ON HIGGS BUNDLES AND HIGGS FUNDAMENTAL GROUP SCHEMES

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ABSTRACT. Let X be a connected reduced proper scheme defined over an algebraically closed field k. We discuss a natural extension of Nori's theory of principal G-bundle as a functor to the case of principal G-Higgs bundles, for G an affine k-group scheme. Then we use this to show invariance of base points for Higgs fundamental group schemes of X.

1. INTRODUCTION

Let X be a connected reduced proper scheme defined over an algebraically closed field k. Let G be an affine group scheme over k, and denote by $\mathcal{R}ep_k(G)$ the category of all k-linear representations of G. In [Nor76], Nori established a one-to-one correspondence between the principal G-bundles on X and the functors $\mathcal{R}ep_k(G) \to \mathfrak{QCoh}(X)$ satisfying certain axioms. In this note, we show that this correspondence can be generalized to the case of principal G-Higgs bundles on X.

Let $\mathcal{H}iggs_G(X)$ the category of all principal G-Higgs bundles on X. Let $\mathcal{H}iggs(X)$ be the category of all quasi-coherent Higgs sheaves on X, and let

$$\mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$$

be the full subcategory of the functor category $\mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$ whose objects satisfies axioms (HF1) – (HF6) as stated in Proposition 2.5.2; see also (2.5.4). Then we have the following.

Theorem 1.0.1. *There is an equivalence of categories*

$$\Phi : \mathcal{H}iggs_G(X) \longrightarrow \mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)).$$

Let $\mathcal{H}iggs_0^{\mathrm{nf}}(X)$ be the full subcategory of $\mathcal{H}iggs(X)$, whose objects are Higgs numerically flat (in short, *H-nflat*) Higgs bundles on X (see Definition 3.2.7). This is a k-linear symmetric monoidal category, and fixing a closed point $x \in X(k)$, we get a k-linear exact faithful tensor functor $\mathscr{F}_x^H : \mathcal{H}iggs_0^{\mathrm{nf}}(X) \to \mathcal{V}ect(k)$ defined by

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sending a H-nflat Higgs bundle (E,θ) to its fiber $E_x \in \mathcal{V}ect(k)$ at x. This gives us a neutral Tannakian category $(\mathcal{H}iggs_0^{\mathrm{nf}}(X), \otimes, \mathcal{O}_X, \mathscr{F}_x^H)$, and the affine k-group scheme $\pi_1^H(X,x)$ Tannakian dual to this is called the *Higgs fundamental group scheme* of X with base point at x. Then using Theorem 1.0.1, we prove the following.

Theorem 1.0.2. *Let X be a connected reduced proper k-scheme. Fix a closed point $x \in X(k)$, and let $G := \pi_1^H(X,x)$. Then there is a principal G-Higgs bundle $\mathfrak{P} := (\mathcal{P}, \theta_{\mathcal{P}})$ on X such that given any object $\mathfrak{E} := (E,\theta) \in \mathcal{H}iggs_0^{\mathrm{nf}}(X)$, there is an object $\rho : G \to \mathrm{GL}(V)$ in $\mathcal{R}ep_k(G)$ such that $\mathfrak{E} = \mathfrak{P} \times^{\rho} V$.*

As an immediate corollary to this, we obtain the following.

Corollary 1.0.3. *Let X be a connected reduced proper k-scheme. For any two points $x_1, x_2 \in X(k)$, the affine k-group schemes $\pi_1^H(X,x_1)$ and $\pi_1^H(X,x_2)$ are isomorphic.*

2. Higgs Bundles

2.1. **Notations.** A k-scheme X is said to be connected if $H^0(X,\mathcal{O}_X) \cong k$. For a k-scheme X, denote by $\mathfrak{QCoh}(X)$ the category of coherent sheaves on X, and let $\mathfrak{Coh}(X)$ (resp., $\mathcal{V}ect(X)$) be the full subcategory of $\mathfrak{QCoh}(X)$, whose objects are coherent sheaves (resp., locally free coherent sheaves) on X. There are natural fully faithful embeddings $\mathcal{V}ect(X) \subset \mathfrak{Coh}(X) \subset \mathfrak{QCoh}(X)$. The objects of $\mathcal{V}ect(X)$ are also referred to as vector bundles on X. When $X = \mathrm{Spec}(k)$, the category $\mathcal{V}ect(\mathrm{Spec}(k))$ coincides with the category of all finite dimensional k-vector spaces $\mathcal{V}ect(k)$, and hence we simply denote it by $\mathcal{V}ect(k)$. For a locally free coherent sheaf (vector bundle) E on X and a point $x \in X$, on contrary to the usual notation of stalk, we denote by E_x the fiber of E at x; whereas the notation $\mathcal{O}_{X,x}$ is preserved to denote the stalk at x of the structure sheaf $\mathcal{O}_X$. For any group scheme G over k, denote by $\mathrm{Lie}(G)$ the *Lie algebra* of G.

2.2. **The category of Higgs sheaves.** Let X ba a connected reduced proper k-scheme.

Definition 2.2.1. A *Higgs sheaf* on X is a pair (E,θ), where E is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X and $\theta : E \longrightarrow E \otimes \Omega_X^1$ is an $\mathcal{O}_X$-module homomorphism such that $\theta \wedge \theta = 0$ in $H^0(X, \mathcal{E}nd(E) \otimes \Omega_X^2)$. When E is coherent we call (E,θ) a *coherent Higgs sheaf* on X. Similarly, for E a locally free coherent sheaf on X, we call (E,θ) a *Higgs bundle* on X.

