(g_1 \circ g_2) \circ (f_2 \circ f_1) \simeq \mathrm{Id}_X$. Therefore, $f_2 \circ f_1 : X \to Z$ is a homotopy equivalence, and hence $X \simeq Z$. Thus “being homotopy equivalent spaces” is a transitive relation, and hence is an equivalence relation. □

Definition 3.1.11. A space X is said to be *contractible* if the identity map $\mathrm{Id}_X : X \to X$ is null homotopic.

Exercise 3.1.12. Show that a contractible space is path-connected.

Corollary 3.1.13. *A space X is contractible if and only if given any topological space T, any two continuous maps $f, g : T \to X$ are homotopic.*

Proof. Suppose that X is contractible. Let T be any topological space, and let $f, g : T \to X$ be any two continuous maps. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ of X is homotopic to a constant map $c_{x_0} : X \to X$ given by $c_{x_0}(x) = x_0$, $\forall\, x \in X$. Then $f = \mathrm{Id}_X \circ f$ is homotopic to the constant map $c_{x_0} \circ f : T \to X$. Similarly, g is homotopic to the constant map $c_{x_0} \circ g : T \to X$. Since $c_{x_0} \circ f = c_{x_0} \circ g$, and being homotopic maps is an equivalence relation by Lemma 3.1.4, we see that f is homotopic to g. Converse part is obvious. $\square$

3.2 Fundamental group

3.2.1 Construction

A *path* in X is a continuous map $\gamma : I \to X$; the point $\gamma(0) \in X$ is called the *initial point* of γ, and $\gamma(1) \in X$ is called the *terminal point* or the *final point* of γ.

Definition 3.2.1. Fix two points $x_0, x_1 \in X$. Two paths $f, g : I \to X$ with the same initial point x_0 and terminal point x_1 are said to be *path homotopic* if

(i) $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, and

(ii) there is a continuous map $F : I \times I \to X$ such that for each $t \in I$, the map

$$\gamma_t : I \to X, \;\; s \mapsto F(s, t)$$

is a path in X from x_0 to x_1, and that $\gamma_0 = f$ and $\gamma_1 = g$.

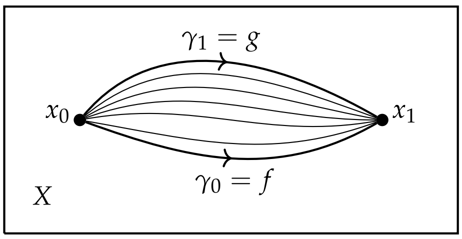


FIGURE 3.3: Path homotopy

Exercise 3.2.2. Given $x_0, x_1 \in X$, let

$$Path(X; x_0, x_1) := \{f : I \to X \,|\, f(0) = x_0,\; f(1) = x_1\}$$

be the set of all paths in X starting at x_0 and ending at x_1. Show that being path homotopic is an equivalence relation on $Path(X; x_0, x_1)$. (*Hint: Follow the proof of Lemma 3.1.4*).

Remark 3.2.3. If $\gamma, \delta : I \to X$ are two paths in X, we use the symbol $\gamma \simeq \delta$ to mean γ and δ are path-homotopic in X in the sense of Definition 3.2.1. Unless explicitly mentioned, by a *homotopy between two paths* we always mean a path-homotopy between them.

A *loop* in X is a path $\gamma : I \to X$ with the same initial and terminal point: i.e., $\gamma(0) = \gamma(1) = x_0 \in X$; the point x_0 is called the base point of the loop γ. For a loop $\gamma : I \to X$ based at $x_0 \in X$, let

$$[\gamma] := \{\delta : I \to X \mid \delta(0) = \delta(1) = x_0 \text{ and } \delta \simeq \gamma\},$$

the homotopy equivalence class of γ. Fix a base point $x_0 \in X$, and let

$$\pi_1(X, x_0) := \{[\gamma] \mid \gamma : I \to X \text{ with } \gamma(0) = \gamma(1) = x_0\}$$

be the set of all equivalence classes of loops in X based at x_0. Next we define a binary operation on $\pi_1(X, x_0)$ and show that it is a group.

Given any two loops $\gamma_1, \gamma_2 : I \to X$ in X with the base point $x_0 \in X$, we define the *product of* γ_1 *with* γ_2 to be the map $\gamma_1 \star \gamma_2 : I \to X$ defined by

$$(\gamma_1 \star \gamma_2)(t) := \begin{cases} \gamma_1(2t), & \text{if } \ t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1), & \text{if } \ t \in [\frac{1}{2}, 1]. \end{cases} \tag{3.2.4}$$

That is, we first travel along γ_1 with double speed from $t = 0$ to $t = \frac{1}{2}$, and then along γ_2 from $t = \frac{1}{2}$ to $t = 1$. Clearly $\gamma_1 \star \gamma_2$ is a continuous map with $(\gamma_1 \circ \gamma_2)(0) = (\gamma_1 \circ \gamma_2)(1) = x_0$, and hence $\gamma_1 \circ \gamma_2$ is a loop in X with the base point x_0. Note that $\gamma_1 \star \gamma_2 \neq \gamma_2 \star \gamma_1$, in general (Find such an example).

Remark 3.2.5. In fact, we shall see later examples of topological spaces X admitting loops $\gamma_1, \gamma_2 : I \to X$ with the same base point $x_0 \in X$ such that $\gamma_1 \star \gamma_2$ is not homotopic to $\gamma_2 \star \gamma_1$.

Proposition 3.2.6. *Let γ_1, γ_2, δ_1, δ_2 be loops in X based at x_0. If $\gamma_1 \simeq \delta_1$ and $\gamma_2 \simeq \delta_2$, then $(\gamma_1 \star \gamma_2) \simeq (\delta_1 \star \delta_2)$. Consequently, the map*

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0), \ \ ([\gamma_1], [\gamma_2]) \mapsto [\gamma_1 \star \gamma_2] \tag{3.2.7}$$

is well-defined, and hence is a binary operation on the set $\pi_1(X, x_0)$.

Proof. Let $F : I \times I \to X$ be a homotopy from $F(-, 0) = \gamma_1$ to $F(-, 1) = \delta_1$, and let $G : I \times I \to X$ be a homotopy from $G(-, 0) = \gamma_2$ to $G(-, 1) = \delta_2$. Define a map $F \star G : I \times I \to X$ by sending $(s, t) \in I \times I$ to

$$(F \star G)(s, t) := \begin{cases} F(2s, t), & \text{if } \ 0 \leq s \leq 1/2, \\ G(2s-1, t), & \text{if } \ 1/2 \leq s \leq 1. \end{cases}$$

Clearly $F \star G$ is a continuous map with $(F \star G)(-, 0) = \gamma_1 \star \gamma_2$ and $(F \star G)(-, 1) = \delta_1 \star \delta_2$. Therefore, $\gamma_1 \star \gamma_2 \simeq \delta_1 \star \delta_2$. $\square$

Theorem 3.2.8. *The set $\pi_1(X, x_0)$ together with the binary operation (3.2.7) defined in Proposition 3.2.6 is a group, known as the fundamental group of X with base point $x_0 \in X$.*

To prove this theorem, we use the following technical tool (Lemma 3.2.10).

Definition 3.2.9. A *reparametrization* of a path $\gamma : I \to X$ is defined to be a composition $\gamma \circ \varphi$, where $\varphi : I \to I$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Lemma 3.2.10. *A reparametrization of a path preserves its homotopy class.*

Proof. Let $\gamma : I \to X$ be a path in X. Let

$$\gamma \circ \varphi : I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

be a reparametrization of γ in X, for some continuous map $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Consider the straight-line homotopy from φ to the identity map of I given by

$$\varphi_t(s) := (1-t)\varphi(s) + ts, \quad \forall\, s, t \in I.$$

Now it is easy to check that the map

$$F : I \times I \to X, \quad (s,t) \mapsto \gamma(\varphi_t(s)),$$

is continuous and satisfies $F(s,0) = (\gamma \circ \varphi)(s)$ and $F(s,1) = \gamma(s)$, for all $s \in I$. Therefore, $\gamma \circ \varphi \simeq \gamma$ via the homotopy F. $\square$

Proof of Theorem 3.2.8. We need to verify group axioms.

Associativity: Given any three loops $\gamma_1, \gamma_2, \gamma_3 : I \to X$ based at x_0, it is enough to show that $(\gamma_1 \star \gamma_2) \star \gamma_3 \simeq \gamma_1 \star (\gamma_2 \star \gamma_3)$.

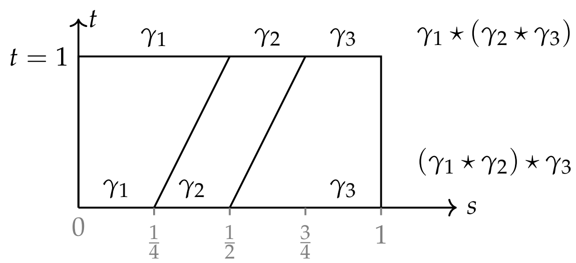


FIGURE 3.4: Homotopy for associativity

Note that

$$((\gamma_1 \star \gamma_2) \star \gamma_3)(t) = \begin{cases} \gamma_1(4t), & \text{if} \quad 0 \leq t \leq 1/4, \\ \gamma_2(4t-1), & \text{if} \quad 1/4 \leq t \leq 1/2, \\ \gamma_3(2t-1), & \text{if} \quad 1/2 \leq t \leq 1, \end{cases}$$

and

$$(\gamma_1 \star (\gamma_2 \star \gamma_3))(t) = \begin{cases} \gamma_1(2t), & \text{if} \quad 0 \leq t \leq 1/2, \\ \gamma_2(4t-2), & \text{if} \quad 1/2 \leq t \leq 3/4, \\ \gamma_3(4t-3), & \text{if} \quad 3/4 \leq t \leq 1. \end{cases}$$

It's an easy exercise to check that $\gamma_1 \star (\gamma_2 \star \gamma_3)$ is a reparametrization of $(\gamma_1 \star \gamma_2) \star \gamma_3$ by a piece-wise linear function (hence, continuous) $\varphi : I \to I$ defined by

$$\varphi(t) = \begin{cases} t/2, & \text{if } 0 \le t \le 1/2, \\ t - \frac{1}{4}, & \text{if } 1/2 \le t \le 3/4, \\ 2t - 1, & \text{if } 3/4 \le t \le 1, \end{cases}$$

(see Figure 3.5). Then using Lemma 3.2.10 we conclude that $\gamma_1 \star (\gamma_2 \star \gamma_3) \simeq (\gamma_1 \star \gamma_2) \star \gamma_3$.

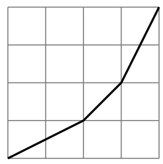


FIGURE 3.5: Graph of φ

Existence of identity: Let $e \in \pi_1(X, x_0)$ be the homotopy class of *constant loop*,

$$c_{x_0} : I \to X, \;\; t \mapsto x_0,$$

at x_0. Let $\gamma : I \to X$ be any loop in X based at x_0. Since $\gamma \star c_{x_0}$ is a reparametrization of γ via the function

$$\psi(t) := \begin{cases} 2t, & \text{if } 0 \le t \le 1/2, \\ 1, & \text{if } 1/2 \le t \le 1, \end{cases}$$

by Lemma 3.2.10 we have $\gamma \star c_{x_0} \simeq \gamma$. Similarly, $c_{x_0} \star \gamma$ is a reparametrization of γ by via the function

$$\eta(t) := \begin{cases} 0, & \text{if } 0 \le t \le 1/2, \\ 2t - 1, & \text{if } 1/2 \le t \le 1, \end{cases}$$

by Lemma 3.2.10 we have $c_{x_0} \star \gamma \simeq \gamma$.

Existence of inverse: Given any loop γ in X based at x_0, we can define its *inverse loop* or *opposite loop* $\overline{\gamma} : I \to X$ by setting $\overline{\gamma}(t) = \gamma(1-t)$, for all $t \in I$. We need to show that $\gamma \star \overline{\gamma} \simeq c_{x_0}$ and $\overline{\gamma} \star \gamma \simeq c_{x_0}$. To show $\gamma \star \overline{\gamma} \simeq c_{x_0}$, consider the map $H : I \times I \to X$ given by

$$H(s,t) := f_t(s) \star g_t(s), \;\; \forall\, (s,t) \in I \times I,$$

where $f_t : I \to X$ is the path defined by

$$f_t(s) = \begin{cases} \gamma(s), & \text{for } 0 \le s \le 1 - t, \\ \gamma(1-t), & \text{for } 1 - t \le s \le 1, \end{cases}$$

and $g_t : I \to X$ is the inverse path of f_t, i.e., $g_t(s) = f_t(1-s)$, $\forall\, s \in I$. It is an easy exercise to check that H is a continuous map satisfying

$$H(s,0) = \gamma \star \overline{\gamma}, \;\text{ and }\; H(s,1) = c_{x_0}, \;\; \forall\, s \in I.$$

The homotopy H can be understood using the Figure 3.6. In the bottom line $t = 0$, we have

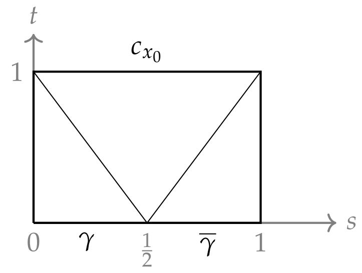


FIGURE 3.6: Homotopy H

$\gamma \star \overline{\gamma}$ while on the top line $t = 1$ we have the constant loop c_{x_0}. And below the 'V' shape we let $H(s, t)$ be independent of t while above the 'V' shape we let $H(s, t)$ be independent of s. Therefore, we have $\gamma \star \overline{\gamma} \simeq c_{x_0}$. Interchanging the roles of γ and $\overline{\gamma}$ in the above construction, we see that $\overline{\gamma} \star \gamma \simeq c_{x_0}$. Therefore, $\pi_1(X, x_0)$ is a group. $\square$

3.2.2 Functoriality

By a *pointed topological space* we mean a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. In the above construction, given a pointed topological space (X, x_0) we attached a group $\pi_1(X, x_0)$, known as the *fundamental group of X with the base point at x_0*. Next we see how fundamental group of a pointed space behaves under continuous maps and their compositions.

Let $f : (X, x_0) \to (Y, y_0)$ be a *continuous map of pointed spaces* (this means, $f : X \to Y$ is a continuous map with $f(x_0) = y_0$). Let $\gamma : I \to X$ be a loop in X based at x_0. Then the composition $f \circ \gamma$,

$$I \xrightarrow{\gamma} X \xrightarrow{f} Y,$$

is a loop in Y based at $f(x_0) = y_0$. Let $\gamma, \delta : I \to X$ be loops in X based at x_0. If $F : I \times I \to X$ is a homotopy from γ to δ, then $f \circ F : I \times I \to Y$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$ (see Lemma 3.1.6). Thus we have a well-defined map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \ \ [\gamma] \mapsto [f \circ \gamma]. \tag{3.2.11}$$

Proposition 3.2.12. *The map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is a group homomorphism.*

Proof. Note that for any two loops $\gamma, \delta : I \to X$ based at x_0, we have

$$\begin{aligned} f_*([\gamma \star \delta]) &= [f \circ (\gamma \star \delta)] \\ &= [(f \circ \gamma) \star (f \circ \delta)] \\ &= [f \circ \gamma] \cdot [f \circ \delta] \\ &= f_*([\gamma]) \cdot f_*([\delta]). \end{aligned}$$

$\square$

Remark 3.2.13. If $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the homomorphism of fundamental group of a pointed topological space (X, x_0) induced by the identity map of (X, x_0) onto itself, then it follows from the construction of the map f_* given in (3.2.11) that $f_* = \mathrm{Id}_{\pi_1(X,x_0)}$, the identity map of $\pi_1(X, x_0)$ onto itself.

Proposition 3.2.14. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps of pointed spaces. Then $g_* \circ f_* = (g \circ f)_*$. In other words, the following diagram commutes.*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \pi_1(Y, y_0) \arrow[d, "g_*"] \\
& \pi_1(Z, z_0)
\end{tikzcd}$$

Proof. Left as an exercise. □

Corollary 3.2.15. *If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism of pointed spaces with its inverse $g : (Y, y_0) \to (X, x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups with its inverse $g_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.*

Proof. Since $g \circ f = \mathrm{Id}_{(X,x_0)}$ and $f \circ g = \mathrm{Id}_{(Y,y_0)}$, applying Proposition 3.2.14 we have $g_* \circ f_* = \mathrm{Id}_{\pi_1(X,x_0)}$ and $f_* \circ g_* = \mathrm{Id}_{\pi_1(Y,y_0)}$. □

Lemma 3.2.16. *Homotopic continuous maps of pointed topological spaces induces the same homomorphism of fundamental groups.*

Proof. Let $f, g : (X, x_0) \to (Y, y_0)$ be two continuous maps of pointed space. If f is homotopic to g in the sense of Definition 3.1.7, then for any loop $\gamma : I \to X$ based at x_0, using Exercise 3.1.8 (c.f. Lemma 3.1.6) we have $f \circ \gamma$ is homotopic to $g \circ \gamma$ in the sense of Definition 3.1.7, and hence $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma])$. Hence the result follows. □

Definition 3.2.17. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \;\; (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \text{ob}(\mathscr{C})$, there exists a morphism $\text{Id}_X \in \text{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \text{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \text{Id}_Y = f$ and $\text{Id}_Z \circ f = f$.

