Notes on derived category

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ABSTRACT. In this note, we discuss basic theory of derived category following [Huy06]. After discussing some basic theories, we are interested to explore some of its applications in algebraic geometry.

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Date: July 25, 2020.
Key words and phrases. Derived category; triangulated category.
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Caution! I have started writing this note (around June 2020) for myself to learn basic theories of derived category, triangulated category, Bridgeland stability conditions, their connections with mathematical physics, and more importantly their applications in various areas of algebraic geometry. The present note is unorganized, incomplete, and may contains many inaccuracies to be fixed. I may post a polished version (as an expository note) in my home page after a year. Any suggestions to improve the exposition are welcome.

Update (July 25, 2020): Since the present note become quite long, I have decided to keep its main focus on basic theories of derived category only. I am planning to write a separate note focusing on Bridgeland stability.

Current version: https://arjunpaul29.github.io/home/notes/derived.pdf.

0. Introduction

One of my preliminary motivation to start this series of discussions is to understand *stability condition* in triangulated category as introduced by Tom Bridgeland in his celebrated 2007 paper published in the Annals of Mathematics [Bri07]. I also would like to learn some of its application in some other areas, like birational geometry, mirror symmetry etc.

Bridgeland's original motivation was to mathematically formulate the concept of Π-*stability* in theoretical physics as formulated by Douglas. In physics, Π-stability is something to relate a *super-symmetric non-linear sigma model* with a *(2, 2) Super Conformal Field Theory (SCFT)*. Let's have a quick tour into an interesting intersection of geometry and physics.

0.1. **Motivation from modern physics.**

I am not an expert in mathematical physics, but am interested to understand its relation with mathematics, in particular with algebraic geometry. After exploring various available sources, what I initially found and become interested in, are summarized below.

Let us start with a tailor of a largely speculating theory, known as *mirror symmetry*. A *super-symmetric non-linear sigma model* consists of a complex Calabi-Yau variety $X = (M, I)$ admitting a Ricci flat Kähler form ω and a "B-field". Let us explain the terminologies:

- M is the underlined real manifold of X and I is a complex structure on it,
- the variety X is *Calabi-Yau* means that the canonical line bundle K_X is trivial,

- the Kähler form ω is *Ricci flat* means that the curvature $F_{\det(\nabla_\omega)} = 0$, where $\det(\nabla_\omega)$ is the connection on $\det(TX)$ induced by the Chern connection ∇_ω on TX with respect to the Kähler form ω, and
- that "B-field" is something mysterious.

  In the context of SYZ mirror symmetry (an attempt to understand mathematically original version of mirror symmetry in physics), a B-field should be a class of a *unitary flat gerbe*, as suggested by Hitchin.

It is expected from physical ground that such a *super-symmetric non-linear sigma model* should give us a *(2, 2) Super Conformal Field Theory (SCFT)*. However, we don't know any precise mathematical formulation of (2, 2) SCFT, except for few cases! Roughly, a (2, 2) SCFT is some physical theory that depends on both complex and symplectic structures of varieties, and using *topological twists* one may separate its parts:

- *A-side*: depend only on symplectic structure, and
- *B-side*: depend only on complex structure.

In his famous ICM talk in 1994, Maxim Kontsevich proposed that the mathematical objects obtained from these topological twists should be in the derived category of coherent sheaves on the B-side (algebraic side), and in the derived Fukaya category of Lagrangian submanifolds on the A-side (symplectic side). Physically, objects of these categories are considered to be boundary conditions, known as *branes*. In this sense, Fukaya category is the category of A-branes and the derived category of coherent sheaves is the category of B-branes.

Conjecture 0.1.1 (Kontsevich)**.** *If two super-symmetric non-linear sigma models (X, ω, B) and (X', ω', B'), as described above, defines mirror symmetric SCFTs, then there are equivalences of categories:*

$$D^b(X) \simeq D^b(\mathrm{Fukaya}(X', \omega')) \quad \textit{and} \quad D^b(\mathrm{Fukaya}(X, \omega)) \simeq D^b(X')\,.$$

This is mathematically quite vague because we don't have precise mathematical formulation of SCFT!

From mathematical point of view, Kontsevich's Conjecture may be considered as a definition of *homological mirror symmetry*. Two super symmetric non-linear sigma models (X, ω, B) and (X', ω', B') are said to be *homological mirror partner* to each other if there are equivalences of such derived categories.

Remark 0.1.2. I think, finding explicit examples of such homological mirror symmetric pairs of super-symmetric non-linear sigma models would be very difficult problem. There is a notion of mirror symmetric varieties in SYZ sense, which identifies X and X' as dual to each other in an appropriate sense; see e.g., works of Hitchin, Hausel-Thaddues, Donagi-Pantev etc. This notion is different from the notion of homological mirror symmetry.

We shall see from construction of $D^b(X)$ that the derived category $D^b(X)$ depends only on complex/algebraic structure of X, and so $D^b(X)$ keeps only half information of the SCFT. Douglas argued that for any Ricci flat Kähler metric ω on X, there is a subcategory of $D^b(X)$, whose objects are physical branes, and these subcategories changes as the Kähler class ω moves in the stringy Kähler moduli.

To get an intuitive idea what this mathematically means, instead of looking at whole $D^b(X)$, consider the abelian category $\mathfrak{Coh}(X)$. Then a choice of Kähler class (or polarization) singles out semistable and stable objects of $\mathfrak{Coh}(X)$, and as we change the polarization, the collection of stable/semistable objects changes. Thus, there might be some way to encode more information of SCFT purely in terms of triangulated category $D^b(X)$ together with some extra structure on it. In a series of papers, Bridgeland set out to put these ideas on a mathematical setting and introduced the notion of *stability conditions* on a triangulated category. He has shown that the space of such stability conditions forms an (infinite) dimensional manifold, and this can be thought of an approximation of the stringy Kähler moduli space.

Mathematically, interesting point is that this new theory associates to a very algebraic object, like a triangulated category, a moduli space with meaningful geometric structure.

Roughly, a *stability condition* on a triangulated category $\mathcal{A}$ is given by a heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{A}$ and an additive group homomorphism $Z : K_0(\mathcal{H}) \to \mathbb{C}$, called the "central charge", satisfying Harder-Narasimhan property.

Well, enough introduction, and we shall see these in detail! Let's first set up some languages from category theory.

1. Some category theory

Joke: Category theory is like Ramayana and Mahabharata — there are lots of arrows!

— Nitin Nitsure

1.1. **Abelian category.**

Definition 1.1.1. A *category* $\mathscr{A}$ consists of the following data:

(i) a class of objects, denoted $\mathrm{Ob}(\mathscr{A})$,
(ii) for $X, Y \in \mathrm{Ob}(\mathscr{A})$, a class of morphisms from X into Y, denoted $\mathrm{Mor}_{\mathscr{A}}(X, Y)$,
(iii) for each $X, Y, Z \in \mathrm{Ob}(\mathscr{A})$, a *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \to \mathrm{Mor}_{\mathscr{A}}(X, Z), \quad (f, g) \mapsto g \circ f\,,$$

which satisfies associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathcal{A}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathcal{A}}(Z, W)$, for all $X, Y, Z, W \in \mathrm{Ob}(\mathscr{A})$.

A category $\mathscr{A}$ is said to be *locally small* if $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ is a set, for all $X, Y \in \mathrm{Ob}(\mathscr{A})$. A category $\mathscr{A}$ is said to be *small* if it is locally small and the class of objects $\mathrm{Ob}(\mathscr{A})$ is a set.

Example 1.1.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{A}$ such that $g \circ f = \mathrm{Id}_{A_1}$ and $f \circ g = \mathrm{Id}_{A_2}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,
(ii) for $X, Y \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, there is $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, which are compatible with the composition maps.

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$$

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ is given by the following data: for each object $A \in \mathscr{A}$, a map $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes.

(1.1.3)
$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_A"'] & \mathcal{F}(A') \arrow[d, "\varphi_{A'}"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A')
\end{tikzcd}$$

Definition 1.1.4. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Mor}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathcal{F}un(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 1.1.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 1.1.6. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Mor}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Notation. For any pre-additive category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ the abelian group $\mathrm{Mor}_{\mathscr{A}}(X, Y)$, for all $X, Y \in \mathrm{Ob}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.1.7 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.
(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0, 0)$ is the trivial group with one element.
(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $\mathrm{j}_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 1.1.8. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A, B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)\,, \quad (f, g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 1.1.9. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1, A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 1.1.10. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and
(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
 & C \arrow[dl, dashed, "\exists!\widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined by reversing the arrows of the above diagram.

Definition 1.1.11. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and
(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists!\widetilde{g}"] \\
 & C &
\end{tikzcd}$$

Definition 1.1.12. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 1.1.13. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.
(ii) If a cokernel of f exists, then it is an epimorphism.
(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.
(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). □

Exercise 1.1.14. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 1.1.15. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$ (see Lemma 1.1.13 (iii)). This completes the proof.

(1.1.16)
$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\
& & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & &
\end{tikzcd}$$

□

Definition 1.1.17. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 1.1.12).

Example 1.1.18.
(1) For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.
(2) Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules on X. Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{QCoh}(X)$ (reps., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of $\mathcal{O}_X$-modules on X, are also abelian. However, the full subcategory $\mathcal{V}ect(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of $\mathcal{O}_X$-modules on X, is not abelian, because kernel of a morphism in $\mathcal{V}ect(X)$ may not be in $\mathcal{V}ect(X)$.

1.2. **Triangulated category.** Let $\mathscr{A}$ be an additive category. A *shift functor* is an additive functor

$$T : \mathscr{A} \longrightarrow \mathscr{A}\,, \tag{1.2.1}$$

which is an equivalence of categories. A *triangle* in $(\mathscr{A}, T)$ is given by a diagram

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,, \tag{1.2.2}$$

with objects and arrows in $\mathscr{A}$. A *morphism of triangles* in $(\mathscr{A}, T)$ is given by a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} \tag{1.2.3}$$

where $f[1] := T(f) \in \mathrm{Hom}_{\mathscr{A}}(A[1], A'[1])$. If, in addition, f, g, h are isomorphisms in $\mathscr{A}$, we say that (1.2.3) is an isomorphism of triangles. We denote by $A[n]$ the object $T^n(A) \in \mathscr{A}$, and denote by $f[n]$ the morphism $T^n(f) \in \mathrm{Hom}_{\mathscr{A}}(A[n], B[n])$, for $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$.

Definition 1.2.4. A *triangulated category* is an additive category $\mathscr{A}$ together with an additive equivalence (*shift functor*)

$$T : \mathscr{A} \longrightarrow \mathscr{A}\,, \tag{1.2.5}$$

and a set of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,, \tag{1.2.6}$$

satisfying the following axioms (TR1) – (TR4) below.

(TR1) (i) Any triangle of the form

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is a distinguished triangle.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]\,.$$

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is a distinguished triangle.

(TR3) Any commutative diagram of distinguished triangles with vertical arrows f and g

(1.2.7)
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dashed, "\exists\, h"'] & A[1] \arrow[d, "f[1]"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

can be completed to a commutative diagram (not necessarily in a unique way).

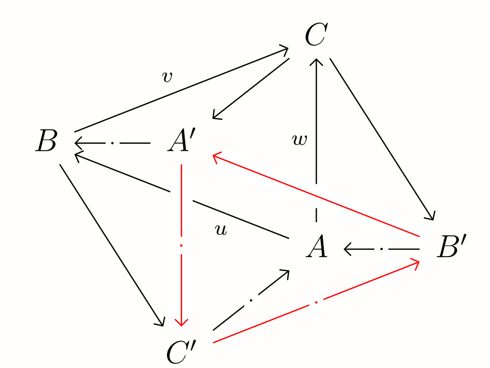
(TR4) (Octahedral axiom) Given any three distinguished triangles

$$A \xrightarrow{u} B \longrightarrow C' \longrightarrow A[1]$$
$$B \xrightarrow{v} C \longrightarrow A' \longrightarrow B[1]$$
$$A \xrightarrow{w} C \longrightarrow B' \longrightarrow A[1]$$

there is a distinguished triangle $C' \longrightarrow B' \longrightarrow A' \longrightarrow C'[1]$ such that the following diagram is commutative.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{Id}"] & B \arrow[r] \arrow[d, "v"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "w"] \arrow[d, "u"] & C \arrow[r] \arrow[d, "\mathrm{Id}"] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "u[1]"] \\
B \arrow[r, "v"] \arrow[d] & C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\mathrm{Id}"] & B[1] \arrow[d] \\
C' \arrow[r] & B' \arrow[r] & A' \arrow[r] & C'[1]
\end{tikzcd}$$

This axiom is called "*octahedron axiom*" because of its original formulation: given composable morphisms $A \xrightarrow{u} B \xrightarrow{v} C$, with $w := v \circ u$, we have the following octahedron diagram.



where any triangle of the form
$$\begin{tikzcd} & Z \arrow[dl, dashed] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$
are distinguished triangles, and the arrow $Z \dashrightarrow X$ stands for $Z \longrightarrow X[1]$.

Remark 1.2.8. A triangulated category need not be abelian, in general. In triangulated category, distinguished triangles play the roles of exact sequences in abelian categories. Examples of triangulated categories, we will be interested in, are derived categories of abelian categories.

Definition 1.2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulated categories. An *exact functor* of triangulated categories $\mathcal{A}$ to $\mathcal{B}$ is a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that for any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\mathcal{A}$, there is an isomorphism $F(A[1]) \xrightarrow{\phi} F(A)[1]$ such that

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{\phi \circ F(h)} F(A)[1]$$

is a distinguished triangle in $\mathcal{B}$. By a *morphism of triangulated categories*, we always mean an exact functor between them.

Definition 1.2.10 (Adjoint functors)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between any two categories. A functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ is said to be *right adjoint to* F, written as $F \dashv G$, if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, G(B)), \ \forall\, A \in \mathcal{A},\ B \in \mathcal{B}\,, \tag{1.2.11}$$

which is functorial in both A and B.

Similarly, a functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ is said to be *left adjoint to* F, written as $H \dashv F$, if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(B, F(A)) \cong \mathrm{Hom}_{\mathcal{A}}(H(B), A), \ \forall\, A \in \mathcal{A},\ B \in \mathcal{B}\,,$$

which is functorial in both A and B.

Remark 1.2.12.
(1) Note that, G is right adjoint to F if and only if F is left adjoint to G.
(2) If $F \dashv G$, then $\mathrm{Id}_{F(A)} \in \mathrm{Hom}_{\mathcal{B}}(F(A), F(A)) \cong \mathrm{Hom}_{\mathcal{A}}(A, (G \circ F)(A))$ induces a morphism $A \longrightarrow (G \circ F)(A)$, for all $A \in \mathcal{A}$. The naturality of this morphism gives us a morphism of functors

$$\mathrm{Id}_{\mathcal{A}} \longrightarrow G \circ F\,.$$

Similarly, taking $A = G(B)$ in (1.2.11), we get a morphism of functors

$$F \circ G \longrightarrow \mathrm{Id}_{\mathcal{B}}\ .$$

In particular, if F and G are quasi-inverse to each other (in case of equivalence of categories), then one is both left and right adjoint to the other one.

(3) Using Yoneda lemma, one can check that, left (resp., right) adjoint of a functor, if it exists, is unique up to isomorphisms.

Proposition 1.2.13. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be an exact functor of triangulated categories. Let $G : \mathcal{D}' \longrightarrow \mathcal{D}$ be a functor. If $F \dashv G$, or $G \dashv F$, then G is also exact.*

1.3. **Semi-orthogonal decomposition.** Let $\mathscr{D}$ be a k-linear triangulated category. An object $E \in \mathscr{D}$ is said to be *exceptional* if

$$\mathrm{Hom}_{\mathscr{D}}(E, E[\ell]) = \begin{cases} k & \text{if} \quad \ell = 0\,, \\ 0 & \text{if} \quad \ell \neq 0\,. \end{cases} \tag{1.3.1}$$

An *exceptional sequence* in $\mathscr{D}$ is a sequence of exceptional objects $E_1, E_2, \ldots, E_n$ of $\mathscr{D}$ such that $\mathrm{Hom}_{\mathscr{D}}(E_i, E_j[\ell]) = 0$, for all $i > j$ and all ℓ. In other words, if

$$\mathrm{Hom}_{\mathscr{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if} \quad i = j \text{ and } \ell = 0\,, \\ 0 & \text{if} \quad i > j, \text{ or if } \ell \neq 0 \text{ and } i = j\,. \end{cases} \tag{1.3.2}$$

An exceptional sequence $\{E_i\}_{i=1}^n$ is said to be *full* if, as a triangulated category, $\mathscr{D}$ is generated by $\{E_i\}_{i=1}^n$; i.e., if $\mathscr{D}'$ is a triangulated full subcategory of $\mathscr{D}$ containing E_i, for all $i = 1, \ldots, n$, then the inclusion morphism $\mathscr{D}' \hookrightarrow \mathscr{D}$ is an equivalence of categories.

1.4. t**-structure and heart.** Let $(\mathscr{A}, T)$ be a triangulated category. Let $\mathscr{B}$ be a subcategory of $\mathscr{A}$. For an integer n, we denote by $\mathscr{B}[n]$ the full subcategory of $\mathscr{A}$, whose objects are of the form $X[n]$, with $X \in \mathscr{B}$. In other words, $\mathscr{B}[n] = T^n(\mathscr{B}) \subset \mathscr{A}$.

Definition 1.4.1. Let $\mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 0}$ be two full subcategories of $\mathscr{A}$. For an integer n, let $\mathscr{A}^{\leq n} := \mathscr{A}^{\leq 0}[-n]$ and $\mathscr{A}^{\geq n} := \mathscr{A}^{\geq 0}[-n]$. A *$t$-structure* on $\mathscr{A}$ is given by a pair $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ of full subcategories of $\mathscr{A}$ satisfying the following axioms.

(t1) $\mathscr{A}^{\leq -1} \subset \mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 1} \subset \mathscr{A}^{\geq 0}$.
(t2) For any $X \in \mathscr{A}^{\leq 0}$ and $Y \in \mathscr{A}^{\geq 1}$, we have $\mathrm{Hom}_{\mathscr{A}}(X, Y) = 0$.
(t3) For any $X \in \mathscr{A}$, there is a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]\,,$$

with $X_0 \in \mathscr{A}^{\leq 0}$ and $X_1 \in \mathscr{A}^{\geq 1}$.

In this case, the full subcategory $\mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ of $\mathscr{A}$ is called the *heart* (or, *core*) of the t-structure $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$.

Example 1.4.2 (Standard t-structure on $D^b(X)$)**.** Consider the full subcategories

$$\mathcal{A}^{\leq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i > 0\} \text{ and}$$
$$\mathcal{A}^{\geq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i < 0\}$$

of $D^b(X)$. The axiom (T1) is easy to see. To check axiom (T2), we need some notations. For an integer $n \in \mathbb{Z}$, let $D^{\leq n}(X)$ (resp., $D^{\geq n}$) be the full subcategory of $D^b(X)$, whose objects are $E^\bullet \in D^b(X)$ satisfying $E^i = 0$, for all $i \leq n$ (resp., for all $i \geq n$). Consider the *truncation functors*

$$\tau^{\leq n} : D^b(X) \longrightarrow D^{\leq n}(X) \quad \text{and} \quad \tau^{\geq n} : D^b(X) \longrightarrow D^{\geq n}(X) \tag{1.4.3}$$

defined by

$$\begin{aligned} &\tau^{\leq n}(E^\bullet) := (\cdots \to E^{n-2} \to E^{n-1} \to \mathrm{Ker}(d^n_{E^\bullet}) \to 0 \to \cdots), \text{ and} \\ &\tau^{\geq n}(E^\bullet) := (\cdots \to 0 \to \mathrm{Ker}(d^n_{E^\bullet}) \to E^n \to E^{n+1} \to \cdots). \end{aligned}$$

where $d^n_{E^\bullet} : E^n \to E^{n+1}$, and $E^\bullet \in D^b(X)$. Now take $E^\bullet \in \mathcal{A}^{\leq 0}$ and $F^\bullet \in \mathcal{A}^{\geq 1}$. If $f \in \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet)$, then f factors as

$$\begin{tikzcd} E^\bullet \arrow[rrr, "f"] \arrow[dr, "\cong", "\tau^{\leq 0}"'] & & & F^\bullet \\ & \tau^{\leq 0}(E^\bullet) \arrow[r, "\tau^{\leq 0}(f)"] & \tau^{\leq 0}(F^\bullet) \arrow[ur] & \end{tikzcd} .$$

Since $\tau^{\leq 0}(F^\bullet) = 0$ in $D^b(X)$, we conclude that $f = 0$. Axiom (T3) follows from the exact triangle

$$\tau^{\leq 0}(E^\bullet) \longrightarrow E^\bullet \longrightarrow \tau^{\geq 1}(E^\bullet) \longrightarrow \tau^{\leq 0}(E^\bullet)[1], \ \forall\, E^\bullet \in D^b(X).$$

Thus $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a t-structure on $D^b(X)$, and the associated heart $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ is isomorphic to $\mathfrak{Coh}(X)$.

The above mentioned t-structure on $D^b(X)$ is not interesting, and somehow useless. We shall be interested in some non-trivial t-structures on $D^b(X)$ giving more interesting and useful hearts different from $\mathfrak{Coh}(X)$.

The next proposition shows that, the truncation functors exists for general triangulated category admitting a t-structure.

Proposition 1.4.4. *Let $\iota : \mathscr{A}^{\geq n} \to \mathscr{A}$ (resp., $\iota' : \mathscr{A}^{\geq n} \to \mathscr{A}$) be the inclusion functor. Then there is a functor $\tau^{\geq n} : \mathscr{A} \to \mathscr{A}^{\geq n}$ (resp., $\tau^{\leq n} : \mathscr{A} \to \mathscr{A}^{\leq n}$) such that for any $X \in \mathscr{A}$ and $Y \in \mathscr{A}^{\leq n}$ (resp., $Y \in \mathscr{A}^{\geq n}$), we have an isomorphism*

$$\mathrm{Hom}_{\mathscr{A}^{\leq n}}(Y, \tau^{\leq n}(X)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{A}}(X, \iota'(Y)) \tag{1.4.5}$$

(resp.,

$$\mathrm{Hom}_{\mathscr{A}^{\geq n}}(\tau^{\geq n}(X), Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{A}}(X, \iota(Y))\,). \tag{1.4.6}$$

Lemma 1.4.7. *Let $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ be a bounded t-structure on a triangulated category $\mathscr{A}$. Then $\mathcal{H} := \mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ is abelian.*

Proof. Let $\mathcal{H} := \mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ be the heart of a bounded t-structure $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ on $\mathscr{D}$. $\square$

Remark 1.4.8. If $D^b(\mathcal{A}) \cong D^b(X)$ for some abelian subcategory $\mathcal{A}$ of $D^b(X)$, then $\mathcal{A}$ is a heart of a t-structure on $D^b(X)$. However, the converse is not true, in general.

1.5. **Tensor Triangulated Category.**

2. Derived category

2.1. **Category of complexes.**
Let $\mathscr{A}$ be an abelian category. A *complex* in $\mathscr{A}$ is given by

$$A^\bullet \quad : \quad \cdots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \cdots , \tag{2.1.1}$$

where A^i are objects of $\mathscr{A}$ and d_A^i are morphisms in $\mathscr{A}$ such that $d_A^i \circ d_A^{i-1} = 0$, for all $i \in \mathbb{Z}$. A complex $A^\bullet$ in $\mathscr{A}$ is said to be *bounded above* (resp., *bounded below*) if there is an integer i_0 such that $A^i = 0$, for all $i \geq i_0$ (resp., if there is an integer j_0 such that $A^j = 0$, for all $j \leq j_0$). If $A^\bullet$ is both bounded above and bounded below, we say that $A^\bullet$ is *bounded*.

A morphism $f^\bullet : A^\bullet \to B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ of objects and morphisms from $\mathscr{A}$ is given by a collection of morphisms $\{f^i : A^i \to B^i\}_{i\in\mathbb{Z}}$ in $\mathscr{A}$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & A^i \arrow[r, "d_B^i"] & A^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots
\end{tikzcd} \tag{2.1.2}$$

Let $\mathcal{K}om(\mathscr{A})$ be the category, whose objects are complexes of objects and morphisms from $\mathscr{A}$, and morphisms are given by morphism of complexes, as defined in (2.1.2). Denote by $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ the full subcategories of $\mathcal{K}om(\mathscr{A})$, whose objects are bounded above complexes, resp., bounded below complexes, resp., bounded complexes. Then we have the following.