Given two Higgs sheaves $\mathfrak{E} = (E,\theta)$ and $\mathfrak{E}' = (E',\theta')$ on X, a morphism from $\mathfrak{E}$ to $\mathfrak{E}'$ is given by an $\mathcal{O}_X$-module homomorphism $f : E \longrightarrow E'$ such that the following diagram commutes

$$
\begin{tikzcd}
E \arrow[r, "\theta"] \arrow[d, "f"] & E \otimes \Omega^1_X \arrow[d, "f\otimes \mathrm{Id}_{\Omega^1_X}"] \\
E' \arrow[r, "\theta'"] & E' \otimes \Omega^1_X .
\end{tikzcd}
\tag{2.2.2}
$$

Moreover, the direct sum and tensor product of two Higgs sheaves $\mathfrak{E}$ and $\mathfrak{E}'$ are again Higgs sheaves, given by

$$
\begin{aligned}
\mathfrak{E} \oplus \mathfrak{E}' &:= (E \oplus E', \theta \oplus \theta'), \quad \text{and} \\
\mathfrak{E} \otimes \mathfrak{E}' &:= (E \otimes E', \theta \otimes \mathrm{Id}_{E'} + \mathrm{Id}_E \otimes \theta').
\end{aligned}
$$

Let $\mathcal{H}iggs(X)$ be the category whose objects are Higgs sheaves on X and morphisms are defined by commutative diagrams as in (2.2.2). Then $\mathcal{H}iggs(X)$ is an abelian category. In fact, $\mathfrak{QCoh}(X)$ admits a natural fully faithful embedding inside $\mathcal{H}iggs(X)$ by considering zero Higgs field. We denote by $\mathcal{H}iggs_{\mathfrak{Coh}}(X)$ the full subcategory of $\mathcal{H}iggs(X)$ whose objects are coherent Higgs sheaves on X. Denote by $\mathcal{H}iggs_0(X)$ the full subcategory of $\mathcal{H}iggs(X)$ whose objects are locally free coherent Higgs sheaves on X. Thus, we have fully faithful embeddings

$$
\begin{tikzcd}
\mathcal{V}ect(X) \arrow[r, hook] \arrow[d, hook] & \mathfrak{Coh}(X) \arrow[r, hook] \arrow[d, hook] & \mathfrak{QCoh}(X) \arrow[d, hook] \\
\mathcal{H}iggs_0(X) \arrow[r, hook] & \mathcal{H}iggs_{\mathfrak{Coh}}(X) \arrow[r, hook] & \mathcal{H}iggs(X) .
\end{tikzcd}
$$

Proposition 2.2.3. *Direct limit of a direct system of coherent Higgs sheaves exists in the category of quasi-coherent Higgs sheaves.*

Proof. Obvious. □

2.3. **Principal G-Higgs bundles.** Let G be a k-group scheme.

Definition 2.3.1. A *principal G-bundle* on X is a k-variety P together with a G-action $\sigma : P \times G \to P$ on P, and a G-invariant morphism of k-schemes $\pi : P \to X$ such that the morphism $(\mathrm{pr}_1, \sigma) : P \times_k G \longrightarrow P \times_X P$ induced by σ and the projection map $\mathrm{pr}_1 : P \times G \to P$, is an isomorphism.

Let P be a principal G-bundle on X. Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional k-linear representation of G. Then G-acts on $P \times V$ by $(z, v) \cdot g := (z \cdot g, \rho(g)^{-1}(v))$, for all $z \in P$, $v \in V$ and $g \in G$. The associated quotient $P \times^\rho V := (P \times V)/G$ is a vector bundle of rank $r = \dim_k(V)$ on X, denoted by $\rho_* P$. Using Grothendieck's theory of flat descent [Gro71], the vector bundle $\rho_* P$ can be constructed as a locally free coherent sheaf on X by taking G-invariants of $\mathcal{O}_P \otimes_k V$. The vector bundle $\mathrm{ad}(P) := P \times^{\mathrm{ad}} \mathfrak{g}$ associated to the adjoint representation

$$\mathrm{ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \tag{2.3.2}$$

of G on its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ is called the *adjoint vector bundle* of P. Note that, $\mathrm{ad}(P)$ is a Lie algebra bundle on X.

Definition 2.3.3. A *principal G-Higgs bundle* on X is a pair $\mathfrak{P} := (P, \theta)$, where P is a principal G-bundle on X and $\theta \in H^0(X, \mathrm{ad}(P) \otimes \Omega^1_X)$ such that $\theta \wedge \theta = 0$ in $H^0(X, \mathrm{ad}(P) \otimes \Omega^2_X)$.

Let P and P' be two principal G-bundles on X. Then any homomorphism of of principal G-bundles $\varphi : P \to P$ induces a homomorphism of their adjoint vector bundles

$$\mathrm{ad}(\varphi) : \mathrm{ad}(P) \to \mathrm{ad}(P') \tag{2.3.4}$$

Tensoring with Ω^1_X and taking global section functor, we have a k-linear homomorphism

$$\widetilde{\varphi} : H^0(X, \mathrm{ad}(P) \otimes \Omega^1_X) \longrightarrow H^0(X, \mathrm{ad}(P') \otimes \Omega^1_X). \tag{2.3.5}$$

Let $\mathfrak{P} = (P, \theta)$ and $\mathfrak{P}' = (P', \theta')$ be two principal G-Higgs bundles on X.

Definition 2.3.6. A morphism of principal G-Higgs bundles $\mathfrak{P} \to \mathfrak{P}'$ is given by a morphism of principal G-bundles $\varphi : P \to P$ such that the induced homomorphism $\widetilde{\varphi}$ in (2.3.5) sends θ to θ'.