Example 3.2.18. (i) Let (Set) be the category of sets; its objects are sets and arrows are map of sets.

(ii) Let $(\mathcal{G}rp)$ be the category of groups; its objects are groups and arrows are group homomorphisms.

(iii) Let $(\mathcal{T}op)$ be the category of topological spaces; its objects are topological spaces and arrows are continuous maps.

(iv) Let $(\mathcal{R}ing)$ be the category of rings; its objects are rings and morphisms are ring homomorphisms.

Definition 3.2.19. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be an isomorphism if there is a morphism $g : Y \to X$ in $\mathscr{C}$ such that $g \circ f = \text{Id}_X$ and $f \circ g = \text{Id}_Y$.

Definition 3.2.20. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *covariant functor* (resp., a *contravariant functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a rule
$$\mathcal{F} : \mathscr{C} \to \mathscr{D}$$
which associate to each object $X \in \mathscr{C}$ an object $\mathcal{F}(X) \in \mathscr{D}$, and to each morphism $f \in \text{Mor}_{\mathscr{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \text{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ (resp., a morphism $\mathcal{F}(f) \in \text{Mor}_{\mathscr{D}}(\mathcal{F}(Y), \mathcal{F}(X))$) such that

(i) $\mathcal{F}(\text{Id}_X) = \text{Id}_{\mathcal{F}(X)}$, for all $X \in \mathscr{C}$, and

(ii) given any objects $X, Y, Z \in \mathscr{C}$ and morphisms $f \in \text{Mor}_{\mathscr{C}}(X, Y)$ and $g \in \text{Mor}_{\mathscr{C}}(Y, Z)$, we have $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (resp., $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$).

If $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is a functor, for each ordered pair of objects $X, Y \in \mathscr{C}$ we denote by $\mathcal{F}_{X,Y}$ the induced map
$$\mathcal{F}_{X,Y} : \text{Mor}_{\mathscr{C}}(X, Y) \to \text{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y)),$$
defined by $\mathcal{F}_{X,Y}(f) := \mathcal{F}(f)$. The same notation is used for contravariant functor.

Definition 3.2.21. A functor $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is said to be

(i) *faithful* if $\mathcal{F}_{X,Y}$ is injective, $\forall\ X, Y \in \mathscr{C}$.

(ii) *full* if $\mathcal{F}_{X,Y}$ is surjective, $\forall\ X, Y \in \mathscr{C}$.

(iii) *fully faithful* if $\mathcal{F}_{X,Y}$ is bijective, $\forall\ X, Y \in \mathscr{C}$.

(iv) *essentially surjective* if given any object $Y \in \mathscr{D}$, there is an object $X \in \mathscr{C}$ and an isomorphism $\varphi : \mathcal{F}(X) \xrightarrow{\simeq} Y$ in $\mathscr{D}$.

(v) *equivalence of categories* if there is a functor $\mathcal{G} : \mathscr{D} \to \mathscr{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \text{Id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \text{Id}_{\mathscr{D}}$. This is equivalent to say that $\mathcal{F}$ is fully faithful and essentially surjective.

Remark 3.2.22. Let $\mathcal{T}op_0$ be the category of pointed topological spaces; its objects are pointed topological space, and given any two pointed topological spaces (X, x_0) and (Y, y_0), a morphism $f : (X, x_0) \to (Y, y_0)$ in $\mathcal{T}op_0$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. Then it follows from Propositions 3.2.12 and 3.2.14 and the Remark 3.2.13 that

$$\begin{aligned} \pi_1 \ : \mathcal{T}op_0 &\longrightarrow (\mathcal{G}rp) \\ (X, x_0) &\mapsto \pi_1(X, x_0) \\ f &\mapsto f_* \end{aligned}$$

is a *covariant functor* from the category of pointed topological spaces to the category of groups. It follows from Lemma 3.2.16 that the functor π_1 is not faithful. It is a non-trivial fact that π_1 is not full. (i.e., there exist pointed topological spaces (X, x_0) and (Y, y_0), and a group homomorphism $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that $\varphi \neq f_*$, for all continuous map $f : (X, x_0) \to (Y, y_0)$.) However, π_1 is *essentially surjective* (i.e., given any group G there is a pointed topological space (X, x_0) such that $\pi_1(X, x_0) \cong G$).

3.2.3 Dependency on base point

Now we investigate relation between fundamental groups of X for different choices of base point. Let $x_0, x_1 \in X$. Let $f : I \to X$ be a path in X joining x_0 to x_1, i.e., f is a continuous map satisfying $f(0) = x_0$ and $f(1) = x_1$. We define the *opposite path* of f to be the map

$$\overline{f} : I \to X, \ \ t \mapsto f(1-t); \tag{3.2.23}$$

note that $\overline{f}(0) = x_1$ and $\overline{f}(1) = x_0$, hence $\overline{f}$ is a path from x_1 to x_0.

Exercise 3.2.24. Show that $f \star \overline{f} \simeq c_{x_0}$ and $\overline{f} \star f \simeq c_{x_1}$, where $\simeq$ stands for path-homotopy relation (see Definition 3.2.1).

Given a loop γ in X based at x_1, we can define $\widetilde{\gamma} := f \star \gamma \star \overline{f}$. Note that $\widetilde{\gamma} : I \to X$ is a continuous map satisfying $\widetilde{\gamma}(0) = f(0) = x_0 = \overline{f}(1) = \widetilde{\gamma}(1)$, and hence is a loop in X based at x_1. Strictly speaking, we have two choices to define this product $\widetilde{\gamma}$, namely $(f \star \gamma) \star \overline{f}$ or $f \star (\gamma \star \overline{f})$, but we are interested in only homotopy classes of paths, and following the proof of associativity as in Theorem 3.2.8 one can easily verify that $(f \star \gamma) \star \overline{f} \simeq f \star (\gamma \star \overline{f})$, therefore, we just fix one ordering of taking products to define $\widetilde{\gamma}$.

If γ and γ' are two loops in X based at x_1 with $\gamma \simeq \gamma'$ via a homotopy $\{h_t\}_{t \in I}$, then $\{f \star h_t \star \overline{f}\}_{t \in I}$ is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma'}$ (*Exercise: Write down the homotopy explicitly and check details*). Thus, we have a well-defined map

$$\beta_f : \pi_1(X, x_1) \to \pi_1(X, x_0), \ \ [\gamma] \mapsto [(f \star \gamma) \star \overline{f}]. \tag{3.2.25}$$

Proposition 3.2.26. *The map β_f defined in (3.2.25) is a group isomorphism.*

Proof. Since $\overline{f} \star f \simeq c_{x_0}$ for any two loops γ and δ in X based at x_1, using Exercise 3.2.24, we have

$$\begin{aligned} f \star (\gamma \star \delta) \star \overline{f} &\simeq f \star \gamma \star c_{x_0} \star \delta \star \overline{f} \\ &\simeq (f \star \gamma \star \overline{f}) \star (f \star \delta \star \overline{f}). \end{aligned}$$

Therefore, $\beta_f([\gamma \star \delta]) = [f \star (\gamma \star \delta) \star \overline{f}] = [f \star \gamma \star \overline{f}][f \star \delta \star \overline{f}] = \beta_f([\gamma])\beta_f([\delta])$, and hence β_f is a group homomorphism. To show β_f an isomorphism of groups, it is enough to show that the group homomorphism

$$\beta_{\overline{f}} : \pi_1(X, x_0) \to \pi_1(X, x_1), \ \ [\gamma] \mapsto [\overline{f} \star \gamma \star f]$$

is the inverse of β_f. Indeed, for any $\gamma \in \pi_1(X, x_0)$ we have

$$\begin{aligned} \beta_f(\beta_{\overline{f}}([\gamma])) &= \beta_f([\overline{f} \star \gamma \star f]) \\ &= [f \star \overline{f} \star \gamma \star f \star \overline{f}] \\ &= [c_{x_0} \star \gamma \star c_{x_0}] = [\gamma], \end{aligned}$$

and similarly, for any $\delta \in \pi_1(X, x_1)$ we have

$$\begin{aligned} \beta_{\overline{f}}(\beta_f([\delta])) &= \beta_{\overline{f}}([f \star \delta \star \overline{f}]) \\ &= [\overline{f} \star f \star \delta \star \overline{f} \star f] \\ &= [c_{x_1} \star \delta \star c_{x_1}] = [\delta]. \end{aligned}$$

Therefore, $\beta_{\overline{f}}$ is the inverse homomorphism of β_f, and hence both of them are isomorphisms. □

Remark 3.2.27. Thus if X is a path connected space, up to isomorphism its fundamental group is independent of choice of base point, and so we may denote it by $\pi_1(X)$ without specifying its base point.

Proposition 3.2.28. *Let $f, g : X \to Y$ be two continuous maps of topological spaces. Fix a point $x_0 \in X$, and let $y_0 = f(x_0)$ and $y_1 = g(x_0)$. Let $F : X \times I \to Y$ be a continuous map such that $F(-, 0) = f$ and $F(-, 1) = g$. Then for any loop γ in X based at x_0, the loop $f \circ \gamma$ is path-homotopic to the loop $F_0 \star (g \circ \gamma) \star \overline{F_0}$ in Y, where $F_0 : I \to Y$ is the path in Y defined by $F_0(t) = F(x_0, t), \ \forall\, t \in I$.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & \pi_1(Y, y_1) \arrow[dl, "\beta_{F_0}"] \\ & \pi_1(Y, y_0) & \end{tikzcd}$$

Proof. Left as an exercise. □

Corollary 3.2.29. *Let $f, g : X \to Y$ be two homotopic continuous maps of topological spaces. Let $x_0 \in X$ be such that $f(x_0) = g(x_0) = y_0 \in Y$. Then the homomorphisms of fundamental groups f_**

and g_*, induced by f and g, respectively, are conjugate by an element of $\pi_1(Y, y_0)$. In other words, there exists an element $[\eta] \in \pi_1(Y, y_0)$ such that $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$.

Proof. Let $F : X \times I \to X$ be a continuous map such that

$$F(x,t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ g(x), & \text{if} \quad t = 1. \end{cases}$$

Then by Proposition 3.2.28 we have $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$, where $\eta : I \to Y$ is the loop defined by $\eta(t) := F(x_0, t)$, $\forall\, t \in I$. □

Corollary 3.2.30. *If $f, g : (X, x_0) \to (Y, y_0)$ are two homotopic continuous maps of pointed topological spaces (see Definition 3.1.7), then $f_* = g_*$.*

Proof. Follows from Corollary 3.2.29. □

Definition 3.2.31. A space X is said to be *simply connected* if X is path connected and $\pi_1(X)$ is trivial.

Corollary 3.2.32. *A contractible space (see Definition 3.1.11) is simply connected.*

Proof. Let X be a contractible space. Then X is path-connected by Exercise 3.1.12. Fix a point $x_0 \in X$, and let $c_{x_0} : X \to X$ be the constant map sending all points to x_0. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ is homotopic to the constant map c_{x_0} in X. Then by Corollary 3.2.29 the identity homomorphism $\mathrm{Id}_{\pi_1(X,x_0)} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is conjugate to the trivial homomorphism $(c_{x_0})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by an element of $\pi_1(X, x_0)$. Therefore, the image of the identity homomorphism $\mathrm{Id}_{\pi_1(X, x_0)}$ is trivial, and hence $\pi_1(X, x_0)$ is trivial. □

Corollary 3.2.33. *A space X is simply connected if and only if there is a unique path-homotopy class of paths connecting any two points of X.*

Proof. Suppose that X is simply connected. Fix $x_0, x_1 \in X$. Since X is path-connected, there is a path in X joining x_0 to x_1. Let $f, g : I \to X$ be any two paths in X from x_0 to x_1. Let $\overline{f}$ and $\overline{g}$ be the opposite paths of f and g, respectively. Since $f \star \overline{g}$ is a loop in X based at x_0 and $\pi_1(X, x_0)$ is trivial, we have $f \star \overline{g}$ is path-homotopic to the constant loop c_{x_0} in X based at x_0. Since $\overline{g} \star g$ is path-homotopic to the constant loop c_{x_1} by Exercise 3.2.24, we have

$$f \simeq f \star c_{x_1} \simeq f \star \overline{g} \star g \simeq c_{x_0} \star g \simeq g.$$

To see the converse part, note that path connectedness of X means any two points of X can be joined by a path in X. Since there is a unique homotopy class of paths connecting any two points of X, path connectedness of X is automatic, and any loop in X based at a given point $x_0 \in X$ is homotopically trivial. Thus, X is path connected with $\pi_1(X, x_0)$ trivial, and hence is simply connected. □

Proposition 3.2.34. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Note that $X \times Y$ naturally acquires product topology induced from X and Y, and the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$, for all $(x, y) \in X \times Y$, are continuous. Moreover, given any space Z and a map $f : Z \to X \times Y$, we have $f = (p_1 \circ f,\, p_2 \circ f)$. From this, it follows that f is continuous if and only if its components $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to Y$ are continuous. Therefore, to give a loop $\gamma : I \to X \times Y$ based at $(x_0,\, y_0) \in X \times Y$ is equivalent to give a pair of loops $(p_1 \circ \gamma,\, p_2 \circ \gamma)$ in the pointed spaces (X, x_0) and (Y, y_0), respectively. Similarly, to give a homotopy $F : I \times I \to X \times Y$ of loops $\gamma, \delta : I \to X \times Y$ based at $(x_0,\, y_0))$ is equivalent to give a pair of homotopies $(p_1 \circ F,\, p_2 \circ F)$ of the corresponding loops $p_j \circ \gamma$ with $p_j \circ \delta$, where $j \in \{1, 2\}$. Thus we have a bijection

$$\phi : \pi_1(X \times Y,\, (x_0,\, y_0)) \to \pi_1(X,\, x_0) \times \pi_1(Y,\, y_0), \quad [\gamma] \mapsto ([p_1 \circ \gamma],\, [p_2 \circ \gamma]).$$

To see ϕ is an isomorphism, note that for any two loops γ and δ in $X \times Y$ based at $(x_0,\, y_0) \in X \times Y$, we have

$$\begin{aligned}
\phi([\gamma] \cdot [\delta]) &= \phi([\gamma \star \delta]) \\
&= ([p_1 \circ (\gamma \star \delta)],\, [p_2 \circ (\gamma \star \delta)]) \\
&= ([(p_1 \circ \gamma) \star (p_1 \circ \delta)],\, [(p_2 \circ \gamma) \star (p_2 \circ \delta)]) \\
&= ([(p_1 \circ \gamma)] \cdot [(p_1 \circ \delta)],\, [(p_2 \circ \gamma)] \cdot [(p_2 \circ \delta)]) \\
&= ([p_1 \circ \gamma],\, [p_2 \circ \gamma]) \cdot ([p_1 \circ \delta],\, [p_2 \circ \delta]) \\
&= \phi([\gamma]) \cdot \phi([\delta]).
\end{aligned}$$

This completes the proof. □

Example 3.2.35. As an immediate application of Proposition 3.2.34 we see that the fundamental group of the 1-torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

We end this subsection with the following useful remark.

Remark 3.2.36. A loop in X based at x_0 can equivalently be defined as a continuous map of pointed spaces $\gamma : (S^1,\, 1) \to (X,\, x_0)$. Indeed, since a loops in X based at $x_0 \in X$ is a continuous map $\gamma : I = [0,\, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, and since S^1 is homeomorphic to the quotient space $[0,\, 1]/\sim$, where only the end points 0 and 1 of the interval I are identified, $\gamma : I \to X$ uniquely factors as

$$\begin{tikzcd}
I \arrow[r, "\gamma"] \arrow[d, "q"'] & X \\
S^1 \arrow[ur, dashed, "\gamma'"'] &
\end{tikzcd}$$

where $q : I \to S^1$ is the quotient map given by $q(t) = e^{2\pi i t}$, for all $t \in I$. Therefore, $\pi_1(X, x_0)$ is the group of all homotopy classes of continuous maps $(S^1, 1) \to (X, x_0)$.

3.2.4 Fundamental group of some spaces

Proposition 3.2.37. $\pi_1(\mathbb{R},\, 0) = \{1\}$.

Proof. Consider the continuous map $F : \mathbb{R} \times I \to \mathbb{R}$ defined by

$$F(x,\, t) = (1-t)x, \;\; \forall\, (x,\, t) \in \mathbb{R} \times I.$$

Note that, for all $x \in \mathbb{R}$ and $t \in I$ we have

- $F(x,\, 0) = x$,
- $F(x,\, 1) = 0$, and
- $F(0,\, t) = 0$.

Therefore, F "contracts" whole $\mathbb{R}$ to the point 0 leaving the point 0 intact at all times. Let $\gamma : I \to \mathbb{R}$ be a loop based at 0. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \overset{\gamma\times\mathrm{Id}_I}{\longrightarrow} \mathbb{R} \times I \overset{F}{\longrightarrow} \mathbb{R}$$

is a homotopy from γ to the constant loop $0 : I \to \mathbb{R}$ which sends all points of S^1 to $0 \in \mathbb{R}$. This completes the proof. □

Proposition 3.2.38. *Let*

$$D^2 := \{z \in \mathbb{C} \,:\, |z| \leq 1\} = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\}$$

be the closed unit disk in the plane. Then $\pi_1(D^2,\, 1) = \{1\}$.