Proposition 2.1.3. *For any abelian category $\mathscr{A}$, the categories $\mathcal{K}om(\mathscr{A})$, $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ are abelian.*

Definition 2.1.4. For any complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ and $k \in \mathbb{Z}$, we define its k^{th}-shift to be the complex $A[k]^\bullet \in \mathcal{K}om(\mathscr{A})$ satisfying

(i) $A[k]^i := A^{k+i}$, for all $i \in \mathbb{Z}$, and
(ii) $d^i_{A[k]^\bullet} := (-1)^k d^{i+k}_{A^\bullet} : A[k]^i \longrightarrow A[k]^{i+1}$, for all $i \in \mathbb{Z}$.

Proposition 2.1.5. *For any integer k, the k^{th}-shift functor*

$$\mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{A})\,, \quad A^\bullet \longmapsto A[k]^\bullet$$

is an equivalence of categories.

Proof. Clearly, the $(-k)^{\text{th}}$-shift functor $A^\bullet \longmapsto A[-k]^\bullet$ defines the inverse functor of the k^{th}-shift functor. □

Remark 2.1.6. We shall see later that the category $\mathcal{K}om(\mathscr{A})$ together with the shift functor do not form a triangulated category, in general. However, we shall construct the derived category $D^b(\mathscr{A})$ from $\mathcal{K}om(\mathscr{A})$, which will turn out to be a triangulated category.

Given a complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, we define its i^{th} cohomology sheaf

$$\mathcal{H}^i(A^\bullet) := \frac{\mathrm{Ker}(d^i_{A^\bullet})}{\mathrm{Im}(d^{i-1}_{A^\bullet})} \in \mathscr{A}\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{2.1.7}$$

A complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ is said to be *acyclic* if $\mathcal{H}^i(A^\bullet) = 0$, for all $i \in \mathbb{Z}$. Any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$ of complexes gives rise to natural homomorphisms

$$\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet)\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{2.1.8}$$

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories. Let

$$\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B} \tag{2.1.9}$$

be an additive functor. Then $\mathcal{F}$ induces a functor, also denoted by the same symbol,

$$\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{B}) \tag{2.1.10}$$

defined by sending $A^\bullet \in \mathcal{K}om(\mathscr{A})$ to the complex $\mathcal{F}(A^\bullet)$, defined by

(i) $\mathcal{F}(A^\bullet)^i := \mathcal{F}(A^i)$, for all $i \in \mathbb{Z}$, and
(ii) $d^i_{\mathcal{F}(A^\bullet)} : \mathcal{F}(A^i) \xrightarrow{F(d^i_{A^\bullet})} \mathcal{F}(A^{i+1})$, for all $i \in \mathbb{Z}$,

and for any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$, we have a natural morphism of complexes

$$\mathcal{F}(f^\bullet) : \mathcal{F}(A^\bullet) \longrightarrow \mathcal{F}(B^\bullet)$$

defined by $\mathcal{F}(f^\bullet)^i := \mathcal{F}(f^i) : \mathcal{F}(A^i) \longrightarrow \mathcal{F}(B^i)$, for all $i \in \mathbb{Z}$.

Definition 2.1.11. An additive functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *exact* if it takes exact sequence to exact sequence.

Remark 2.1.12. Note that $\mathcal{F}$ is exact if and only if for any acyclic complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, its image $\mathcal{F}(A^\bullet) \in \mathcal{K}om(\mathscr{B})$ is acyclic.

Since $\mathcal{K}om(\mathscr{A})$ is abelian for $\mathscr{A}$ abelian, we can talk about short exact sequences in $\mathcal{K}om(\mathscr{A})$. Then by standard techniques from homological algebra, any short exact sequence

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0 \tag{2.1.13}$$

gives rise to a long exact sequence of cohomologies (which are objects of $\mathscr{A}$)

$$\cdots \longrightarrow \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \longrightarrow \mathcal{H}^i(C^\bullet) \longrightarrow \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \cdots, \ \forall\, i \in \mathbb{Z}. \tag{2.1.14}$$

Definition 2.1.15. A morphism of complexes $f^\bullet : A^\bullet \longrightarrow B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is called *quasi-isomorphism* if the induced morphism

$$\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \tag{2.1.16}$$

is an isomorphism, for all $i \in \mathbb{Z}$.

Example 2.1.17. Let X be a smooth projective k-variety and let E be a coherent sheaf on X. Then we can find a finite resolution

$$0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0 \to E \to 0.$$

of E with E^i projective (locally free) $\mathcal{O}_X$-modules. (We can use this to study many properties of E in terms of locally free coherent sheaves.) This gives rise to a morphism of complexes

$$f^\bullet : (0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0) \longrightarrow (\cdots \to 0 \to E \to 0 \to \cdots),$$

which is a quasi-isomorphism.

2.2. **What is a derived category?** The main idea for definition of derived category is: *quasi-isomorphism of complexes should become isomorphism in the derived category*. Therefore, the derived category $D(\mathscr{A})$ is the localization of $\mathcal{K}om(\mathscr{A})$ by quasi-isomorphisms. This can be done by passing to the appropriate homotopy category.

Theorem 2.2.1. *Let $\mathscr{A}$ be an abelian category, and $\mathcal{K}om(\mathscr{A})$ the category of complexes in $\mathscr{A}$. Then there is a category $D(\mathscr{A})$, known as the derived category of $\mathscr{A}$, together with a functor*

$$Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A}) \tag{2.2.2}$$

such that:

(i) If $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a quasi-isomorphism, then $Q(f^\bullet)$ is an isomorphism in $D(\mathscr{A})$,
(ii) if a functor $\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathscr{D}$ satisfies property (i), there is a unique functor $\widetilde{\mathcal{F}} : D(\mathscr{A}) \longrightarrow \mathscr{D}$ such that $\widetilde{\mathcal{F}} \circ Q \cong \mathcal{F}$.

(2.2.3)
$$\begin{tikzcd} \mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "\mathcal{F}"'] & & D(\mathscr{A}) \arrow[dl, dashed, "\exists\,!\,\widetilde{\mathcal{F}}"] \\ & \mathscr{D} & \end{tikzcd}$$

Now we go ahead for construction of the derived category $D(\mathscr{A})$ of $\mathscr{A}$. Since we want any quasi-isomorphism $C^\bullet \to A^\bullet$ of complexes in $\mathcal{K}om(\mathscr{A})$ to become isomorphism in the derived category $D(\mathscr{A})$, any morphism of complexes $C^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ should give rise to a morphism $A^\bullet \to B^\bullet$ in $D^b(\mathscr{A})$. This leads to the definition of morphisms in $D^b(\mathscr{A})$ as diagrams of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

where "qis" stands for "quasi-isomorphism" of complexes. To make this more precise, we need to define when two such roofs should considered to be equal, and how to define their compositions. Then natural context for both problems is to consider the homotopy category $K(\mathscr{A})$ of complexes in $\mathcal{K}om(\mathscr{A})$, which is an intermediate step for going from $\mathcal{K}om(\mathscr{A})$ to $\mathscr{D}(\mathscr{A})$.

$$\begin{tikzcd} \mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, bend right] & & D(\mathscr{A}) \\ & K(\mathscr{A}) \arrow[ur, bend right, "Q'"'] & \end{tikzcd}$$

Definition 2.2.4. Two morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ are said to be *homotopically equivalent*, written as $f^\bullet \sim g^\bullet$, if there is a morphism of complexes $h^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

$$\begin{tikzcd} A^\bullet \arrow[d, shift right, "g^\bullet"'] \arrow[d, shift left, "f^\bullet"] & \cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, shift right, "g^{i-1}"'] \arrow[d, shift left, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[dl, "h^i"'] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "g^{i+1}"'] \arrow[d, shift left, "f^{i+1}"] & \cdots \\ B^\bullet & \cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots \end{tikzcd}$$

Let $K(\mathscr{A})$ be the category, whose objects are the same as objects of $\mathcal{K}om(\mathscr{A})$ and morphisms are given by $\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, for all $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$.

Following proposition is an easy consequence of the above definition.

Proposition 2.2.5. *(i) Homotopy equivalence of morphisms $A^\bullet \to B^\bullet$ of complexes is an equivalence relation.*

(ii) Homotopically trivial morphisms of complexes form an 'ideal' in the morphisms of $\mathcal{K}om(\mathscr{A})$.

(iii) If $f^\bullet$ and $g^\bullet$ are two homotopically equivalent morphisms of complexes in $\mathcal{K}om(\mathscr{A})$, then the induced morphisms $\mathcal{H}^i(f^\bullet)$ and $\mathcal{H}^i(g^\bullet)$ from $\mathcal{H}^i(A^\bullet)$ to $\mathcal{H}^i(B^\bullet)$ coincides.

(iv) Let $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to A^\bullet$ be two morphisms of complexes. If $f^\bullet \circ g^\bullet \sim \mathrm{Id}_{B^\bullet}$ and $g^\bullet \circ f^\bullet \sim \mathrm{Id}_{A^\bullet}$, then $f^\bullet$ and $g^\bullet$ are quasi-isomorphisms, and $\mathcal{H}^i(f^\bullet)^{-1} = \mathcal{H}^i(g^\bullet)$, for all $i \in \mathbb{Z}$.

Now we complete the construction of derived category $D(\mathscr{A})$. Take $\mathrm{Ob}(D(\mathscr{A})) := \mathrm{Ob}(\mathcal{K}om(\mathscr{A}))$. As discussed before, a morphism $f : A^\bullet \to B^\bullet$ in $D(\mathscr{A})$ is given by equivalence class of roofs of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \xrightarrow{qis} A^\bullet$ is a quasi-isomorphism of complexes in $\mathcal{K}om(\mathscr{A})$, and two such roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

are considered to be equivalent if they are dominated by a third one of the same type

$$\begin{tikzcd} & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] \arrow[ddll, dashed, bend right, "qis"'] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "qis"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd} \tag{2.2.6}$$

such that the above diagram commutes in the homotopy category $K(\mathscr{A})$. In particular, the compositions $C_0^\bullet \to C_1^\bullet \to A^\bullet$ and $C_0^\bullet \to C_2^\bullet \to A^\bullet$ are homotopically equivalent. To define composition of morphisms in $D(\mathscr{A})$, consider two roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & B^\bullet & & C^\bullet \end{tikzcd}$$

representing two morphisms in $D(\mathscr{A})$. It is natural to guess that, one should be able to define their composition to be a morphism represented by 'the' following roof

(2.2.7)
$$\begin{tikzcd} & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet , \end{tikzcd}$$

which commutes in the homotopy category $K(\mathscr{A})$. Now we need to ensure that such a diagram exists uniquely, up to *equivalence of roofs* as defined in (2.2.6) (c.f., Proposition 2.2.12). For this, we need the concept of "mapping cone".

Definition 2.2.8. The *mapping cone* of a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a complex $C(f^\bullet)$ defined as follow:

$$C(f^\bullet)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f^\bullet)} = \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix}, \ \forall\, i \in \mathbb{Z}\,.$$

Note that $C(f^\bullet)$ is a complex. Moreover, there are natural morphisms of complexes

(2.2.9)
$$\tau : B^\bullet \to C(f^\bullet) \quad \text{and} \quad \pi : C(f^\bullet) \to A^\bullet[1]$$

given by natural injection $B^i \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively, for all i. Then we have the following.

(i) The composition $B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$ is trivial. In fact,
$$0 \longrightarrow B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1] \longrightarrow 0$$
is a short exact sequence in $\mathcal{K}om(\mathscr{A})$, and gives us a long exact sequence of cohomologies
$$\cdots \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C(f^\bullet)) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots .$$

(ii) The composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet)$ is homotopic to the trivial morphism. Indeed, take $h^\bullet : A^\bullet \longrightarrow C(f^\bullet)$ to be morphism of complexes defined by the natural injective morphism $h^i : A^i \to C(f^\bullet)^{i-1} = A^i \oplus B^{i-1}$, for all i. Then we have
$$h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{C(f^\bullet)} \circ h^i = \tau^i \circ f^i, \quad \forall\, i \in \mathbb{Z}\,.$$

Remark 2.2.10. It follows from the above construction that any commutative diagram of complexes

$$\begin{tikzcd}
A_1^\bullet \arrow[r, "f_1^\bullet"] \arrow[d, "\phi"] & B_1^\bullet \arrow[r, "\tau_1"] \arrow[d, "\psi"] & C(f_1^\bullet) \arrow[r, "\pi_1"] \arrow[d, dashed, "\exists\ \phi[1]\oplus\psi"] & A_1^\bullet[1] \arrow[d, "\phi[1]"] \\
A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r, "\tau_2"] & C(f_2) \arrow[r, "\pi_2"] & A_2^\bullet[1]
\end{tikzcd}$$

can be completed by a dashed arrow as above (c.f., axiom (TR3) in Definition 1.2.4).

Proposition 2.2.11. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone. Let $\tau : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$ be the natural morphisms as in (2.2.9). Then there is a morphism of complexes $g : A^\bullet[1] \longrightarrow C(\tau)$ which is an isomorphism in $K(\mathscr{A})$ such that the following diagram commutes in the homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \\
& & C(\tau) \arrow[ur, "\pi_\tau"'] &
\end{tikzcd}$$

Proof. Define $g : A^\bullet[1] \longrightarrow C(\tau)$ by setting

$$g^i : A^\bullet[1]^i = A^{i+1} \longrightarrow C(\tau)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$$

to be the morphism $g^i = (-f^{i+1}, \mathrm{Id}_{A^{i+1}}, 0)$, for all i. Clearly, g is a morphism of complexes, and its inverse (in $K(\mathscr{A})$) is given by the morphism of complexes $g^{-1} : C(\tau) \to A^\bullet[1]$ defined by projection onto the middle factor. Clearly, $\pi_\tau \circ g = -f$ in $\mathcal{K}om(\mathscr{A})$. However, the diagram

$$\begin{tikzcd}
C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[d, "g"] \\
& C(\tau)
\end{tikzcd}$$

does not commute in $\mathcal{K}om(\mathscr{A})$. We show that, $\pi \circ g \sim \tau_\tau$. For this, note that $g^{-1} \circ \tau_\tau = \pi$ in $\mathcal{K}om(\mathscr{A})$. Since $g \circ g^{-1} \sim \mathrm{Id}_{C(\tau)}$, we have $\tau_\tau \sim g \circ \pi$. This completes the proof. □

Now we use the above proposition to complete the proof of existence and uniqueness of composition of morphisms in $D(\mathscr{A})$.

Proposition 2.2.12. *Let $f : A^\bullet \to B^\bullet$ and $g : C^\bullet \to B^\bullet$ be morphism of complexes with f a quasi-isomorphism. Then there is a complex $C_0^\bullet$ together with a quasi-isomorphism $C_0^\bullet \to C^\bullet$ and a morphism $C_0^\bullet \to A^\bullet$ such that the following diagram commutes in the* *homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd} C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "qis"'] & B^\bullet \end{tikzcd}$$

Proof. Note that, there is a natural morphism of complexes $\phi^i : C(\tau \circ g) \to A^\bullet[1]$ given by the natural projection

$$\phi^i : C(\tau \circ g)^i = C^{i+1} \oplus C(f)^i = C^{i+1} \oplus A^{i+1} \oplus B^i \xrightarrow{pr_2} A^{i+1} = A^\bullet[1]^i$$

onto the middle factor, for each i. By Proposition 2.2.11, there is morphism of complexes $\psi : C(\tau) \xrightarrow{\sim} A^\bullet[1]$ which is an isomorphism in $K(\mathscr{A})$. Then the following diagram is commutative in $K(\mathscr{A})$.

$$\begin{tikzcd} C_0^\bullet := C(\tau \circ g)[-1] \arrow[r] \arrow[d, dashed, "\phi[-1]"] & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, "g"] & C(f) \arrow[rr, "\tau_{\tau\circ g}"] \arrow[d, equal] & & C(\tau \circ g) \arrow[d, "\phi"] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & C(\tau) \arrow[r, "\psi", "\sim"'] & A^\bullet[1] \end{tikzcd}$$

Define $C_0^\bullet := C(\tau \circ g)[-1]$. Since f is a quasi-isomorphism, it follows from the commutativity of the above diagram that $C_0^\bullet \to C^\bullet$ is a quasi-isomorphism. $\square$

Remark 2.2.13. Existence and uniqueness of composition of morphisms in $D(\mathscr{A})$ follows from the above Proposition 2.2.12 (c.f., diagram (2.2.7)). This completes the construction of the derived category $D(\mathscr{A})$.

Proposition 2.2.14. *The categories $K(\mathscr{A})$ and $D(\mathscr{A})$ are additive.*

Proof. Let $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$. Since $\mathcal{K}om(\mathscr{A})$ is an abelian category, it follows from Proposition 2.2.5 that the quotient

$$\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim,$$

is an abelian group. Thus $K(\mathscr{A})$ is an additive category.

To see $D(\mathscr{A})$ is an additive category, let $f_1, f_2 \in \mathrm{Hom}_{D(\mathscr{A})}(A^\bullet, B^\bullet)$ be two morphisms in $D(\mathscr{A})$ represented by following equivalence classes of roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\phi_1"', "qis"] \arrow[dr, "\psi_1"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^\bullet \arrow[dl, "\phi_2"', "qis"] \arrow[dr, "\psi_2"] & \\ A^\bullet & & B^\bullet \end{tikzcd},$$

respectively. It follows from Proposition 2.2.12 that there is an object $C^\bullet \in D(\mathscr{A})$ and quasi-morphisms $\delta_i : C^\bullet \to C_i^\bullet$, for $i = 1, 2$, such that the following diagram commutes in the homotopy category.

(2.2.15)
$$\begin{tikzcd} C^\bullet \arrow[r, "\delta_2", "qis"'] \arrow[d, "\delta_1", "qis"'] & C_2^\bullet \arrow[d, "\phi_2"] \\ C_1^\bullet \arrow[r, "\phi_1", "qis"'] & A^\bullet \end{tikzcd}$$

Note that, both $\phi_1 \circ \delta_1$ and $\phi_2 \circ \delta_2$, are quasi-isomorphisms, and are equal in $K(\mathscr{A})$. Let $\delta = \phi_1 \circ \delta_1 = \phi_2 \circ \delta_2$ in $K(\mathscr{A})$. Then in $D(\mathscr{A})$, we can write

$$f_1 + f_2 = \psi_1 \circ \phi_1^{-1} + \psi_2 \circ \phi_2^{-1} = (\psi_1 \circ \delta_1 + \psi_2 \circ \delta_2) \circ \delta^{-1}.$$

This defines $f_1 + g_1$ in $D(\mathscr{A})$. One can check that, $f + g$ as defined above, is well-defined in $D(\mathscr{A})$. Note that, the roof $A^\bullet \xleftarrow[qis]{-\phi_1} C_1^\bullet \xrightarrow{\psi_1} B^\bullet$ is the additive inverse of f_1 in $D(\mathscr{A})$. Now one can check that $\text{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ is an abelian group. $\square$

Definition 2.2.16. A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $K(\mathscr{A})$ (resp., in $D(\mathscr{A})$) is said to be a *distinguished triangle* if it is isomorphic in $K(\mathscr{A})$ (resp., $D(\mathscr{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]\,,$$

where f is a morphism of complexes with mapping cone $C(f)$, and τ and π are natural morphisms as defined in (2.2.9).

Proposition 2.2.17. *The categories $D(\mathscr{A})$ and $K(\mathscr{A})$ together with the shift functor are triangulated. Moreover, the natural functor $Q : K(\mathscr{A}) \longrightarrow D(\mathscr{A})$ is an exact functor of triangulated categories.*

Proof. Triangulated structure on both $K(\mathscr{A})$ and $D(\mathscr{A})$ are given by shift functor $A^\bullet \mapsto A^\bullet[1]$ together with the collection of 'distinguished triangles' as defined above. Verification of axioms (TR1) – (TR4) requires crucial use of mapping cone. $\square$

Example 2.2.18. Let $\mathscr{A} = \mathcal{V}ect_{fd}(k)$ be the category, whose objects are finite dimensional k-vector spaces, and morphisms between objects are k-linear homomorphisms. Then $D(\mathscr{A})$ is equivalent to the category $\prod_{i\in\mathbb{Z}} \mathscr{A}$ of graded k-vector spaces. Note that, any complex of k-vector spaces $A^\bullet \in D(\mathcal{V}ect_{fd}(k))$ is isomorphic to its cohomology complex $\bigoplus\limits_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$ with trivial differentials. More generally, this holds for any *semisimple* abelian category $\mathscr{A}$ (i.e., when $\mathscr{A}$ is abelian and any short exact sequence in $\mathscr{A}$ splits).

Remark 2.2.19. Contrary to the category $\mathcal{K}om(\mathscr{A})$ of complexes in $\mathscr{A}$, the derived category $D(\mathscr{A})$ is not abelian, in general. However, $D(\mathscr{A})$ is always triangulated. $D^b(\mathscr{A})$ is abelian if and only if $\mathscr{A}$ is semisimple
(see https://math.stackexchange.com/questions/189769).

Corollary 2.2.20. *(a) The functor $Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A})$ identifies set underlying set of objects of both categories (Apply property (ii) to the identity functor $\mathcal{K}om(\mathscr{A}) \to \mathcal{K}om(\mathscr{A})$).*
(b) For any complex $A^\bullet \in D(\mathscr{A})$, its cohomology objects $\mathcal{H}^i(A^\bullet)$ are well-defined objects in $\mathscr{A}$. (This is because, quasi-isomorphisms of $\mathcal{K}om(\mathscr{A})$ turns into isomorphisms in $D(\mathscr{A})$.)
(c) Considering $A \in \mathscr{A}$ as a complex $A^\bullet \in D(\mathscr{A})$ concentrated at degree zero only, gives an equivalence between $\mathscr{A}$ and the full subcategory of objects of $D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$.

Proposition 2.2.21. *Let $\mathscr{A}$ be an abelian category and $K(\mathscr{A})$ its homotopy category. Let $\mathscr{C}$ be any additive category.*

(1) An additive functor $F : K(\mathscr{A}) \longrightarrow \mathscr{C}$ factors through an additive functor $\widetilde{F} : D(\mathscr{A}) \longrightarrow \mathscr{C}$ if and only if F send quasi-isomorphisms to isomorphisms.
(2) Let $\mathscr{B}$ be an abelian category, and $G : K(\mathscr{A}) \longrightarrow K(\mathscr{B})$ an additive functor which maps quasi-isomorphism to quasi-isomorphism. Then G induces an additive functor $\widetilde{G} : D(\mathscr{A}) \longrightarrow D(\mathscr{B})$ such that the following diagram commutes.

$$\begin{tikzcd} K(\mathscr{A}) \arrow[r, "G"] \arrow[d, "Q_{\mathscr{A}}"'] & K(\mathscr{B}) \arrow[d, "Q_{\mathscr{B}}"] \\ D(\mathscr{A}) \arrow[r, "\widetilde{G}"] & D(\mathscr{B}) \end{tikzcd}$$

Proof. If $F : K(\mathscr{A}) \to \mathscr{C}$ sends quasi-isomorphisms to quasi-isomorphism, we define $\widetilde{F} : D(\mathscr{A}) \longrightarrow \mathscr{C}$ by sending an object $E^\bullet \in D(\mathscr{A})$ to $F(E^\bullet) \in \mathscr{C}$, and any morphism $f/\phi : A^\bullet \underset{\sim}{\xleftarrow{\phi}} C^\bullet \xrightarrow{f} B^\bullet$ in $D(\mathscr{A})$ to the morphism

$$F(f) \circ F(\phi)^{-1} : F(A^\bullet) \underset{\simeq}{\xrightarrow{F(\phi)^{-1}}} F(C^\bullet) \xrightarrow{F(f)} F(B^\bullet).$$

in $\mathscr{C}$. Converse follows from the construction of $D(\mathscr{A})$ and Theorem 2.2.1.