2.4. **Principal G-Higgs bundle as a functor.** Let $\mathcal{H}iggs(X)$ be the category of coherent Higgs sheaves on X (see §2.2). Let G be an affine k-group scheme. Let $\mathcal{R}ep^{\mathrm{fd}}_k(G)$ be the category of finite dimensional k-linear representations of G; its objects are pair (V, ρ), where V is a finite dimensional k-vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. A morphism $(V, \rho) \to (V', \rho')$ in $\mathcal{R}ep^{\mathrm{fd}}_k(G)$ is given by a G-equivariant homomorphism of k-vector spaces $V \to V'$. The category $\mathcal{R}ep^{\mathrm{fd}}_k(G)$ admits finite direct sum and tensor products.

Let $\mathfrak{P} = (P, \theta)$ be a principal G-Higgs bundle on X. Any finite dimensional k-linear representation $\rho : G \to \mathrm{GL}(V)$ give rise to a G-module homomorphism

$$d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) := \mathrm{Lie}(\mathrm{GL}(V)), \tag{2.4.1}$$

which in turn give rise to a homomorphism of vector bundles

$$(d\rho)_P : \mathrm{ad}(P) := P \times^{\mathrm{ad}} \mathfrak{g} \longrightarrow \mathcal{E}nd(\rho_* P), \tag{2.4.2}$$

where $\rho_*P := P\times^\rho V$ is the vector bundle on X associated to P and the representation (V,ρ). This gives a k-linear homomorphism

$$\widetilde{\rho}_P : H^0(X, \mathrm{ad}(P)\otimes\Omega^1_X) \longrightarrow H^0(X, \mathcal{E}nd(\rho_*P)\otimes\Omega^1_X). \tag{2.4.3}$$

Thus we obtain a Higgs bundle

$$\rho_*\mathfrak{P} := (\rho_*P, \rho_*\theta) \tag{2.4.4}$$

on X, where $\rho_*P := P\times^\rho V$ and $\rho_*\theta = \widetilde{\rho}_P(\theta) \in H^0(X, \mathcal{E}nd(\rho_*P)\otimes\Omega^1_X)$.

A morphism $\varphi : (V,\rho) \longrightarrow (V',\rho')$ in $\mathcal{R}ep_k^{\mathrm{fd}}(G)$ give rise to the following commutative diagram of (Lie algebras) G-module homomorphisms

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\rho"] \arrow[d, equal] & \mathfrak{gl}(V) \arrow[d, "\widehat{\varphi}"] \\ \mathfrak{g} \arrow[r, "d\rho'"] & \mathfrak{gl}(V'), \end{tikzcd} \tag{2.4.5}$$

which makes the following diagram of k-linear maps commutative

$$\begin{tikzcd} H^0(X, \mathrm{ad}(P)\otimes\Omega^1_X) \arrow[r, "\widetilde{\rho}_P"] \arrow[d, equal] & H^0(X, \mathcal{E}nd(\rho_*(P))\otimes\Omega^1_X) \arrow[d, "\widetilde{\varphi}"] \\ H^0(X, \mathrm{ad}(P)\otimes\Omega^1_X) \arrow[r, "\widetilde{\rho'}_P"] & H^0(X, \mathcal{E}nd(\rho'_*(P))\otimes\Omega^1_X). \end{tikzcd} \tag{2.4.6}$$

Thus we get a homomorphism of Higgs bundles

$$\varphi_P : \rho_*\mathfrak{P} \longrightarrow \rho'_*\mathfrak{P}\,; \tag{2.4.7}$$

(see (2.4.4)). The above construction is functorial, and hence give rise to a covariant functor

$$\Phi_{\mathfrak{P}} : \mathcal{R}ep_k^{\mathrm{fd}}(G) \longrightarrow \mathcal{H}iggs_0(X), \tag{2.4.8}$$

which sends an object $(V,\rho) \in \mathcal{R}ep_k^{\mathrm{fd}}(G)$ to the Higgs bundle $\rho_*\mathfrak{P} := (\rho_*P, \rho_*\theta)$ as defined in (2.4.4), and a morphism $\varphi : (V,\rho) \to (V',\rho')$ to φ_P as defined in (2.4.7).

Proposition 2.4.9. *The functor $\Phi_{\mathfrak{P}}$ defined in* (2.4.8) *preserve finite direct sums and tensor products.*

Proof. It is well-known that $(V,\rho) \mapsto \rho_*P = P\times^\rho V$ is a covariant additive tensor functor of tensor abelian categories $\mathcal{R}ep_k^{\mathrm{fd}}(G) \to \mathcal{V}ect(X)$. Therefore, it is enough to check what happens to the Higgs fields.

Let $(V_1,\rho_1),(V_2,\rho_2) \in \mathcal{R}ep_k^{\mathrm{fd}}(G)$. It follows from the commutative diagram of G-module homomorphisms

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\rho_1"] \arrow[d, "d\rho_2"] & \mathfrak{gl}(V_1) \arrow[d] \\ \mathfrak{gl}(V_2) \arrow[r] & \mathfrak{gl}(V_1 \oplus V_2) \end{tikzcd} \tag{2.4.10}$$

and the corresponding induced homomorphisms of vector bundles induced by P that $(\rho_1 \oplus \rho_2)_*\theta = (\rho_{1*}\theta) \oplus (\rho_{2*}\theta)$. Similarly, for the case of tensor product representation $\rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$, we have $(\rho_1 \otimes \rho_2)_*\theta = (\rho_{1*}\theta \otimes \mathrm{Id}) + (\mathrm{Id} \otimes \rho_{2*}\theta)$. Hence the result follows. $\square$

2.5. **Recovering G-Higgs bundle from the associated functor.** Let $\mathcal{H}iggs(G,X)$ be the category whose objects are principal G-Higgs bundles on X, and morphisms are morphisms of principal G-Higgs bundles (see Definition 2.3.6). Given any two categories $\mathscr{C}$ and $\mathscr{D}$, we denote by $\mathcal{F}un(\mathscr{C},\mathscr{D})$ the category whose objects are functors $\mathscr{C} \to \mathscr{D}$ and morphisms are natural transformations of those functors.