Proof. Consider the map $F : D^2 \times I \to D^2$ defined by

$$F(z,t) = (1-t)z + t, \;\; \forall\, (z,t) \in D^2 \times I.$$

Note that F is continuous and for all $z \in D^2$ and $t \in I$ we have

- $F(z,\, 0) = z$,
- $F(z,\, 1) = 1$, and
- $F(1,\, t) = 1$.

Therefore, F contracts D^2 to the point 1 leaving 1 intact at all times. Let $\gamma : I \to D^2$ be a loop based at 1. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \overset{\gamma\times\mathrm{Id}_I}{\longrightarrow} D^2 \times I \overset{F}{\longrightarrow} D$$

is a homotopy from γ to the constant loop $1 : I \to D^2$ which sends all points of I to $1 \in D^2$. This completes the proof. □

3.3 Covering Space

3.3.1 Covering map

We begin this section with an aim to compute fundamental group of the unit circle in plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and we'll see how the idea of a 'covering map' could help us.

Let $\omega : I \to S^1$ be the map defined by $\omega(t) = e^{2\pi i t}$, $\forall\, t \in I$, where $i = \sqrt{-1}$. Then ω is a loop in S^1 based at $x_0 := 1 \in S^1$. For each integer n, let $\omega_n : I \to S^1$ be the loop based at x_0 defined by $\omega_n(t) = e^{2\pi i n t}$, $\forall\, t \in I$. So ω_n winds around the circle $|n|$-times in the anti-clockwise direction if $n > 0$, and in the clockwise direction if $n < 0$. We shall see later that $[\omega]^n = [\omega_n]$ in $\pi_1(S^1, 1)$, for all $n \in \mathbb{Z}$. The following is the main theorem of this section.

Theorem 3.3.1. *$\pi_1(S^1, x_0)$ is the infinite cyclic group $\mathbb{Z}$ generated by the loop ω.*

To prove this theorem, we compare paths in S^1 with paths in $\mathbb{R}$ via the map

$$p : \mathbb{R} \to S^1 \text{ given by } p(s) = (\cos 2\pi s, \sin 2\pi s),\ \forall\, s \in \mathbb{R}.$$

We can visualize this map geometrically by embedding $\mathbb{R}$ inside $\mathbb{R}^3$ as the helix parametrized as

$$s \mapsto (\cos 2\pi s, \sin 2\pi s, s),$$

and then p is the restriction of the projection map

$$\mathbb{R}^3 \to \mathbb{R}^2,\ \ (x, y, z) \mapsto (x, y)$$

from this helix onto $S^1 \subset \mathbb{R}^2$, as shown in the Figure 3.7.

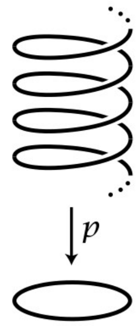


FIGURE 3.7

In this setup, the loop

$$\omega_n : I \to S^1,\ \ s \mapsto (\cos 2n\pi s, \sin 2n\pi s)$$

is the composition $p \circ \widetilde{\omega}_n$, where

$$\widetilde{\omega}_n : I \to \mathbb{R},\ \ s \mapsto ns$$

is the path in $\mathbb{R}$ starting at 0 and ending at n, winding around the helix $|n|$-times, upward direction if $n > 0$, and downward direction if $n < 0$.

Before proceeding further, we introduce notion of a *covering map*, and discuss some of its useful properties.

Definition 3.3.2. Let $f : X \to Y$ be a continuous map. An open subset $V \subseteq Y$ is said to be *evenly covered by f* if $f^{-1}(V)$ is a union of pairwise disjoint open subsets of X each of which are homeomorphic to V by f (meaning that, $f^{-1}(V) = \bigcup_{i \in I} U_i$, where $U_i \subseteq X$ is an open subset of X with $U_i \cap U_j = \emptyset$, for all $i \neq j$ in I, and $f\big|_{U_i} : U_i \to V$ is a homeomorphism, for all $i \in I$).

Example 3.3.3. (i) Let $f : \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ be the map defined by $f(t) = e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$, for all $t \in \mathbb{R}$. For $a, b \in \mathbb{R}$ with $a < b$, we define an open subset

$$V_{a,b} := \{f(t) : a \leq t \leq b\} \subseteq S^1.$$

If $b - a < 1$, then $V_{a,b}$ is evenly covered by f. In fact, in this case, we have $f^{-1}(V_{a,b}) =$

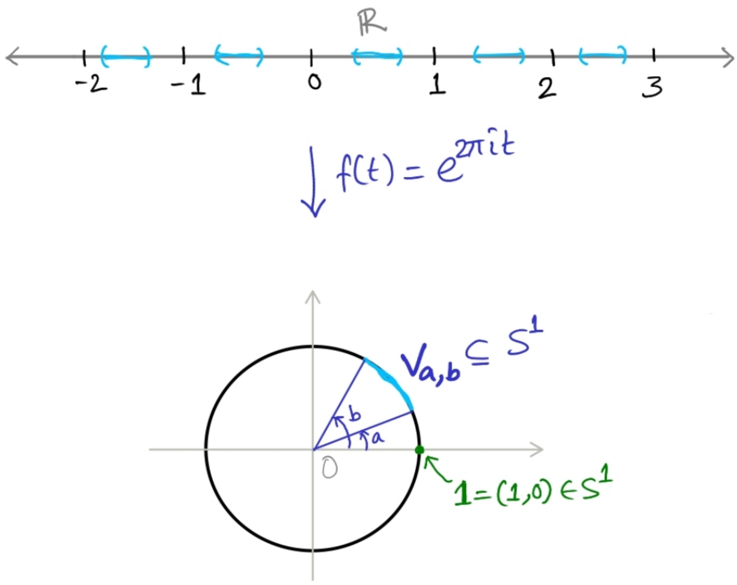


FIGURE 3.8

$\bigsqcup_{n \in \mathbb{Z}} (a+n, b+n)$, and $f : (a+n, b+n) \xrightarrow{\simeq} V_{a,b}$ is a homeomorphism, $\forall\, n \in \mathbb{Z}$. See Figure 3.8.

If $b - a \geq 1$, then $V_{a,b} = S^1$, and hence $f^{-1}(V_{a,b}) = \mathbb{R}$. In this case, $V_{a,b}$ is not evenly covered by f, for otherwise we would have $\mathbb{R} = \bigsqcup_{i \in I} U_i$ with each U_i open subset of $\mathbb{R}$ and $f\big|_{U_i} : U_i \to S^1$ is a homeomorphism, which is not possible because S^1 is compact, whereas an open subset of $\mathbb{R}$ cannot be compact.

(ii) Let $\mathbb{R}_{>0} := \{t \in \mathbb{R} : t > 0\}$ be the positive part of the real line. Let

$$f : \mathbb{R}_{>0} \to S^1, \ \ t \mapsto e^{2\pi i t}. \tag{3.3.4}$$

For any point $x \in S^1$ with $x \neq \mathbb{1} := (1, 0) \in S^1$, we can choose a small enough open neighbourhood V of x in S^1 with $\mathbb{1} \notin V$. Then it is easy to see that V is evenly covered by f. However, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$. To see this, note that if $U \subseteq V$ is an open subset of an evenly covered neighbourhood V, then U is also evenly covered. Thus, if there is a neighbourhood V of $\mathbb{1}$ which is evenly covered, then we may find $\epsilon \in (0, 1/2)$ small enough such that $V_{-\epsilon,\epsilon} \subseteq V$, and hence $V_{-\epsilon,\epsilon}$ is evenly covered. Then we must have $f^{-1}(V_{-\epsilon,\epsilon}) = \bigsqcup_{i \in I} U_i$, with $f\big|_{U_i} : U_i \to V_{-\epsilon,\epsilon}$ homeomorphism, for all

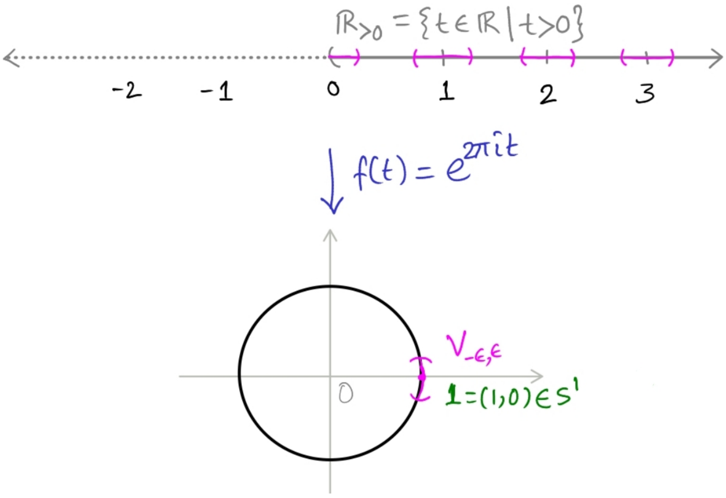


FIGURE 3.9

$i \in I$. In particular, each U_i is connected and are path components of $f^{-1}(V_{-\epsilon,\epsilon})$. Let U_0 be the path component of $\epsilon/2 \in \mathbb{R}_{>0}$. Since

$$f^{-1}(V_{-\epsilon,\epsilon}) = (0,\epsilon) \bigcup \left(\bigcup_{n \geq 1} (n-\epsilon, n+\epsilon) \right),$$

we must have $U_0 = (0,\epsilon)$. But $f\big|_{(0,\epsilon)} : (0,\epsilon) \to V_{-\epsilon,\epsilon}$ cannot be surjective because only possible preimage of $\mathbb{1} \in V_{-\epsilon,\epsilon}$ in $\mathbb{R}^+$ could be positive integers, and none of which are in the domain of $f\big|_{(0,\epsilon)}$. Thus we get a contradiction. See Figure 3.9. Therefore, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$ for the map f in (3.3.4).

Definition 3.3.5. A continuous map $f : X \to Y$ is called a *covering map* if each point $y \in Y$ has an open neighbourhood $V_y \subseteq Y$ that is evenly covered by f.

Note that, a covering map is always surjective. This follows immediately from the Definition 3.3.5.

Example 3.3.6. (i) Let F be a non-empty discrete topological space, and let X be any topological space. Give $X \times F$ the product topology. Then the projection map $pr_1 : X \times F \to X$ defined by $pr_1(x,v) = x$, $\forall\ (x,v) \in X \times F$, is a covering map. Such a covering map is called a *trivial cover* of X.

(ii) The continuous map
$$f : \mathbb{R} \to S^1,\ \ t \mapsto e^{2\pi i t},$$
as discussed in Example 3.3.3 (i), is a covering map, while its restriction $f\big|_{\mathbb{R}^+} : \mathbb{R}^+ \to S^1$, in Example 3.3.3 (ii), is not a covering map.

(iii) The map $f : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by $f(z) = e^z$, for all $z \in \mathbb{C}$, is a covering map.

(iv) Fix an integer $n \geq 1$. Then the map $f : \mathbb{C}^* \to \mathbb{C}^*$ defined by $f(z) = z^n$, for all $z \in \mathbb{C}$, is a covering map, known as the *n-sheeted covering map* of $\mathbb{C}^*$.

Exercise 3.3.7. If $f_i : X_i \to Y_i$ is a covering map, for $i = 1, 2$, show that the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by sending $(x_1, x_2) \in X_1 \times X_2$ to $(f_1(x_1), f_2(x_2)) \in Y_1 \times Y_2$, is a covering map.

Exercise 3.3.8. If $f : X \to Y$ is a covering map, for any subspace $Z \subseteq Y$, the restriction of f on $f^{-1}(Z) \subseteq X$ is a covering map.

Definition 3.3.9. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covering maps. A *morphism of covering maps* from p_1 to p_2 is a continuous map $\phi : Y_1 \to Y_2$ such that $p_2 \circ \phi = p_1$. In other words, the following diagram commutes.

$$\begin{tikzcd} Y_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A morphism of covering maps $\phi : Y_1 \to Y_2$ is said to be an *isomorphism of covering maps* if there is a covering map $\psi : Y_2 \to Y_1$ such that $\phi \circ \psi = \mathrm{Id}_{Y_2}$ and $\psi \circ \phi = \mathrm{Id}_{Y_1}$. In other words, an isomorphism of covering spaces is a homeomorphism of the covers compatible with the base. An isomorphism of a covering map $p : Y \to X$ to itself is called a *Deck transformation* or a *covering transformation*.

Exercise 3.3.10. Show that any covering map $p : Y \to X$ is locally trivial (i.e., each point $x \in X$ has an open neighbourhood $U_x \subseteq X$ such that the restriction map $p : p^{-1}(U_x) \to U_x$ is isomorphic to a trivial covering map over U_x).

A continuous map $f : X \to Y$ is said to be an *open map* if for any open subset U of X, $f(U)$ is open in Y.

Proposition 3.3.11. *If $f : X \to Y$ is a covering map, then f is an open map.*

Proof. Let $U \subseteq X$ be an open subset of X, and let $y \in f(U)$. Then there is $x_0 \in U$ such that $f(x_0) = y$. Since f is a covering map, there is an open neighbourhood $V \subseteq Y$ of y such that $f^{-1}(V) = \bigcup\limits_{j \in J} W_i$ is a union of pairwise disjoint open subsets $W_j \subseteq X$, and that $f\big|_{W_j} : W_j \to V$ is a homeomorphism, for all $j \in J$. Then $x_0 \in U \cap W_{j_0}$, for some unique $i_0 \in I$. Since $f\big|_{W_i}$ is a homeomorphism, $f(U \cap W_{j_0}) \subseteq V$ is an open neighbourhood of $f(x_0) = y$. Since V is open in Y, $f(U \cap W_{j_0})$ is open in Y. Thus $f(U)$ is open in Y, and hence f is an open map. $\square$

Theorem 3.3.12 (Lifting path to a cover)**.** *Let $f : X \to Y$ be a covering map. Let $\gamma : [0, 1] \to Y$ be a path in Y. Fix a point $x_0 \in X$ such that $f(x_0) = y_0 := \gamma(0)$. Then there is a unique path $\widetilde{\gamma} : [0, 1] \to X$ with $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$.*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\widetilde{\gamma}"] & Y \end{tikzcd}$$

The path $\widetilde{\gamma}$ is called a lift of γ in X starting at x_0.

Proof. We first prove uniqueness of lift of γ, if it exists. Let $\eta_1, \eta_2 : [0,1] \to X$ be any two continuous maps such that $\eta_1(0) = x_0 = \eta_2(0)$ and $f \circ \eta_1 = \gamma = f \circ \eta_2$. We need to show that $\eta_1 = \eta_2$ on $[0,1]$. Let

$$S = \{t \in [0,1] \,:\, \eta_1(t) = \eta_2(t)\}.$$

Since both η_1 and η_2 are continuous, S is a closed subset of $[0, 1]$. Note that $S \neq \emptyset$ since $0 \in S$. Since $[0, 1]$ is connected, it is enough to show that S is both open and closed in $[0,1]$, so that S is a connected component of $[0, 1]$, and hence $S = [0, 1]$.

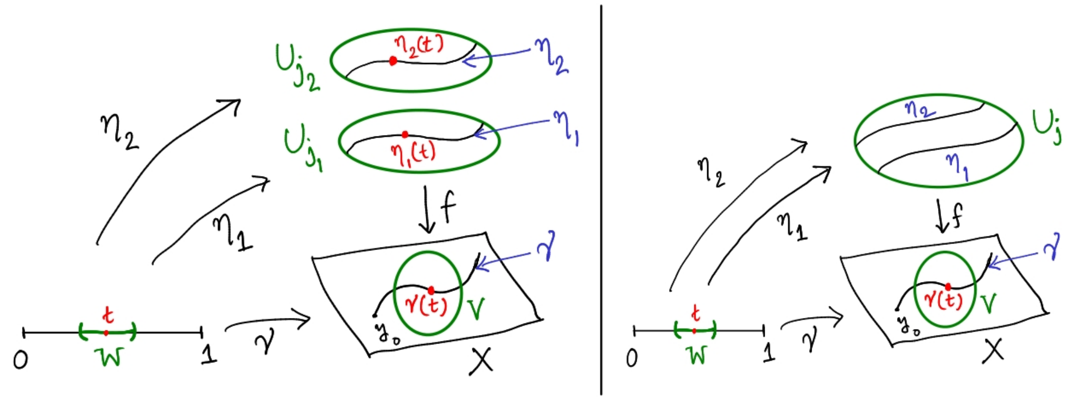


FIGURE 3.10

Fix a $t \in S$, and let $V \subseteq Y$ be an open neighbourhood of $y := \gamma(t)$ that is evenly covered by f. So $f^{-1}(V) = \bigcup\limits_{j \in J} U_j$, where $\{U_j\}_{j \in J}$ is a collection of pairwise disjoint open subsets of X each of which gets mapped homeomorphically onto V by f. Then there are $j_1, j_2 \in J$ such that $\eta_1(t) \in U_{j_1}$ and $\eta_2(t) \in U_{j_2}$. Since η_1 and η_2 are continuous at $t \in [0, 1]$, there is an open neighbourhood $W \subseteq [0,1]$ of t such that $\eta_1(W) \subseteq U_{j_1}$ and $\eta_2(W) \subseteq U_{j_2}$. Since $U_{j_1} \cap U_{j_2} = \emptyset$ for $j_1 \neq j_2$, and since $\eta_1(t) = \eta_2(t)$ by assumption, we must have $j_1 = j_2$ and $U_{j_1} = U_{j_2}$. Since $f|_{U_j} : U_j \to V$ is injective (in fact, homeomorphism), for all $j \in J$, and $f \circ \eta_1 = f \circ \eta_2$, we must have $\eta_1|_W = \eta_2|_W$. Therefore, $W \subseteq S$. Thus S is both open and closed in $[0, 1]$, and hence is the connected component of $[0, 1]$. Therefore, $S = [0, 1]$, and hence $\eta_1 = \eta_2$ on $[0, 1]$.