The second assertion follows by applying the first one to the composition $F : K(\mathscr{A}) \xrightarrow{G} K(\mathscr{B}) \xrightarrow{Q_{\mathscr{B}}} D(\mathscr{B})$. ∎

2.3. **Derived categories: $D^-(\mathscr{A}), D^+(\mathscr{A})$, and $D^b(\mathscr{A})$.**

Definition 2.3.1. Let $\mathcal{K}om^*(\mathscr{A})$, with $* = +, -$, or b, be the full subcategory of $\mathcal{K}om(\mathscr{A})$, whose objects are complexes $A^\bullet \in \mathcal{K}om(\mathscr{A})$ with $A^i = 0$ for all $i \ll 0$, $i \gg 0$, or $|i| \ll 0$, respectively.

Note that, $\mathcal{K}om^*(\mathscr{A})$ is again abelian, for $* \in \{+, -, b\}$. So following similar construction (i.e., by dividing out first by homotopy equivalence, and then by localizing with respect to quasi-isomorphisms), we can construct a category, denoted by $D^*(\mathscr{A})$. There is a natural forgetful functor

$$\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A}),$$

which just forgets boundedness condition.

Proposition 2.3.2. *The natural forgetful functor $\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A})$, where $* = +, -$, or b, gives an equivalence of $D^*(\mathscr{A})$ with the full triangulated subcategories of all complexes $A^\bullet \in D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for $i \ll 0$, for $i \gg 0$, or for $|i| \ll 0$, respectively.*

To give an idea how this works, let $A^\bullet \in \mathcal{K}om(\mathscr{A})$ be such that $\mathcal{H}^i(A^\bullet) = 0$, for $i > n$. Then the commutative diagram

$$\begin{tikzcd}
B^\bullet : & \cdots \arrow[r] & A^{n-2} \arrow[r] \arrow[d, equal] & A^{n-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d_A^n) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : & \cdots \arrow[r] & A^{n-2} \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.3.3}$$

defines a quasi-isomorphism between a complex $B^\bullet \in \mathcal{K}om^-(\mathscr{A})$ and $A^\bullet$. Similarly, if $\mathcal{H}^i(A^\bullet) = 0$ for $i < m$, then the commutative diagram

$$\begin{tikzcd}
A^\bullet : & \cdots \arrow[r] & A^{m-1} \arrow[r] \arrow[d] & A^m \arrow[r] \arrow[d] & A^{m+1} \arrow[r] \arrow[d, equal] & \cdots \\
C^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_A^{m-1}) \arrow[r] & A^{m+1} \arrow[r] & \cdots
\end{tikzcd} \tag{2.3.4}$$

defines a quasi-isomorphism of a complex $C^\bullet \in \mathcal{K}om^+(\mathscr{A})$ and $A^\bullet$. Similar idea works for $\mathcal{K}om^b(\mathscr{A})$. However, one need to pass from $\mathcal{K}om^*(\mathscr{A})$ to the derived category $D^*(\mathscr{A})$ by inverting quasi-isomorphisms. This needs some technical care.

Let $\mathscr{A} \subset \mathscr{B}$ be full abelian subcategory of an abelian category $\mathscr{B}$. Then there is an obvious functor $\iota : D(\mathscr{A}) \longrightarrow D(\mathscr{B})$. One might expect that this is an equivalence of $D(\mathscr{A})$ with the full subcategory $D_{\mathscr{A}}(\mathscr{B})$ of $D(\mathscr{B})$ consisting of objects $E^\bullet \in D(\mathscr{B})$ with $\mathcal{H}^i(E^\bullet) \in \mathscr{A}$, for all $i \in \mathbb{Z}$. However, this is not true, in general. There are several issues.

- $D_{\mathscr{A}}(\mathscr{B})$ need not be triangulated!
- The functor ι is neither faithful nor full, in general.

However, the next proposition answers when the above expectation holds true. First, we need a definition.

Definition 2.3.5. Let $\mathscr{A}$ be an abelian category. A *thick subcategory* of $\mathscr{A}$ is a full abelian subcategory $\mathscr{B} \subset \mathscr{A}$ of $\mathscr{A}$ such that for any short exact sequence (in $\mathscr{A}$)

$$0 \to A \to B \to C \to 0$$

with $A, C \in \mathscr{B}$, we have $B \in \mathscr{B}$.

Let $E, F \in \mathscr{A}$. We say that F is *embedded* in E (or, F is a *subobject* of E) if there is a monomorphism $F \to E$ in $\mathscr{A}$. An object $I \in \mathscr{A}$ is called *injective* if the functor

$$\mathrm{Hom}_{\mathscr{A}}(-, I) : \mathscr{A} \longrightarrow Ab$$

is exact.

Proposition 2.3.6. *Let $\mathscr{A} \subset \mathscr{B}$ be a thick abelian subcategory of an abelian category $\mathscr{B}$. Assume that any object $A \in \mathscr{A}$ is a suboject of an object $I_A \in \mathscr{A}$, which is injective as an object of $\mathscr{B}$. Then the natural functor $D(\mathscr{A}) \longrightarrow D(\mathscr{B})$ induces an equivalence*

$$D^*(\mathscr{A}) \longrightarrow D^*_{\mathscr{A}}(\mathscr{B}), \quad \text{where } * = + \text{ or } b,$$

of $D^*(\mathscr{A})$ and the full triangulated subcategory $D^*_{\mathscr{A}}(\mathscr{B}) \subset D^*(\mathscr{B})$ of complexes with cohomologies in $\mathscr{A}$.

P.S.: I have not seen similar statement for $* = \emptyset$ or $-$.

Next, we want to get a computable description of Hom's in the derived category. In the next section, using derived functor, we show that

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]) = \mathrm{Ext}^i(A^\bullet, B^\bullet), \quad \forall\, i\,.$$

3. Derived functors

3.1. **What is it?** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. We want to know when such a functor give rise to a natural functor between derived categories. Note that, if F is not exact, then image of an acyclic complex ($\mathcal{H}^i(A^\bullet) = 0,\ \forall\, i$) may not be acyclic. So to get a induced functor at the level of derived categories, F should be exact.

Lemma 3.1.1. *Let $F : K^*(\mathcal{A}) \to K^*(\mathcal{B})$, where $* \in \{\emptyset, -, +, b\}$, be an exact functor of triangulated categories. Then F induces a functor $\widetilde{F} : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ making the* *following diagram commutative*

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d] & K^*(\mathcal{B}) \arrow[d] \\ D^*(\mathcal{A}) \arrow[r, "\widetilde{F}"] & D^*(\mathcal{B}) \end{tikzcd}$$

if and only if one of the following (equivalent) conditions holds:

(i) F *sends a quasi-isomorphism to a quasi-isomorphism.*

(ii) F *sends any acyclic complex to an acyclic complex.*

However, if the functor F is not exact or F does not satisfies one of the equivalent conditions in (i) and (ii) above, still there is a bit complicated way to induce a natural functor between derived categories. This new functor is called the derived functor of F, but they will not produce a commutative diagram as in the above lemma. However, derived functor encodes more information about objects of the abelian categories.

To ensure existence of a derived functors, we need to assume some kind of exactness of F. If F is left exact (resp., right exact), we generally get a right derived functor (resp., let derived functor)

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \quad (\text{ resp.,} \quad LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})\,).$$

Both constructions are similar, and we only discuss the case of left exact functor.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume that $\mathcal{A}$ has *enough injective* (meaning that, for any $A \in \mathcal{A}$, there is an injective object I of $\mathcal{A}$ together with a monomorphism $\iota : A \hookrightarrow I$ in $\mathcal{A}$). Let $\mathcal{I}_{\mathcal{A}} \subset \mathcal{A}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects of $\mathcal{A}$. Note that, $\mathcal{I}_{\mathcal{A}}$ is additive, but not necessarily abelian. However, the construction of homotopy category works for any additive category. Therefore, $K^*(\mathcal{I}_{\mathcal{A}})$ is defined, and is a triangulated category. Now the inclusion functor $\mathcal{I}_{\mathcal{A}} \hookrightarrow \mathcal{A}$ induces a natural exact functor $K^*(\mathcal{I}_{\mathcal{A}}) \longrightarrow K^*(\mathcal{A})$, and composing it with the exact functor $Q_{\mathcal{A}} : K^+(\mathcal{A}) \to D^+(\mathcal{A})$, we get a natural exact functor $K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$.

Proposition 3.1.2. *The functor* $Q_{\mathcal{A}} : K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$ *induces a natural equivalence of categories* $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$.

Then we have the following diagram

$$\begin{tikzcd} K^+(\mathcal{I}_{\mathcal{A}}) \arrow[r, hook] \arrow[dr, "\iota"] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ & D^+(\mathcal{A}) \arrow[ul, bend left, "\iota^{-1}"] & D^+(\mathcal{B})\,. \end{tikzcd}$$

where ι^{-1} is the quasi-inverse functor. Such a quasi-isomorphism (ι^{-1}) is obtained by choosing a complex of injective objects quasi-isomorphic to a given bounded below complex in $D^+(\mathcal{A})$. Note that, the functor $K(F)$ is well-defined at the level of homotopy category, because F is left exact and we are working with bounded below complexes.

Definition 3.1.3. The *right derived functor* of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ is the functor

$$RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1} : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})\,.$$

Proposition 3.1.4. *(i) There is a natural morphism of functors*

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}\,.$$

(ii) The right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ is an exact functor of triangulated categories.

(iii) Let $G : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ be an exact functor of triangulated categories. Then any morphism of functors

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}},$$

factorize through a unique morphism of functors $RF \longrightarrow G$.

Proof. (i) Let $A^\bullet \in K^+(\mathcal{A})$. Note that, $Q_{\mathcal{A}}(A^\bullet) = A^\bullet$. Let $I^\bullet := \iota^{-1}(A^\bullet) \in K^+(\mathcal{I}_{\mathcal{A}})$. Then the natural isomorphism of functors $\mathrm{Id}_{D^+(\mathcal{A})} \xrightarrow{\simeq} \iota \circ \iota^{-1}$ gives rise to a functorial isomorphism $A^\bullet \xrightarrow{\simeq} \iota(I^\bullet) \cong I^\bullet$. Since $I^\bullet$ is a complex of injective objects, the above isomorphism in $D^+(\mathcal{A})$ gives rise to a unique morphism $A^\bullet \longrightarrow I^\bullet$ in the homotopy category $K(\mathcal{A})$ by Proposition 3.1.13 (see below). Therefore, we have a functorial morphism

$$K(F)(A^\bullet) \longrightarrow K(F)(I^\bullet) = RF(A^\bullet),$$

which gives the required morphism of functors.

(ii) Note that, $K^+(\mathcal{I}_{\mathcal{A}})$ is a triangulated category and $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$ is an exact equivalence of categories. Then ι^{-1} being the adjoint of ι, it is also exact (c.f. Proposition 1.2.13). Now $RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1}$ being a composition of exact functors, is exact.

(iii) See [GM03, III.6.11].

□

One can rephrase the Proposition 3.1.4 as a universal property to define derived functor of a left exact functor as follow.

Definition 3.1.5 (Universal property of derived functor)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then the right derived functor of F, if it exists, is an exact functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ of triangulated categories such that

(i) there is a natural morphism of functors $Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}$, and
(ii) for any exact functor $G : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$, there is a natural bijection

$$\mathrm{Hom}(RF, G) \xrightarrow{\simeq} \mathrm{Hom}(Q_{\mathcal{B}} \circ F, G \circ Q_{\mathcal{A}}) .$$

Definition 3.1.6. Let $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ be a right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$, we define

$$R^i F(A^\bullet) := \mathcal{H}^i(RF(A^\bullet)) \in \mathcal{B} , \quad \forall\, i \in \mathbb{Z} .$$

The induced functors $R^i F : \mathcal{A} \longrightarrow \mathcal{B}$ given by composition

$$\mathcal{A} \hookrightarrow D^+(\mathcal{A}) \xrightarrow{R^i F} \mathcal{B}$$

are known as *higher derived functors* of F.

Given any $A \in \mathcal{A}$, choosing an injective resolution

$$A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

we see that, $R^i F(A) = \mathcal{H}^i(\cdots \to F(I^0) \to F(I^1) \to \cdots)$. In particular,

$$R^0 F(A) = \mathrm{Ker}(F(I^0) \to F(I^1)) = F(A),$$

since F is left exact.

Definition 3.1.7. An object $A \in \mathcal{A}$ is called *F-acyclic* if $R^i F(A) \cong 0$, for all $i \neq 0$.

Corollary 3.1.8. *With the above assumptions, any short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ give rise to a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to R^1 F(B) \to R^1 F(C) \to R^2(A) \to \cdots .$$

To see how it works, note that any short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ in $\mathcal{A}$ gives rise to a distinguished triangle $A \to B \to C \to A[1]$ in $D(\mathcal{A})$. Again any distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D(\mathcal{A})$ give rise to a long exact sequence of cohomologies

$$\cdots \to \mathcal{H}^{i-1}(C^\bullet) \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C^\bullet) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots .$$

Now the above corollary follows by considering the distinguished triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ in $D(\mathcal{B})$.

Example 3.1.9. Let $\mathcal{A}$ be an abelian category, and let Ab be the category of abelian groups. Consider the covariant functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \longrightarrow \mathrm{Ab}.$$

Clearly, $\mathrm{Hom}(A, -)$ is left exact. If $\mathcal{A}$ contains enough injectives (for example, if $\mathcal{A}$ is $\mathfrak{Mod}(\mathcal{O}_X)$ or $\mathfrak{QCoh}(X)$ for X a noetherian scheme; c.f., [Har77, Exercise III. 3.6]), then we define

$$\mathrm{Ext}^i(A, -) := H^i(R\,\mathrm{Hom}_{\mathcal{A}}(A, -))\,. \tag{3.1.10}$$

Proposition 3.1.11. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A, B \in \mathcal{A}$. Then there are natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]), \quad \forall\ i,$$

where A and B are considered as complexes in $D(\mathcal{A})$ concentrated at degree 0 place.

Proof. Let

$$B \to I^0 \to I^1 \to \cdots$$

be an injective resolution of B in $\mathcal{A}$. It follows from the construction of right derived functor that $R\,\mathrm{Hom}(A, B)$, as an object of $D^+(\mathrm{Ab})$, is isomorphic to the complex

$$\mathrm{Hom}(A, I^\bullet): \quad \mathrm{Hom}(A, I^0) \to \mathrm{Hom}(A, I^1) \to \mathrm{Hom}(A, I^2) \to \cdots .$$

Therefore, $\mathrm{Ext}^i(A, B) = H^i(\mathrm{Hom}(A, I^\bullet))$.

Note that, a morphism $f \in \mathrm{Hom}(A, I^i)$ is a *cycle* (i.e., $f \in \mathrm{Ker}(\mathrm{Hom}(A, I^i) \to \mathrm{Hom}(A, I^{i+1}))$) if and only if f defines a morphism of complexes

$$f : A \longrightarrow I^\bullet[i].$$

This morphism of complexes f is homotopically trivial if and only if f is a *boundary* (i.e., $f \in \mathrm{image}(\mathrm{Hom}(A, I^{i-1}) \to \mathrm{Hom}(A, I^i))$). Therefore, we have

$$\mathrm{Ext}^i(A, B) \cong H^i(\mathrm{Hom}(A, I^\bullet)) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]).$$

Since $I^\bullet$ is a complex of injective objects, by Lemma 3.1.12 (see below) we have

$$\mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, I^\bullet[i]).$$

Since $B \cong I^\bullet$ in $D^+(\mathcal{A})$, we have $\mathrm{Ext}^i(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$. ∎

Lemma 3.1.12. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $A^\bullet, I^\bullet \in \mathcal{K}om^+(\mathcal{A})$ be two bounded below complexes such that I^i is injective, for all i. Then*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

Proof. Clearly, there is a natural morphism

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet)\,.$$

We need to show that given any morphism $f \in \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet)$ represented by a roof of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr, "\psi"] & \\ A^\bullet \arrow[rr, dashed, "\psi'"'] \arrow[rr, phantom, "\phi", shift left=3ex] & & I^\bullet \end{tikzcd}$$

there is a unique morphism of complexes $\psi' : A^\bullet \to I^\bullet$ making the above diagram commutative in the homotopy category $K(\mathcal{A})$. Now the result follows from the Proposition 3.1.13 below. $\square$

Proposition 3.1.13. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for any quasi-isomorphism $\phi : B^\bullet \to A^\bullet$ in $\mathcal{K}om^+(\mathcal{A})$, the induced map*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, I^\bullet),$$

obtained by precomposing with ϕ, is bijective.

Sketch of a proof. Complete the morphism $\phi : B^\bullet \to A^\bullet$ to a distinguished triangle in the triangulated
category $K^+(\mathcal{A})$. Applying the functor $\mathrm{Hom}(-, I^\bullet)$ and then taking the associated long exact $\mathrm{Hom}(-, I^\bullet)$-sequence, we see that it is enough to show that $\mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, I^\bullet) = 0$, for any acyclic complex $E^\bullet$.

Next, we take any morphism of complexes $f : E^\bullet \to I^\bullet$, and construct a homotopy between f and the zero morphism. This can be done by induction. Assume that $h^i : E^i \to I^{i-1}$ is constructed by induction. If h^j is constructed for all $j \leq i$, then the morphism

$$f^i - d^{i-1}_{I^\bullet} \circ h^i : E^i \longrightarrow I^i$$

factors through $E^i/E^{i-1} \longrightarrow I^i$. Since I^i is injective, this lifts to a morphism $h^{i+1} : E^{i+1} \longrightarrow I^i$ so that $f^i - d^{i-1}_{I^\bullet} \circ h^i = h^{i+1} \circ d^i_{E^\bullet}$. Thus the induction works! $\square$

Remark 3.1.14. In practice, we need to deal with many important abelian categories without enough injective objects, or sometimes the functor F is defined at the level of homotopy categories only. However, one can still construct derived functors in that setup under certain assumption. Let us explain briefly.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

Case I. F is defined only at the level of homotopy category: let

$$F : K^+(\mathcal{A}) \longrightarrow K(\mathcal{B}) \tag{3.1.15}$$

be an exact functor of triangulated categories. Then the right derived functor

$$RF : D^+(\mathcal{A}) \longrightarrow D(\mathcal{B}) \tag{3.1.16}$$

of F satisfying the properties (i)–(iii) of Proposition 3.1.4 exists if there is a triangulated subcategory $K_F \subset K^+(\mathcal{A})$ *adapted to* F, meaning that K_F satisfies the following conditions:

(i) if $A^\bullet \in K_F$ is acyclic, then so is $F(A^\bullet)$, and
(ii) any $A^\bullet \in K^+(\mathcal{A})$ is quasi-isomorphic to a complex in K_F.

Roughly, with the above hypotheses (i)–(iii), we may localize the subcategory $K^+(K_F)$ with respect to the quasi-isomorphisms of objects from K_F to produce an equivalence of categories $K^+(K_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A})$. Moreover, the functor $K(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ give rise to a functor $K^+(K_F)_{qis} \longrightarrow K^+(\mathcal{B})$. Then by choosing a quasi-inverse of the above equivalence, we get the required derived functor $D^+(\mathcal{A}) \to D(\mathcal{B})$.

Case II. $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, but $\mathcal{A}$ has not enough injectives. In this situation, we may construct the right derived functor of F by looking at the F*-adapted class of objects* $\mathcal{I}_F \subset \mathcal{A}$, which is defined by the following properties.

(a) $\mathcal{I}_F$ is stable under finite sums,
(b) if $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^i \in \mathcal{I}_F$, for all i, then $F(A^\bullet)$ is acyclic, and
(c) any object of $\mathcal{A}$ can be embedded inside an object of $\mathcal{I}_F$.

Let $K^+(\mathcal{I}_F)_{qis}$ be the localization of $K^+(\mathcal{I}_F)$ by quasi-isomorphism of complexes with objects from $\mathcal{I}_F$. Then the above hypotheses (a)–(c) gives rise to an equivalence of categories

$$\iota_q : K^+(\mathcal{I}_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A}). \tag{3.1.17}$$

Then $K(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ induces a functor $K(F)_{qis} : K^+(\mathcal{I}_F)_{qis} \to K^+(\mathcal{B})$. Now choosing a quasi-inverse ι_q^{-1} of (3.1.17) and composing with $Q_{\mathcal{B}} \circ K(F)_{qis}$, we get the required right derived functor $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ of F as discussed before.

The definition of Ext group as given in Example (3.1.9) can be generalized for complexes as follow. Given $A^\bullet, B^\bullet \in \mathcal{K}om(\mathcal{A})$, let $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ be the complex defined by

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(A^j, B^{i+j}) \tag{3.1.18}$$

with the differential

$$d(f) := d_B \circ f - (-1)^i f \circ d_A, \quad \forall\, f \in \mathrm{Hom}(A^*, B^{i+*})\,.$$

$$\begin{tikzcd} A^j \arrow[r, "f_i^j"] \arrow[d, "d_A^j"'] & B^{i+j} \arrow[d, "d_B^{i+j}"] \\ A^{j+1} \arrow[r, "f_i^{j+1}"] & B^{i+j+1} \end{tikzcd}$$

The complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is known as the *internal hom of $A^\bullet$ into $B^\bullet$*.

Note that, any $A^\bullet \in \mathcal{K}om(\mathcal{A})$ gives rise to an exact functor

$$\mathrm{Hom}^\bullet(A^\bullet, -) : K^+(\mathcal{A}) \longrightarrow K(\mathbf{A}b), \quad B^\bullet \longmapsto \mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\,. \tag{3.1.19}$$

Let $\mathcal{I} \subset K^+(\mathcal{A})$ be the full triangulated subcategory, whose objects are complexes $I^\bullet$ with I^i injective object of $\mathcal{A}$, for all i. Then $\mathcal{I}$ is F-adapted, where $F = \mathrm{Hom}^\bullet(A^\bullet, -)$, as defined in Remark 3.1.14. Then the right derived functor

$$R\,\mathrm{Hom}^\bullet(A^\bullet, -) : D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.20}$$

of $\mathrm{Hom}^\bullet(A^\bullet, -)$ exists. Then we define

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := H^i\big(R\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\big)\,, \quad \forall\, i\,. \tag{3.1.21}$$

Now the proof of Proposition 3.1.11 can be modified to prove the following.

Theorem 3.1.22. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A^\bullet, B^\bullet \in \mathcal{K}om(\mathcal{A})$ be two bounded (or bounded below) complexes. Then there are natural isomorphisms of abelian groups*

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]), \quad \forall\ i.$$

It follows from Theorem 3.1.22 that the abelian group $\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet)$ depends on the "isomorphism classes" of $A^\bullet$ and $B^\bullet$ in the derived category, not on the complexes. If $A_1^\bullet \stackrel{\sim}{\longrightarrow} A_2^\bullet$ is a quasi-isomorphism of complexes, then the induced morphism

$$R\,\mathrm{Hom}^\bullet(A_1^\bullet, B^\bullet) \longrightarrow R\,\mathrm{Hom}^\bullet(A_2^\bullet, B^\bullet)$$

is an isomorphism in $D(\mathbf{Ab})$, because their cohomologies are isomorphic. Therefore, the functor $\mathrm{Hom}^\bullet(-, B^\bullet)$ descends to the derived category to give a bifunctor

$$D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,, \tag{3.1.23}$$

which is exact in each variable.

Definition 3.1.24. An abelian category $\mathcal{A}$ is said to have *enough projectives* if for each object $A \in \mathcal{A}$ there is a projective object P in $\mathcal{A}$ together with an epimorphism $A \to P$ in $\mathcal{A}$.