Following [Nor76], let

$$\mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k^{\mathrm{fd}}(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k^{\mathrm{fd}}(G), \mathcal{H}iggs(X)) \tag{2.5.1}$$

be the full subcategory of $\mathcal{F}un(\mathcal{R}ep_k^{\mathrm{fd}}(G), \mathcal{H}iggs(X))$ whose objects are functors

$$\mathscr{F} : \mathcal{R}ep_k^{\mathrm{fd}}(G) \longrightarrow \mathcal{H}iggs(X)$$

satisfying the following axioms (HF1) – (HF6):

(HF1) $\mathscr{F}$ is a faithful k-linear exact functor,
(HF2) $\mathscr{F}$ sends trivial G-module to $(\mathcal{O}_X, 0)$ in $\mathcal{H}iggs(X)$,
(HF3) $\mathscr{F} \circ \otimes = \otimes \circ (\mathscr{F} \times \mathscr{F})$,
(HF4) $\otimes$ in $\mathcal{R}ep_k^{\mathrm{fd}}(G)$ is associative and compatible with $\mathscr{F}$,
(HF5) $\otimes$ in $\mathcal{R}ep_k^{\mathrm{fd}}(G)$ is commutative and compatible with $\mathscr{F}$, and
(HF6) if $V \in \mathcal{R}ep_k^{\mathrm{fd}}(G)$ is of rank n, then $\mathscr{F}(V)$ is a Higgs bundle of rank n over X.

Let $\mathcal{R}ep_k(G)$ be the category of all (including infinite dimensional) k-linear representations of G. Note that, $\mathcal{R}ep_k^{\mathrm{fd}}(G)$ is a full subcategory of $\mathcal{R}ep_k(G)$, and [Nor76, Lemma 2.1] generalizes to the following.

Proposition 2.5.2. *Any functor $\mathscr{F} : \mathcal{R}ep_k^{\mathrm{fd}}(G) \longrightarrow \mathcal{H}iggs(X)$ satisfying axioms (HF1) – (HF6) extends uniquely to a functor $\widehat{\mathscr{F}} : \mathcal{R}ep_k(G) \longrightarrow \mathcal{H}iggs(X)$ such that*

(i) the axioms (HF1) – (HF5) holds for $\widehat{\mathscr{F}}$,
(ii) $\widehat{\mathscr{F}}$ restricts to $\mathscr{F}$ on $\mathcal{R}ep_k^{\mathrm{fd}}(G)$,

(iii) *the underlined $\mathcal{O}_X$-module of $\widehat{\mathscr{F}}(V)$ is flat, for all $V \in \mathcal{R}ep_k(G)$, and is faithfully flat if $V = 0$, and*
(iv) *$\widehat{\mathscr{F}}$ preserves direct limits.*

Proof. In view of Proposition 2.2.3, given any object $V \in \mathcal{R}ep_k(G)$, we define $\widehat{\mathscr{F}}(V) := \varinjlim \mathscr{F}(W)$, where W runs through the directed system of all finite dimensional G-invariant k-linear subspaces of V. Then the result follows. □

Henceforth, we use the same notation $\mathscr{F}$ to denote the extended functor $\widehat{\mathscr{F}}$ as in Proposition 2.5.2. The category under consideration would be clear from the context.

It follows from the construction discussed in the subsection §2.4 that given any principal G-Higgs bundle $\mathfrak{P} = (P, \theta)$ on X, the associated covariant functor

$$\Phi_{\mathfrak{P}} : \mathcal{R}ep_k^{\mathrm{fd}}(G) \longrightarrow \mathcal{H}iggs_0(X)$$

defined in (2.4.8) satisfies the axioms (HF1) – (HF6), and hence extends uniquely to a covariant functor, also denoted by

$$\Phi_{\mathfrak{P}} : \mathcal{R}ep_k(G) \longrightarrow \mathcal{H}iggs(X) \tag{2.5.3}$$

satisfying the conditions (i) – (iv) as in Proposition 2.5.2. We want to show that the converse also holds. More precisely, we construct a natural equivalence between the category of principal G-Higgs bundles on X and the full subcategory

$$\mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)) \subset \mathcal{F}un(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)) \tag{2.5.4}$$

of functors $\mathscr{F} : \mathcal{R}ep_k(G) \to \mathcal{H}iggs(X)$ as described in Proposition 2.5.2.