Remark 3.3.13. Note that, by replacing $[0,1]$ with any connected topological space T in the above proof of uniqueness of lift of γ, we get the following result:

Lemma 3.3.13. *Let $f : X \to Y$ be a covering map. Let η_1, $\eta_2 : T \to X$ be any continuous maps such that $f \circ \eta_1 = f \circ \eta_2$. If T is connected and $\eta_1(t) = \eta_2(t)$, for some $t \in T$, then $\eta_1 = \eta_2$ on whole T.*

To complete the proof of Theorem 3.3.12, it remains to construct an explicit lift of γ to the cover $f : X \to Y$ starting at x_0. For this we use a result from basic topology course, called *Lebesgue number lemma*.

Lemma 3.3.14 (Lebesgue number lemma). *Let $\{U_j\}_{j \in J}$ be an open cover of a compact metric space (X, d). Then there is a $\delta > 0$ such that for each $x_0 \in X$, the open ball $B_\delta(x_0)$ is contained in U_{j_0}, for some $j_0 \in J$.*

Since $f : X \to Y$ is a covering map, we can write $Y = \bigcup_{y\in Y} V_y$, where $V_y \subseteq Y$ is an open neighbourhood of y that is evenly covered by f, for all $y \in Y$. Since $[0,1] = \bigcup_{y\in Y} \gamma^{-1}(V_y)$, by Lebesgue covering lemma (c.f. Lemma 3.3.14) we can find a $\delta > 0$ such that for each $t \in (0,1)$ there is a $y_t \in Y$ such that $\gamma\big([t-\frac{\delta}{2},\, t+\frac{\delta}{2}] \cap [0,1]\big) \subseteq V_{y_t}$. Choose $n \gg 0$ such that $\frac{1}{n} < \delta$, and write

$$[0,1] = \bigcup_{k=0}^{n-1} \left[\frac{k}{n},\, \frac{k+1}{n}\right].$$

Now $\gamma([0,1/n]) \subseteq V_0$, for some open subset $V_0 \subset Y$ evenly covered by f, and $y_0 = \gamma(0) \in V_0$. Write

$$f^{-1}(V_0) = \bigsqcup_{j\in J} U_{0,j}\,,$$

where $\{U_{0,j}\}_{j\in J}$ is a collection of pair-wise disjoint open subsets of X each of which are homeomorphic to V_0 via the restriction of f onto them. Since $x_0 \in f^{-1}(V_0)$, there is a unique $j_0 \in J$ such that $x_0 \in U_{0,j_0}$. Let $s_0 : V_0 \to U_{0,j_0}$ be the inverse of the homeomorphism $f\big|_{U_{0,j_0}}$. Clearly $s_0(y_0) = x_0$. Consider the map $\widetilde{\gamma}_0 : [0,\, \frac{1}{n}] \to U_{0,j_0}$ defined by

$$\widetilde{\gamma}_0(t) := s_0(\gamma(t)),\ \ \forall\ t \in [0,\, 1/n]\,.$$

Then $\widetilde{\gamma}_0$ satisfies $\widetilde{\gamma}_0(0) = x_0$ and $f \circ \widetilde{\gamma}_0 = \gamma$ on $[0,\, \frac{1}{n}]$.

Let $x_1 = \widetilde{\gamma}_0(\frac{1}{n})$ and $y_1 = \gamma(\frac{1}{n}) = (f \circ \widetilde{\gamma}_0)(\frac{1}{n})$. Then there is an open subset $V_1 \subseteq Y$ which is evenly covered by f and $\gamma([\frac{1}{n},\, \frac{2}{n}]) \subseteq V_1$. Proceeding in the same way as above, we can write

$$f^{-1}(V_1) = \bigsqcup_{j\in J} U_{1,j},$$

where $U_{1,j}$ are pairwise disjoint open subsets of X each of which are homeomorphic to V_1 by the restriction of f onto them. Since $x_1 = \widetilde{\gamma}_0(\frac{1}{n}) \in f^{-1}(V_1)$, there is a $j_1 \in J$ such that $x_1 \in U_{1,j_1}$. Let $s_1 : V_1 \to U_{1,j_1}$ be the inverse of the homeomorphism $f : U_{1,j_1} \to V_1$. Clearly $s_1(y_1) = x_1$. Then the continuous map $\widetilde{\gamma}_1 : [\frac{1}{n},\, \frac{2}{n}] \to U_{1,j_1}$ defined by

$$\widetilde{\gamma}_1(t) = s_1(\gamma(t)),\ \ \forall\, t \in [1/n,\, 2/n]$$

satisfies $\widetilde{\gamma}_1(\frac{1}{n}) = x_1$ and $f \circ \widetilde{\gamma}_1 = \gamma$ on $[\frac{1}{n},\, \frac{2}{n}]$. Since the maps $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ agrees on $[0,\, \frac{1}{n}] \cap [\frac{1}{n},\, \frac{2}{n}] = \{\frac{1}{n}\}$, by Lemma 3.1.3 we can join them to get a continuous map $\widetilde{\gamma} : [0,\, \frac{2}{n}] \to X$ such that $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$ on $[0,\, \frac{2}{n}]$. Proceeding in this way we can construct a lift $\widetilde{\gamma}$ of γ to the whole $[0,1]$ as required. □

Next we lift homotopy from a base to its cover.

Lemma 3.3.15 (Glueing continuous maps)**.** *Let X and Y be two topological spaces. Let $\{U_j\}_{j\in J}$ be an open covering of X. Then given a family of continuous maps $\{f_j : U_j \to Y\}_{j\in J}$ satisfying $f_j\big|_{U_j\cap U_k} = f_k\big|_{U_j\cap U_k}$, for all $j, k \in J$, there is a unique continuous map $f : X \to Y$ such that $f\big|_{U_j} = f_j$, for all $j \in J$.*

Proof. Left as an exercise. □

Theorem 3.3.16 (Lifting homotopy to covers)**.** *Let $I := [0,1] \subset \mathbb{R}$. Let $f : X \to Y$ be a covering map. Let $F : I \times I \to Y$ be a continuous map. Let $y_0 := F(0,0)$ and fix a point $x_0 \in f^{-1}(y_0)$. Then there is a unique continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0,0) = x_0$ and $f \circ \widetilde{F} = F$.*

Proof. Since $I \times I$ is connected, uniqueness of $\widetilde{F}$, if it exists, follows from Remark 3.3.13. We only show a construction of such a lift $\widetilde{F}$.

It is enough to show that, for each $s \in I$ there is a connected open neighbourhood $U_s \subseteq I$ of $s \in I$ such that $\widetilde{F}$ can be constructed on $U_s \times I$. Indeed, since $\{U_s \times I : s \in I\}$ is a connected open covering of $I \times I$ and those $\widetilde{F}$'s agree on their intersections $(U_s \times I) \cap (U_{s'} \times I) = (U_s \cap U_{s'}) \times I$, which are connected (because U_s' are open intervals), uniqueness of liftings $\widetilde{F}$'s defined on connected domains ensures that they can be glued together to get a well-defined continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0,0) = x_0$ and $f \circ \widetilde{F} = F$ on $I \times I$.

Now we construct such a lift $\widetilde{F} : U \times I \to X$, for some open neighbourhood $U \subseteq I$ of a given point $s_0 \in I$. Since F is continuous, each point $(s_0, t) \in I \times I$ has an open neighbourhood $U_t \times (a_t, b_t) \subset I \times I$ such that $F(U_t \times (a_t, b_t))$ is contained in some open neighbourhood of $F((s_0, t)) \in Y$ that is evenly covered by f. Since $\{s_0\} \times I$ is compact, finitely many such open subsets $U_t \times (a_t, b_t)$ cover $\{s_0\} \times I$. Taking intersection of those finitely many open subsets $U_t \subseteq I$, we can find a single open neighbourhood $U \subset I$ of s_0 and a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $I = [0,1]$ such that for each $i \in \{0, 1, \ldots, m\}$, $F(U \times [t_i, t_{i+1}]) \subseteq V_i$, for some open subset $V_i \subset Y$ that is evenly covered by f.

By Theorem 3.3.12 (Lifting paths to a cover), we can find a unique continuous function $\widetilde{F} : I \times \{0\} \to X$ with $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F\big|_{I \times \{0\}}$. Assume inductively that $\widetilde{F}$ has been constructed on $U \times [0, t_i]$, starting with the given $\widetilde{F}$ on $U \times \{0\} \subseteq I \times \{0\}$. Since $F(U \times [t_i, t_{i+1}]) \subseteq V_i$, and V_i is evenly covered by f, there is an open subset $W_i \subseteq X$ such that $\widetilde{F}(s_0, t_i) \in W_i$ and $f\big|_{W_i} : W_i \to V_i$ is a homeomorphism. Replacing U by a smaller open neighbourhood of $s_0 \in I$, if required, we may assume that $\widetilde{F}(U \times \{t_i\}) \subseteq W_i$; for instance, it is enough to replace $U \times \{t_i\}$ with $\big(U \times \{t_i\}\big) \bigcap \big(\widetilde{F}\big|_{U \times \{t_i\}}\big)^{-1}(W_i)$. Then we can define $\widetilde{F}$ on $U \times [t_i, t_{i+1}]$ to be the composition $\varphi \circ F$, where $\varphi : V_i \to W_i$ is the inverse of the homeomorphism $f\big|_{W_i} : W_i \to V_i$. Continuing in this way, after a finite number of steps, we get a continuous map $\widetilde{F} : U \times I \to X$ with $\widetilde{F}(0,0) = x_0$ and $f \circ \widetilde{F} = F\big|_{U \times I}$, as required. □

Lemma 3.3.17. *Let $f : X \to Y$ be a covering map, and let $\gamma : I \to X$ be a continuous map. If $f \circ \gamma$ is a constant map, so is γ.*

Proof. Suppose that $(f \circ \gamma)(t) = y_0$, for all $t \in I$. Let $V \subseteq Y$ be an open neighbourhood of y_0 that is evenly covered by f. Then $f^{-1}(V) = \bigsqcup\limits_{\alpha \in \Lambda} U_\alpha$, where $\{U_\alpha\}_{\alpha \in \Lambda}$ is a family of pairwise disjoint open subsets of X with $f\big|_{U_\alpha} : U_\alpha \to V$ a homeomorphism, for all $\alpha \in \Lambda$. Since $\gamma(t) \in f^{-1}(V)$, for all $t \in I$, and I is connected, there is a unique $\alpha_0 \in \Lambda$ such that $\gamma(t) \in U_{\alpha_0}$, for all $t \in I$. Since $f\big|_{U_{\alpha_0}}$ is a homeomorphism, its restriction on the image of γ must be a homeomorphism; this is not possible since $f \circ \gamma$ is a constant map. □

Corollary 3.3.18 (Lifting of path-homotopy). *Let $f : X \to Y$ be a covering map. Let $\gamma_0, \gamma_1 : I \to Y$ be two paths in Y with $\gamma_0(0) = \gamma_1(0) = y_0$ and $\gamma_0(1) = \gamma_1(1) = y_1$. Let $F : I \times I \to Y$ be a path-homotopy from γ_0 to γ_1 in X. If $\widetilde{F} : I \times I \to X$ is a lifting of F on X, then $\widetilde{F}$ is a path-homotopy.*

Proof. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{F} : I \times I \to X$ be the lifting of F on X with $\widetilde{F}(0,0) = x_0$. Then by Theorem 3.3.16, $\widetilde{F}$ is a homotopy of maps from $\widetilde{\gamma}_0 := \widetilde{F}(-,0)$ to $\widetilde{\gamma}_1 := \widetilde{F}(-,1)$. Let $x_1 := \widetilde{\gamma}_0(1) = \widetilde{F}(1,0)$. To show $\widetilde{F}$ is a path-homotopy, we need to ensure that $\widetilde{F}(0,t) = x_0$ and $\widetilde{F}(1,t) = x_1$, for all $t \in I$. This follows from the Lemma 3.3.17 applied to the paths $t \mapsto \widetilde{F}(0,t)$ and $t \mapsto \widetilde{F}(1,t)$. □

Corollary 3.3.19. *Let $f : X \to Y$ be a covering map. Let $y_0 \in Y$ and fix a point $x_0 \in f^{-1}(y_0)$. Then the group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is injective. The image subgroup $f_*(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$ consists of the homotopy classes of loops in Y based at y_0 whose lifts to X starting at x_0 are loops.*

Proof. Let $[\gamma], [\delta] \in \pi_1(X, x_0)$ be such that $f_*([\gamma]) = f_*([\delta])$. Then $f \circ \gamma$ is homotopic to $f \circ \delta$. Let $F : I \times I \to Y$ be a path homotopy from $f \circ \gamma$ to $f \circ \delta$. Then by Theorem 3.3.16 and its Corollary 3.3.18, we can lift F to a path-homotopy $\widetilde{F} : I \times I \to X$ with $\widetilde{F}(0,0) = x_0$. By uniqueness of path-lifting (see Theorem 3.3.12), $\widetilde{F}$ must be a path-homotopy from γ to δ (verify!). Therefore, f_* is injective.

To see the second part, note that any element of $f_*(\pi_1(X, x_0))$ is of the form $[f \circ \gamma]$, for some loop $\gamma : I \to X$ in X based at x_0. By Theorem 3.3.12 (Path-lifting) we can lift $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ starting at $\gamma(0)$. Then by uniqueness of path-lifting, we have $\widetilde{f \circ \gamma} = \gamma$. Conversely, if δ is a loop in Y based at x_0 such that its lift $\widetilde{\delta}$ in X is a loop in X based at x_0, then $f_*([\widetilde{\delta}]) = [f \circ \widetilde{\delta}] = [\delta]$. This completes the proof. □

Exercise 3.3.20 (Lifting of opposite path). Let $f : X \to Y$ be a covering map. Let $\gamma : I \to Y$ be a path in Y from y_0 to y_1. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{\gamma}$ be the lift of γ in X starting at x_0. Let $\overline{\gamma}$ be the opposite path of γ. If $\widetilde{\overline{\gamma}}$ is the lift of $\overline{\gamma}$ in X starting at $\widetilde{\gamma}(1)$, then show that $\overline{\widetilde{\gamma}} = \widetilde{\overline{\gamma}}$.

Exercise 3.3.21 (Lifting of product of paths). Let $f : X \to Y$ be a covering map. Let $\gamma, \delta : I \to Y$ be two paths in Y such that $\gamma(1) = \delta(0)$. Fix a point $x_0 \in f^{-1}(\gamma(0))$, and let $\widetilde{\gamma}$ and $\widetilde{\gamma \star \delta}$ be the liftings of the paths γ and $\gamma \star \delta$, respectively, in X starting at x_0. If $\widetilde{\delta}$ is the lifting of δ in X starting at $x_1 := \widetilde{\gamma}(1)$, show that $\widetilde{\gamma} \star \widetilde{\delta} = \widetilde{\gamma \star \delta}$.

