1. MODEL ANSWER

Question. Use Plancherel's identity to find a relation between the solution and the initial data in terms of mean square norm called L^2 norm of the Cauchy problem:

$$u_t + u_{xxxx} + 2u = 0, \ -\infty < x < \infty, \ t > 0$$

with initial data $u(x, 0) = u_0(x), \ \ -\infty < x < \infty$; and also find $\lim_{t\to\infty} \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx$.

Model Answer: We have given

$$u_t + u_{xxxx} + 2u = 0, \ -\infty < x < \infty, \ t > 0, \tag{1.1}$$

with initial data $u(x, 0) = u_0(x)$, $-\infty < x < \infty$. Taking Fourier transform with respect to x, from equation (1.1), we get

$$\begin{aligned} &\widehat{u}_t(w,t) + (iw)^4 \widehat{u} + 2\widehat{u} = 0 \\ \implies &\widehat{u}_t(w,t) + (w^4 + 2)\widehat{u} = 0 \\ \implies &\widehat{u}(w,t) = \widehat{u}_0(w) e^{-t(w^4+2)} \end{aligned} \tag{1.2}$$

Using Plancherel's identity and (1.1), we get

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx = \int_{-\infty}^{\infty} |\widehat{u}(w,t)|^2 dw = e^{-4t} \int_{-\infty}^{\infty} |e^{-2tw^4}| \cdot |\widehat{u}_0(w)|^2 dw.$$

Since $|e^{-2tw^4}| \le 1, \forall\, t > 0$ and $w \in \mathbb{R}$, from above equation we have

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \le e^{-4t} \int_{-\infty}^{\infty} |\widehat{u}_0(w)|^2 dw. \tag{1.3}$$

Since $e^{-4t} \le 1$, for all $t > 0$, again using Plancherel's identity, we find that the relation between the solution and the initial data is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \le \int_{-\infty}^{\infty} |\widehat{u_0}(w)|^2 dw = \int_{-\infty}^{\infty} |u_0(x)|^2 dx, \ \ \forall\, t > 0.$$

Taking limit as $t \to \infty$ in (1.3), we get

$$\begin{aligned} 0 \le \lim_{t\to\infty} \int_{-\infty}^{\infty} |u(x,t)|^2 dx &\le \lim_{t\to\infty} e^{-4t} \int_{-\infty}^{\infty} |\widehat{u_0}(w)| dw = 0. \\ \implies \lim_{t\to\infty} \int_{-\infty}^{\infty} |u(x,t)|^2 dx &= 0. \end{aligned}$$

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