Let $\mathfrak{P} = (P, \theta)$ and $\mathfrak{P}' = (P', \theta')$ be two principal G-Higgs bundles on X. Let $\varphi : \mathfrak{P} \longrightarrow \mathfrak{P}'$ be a morphism of principal G-Higgs bundles (see Definition 2.3.6). Then for an object $(V, \rho) \in \mathcal{R}ep_k(G)$, we have a homomorphism of vector bundles

$$\varphi_\rho : \rho_* P \longrightarrow \rho_* P'. \tag{2.5.5}$$

In particular, for the adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$, we have a homomorphism of adjoint vector bundles $\mathrm{ad}(P) \to \mathrm{ad}(P')$. Since the induced homomorphism of Lie algebras $d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a G-module homomorphism, we have a commutative diagram of vector bundle homomorphisms

$$\begin{array}{ccc} \mathrm{ad}(P) & \longrightarrow & \mathcal{E}nd(\rho_* P) \\ \downarrow & & \downarrow \\ \mathrm{ad}(P') & \longrightarrow & \mathcal{E}nd(\rho_* P'), \end{array} \tag{2.5.6}$$

where the horizontal homomorphisms are induced by the G-module homomorphism $d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, and the left and the right vertical homomorphisms are induced by the adjoint action of G on $\mathfrak{g}$ and the induced G-action on $\mathfrak{gl}(V)$, respectively. Now tensoring the commutative diagram (2.5.6) with Ω^1_X, it follows that φ_ρ sends the Higgs field $\rho_*\theta \in H^0(X, \mathcal{E}nd(\rho_* P) \otimes \Omega^1_X)$ to $\rho_*\theta' \in H^0(X, \mathcal{E}nd(\rho_* P') \otimes \Omega^1_X)$. Thus we have a homomorphism of Higgs bundles

$$\Phi_\varphi(\rho) : \Phi_{\mathfrak{P}}(\rho) \longrightarrow \Phi_{\mathfrak{P}'}(\rho) \tag{2.5.7}$$

where $\Phi_{\mathfrak{P}}(\rho) := \rho_*\mathfrak{P} = (\rho_* P, \rho_*\theta)$ and $\Phi_{\mathfrak{P}'}(\rho) := \rho_*\mathfrak{P}' = (\rho_* P', \rho_*\theta')$.

Given a morphism

$$\eta : (V_1, \rho_1) \longrightarrow (V_2, \rho_2) \tag{2.5.8}$$

in $\mathcal{R}ep_k(G)$ and any principal G-Higgs bundle $\mathfrak{P} = (P, \theta)$ on X, the construction just before the Proposition 2.4.9 give rise to a homomorphism of flat Higgs sheaves

$$\Phi_{\mathfrak{P}}(\eta) : \Phi_{\mathfrak{P}}(\rho_1) \longrightarrow \Phi_{\mathfrak{P}}(\rho_2)\,. \tag{2.5.9}$$

Now it follows from the construction in the preceding paragraph that, given any morphism of principal G-Higgs bundles $\varphi : \mathfrak{P} \longrightarrow \mathfrak{P}'$ on X, the following diagram is commutative.

$$\begin{tikzcd}
\rho_{1*}\mathfrak{P} = (\rho_{1*}P, \rho_{1*}\theta) \arrow[r, "\Phi_\varphi(\rho_1)"] \arrow[d, "\Phi_{\mathfrak{P}}(\eta)"] & \rho_{1*}\mathfrak{P}' = (\rho_{1*}P', \rho_{1*}\theta) \arrow[d, "\Phi_{\mathfrak{P}'}(\eta)"] \\
\rho_{2*}\mathfrak{P} = (\rho_{2*}P, \rho_{2*}\theta) \arrow[r, "\Phi_\varphi(\rho_2)"] & \rho_{2*}\mathfrak{P}' = (\rho_{2*}P', \rho_{2*}\theta')\,.
\end{tikzcd} \tag{2.5.10}$$

In other words, $\varphi : \mathfrak{P} \longrightarrow \mathfrak{P}'$ induces a morphism of functors $\Phi_\varphi : \Phi_{\mathfrak{P}} \longrightarrow \Phi_{\mathfrak{P}'}$.

$$\begin{tikzcd}
\mathcal{R}ep_k(G) \arrow[rr, bend left, "\Phi_{\mathfrak{P}}"{name=U}] \arrow[rr, bend right, "\Phi_{\mathfrak{P}'}"'{name=D}] & & \mathcal{H}iggs(X) \arrow[from=U, to=D, "\Phi_\varphi"]
\end{tikzcd} \tag{2.5.11}$$

Thus the above construction give rise to a functor

$$\Phi : \mathcal{H}iggs_G(X) \longrightarrow \mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X)). \tag{2.5.12}$$

defined by sending a principal G-Higgs bundle $\mathfrak{P} \in \mathcal{H}iggs_G(X)$ on X to the functor $\Phi_{\mathfrak{P}}$ as defined in (2.5.3), and a morphism of principal G-Higgs bundles $\varphi : \mathfrak{P} \to \mathfrak{P}'$ on X to the morphism of functors Φ_φ defined in (2.5.11).

Theorem 2.5.13. *The functor Φ defined in* (2.5.12) *is an equivalence of categories.*

Proof. We first show that Φ is essentially surjective. Let $\mathscr{F} : \mathcal{R}ep_k(G) \to \mathcal{H}iggs(X)$ be a functor satisfying axioms (HF1) – (HF6). We need to show that there is a (unique) principal G-Higgs bundle $\mathfrak{P} = (P, \theta)$ on X such that $\Phi_{\mathfrak{P}} \cong \mathscr{F}$. Let $k[G]$ be the function k-algebra of the affine k-group scheme G. There is a natural regular G-action on $k[G]$ given by

$$(g \cdot f)(a) := f(ga), \ \forall \ g, \ a \in G \ \text{ and } f \in k[G]. \tag{2.5.14}$$