Lemma 3.3.22. *Let $f : X \to Y$ be a covering space. If both X and Y are path-connected, then the cardinality of the fiber $f^{-1}(y)$ is independent of $y \in Y$.*

Proof. Fix a point $y_0 \in Y$, and a point $x_0 \in f^{-1}(y_0) \subseteq X$. Let $G = \pi_1(Y, y_0)$ and $H = f_*(\pi_1(X, x_0))$. Let $H \backslash G := \{Hg : g \in G\}$ be the set of all right cosets of H in G. Since both X and Y are path-connected, the cardinality of the set $H \backslash G$ is independent of choices of $y_0 \in Y$ and $x_0 \in f^{-1}(y_0)$. Therefore, to show the cardinality of the fibers $f^{-1}(y)$ is independent of $y \in Y$, it is enough to construct a bijective map

$$\Phi : H \backslash G \longrightarrow f^{-1}(y_0). \tag{3.3.23}$$

Given a loop γ in Y based at y_0, let $\widetilde{\gamma}$ be the lifting of γ in X starting at x_0. Note that, $x_1 := \widetilde{\gamma}(1) \in f^{-1}(y_0)$. Then we define

$$\Phi(H[\gamma]) := \widetilde{\gamma}(1). \tag{3.3.24}$$

We need to show that x_1 is independent of choice of γ. Let δ be a loop in Y based at y_0 with $H[\gamma] = H[\delta]$. Then $[\gamma \star \overline{\delta}] = [\gamma][\delta]^{-1} \in H = f_*\big(\pi_1(X, x_0)\big)$, where $\overline{\delta}$ is the opposite path of δ. Then by Corollary 3.3.19 the loop $\gamma \star \overline{\delta}$ lifts to a unique loop $\widetilde{(\gamma \star \overline{\delta})}$ in X based at x_0. Let $\widetilde{\overline{\delta}}$ be the lifting of $\overline{\delta}$ in X starting at $x_1 := \widetilde{\gamma}(1)$. Then by Exercises 3.3.21 we have $\widetilde{\gamma \star \overline{\delta}} = \widetilde{\gamma} \star \widetilde{\overline{\delta}}$. Since $\widetilde{\gamma \star \overline{\delta}}$ is a loop in X based at x_0, we have $\widetilde{\overline{\delta}}(1) = x_0$. Let η be the opposite path of $\widetilde{\overline{\delta}}$ in X. Since

$$\begin{aligned}(f \circ \eta)(t) = f(\eta(t)) &= f\big(\widetilde{\overline{\delta}}(1-t)\big) \\ &= \overline{\delta}(1-t) = \delta(t), \ \ \forall\, t \in I,\end{aligned}$$

η is a lift of δ in X starting at $\eta(0) = \widetilde{\overline{\delta}}(1) = x_0$. Then by uniqueness of path-lifting (Theorem 3.3.12) we have $\eta = \widetilde{\delta}$. Then $\widetilde{\delta}(1) = \eta(1) = \widetilde{\overline{\delta}}(0) = x_1$. Therefore, the map Φ in (3.3.24) is well-defined. Since X is path connected, given any $x_1 \in f^{-1}(y_0)$, there is a path φ in X from x_0 to x_1. Then $f \circ \varphi$ is a loop in Y based at y_0 whose lift $\widetilde{f \circ \varphi}$ starting at x_0 is the unique path φ ending at $x_1 = \varphi(1)$. Therefore, Φ is surjective. Let $[\gamma], [\delta] \in \pi_1(Y, y_0) = G$ be such that $\Phi(H[\gamma]) = \Phi(H[\delta])$. Let $\widetilde{\gamma}$ and $\widetilde{\delta}$ be the lifts of γ and δ, respectively, in X starting at x_0. Let $\overline{\widetilde{\delta}}$ be the opposite path of $\widetilde{\delta}$ in X. Since $\Phi(H[\gamma]) = \Phi(H[\delta])$, we have $\widetilde{\gamma}(1) = \widetilde{\delta}(1)$, and hence $\widetilde{\gamma} \star \overline{\widetilde{\delta}}$ is a loop in X based at x_0. Since $f \circ \big(\widetilde{\gamma} \star \overline{\widetilde{\delta}}\big) = \gamma \circ \overline{\delta}$, by uniqueness of path-lifting and Corollary 3.3.19, we conclude that $[\gamma \star \overline{\delta}] \in f_*\big(\pi_1(X, x_0)\big) = H$. Therefore, $H[\gamma] = H[\delta]$, and hence Φ is injective. Therefore, $\Phi : H\backslash G \to f^{-1}(y_0)$ is a bijection. $\square$

Exercise 3.3.25. Give an example to show that the Lemma 3.3.22 fails if X and Y are not path-connected.

Theorem 3.3.26 (General Lifting Criterion)**.** *Let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let T be a path-connected and locally path-connected space. A continuous map $g : (T, t_0) \to (Y, y_0)$ lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ if and only if $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Note that, such a lift $\widetilde{g}$ of g, if it exists, is unique by Lemma 3.3.13.*

Proof. If g lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ such that $f \circ \widetilde{g} = g$, then $g_*\big(\pi_1(T, t_0)\big) = f_*\big(\widetilde{g}_*(\pi_1(T, t_0))\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$.

To see the converse, suppose that $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Since T is path-connected, given a point $t_1 \in T$, there is a path $\gamma : I \to T$ with $\gamma(0) = t_0$ and $\gamma(1) = t_1$. Then $g \circ \gamma : I \to Y$ is a path in Y from $g(t_0) = y_0$ to $y_1 := g(t_1) = (g \circ \gamma)(1)$. Since $f : (X, x_0) \to (Y, y_0)$ is a covering map, by Theorem 3.3.12 (Path-lifting) the path $g \circ \gamma$ lifts to a unique path $\widetilde{g \circ \gamma}$ in X starting at x_0. Define a map

$$\widetilde{g} : T \to X \tag{3.3.27}$$

by sending t_1 to $x_1 := \widetilde{g \circ \gamma}(1) \in X$. To show the map $\widetilde{g}$ is independent of choice of a path γ in T from t_0 to t_1, note that given any path $\delta : I \to T$ from t_0 to t_1, the product path $\gamma \star \overline{\delta}$ is a loop in T based at t_0. Since $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$, by the second part of the Corollary 3.3.19 the loop $g \circ (\gamma \star \overline{\delta}) = (g \circ \gamma) \star (g \circ \overline{\delta})$ lifts to a unique loop, say φ, in X based at x_0. Let

$\widetilde{(g \circ \overline{\delta})}$ be the lifting of $g \circ \overline{\delta}$ in X starting at $x_1 := \widetilde{g \circ \gamma}(1)$. Then by Exercises 3.3.21 we have $\varphi = \widetilde{(g \circ \gamma)} \star \widetilde{(g \circ \overline{\delta})}$. Since φ is a loop in X based at x_0, we have $\widetilde{(g \circ \overline{\delta})}(1) = x_0$. Let η be the opposite path of $\widetilde{(g \circ \overline{\delta})}$. Since

$$\begin{aligned}(f \circ \eta)(t) &= f(\widetilde{(g \circ \overline{\delta})}(1-t)) \\ &= (g \circ \overline{\delta})(1-t) \\ &= (g \circ \delta)(t), \ \forall\, t \in I,\end{aligned}$$

and $\eta(0) = \widetilde{(g \circ \overline{\delta})}(1) = x_0$, by uniqueness of path-lifting, we have $\eta = \widetilde{(g \circ \delta)}$. Then $\widetilde{(g \circ \delta)}(1) = \eta(1) = \widetilde{(g \circ \overline{\delta})}(0) = \widetilde{(g \circ \gamma)}(1) = x_1$. Therefore, the map $\widetilde{g}$ in (3.3.27) is well-defined. It follows from the construction of $\widetilde{g}$ that $f \circ \widetilde{g} = g$. It remains to show that $\widetilde{g}$ is continuous. Here we need to use local path-connectedness of T.

Fix a point $t_1 \in T$ and let $y_1 = g(t_1) \in Y$ and $x_1 := \widetilde{g}(t_1) \in f^{-1}(y_1)$. Since f is a covering map, there is an open neighbourhood $U \subseteq Y$ of y_1 and an open neighbourhood $\widetilde{U} \subseteq X$ of x_1 such that

$$f|_{\widetilde{U}} : \widetilde{U} \to U \tag{3.3.28}$$

is a homeomorphism. Since T is locally path-connected and g is continuous, there is a path-connected neighbourhood $V \subseteq T$ of t_1 such that $g(V) \subseteq U$. To show $\widetilde{g} : T \to X$ continuous, it is enough to show that $\widetilde{g}(V) \subseteq \widetilde{U}$. Given $t' \in V$, choose a path α inside V joining t_1 to t'. Then $\gamma \star \alpha$ is a path in T joining t_0 to t', and its image $g \circ (\gamma \star \alpha)$ has a lifting, say β, in X starting at x_0. Let $\widetilde{\alpha} := s \circ (g \circ \alpha)$, where $s : U \to \widetilde{U}$ is the inverse of the homeomorphism $f|_{\widetilde{U}}$ given in (3.3.28). Since $\widetilde{\gamma}(1) = (s \circ g \circ \alpha)(0)$, by uniqueness of path-lifting, β coincides with $\widetilde{\gamma} \star (s \circ g \circ \alpha)$. Then $\widetilde{g}(t') = \beta(1) = (\widetilde{\gamma} \star (s \circ g \circ \alpha))(1) = (s \circ g \circ \alpha)(1) \in \widetilde{U}$. Therefore, $\widetilde{g}(V) \subseteq \widetilde{U}$, and hence $\widetilde{g}$ is continuous. □

3.3.2 Fundamental group of S^1

Now we are in a position to compute fundamental group of the unit circle

$$S^1 := \{z \in \mathbb{C} \, : \, |z| = 1\} = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = 1\}.$$

Assuming that the reader has forgotten the statement of Theorem 3.3.1 by now, let's recall it once again.

Theorem 3.3.1. *The fundamental group $\pi_1(S^1, 1)$ of the unit circle S^1 with the base point $1 \in S^1$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ generated by the loop $\omega : I \to S^1$ defined by $\omega(t) = e^{2\pi i t}$, for all $t \in I = [0, 1]$.*

Proof. Let $\gamma : I \to S^1$ be a loop based at $x_0 = 1 \in S^1$. Since

$$p : \mathbb{R} \to S^1, \;\; t \mapsto e^{2\pi i t}$$

is a covering map (c.f. Example 3.3.6 (i)), there is a unique continuous map $\widetilde{\gamma} : I \to \mathbb{R}$ such that $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} = \gamma$. Since $p^{-1}(\gamma(1)) = \mathbb{Z}$, the path $\widetilde{\gamma}$ ends at some integer, say n. Note that, we have a path

$$\widetilde{\omega}_n : I \to \mathbb{R}, \ \ s \mapsto ns,$$

starting at 0 and ending at n. Clearly the path $\widetilde{\gamma}$ is homotopic to $\widetilde{\omega}_n$ by the linear homotopy

$$F : I \times I \to \mathbb{R}, \ \ (s,t) \mapsto (1-t)\widetilde{\gamma}(s) + t\,\widetilde{\omega}_n(s).$$

Then the composition $p \circ F : I \times I \to S^1$ is a homotopy from γ to ω_n, where $\omega_n : I \to S^1$ is the loop based at $1 \in S^1$ defined by

$$\omega_n(s) = e^{2\pi i n s}, \ \ \forall\, s \in I.$$

Therefore, $[\gamma] = [\omega_n]$ in $\pi_1(S^1, 1)$.

Define a map

$$\varphi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1), \ \ n \mapsto [\omega_n].$$

It follows from the above construction that φ is surjective. To show that φ is a group homomorphism, we need to show that $\omega_m \star \omega_n \simeq \omega_{m+n}$, for all $m, n \in \mathbb{Z}$. To see this, consider the *"translation by m"* map

$$\tau_m : \mathbb{R} \to \mathbb{R}, \ \ x \mapsto x + m.$$

Note that $\tau_m \circ \widetilde{\omega}_n$ is a path in $\mathbb{R}$ starting at m and ending at $m+n$, and hence the path $\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)$ in $\mathbb{R}$ starts at 0 and ends at $m+n$. Then it follows from the first paragraph that $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n))$ is homotopic to ω_{m+n}. Since $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)) = \omega_m \star \omega_n$, we conclude that φ is a group homomorphism.

To show that φ is injective, it is enough to show if a loop $\gamma : I \to S^1$ based at 1 is homotopic to both ω_n and ω_m, for some $m, n \in \mathbb{Z}$, then $m = n$. Indeed, if $\gamma \simeq \omega_m$ and $\gamma \simeq \omega_n$, then $\omega_m \simeq \omega_n$ by Lemma 3.1.4. Let $G : I \times I \to S^1$ be a homotopy from ω_m to ω_n in S^1. By Theorem 3.3.16 there is a unique continuous map $\widetilde{G} : I \times I \to \mathbb{R}$ such that $p \circ \widetilde{G} = G$ and $\widetilde{G}(0,0) = 0$. Then by uniqueness of path lifting (c.f. Theorem 3.3.12) we have $\widetilde{G}\big|_{\{0\}\times I} = \widetilde{\omega}_n$ and $\widetilde{G}\big|_{\{1\}\times I} = \widetilde{\omega}_m$. Since $\big\{\widetilde{G}\big|_{\{t\}\times I} : I \to \mathbb{R}\big\}_{t \in I}$ is a homotopy of paths, the end points $\widetilde{G}\big|_{\{t\}\times I}(1)$ are independent of t. Thus, $m = \widetilde{G}\big|_{\{0\}\times I}(1) = \widetilde{G}\big|_{\{1\}\times I}(1) = n$, and hence φ is injective. This completes the proof. □

3.3.3 Fundamental group of S^n, for $n \geq 2$

In this subsection we show that S^n is simply connected, for $n \geq 2$. First we need the following.

Lemma 3.3.29. *Let (X, x_0) be a pointed topological space. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X such that*

1. *each U_α is path-connected,*

2. $x_0 \in U_\alpha$, *for all* $\alpha \in \Lambda$,

3. $U_\alpha \cap U_\beta$ *is path-connected, for all* α, $\beta \in \Lambda$.

Then any loop in X *based at* x_0 *is homotopic to a finite product of loops each of which is contained in a single* U_α*, for finitely many* α*'s.*

Proof. Let $\gamma : I \to X$ be a loop based at x_0. Since γ is continuous, each $s \in I$ is contained in an open neighbourhood $V_s := (s - \delta_s, s + \delta_s) \subseteq I$ of s such that $\gamma(\overline{V}_s) \subseteq U_{\alpha_s}$, for some $\alpha_s \in \Lambda$. Since I is compact, we can choose finitely many such open neighbourhoods V_s's to cover I. Thus we get a finite partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $I = [0, 1]$ such that $\gamma([s_{j-1}, s_j]) \subseteq U_{\alpha_j}$, for some $\alpha_j \in \Lambda$, for all $j = 1, \ldots, m$. Therefore, the restriction

$$\gamma_j := \gamma\big|_{[s_{j-1}, s_j]} : [s_{j-1}, s_j] \to U_{\alpha_j} \subseteq X$$

is a path in U_{α_j}, for each $j = 1, \ldots, m$, and that $\gamma = \gamma_1 \star \cdots \star \gamma_m$. Since $U_j \cap U_{j+1}$ is path-

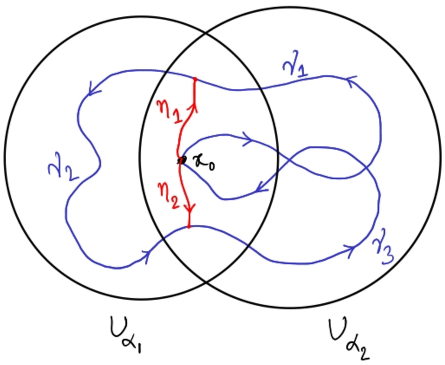


FIGURE 3.11

connected, we may choose a path η_i in $U_{\alpha_j} \cap U_{\alpha_{j+1}}$ from the base point x_0 to the point $\gamma(s_j) \in U_{\alpha_j} \cap U_{\alpha_{j+1}}$, for all j (see Figure 3.11). Denote by $\overline{\eta}_j$ the opposite path of η_j, for all j (see definition (3.2.23) in §3.2.3). Then the product loop

$$(\gamma_1 \star \overline{\eta_1}) \star (\eta_1 \star \gamma_2 \star \overline{\eta_2}) \star (\eta_2 \star \gamma_3 \star \overline{\eta_3}) \star \cdots \star (\eta_{m-1} \star \gamma_m) \tag{3.3.30}$$

is homotopic to γ (see Exercise 3.2.24). Clearly this loop is a composition of the loops $\gamma_1 \star \overline{\eta_1}$, $\eta_1 \star \gamma_2 \star \overline{\eta_2}$, $\eta_2 \star \gamma_3 \star \overline{\eta_3}, \cdots, \eta_{m-1} \star \gamma_m$ based at x_0, each lying inside a single U_{α_j}, for all $j = 1, \ldots, m$. This completes the proof. □

Exercise 3.3.31. Fix an integer $n \geq 1$.

(i) For any $x_0 \in S^n$, show that $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbb{R}^n$.

(ii) For a pair of antipodal points $x_1, x_2 \in S^n$, let $U_j := S^n \setminus \{x_j\}$, for $j = 1, 2$. Show that $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Proposition 3.3.32. *For an integer* $n \geq 2$*, we have* $\pi_1(S^n) = \{1\}$.

Proof. Fix a pair of antipodal points x_1, x_2 in S^n. Then we have two open subsets $U_1 = S^n \setminus \{x_1\}$ and $U_2 = S^n \setminus \{x_2\}$ each homeomorphic to $\mathbb{R}^n$. Clearly $S^n = U_1 \cup U_2$ and $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. Then by Exercise 3.3.31 we have $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, which is path-connected because $n \geq 2$. Fix a base point $x_0 \in U_1 \cap U_2$. Let γ be a loop in S^n based at x_0. Then by Lemma 3.3.29 γ is homotopic to a product of finitely many loops in S^n based at x_0 each of which are contained in either U_1 or U_2. Since both U_1 and U_2 are homeomorphic to $\mathbb{R}^n$ by Exercise 3.3.31, we have $\pi_1(U_j) = \pi_1(\mathbb{R}^n) = \{1\}$, for $j = 1, 2$. Therefore, γ is homotopic to a finite product of loops based at x_0 each of which are null-homotopic, and hence γ is null-homotopic. □

Corollary 3.3.33. *S^n is simply connected, for $n \geq 2$.*

Exercise 3.3.34. For a point $x_0 \in \mathbb{R}^n$, show that the space $\mathbb{R}^n \setminus \{x_0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Corollary 3.3.35. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for $n \neq 2$.*

Proof. If possible let $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. For $n = 1$, since $\mathbb{R}^2 \setminus \{0\}$ is path-connected while $\mathbb{R} \setminus \{f(0)\}$ is disconnected, there is no such homeomorphism in this case. Suppose that $n > 2$. In this case, we cannot distinguish $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{R}^n \setminus \{f(0)\}$ in terms of number of path-components; but we can distinguish them by their fundamental groups.