If the abelian category $\mathcal{A}$ has enough projectives, then for any complex $B^\bullet \in \mathcal{K}om(\mathcal{A})$, the left exact functor

$$\mathrm{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathbf{Ab})$$

admits a right derived functor

$$R\,\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab})\,.$$

One can check that, this depends only on $B^\bullet$ as an object of derived category. Therefore, it defines a bifunctor

$$D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.25}$$

If $\mathcal{A}$ has enough injectives and enough projectives, both bifunctors in (3.1.23) and (3.1.25) give rise to the same bifunctor

$$R\operatorname{Hom}^\bullet(-,-) : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.26}$$

Remark 3.1.27. If $\mathcal{A}$ has enough injectives, but not necessarily have enough projectives, using (3.1.23) we can get the derived functor

$$R\operatorname{Hom}^\bullet(-,B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.28}$$

Note that, thanks to Theorem 3.1.22, composition of morphisms in the derived category can be used to define composition for Ext groups:

$$\operatorname{Ext}^i_{\mathcal{A}}(A^\bullet,B^\bullet) \times \operatorname{Ext}^j_{\mathcal{A}}(B^\bullet,C^\bullet) \longrightarrow \operatorname{Ext}^{i+j}_{\mathcal{A}}(A^\bullet,C^\bullet)\,, \tag{3.1.29}$$

for all $A^\bullet, B^\bullet, C^\bullet \in D^+(\mathcal{A})$. This follows because

$$\operatorname{Ext}^j_{\mathcal{A}}(B^\bullet,C^\bullet) \cong \operatorname{Hom}_{D(\mathcal{A})}(B^\bullet, C^\bullet[j]) \cong \operatorname{Hom}_{D(\mathcal{A})}(B^\bullet[i], C^\bullet[i+j])\,.$$

Proposition 3.1.30 (Grothendieck's composite functor theorem)**.** *Let $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories. Suppose that there are adapted classes $\mathcal{I}_{F_1} \subset \mathcal{A}$ and $\mathcal{I}_{F_2} \subset \mathcal{B}$ for F_1 and F_2, respectively, such that $F_1(\mathcal{I}_{F_1}) \subseteq \mathcal{I}_{F_2}$. Then the derived functors $RF_1 : D^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$, $RF_2 : D^+(\mathcal{B}) \to D^+(\mathcal{C})$ and $R(F_2 \circ F_1) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ exists, and there is a natural isomorphism of functors*

$$R(F_2 \circ F_1) \xrightarrow{\simeq} RF_2 \circ RF_1\,.$$

Proof. Clearly RF_1 and RF_2 exists by given assumptions (c.f., Remark 3.1.14). Since $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, we see that $\mathcal{I}_{F_1}$ is $(F_2 \circ F_1)$-adapted. Therefore, $R(F_2 \circ F_1)$ exists. Then the natural morphism of functors

$$R(F_2 \circ F_1) \longrightarrow RF_2 \circ RF_1 \tag{3.1.31}$$

follows from the universal property of derived functor $R(F_2 \circ F_1)$ (c.f., Definition 3.1.5). To see (3.1.31) is an isomorphism, given any complex $A^\bullet \in D^+(\mathcal{A})$ we choose a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$ quasi-isomorphic to $A^\bullet$. Then we have

$$R(F_2 \circ F_1)(A^\bullet) \cong (K(F_2) \circ K(F_1))(I^\bullet) \tag{3.1.32}$$

and

$$RF_2(RF_1(A^\bullet)) \cong RF_2(K(F_1)(I^\bullet)) \cong K(F_2)(K(F_1)(I^\bullet))\,. \tag{3.1.33}$$

Now it follows from the natural isomorphism between (3.1.32) and (3.1.33) that the morphism of functor in (3.1.31) is an isomorphism. $\square$

Remark 3.1.34. If both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, then the hypotheses of the above Proposition 3.1.30 are satisfied if $F_1(\mathcal{I}_{\mathcal{A}}) \subset \mathcal{I}_{\mathcal{B}}$.

4. Serre functor

4.1. Abstract Serre functor.

Let k be a field. Let $\mathcal{A}$ be a k-linear additive category.

Definition 4.1.1. A *Serre functor* on $\mathcal{A}$ is a k-linear equivalence of categories

$$S : \mathcal{A} \longrightarrow \mathcal{A}$$

such that for any two objects $A, B \in \mathcal{A}$, there is a natural k-linear isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(B, S(A))^*,$$

which is functorial in both A and B. We write the induced k-bilinear pairing as

$$\mathrm{Hom}(B, S(A)) \times \mathrm{Hom}(A, B) \longrightarrow k\,, \quad (f, g) \longmapsto \langle f|g\rangle\,.$$

Proposition 4.1.2. *Let $\mathcal{A}$ be a k-linear additive category together with a Serre functor $S : \mathcal{A} \to \mathcal{A}$. Then for any $A, B \in \mathcal{A}$, the following diagram commutes.*

(4.1.3)
$$\begin{tikzcd}
\mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] \arrow[d, "S_{A,B}"'] & \mathrm{Hom}(B,S(A))^* \\
\mathrm{Hom}(S(A),S(B)) \arrow[r, "\eta_{S(A),S(B)}"] \arrow[ur, dashed, "\exists\, \eta^*_{B,S(A)}"] & \mathrm{Hom}(S(B),S^2(A))^* \arrow[u, "S^*_{B,S(A)}"']
\end{tikzcd}$$

Proof. By abuse of notation, we denote by $\eta^*_{B,S(A)}$ the composite k-linear homomorphism

$$\eta^*_{B,S(A)} : \mathrm{Hom}(S(A), S(B)) \hookrightarrow \mathrm{Hom}(S(A), S(B))^{**} \xrightarrow{\eta^*_{B,S(A)}} \mathrm{Hom}(B, S(A))^*.$$

Therefore, it suffices to show that both upper and lower triangles in (4.1.3) commutes. Note that, commutativity of upper triangle is equivalent to

$$\langle f|g\rangle = \langle S_{A,B}(g)|f\rangle, \quad \forall\; f \in \mathrm{Hom}(B, S(A)),\; g \in \mathrm{Hom}(A, B)\,.$$

Applying functoriality of η in the second variable, we have the following commutative diagram.

(4.1.4)
$$\begin{tikzcd}
\mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] & \mathrm{Hom}(B,S(A))^* \\
\mathrm{Hom}(B,B) \arrow[r, "\eta_{B,B}"] \arrow[u, "-\circ g"] & \mathrm{Hom}(B,S(B))^* \arrow[u, "(S(g)\circ -)^*"']
\end{tikzcd}$$

Applying commutativity of (4.1.4) to $\mathrm{Id}_B \in \mathrm{Hom}(B, B)$ we have $\langle f|g\rangle = \langle S(g)\circ f|\,\mathrm{Id}\rangle$. Applying functoriality of η in the first variable, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(B,B) \arrow[r, "\eta_{B,B}"] \arrow[d, "f\circ -"'] & \mathrm{Hom}(B,S(B))^* \arrow[d, "(-\circ f)^*"] \\ \mathrm{Hom}(B,S(A)) \arrow[r, "\eta_{B,S(A)}"] & \mathrm{Hom}(S(A),S(B))^*\,, \end{tikzcd} \tag{4.1.5}$$

which gives $\langle (S(g)\circ f) | \,\mathrm{Id}_B\rangle = \langle S(g)|f\rangle$. This completes the proof. □

Remark 4.1.6. In order to avoid trouble with identifying $\mathrm{Hom}(A,B)$ with its double dual $\mathrm{Hom}(A,B)^{**}$, we always assume that a k-linear additive category $\mathcal{A}$ has finite dimensional Hom's (i.e., $\dim_k \mathrm{Hom}_{\mathcal{A}}(A,B) < \infty$, for all $A, B \in \mathrm{Ob}(\mathcal{A})$).

Lemma 4.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear additive categories with finite dimensional Hom's. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively, then any k-linear equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ commutes with Serre functors (i.e., there is an isomorphism of functors $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$).*

Proposition 4.1.8. *Let $\mathcal{A}$ be a k-linear additive category. Then any two Serre functors on $\mathcal{A}$ are isomorphic.*

4.2. **Serre duality in $D^b(X)$.** Let X be a smooth projective k-variety of dimension $n \geq 1$. Note that, for any locally free coherent sheaf E on X, the functor

$$-\otimes E : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(X)\,, \quad F \longmapsto F \otimes E$$

is exact. Let ω_X be the dualizing sheaf on X. Let $D^*(X) = D^*(\mathfrak{Coh}(X))$, where $* \in \{\emptyset, b, -, +\}$. Consider the composite functor

$$S_X : D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X), \tag{4.2.1}$$

where $[n] : D^*(X) \to D^*(X)$ is the n-th shift functor given by sending a complex $E^\bullet$ to $E^\bullet[n]$. Since both the functors $\omega_X \otimes -$ and $[n]$ are exact, their composite functor $S_X := \omega_X \otimes (-)[n]$ is exact.

Theorem 4.2.2 (Grothendieck-Serre duality). *Let X be a smooth projective variety over a field k. Then the functor $S_X : D^b(X) \longrightarrow D^b(X)$ as defined in (4.2.1) is a Serre functor in the sense of Definition 4.1.1.*

Proof. Given any two objects $E^\bullet, F^\bullet \in D^b(X)$, we need to give an isomorphism of k-vector spaces

$$\eta_{E^\bullet,F^\bullet} : \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet))^* \tag{4.2.3}$$

which is functorial in both $E^\bullet$ and $F^\bullet$. Thanks to Theorem 3.1.22, we have

$$\mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i]) = \mathrm{Ext}^i(E^\bullet, F^\bullet), \quad \forall\, i\,.$$

Since X is smooth and projective, choosing a resolution by complex of locally free sheaves on X, we may assume that E^i is locally free, for all i. Then we have functorial isomorphisms

$$\begin{aligned}\operatorname{Hom}^i(E^\bullet, F^\bullet) &= \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(E^j, F^{i+j}) = \bigoplus_{j\in\mathbb{Z}} H^0(X, \mathcal{H}om(E^j, F^{i+j}))\\ &\cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}^n(F^{i+j}, E^j \otimes \omega_X)^*, \quad \text{by classical Serre duality theorem.}\\ &\cong \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_{D^b(X)}(F^{i+j}, E^j \otimes \omega_X[n])^*, \quad \text{by Proposition 3.1.11.}\\ &\cong \operatorname{Hom}^{n-i}(F^\bullet, E^\bullet \otimes \omega_X)^*.\end{aligned}$$

Since for any two complexes $A^\bullet, B^\bullet$, we have

$$\operatorname{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := H^i\big(R\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)\big)\,, \quad \forall\, i\,,$$

the theorem follows. □

Remark 4.2.4. Theorem 4.2.2 is a special case of Grothendieck-Verdier duality (c.f. Section §5.6). We shall see some interesting applications of the Serre functor $\omega_X \otimes (-)[n]$ on $D^b(X)$ in Section §7. For this, we need concept of local Hom complex, and spectral sequences to be explained in the next two sections.

5. Derived functors in algebraic geometry

5.1. **Local $\mathcal{H}om^\bullet$ complex.** Let X be a noetherian scheme. For $E \in \mathfrak{QCoh}(X)$, the functor

$$\mathcal{H}om(E, -) : \mathfrak{QCoh}(X) \longrightarrow \mathfrak{QCoh}(X), \quad F \mapsto \mathcal{H}om(E, F)\,, \tag{5.1.1}$$

is left exact. Moreover, $\mathcal{H}om(E, F) \in \mathfrak{Coh}(X)$ if both E, F are coherent. Since $\mathfrak{QCoh}(X)$ has enough injectives (c.f., [Har77, Chapter III, Exercise 3.6]), its right derived functor

$$R\,\mathcal{H}om(E, -) : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X)) \tag{5.1.2}$$

exists. Then for any $E, F \in \mathfrak{QCoh}(X)$ and any integer i, we define

$$\mathcal{E}xt^i(E, F) := R^i\,\mathcal{H}om(E, F) := \mathcal{H}^i(R\,\mathcal{H}om(E, F)) \in \mathfrak{QCoh}(X)\,. \tag{5.1.3}$$

If $E \in \mathfrak{Coh}(X)$, the above definition is local in the sense that its stalk at $x \in X$ can be computed as

$$\mathcal{E}xt^i(E, F)_x = \operatorname{Ext}^i_{\mathcal{O}_{X,x}}(E_x, F_x)\,, \tag{5.1.4}$$

which follows from commutativity of the following diagram.

$$
\begin{tikzcd}
\mathfrak{QCoh}(X) \arrow[r, "\mathcal{H}om(E,-)"] \arrow[d] & \mathfrak{QCoh}(X) \arrow[d] \\
\mathrm{Mod}(\mathcal{O}_{X,x}) \arrow[r, "\mathrm{Hom}(E_x,-)"] & \mathrm{Mod}(\mathcal{O}_{X,x})\,.
\end{tikzcd}
\tag{5.1.5}
$$

Note that, $\mathcal{E}xt^i(E,F) \in \mathfrak{Coh}(X)$ whenever both $E, F \in \mathfrak{Coh}(X)$.

When $E \in \mathfrak{Coh}(X)$, the functor (5.1.2) restricts to the bounded below derived category of coherent sheaves

$$R\,\mathcal{H}om(E,-) : D^+(X) \longrightarrow D^+(X)\,. \tag{5.1.6}$$

Since for a non-regular local ring A, the groups $\mathrm{Ext}^i_A(M,-)$ can be non-trivial even for $i \gg 0$, only for non-singular schemes X, the above functor $R\,\mathcal{H}om(E,-)$ restricts to $D^b(X)$, the bounded derived category of coherent sheaves on X.

As discussed before, the above construction easily generalizes for bounded above complexes of coherent sheaves $E^\bullet \in D^-(X)$. For this, we note that the following functor is exact.

$$\mathcal{H}om^\bullet(E^\bullet,-) : K^+(\mathfrak{QCoh}(X)) \longrightarrow K^+(\mathfrak{QCoh}(X))$$

given by sending a complex $F^\bullet \in K^+(\mathfrak{QCoh}(X))$ to the complex $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$, where

$$\mathcal{H}om^i(E^\bullet, F^\bullet) := \prod_{p\in\mathbb{Z}} \mathcal{H}om(E^p, F^{i+p})$$

and the differentials are given by $d^i = d_{E^\bullet} - (-1)^i d_{F^\bullet}$, for all $i \in \mathbb{Z}$. The following lemma follows form corresponding local statement for modules over a ring.

Lemma 5.1.7. *Let $F^\bullet \in D^-(X)$ be a complex of injective sheaves. If $F^\bullet$ or $E^\bullet \in K^+(\mathfrak{QCoh}(X))$ is acyclic, then $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$ is acyclic.*

The above Lemma 5.1.7 applied to the class

$$\mathcal{I} := \{I^\bullet \in K^+(\mathfrak{QCoh}(X)) : I^i \text{ is injective } \mathcal{O}_X\text{-module}\}$$

shows that $\mathcal{I}$ is adapted to the functor $\mathcal{H}om^\bullet(E^\bullet,-)$ (see Remark 3.1.14), and hence, the right derived functor

$$R\,\mathcal{H}om^\bullet(E^\bullet,-) : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X)) \tag{5.1.8}$$

exists. Note that, we are working with $\mathfrak{QCoh}(X)$, because $\mathfrak{Coh}(X)$ has not enough injectives. Similarly, to see that the functor

$$\mathcal{H}om^\bullet(-,F^\bullet) : D^-(\mathfrak{QCoh}(X))^{\mathrm{op}} \longrightarrow D^+(\mathfrak{QCoh}(X))$$

descends to the derived category for any $F^\bullet \in D^+(\mathfrak{QCoh}(X))$. Therefore, we get a bifunctor

$$R\mathcal{H}om^\bullet(-,-) : D^-(\mathfrak{QCoh}(X))^{\mathrm{op}} \times D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X))\,. \tag{5.1.9}$$

This enables us to define

$$\mathcal{E}xt^i(E^\bullet, F^\bullet) := R^i\,\mathcal{H}om^\bullet(E^\bullet, F^\bullet) := \mathcal{H}^i(R\,\mathcal{H}om^\bullet(E^\bullet, F^\bullet)) \in \mathfrak{QCoh}(X)\,, \quad \forall\, i. \tag{5.1.10}$$

Proposition 5.1.11. *Let X be a non-singular noetherian k-scheme. Then any bounded complex $E^\bullet \in D^b(X)$ is isomorphic to a bounded complex $\mathcal{E}^\bullet \in D^b(X)$ of locally free coherent sheaves of $\mathcal{O}_X$-modules on X.*

Proof. Since X is a noetherian non-singular k-scheme, $\mathfrak{Coh}(X)$ has enough projectives, So any $E^i \in \mathfrak{Coh}(X)$ admits a finite resolution

$$0 \to F_i^\ell \to F_i^{\ell-1} \to \cdots \to F_i^0 \to E \to 0,$$

with F_i^j a locally free coherent sheaves of $\mathcal{O}_X$-modules on X, and $\ell \leq \dim_k(X)$.

□

5.2. Trace map.

5.3. **Derived tensor product.** Let X be a projective k-scheme. Then any coherent sheaf $E \in \mathfrak{Coh}(X)$ admits a resolution (not necessarily finite) by locally free coherent sheaves of $\mathcal{O}_X$-modules

$$\mathcal{E}^\bullet \to E. \tag{5.3.1}$$

Let $F \in \mathfrak{Coh}(X)$. Since tensor product functor $F \otimes - : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X)$ is right exact, for any bounded above acyclic complex $E^\bullet$ of locally free coherent sheaves of $\mathcal{O}_X$-modules, $F \otimes E^\bullet$ is also acyclic.

5.4. Defived pullback.

5.5. Compatibilities.

5.6. Grothendieck-Verdier duality.

Theorem 5.6.1 (Grothendieck-Verdier duality). *Let $f : X \longrightarrow Y$ be a morphism of smooth schemes over a field k of relative dimension $\dim(f) := \dim(X) - \dim(Y)$. Let*

$$\omega_f := \omega_X \otimes f^*\omega_Y \tag{5.6.2}$$

be the relative dualizing sheaf of f. Then for any $F^\bullet \in D^b(X)$ and $E^\bullet \in D^b(Y)$, there is a a functorial isomorphism

$$Rf_*R\,\mathcal{H}om(F^\bullet, Lf^*(E^\bullet)\otimes\omega_f[\dim(f)]) \xrightarrow{\simeq} R\,\mathcal{H}om(Rf_*F^\bullet, E^\bullet)\,. \tag{5.6.3}$$

6. SPECTRAL SEQUENCE

6.1. **What is it?** In this subsection, we explain how spectral sequence occur when we compose two derived functors. Let $\mathcal{A}$ be an abelian category.

Definition 6.1.1. A *spectral sequence in* $\mathcal{A}$ is given by a collection of objects

$$(E_r^{p,q}, E^n), \;\; n, p, q, r \in \mathbb{Z}, \; r \geq 1$$

and morphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

satisfying that the following conditions.

(i) $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$, for all p, q, r. This yields a complex $E_r^{p+\bullet r, q-\bullet r+\bullet}$, for all $r \geq 1$.
(ii) There are isomorphisms

$$E_{r+1}^{p,q} \cong H^0(E_r^{p+\bullet r, q-\bullet r+\bullet}),$$

which are part of the data.
(iii) For any (p, q), there is an r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$, for all $r \geq r_0$. In particular, $E_r^{p,q} \cong E_{r_0}^{p,q}$, for all $r \geq r_0$. This object is denoted by $E_\infty^{p,q}$.
(iv) There is a decreasing filtration

$$\cdots \subset F^{p+1}E^n \subset F^pE^n \subset F^{p-1}E^n \subset \cdots \subset F^0E^n := E^n$$

such that $\bigcap_{p\in\mathbb{Z}} F^pE^n = 0$ and $\bigcup_{p\in\mathbb{Z}} F^pE^n = E^n$, and isomorphisms

$$E_\infty^{p,q} \cong F^pE^{p+q}/F^{p+1}E^{p+q}.$$

Let us try to visualize a spectral sequence. In page E_1, we have the following data.

$$\begin{array}{ccccccccc}
E_1^{p-2,q+1} & \longrightarrow & E_1^{p-1,q+1} & \longrightarrow & E_1^{p,q+1} & \longrightarrow & E_1^{p+1,q+1} & & \cdots \\
\\
E_1^{p-2,q} & \longrightarrow & E_1^{p-1,q} & \longrightarrow & E_1^{p,q} & \longrightarrow & E_1^{p+1,q} & & \cdots \\
\\
E_1^{p-2,q-1} & \longrightarrow & E_1^{p-1,q-1} & \longrightarrow & E_1^{p,q-1} & \longrightarrow & E_1^{p+1,q-1} & & \cdots
\end{array}$$

In page E_2, we have the following data.

$$\begin{tikzcd}
E_2^{p-2,q+1} \arrow[drr] & E_2^{p-1,q+1} \arrow[drr] & E_2^{p,q+1} \arrow[drr] & E_2^{p+1,q+1} & \cdots \\
E_2^{p-2,q} \arrow[drr] & E_2^{p-1,q} \arrow[drr] & E_2^{p,q} \arrow[drr] & E_2^{p+1,q} & \cdots \\
E_2^{p-2,q-1} & E_2^{p-1,q-1} & E_2^{p,q-1} & E_2^{p+1,q-1} & \cdots
\end{tikzcd}$$

In some sense, the condition (iv) says that the objects $E_r^{p,q}$ converges towards a subquotient of certain filtration of E^n. Usually objects of one layer, say $E_r^{p,q}$ with r fixed, are given, and objects of the next layer can be obtained using (ii). It is enough to give objects $E_r^{p,q}$ with $r \geq m$, for some m; the information is just the same. We express the spectral sequence by writing

$$E_r^{p,q} \Longrightarrow E^{p+q}.$$

In most of the applications, only $E_r^{p,q}$ are given for $r \geq 2$, and in most of the cases, we don't need to go beyond page E_2 or E_3.

Definition 6.1.2. A *double complex* $K^{\bullet,\bullet}$ is given by the following data: for each pair of integers (i,j), an object $K^{i,j} \in \mathcal{A}$ and morphisms

$$d_I^{i,j} : K^{i,j} \longrightarrow K^{i+1,j} \quad \text{and} \quad d_{II}^{i,j} : K^{i,j} \longrightarrow K^{i,j+1}$$

such that

$$d_I^2 = d_{II}^2 = d_I d_{II} + d_{II} d_I = 0.$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & K^{i-1,j-1} \arrow[r, "d_I^{i-1,j-1}"] \arrow[d, "d_{II}^{i-1,j-1}"] & K^{i,j-1} \arrow[r, "d_I^{i,j-1}"] \arrow[d, "d_{II}^{i,j-1}"] & K^{i+1,j-1} \arrow[r, "d_I^{i+1,j-1}"] \arrow[d, "d_{II}^{i+1,j-1}"] & \cdots \\
\cdots \arrow[r] & K^{i-1,j} \arrow[r, "d_I^{i-1,j}"] \arrow[d, "d_{II}^{i-1,j}"] & K^{i,j} \arrow[r, "d_I^{i,j}"] \arrow[d, "d_{II}^{i,j}"] & K^{i+1,j} \arrow[r, "d_I^{i+1,j}"] \arrow[d, "d_{II}^{i+1,j}"] & \cdots \\
\cdots \arrow[r] & K^{i-1,j+1} \arrow[r, "d_I^{i-1,j+1}"] \arrow[d, "d_{II}^{i-1,j+1}"] & K^{i,j+1} \arrow[r, "d_I^{i,j+1}"] \arrow[d, "d_{II}^{i,j+1}"] & K^{i+1,j+1} \arrow[r, "d_I^{i+1,j+1}"] \arrow[d, "d_{II}^{i+1,j+1}"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The associated *total complex* $K^\bullet := \operatorname{tot}(K^{\bullet,\bullet})$ is defined by $K^n := \bigoplus_{i+j=n} K^{i,j}$ with differentials $d = d_I + d_{II}$.

The total complex $K^\bullet = \operatorname{tot}(K^{\bullet,\bullet})$ admits a natural decreasing filtration $\{F^\ell K^n\}_\ell$ given by

$$F^\ell K^n := \bigoplus_{j \geq \ell} K^{n-j,j}\,, \tag{6.1.3}$$

which satisfies $d_I(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n. Due to symmetry of the situation, there is another such natural filtration.

Example 6.1.4. The complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is an example of a total complex of the double complex $K^{i,j} := \mathrm{Hom}(A^{-i}, B^j)$ together with the differentials $d_I = (-1)^{j-i+1} d_A$ and $d_{II} = d_B$ (there are different sign conventions in the literature; however one can choose one sign convention, and final conclusion would be the same).