Let E be the underlined $\mathcal{O}_X$-module of the Higgs sheaf $\mathscr{F}(k[G])$. Then the relative spectrum $\mathcal{P} := \mathrm{Spec}_{\mathcal{O}_X}(E)$ together with the natural projection $\mathcal{P} \to X$ (affine morphism) is a principal G-bundle on X (see proof of [Nor76, Lemma 2.3, p. 32]). Since the associated locally free adjoint vector bundle (sheaf) $\mathrm{ad}(\mathcal{P}) = (\mathcal{O}_{\mathcal{P}} \otimes \mathfrak{g})^G$ is naturally isomorphic to the locally free coherent sheaf $\mathcal{E}nd(E)$, the Higgs field

$$\theta \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_X) = H^0(X, \mathrm{ad}(\mathcal{P}) \otimes \Omega^1_X)$$

of $\mathscr{F}(k[G])$ can be considered as a Higgs field on $\mathcal{P}$. Now with this $\mathfrak{P} := (\mathcal{P}, \theta)$, we have $\Phi_{\mathfrak{P}} \cong \mathscr{F}$. Thus Φ is essentially surjective.

To see Φ is faithful, note that if $\Phi_\varphi = \Phi_\psi$, for some morphisms of principal G-Higgs bundles $\varphi, \psi : \mathfrak{P}_1 \to \mathfrak{P}_2$, then for any k-linear representation $\rho : G \to \mathrm{GL}(V)$ we have $\Phi_\varphi(\rho) = \Phi_\psi(\rho)$; see (2.5.7). In particular, taking $V = k[G]$ together with the natural regular G-action described in (2.5.14), we see that $\varphi = \psi$. To see Φ is full, given morphism of functors $\mathscr{F} : \Phi_{\mathfrak{P}_1} \to \Phi_{\mathfrak{P}_1}$ in $\mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$, we can use the G-module $k[G]$ as above to get a morphism of G-Higgs bundles $\psi : \mathfrak{P}_1 \to \mathfrak{P}_2$ on X such that $\Phi_\psi = \mathscr{F}$. Thus Φ is an equivalence of categories. □

Let $\mathfrak{P} := (P, \theta)$ be a principal G-Higgs bundle on X. Given any morphism of k-schemes $f : Y \to X$, we can pullback P along f to get a principal G-bundle $f^*P := P \times_X Y$ on Y. Then the image of θ under the induced natural k-linear homomorphism

$$H^0(X, \mathrm{ad}(P) \otimes \Omega^1_X) \longrightarrow H^0(Y, \mathrm{ad}(f^*P) \otimes \Omega^1_Y) \tag{2.5.15}$$

gives a Higgs field $f^*\theta$ on f^*P. Thus we obtain a principal G-Higgs bundle $f^*\mathfrak{P} := (f^*P, f^*\theta)$ on Y.

Let $\sigma : G \to H$ is a homomorphism of affine k-group schemes. Given a principal G-bundle P on X, we can extend the structure group of P by σ to get a principal H-bundle on X as follow: take quotient of $P \times H$ by the equivalence relation

$$(z, \, h) \cdot g \, \sim \, (z \cdot g, \, \sigma(g)^{-1} h), \ \ \forall \, z \in P, \, g \in G, \text{ and } h \in H,$$

induced by the twisted G-action on $P \times H$ to obtain a principal H-bundle

$$\sigma_* P := (P \times H)/\sim$$

on X. Let $\mathcal{R}_\sigma : \mathcal{R}ep_k(H) \longrightarrow \mathcal{R}ep_k(G)$ be the functor obtained by sending an object $\rho : H \to \mathrm{GL}(V)$ of $\mathcal{R}ep_k(H)$ to the object $\rho \circ \sigma : G \to \mathrm{GL}(V)$ of $\mathcal{R}ep_k(G)$. Considering the adjoint representations of both G and H to their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and the Lie algebra homomorphism $d\sigma : \mathfrak{g} \to \mathfrak{h}$ induced by σ, we get a k-linear homomorphism (denoted by the same symbol)

$$\sigma_* : H^0(X, \mathrm{ad}(P) \otimes \Omega^1_X) \to H^0(X, \mathrm{ad}(\sigma_* P) \otimes \Omega^1_X). \tag{2.5.16}$$

Thus, given a principal G-Higgs bundle $\mathfrak{P} := (P, \theta)$ on X, we obtain a principal H-Higgs bundle $\sigma_*\mathfrak{P} := (\sigma_* P, \sigma_*\theta)$ on X. Then we have the following.

Proposition 2.5.17. *With the above notations, if $\mathfrak{P}$ is a principal G-Higgs bundle on X, then the following hold.*

(i) For any morphism $f : Y \to X$ of k-schemes, pulled-back of $\Phi_{\mathfrak{P}}$ along f is the functor $f^* \circ \Phi_{\mathfrak{P}} = \Phi_{f^*\mathfrak{P}}$, and hence $f^* \circ \Phi_{\mathfrak{P}} \in \mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(Y))$.
(ii) For any homomorphism $\sigma : G \to H$ of affine k-group schemes, $\Phi_{\mathfrak{P}} \circ \mathcal{R}_\sigma = \Phi_{\sigma_*\mathfrak{P}}$.

Proof. Follows by chasing construction of the functor Φ in (2.5.12). □

3. Higgs Fundamental Group Schemes

3.1. **S-fundamental group scheme.** Let C be a smooth projective curve defined over k, and let E a vector bundle on C. The *degree* of E is by definition

$$\deg(E) := \int_X c_1(\det(E)) \in \mathbb{Z},$$

where $c_1(\det(E))$ denotes the first Chern class of the determinant line bundle of E. The ratio

$$\mu(E) := \deg(E)/\operatorname{rk}(E)$$

is called the *slope* of E. A vector bundle E on C is said to be semistable if for any non-zero subsheaf $F \subset E$, we have $\mu(F) \leq \mu(E)$.