Since for any point $x \in \mathbb{R}^n$ the space $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ by Exercise 3.3.34, we have

$$\begin{aligned}\pi_1(\mathbb{R}^n \setminus \{x\}) &\cong \pi_1(S^{n-1} \times \mathbb{R}) \\ &\cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \\ &\cong \pi_1(S^{n-1}),\end{aligned}$$

because $\pi_1(\mathbb{R})$ is trivial. Since $\pi_1(S^1) \cong \mathbb{Z}$ by Theorem 3.3.1 while $\pi_1(S^{n-1}) \cong \{1\}$, for $n > 2$, by Proposition 3.3.32, such a homeomorphism cannot exists. □

Remark 3.3.36. A more general result that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$ can be proved in a similar fashion using higher homotopy groups or homology groups. In fact, using homology groups one can show that *non-empty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ can be homeomorphic if and only if $m = n$.*

3.3.4 Some applications

Theorem 3.3.37 (Fundamental theorem of algebra)**.** *Every non-constant polynomial with coefficients from $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Take a non-constant polynomial $p(z) \in \mathbb{C}[z]$. Diving $p(z)$ by its leading coefficient, if required, we may assume that

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z].$$

If $p(z)$ has no roots in $\mathbb{C}$, then for each real number $r \geq 0$, the map $\gamma_r : I \to S^1 \subset \mathbb{C}$ defined by

$$\gamma_r(s) := \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}, \quad \forall\, s \in I, \tag{3.3.38}$$

is a loop in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ with the base point $1 \in \mathbb{C}$. As r varies, the collection $\{\gamma_r\}_{r \geq 0}$ defines a homotopy of loops in S^1 based at 1. Since γ_0 is the constant loop 1 in S^1, we see that the homotopy class $[\gamma_r] \in \pi_1(S^1, 1)$ is trivial, for all $r \geq 0$.

Choose any $r \in \mathbb{R}$ with $r > \max\{1, |a_1| + \cdots + |a_n|\}$. Then for $|z| = r$ we have

$$\begin{aligned} |z^n| = r^n = r \cdot r^{n-1} &> \big(|a_1| + \cdots + |a_n|\big)|z^{n-1}| \\ &\geq |a_1 z^{n-1} + \cdots + a_n| \end{aligned}$$

From this inequality, it follows that for each $t \in [0, 1]$, the polynomial

$$p_t(z) := z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = r$. Replacing $p(z)$ with $p_t(z)$ in the expression of γ_r in (3.3.38) and letting t vary from 1 to 0, we get a homotopy from the loop γ_r to the loop

$$\omega_n : I \to S^1, \;\; s \mapsto e^{2\pi ins}.$$

Since the loop ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1, 1) \cong \mathbb{Z}$, and that $[\omega_n] = [\gamma_t] = 0$, we must have $n = 0$. Thus the only polynomials without roots in $\mathbb{C}$ are constants. □

Definition 3.3.39. A *deformation retraction* of X onto its subspace A is a continuous map $F : X \times I \to X$ such that the associated family of continuous maps

$$\Big\{ f_t := F\big|_{X \times \{t\}} : X \to X \Big\}_{t \in I}$$

obtained by restricting F on the slices $X \times \{t\} \hookrightarrow X \times I$, for each $t \in I$, satisfies $f_0 = \mathrm{Id}_X$, $f_1(X) = A$, and $f_t\big|_A = \mathrm{Id}_A$, $\forall\, t \in I$. In this case, we say that A is a deformation retract of X.

Example 3.3.40. (i) Let $D = \{re^{i\theta} \in \mathbb{C} : 0 < r \leq 1,\ 0 \leq \theta < 2\pi\}$ be the punctured disk of radius 1 in the plane $\mathbb{C}$, and let $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subseteq X$ be the unit circle. For each $t \in I = [0, 1]$, we define a map

$$f_t : D \longrightarrow D$$

by sending $re^{i\theta} \in D$ to $\big(t + (1-t)r\big)e^{i\theta} \in D$. It is easy to verify that $\{f_t\}_{t \in I}$ is a family of continuous maps from D into itself, and satisfies $f_0 = \mathrm{Id}_D$, $f_1(D) = S^1$ and $f_t\big|_{S^1} = \mathrm{Id}_{S^1}$. Therefore, $\{f_t\}_{t \in I}$ is a deformation retraction of D onto S^1.

(ii) Let X be the Möbius strip (see Figure 3.12) and $A \subset X$ be the central simple loop of X. Then there is a deformation retraction of X onto A.

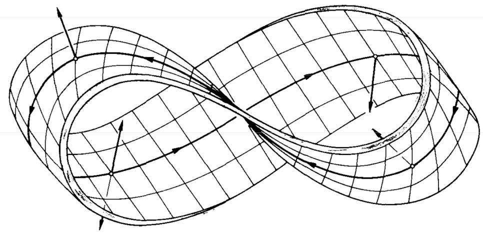


FIGURE 3.12: Möbius strip

Definition 3.3.41. A *retraction* of X onto a subspace $A \subset X$ is a continuous map $f : X \to X$ such that $f(X) = A$ and $f\big|_A = \mathrm{Id}_A$. A subspace $A \subseteq X$ is said to be a *retract* of X if there is a retraction of X onto A.

Note that a retraction $f : X \to X$ of X onto a subspace $A \subseteq X$ can be characterized by its property $f \circ f = f$, and hence we can think of it as a topological analogue of a *projection operator* in algebra.

Lemma 3.3.42. *If $A \subseteq X$ is a retract of X, for any $a_0 \in A$ the homomorphism of fundamental groups*

$$\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0),$$

induced by the inclusion map $\iota : A \hookrightarrow X$, is injective.

Proof. Let $f : X \to X$ be a retraction of X onto A. Then $f \circ \iota = \mathrm{Id}_A$, the identity map of A. Then by Proposition 3.2.12 and Remark 3.2.13 we have $f_* \circ \iota_* = \mathrm{Id}_{\pi_1(A,a_0)}$. Thus ι_* admits a left inverse, and hence is injective. □

Proposition 3.3.43. *If $A \subseteq X$ is a deformation retract of X, then X is homotopically equivalent to A (see Definition 3.1.9).*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto its subspace A. Since

$$f_0 : X \to X, \ x \mapsto F(x, 0)$$

is the identity map $\mathrm{Id}_X : X \to X$, and

$$f_1 : X \to X, \ x \mapsto F(x, 1)$$

is a retraction of X onto A, we conclude that F is a homotopy from Id_X to a retraction of X onto A. Since $f_1 \circ \iota = \mathrm{Id}_A$ and $\iota \circ f_1$ is homotopic to the identity map of X, we conclude that X and A are homotopically equivalent. □

Corollary 3.3.44. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$ we have an isomorphism of fundamental groups $\pi_1(A, a_0) \cong \pi_1(X, a_0)$.*

Proof. Follows from Lemma 3.2.16. □

Remark 3.3.45. Note that the constant map $X \to \{x_0\} \subseteq X$ being continuous, every space X admits a retraction onto a point of it. However, the next Proposition 3.3.46 and Lemma 3.3.47 produce examples of topological spaces that do not admit any deformation retract onto a point of it.

Proposition 3.3.46. *If there is a deformation retract of X onto a point $x_0 \in X$, then X is path connected.*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto a point $x_0 \in X$. Since for any point $x \in X$, the continuous map

$$\phi_x : I \to X,\ t \mapsto F(x,t)$$

is a path joining $F(x,0) = x$ and $F(x,1) = x_0$, X is path connected. □

Lemma 3.3.47. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$, the homomorphism of fundamental groups $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ induced by the inclusion map $\iota : A \hookrightarrow X$ is an isomorphism.*

Proof. Let $F : X \times I \to X$ be a deformation retraction of X onto X. Then $f_1 := F\big|_{X\times\{1\}} : X \to X$ is a retraction of X onto A. Then by Lemma 3.3.42 the homomorphism $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective. To show ι_* is an isomorphism, it enough to show that it is surjective. Note that, given any loop $\gamma : I \to X$ in X based at a_0, the composite map

$$G : I \times I \stackrel{\gamma\times \mathrm{Id}_I}{\longrightarrow} X \times I \stackrel{F}{\longrightarrow} X$$

is a path-homotopy from $G\big|_{I\times\{0\}} = \gamma$ to a loop $g := G\big|_{I\times\{1\}} : I \to A$ based at a_0. Thus, $\iota_*([g]) = [g] = [\gamma]$, and hence ι_* is surjective. □

Remark 3.3.48. The notion of deformation retraction of a space X onto a subspace $A \subseteq X$ is a way to continuously deform X onto A in a very strong sense, while the notion of homotopy equivalence seems to be a weaker notion of being able to deform a space into another space. However, if two spaces X and Y are homotopically equivalent, then there is a space Z such that both X and Y are deformation retracts of Z. Such a space Z can be constructed as a mapping cylinder

$$M_f := \big((X \times I) \sqcup Y\big)/(x, 1) \sim f(x)$$

of a homotopy equivalence $f : X \to Y$. We shall not go into details for its proof in this course.

Exercise 3.3.49. Show that the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ do not admit any deformation retraction onto a point of it.

Exercise 3.3.50. Show that $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \mathbb{Z}$.

For an integer $n \geq 1$, let

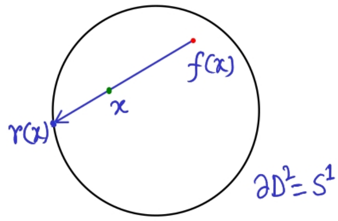
$$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1\}$$

be the *closed unit disk* in $\mathbb{R}^n$. Its boundary ∂D^n is the *unit sphere* in $\mathbb{R}^n$ given by

$$S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ \sum_{j=1}^{n} x_j^2 = 1\}.$$

Theorem 3.3.51 (Brouwer's fixed point theorem)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Suppose on the contrary that $f : D^2 \to D^2$ has no fixed point, i.e., $f(x) \neq x$, $\forall\, x \in D^2$. Then for each $x \in D^2$, the ray in $\mathbb{R}^2$ starting at $f(x)$ and passing through x hits a unique point, say $r(x) \in S^1$. This defines a map $r : D^2 \to S^1$. Since f is continuous, small perturbations of



x produce small perturbations of $f(x)$, and hence small perturbations of the ray starting from $f(x)$ and passing through x, it follows that the function $x \mapsto r(x)$ is continuous. Explicit proof of continuity could be given by writing down the explicit expression for $r(x)$ in terms of $f(x)$. Note that $r(x) = x$, for all $x \in S^1$. Therefore, $r : D^2 \to S^1$ is a retraction of D^2 onto its subspace $S^1 = \partial D^2$. Then by Lemma 3.3.42 the homomorphism of fundamental groups

$$\iota_* : \pi_1(S^1, (1,0)) \longrightarrow \pi_1(D^2, (1,0))$$

induced by the inclusion map $\iota : S^1 \hookrightarrow D^2$, is injective. Since $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1,0))$ is trivial, we get a contradiction. $\square$

Remark 3.3.52. The corresponding statement for Brouwer's fixed point theorem holds, more generally, for a closed unit disk $D^n \subset \mathbb{R}^n$, for all $n \geq 2$. If time permits, we shall give a proof of it using homology. However, the original proof of it, due to Brouwer, neither uses homology nor uses homotopy groups, which was not invented at that time. Instead, Brouwer's proof uses the notion of degree of maps $S^n \to S^n$, which could be defined later using homology, but Brouwer defined it more directly in a geometric way.

Definition 3.3.53. For $x = (x_1, \ldots, x_{n+1}) \in S^n$, we define its *antipodal point* to be the point $-x := (-x_1, \ldots, -x_{n+1}) \in S^n$.

Theorem 3.3.54 (Borsuk-Ulam)**.** *Let $n \in \{1, 2\}$. Then for every continuous map $f : S^n \to \mathbb{R}^n$, there is a pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$.*

Proof. The case $n = 1$ is easy. Indeed, since the function

$$g : S^1 \to \mathbb{R}, \quad x \mapsto f(x) - f(-x)$$

changes its sign after the point $x \in S^1$ moves half way along the circle S^1, there must be a pint $x \in S^1$ such that $f(x) = f(-x)$.

Assume that $n = 2$. We use the same technique used to compute the fundamental group of S^1. Suppose on the contrary that there is a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$, for all $x \in S^2$. Then we can define a map $g : S^2 \to \mathbb{R}^2$ by

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}, \ \forall\, x \in S^2,$$

where $\|(y_1, y_2)\| := \sqrt{y_1^2 + y_2^2}$ is the *norm* of $(y_1, y_2) \in \mathbb{R}^2$. Since $\|g(x)\| = 1$, the image of the map g lands inside $S^1 \subset \mathbb{R}^2$. Note that the map $g : S^2 \to S^1$ is continuous. Define a loop $\eta : I = [0, 1] \to S^2$ by

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0), \ \forall\, s \in I. \tag{3.3.55}$$

Then η circles around the equator of the sphere $S^2 \subset \mathbb{R}^3$. Let $h : I \to S^1$ be the composite map $h := g \circ \eta$.

$$h : I \xrightarrow{\eta} S^2 \xrightarrow{g} S^1.$$

Since $g(x) = -g(-x)$, we have

$$h(s + \frac{1}{2}) = -h(s), \ \forall\, s \in [0, 1/2]. \tag{3.3.56}$$

Now consider the covering map

$$p : \mathbb{R} \to S^1, \ \ s \mapsto e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s).$$

Lift the loop $h : I \to S^1$ to this cover to get a unique path $\widetilde{h} : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$ (see Theorem 3.3.12). Then it follows from the relation (3.3.56) that

$$\widetilde{h}(s + \frac{1}{2}) = \widetilde{h}(s) + \frac{q(s)}{2}, \tag{3.3.57}$$

for some odd integer $q(s)$ depending on $s \in [0, \frac{1}{2}]$. Since $\widetilde{h}$ is continuous, it follows from the equation (3.3.57) that the map

$$I \to \mathbb{R}, \ \ s \mapsto q(s),$$

is continuous on $[0, \frac{1}{2}]$. Since q is a discrete function taking values in odd integers, we must have $q(s) = q$, for some odd integer q, for all $s \in [0, \frac{1}{2}]$. In particular, putting $s = 1/2$ and 0 in (3.3.57) we have

$$\widetilde{h}(1) = \widetilde{h}(1/2) + \frac{q}{2} = \widetilde{h}(0) + q.$$

This means that the loop h represents q times a generator of $\pi_1(S^1)$. Since q is an odd integer, h cannot be null homotopic. But this cannot happen because the loop $\eta : I \to S^2$ being null-homotopic, the loop $h := g \circ \eta : I \to S^2 \to S^1$ should be null-homotopic. Thus we get a contradiction. This completes the proof. □

Remark 3.3.58. (i) Borsuk-Ulam theorem (Theorem 3.3.54) holds for all integer $n \geq 1$. A general proof could be given using homology theory later.

(ii) Theorem 3.3.54 says that there is no one-to-one continuous map from S^n into $\mathbb{R}^n$. As a result, S^n cannot be homeomorphic to a subspace of $\mathbb{R}^n$.

3.4 Galois theory for covering spaces

3.4.1 Universal cover

Since we shall work with paths in X, and a locally path-connected space is connected if and only if it is path-connected, and path-connected components of X are the same as connected components of X, there is no harm in assuming that X is connected or equivalently path-connected. Unless explicitly mentioned, in this section, we always assume that X is path-connected and locally path-connected.

Proposition 3.4.1. *Let X be a connected and locally path-connected topological space. Fix a point $x_0 \in X$. Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a simply connected covering. Then for any connected covering $f : (Y, y_0) \to (X, x_0)$, there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $p \circ F = f$.*

$$\begin{tikzcd}
& (\widetilde{X}, \widetilde{x_0}) \arrow[dl, dashed, "F"'] \arrow[d, "p"] \\
(Y, y_0) \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}$$

Proof. Since X is locally path-connected and $\widetilde{X}$ is a simply connected covering of X, $\widetilde{X}$ is path-connected and locally path-connected. Since $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial and $f : (Y, y_0) \to (X, x_0)$ is a covering map, by general lifting criterion (see Theorem 3.3.26) there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $f \circ F = p$. □

Proposition 3.4.2. *Let (X, x_0) be a locally path-connected and path-connected topological space. Let $p_1 : (\widetilde{X_1}, \widetilde{x_1}) \to (X, x_0)$ and $p_2 : (\widetilde{X_2}, \widetilde{x_2}) \to (X, x_0)$ be two simply connected covering spaces of (X, x_0). Then there is a unique homeomorphism of pointed topological spaces $F : (\widetilde{X_1}, \widetilde{x_1}) \to (\widetilde{X_2}, \widetilde{x_2})$ such that $p_2 \circ F = p_1$.*

$$\begin{tikzcd}
(\widetilde{X_1}, \widetilde{x_1}) \arrow[rr, "F"] \arrow[dr, "p_1"'] & & (\widetilde{X_2}, \widetilde{x_2}) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

Proof. Follows from Proposition 3.4.1. □

Definition 3.4.3. A simply connected covering space of a path-connected locally path-connected topological space (X, x_0) is called the *universal cover* of (X, x_0). This name is due to its universal property (c.f. Proposition 3.4.1) and uniqueness upto a unique homeomorphism (c.f. Proposition 3.4.2).

It is not yet clear if universal cover of a path-connected locally path-connected topological space exists or not, however if it exists, it is unique up to a unique homeomorphism of pointed topological space by Proposition 3.4.2. The following Lemma 3.4.4 gives a necessary condition on (X, x_0) for existence of a universal covering space.