Definition 6.1.5. A *filtered complex* is a complex $K^\bullet$ together with a decreasing filtration

$$\cdots \subset F^\ell K^n \subset F^{\ell-1} K^n \subset \cdots \subset F^0 K^n := K^n, \ \ \forall\, n,$$

such that $d^n(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F^\ell(K^{n-1}) \arrow[r] \arrow[u, hook] & F^\ell(K^n) \arrow[r] \arrow[u, hook] & F^\ell(K^{n+1}) \arrow[r] \arrow[u, hook] & \cdots \\
\cdots \arrow[r] & F^{\ell+1}(K^{n-1}) \arrow[r] \arrow[u, hook] & F^{\ell+1}(K^n) \arrow[r] \arrow[u, hook] & F^{\ell+1}(K^{n+1}) \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

Consider the filtrations $\{F^\ell K^n\}_\ell$ of the total complex $K^\bullet = \operatorname{tot}(K^{\bullet,\bullet})$ in (6.1.3). The associated graded objects

$$\mathrm{gr}^\ell(K^n) := F^\ell(K^n)/F^{\ell+1}(K^n) = K^{n-\ell,\ell}$$

forms a complex $K^{\bullet,\ell}[-\ell]$ (up to a global sign $(-1)^\ell$). Hence $\mathcal{H}^k(\mathrm{gr}^\ell(K^\bullet)) = \mathcal{H}^{k-\ell}(K^{\bullet,\ell})$, for all ℓ, and the cohomology of the complex $\mathcal{H}^n_I(K^{\bullet,\bullet}) := (\mathcal{H}^n(K^{\bullet,j}))_{j\in\mathbb{Z}}$, with respect to d_{II}, gives $\mathcal{H}^\ell_{II}(\mathcal{H}^{k-\ell}_I(K^{\bullet,\bullet}))$.

Assuming the following finiteness condition: for each n, there is $\ell_+(n)$ and $\ell_-(n)$ such that $F^\ell K^n = 0$, for all $\ell \geq \ell_+(n)$ and $F^\ell K^n = K^n$, for all $\ell \leq \ell_-(n)$, one can show that any filtered complex gives rise to a spectral sequence. In case of double complex, we have the following.

Proposition 6.1.6. *Let $K^{\bullet,\bullet}$ be a double complex such that for any n, $K^{n-\ell,\ell} = 0$, for $|\ell| \gg 0$. Then there is a spectral sequence*

$$E_2^{p,q} := \mathcal{H}^p_{II}\mathcal{H}^q_I(K^{\bullet,\bullet}) \Longrightarrow \mathcal{H}^{p+q}(K^\bullet).$$

Definition 6.1.7. Let $A^\bullet \in K^+(\mathcal{A})$. A *Cartan-Eilenberg resolution* of $A^\bullet$ is a double complex $C^{\bullet,\bullet}$ together with a morphism of complexes $A^\bullet \longrightarrow C^{\bullet,0}$ such that

(i) $C^{i,j} = 0$, for $j < 0$,

(ii) the sequences $A^n \to C^{n,0} \to C^{n,1} \to \cdots$ are injective resolutions of A^n, and the induced sequences

$$\begin{aligned}
&\mathrm{Ker}(d_A^n) \to \mathrm{Ker}(d_I^{n,0}) \to \mathrm{Ker}(d_I^{n,1}) \to \cdots\\
&\mathrm{Im}(d_A^n) \to \mathrm{Im}(d_I^{n,0}) \to \mathrm{Im}(d_I^{n,1}) \to \cdots\\
&\mathcal{H}^n(A^\bullet) \to \mathcal{H}^n_I(C^{\bullet,0}) \to \mathcal{H}^n_I(C^{\bullet,1}) \to \cdots
\end{aligned}$$

are injective resolutions of $\mathrm{Ker}(d_A^n)$, $\mathrm{Im}(d_A^n)$ and $\mathcal{H}^n(A^\bullet)$, respectively, and

(iii) any short exact sequences of the form

$$0 \to \mathrm{Ker}(d_I^{i,j}) \to C^{i,j} \to \mathrm{Im}(d_I^{i,j}) \to 0$$

split.

Proposition 6.1.8. *If $\mathcal{A}$ has enough injectives, then any $A^\bullet \in K^+(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

Theorem 6.1.9 (Grothendieck spectral sequence)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $F_1 : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$ and $F_2 : K^+(\mathcal{B}) \longrightarrow K(\mathcal{C})$ be exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ contains enough injectives, and for any complex $I^\bullet \in K^+(\mathcal{A})$ of injective objects from $\mathcal{A}$, its image $F_1(I^\bullet)$ is inside an F_2-adapted triangulated subcategory $\mathcal{K}_{F_2}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$, there is a spectral sequence*

$$E_2^{p,q} := R^pF_2(R^qF_1(A^\bullet)) \Longrightarrow E^{p+q} := R^{p+q}(F_2 \circ F_1)(A^\bullet)\,. \tag{6.1.10}$$

Proof. Note that, if $F_1 = \mathrm{Id}$, then for a left exact functor $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$, the above spectral sequence reads

$$E_2^{p,q} := R^pF(\mathcal{H}^q(A^\bullet)) \Longrightarrow E^{p+q} := R^{p+q}F(A^\bullet)\,. \tag{6.1.11}$$

It follows from construction of derived functors that, given $A^\bullet \in D^+(\mathcal{A})$ isomorphic to a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$, we have $RF_1(A^\bullet) \cong F_1(I^\bullet)$ and

$$R^pF_2(R^qF_1(A^\bullet)) = R^pF_2(\mathcal{H}^q(F_1(I^\bullet))). \tag{6.1.12}$$

Since

$$\begin{aligned}
R^n(F_2 \circ F_1)(A^\bullet) = \mathcal{H}^n(R(F_2 \circ F_1)(I^\bullet)) &\cong \mathcal{H}^n(RF_2(RF_1(A^\bullet))\\
&\cong \mathcal{H}^n(RF_2(F_1(I^\bullet))) \cong R^nF_2(F_1(I^\bullet))\,,
\end{aligned} \tag{6.1.13}$$

using (6.1.12), the general case (6.1.10) follows from the special case (6.1.11) above.

Therefore, it suffices to prove the result with $F_1 = \mathrm{Id}$ and $F := F_2$. For this we need an appropriate double complex, which is provided by a Cartan-Eilenberg resolution of $A^\bullet$. Let $C^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A^\bullet$, and set $K^{\bullet,\bullet} := F(C^{\bullet,\bullet})$. Since F is additive, it preserve direct sums, and since $C^{i,j} \cong \mathrm{Ker}(d_I^{i,j}) \oplus \mathrm{Im}(d_I^{i,j})$, we have $\mathcal{H}_I^q(K^{\bullet,p}) = F\mathcal{H}_I^q(C^{\bullet,p})$. Fixing q, and allowing p to vary, we see that $\mathcal{H}_I^q(C^{\bullet,p})$ defines an injective resolution of $\mathcal{H}^q(K^{\bullet,p}) = \mathcal{H}^q(A^\bullet)$. Then we have

$$\mathcal{H}_{II}^p\mathcal{H}_I^q(K^{\bullet,\bullet}) = R^pF(\mathcal{H}^q(A^\bullet)).$$

Now applying spectral sequence in Proposition 6.1.6 and using the fact that the natural morphism $A^\bullet \longrightarrow \mathrm{tot}(C^{\bullet,\bullet})$ is a quasi-isomorphism, we see that

$$\begin{aligned}\mathcal{H}^{p+q}(\mathrm{tot}(K^{\bullet,\bullet})) &= \mathcal{H}^{p+q}(F(\mathrm{tot}(C^{\bullet,\bullet}))) \\ &= \mathcal{H}^{p+q}(RF(A^\bullet)) \\ &= R^{p+q}F(A^\bullet).\end{aligned}$$

This completes the proof. □

6.2. **Examples of spectral sequence.** Here we give some useful examples of Grothendieck spectral sequences, which will appear in next sections.

Example 6.2.1. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $A^\bullet \in D(\mathcal{A})$ and $B^\bullet \in D^+(\mathcal{A})$. Take $F_1 = \mathrm{Id}$ so that $R^qF_1(B^\bullet) = \mathcal{H}^q(B^\bullet)$, and take $F_2 = \mathrm{Hom}^\bullet(A^\bullet, -)$. Then we have,

$$R^pF_2(R^qF_1(B^\bullet) = R^p\,\mathrm{Hom}^\bullet(A^\bullet, \mathcal{H}^q(B^\bullet)) = \mathrm{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)),$$

and

$$R^{p+q}(F_2 \circ F_1)(B^\bullet) = R^{p+q}\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet).$$

Since $\mathrm{Ext}^{p+q}(A^\bullet, B^\bullet) = \mathrm{Hom}(A^\bullet, B^\bullet[p+q])$ and $\mathrm{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)) \cong \mathrm{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p])$, by Theorem 6.1.9, we have a spectral sequence

(6.2.2) $$E_2^{p,q} := \mathrm{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p]) \Longrightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q]).$$

Similarly, if $\mathcal{A}$ has enough projectives so that we can compute $R^p\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ for $A^\bullet \in D^-(\mathcal{A})$ as the right derived functor of the contravariant functor $\mathrm{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathbf{Ab})$, we have the spectral sequence

(6.2.3) $$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(A^\bullet), B^\bullet[p]) \Longrightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q])\,.$$

If $B^\bullet \in D^+(\mathcal{A})$ is bounded below, and $\mathcal{A}$ has enough injectives, but may not have enough projectives, then also we have this spectral sequence.

Remark 6.2.4. It should be noted that, a spectral sequence $E_2^{p,q} \Rightarrow E^{p+q}$ given at page E_2 **does not imply** that the term $E_\infty^{p,q}$ lies in page E_2.

Example 6.2.5. Let X be a noetherian scheme so that $\mathfrak{QCoh}(X)$ has enough injectives. Then for any $E^\bullet, F^\bullet \in D^b(X)$, we have the following spectral sequences

$$E_2^{p,q} := \mathrm{Ext}^p(E^\bullet, \mathcal{H}^q(F^\bullet)) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet)\,, \tag{6.2.6}$$

and

$$E_2^{p,q} := \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), F^\bullet) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet)\,, \tag{6.2.7}$$

Remark 6.2.8. If X is a projective k-variety, $\dim_k H^i(X, E) < \infty$ for any coherent sheaf E on X. Using this, one can deduce that $\dim_k \mathrm{Ext}^i(E, F) < \infty$, for any $E, F \in \mathfrak{Coh}(X)$. Then using the spectral sequences (6.2.6) and (6.2.7) one can show that $\mathrm{Ext}^i(E^\bullet, F^\bullet)$ has finite dimension, for all $E^\bullet, F^\bullet \in D^b(X)$.

Example 6.2.9. Let $E^\bullet \in D^-(X)$. Then by definition of local Hom complex, we have $\Gamma \circ \mathcal{H}om^\bullet(E^\bullet, -) = \mathrm{Hom}^\bullet(E^\bullet, -)$. Since for a complex $I^\bullet$ of injective sheaves of $\mathcal{O}_X$-modules, the complex $\mathcal{H}om^\bullet(E^\bullet, I^\bullet)$ is Γ-acyclic (meaning that, $\mathrm{Ext}^i(E^\bullet, I^\bullet) = R^i\Gamma(\mathcal{H}om^\bullet(E^\bullet, I^\bullet)) = 0$ for all $i \neq 0$, which indeed holds), we have

$$R\Gamma \circ R\,\mathcal{H}om^\bullet(E^\bullet, -) = R\,\mathrm{Hom}^\bullet(E^\bullet, -).$$

Therefore, applying Theorem 6.1.9 we have the following spectral sequence relating local and global Ext:

$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q(E^\bullet, F^\bullet)) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet). \tag{6.2.10}$$

7. Bondal–Orlov's reconstruction theorem

7.1. **What is it?** A famous theorem of Gabriel says that two k-varieties X and Y are isomorphic if and only if there is an equivalence of categories $\mathfrak{Coh}(X)$ with $\mathfrak{Coh}(Y)$. In [Muk81], Mukai established an equivalence $D^b(A) \simeq D^b(\check{A})$, where A is an abelian variety and $\check{A}$ its dual abelian variety. Therefore, equivalence between bounded derived category of coherent sheaves fails to ensure isomorphism of varieties, in general. In their famous paper [BO01], Bondal and Orlov shows how to reconstruct a smooth projective variety X from $D^b(X)$ when ω_X or its dual is ample (see Theorem 7.1.1). More precisely,

Theorem 7.1.1 (Bondal–Orlov)**.** *Let X be a smooth projective variety over k with canonical line bundle ω_X. Assume that ω_X (resp., $\omega_X^\vee$) is ample. Let Y be any smooth projective variety over k. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k-varieties. In particular, ω_Y (resp., $\omega_Y^\vee$) is ample.*

The main idea behind the proof is to "cohomologically" characterize closed points, invertible sheaves and Zariski topology of a smooth projective k-variety. For this we need "Serre functor" as defined in Definition 4.1.1. Note that, both $D^b(X)$ and $D^b(Y)$

admits Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim(Y)]$, respectively. As a first step towards this theorem, we now establish equality of dimensions of X and Y.

7.2. **Equality of dimensions.** Let k be a field. A *k–variety* is an integral separated finite type k-scheme. For any smooth projective k-variety X, we define $D^b(X) := D^b(\mathfrak{Coh}(X))$. A rank one invertible sheaf L on X is said to have *finite order* if $L^r \cong \mathcal{O}_X$ for some integer $r > 0$. The smallest positive integer r such that $L^r \cong \mathcal{O}_X$ is called the *order* of L. If $L^r \ncong \mathcal{O}_X$, $\forall\, r > 0$, we say that L has *infinite order*. For any $x \in X$, let $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at x. For any closed point $x \in X$, we can consider $k(x)$ as a coherent sheaf on X supported at x by taking its push-forward along the closed embedding $\iota_x : \operatorname{Spec}(k(x)) \hookrightarrow X$. This is the skyscraper sheaf supported at x given by

$$k(x)(U) = \begin{cases} k(x), & \text{if } x \in U, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 7.2.1. *Let X and Y be smooth projective varieties over k. If there is an exact equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ of bounded derived categories, then $\dim_k(X) = \dim_k(Y)$. In this case, both ω_X and ω_Y have the same order (can be infinity too).*

Proof. Since both X and Y are smooth projective k-varieties, by Theorem 4.2.2, they admit natural Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim_k(Y)]$, respectively. By Lemma 4.1.7, any k-linear equivalence $F : D^b(X) \longrightarrow D^b(Y)$ commutes with Serre functors S_X and S_Y (i.e., there is a natural isomorphism of functors $F \circ S_X \cong S_Y \circ F$).

For a closed point $x \in X$, we have $k(x) \cong k(x) \otimes \omega_X \cong S_X(k(x))[-\dim_k(X)]$. So,

$$\begin{aligned} F(k(x)) &\cong F(k(x) \otimes \omega_X) = F(S_X(k(x))[-\dim_k(X)]) \\ &\cong F(S_X(k(x)))[-\dim_k(X)], \quad \text{since } F \text{ is exact.} \\ &\cong S_Y(F(k(x)))[-\dim_k(X)], \quad \text{since } F \circ S_X \cong S_Y \circ F. \\ &\cong F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]\,. \end{aligned} \tag{7.2.2}$$

Since F is an equivalence of categories, $F(k(x))$ is a non-trivial bounded complex. Let i be the maximal (resp., minimal) integer such that $\mathcal{H}^i(F(k(x))) \neq 0$. Now from (7.2.2) we have

$$\begin{aligned} 0 \neq \mathcal{H}^i(F(k(x))) &\cong \mathcal{H}^i(F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x)) \otimes \omega_Y) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x))) \otimes \omega_Y\,. \end{aligned} \tag{7.2.3}$$

Since ω_Y is a line bundle, (7.2.3) contradicts maximality (resp., minimality) of i whenever $\dim_k(X) < \dim_k(Y)$ (resp., $\dim_k(X) > \dim_k(Y)$). Therefore, $\dim_k(X) = \dim_k(Y)$.

To see that both ω_X and ω_Y have the same order, assume that $\omega_X^k \cong \mathcal{O}_X$. Let $n = \dim_k(X) = \dim_k(Y)$. Note that, $S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)}$. Since $F \circ S_X \cong S_Y \circ F$, choosing a quasi-inverse of the equivalence F, we have

$$\begin{aligned} &F^{-1} \circ S_Y^k[-kn] \circ F \cong S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)} \\ \Rightarrow &S_Y^k[-kn] \cong \mathrm{Id}_{D^b(Y)} \ . \end{aligned}$$

Applying $\mathcal{O}_Y$ to the above isomorphism of functors, we get $\omega_Y^k \cong \mathcal{O}_Y$. □

Remark 7.2.4. In the proof of above Proposition, to show both ω_X and ω_Y have the same order, under the assumption that $\dim(X) = \dim(Y)$, we don't need F to be exact.

7.3. **Point like objects.**

Definition 7.3.1. A *graded category* is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ and an equivalence functor $T_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, known as *shift functor*. A functor $F : \mathcal{D} \to \mathcal{D}'$ between graded categories is called *graded* if there is an isomorphism of functors $F \circ T_{\mathcal{D}} \xrightarrow{\simeq} T_{\mathcal{D}'} \circ F$.

Example 7.3.2. Any triangulated category is a graded category, and any morphism between two triangulated categories is a graded morphism.

Definition 7.3.3. Let $\mathcal{D}$ be a k-linear triangulated category with Serre functor S. An object $P \in \mathcal{D}$ is said to be *point like* of codimension s if

(i) $S(P) \cong P[s]$,
(ii) $\mathrm{Hom}(P, P[i]) = 0$, for $i < 0$, and
(iii) $k(P) := \mathrm{Hom}(P, P)$ is a field.

An object E of an additive category is called *simple* if $\mathrm{Hom}(E, E)$ is a field.

Example 7.3.4. Let X be a smooth projective k-variety of dimension n.

(i) For any closed point $x \in X$, we have $S_X(k(x)) = (k(x) \otimes \omega_X)[n] \cong k(x)[n]$. Therefore, $k(x) \in D^b(X)$ is a point like object of codimension d.
(ii) Let $\omega_X \cong \mathcal{O}_X$ (for example when X is an abelian variety or a K3 surface). Then any simple object $E \in \mathfrak{Coh}(X)$ defines a point like object of codimension n in $D^b(X)$.

Proposition 7.3.5. *Let $\mathcal{A}$ be an abelian category, and $A^\bullet \in D^b(\mathcal{A})$. Let*

$$i^+ := \max\{i : \mathcal{H}^i(A^\bullet) \neq 0\} \quad \textit{and} \quad i^- := \min\{i : \mathcal{H}^i(A^\bullet) \neq 0\}.$$

Then in $D^b(\mathcal{A})$, there are morphisms $\phi : A^\bullet \to \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ and $\psi : \mathcal{H}^{i^-}(A^\bullet)[-i^-] \to A^\bullet$ such that $\mathcal{H}^{i^+}(\phi) = \mathrm{Id}_{\mathcal{H}^{i^+}(A^\bullet)}$ and $\mathcal{H}^{i^-}(\psi) = \mathrm{Id}_{\mathcal{H}^{i^-}(A^\bullet)}$.

Proof. There is a natural quasi-isomorphism of complexes

$$\begin{tikzcd}
A^\bullet_- : \arrow[d, "qis"'] & \cdots \arrow[r] & A^{i^+-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d^{i^+}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : & \cdots \arrow[r] & A^{i^+-1} \arrow[r] & A^{i^+} \arrow[r, "d^{i^+}"] & A^{i^++1} \arrow[r] & \cdots .
\end{tikzcd}$$

Since the natural morphism of complexes $A^\bullet_- \longrightarrow \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ induces identity morphism at i^+-th cohomology, the first part follows. The second part is similar. $\square$

Corollary 7.3.6. *With the above notations, for any $B \in \mathcal{A}$, we have the following natural isomorphisms*

(i) $\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[-i^+])$, *and*
(ii) $\mathrm{Hom}_{D^b(\mathcal{A})}(B, \mathcal{H}^{i^-}(A^\bullet)) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(B[-i^-], A^\bullet)$.

Proof. Send $f \in \mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B)$ to $f[-i^+]$ and use above Proposition 7.3.5. To get the inverse map, send any $\phi \in \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[i^+])$ to $\mathcal{H}^{i^+}(\phi)[-i^+]$. The second part is similar. $\square$

Exercise 7.3.7. Let $A^\bullet \in D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for all $i < m$. Then there is a distinguished triangle

$$\mathcal{H}^m(A^\bullet)[-m] \longrightarrow A^\bullet \xrightarrow{\varphi} B^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[1-m]$$

in the derived category $D(\mathcal{A})$ such that

$$\mathcal{H}^i(B^\bullet) \cong \begin{cases} \mathcal{H}^i(A^\bullet) & \text{if } i \leq m, \text{ and} \\ 0, & \text{if } i > m. \end{cases}$$

Remark 7.3.8. Let X be a smooth projective k-variety of dimension d. Then any point like object $P \in D^b(X)$ has codimension d. This follows from assumption (i) in the Definition 7.3.3, because looking at minimal i with non-zero cohomologies, the isomorphism $P \otimes \omega_X[d] \cong P[s]$ implies

$$\mathcal{H}^i(P) \otimes \omega_X[d] \cong \mathcal{H}^i(P)[s]. \tag{7.3.9}$$

This forces $d = s$.

Lemma 7.3.10. *Let M be a finitely generated non-zero module over a noetherian ring A. Then there is a finite chain of A-submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ (as A-modules), for some $\mathfrak{p}_i \in \mathrm{Supp}(M)$.

Proof. Denote by $\mathrm{Ass}(M)$ the set of all associated primes of M. Recall that, $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ for any finitely generated A-module M. Since $M \neq 0$, we can choose a $\mathfrak{p}_1 \in \mathrm{Ass}(M)$ to get an A-submodule

$$M_1 := \mathrm{image}(A/\mathfrak{p}_1 \hookrightarrow M) \subset M.$$

If $M_1 \neq M$, we do the same for M/M_1 to choose a $\mathfrak{p}_2 \in \mathrm{Ass}(M/M_1)$ and apply the same to obtain a sequence $M_1 \subsetneq M_2 \subseteq M$ with $M_2/M_1 \cong A/\mathfrak{p}_2$. Since $(M/M_1)_{\mathfrak{p}_2} \neq 0$, we see that $\mathfrak{p}_2 \in \mathrm{Supp}(M)$. Since M is finitely generated, the result follows by induction. □

Corollary 7.3.11. *With the above notation, if $\mathrm{Supp}(M) = \{\mathfrak{m}\}$, for some maximal ideal $\mathfrak{m}$ of A, there is a surjective (resp., injective) A-module homomorphism $M \twoheadrightarrow A/\mathfrak{m}$ (resp., $A/\mathfrak{m} \hookrightarrow M$).*

Proof. Since $\mathrm{Ass}(M) = \{\mathfrak{m}\}$, the result follows from the above Lemma 7.3.10. □

Definition 7.3.12. Support of a complex $E^\bullet \in D^b(X)$ is the union of the supports of its cohomologies. In other words, $\mathrm{Supp}(E^\bullet)$ is the closed subset of X defined by

$$\mathrm{Supp}(E^\bullet) := \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(\mathcal{H}^i(E^\bullet))\,.$$

Lemma 7.3.13. *Let $E^\bullet \in D^b(X)$ with $\mathrm{Supp}(E^\bullet) = Z_1 \cup Z_2$, for some disjoint closed subsets Z_1 and Z_2 in X. Then $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, for some non-zero objects $E_j^\bullet \in D^b(X)$ with $\mathrm{Supp}(E_j^\bullet) \subseteq Z_j$, for all $j = 1, 2$.*

Proof. This is clear for any $E \in \mathfrak{Coh}(X)$, and hence the result follows for $E^\bullet \cong E[n] \in D^b(X)$, for $E \in \mathfrak{Coh}(X)$ and $n \in \mathbb{Z}$. Let

$$i^+_{E^\bullet} := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\} \quad \text{and} \quad i^-_{E^\bullet} := \min\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\};$$

and we drop the subscript $E^\bullet$ when there is no confusion likely to arise. The *length* of an object $E^\bullet \in D^b(X)$ is the difference $i^+ - i^-$. For general case, we use induction on the length of a complex.