Let X be a connected reduced proper k-scheme.

Definition 3.1.1. A vector bundle E on X is said to be *numerically flat* if for any smooth projective curve C and any morphism $f : C \longrightarrow X$, the pullback f^*E is a semistable vector bundle of degree 0 on C.

It follows from the definition that the pullback of a numerically flat vector bundle $E \in \mathcal{V}ect(X)$ by a morphism of connected reduced proper k-schemes $f : Y \longrightarrow X$ is again numerically flat, and the converse holds if f is finite and surjective. Denote by $\mathscr{C}^{\mathrm{nf}}(X)$ the full subcategory of $\mathcal{V}ect(X)$ whose objects are numerically flat vector bundles on X. It is well-known that $\mathscr{C}^{\mathrm{nf}}(X)$ is an abelian category closed under tensor product [Lan11]. Moreover, choosing a closed point $x \in X(k)$, we get a *fiber functor* (k-linear exact faithful tensor functor)

$$\mathscr{F}_x : \mathscr{C}^{\mathrm{nf}}(X) \longrightarrow \mathcal{V}ect(k) \tag{3.1.2}$$

given by sending a vector bundle $E \in \mathscr{C}^{\mathrm{nf}}(X)$ to its fiber $E_x \in \mathcal{V}ect(k)$ at x. It turns out that, the quadruple $(\mathscr{C}^{\mathrm{nf}}(X), \otimes, \mathcal{O}_X, \mathscr{F}_x)$ forms a k-linear neutral Tannakian category, and the associated functor $\underline{\mathrm{Aut}}^{\otimes}(\mathscr{F}_x)$ of k-algebras of automorphisms of the fiber functor $\mathscr{F}_x$ is represented by an affine k-group scheme $\pi_1^S(X, x)$ (see [DM82]), known as the *S-fundamental group scheme of X with base point x*; [BPS06], [Lan11].

3.2. **Numerically flat Higgs bundles.** Let us first recall some definitions from [BBG19] that we need. Let E be a locally free coherent sheaf of rank r (≥ 2) on X. Fix a positive integer s with $s < r$, and consider the functor:

$$\mathcal{G}r(E, s) : (\mathrm{Sch}/X)^{\mathrm{op}} \longrightarrow (\mathrm{Set}) \tag{3.2.1}$$

given by sending $g : T \to X \in (\mathrm{Sch}/X)$ to the set

$$\begin{aligned}\mathcal{G}r(E, s)(T) :=\{q : g^*E \twoheadrightarrow F \mid q \text{ is surjective and } F \text{ is a locally free} \\ \text{coherent sheaf of rank } s \text{ on } T\}/\sim,\end{aligned}$$

where two such quotients $q : g^*E \twoheadrightarrow F$ and $q' : g^*E \twoheadrightarrow F'$ are said to be equivalent, denoted $q \sim q'$, if $\mathrm{Ker}(q) = \mathrm{Ker}(q')$. There is a projective X-scheme

$$p : \mathrm{Gr}(E, s) \longrightarrow X \tag{3.2.2}$$

which represents the functor $\mathcal{G}r(E, s)$, meaning that there is a natural isomorphism of functors $\mathcal{G}r(E, s) \xrightarrow{\simeq} \mathrm{Mor}_{(\mathrm{Sch}/X)}(-, \mathrm{Gr}(E, s))$. In particular, the identity morphism of $\mathrm{Gr}(E, s)$ corresponds to an exact sequence of locally free coherent sheaves

$$0 \longrightarrow \mathcal{S}(E, s) \xrightarrow{\Psi} p^*E \xrightarrow{\mathscr{F}} \mathcal{Q}(E, s) \longrightarrow 0, \tag{3.2.3}$$

known as the universal exact sequence over $\mathrm{Gr}(E, s)$.

If $\mathfrak{E} := (E, \theta)$ is a Higgs bundle on X, then its pullback $p^*\mathfrak{E} := (p^*E, p^*\theta)$ is a Higgs bundle on $\mathrm{Gr}(E, s)$, where $p : \mathrm{Gr}(E, s) \to X$ is the Grassmannian as described in (3.2.2). The Higgs field $p^*\theta$ naturally induces a Higgs field on the universal quotient $\mathcal{Q}(E, s)$ making it a quotient Higgs bundle of $(p^*E, p^*\theta)$ if and only if the universal kernel bundle $\mathcal{S}(E, s)$ is preserved under $p^*\theta$ in the sense that

$$p^*\theta(\mathcal{S}(E, s)) \subseteq \mathcal{S}(E, s) \otimes \Omega^1_{\mathrm{Gr}(E,s)}.$$

Let $\mathfrak{Gr}(E, s) \subseteq \mathrm{Gr}(E, s)$ be the subscheme defined by the vanishing locus of the following composite homomorphism

$$\mathcal{S}(E, s) \xrightarrow{\Psi} p^*E \xrightarrow{p^*\theta} p^*E \otimes p^*\Omega^1_X \xrightarrow{\mathscr{F}\otimes \mathrm{Id}} \mathcal{Q}(E, s) \otimes p^*\Omega^1_X, \tag{3.2.4}$$

(see (3.2.3)). It follows that $\mathfrak{Gr}(E,s)$, is the closed subscheme of $\mathrm{Gr}(E,s)$ parametrizing the quotient Higgs bundles of (E,θ), and we call it the *Higgs Grassmannian of* (E,θ). This closed embedding of $\mathfrak{Gr}(E,s)$ into $\mathrm{Gr}(E,s)$ gives rise to an exact sequence on $\mathfrak{Gr}(E,s)$

$$0 \longrightarrow \mathcal{S}(\mathfrak{E},s) \xrightarrow{\Psi} p^*\mathfrak{E} \xrightarrow{\mathscr{F}} \mathcal{Q}(\mathfrak{E},s) \longrightarrow 0, \tag{3.2.5}$$

where $\mathcal{Q}(\mathfrak{E},s)$ may be called the universal Higgs quotient for $\mathfrak{E}$ (c.f., (3.2.3)).