Lemma 3.4.4. *Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be the universal cover of (X, x_0). Then each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.*

Proof. Fix $x \in X$. Then there is a path-connected open neighbourhood $U \subseteq X$ which is evenly covered by the covering map p. Let $\widetilde{U} \subseteq \widetilde{X}$ be the path-connected open subset such that $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Let γ be a loop in U based at x. Using the homeomorphism $p|_{\widetilde{U}}$, we can lift it to a loop $\widetilde{\gamma}$ in $\widetilde{X}$ based at the point $\widetilde{x} \in \widetilde{U} \cap p^{-1}(x)$. Since $\widetilde{X}$ is simply-connected, we have a path-homotopy $F : I \times I \to \widetilde{X}$ from $\widetilde{\gamma}$ to the constant loop $c_{\widetilde{x}}$ at $\widetilde{x}$ in $\widetilde{X}$. Composing F with p we get a path-homotopy $p \circ F$ from γ to the constant loop c_x at x in X. This shows that the homomorphism $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $\iota : U \hookrightarrow X$ is trivial. $\square$

Definition 3.4.5. A path-connected and locally path-connected topological space X is said to be *semi-locally simply connected* if each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.

3.4.2 Construction of universal cover

The following theorem shows that the condition on (X, x_0) for existence of its universal covering space given in Lemma 3.4.4 is, in fact, sufficient.

Theorem 3.4.6. *Let X be a path-connected, locally path-connected topological space. Fix a point $x_0 \in X$. Then a simply connected covering space of (X, x_0) exists if and only if X is semi-locally simply connected.*

Proof. If a simply connected covering space for X exists, then X is semi-locally simply connected by Lemma 3.4.4.

Suppose that X is semi-locally simply connected. We give an explicit construction of a simply connected covering space of X. Note that, if $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a simply connected covering space for (X, x_0), then for each $\widetilde{x} \in \widetilde{X}$, there is a unique path-homotopy class of paths in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x}$ (see Corollary 3.2.33). Thus, points of $\widetilde{X}$ can be thought of as homotopy classes of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$, and hence can be thought of as the homotopy classes of paths in X starting at x_0 thanks to the homotopy lifting property. This motivates us to construct the underlined set of points of $\widetilde{X}$ as

$$\widetilde{X} := \{[\gamma] : \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

where $[\gamma]$ denotes the path-homotopy class of a path γ in X. Define

$$p : \widetilde{X} \to X \tag{3.4.7}$$

by sending a $[\gamma] \in \widetilde{X}$ to the end point $\gamma(1) \in X$ of γ; this map is well-defined because of the definition of path-homotopy (see Definition 3.2.1). Since X is path-connected, given any $x_1 \in X$ there is a path γ in X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $[\gamma] \in \widetilde{X}$ with $p([\gamma]) = x_1$. Thus, p is surjective. If we set $\widetilde{x_0} \in \widetilde{X}$ to be the path-homotopy class of the constant path $c_{x_0} : I \to X$ given by $c_{x_0}(t) = x_0$, $\forall\, t \in I$, then $p(\widetilde{x_0}) = x_0$.

It remains to give a suitable topology on $\widetilde{X}$ to make $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ a simply connected covering space of (X, x_0). Let

$$\begin{aligned} \mathscr{U} := \{V \overset{\iota}{\hookrightarrow} X \mid\ & V \text{ is a path-connected open subset of } X \text{ such that} \\ & \text{the homomorphism } \iota_* : \pi_1(V) \to \pi_1(X) \text{ is trivial } \}. \end{aligned}$$

Note that, if the homomorphism $\iota_* : \pi_1(V, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : V \hookrightarrow X$, is trivial for some $x \in V$, then it is trivial for all points of V, whenever V is path-connected. Moreover, if U and V are two path-connected open subsets of X with $V \subseteq U$ and $U \in \mathscr{U}$, then it follows from the following commutative diagram

$$\begin{tikzcd} \pi_1(V) \arrow[rr, "\iota_{V,U*}"] \arrow[dr, "\iota_{V*}"'] & & \pi_1(U) \arrow[dl, "\iota_{U*}"] \\ & \pi_1(X) & \end{tikzcd}$$

that $V \in \mathscr{U}$, where $\iota_U : U \hookrightarrow X$, $\iota_V : V \hookrightarrow X$ and $\iota_{V,U} : V \hookrightarrow U$ are inclusion maps. Since X is locally-path-connected, path-connected and semi-locally simply connected, now it follows that $\mathscr{U}$ is a basis for the topology on X (verify!).

We now use the collection $\mathscr{U}$ to construct a collection $\mathscr{B}$ of subsets of $\widetilde{X}$ which forms a basis for the desired topology on $\widetilde{X}$. Given $U \in \mathscr{U}$ and a path γ in X starting at x_0 and ending at a point in U, consider the subset

$$U_{[\gamma]} := \{[\gamma \star \eta] : \eta \text{ is a path in } U \text{ starting at } \gamma(1)\} \subseteq \widetilde{X}.$$

Note that, if γ is path-homotopic to γ' in X, then $\gamma(1) = \gamma'(1)$, and hence for any path η in U starting at $\gamma(1) = \gamma'(1)$, we have $[\gamma \star \eta] = [\gamma' \star \eta]$. Therefore, the subset $U_{[\gamma]} \subseteq \widetilde{X}$ depends only on U and the path-homotopy class of γ in X.

<u>Observation 1</u>: The restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U \tag{3.4.8}$$

is bijective. Indeed, it is surjective because U is path-connected. To see it is injective, note that if $p([\gamma \star \eta]) = p([\gamma \star \eta'])$, then $\eta(1) = \eta'(1)$ and so the loop $\eta \star \overline{\eta'}$ is path-homotopic to the constant path $c_{\eta(0)}$ inside X, because the homomorphism $\iota_* : \pi_1(U) \to \pi_1(X)$ is trivial. Then it follows that $[\gamma \star \eta] = [\gamma \star \eta']$. Therefore, the restriction of p on $U_{[\gamma]}$ (see (3.4.8)) is injective, and hence is bijective.

Observation 2: Given $U \in \mathscr{U}$ and any two paths γ and δ in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1), \delta(1) \in U$, if $[\delta] \in U_{[\gamma]}$, then we must have $U_{[\gamma]} = U_{[\delta]}$. Indeed, if $[\delta] \in U_{[\gamma]}$, then $[\delta] = [\gamma \star \eta]$, for some path η in U with $\eta(0) = \gamma(1)$. Then for any path α in U with $\alpha(0) = \delta(1)$, we have $[\delta \star \alpha] = [(\gamma \star \eta) \star \alpha] = [\gamma \star (\eta \star \alpha)] \in U_{[\gamma]}$. Thus $U_{[\delta]} \subseteq U_{[\gamma]}$. Conversely, given any $[\gamma \star \alpha] \in U_{[\gamma]}$ we have $[\gamma \star \alpha] = [\gamma \star \eta \star \overline{\eta} \star \alpha] = [\delta \star (\overline{\eta} \star \alpha)] \in U_{[\delta]}$, which shows that $U_{[\gamma]} \subseteq U_{[\delta]}$. Therefore, we conclude that $U_{[\gamma]} = U_{[\delta]}$ if $[\delta] \in U_{[\gamma]}$.

Now we use the above two observations to show that the collection

$$\mathscr{B} := \{U_{[\gamma]} : U \in \mathscr{U} \text{ and } \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

forms a basis for a topology on $\widetilde{X}$. Note that, X being path-connected, we have $\widetilde{X} = \bigcup\limits_{U_{[\gamma]} \in \mathscr{B}} U_{[\gamma]}$. To check the second property for $\mathscr{B}$ to be a basis for a topology on $\widetilde{X}$, suppose that we are given two objects $U_{[\gamma]}, V_{[\delta]} \in \mathscr{B}$ and an element

$$[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}. \tag{3.4.9}$$

Now $U, V \in \mathscr{U}$, and γ and δ are paths in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) \in U$, $\delta(1) \in V$. We claim that

$$U_{[\gamma]} = U_{[\alpha]} \quad \text{and} \quad V_{[\delta]} = V_{[\alpha]}. \tag{3.4.10}$$

Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, we have $[\alpha] = [\gamma \star \eta] = [\delta \star \eta']$, for some paths η and η' in U and V respectively, with $\eta(0) = \gamma(1)$ and $\eta'(0) = \delta(1)$. Since $\gamma \star \eta$ is path-homotopic to $\delta \star \eta'$, both of them have the same end point, and hence $\alpha(1) = \eta(1) = \eta'(1) \in U \cap V$. Then the claim in (3.4.10) follows from the Observation 2. Since $\mathscr{U}$ is a basis for the topology on X, and $\alpha(1) \in U \cap V$, there is an object $W \in \mathscr{U}$ such that $\alpha(1) \in W$ and $W \subseteq U \cap V$. Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, the argument given in Observation 2 shows that

$$W_{[\alpha]} \subseteq U_{[\alpha]} \cap V_{[\alpha]} = U_{[\gamma]} \cap V_{[\delta]},$$

where the equality of sets on the right side is by (3.4.10). Clearly $[\alpha] \in W_{[\alpha]}$. Therefore, $\mathscr{B}$ is a basis for a topology on $\widetilde{X}$. Give $\widetilde{X}$ the topology generated by this basis $\mathscr{B}$.

Now it remains to show that $p : \widetilde{X} \to X$ in (3.4.7) is a covering map and that $\widetilde{X}$ is simply connected. We first show that, for each $U_{[\gamma]} \in \mathscr{B}$, the restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U$$

is a homeomorphism. We already have shown that $p\big|_{U_{[\gamma]}}$ is bijective. Note that, for any $V'_{[\delta]} \in \mathscr{B}$ with $V'_{[\delta]} \subseteq U_{[\gamma]}$ we have $p(V'_{[\delta]}) = V' \subseteq U$. Since both $\mathscr{U}$ and $\mathscr{B}$ are basis for the topologies of X and $\widetilde{X}$, respectively, this shows that the restriction map $p\big|_{U_{[\gamma]}}$ is open. To show that $p\big|_{U_{[\gamma]}}$ is continuous, it suffices to show that for any $V \in \mathscr{U}$ with $V \subseteq U$, we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma]}$. Indeed, for any $[\alpha] \in p^{-1}(V) \cap U_{[\gamma]}$, we have $\alpha(1) \in V \cap U$, and so $V_{[\alpha]} \subseteq U_{[\alpha]} = U_{[\gamma]}$ by

Observation 2. Since $p(V_{[\alpha]}) = V$, it follows that $p^{-1}(V) \cap U_{[\gamma]} = V_{[\alpha]}$.

Since $\mathscr{B}$ is a basis for the topologies on $\widetilde{X}$, it follows that $p^{-1}(V)$ is open in $\widetilde{X}$, for all $V \in \mathscr{U}$. Since $\mathscr{U}$ is a basis for the topology on X, it follows that $p : \widetilde{X} \to X$ is continuous. Given a point $x \in X$, choose an object $U \in \mathscr{U}$ with $x \in U$. We claim that the collection

$$\mathscr{C}_U := \{U_{[\gamma]} : \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

is a partition of $p^{-1}(U)$. Since $p^{-1}(U) = \bigcup_{U_{[\gamma]} \in \mathscr{C}_U} U_{[\gamma]}$, it suffices to show that objects of the collection $\mathscr{C}_U$ are either disjoint or identical. If $[\alpha] \in U_{[\gamma]} \cap U_{[\delta]}$, then α is a path in X with $\alpha(0) = x_0$ and $\alpha(1) \in U$, and hence by Observation 2 we have $U_{[\gamma]} = U_{[\alpha]} = U_{[\delta]}$. Since the restriction of p on each of $U_{[\gamma]}$ is a homeomorphism, $p : \widetilde{X} \to X$ is a covering map.

It remains to show that $\widetilde{X}$ is simply connected. Given a point $[\gamma] \in \widetilde{X}$ and $t \in I$, consider the map $\gamma_t : I \to X$ defined by

$$\gamma_t(s) := \begin{cases} \gamma(s), & \text{if} \quad 0 \leq s \leq t, \text{ and} \\ \gamma(t), & \text{if} \quad t \leq s \leq 1. \end{cases} \tag{3.4.11}$$

Note that, each γ_t is a path in X starting at x_0, and hence its path-homotopy class is an element of $\widetilde{X}$. Then the map $\phi_{[\gamma]} : I \to \widetilde{X}$ defined by

$$\phi_{[\gamma]}(t) = [\gamma_t], \;\; \forall\, t \in I,$$

is a path (why it is continuous?) in $\widetilde{X}$ starting at $\widetilde{x_0} = [c_{x_0}] \in \widetilde{X}$ and ending at $[\gamma] \in \widetilde{X}$. Therefore, $\widetilde{X}$ is path-connected. Since $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a covering map, the homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ induced by the map p is injective by Corollary 3.3.19. Therefore, to show $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial it suffices to show that $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big)$ is the trivial subgroup of $\pi_1(X, x_0)$. By Corollary 3.3.19 elements of $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big) \subseteq \pi_1(X, x_0)$ are given by loops γ in X based at x_0 whose lift to the cover $p : \widetilde{X} \to X$ starting at $\widetilde{x_0}$ is a loop in $\widetilde{X}$ based at $\widetilde{x_0}$. Since $\phi_{[\gamma]}$ is a path in $\widetilde{X}$ starting at $\widetilde{x_0}$ and $p \circ \phi_{[\gamma]} = \gamma$, we must have $[\gamma] = \phi_{[\gamma]}(1) = \widetilde{x_0} = [c_{x_0}]$. In other words, γ is path-homotopic to the constant loop c_{x_0} in X. This completes the proof. $\square$

We now go towards establishing Galois correspondence for covering spaces. Whenever we talk about simply connected covering space of X, we assume that X is semi-locally simply connected in addition to be it path-connected and locally path-connected.

3.4.3 Group action and covering map

Before proceeding further, let's recall some standard terminologies related to group action. Let G be a group, and let $\sigma : G \times X \to X$ be a left G-action on X. For notational simplicity, we denote by $g \cdot x$ the element $\sigma(g, x) \in X$, for all $(g, x) \in G \times X$. Given $x \in X$, the subset

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\} \subseteq G$$

is a subgroup of G, known as the *stabilizer of* x or the *isotropy subgroup* for x. The G-action σ is said to be *free* if $\mathrm{Stab}_G(x) = \{e\}$, for all $x \in X$. This means that, for each $x \in X$, given $g_1, g_2 \in G$, we have $g_1 \cdot x = g_2 \cdot x$ if and only if $g_1 = g_2$. Note that the G-action σ on X defines an equivalence relation on X; for $x \in X$, its equivalence class is the subset

$$\mathrm{Orb}_G(x) := \{g \cdot x : g \in G\} \subseteq X,$$

called the *G-orbit* of x in X. The G-action σ is said to be *transitive* if there is exactly one G-orbit in X. In other words, given any two points $x_1, x_2 \in X$, there exists $g \in G$ such that $x_2 = g \cdot x_1$.

Definition 3.4.12. Let G be a group. A G-action $\sigma : G \times X \to X$ on X is said to be *even* (or, *properly discontinuous* according to old texts) if the G-action map σ is continuous, and each point $x_0 \in X$ has an open neighbourhood $V \subseteq X$ such that $(g \cdot V) \cap V = \emptyset$, for all $g \neq e$ in G, where $g \cdot V := \{g \cdot x : x \in V\} \subseteq X$.

Remark on old notation: Most of the old texts uses the term *properly discontinuous G-action* to mean an even G-action. This terminology is awkward because the G-action on X itself is a continuous map.

Proposition 3.4.13. *If a group G is acting evenly on a path-connected and locally path-connected topological space Y, then the associated quotient map $q : Y \to Y/G$ is a covering map.*

Proof. Clearly the quotient map $q : Y \to Y/G$ is continuous. Note that, for any subset $V \subseteq Y$ we have

$$q^{-1}(q(V)) = \bigcup_{g \in G} g \cdot V, \tag{3.4.14}$$

where $g \cdot V = \{g \cdot v : v \in V\} \subseteq X$, for all $g \in G$. Since the left translation map $L_g : Y \to Y$ given by

$$L_g(y) = g \cdot y := \sigma(g, y), \ \forall\, y \in Y$$

is a homeomorphism, V is open in Y if and only if $g \cdot V = L_g(V)$ is open in Y, for all $g \in G$. Since q is a quotient map, it follows that $q(V)$ is open in Y/G if V is open in Y. Therefore, q is an open map.

To see $q : Y \to Y/G$ is a covering map, let's fix a point $v \in Y/G$, and a point $y \in q^{-1}(v)$. Since the G-action on Y is even, y has an open neighbourhood $U_y \subseteq Y$ such that $(g \cdot U_y) \cap U_y = \emptyset$, for all $g \neq e$ in G. Take $V_y := q(U_y)$. Then it follows that

$$q^{-1}(V_y) = \bigsqcup_{g \in G} g \cdot U_y.$$

It remains to show that the restriction map

$$q\big|_{g \cdot U_y} : g \cdot U_y \to V_y = q(U_y)$$

is a homeomorphism, for all $g \in G$. Since q is continuous and open, it suffices to show that $q\big|_{g \cdot U_y}$ is bijective, for all $g \in G$.