Let $E^\bullet \in D^b(X)$ be a complex of length at least 2. Let $m = i^-_{E^\bullet}$, and write $\mathcal{H} := \mathcal{H}^m(E^\bullet)$. The sheaf $\mathcal{H}$ can be decomposed as $\mathcal{H} \cong \mathcal{H}_1 \bigoplus \mathcal{H}_2$, with $\mathrm{Supp}(\mathcal{H}_j) \subset Z_j$, for $j = 1, 2$. By Proposition 7.3.5, we have a natural morphism $\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet$ inducing identity morphism on the m-th cohomology; complete it to a distinguished triangle

$$\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet \longrightarrow F^\bullet := C(\varphi) \longrightarrow \mathcal{H}[1-m]\,.$$

Then from long exact sequence of cohomologies we have

$$\mathcal{H}^i(F^\bullet) = \begin{cases} \mathcal{H}^i(E^\bullet), & \text{if } i > m, \text{ and} \\ 0, & \text{if } i \leq m; \end{cases}$$

(c.f. Exercise 7.3.7). Since the length of $F^\bullet$ is less than the length of $E^\bullet$, induction hypothesis applied to $F^\bullet$ gives a decomposition $F^\bullet \cong F_1^\bullet \bigoplus F_2^\bullet$ with $\mathrm{Supp}(\mathcal{H}^i(F_j^\bullet)) \subset Z_j$, for all $j = 1, 2$, and $i \in \mathbb{Z}$. Since $\mathcal{H}^{-q}(F_1^\bullet)$ and $\mathcal{H}_2$ are coherent sheaves of $\mathcal{O}_X$-modules with disjoint supports, we have

$$\mathrm{Hom}_{D^b(X)}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2) = 0, \ \ \forall\, p \in \mathbb{Z},$$

which can be verified locally. Then $\mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[1-m]) = 0$ follows from the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) \Longrightarrow E^{p+q} := \mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[p+q]);$$

c.f., Example 6.2.1. Similarly, we have $\mathrm{Hom}(F_2^\bullet, \mathcal{H}_1[1-m]) = 0$. Choose a complex $E_j^\bullet$ to complete a distinguished triangle

$$E_j^\bullet \longrightarrow F_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow E_j^\bullet[1], \ \ \forall\, j = 1, 2,$$

we have a decomposition $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$. Since $\mathrm{Supp}(F_j^\bullet) \subset Z_j$, it follows that $\mathrm{Supp}(E_j^\bullet) \subset Z_j$, for all $j = 1, 2$. □

Lemma 7.3.14. *Let $E^\bullet$ be a simple object in $D^b(X)$ with zero dimensional support. If $\mathrm{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, then $E^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer m.*

Proof. Since $E^\bullet$ is supported in dimension zero, $\mathrm{Supp}(E)$ is a finite subset of closed points in X. If $\mathrm{Supp}(E)$ is not a singleton set, then it has disjoint components. Then in $D^b(X)$, we have an isomorphism $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, with $E_j^\bullet \not\cong 0$, $\forall\, i = 1, 2$, which contradicts simplicity of $E^\bullet$. Therefore, $\mathrm{Supp}(E^\bullet)$ is a closed point, say $x \in X$. Let $i^+ := \max\{i : \mathcal{H}^i(E^\bullet) \neq 0\}$ and $i^- := \min\{j : \mathcal{H}^j(E^\bullet) \neq 0\}$. Since both $\mathcal{H}^{i^+}(E^\bullet)$ and $\mathcal{H}^{i^-}(E^\bullet)$ have support $\{x\}$, they are given by finite modules over the noetherian local ring $\mathcal{O}_{X,x}$ supported at $\mathfrak{m}_x$. Then applying Corollary 7.3.11, we get a non-trivial $\mathcal{O}_{X,x}$-module homomorphism $\phi : \mathcal{H}^{i^+}(E^\bullet) \longrightarrow \mathcal{H}^{i^-}(E^\bullet)$ given by the composition

$$\mathcal{H}^{i^+}(E^\bullet) \twoheadrightarrow k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \hookrightarrow \mathcal{H}^{i^-}(E^\bullet).$$

Now it follows from Proposition 7.3.5 that the following composite morphism is non-trivial.

$$E^\bullet[i^+] \longrightarrow \mathcal{H}^{i^+}(E^\bullet) \xrightarrow{\phi} \mathcal{H}^{i^-}(E^\bullet) \longrightarrow E^\bullet[i^-]\,.$$

Since $\mathrm{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, we must have $i^- - i^+ \geq 0$. Hence, $i^- = i^+ =: m$ (say). Therefore, $E^\bullet \cong E[m]$, for some $E \in \mathfrak{Coh}(X)$ with $\mathrm{Supp}(E) = \{x\}$. Since $\mathrm{Hom}(E[m], E[m]) \cong \mathrm{Hom}(E, E)$, so E is simple. Then the natural surjective homomorphism $E \to k(x)$ must be isomorphism. Therefore, $E^\bullet \cong k(x)[m]$. □

Proposition 7.3.15 (Bondal–Orlov). *Let X be a smooth projective k-variety with ω_X or $\omega_X^\vee$ ample. Then any point like object in $D^b(X)$ is isomorphic to an object of the form $k(x)[m]$, for some closed point $x \in X$ and some integer m.*

Remark 7.3.16. Above result fails if neither ω_X nor $\omega_X^\vee$ is ample; c.f. Example 7.3.4.

Proof. Note that X is projective because there is an ample line bundle on X. Clearly for any closed point $x \in X$ and any integer m, the shifted skyscraper sheaf $k(x)[m] \in D^b(X)$ is a point like object of codimension $d = \dim(X)$ (c.f., Example 7.3.4).

To see the converse, let $P \in D^b(X)$ be a point like object of codimension n. It follows from $P \otimes \omega_X[d] \cong P[n]$ that $n = d$ (c.f., Remark 7.3.8). Then we have,

$$\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P), \quad \forall\, i \in \mathbb{Z}\,. \tag{7.3.17}$$

Suppose that ω_X is ample. Let

$$m \mapsto P_E(m) := \chi(E \otimes \omega_X^m)$$

be the Hilbert polynomial of $E \in \mathfrak{Coh}(X)$. Since $\deg(P_E(m)) = \dim(\mathrm{Supp}(E))$, taking tensor product with ω_X makes difference only if $\dim(\mathrm{Supp}(E)) > 0$. Therefore, form (7.3.17) we conclude that $\mathcal{H}^i(P)$ is supported in dimension zero. Since P is simple, the result follows from Lemma 7.3.14. The same argument applies for $\omega_X^\vee$ ample. □

7.4. **Invertible objects.** Now we realize line bundles on X as objects of $D^b(X)$.

Definition 7.4.1. Let $\mathcal{D}$ be a triangulated category together with a Serre functor $T_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$. An object $L \in \mathcal{D}$ is said to be *invertible* if for each point like object $P \in \mathcal{D}$, there is an integer n_P (which also depends on L) such that

$$\mathrm{Hom}_{\mathcal{D}}(L, P[i]) = \begin{cases} k(P), & \text{if } i = n_P, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we characterize invertible objects in $D^b(X)$. For this, we need the following well-known result form commutative algebra.

Lemma 7.4.2. *Let M be a finitely generated module over a noetherian local ring $(A, \mathfrak{m})$. If $\mathrm{Ext}^1(M, A/\mathfrak{m}) = 0$, then M is free.*

Proof. Let $k = A/\mathfrak{m}$. Then any k-basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for the A-module M by Nakayama lemma. Thus we get a short exact sequence of A-modules

$$0 \longrightarrow N \xrightarrow{\iota} A^n \xrightarrow{\phi} M \longrightarrow 0.$$

Note that, $N = \mathrm{Ker}(\phi)$ is finitely generated, and ι induces a trivial homomorphism $\widetilde{\iota} \colon N/\mathfrak{m}N \longrightarrow k^n$. Since $\mathrm{Ext}^1(M, k) = 0$, the induced homomorphism

$$\mathrm{Hom}(A^n, k) \longrightarrow \mathrm{Hom}(N, k)$$

is surjective. Since $\mathrm{Hom}_A(A^n, k) \cong \mathrm{Hom}_k(k^n, k)$ and $\mathrm{Hom}_A(N, k) \cong \mathrm{Hom}_k(N/\mathfrak{m}N, k)$, the homomorphism $\mathrm{Hom}_k(k^n, k) \longrightarrow \mathrm{Hom}_k(N/\mathfrak{m}N, k)$ induced by $\widetilde{\iota}$ is surjective.

Since $\widetilde{\iota} = 0$, this forces $N/\mathfrak{m}N = 0$. Then $N = 0$ by Nakayama lemma, and hence M is a free A-module. □

Proposition 7.4.3 (Bondal–Orlov). *Let X be a smooth projective k-variety. Any invertible object in $D^b(X)$ is of the form $L[m]$, for some line bundle L on X and some integer m. Conversely, if any point like object of $D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and some integer ℓ, then for any line bundle L on X and any integer m, $L[m] \in D^b(X)$ is invertible.*

Remark 7.4.4. Note that, by Proposition 7.3.15 the condition in the converse part of the above Proposition is satisfied when ω_X or $\omega_X^\vee$ is ample.

Proof of Proposition 7.4.3. **Step 1.** Let $E^\bullet \in D^b(X)$ be an invertible object. Let $m = \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then by Proposition 7.3.5, there is a morphism

$$E^\bullet \longrightarrow \mathcal{H}^m(E^\bullet)[-m]$$

in $D^b(X)$ inducing identity morphism at m-th cohomology $\mathcal{H}^m(E^\bullet)$. This gives

$$\mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,, \tag{7.4.5}$$

(c.f., Corollary 7.3.6). Fix a closed point $x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$. Then by Lemma 7.3.10, there is an associated prime ideal $\mathfrak{p} \subseteq \mathfrak{m}_{x_0}$ and a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow \mathcal{O}_{X,x_0}/\mathfrak{p}$, which gives a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x_0)$. Therefore, by (7.4.5), we have

$$0 \neq \mathrm{Hom}_{D^b(X)}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,.$$

This forces $n_{k(x_0)} = -m$ (c.f., Definition 7.4.1).

Step 2. *We show that,* $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $n_{k(x_0)} = -m$, it follows from the definition of invertible object $E^\bullet \in D^b(X)$ that

$$\mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = \mathrm{Hom}(E^\bullet, k(x_0)[1 + n_{k(x_0)}]) = 0. \tag{7.4.6}$$

Consider the spectral sequence (see Example 6.2.5)

$$\begin{aligned} E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(E^\bullet), k(x_0)[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), k(x_0)) \\ \Longrightarrow E^{p+q} := \mathrm{Hom}(E^\bullet, k(x_0)[p+q]). \end{aligned} \tag{7.4.7}$$

Since $\mathcal{H}^{m+1}(E^\bullet) = 0$, we have

$$E_2^{3,-m-1} = \mathrm{Hom}(\mathcal{H}^{m+1}(E^\bullet), k(x_0)[3]) = 0. \tag{7.4.8}$$

Also

$$E_2^{-1,-m+1} = \mathrm{Hom}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)[-1]) = \mathrm{Ext}^{-1}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)) = 0. \tag{7.4.9}$$

Now using (7.4.8) and (7.4.9), and taking H^0 of the complex

$$\cdots \longrightarrow 0 = E_2^{-1,-m+1} \xrightarrow{d} E_2^{1,-m} \xrightarrow{d} E_2^{3,-m-1} = 0 \longrightarrow \cdots ,$$

we see that $E_3^{1,-m} = E_2^{1,-m}$; similarly, $E_r^{1,-m} = E_2^{1,-m}$, for all $r \geq 2$. The following picture of page E_2 could be useful to understand the situation.

$$\begin{tikzcd}
0 \arrow[drr] & E_2^{0,-m+1} \arrow[drr] & E_2^{1,-m+1} \arrow[drr] & E_2^{2,-m+1} & E_2^{3,-m+1} \\
0 & E_2^{0,-m} \arrow[drr] & E_2^{1,-m} \arrow[drr] & E_2^{2,-m} & E_2^{3,-m} \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

This shows that,

$$E_2^{1,-m} = E_\infty^{1,-m}. \tag{7.4.10}$$

Since $E_\infty^{1,-m}$ is isomorphic to a subquotient of

$$E^{1-m} = \mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = 0 \tag{7.4.11}$$

(see, (7.4.6) and (7.4.7)), using (7.4.10) we conclude that $E_2^{1,-m} = 0$. Therefore,

$$\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0, \ \ \forall\, x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))\,. \tag{7.4.12}$$

Step 3. *We show that $\mathcal{H}^m(E^\bullet)$ is a locally free $\mathcal{O}_X$-module.*

For this, we consider the *local-to-global* spectral sequence (see Example 6.2.9)

$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q(\mathcal{H}^m(E^\bullet), k(x_0))) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m(E^\bullet), k(x_0))\,, \tag{7.4.13}$$

which allow us to pass from the global vanishing $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ to the local one $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $\mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))$ is a skyscraper sheaf supported at x_0, it is flasque, and hence is Γ-acyclic. Then form (7.4.13), we have

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{7.4.14}$$

Again,

$$E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{7.4.15}$$

Since at page E_2, we have morphisms

$$0 = E_2^{-2,2} \xrightarrow{d} E_2^{0,1} \xrightarrow{d} E_2^{2,0} = 0,$$

we have $E_3^{0,1} = \mathcal{H}^0(\cdots \to 0 \to E_2^{0,1} \to 0 \to \cdots) = E_2^{0,1}$. Similar computations shows that $E_r^{0,1} = E_2^{0,1}$, for all $r \geq 2$. Hence we conclude that,

$$E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))) = E_\infty^{0,1}\,. \tag{7.4.16}$$

Since $E^1 = \mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ by Step 2, we have $E_2^{0,1} = E_\infty^{0,1} = 0$. Since $k(x_0)$ is a skyscraper sheaf supported at x_0, we see that $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))$ is supported over $\{x_0\}$, and hence is globally generated. Since

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = E_2^{0,1} = 0,$$

we have $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$. Since $\mathcal{H}^m(E^\bullet) \in \mathfrak{Coh}(X)$, we have

$$\mathrm{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))_{x_0} = 0. \tag{7.4.17}$$

The by Lemma 7.4.2, $\mathcal{H}^m(E^\bullet)_{x_0}$ is free $\mathcal{O}_{X,x_0}$-module. Since freeness is an open property, there is a non-empty open (dense) subset U of X containing x_0 such that $U \subseteq \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$ and $\mathcal{H}^m(E^\bullet)\big|_U$ is a free $\mathcal{O}_U$-module. Since X is irreducible, $\mathcal{H}^m(E^\bullet)$ is locally free on X.

Step 4. *We show that, $\mathcal{H}^m(E^\bullet)$ is a line bundle on X.*

Since $\mathrm{Supp}(\mathcal{H}^m(E^\bullet)) = X$, there is a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x)$, for each $x \in X$. Then following argument of Step 1, we have

$$\mathrm{Hom}(E^\bullet, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \neq 0, \ \ \forall\, x \in X. \tag{7.4.18}$$

Now it follows from Definition 7.4.1 of invertible objects that

$$n_{k(x)} = -m, \ \forall \ x \in X. \tag{7.4.19}$$

If r is the rank of $\mathcal{H}^m(E^\bullet)$, we have

$$\begin{aligned} k(x) = \mathrm{Hom}(E^\bullet, k(x)[-m]) &= \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \\ &= \mathrm{Hom}(\mathcal{O}_{X,x}^{\oplus r}, k(x)) \cong k(x)^{\oplus r}. \end{aligned} \tag{7.4.20}$$

Therefore, $r = 1$, and hence $\mathcal{H}^m(E^\bullet)$ is a line bundle on X.

Step 5. *We show that, $\mathcal{H}^i(E^\bullet) = 0$, for all $i < m$.*

From the spectral sequence in (7.4.7), we have

$$\begin{aligned} E_2^{q,-m} &= \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)[q]) \\ &= \mathrm{Ext}^q(\mathcal{H}^m(E^\bullet), k(x)) \\ &\cong H^q(X, \mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))) = 0, \quad \forall\, q > 0, \end{aligned} \tag{7.4.21}$$

because $\mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))$ is a skyscraper sheaf supported on $\{x\}$, and hence is Γ-acyclic.

Suppose that $i < m$. Then it follows from Definition 7.4.1 and (7.4.19) that

$$E^{-i} = \mathrm{Hom}(E^\bullet, k(x)[-i]) = 0, \ \ \forall\, x \in X. \tag{7.4.22}$$

Now to show $\mathcal{H}^i(E^\bullet) = 0$, it is enough to show that

$$E_2^{0,-i} = \mathrm{Hom}(\mathcal{H}^i(E^\bullet), k(x)) = 0, \forall\, x \in X. \tag{7.4.23}$$

Since $E^{-i} = 0$, if we can show that

$$E_2^{0,-i} = E_\infty^{0,-i}, \tag{7.4.24}$$

then from the spectral sequence (7.4.7) we would get $E_2^{0,-i} = 0$. We prove this by induction on i.

If $i = m-1$, then $E_2^{2,-i-1} = E_2^{2,-m} = 0$ by (7.4.21). Since negative indexed Ext groups between two coherent sheaves are zero, we have $E_2^{-2,-(m-2)} = 0$. Then (7.4.24), for the case $i = m-1$, follows from the complex

$$\cdots \to 0 = E_2^{-2,-(m-2)} \xrightarrow{d} E_2^{0,1-m} \xrightarrow{d} E_2^{2,-m} = 0 \to \cdots .$$

Therefore, $\mathcal{H}^{m-1}(E^\bullet) = 0$. Assume inductively that $\mathcal{H}^i(E^\bullet) = 0$, for all $i \in \mathbb{Z}$, with $i_0 < i \leq m-1$. Then putting $m = i_0 + 1$ in (7.4.21) and using $\mathcal{H}^{i_0+1}(E^\bullet) = 0$, we have $E_2^{2,-i_0-1} = 0$. Then (7.4.24) follows from the complex

$$\cdots \to 0 = E_2^{-2,1-i_0} \xrightarrow{d} E_2^{0,-i_0} \xrightarrow{d} E_2^{2,-i_0-1} = 0 \to \cdots .$$

This completes induction. Therefore, $\mathcal{H}^i(E^\bullet) = 0,\ \forall\, i < m$, and hence for all $i \neq m$.

Step 6. Now we prove converse part of the Proposition 7.4.3. Suppose that any point like object $P \in D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and $\ell \in \mathbb{Z}$. Let L be a line bundle on X, and $m \in \mathbb{Z}$. Then from Definition 7.4.1 we get

$$\begin{aligned} \mathrm{Hom}(L[m], P[i]) &\cong \mathrm{Hom}(L, k(x)[\ell + i - m]) \\ &= \mathrm{Ext}^{\ell+i-m}(\mathcal{O}_X, L^\vee \otimes k(x)) \\ &\cong H^{\ell+i-m}(X, L^\vee \otimes k(x))\,, \end{aligned} \tag{7.4.25}$$

which vanishes except for $i = m - \ell$. Then we set $n_P := m - \ell$. This completes the proof. □

Remark 7.4.26. Let $\mathcal{D}$ be a (tensor) triangulated category admitting a Serre functor S. If we naively define *Picard group* of $\mathcal{D}$ to be the set $\mathrm{Pic}(\mathcal{D})$ of all invertible objects in $\mathcal{D}$, then for a smooth projective k-variety X with ω_X or $\omega_X^\vee$ ample, we have $\mathrm{Pic}(D^b(X)) = \mathrm{Pic}(X) \times \mathbb{Z}$.

7.5. **Spanning class of $D^b(X)$.**

Definition 7.5.1. A collection Ω of objects in a triangulated category $\mathcal{D}$ is called a *spanning class of* $\mathcal{D}$ (or *spans* $\mathcal{D}$) if for all $B \in \mathcal{D}$ the following conditions hold.

(i) If $\mathrm{Hom}(A, B[i]) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
(ii) If $\mathrm{Hom}(B[i], A) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Remark 7.5.2. If a triangulated category $\mathcal{D}$ admits a Serre functor, then the conditions (i) and (ii) in the above Definition 7.5.1 are equivalent.

Proposition 7.5.3. *Let X be a smooth projective k-variety. Then the objects of the form $k(x)$, with $x \in X$ a closed point, spans $D^b(X)$.*

Proof. It is enough to show that, for any non-zero object $E^\bullet \in D^b(X)$ there exists closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0 \quad \text{and} \quad \mathrm{Hom}(k(x_2), E^\bullet[i_2]) \neq 0\,.$$

Since $\mathrm{Hom}(k(x_2), E^\bullet[i_2]) \cong \mathrm{Hom}(E^\bullet, k(x_2)[\dim(X) - i_2])^*$ by Serre duality, it is enough to show that $\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0$, for some closed point $x \in X$ and some $i \in \mathbb{Z}$. Let $m := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then $\mathrm{Hom}(E^\bullet, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x))$ by Corollary 7.3.6. Now choosing a closed point x in the support of $\mathcal{H}^m(E^\bullet)$, we see that $\mathrm{Hom}(E^\bullet, k(x)[-m]) \neq 0$. This completes the proof. □

Remark 7.5.4. Spanning class in $D^b(X)$ is not unique. For a smooth projective k-variety X, for a choice of an ample line bundle L on X, we shall see later that, $\{L^{\otimes i} : i \in \mathbb{Z}\}$ forms a spanning class in $D^b(X)$.

7.6. **Proof of the reconstruction theorem.** Now we are in a position to prove the reconstruction theorem of Bondal and Orlov in the light of the following well-known results.

Proposition 7.6.1. [Sta20, Tag01PR] *Let X be a quasi-compact scheme. Let L be an invertible sheaf of $\mathcal{O}_X$-modules on X. Consider the graded algebra $S := \bigoplus_{i \geq 0} H^0(X, L^i)$, and its ideal $S_+ = \bigoplus_{i > 0} H^0(X, L^i)$. For each homogeneous element $s \in H^0(X, L^i)$, for $i > 0$, let $X_s := \{x \in X : s_x \notin \mathfrak{m}_x L^i_x\}$. Then the following are equivalent.*

(i) L is ample.
(ii) The collection of open sets X_s, with $s \in S_+$ homogeneous, covers X, and the natural morphism $X \longrightarrow \mathbf{Proj}(S)$ is an open immersion.
(iii) The collection of open sets X_s, with $s \in S_+$ homogeneous, forms a basis for the Zariski topology on X.

Proposition 7.6.2. *Let X be a smooth projective k-variety. Let L be a line bundle on X. If L or $L^\vee$ is ample, then the natural morphism of k-schemes*

$$X \longrightarrow \mathbf{Proj}\left(\bigoplus_n H^0(X, L^n)\right)$$

is an isomorphism.

Theorem 7.1.1 (Bondal–Orlov)**.** *Let X be a smooth projective variety over k with canonical line bundle ω_X. Assume that ω_X (resp., $\omega_X^\vee$) is ample. Let Y be any smooth projective variety* *over k. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k-varieties. In particular, ω_Y (resp., $\omega_Y^\vee$) is ample.*

Proof. **Step 1.** If $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or $\omega_Y^\vee$ is ample, the theorem follows.

Indeed, assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Since F is an exact equivalence of categories, $F \circ S_X \cong S_Y \circ F$ and $\dim(X) = \dim(Y) = n$ (say), (see Proposition 7.2.1). Then we have

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X))[-kn] = S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k, \ \forall\, k\,. \tag{7.6.3}$$

Since F is fully faithful, we have

$$H^0(X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k), \ \forall\, k\,. \tag{7.6.4}$$

The product structure on the graded k-algebra $\bigoplus\limits_k H^0(X, \omega_X^k)$ can be expressed in terms of following composition: for $s_i \in H^0(X, \omega_X^{k_i})$, $i = 1, 2$, we have

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1\,.$$

Note that, $s_1 \cdot s_2 = s_2 \cdot s_1$ follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "s_1"] \arrow[d, "s_2"'] & \omega_X^{k_1} \arrow[d, "S_X^{k_1}(s_2)[-k_1 n]"] \\ \omega_X^{k_2} \arrow[r, "S_X^{k_2}(s_1)[-k_2 n]"] & \omega_X^{k_1+k_2} \end{tikzcd} \tag{7.6.5}$$

Similarly, we have product structure on $\bigoplus\limits_k H^0(Y, \omega_Y^k)$. Therefore, F naturally induces an isomorphism of graded k-algebras

$$\widetilde{F} : \bigoplus_k H^0(X, \omega_X^k) \longrightarrow \bigoplus_k H^0(Y, \omega_Y^k)\,, \tag{7.6.6}$$

which induces isomorphism of k-schemes

$$X \xrightarrow{\cong} \mathbf{Proj}\Big(\bigoplus_k H^0(X, \omega_X^k)\Big) \xrightarrow{\cong} \mathbf{Proj}\Big(\bigoplus_k H^0(Y, \omega_Y^k)\Big) \xrightarrow{\cong} Y, \tag{7.6.7}$$

whenever ω_Y or its dual $\omega_Y^\vee$ is ample (c.f., Proposition 7.6.2). Therefore, it is enough to show that $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or $\omega_Y^\vee$ is ample whenever ω_X or $\omega_X^\vee$ is ample.