Definition 3.2.6. Let $\mathfrak{E} = (E,\theta)$ be a Higgs bundle on X. If $\mathrm{rk}(E) = 1$ and E is numerically effective, we say that (E,θ) is *Higgs numerically effective* (in short, *H-nef*). When $r := \mathrm{rk}(E) > 1$, we define H-nefness inductively by requiring that

(1) the universal Higgs quotient $\mathcal{Q}(\mathfrak{E},s)$ is H-nef, for all $s = 1,\dots,r-1$, and
(2) $\det(E) := \bigwedge^r E$ is nef.

Definition 3.2.7. A Higgs bundle $\mathfrak{E} := (E,\theta)$ on X is said to be *Higgs numerically flat* (in short, *H-nflat*) if both $\mathfrak{E}$ and its dual Higgs bundle $\mathfrak{E}^\vee$ are H-nef.

Lemma 3.2.8. *Let $f : X \to Y$ be a surjective morphism of connected reduced proper k-schemes. A Higgs bundle $\mathfrak{E} = (E,\theta)$ on Y is H-nflat if and only if $f^*\mathfrak{E}$ is H-nflat*

Proof. The same proof given in [BBG19, Lemma 3.4] works without smoothness assumptions on X and Y, and hence the result follows. □

Let $\mathcal{H}iggs_0(X) \subset \mathcal{H}iggs(X)$ be the full subcategory of Higgs bundles (locally free) on X. Let $\mathcal{H}iggs_0^{\mathrm{nf}}(X)$ be the full subcategory of $\mathcal{H}iggs_0(X)$ whose objects are Higgs numerically flat in the sense of Definition 3.2.7. It is known that $\mathcal{H}iggs_0^{\mathrm{nf}}(X)$ is an abelian category closed under tensor product, and has a structure of a k-linear symmetric monoidal category (the same lines of arguments given in [BBG19] works when X is a connected reduced proper k-scheme which is not necessarily smooth). As before, fixing a closed point $x \in X(k)$, we have a faithful exact k-linear tensor functor

$$\mathscr{F}_x^H : \mathcal{H}iggs_0^{\mathrm{nf}}(X) \longrightarrow \mathcal{V}ect(k) \tag{3.2.9}$$

given by sending $(E,\theta) \in \mathcal{H}iggs_0^{\mathrm{nf}}(X)$ to its fiber $E_x \in \mathcal{V}ect(k)$ at x. It turns out that the quadruple $(\mathcal{H}iggs_0^{\mathrm{nf}}(X), \otimes, \mathcal{O}_X, \mathscr{F}_x^H)$ is a neutral Tannakian category, and the associated affine k-group scheme $\pi_1^H(X,x)$ representing the functor of k-algebras $\underline{\mathrm{Aut}}^\otimes(\mathscr{F}_x^H)$ is called the *Higgs fundamental group scheme of X with base point at x.*

Theorem 3.2.10. *Let X be a connected reduced proper k-scheme. Fix a closed point $x \in X(k)$, and let $G := \pi_1^H(X,x)$. Then there is a principal G-Higgs bundle $\mathfrak{P} := (\mathcal{P}, \theta_{\mathcal{P}})$ on X such that given any object $\mathfrak{E} := (E,\theta) \in \mathcal{H}iggs_0^{\mathrm{nf}}(X)$, there is an object $\rho : G \to \mathrm{GL}(V)$ in $\mathcal{R}ep_k(G)$ such that $\mathfrak{E} = \mathfrak{P} \times^\rho V$.*

Proof. It follows from [DM82, Theorem 2.11] that the fiber functor $\mathscr{F}_x^H$ in (3.2.9) defines an equivalence of k-linear tensor abelian categories

$$\widehat{\mathscr{F}_x^H} : \mathcal{H}iggs_0^{\mathrm{nf}}(X) \longrightarrow \mathcal{R}ep_k^{\mathrm{fd}}(G), \tag{3.2.11}$$

whose composition with the forgetful functor $\mathcal{R}ep_k^{\mathrm{fd}}(G) \to \mathcal{V}ect(k)$ gives the fiber functor $\mathscr{F}_x^H$. Now one can check that the inverse of the equivalence $\widehat{\mathscr{F}_x^H}$ in (3.2.11) give rise to an object of $\mathcal{F}un_{\mathrm{HF}}(\mathcal{R}ep_k(G), \mathcal{H}iggs(X))$ (see (2.5.4) for the definition of this category), and hence by Theorem 2.5.13, it is isomorphic to a functor $\Phi_{\mathfrak{P}}$ for some unique principal G-Higgs bundle $\mathfrak{P}$ on X. From this the result follows. □

Corollary 3.2.12. *Let X be a connected reduced proper k-scheme. For any two points $x_1, x_2 \in X(k)$, the affine k-group schemes $\pi_1^H(X, x_1)$ and $\pi_1^H(X, x_2)$ are isomorphic.*

Proof. Using Theorem 3.2.10 above, the result follows from the proof of [PS20, Lemma 2.2.2], mutatis mutandis. □

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