If $q\big|_{g\cdot U_y}$ were not injective, then there exist $y_1, y_2 \in g \cdot U_y$ with $y_1 \neq y_2$ such that $q(y_1) = q(y_2)$. Then there exists $h \in G$ such that $y_2 = h \cdot y_1$. Then $y_2 = h \cdot y_1 \in U_y \cap (h \cdot U_1)$ implies $h = e$ because the G-action on Y is even. This contradicts our assumption that $y_1 \neq y_2 = h \cdot y_1$. Therefore, $q\big|_{g\cdot U_y}$ must be injective. To show $q\big|_{g\cdot U_y}$ is surjective, note that a typical element of $V_y = q(U_y)$ is of the form $q(y_1)$, for some $y_1 \in U_y$. Since $q(y_1) = \mathrm{Orb}_G(y_1) = \{a \cdot y_1 : a \in G\}$, we see that $g \cdot y_1 \in g \cdot U_y$ satisfies $q\big|_{g\cdot U_y}(g \cdot y_1) = q(y_1)$. Therefore, $q\big|_{g\cdot U_y}$ is surjective. □

Proposition 3.4.13 allow us to construct a lot of examples of covering maps.

3.4.4 Group of Deck transformations

Let $f : Y \to X$ be a covering map. An *automorphism of* $f : Y \to X$ is a homeomorphisms $\phi : Y \to Y$ satisfying $f \circ \phi = f$. The set

$$\mathrm{Aut}(Y/X) := \{\phi : Y \to Y \mid \phi \text{ is a homeomorphism satisfying } f \circ \phi = f\}$$

of all automorphisms of $f : Y \to X$ forms a group with respect to the binary operation on $\mathrm{Aut}(Y/X)$ given by composition of homeomorphisms. The group $\mathrm{Aut}(Y/X)$ is also known as the group of *Deck transformations* or *covering transformations* of $f : Y \to X$. Note that, $\mathrm{Aut}(Y/X)$ acts on Y from the left by automorphisms:

$$a : \mathrm{Aut}(Y/X) \times Y \to Y, \quad (\phi, y) \mapsto \phi(y). \tag{3.4.15}$$

We shall show in Proposition 3.4.19 that if we equip $\mathrm{Aut}(Y/X)$ with discrete topology, then the action map in (3.4.15) become continuous.

Proposition 3.4.16. *Fix a point $x_0 \in X$, and a path-connected covering space $f : Y \to X$ of X. Then the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to give a free $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If Y is simply connected, then the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive.*

Proof. Let $y_0 \in f^{-1}(x_0)$ be given. Since $\phi \in \mathrm{Aut}(Y/X)$ satisfies $f \circ \phi = f$, we have $f(\phi(y_0)) = f(y_0) = x_0$, and hence $\phi(y_0) \in f^{-1}(x_0)$. Therefore, the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to an $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If $\phi(y_0) = y_0$, for some $\phi \in \mathrm{Aut}(Y/X)$, then by uniqueness of lifting of maps (see Theorem 3.3.26 or Lemma 3.3.13) we must have $\phi = \mathrm{Id}_Y$. Therefore, the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is free.

Now assume that Y is simply connected. To show that $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive, choose two points $y_0, y_1 \in f^{-1}(x_0)$. Since X is locally path-connected and $f : Y \to X$ is a covering map, Y is locally path-connected. Since by assumption Y is path-connected and locally path-connected (since X is so) with $\pi_1(Y)$ trivial, by general lifting criterion (Theorem 3.3.26) there is a unique continuous map $\phi : (Y, y_0) \to (Y, y_1)$ such that $f \circ \phi = f$. Similarly, there is a unique continuous map $\psi : (Y, y_1) \to (Y, y_0)$ such that $f \circ \psi = f$. Then by uniqueness of lifting (see Theorem 3.3.26), we must have $\phi \circ \psi = \mathrm{Id}_{(Y,y_1)}$ and $\psi \circ \phi = \mathrm{Id}_{(Y,y_0)}$. Therefore, both ϕ and ψ are homeomorphisms, and that $\phi(y_0) = y_1$. Thus, the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive. □

Let $f : Y \to X$ be a covering map. Fix a point $x_0 \in X$. Since X is locally path-connected, there is a path-connected open neighbourhood $U \subset X$ of x_0 which is evenly covered by f. Then we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y, \tag{3.4.17}$$

where $V_y \subset Y$ is the path-connected open neighbourhood of $y \in f^{-1}(y_0)$ such that $f\big|_{V_y} : V_y \to U$ is a homeomorphism. Note that, $\{V_y : y \in f^{-1}(x_0)\}$ is precisely the set of all path-components of $f^{-1}(U)$.

Proposition 3.4.18. *With the above notations,* $\mathrm{Aut}(Y/X)$ *acts freely on the set of all path-components* $\{V_y : y \in f^{-1}(x_0)\}$ *of* $f^{-1}(U)$*. Moreover, this action is transitive when Y is simply connected.*

Proof. Since $f : Y \to X$ is a covering map, the restricted map

$$f_U := f\big|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

is a covering map. Since for any $\phi \in \mathrm{Aut}(Y/X)$ we have $f \circ \phi = f$, image of the restriction map $\phi\big|_{f^{-1}(U)} : f^{-1}(U) \to Y$ lands inside $f^{-1}(U)$, and hence gives rise to an automorphism of the covering space $f_U : f^{-1}(U) \to U$, i.e., $\phi\big|_{f^{-1}(U)} \in \mathrm{Aut}(f^{-1}(U)/U)$. Clearly $\phi \in \mathrm{Aut}(Y/X)$ takes path-components of $f^{-1}(U)$ to path-components of $f^{-1}(U)$. In particular, for each $y \in f^{-1}(x_0)$, the induced map

$$\phi : V_y \to V_{\phi(y)}$$

is a homeomorphism. Since $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is free by Proposition 3.4.16, if for some $y \in f^{-1}(x_0)$, the automorphism $\phi \in \mathrm{Aut}(Y/X)$ takes V_y to itself, then we must have $\Phi = \mathrm{Id}_Y$.

Now assume that Y is simply connected. Since the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive by Proposition 3.4.16, and the path-components of $f^{-1}(U)$ are uniquely determined by the conditions that $V_y \cap f^{-1}(x_0) = \{y\}$ and $V_{y_1} \cap V_{y_2} = \emptyset$ for $y_1 \neq y_2$ in $f^{-1}(x_0)$, given any two path-components V_{y_1}, V_{y_2} of $f^{-1}(U)$, there exists $\phi \in \mathrm{Aut}(Y/X)$ such that $\phi(y_1) = y_2$, and hence $\phi(V_{y_1}) = V_{y_2}$. Thus, the $\mathrm{Aut}(Y/X)$-action on the set of all path-components of $f^{-1}(U)$ is transitive. $\square$

Proposition 3.4.19. *Let $f : Y \to X$ be a path-connected covering space of X. Equip* $\mathrm{Aut}(Y/X)$ *with discrete topology. Then there is a continuous map (action map)*

$$a : \mathrm{Aut}(Y/X) \times Y \to Y, \tag{3.4.20}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}(Y/X) \times Y \arrow[r, "a"] \arrow[d, "pr_2"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X, \end{tikzcd} \tag{3.4.21}$$

where pr_2 : $\mathrm{Aut}(Y/X) \times Y \to Y$ *is the projection map onto the second factor.*

Proof. Clearly $a : \mathrm{Aut}(Y/X) \times Y \to X$ is defined by

$$a(\phi, y) = \phi(y), \ \forall\, (\phi, y) \in \mathrm{Aut}(Y/X) \times Y,$$

makes the above diagram commutative. We only need to show that the action map a is continuous.

Let

$$\mathcal{B} := \{V \subseteq Y \ : V \text{ is path-connected, open and } f(V) \text{ is evenly covered by } f\}.$$

Since Y is path-connected and locally path-connected covering space for X, it is easy to check that $\mathcal{B}$ is a basis for the topology on Y. Therefore, to show the action map a is continuous, it is enough to show that $a^{-1}(V)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $V \in \mathcal{B}$. Fix $V \in \mathcal{B}$. Since V is path-connected and $f : Y \to X$ is a covering map, $U := f(V)$ is path-connected and open in X. Fix a point $x_0 \in U$. Since $U = f(V)$ is evenly covered by f, we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y,$$

where $V_y \subseteq Y$ is an open neighbourhood of $y \in f^{-1}(x_0)$ such that $f\big|_{V_y} : V_y \to U$ is a homeomorphism. Since V is path-connected and $p(V) = U$, we have $V \subseteq V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Since $f\big|_{V_{y_0}} : V_{y_0} \to U$ is a homeomorphism, we must have $V = V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Therefore, it is enough to show that $a^{-1}(V_y)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $y \in f^{-1}(x_0)$.

Let $(\phi, y) \in a^{-1}(V_{y_0}) = \{(\psi, y') \in \mathrm{Aut}(Y/X) \times Y : \psi(y') \in V_{y_0}\}$ be arbitrary. Then $\phi(y) \in V_{y_0}$. Since ϕ is an automorphism of Y, there is a unique $y_1 \in Y$ such that $\phi(y_1) = y_0$. Then $\phi : V_{y_1} \to V_{y_0}$ is a homeomorphism. Since $\phi(y) \in V_{y_0}$, we must have $y \in V_{y_1}$. Then $\{\phi\} \times V_{y_1}$ is an open neighbourhood of (ϕ, y) in $\mathrm{Aut}(Y/X) \times Y$ such that $a\big(\{\phi\} \times V_{y_1}\big) \subseteq V_{y_0}$. Therefore, $a^{-1}(V_{y_0})$ is open in $\mathrm{Aut}(Y/X) \times Y$. This completes the proof. □

Corollary 3.4.22. *If $f : Y \to X$ is a connected cover of X, the action of* $\mathrm{Aut}(Y/X)$ *on Y is even (see Definition 3.4.12).*

Proof. Follows from Proposition 3.4.19 and 3.4.18. □

Proposition 3.4.23. *If a group G acts evenly on a connected topological space Y, then the automorphism group* $\mathrm{Aut}(Y/X)$ *of the covering map $q : Y \to X := Y/G$ is naturally isomorphic to G.*

Proof. Let $\sigma : G \times Y \to Y$ be the left G-action which is even. Since σ is continuous, for each $g \in G$, the induced map

$$\sigma_g : Y \to Y, \ \ y \mapsto g \cdot y := \sigma(g, y)$$

is a homeomorphism of Y onto itself. Since the quotient map $q : Y \to X := Y/G$ sends a point $y \in Y$ to its G-orbit $\mathrm{Orb}_G(y) \in Y/G$, it follows that $q(\sigma_g(y)) = q(y)$, for all $g \in G$. Therefore,

$\sigma_g \in \mathrm{Aut}(Y/X)$. Thus we have a natural map

$$\Phi : G \longrightarrow \mathrm{Aut}(Y/X), \ \ g \longmapsto \sigma_g. \tag{3.4.24}$$

Note that, for any $g, h \in G$ we have

$$\sigma_{gh}(y) = (gh) \cdot y = g \cdot (h \cdot y) = \sigma_g(\sigma_h(y)), \ \forall\, y \in Y.$$

Therefore, Φ is a group homomorphism. Since the G-action on Y is even (see Definition 3.4.12), it follows that $\mathrm{Ker}(\Phi)$ is trivial, and hence Φ is injective. Let $\varphi \in \mathrm{Aut}(Y/X)$ be arbitrary. Fix a pint $y \in Y$, and let $x := q(y) \in X$. Since $\varphi(y) \in q^{-1}(x) = \mathrm{Orb}_G(y)$, we have $\varphi(y) = g \cdot y = \sigma_g(y)$, for some $g \in G$. Since both $\varphi, \sigma_g \in \mathrm{Aut}(Y/X)$ and they agree at a point of Y and Y is connected, by uniqueness of lifting (see Lemma 3.3.13) we have $\varphi = \sigma_g$. Therefore, Φ is surjective, and hence is an isomorphism. □

3.4.5 Galois covers

Let $f : Y \to X$ be a path-connected covering space of X. Then the natural left $\mathrm{Aut}(Y/X)$-action on Y gives rise to an equivalence relation on Y, where the equivalence classes are $\mathrm{Aut}(Y/X)$-orbits of points of Y. Given $y \in Y$, its $\mathrm{Aut}(Y/X)$-orbit is the subset

$$\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y) = \{\phi(y) : \phi \in \mathrm{Aut}(Y/X)\} \subseteq Y.$$

Fix $y_0 \in Y$, and let $x_0 = f(y_0)$. Clearly, $\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y_0) \subseteq f^{-1}(x_0)$, and equality holds if and only if the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive. By the universal property of quotient space, there is a unique continuous map

$$\widetilde{f} : Y/\,\mathrm{Aut}(Y/X) \to X \tag{3.4.25}$$

such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \\ & Y/\,\mathrm{Aut}(Y/X) \arrow[ur, dashed, "\exists!\,\widetilde{f}"'] & \end{tikzcd} \tag{3.4.26}$$

where $q : Y \to Y/\,\mathrm{Aut}(Y/X)$ is the quotient map.

Definition 3.4.27 (Galois cover)**.** A covering map $f : Y \to X$ is said to be a *Galois cover* of X if Y is path-connected and the continuous map $\widetilde{f} : Y/\,\mathrm{Aut}(Y/X) \to X$ in (3.4.25), induced by f, is a homeomorphism (see the diagram (3.4.26)).

Proposition 3.4.28. *A connected covering map $p : Y \to X$ is Galois if and only if* $\mathrm{Aut}(Y/X)$ *acts transitively on each fiber of the covering map p.*

Proof. Suppose that $p : Y \to X$ is Galois cover. Consider the commutative diagram.

$$\begin{tikzcd} Y \arrow[rr, "p"] \arrow[dr, "q"'] & & X \\ & Y/\operatorname{Aut}(Y/X) \arrow[ur, "\widetilde{p}"'] & \end{tikzcd}$$

Since the induced map $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is a homeomorphism (by definition), for each $x \in X$, the fiber $p^{-1}(x)$ coincides with the $\operatorname{Aut}(Y/X)$-orbit of a point of the fiber $p^{-1}(x)$. In other words, the $\operatorname{Aut}(Y/X)$-action on each of the fibers of p is transitive.

Conversely, if the $\operatorname{Aut}(Y/X)$-action on each of the fibers of p is transitive, then the induced continuous map $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is bijective. Therefore, to show that $p : Y \to X$ a Galois cover, it suffices to show that $\widetilde{p}$ is an open map. Let $U \subseteq Y/\operatorname{Aut}(Y/X)$ be an open subset. Since the quotient map $q : Y \to Y/\operatorname{Aut}(Y/X)$ is continuous, $q^{-1}(U)$ is open in Y. Since the covering map $p : Y \to X$ is an open map, $p\big(q^{-1}(U)\big)$ is open in X. Since q is surjective, we have $q(q^{-1}(U)) = U$. Since $p = \widetilde{p} \circ q$, we have

$$\widetilde{p}(U) = \widetilde{p}\big(q(q^{-1}(U))\big) = p(q^{-1}(U)).$$

Therefore, $\widetilde{p}(U)$ is open in X. This completes the proof. $\square$

If Y is simply connected, as remarked above, the $\operatorname{Aut}(Y/X)$-orbit of y_0 is precisely the fiber $f^{-1}(x_0)$, for all $y_0 \in f^{-1}(x_0)$. Therefore, in that case, the map $\widetilde{f}$ is bijective. This leads to the following.

Corollary 3.4.29. *A simply-connected covering map $p : \widetilde{X} \to X$ is Galois cover.*

Proof. $\widetilde{X}$ being simply connected, $\operatorname{Aut}(\widetilde{X}/X)$ acts transitively on each fiber of p by Proposition 3.4.18. Therefore, the result follows from Proposition 3.4.28. $\square$

Remark 3.4.30. If $p : Y \to X$ is a covering map with Y connected, then to show $p : Y \to X$ is a Galois cover it suffices to show that $\operatorname{Aut}(Y/X)$ acts transitively on one fibre. Indeed, since in this case $Y/\operatorname{Aut}(Y/X)$ is a connected cover of X where one of the fibres is singleton, it follows that $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is a homeomorphism.

3.4.6 Galois correspondence for covering spaces

Theorem 3.4.31. *Let $p : Y \to X$ be a Galois cover. For each subgroup H of the Galois group $G := \operatorname{Aut}(Y/X)$, the projection map p induces a natural continuous map $\widetilde{p}_H : Y/H \to X$ which is a covering map. Conversely, if $f : Z \to X$ is a connected cover of X fitting into a commutative diagram*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "p"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

then $\phi : Y \to Z$ is a Galois cover and Z is homeomorphic to Y/H. The maps $H \mapsto Y/H$ and $Z \mapsto \mathrm{Aut}(Y/Z)$ induces a natural one-to-one correspondence between the collection of subgroups of G and the intermediate covers of $p : Y \to X$ as above. Moreover, the cover $f : Z := Y/H \to X$ is Galois if and only if H is a normal subgroup of G; and in this case we have $\mathrm{Aut}(Z/X) \cong G/H$.

[Need to be added!] □

3.4.7 Monodromy action

[[Need to be added]]

3.5 Homology

3.5.1 Simplicial Complex

3.5.2 Homology group

3.5.3 Homology group for surfaces

3.5.4 Applications

3.6 Cohomology

Chapter 4

Appendix

4.1 Category Theory

Definition 4.1.1. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from X to Y in $\mathscr{C}$*; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

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