Step 2. We can assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Indeed, it follows from Definition 7.3.3 and Definition 7.4.1 that an exact equivalence $F : D^b(X) \to D^b(X)$ induce bijections

$$\begin{tikzcd} \{\text{point like objects of } D^b(X)\} \arrow[r, "F", "\simeq"'] \arrow[d, equal] & \{\text{point like objects of } D^b(Y)\} \\ \{k(x)[m] : x \in X_{closed} \text{ and } m \in \mathbb{Z}\} & \{k(y)[m] : y \in Y_{closed} \text{ and } m \in \mathbb{Z}\} \arrow[u, hook, "(*)"'] \end{tikzcd} \tag{7.6.8}$$

and

$$
\begin{tikzcd}
\{\text{invertible objects of } D^b(X)\} \arrow[r, "F", "\simeq"'] \arrow[d, equal] & \{\text{invertible objects of } D^b(Y)\} \\
\{L[m] : L \in \mathrm{Pic}(X) \text{ and } m \in \mathbb{Z}\} & \{M[m] : M \in \mathrm{Pic}(Y) \text{ and } m \in \mathbb{Z}\}, \arrow[u, hook, "(**)"']
\end{tikzcd} \tag{7.6.9}
$$

where X_{closed} (resp., Y_{closed}) is the set of all closed points of X (resp., Y), and the vertical inclusions and equalities are given by Proposition 7.3.15 and Proposition 7.4.3. Therefore, $F(\mathcal{O}_X) = M[m]$, for some $M \in \mathrm{Pic}(Y)$ and some $m \in \mathbb{Z}$.

If $F(\mathcal{O}_X) \neq \mathcal{O}_Y$, replacing F with the following composite functor

$$D^b(X) \xrightarrow{F} D^b(Y) \xrightarrow{(M^\vee \otimes -)[-m]} D^b(Y)\,, \tag{7.6.10}$$

which is an exact equivalence sending $\mathcal{O}_X$ to $\mathcal{O}_Y$, we may assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Therefore, it remains to show is that ω_Y or its dual is ample.

Step 3. We establish bijections $X_{closed} \xleftrightarrow{F} Y_{closed}$ and $\mathrm{Pic}(X) \xleftrightarrow{F} \mathrm{Pic}(Y)$.

Using the equivalence F, we first show that the vertical inclusion $(*)$ in the diagram (7.6.8) is a bijection. This immediately imply that the vertical inclusion $(**)$ in the diagram (7.6.9) is bijective by Proposition 7.4.3. Then Step 3 will follow.

By horizontal bijection in the diagram (7.6.8), for any closed point $y \in Y$ there is a closed point $x_y \in X$ and $m_y \in \mathbb{Z}$ such that $F(k(x_y)[m_y]) \cong k(y)$. Suppose on the contrary that there is a point like object $P \in D^b(Y)$, which is not of the form $k(y)[m]$, for any closed point $y \in Y$ and integer m. Because of bijection in (7.6.8), there is a unique closed point $x_P \in X$ and integer m_P such that $F(k(x_P)[m_P]) \cong P$. Then $x_P \neq x_y$, for all closed point $y \in Y$. Hence, for any closed point $y \in Y$ and any integer m, we have

$$
\begin{aligned}
\mathrm{Hom}(P, k(y)[m]) &= \mathrm{Hom}(F(k(x_P)[m_P]), k(y)[m]) \\
&= \mathrm{Hom}(k(x_P)[m_P], k(x_y)[m_y + m]) \\
&= \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) = 0,
\end{aligned} \tag{7.6.11}
$$

because $k(x_P)$ and $k(x_y)$ being skyscraper sheaves supported at different points, $\mathrm{Ext}^i(k(x_P), k(x_y)) = 0$, for all i. Since the objects $k(y)$, with $y \in Y$ a closed point, form a spanning class of $D^b(X)$ (c.f. Definition 7.5.1), $P \cong 0$ by Proposition 7.5.3, which contradicts our assumption that P is a point like object in $D^b(Y)$. Therefore, point like objects of $D^b(Y)$ are exactly of the form $k(y)[m]$, for $y \in Y$ a closed point and $m \in \mathbb{Z}$.

Note that, for any closed point $x \in X$, there is a closed point $y_x \in Y$ such that $F(k(x)) \cong k(y_x)[m_x]$, for some $m_x \in \mathbb{Z}$. Since F is fully faithful and $F(\mathcal{O}_X) = \mathcal{O}_Y$, we have $\mathrm{Hom}(\mathcal{O}_X, k(x)) = \mathrm{Hom}(\mathcal{O}_Y, k(y_x)[m_x]) = \mathrm{Ext}^{m_x}(\mathcal{O}_Y, k(y_x)) \neq 0$. This forces

$m_x = 0$, and hence $F(k(x)) \cong k(y_x)$ (no shift!). This immediately imply that, for any $L \in \text{Pic}(X)$, $F(L) \cong M$, for some $M \in \text{Pic}(Y)$. Indeed, from bijections in the diagram (7.6.9), we find unique $M \in \text{Pic}(Y)$ and $m_L \in \mathbb{Z}$ such that $F(L) \cong M[m_L]$. Take closed points $x \in X$ and $y_x \in Y$ such that $F(k(x)) \cong k(y_x)$. Then

$$\begin{aligned}\text{Ext}^{-m_L}(M, k(y_x)) &= \text{Hom}(M, k(y_x)[-m_L]) = \text{Hom}(M[m_L], k(y_x)) \\ &= \text{Hom}(F(L), F(k(x))) = \text{Hom}(L, k(x)) \neq 0.\end{aligned}$$

This forces $m_L = 0$.

Step 4. Recovering Zariski topology from derived category to conclude ampleness.

Let Z be a quasi-compact k-scheme. Denote by Z_0 the subset of all closed points of Z. Take line bundles L_1 and L_2 on Z, and take $\alpha \in \text{Hom}(L_1, L_2) = H^0(X, L_1^\vee \otimes L_2)$. For each closed point $z \in Z$, let

$$\alpha_z^* : \text{Hom}(L_2, k(z)) \longrightarrow \text{Hom}(L_1, k(z)) \tag{7.6.12}$$

be the homomorphism induced by α. Then $U_\alpha := \{z \in Z : \alpha_z^* \neq 0\}$ is a Zariski open subset of Z, and hence $U_\alpha \cap Z$ is open in Z_0.

Fix a line bundle $L_0 \in \text{Pic}(X)$. Then it follows from Proposition 7.6.1 that the collection of all such U_α, where $\alpha \in H^0(X, L_0^n)$ and $n \in \mathbb{Z}$, forms a basis for the Zariski topology on Z if and only if either L_0 or $L_0^\vee$ is ample.

By Step 3, the exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$ sends closed points of X to closed points of Y bijectively, and sends line bundles on X to line bundles on Y bijectively. In particular, $F(\omega_X^i) \cong \omega_Y^i$, for all $i \in \mathbb{Z}$. Then the natural isomorphisms $H^0(X, \omega_X^i) \cong H^0(Y, \omega_Y^i)$, $\forall\ i \in \mathbb{Z}$, give rise to a bijection between the collection of open subsets

$$\begin{aligned}\mathcal{B}_X &:= \{U_\alpha : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}, \text{ and} \\ \mathcal{B}_Y &:= \{V_\alpha : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}.\end{aligned}$$

Since ω_X (resp., $\omega_X^\vee$) is ample, $\mathcal{B}_X$ is a basis for the Zariski topology on X, and hence $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}$ is a basis for the Zariski topology on X_0. Therefore, $\mathcal{B}_{Y_0} := \{V_\alpha \cap Y_0 : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}$ is a basis for the Zariski topology on Y_0, and hence $\mathcal{B}_Y$ is a basis for the Zariski topology on Y (see Lemma 7.6.13 below). Therefore, ω_Y (resp., $\omega_Y^\vee$) is ample. This completes the proof. □

I thank Arideep Saha for useful discussion leading to the following Lemma.

Lemma 7.6.13. *Let X be a scheme locally of finite type over* $\text{Spec}(\Bbbk)$*, where $\Bbbk$ is a field or $\mathbb{Z}$. Let X_0 be a subset of X containing all closed points of X. Let $\mathcal{B}_X := \{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of X such that $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in \Lambda\}$ is a basis for the subspace Zariski topology on X_0. Then $\mathcal{B}$ is a basis for the Zariski topology on X.*

Proof. **Step 1.** First we show that, *if an open set $U \subset X$ contains a closed point x_0, then for any point $x \in X$ which contains x_0 in its closure (i.e., $x_0 \in \overline{\{x\}}$), we have $x \in U$*. Since $\mathcal{B}_{X_0}$ is a basis, there is $\alpha \in \Lambda$ such that $x_0 \in U_\alpha \cap X_0 \subseteq U \cap X_0$. If $x \notin U_\alpha$, then x belongs to the closed set $X \setminus U_\alpha$, and hence $\overline{\{x\}} \subseteq X \setminus U_\alpha$, which contradicts the assumption that $x_0 \in \overline{\{x\}}$. Therefore, $x \in U_\alpha$. Since closure of any point in X contains a closed point, it follows that $\mathcal{B}_X$ is an open cover for X.

It remains to show that for $x \in U_\alpha \cap U_\beta$, there is $\gamma \in \Lambda$ such that $x \in U_\gamma \subseteq U_\alpha \cap U_\beta$.

Step 2. Assume that, *for any open subset U of X with $x \in U$, there is a closed point $x_0 \in \overline{\{x\}} \cap U$*. For then, taking $U = U_\alpha \cap U_\beta$, we can find a $\gamma \in \Lambda$ such that

$$x_0 \in U_\gamma \cap X_0 \subseteq U_\alpha \cap U_\beta \cap X_0.$$

Then we will have $U_\gamma \subseteq U_\alpha \cap U_\beta$. Indeed, for each $z \in U_\gamma$, by *above assumption* there is a closed point $z_0 \in \overline{\{z\}} \cap U_\alpha \cap U_\beta$. Then by Step 1, we have $z \in U_\alpha \cap U_\beta$.

Step 3. We now prove the *assumption* of Step 2. Since the statement is local, we may assume that $X = \text{Spec}(A)$, for some finitely generated $\Bbbk$-algebra A. For each $f \in A$, let $D_f := \{\mathfrak{q} \in \text{Spec}(A) : f \notin \mathfrak{q}\}$. Since $\{D_f : f \in A\}$ forms a basis for the Zariski topology on $\text{Spec}(A)$, any point $\mathfrak{p} \in \text{Spec}(A)$ is contained in D_f, for some $f \in A \setminus \{0\}$. We claim that, there is a closed point (maximal ideal) $\mathfrak{m} \in D_f$ with $\mathfrak{p} \subset \mathfrak{m}$. If not, then all closed points (maximal ideal) $\mathfrak{m} \in \text{Max}(A/\mathfrak{p}) \subset \text{Spec}(A/\mathfrak{p})$ lies outside D_f. Since $A/\mathfrak{p}$ is a finitely generated $\Bbbk$-algebra, we have

$$\text{Jac}(A/\mathfrak{p}) = \bigcap_{\mathfrak{m} \in \text{Max}(A/\mathfrak{p})} \mathfrak{m} = \bigcap_{\mathfrak{q} \in \text{Spec}(A/\mathfrak{p})} \mathfrak{q} = \text{Nil}(A/\mathfrak{p}),$$

which is zero because $A/\mathfrak{p}$ is an integral domain. This contradicts the fact that $f \neq 0$ in $A/\mathfrak{p}$. This completes the proof. $\square$

Although we don't need full strength of the following Lemma 7.6.14 here, let me mention it here since it can be useful in may purpose. I thank Saurav Bhaumik for explaining it to me.

Lemma 7.6.14. *Any polarized reduced finite type projective scheme defined over any field can be reconstructed from its set of closed points.*

Proof. If $\mathscr{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is the ideal sheaf of a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, for some integer $n \geq 1$, then $X \cong \text{Proj}(S/I)$, where $I := \bigoplus_{i \geq 0} H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ is the homogeneous ideal of the graded k-algebra $S := k[x_0, \ldots, x_n]$. Therefore, it suffices to show that, I coincides with the ideal of homogeneous polynomials in S vanishing at each closed point of X. It follows from the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n_k, \mathscr{I}_X(i)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_X(i)) \longrightarrow H^0(X, \mathcal{O}_X(i)) \tag{7.6.15}$$

that $H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ can be identified with the set of all homogeneous polynomials of degree i in S that vanishes at each point of X. Therefore, it suffices to show that, if X is a finite type reduced k-subscheme of a k-scheme $\widetilde{X}$, a section $s \in H^0(\widetilde{X}, L)$ of a line bundle L on $\widetilde{X}$ vanishes at every closed points of X if and only if $s|_X = 0$. This can be checked locally. Take an affine open subset $U = \mathrm{Spec}(A)$ of X such that $L\big|_U$ is trivial. Then $s\big|_U$ is given by an element $f \in A$. Since s vanishes at every closed points of X, $f \in \mathrm{Jac}(A)$. Since X is a finite type k-scheme, $\mathrm{Jac}(A) = \mathrm{Nil}(A)$, which is zero because X is reduced. Therefore, $f = 0$, and hence $s\big|_X = 0$. Hence the result follows. □

Remark 7.6.16. There is a more geometric proof of ampleness of ω_Y or its dual in Theorem 7.1.1 when k is algebraically closed. The idea is to use the fact that line bundle is very ample if and only if it separates points and tangent vectors.

Alternative proof of ampleness of ω_Y or its dual, for $k = \overline{k}$. In this subsection, we assume that k is algebraically closed. Let X be a projective k-scheme.

Definition 7.6.17. An invertible sheaf of $\mathcal{O}_X$-modules L on X is said to be *very ample* if there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, for some $n \geq 1$, such that $L \cong \iota^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$.

It should be noted that, an invertible sheaf L on X is ample if and only if L^m is very ample, for some integer $m \gg 0$; [Har77].

Definition 7.6.18. Let L be an invertible sheaf of $\mathcal{O}_X$-modules on X. We say that,

(i) L *separates points* if for any two closed points $p, q \in X$, there is a section $s \in H^0(X, L)$ such that $s_p \in \mathfrak{m}_p L_p$ and $s_q \notin \mathfrak{m}_q L_q$.
(ii) L *separates tangent vectors* if for any closed point $p \in X$ and any tangent vector $v \in T_pX = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$, there is a non-zero section $s \in H^0(X, L)$ such that $s_p \in \mathfrak{m}_p L_p$ and $v \notin T_pV$, where V is the divisor of zero locus of s.

Note that, an invertible sheaf L on X separate tangent vectors if and only if for each closed point $x \in X$, the set $\{s \in H^0(X, L) : s_x \in \mathfrak{m}_x L_x\}$ spans the k-vector space $L_x \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$.

Theorem 7.6.19. [Har77, Proposition II.7.3] *Let X be a projective k-scheme. An invertible sheaf of $\mathcal{O}_X$-modules on X is very ample if and only if it separate points and tangent vectors.*

Continuing with above notations, it follows from the Definition 7.6.18 that $L \in \mathrm{Pic}(X)$ separates points if and only if for any two closed points $x_1, x_2 \in X$ with $x_1 \neq x_2$, the restriction homomorphism (to the fibers)

$$r_{x_1,x_2} : L \longrightarrow (L \otimes k(x_1)) \oplus (L \otimes k(x_2)) \cong k(x_1) \oplus k(x_2) \tag{7.6.20}$$

induces a surjective homomorphism

$$H^0(r_{x_1,x_2}) : H^0(X, L) \longrightarrow H^0(X, k(x_1) \oplus k(x_2)). \tag{7.6.21}$$

Let Y be a smooth projective k-variety, and $F : D^b(X) \longrightarrow D^b(Y)$ be an exact equivalence of k-linear graded categories. Then we have the following commutative diagram

$$\begin{tikzcd} H^0(X, \omega_X^i) \arrow[r, "H^0(r_{x_1,r_2})"] \arrow[d, "F"', "\cong"] & H^0(X, k(x_1) \oplus k(x_2)) \arrow[d, "\cong"', "F"] \\ H^0(Y, \omega_Y^i) \arrow[r, "H^0(r_{y_1,y_2})"] & H^0(Y, k(y_1) \oplus k(y_2)) \end{tikzcd} \tag{7.6.22}$$

where $y_j \in Y$ is the closed point such that $F(k(x_j)) = k(y_j)$, for all $j = 1, 2$. Therefore, ω_X^i separates points if and only if ω_Y^i separates points.

To see ω_Y^i separates tangent vectors if and only if ω_X^i do the same, first we need the following observation.

Lemma 7.6.23. *Let X be a scheme over any field k. To give a point $x \in X$ with residue field $k(x) = k$ and a tangent vector $v \in T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is equivalent to give a subscheme $Z_x \subset X$, supported at x, of length 2 (i.e., $\dim_k H^0(Z_x, \mathcal{O}_{Z_x}) = 2$).*

Let $Z_y \subset Y$ be a subscheme of length 2 supported at a closed point $y \in Y$. Then we have an exact sequence

$$0 \longrightarrow k(y) \longrightarrow \mathcal{O}_{Z_y} \longrightarrow k(y) \longrightarrow 0\,. \tag{7.6.24}$$

Therefore, Z_y is given by an non-trivial extension class

$$\Phi_{Z_y} \in \mathrm{Ext}^1(k(y), k(y)). \tag{7.6.25}$$

Since F is fully faithful, $\Phi_{Z_y} \in \mathrm{Ext}^1(k(y), k(y))$ corresponds to a non-trivial extension class

$$F(\Phi_{Z_y}) \in \mathrm{Ext}^1(k(x_y), k(x_y)), \tag{7.6.26}$$

where $x_y \in X$ is the closed point satisfying $F(k(x_y)) = k(y)$. Then $F(\Phi_{Z_y})$ defines a subscheme $Z_x \subset X$ of length 2 supported at $x \in X$. Therefore, $F(\mathcal{O}_{Z_x}) = \mathcal{O}_{Z_y}$. Moreover, F gives an isomorphism

$$F : \mathrm{Hom}(\omega_X^i, \mathcal{O}_{Z_x}) \xrightarrow{\simeq} \mathrm{Hom}(\omega_Y^i, \mathcal{O}_{Z_y}).$$

It follows from the Lemma 7.6.23 that a line bundle L on X separate tangent vectors if and only if the homomorphism (induced by the restriction morphism)

$$H^0(X, L) \longrightarrow H^0(X, \mathcal{O}_{Z_x}) \tag{7.6.27}$$

is surjective. Now it follows from the commutative diagram

$$\begin{tikzcd} H^0(X, \omega_X^i) \arrow[r] \arrow[d, "F"', "\simeq"] & H^0(X, \mathcal{O}_{Z_x}) \arrow[d, "\simeq"', "F"] \\ H^0(Y, \omega_Y^i) \arrow[r] & H^0(Y, \mathcal{O}_{Z_y}) \end{tikzcd} \tag{7.6.28}$$

that ω_X^i separate tangent vectors if and only if ω_Y^i separate tangent vectors. Hence, ω_X (resp., $\omega_X^\vee$) is ample if and only if ω_Y^i (resp., $\omega_Y^\vee$) is ample.

7.7. **Auto equivalence of derived category.** Let X be a smooth projective next section$k$-variety.

8. Fourier-Mukai transforms

8.1. **Integral functor.** Let X be a smooth projective scheme defined over a field k. Consider the two projections

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y\,. \tag{8.1.1}$$

Definition 8.1.2. An *integral functor* with *kernel* $P \in D^b(X \times Y)$ is a functor

$$\Phi_P^{X \to Y} : D^b(X) \longrightarrow D^b(Y) \tag{8.1.3}$$

defined by

$$\Phi_P^{X \to Y}(E) := p_{Y*}(p_X^* E \otimes P), \;\; \forall\, E \in D^b(X);$$

where p_{Y*}, p_X^* and $\otimes$ are derived functors.

When there is no confusion regarding the direction of the functor likely to arise, we just drop the superscript $X \to Y$ from $\Phi_P^{X \to Y}$, and simply denote it by Φ_P. An integral functor Φ_P, which is an equivalence of categories, is called a *Fourier-Mukai functor* with kernel P. We say that X and Y are *Fourier-Mukai partner* if there is a Fourier-Mukai transform $\Phi_P : D^b(X) \to D^b(Y)$. Since the derived functors p_X^*, p_{Y*} and $\otimes$ are exact, Φ_P is an exact functor.

8.2. **Orlov's theorem.** In this section, we prove the following version of Orlov's theorem using a deep result due to Bondal.

Theorem 8.2.1 (Orlov, Bondal)**.** *Let X and Y be two smooth projective k-varieties. Let $F : D^b(X) \longrightarrow D^b(Y)$ be an exact fully faithful functor (resp., exact equivalence of categories). Then there is an object $P \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_P$, where Φ_P is the integral functor with kernel P.*

Theorem 8.2.2 (Orlov). *Let X and Y be two smooth projective k-varieties. Let $F : D^b(X) \to D^b(Y)$ be an exact fully faithful functor admitting both left and right adjoint. Then there is an object $P \in D^b(X \times Y)$, unique up to isomorphisms, such that F is isomorphic to the integral functor Φ_P with kernel P.*

8.3. Bondal-Orlov's reconstruction theorem revisited.

9. Grothendieck group

Definition 9.0.1. Let $\mathscr{A}$ be a small abelian category. The *Grothendieck group* of $\mathscr{A}$, denoted by $K_0(\mathscr{A})$, is the quotient of the free abelian group generated by the set of all isomorphism classes of objects of $\mathscr{A}$ by its normal subgroup generated by all elements $[B]-[A]-[C]$, where $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathscr{A}$.

Let $\mathscr{A}$ be an abelian category. For an object $A \in \mathscr{A}$, we denote by $A^\bullet \in \mathcal{K}om(\mathscr{A})$ the complex defined by

$$A^i = \begin{cases} A, & \text{if} \quad i = 0, \\ 0, & \text{if} \quad i \neq 0\,. \end{cases}$$

We also denote by $A^\bullet \in D^b(\mathscr{A})$, the object in $D^b(\mathscr{A})$ represented by the complex $A^\bullet$.

Definition 9.0.2. Let $\mathscr{T}$ be a small triangulated category. For example, $\mathscr{T}$ can be the bounded derived category $D^b(\mathscr{A})$ of a small abelian category $\mathscr{A}$. Let $F(\mathscr{T})$ be the free abelian group generated by the set of all objects of $\mathscr{T}$. Let $\mathcal{R}(\mathscr{T})$ be the normal subgroup of $F(\mathscr{T})$ generated by the elements $[A]+[C]-[B] \in F(\mathscr{A})$, whenever there is a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathscr{T}$. The quotient abelian group

$$K_0(\mathscr{T}) := F(\mathscr{T})/\mathcal{R}(\mathscr{T}) \tag{9.0.3}$$

is called the *Grothendieck group* of $\mathscr{T}$.

Remark 9.0.4. It follows from the axiom (TR3) in Definition 1.2.4 that $A[1] = 0$ in $K_0(\mathscr{T})$.

The following proposition establishes relation between $K_0(\mathscr{A})$ and $K_0(D^b(\mathscr{A}))$.

Proposition 9.0.5. *Let $\mathscr{A}$ be a small abelian category, and let $D^b(\mathscr{A})$ be the bounded derived category of $\mathscr{A}$. Then the natural functor $\iota : \mathscr{A} \longrightarrow D^b(\mathscr{A})$ defined by sending an object $A \in \mathscr{A}$ to $A^\bullet \in D^b(\mathscr{A})$ induces an isomorphism of their Grothendieck groups $K_0(\mathscr{A}) \xrightarrow{\simeq} K_0(D^b(\mathscr{A}))$.*

10. Bridgeland Stability

The definition of stability condition generalized in different ways from curve to higher dimensional varieties. We follow [MS17, BBHR09, Huy06].

10.1. **Stability condition in an abelian category.** Let $\mathcal{A}$ be an abelian category.

Definition 10.1.1. A *subobject* of an object $E \in \mathcal{A}$ is a monomorphism $\iota : F \hookrightarrow E$.

Let $\iota_1 : F_1 \hookrightarrow E$ and $\iota_2 : F_2 \hookrightarrow E$ be two subobjects of $E \in \mathcal{A}$. Since direct sum of two objects in $\mathcal{A}$ is both product and coproduct in $\mathcal{A}$, by the universal property of coproduct there is a unique morphism $\varphi : F_1 \oplus F_2 \to E$ such that the following diagram commutes.

$$\begin{tikzcd} & E & \\ F_1 \arrow[ur, "\iota_1"] \arrow[r] & F_1 \oplus F_2 \arrow[u, dashed, "\varphi"] & F_2 \arrow[ul, "\iota_2"'] \arrow[l] \end{tikzcd}$$

Since the image of $\varphi : F_1 \oplus F_2 \to E$ is the kernel of the cokernel $E \to \mathrm{Coker}(\varphi)$, $\mathrm{im}(\varphi) \hookrightarrow E$ is a monomorphism, and hence is a *subobject* of E, denoted by $F_1 + F_2$. We denote by $F_1 \cap F_2$ the kernel of the epimorphism $F_1 \oplus F_2 \longrightarrow \mathrm{im}(\varphi)$ in $\mathcal{A}$. Thus, we have an exact sequence

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0. \tag{10.1.2}$$

Note that, $F_1 \cap F_2$ is the kernel of both composite morphisms

$$F_1 \xrightarrow{\iota_1} E \longrightarrow \mathrm{Coker}(\iota_2) \quad \text{and} \quad F_2 \xrightarrow{\iota_2} E \longrightarrow \mathrm{Coker}(\iota_1). \tag{10.1.3}$$

Therefore, $F_1 \cap F_2$ is a subobject of both F_1 and F_2, and hence of E. Note that, for any two morphisms $f_i : A_i \to B$, $i = 1, 2$, their fiber product $A_1 \times_{f_1,B,f_2} A_2$ exists in $\mathcal{A}$, and can be described as $\mathrm{Ker}((f_1, -f_2) : A_1 \oplus A_2 \to B)$. In particular, the *preimage* (fiber product) of a subobject $C \subset B$ along a morphism $f : A \to B$ exists uniquely as a subobject of A.

Remark 10.1.4. Note that, "being a subobject" is a transitive relation on $\mathrm{Ob}(\mathscr{C})$. (This may fails to hold if we consider equivalence class of monomorphisms instead of just monomorphism).

Let $\mathcal{A}$ be an abelian category. Denote by $K_0(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$ (c.f., Definition 9.0.1). For any complex number z, we denote by $\mathrm{Im}(z)$ (resp., $\mathrm{Re}(z)$) the *imaginary part* (resp., the *real part*) of z.

Definition 10.1.5. A *stability function* on $\mathcal{A}$ is an additive group homomorphism

$$Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

such that for any non-zero object $A \in \mathcal{A}$, we have $\mathrm{Im}(Z(A)) \geq 0$, and if $\mathrm{Im}(Z(A)) = 0$, then $\mathrm{Re}(Z(A)) < 0$.

Note that, "$\mathrm{Im}(Z(A)) \geq 0,\ \forall\ A \in \mathcal{A}$" does not imply that "$\mathrm{Im}(Z(B)) \geq 0,\ \forall\ B \in K_0(\mathcal{A})$".

Given a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$, we may think of

$$\deg_Z(A) := -\mathrm{Re}(Z(A)) \quad \text{and} \quad \mathrm{rk}_Z(A) := \mathrm{Im}(Z(A))$$

to be the *degree* and the *rank* of A with respect to the stability function Z, respectively. We may define the *slope* of $A \in \mathcal{A} \setminus \{0\}$ with respect to Z by

$$\mu_Z(A) := \begin{cases} \dfrac{\deg_Z(A)}{\mathrm{rk}_Z(A)}, & \text{if } \mathrm{rk}_Z(A) \neq 0, \text{ and} \\ +\infty, & \text{otherwise.} \end{cases} \tag{10.1.6}$$

Example 10.1.7. Let X be a smooth projective curve defined over an algebraically closed field k. Let $\mathfrak{Coh}(X)$ be the category of coherent sheaves of $\mathcal{O}_X$-modules on X. Let $Z : K_0(\mathfrak{Coh}(X)) \longrightarrow \mathbb{C}$ be the additive group homomorphism defined by sending a non-zero object $E \in \mathfrak{Coh}(X)$ to

$$Z(E) := -\deg(E) + \sqrt{-1} \cdot \mathrm{rk}(E) \in \mathbb{C}\,.$$

Clearly, Z is a stability function on $\mathfrak{Coh}(X)$. Note that, $\mu_Z(E)$ coincides with the usual slope $\mu(E) := \deg(E)/\,\mathrm{rk}(E)$ of E, and hence in this case, Z-(semi)stability coincides with the usual slope (semi)stability of coherent sheaves on X.

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k. Fix an ample class $\omega \in \mathrm{Amp}(X) \subseteq N^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and a divisor class $B \in N^1(X)$.

Definition 10.1.8 (K. Matsuki, R. Wentworth)**.** The *B-twisted Chern character* of $E \in \mathfrak{Coh}(X)$ is defined by

$$\mathrm{ch}^B(E) := \mathrm{ch}(E) \cdot e^{-B} = \sum_{i \geq 0} \mathrm{ch}_i(E) \cdot \sum_{j \geq 0} \frac{(-1)^j}{j!} B^j\,.$$

Thus, for $i \geq 0$, the *B-twisted i-th Chern character of E*, denoted $\mathrm{ch}_i^B(E)$, are given by

$$\begin{aligned} \mathrm{ch}_0^B(E) &= \mathrm{ch}_0(E) = \mathrm{rk}(E), \\ \mathrm{ch}_1^B(E) &= \mathrm{ch}_1(E) - \mathrm{ch}_0(E) \cdot B, \\ \mathrm{ch}_2^B(E) &= \mathrm{ch}_2(E) - \mathrm{ch}_1(E) \cdot B + \frac{1}{2}\,\mathrm{rk}(E) \cdot B^2\,, \end{aligned}$$

and so on. Note that, taking $B = 0$, we get back the usual Chern characters.

Define an additive group homomorphism

$$Z_{\omega,B} : K_0(\mathfrak{Coh}(X)) \longrightarrow \mathbb{C}$$

by sending a non-zero object E of $\mathfrak{Coh}(X)$ to the complex number

$$Z_{\omega,B}(E) := -\omega^{n-1} \cdot \mathrm{ch}_1^B(E) + \sqrt{-1} \cdot \omega^n \cdot \mathrm{ch}_0^B(E). \tag{10.1.9}$$

If T is a torsion coherent sheaf on X supported in dimension $\leq n-2$, then $\mathrm{rk}(T) = 0$, and the line bundle $\det(E)$ admits a nowhere vanishing global section (c.f., [Kob87, Proposition 5.6.14]). Then $\det(T) \cong \mathcal{O}_X$, and hence $\mathrm{ch}_1(T) = 0$. Therefore, $Z_{\omega,B}(T) = 0$. This shows that, $Z_{\omega,B}$ is not a stability function.

Remark 10.1.10. Let $\mathfrak{Coh}_{\leq n-2}(X)$ be the full subcategory of coherent sheaves on X whose supports have dimension $\leq n-2$, and let $\mathcal{A}$ be the localized category

$$\mathfrak{Coh}_{n,n-2}(X) = \mathfrak{Coh}(X)/\, \mathfrak{Coh}_{\leq n-2}(X)\,.$$

Then the function

$$Z_{\omega,B} : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

as defined in (10.1.9) above, is a stability function.

Definition 10.1.11. A *stability condition* is a pair $(\mathcal{A}, Z)$, where $\mathcal{A}$ is an abelian category and $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a stability function such that for any non-zero object E of $\mathcal{A}$, there is a filtration of E by its subobjects

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E\,, \tag{10.1.12}$$

such that all E_i/E_{i-1} are Z-semistable, and their Z-slopes satisfies

$$\mu_Z(E_1) > \mu_Z(E_2/E_1) > \cdots > \mu_Z(E_\ell/E_{\ell-1}).$$

Such a filtration (10.1.12) is known as *Harder-Narasimhan filtration* of E.

Proposition 10.1.13. *Let $\mathcal{A}$ be an abelian category. Given a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$, Harder-Narasimhan filtration of an object $E \in \mathcal{A}$, if it exists, is unique up to isomorphism in $\mathcal{A}$.*

Existence of Harder-Narasimhan filtration requires some additional assumption on the category $\mathscr{A}$ and the stability function Z. For this we need some definitions.

Definition 10.1.14. An additive category $\mathcal{A}$ is said to be *noetherian* if for any object $E \in \mathcal{A}$, and any any ascending chain of subobjects

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots\cdots \subseteq E,$$

of E, there is an integer $i_0 \geq 0$ such that $E_i = E_{i+1}$, for all $i \geq i_0$.

Lemma 10.1.15. *Let $\mathcal{A}$ be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function. If the image of the imaginary part of Z,*

$$\operatorname{Im}(Z) : K_0(\mathcal{A}) \to \mathbb{R},$$

is discrete in $\mathbb{R}$, then for any object $E \in \mathcal{A}$, there is a number $D_E \in \mathbb{R}$ such that for any subobject F of E in $\mathcal{A}$, we have $D(F) \leq D_E$.

Proof. Since the image of R is discrete in $\mathbb{R}$, we can do induction on $R(E)$. Our induction hypothesis would be the following: *if $E' \in \mathcal{A}$ with $R(E') < R(E)$, then there is $D_{E'} \in \mathbb{R}$ such that for any subobject $F' \subset E'$ we have $D(F') \leq D_{E'}$.*

If $R(E) = 0$, then $D(E) > 0$. Then for any subobject $F \subset E$, from the exact sequence $0 \to F \to E \to E/F \to 0$, we have $R(E) = R(F) + R(E/F)$ and $D(E) = D(F) + D(E/F)$ (since both $D := -\operatorname{Re} Z$ and $R := \operatorname{Im} Z$ are homomorphisms of additive groups). Since Z is a stability function, we have $R(F), R(E/F) \geq 0$. Then $R(F) = 0$, and hence $D(F) > 0$. Thus, $0 < D(F) \leq D(E)$; note that, $D(E/F) \geq 0$, and the inequality is strict if $E/F \neq 0$.

Assume that $R(E) > 0$. Suppose on the contrary that there is an infinite sequence of subobjects $\{F_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} D(F_n) = +\infty. \tag{10.1.16}$$

If for some $n \in \mathbb{N}$, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ implies $D(E/F_n) \geq 0$, and so $D(F_n) \leq D(E)$. Therefore, we may assume that

$$R(F_n) < R(E), \;\; \forall\, n \in \mathbb{N}\,. \tag{10.1.17}$$

Note that, (by induction) it suffices to construct *an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that*

$$D\Big(\sum_{i=1}^{k} F_{n_i}\Big) \geq k \quad \textit{and} \quad R\Big(\sum_{i=1}^{k} F_{n_i}\Big) < R(E). \tag{10.1.18}$$

Because then, $\{\sum_{i=1}^{k} F_{n_i}\}_{k \in \mathbb{N}}$ would form an increasing sequence of proper subobjects of E, contradicting the fact that $\mathcal{A}$ is noetherian.

By our assumption (10.1.16), there is $n_1 \in \mathbb{N}$ such that $D(F_{n_1}) \geq 1$. Suppose that we have constructed $n_1, \ldots, n_{k-1}$. Then we have an exact sequence

$$0 \longrightarrow F_n \cap \sum_{i=1}^{k-1} F_{n_i} \longrightarrow F_n \oplus \sum_{i=1}^{k-1} F_{n_i} \longrightarrow F_n + \sum_{i=1}^{k-1} F_{n_i} \longrightarrow 0\,, \tag{10.1.19}$$

where the sum and intersection are taken inside of E in $\mathcal{A}$ (see (10.1.2)). This gives

$$D(F_n + \sum_{i=1}^{k-1} F_{n_i}) = D(F_n) + D(\sum_{i=1}^{k-1} F_{n_i}) - D(F_n \cap \sum_{i=1}^{k-1} F_{n_i}). \tag{10.1.20}$$

Since $R(\sum\limits_{i=1}^{k-1} F_{n_i}) < R(E)$ by induction hypothesis (10.1.18) and $F_n \cap \sum\limits_{i=1}^{k-1} F_{n_i}$ is a subobject of $\sum\limits_{i=1}^{k-1} F_{n_i}$, by induction hypothesis $D(F_n \cap \sum\limits_{i=1}^{k-1} F_{n_i}) \leq D_0$, for some $D_0 \in \mathbb{R}$, which depends only on $\sum\limits_{i=1}^{k-1} F_{n_i}$. Then from (10.1.20) and (10.1.18), we have

$$D(F_n + \sum_{i=1}^{k-1} F_{n_i}) \geq D(F_n) + (k-1) - D_0. \tag{10.1.21}$$

Taking limit as $n \to +\infty$ in (10.1.21) and using (10.1.16), we have

$$\lim_{n \to +\infty} D(F_n + \sum_{i=1}^{k-1} F_{n_i}) = +\infty. \tag{10.1.22}$$

As before, it follows from the exact sequence

$$0 \longrightarrow F_n + \sum_{i=1}^{k-1} F_{n_i} \longrightarrow E \longrightarrow E/(F_n + \sum_{i=1}^{k-1} F_{n_i}) \longrightarrow 0$$

that, if $R(F_n + \sum\limits_{i=1}^{k-1} F_{n_i}) = R(E)$ then $D(F_n + \sum\limits_{i=1}^{k-1} F_{n_k}) \leq D(E)$. Therefore, in view of the limit in (10.1.22), we must have

$$R(F_n + \sum_{i=1}^{k-1} F_{n_i}) < R(E), \quad \forall\, n \gg 0. \tag{10.1.23}$$

Therefore, we can choose n_k to be some integer $n > n_{k-1}$ for which (10.1.23) holds, as claimed in (10.1.18). This completes the proof. $\square$

Theorem 10.1.24. *Let $\mathcal{A}$ be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ be a stability function. Assume that $R := \mathrm{Im}(Z) : K_0(\mathcal{A}) \to \mathbb{R}$ has discrete image in $\mathbb{R}$. Then any non-zero object $E \in \mathcal{A}$ admits a unique Harder-Narasimhan filtration.*

Proof. Since we have nothing to prove in case E is Z-semistable, we assume that E is not Z-semistable. Suppose that the image of $D := -\mathrm{Re}(Z)$ is discrete in $\mathbb{R}$ (and hence the image of Z is discrete in $\mathbb{C}$). Let $\mathcal{H}(E)$ be the convex hull in $\mathbb{C}$ of the (discrete) subset $\{Z(F) \in \mathbb{C} : F \text{ is a subobject of } E\}$. Then by Lemma 10.1.15, $\mathcal{H}(E)$ is bounded from the left side in $\mathbb{C}$. Let $\mathcal{H}_\ell$ be the half plane to the left of the straight line passing through $Z(E)$ and 0 in $\mathbb{C}$. Since the image of Z is discrete, $\mathcal{P}(E) := \mathcal{H}_\ell \cap \mathcal{H}(E)$ is a convex polygon in $\mathbb{C}$. Let $v_0 = 0, v_1, \ldots, v_{n-1}, v_n := Z(E)$

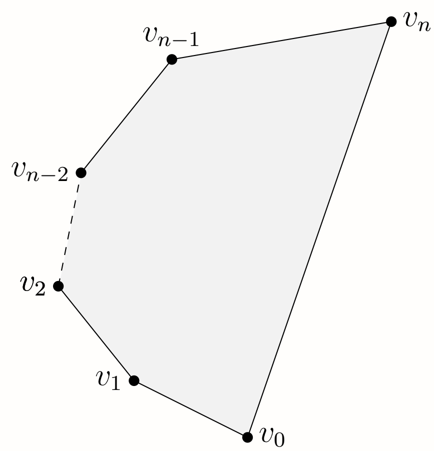


FIGURE 1. The polygon $\mathcal{P}(E)$

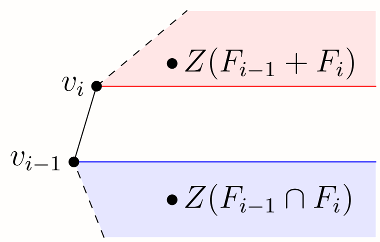


FIGURE 2. Locations of $Z(F_{i-1} \cap F_i)$ and $Z(F_{i-1} + F_i)$

be the extremal vertices of $\mathcal{P}(E)$ in the ascending order of their imaginary parts. For each $i \in \{1, \ldots, n-1\}$, choose a subobject $F_i \subset E$ with $Z(F_i) = v_i$. We claim that,

(i) $F_{i-1} \subset F_i$, for all $i = 1, \ldots, n$, with $F_0 := 0$ and $F_n := E$,
(ii) $Q_i := F_i / F_{i-1}$ is Z-semistable, for all $i = 1, \ldots, n$, and
(iii) $\mu_Z(Q_1) > \cdots > \mu_Z(Q_{n-1})$.

Since $F_{i-1} \cap F_i$ and $F_{i-1} + F_i$ are subobjects of E, we have

$$Z(F_{i-1} \cap F_i), Z(F_{i-1} + F_i) \in \mathcal{H}(E). \tag{10.1.25}$$

Moreover, we have

$$R(F_{i-1} \cap F_i) \leq R(F_{i-1}) < R(F_i) \leq R(F_{i-1} + F_i). \tag{10.1.26}$$

Therefore, $Z(F_{i-1} \cap F_i) \in \mathcal{P}(E)$ lies on or below the line $\mathrm{Im}(z) = \mathrm{Im}(v_{i-1})$, and $Z(F_{i-1} + F_i) \in \mathcal{P}(E)$ lies on or above the line $\mathrm{Im}(z) = \mathrm{Im}(v_i)$; see Figure 2. From the exact sequence $0 \to F_{i-1} \cap F_i \to F_{i-1} \oplus F_i \to F_{i-1} + F_i \to 0$, we have

$$Z(F_{i-1} \cap Z_i) + Z(F_{i-1} + F_i) = v_{i-1} + v_i\,; \tag{10.1.27}$$

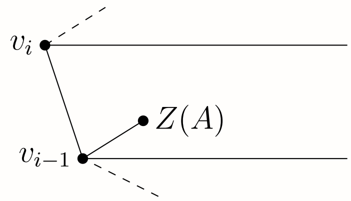


FIGURE 3. Slope of subobjects of F_i/F_{i-1}

which gives

$$Z(F_{i-1}+F_i)-Z(F_{i-1}\cap F_i)=(v_{i-1}-Z(F_{i-1}\cap F_i))+(v_i-Z(F_{i-1}\cap F_i)). \tag{10.1.28}$$

Comparing the real parts, in view of the Figure 2, we conclude that

$$Z(F_{i-1}\cap F_i)=v_{i-1}\quad\text{and}\quad Z(F_{i-1}+F_i)=v_i. \tag{10.1.29}$$

Therefore, $Z(F_{i-1}/(F_{i-1}\cap F_i))=0$. Since Z is a stability function, $F_{i-1}/(F_{i-1}\cap F_i)=0$, and hence $F_{i-1}\cap F_i=F_{i-1}$. This proves our claim (i).

Note that, the Z-slope $\mu_Z(F_i/F_{i-1})$ is given by the slope of the line segment joining v_{i-1} to v_i. Hence the convexity of the polygon $\mathcal{P}(E)$ proves claim (iii); see Figure 1. Let $\overline{A}$ be a non-zero subobject of $Q_i:=F_i/F_{i-1}$. Let $A\subset F_i$ be the preimage of $\overline{A}$ in F_i along the epimorphism $F_i\to F_i/F_{i-1}$. Then $Z(A)\in\mathcal{H}(E)$ and $R(F_{i-1})\le R(A)\le R(F_i)$, since F_{i-1} is a subobject of A. Then $Z(A)-Z(F_{i-1})=Z(A)-v_{i-1}$ has smaller of equal slope than that of $Z(F_i)-Z(F_{i-1})=v_i-v_{i-1}$. In other words, $\mu_Z(\overline{A})\le\mu_Z(Q_i)$, which proves claim (ii); c.f. Figure 3.

It remains to prove the theorem without assuming that the image of $D:=-\mathrm{Re}(Z)$ is discrete in $\mathbb{R}$. Since the image of Z need not be discrete in $\mathbb{C}$ anymore, we cannot directly conclude if the extremal vertices of $\mathcal{P}(E)$ are of the form $Z(F_i)$, for some subobject F_i of E. Suppose on the contrary that, there is an integer $i\in\{1,\ldots,n-1\}$ for which there is no subobject $F_i\subset E$ with $Z(F_i)=v_i$. By definition of $\mathcal{P}(E)$, there is a sequence of subobjects $\{F_j\}_{j\in\mathbb{N}}$ of E such that $\lim\limits_{j\to+\infty}Z(F_j)=v_i$. Since the image of $R:=\mathrm{Im}(Z)$ is discrete in $\mathbb{R}$, there is an integer $n_0(i)\ge 1$ such that $\mathrm{Im}(Z(F_j))=\mathrm{Im}(v_i)$, for all $j\ge n_0(i)$. Therefore, we can find a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that

$$R(v_i)=R(F_{n_k})\quad\text{and}\quad D(v_i)-D(F_{n_k})<\frac{1}{k},\ \ \forall\, k\in\mathbb{N}. \tag{10.1.30}$$

Therefore, we may assume that all terms in the sequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ are distinct. From the exact sequence

$$0\longrightarrow F_{n_k}\cap\sum_{j=1}^{k-1}F_{n_j}\longrightarrow F_{n_k}\oplus\sum_{j=1}^{k-1}F_{n_j}\longrightarrow\sum_{j=1}^{k}F_{n_j}\longrightarrow 0 \tag{10.1.31}$$

we have

$$D(\sum_{j=1}^{k} F_{n_k}) - D(\sum_{j=1}^{k-1} F_{n_k}) = D(F_{n_k}) - D(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) = D(F_{n_k}/F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) \tag{10.1.32}$$

and

$$R(\sum_{j=1}^{k} F_{n_k}) - R(\sum_{j=1}^{k-1} F_{n_k}) = R(F_{n_k}) - R(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) = R(F_{n_k}/F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}). \tag{10.1.33}$$

Since $F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}$ is a subobject of F_{n_k}, and Z is a stability condition,

$$R(F_{n_k}) - R(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) \geq 0, \tag{10.1.34}$$

and if equality holds in (10.1.34) then $D(F_{n_k}) - D(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) > 0$. Since $\sum_{j=1}^{k-1} F_{n_j}$ is a subobject of $\sum_{j=1}^{k} F_{n_j}$, now it follows from (10.1.32) and (10.1.33) that $\{\sum_{j=1}^{k} F_{n_j}\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of subobjects of E, which contradicts our assumption that $\mathcal{A}$ is noetherian. This completes the proof. □

11. Birational Geometry

12. Mirror Symmetry

12.1. **Fukaya category.**

Acknowledgments

The present note is growing out of a series of online discussion meetings on derived category and related topics. I would like to thank Suratno Basu, Saurav Bhaumik, Sujoy Chakraborty, Chandranandan Gangopadhyay, Arideep Saha and Ronnie Sebastian for their questions, comments, and sharing their thoughts which help and encourage me learn this subject.

References

[BBHR09] Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández Ruipérez. *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, volume 276 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2009. doi:10.1007/b11801. [↑ 65.]

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Math.*, 125(3):327–344, 2001. doi:10.1023/A:1002470302976. [↑ 45.]

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007. doi:10.4007/annals.2007.166.317. [↑ 3.]

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008. doi:10.1215/S0012-7094-08-14122-5. [Not cited.]

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. doi:10.1007/978-3-662-12492-5. [↑ 28.]

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. [↑ 30, 37, and 61.]

[Huy06] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. doi:10.1093/acprof:oso/9780199296866.001.0001. [↑ 1 and 65.]

[Kob87] Shoshichi Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987. doi:10.1515/9781400858682. Kanô Memorial Lectures, 5. [↑ 67.]

[MS17] Emanuele Macrì and Benjamin Schmidt. Lectures on Bridgeland stability. In *Moduli of curves*, volume 21 of *Lect. Notes Unione Mat. Ital.*, pages 139–211. Springer, Cham, 2017. doi:10.1007/978-3-319-59486-6_5. [↑ 65.]

[Muk81] Shigeru Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981. URL http://projecteuclid.org/euclid.nmj/1118786312. [↑ 45.]

[Sta20] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2020. [↑ 56.]