Fundamental Group Schemes

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Fundamental groups in algebraic geometry

- Topological fundamental group of a space X, together with a marked point $x_0 \in X$, is the group $\pi_1^{\mathrm{top}}(X, x_0)$ of homotopy classes of loops in X based at x_0.
- The notion of homotopy classes of loops is difficult to adopt when X is an algebraic variety or a scheme.
- However, there is a different way of looking at $\pi_1^{\mathrm{top}}(X, x_0)$, namely the group of automorphisms of the universal cover of X.
- This approach gives a better way to extend the notion of fundamental group to the case of algebraic varieties or schemes.
- In [SGA1], Alexander Grothendieck introduced étale fundamental group $\pi_1^{\text{ét}}(X, x_0)$ of a scheme X with a geometric point x_0 of X as a replacement of the notion of topological fundamental group.

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Definition of $\pi_1^{\text{ét}}(X,x)$

- A local homomorphism of local rings $\phi : A \to B$ is said to be unramified if $B/\phi(m_A)B$ is a finite separable field extension of A/m_A, where m_A is the maximal ideal of A. A morphism of schemes $f : Y \to X$ is said to be étale if the homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is flat and unramified.
- The definition of $\pi_1^{\text{ét}}(X,x_0)$ involves a category $\text{FÉt}(X)$ of finite étale covers of X, together with a fiber functor $F_{x_0} : \text{FÉt}(X) \longrightarrow (Set)$ given by sending a finite étale cover $f : Y \to X$ to the underlined set of points of the fiber $Y_{x_0} := Y \times_X \{x_0\}$; and then $\pi_1^{\text{ét}}(X,x_0) := \text{Aut}(F_{x_0})$.
- When X is a smooth projective variety over $\mathbb{C}$, it turns out that

$$\pi_1^{\text{ét}}(X,x_0) \cong \widehat{\pi_1^{\text{top}}(X_{\text{an}},x_0)},$$

the profinite completion of the topological fundamental group of the underlined complex manifold X_{an} of X.

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Nori's Fundamental group schemes

- Since the definition of $\pi_1^{\text{ét}}(X, x_0)$ involves only finite étale covers of X, it does not take care of inseparable covers of X in positive characteristics.
- To remedy the situation, for a connected proper reduced scheme X defined over a perfect field k, in [Nor76] Madhav Nori considered more general kind of covers of X, namely "essentially finite vector bundles" over X.
- Let $\mathrm{EF}(X)$ be the category of essentially finite vector bundles on X. Fix a point $x_0 \in X$, and let $T_{x_0} : \mathrm{EF}(X) \longrightarrow \mathcal{V}ect_k$ be the functor which sends a vector bundle $E \in \mathrm{EF}(X)$ to its fiber $E_{x_0} \in \mathcal{V}ect_k$ at x_0.
- The the quadruple $(\mathrm{EF}(X), \mathcal{O}_X, \otimes, T_{x_0})$ forms a "neutral Tannakian category".
- The affine k-group scheme $\pi_1^N(X, x_0)$ representing the functor $\underline{\mathrm{Aut}}^{\otimes}(T_{x_0})$ is called Nori's fundamental group scheme of X with base point at $x_0 \in X$.
- We have an equivalence of categories $\mathcal{R}ep_k^{\mathrm{fd}}(\pi_1^N(X, x_0)) \cong \mathrm{EF}(X)$.

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S-Fundamental group schemes

- Let C be a smooth projective curve over a perfect field k. In [BPS06], Indranil Biswas, A. J. Parameswaran and S. Subramanian introduced a new group scheme $\pi_1^S(C, x_0)$ by considering more general kind of covers of C, namely numerically flat vector bundles over C. This group scheme $\pi_1^S(C, x_0)$ is called the S-fundamental group scheme of C with base point at $x_0 \in C$.
- The definition of $\pi_1^S(C, x_0)$ is extended for higher dimensional connected proper k-varieties X independently by V. Mehta and A. Langer [Lan11, Lan12].
- We have an exact k-linear tensor equivalence of categories $\mathcal{R}ep_k^{\mathrm{fd}}(\pi_1^S(X, x_0)) \cong \mathscr{C}_X^{\mathrm{nf}}$, where $\mathscr{C}_X^{\mathrm{nf}}$ is the category of numerically flat vector bundles on X.
- In general $\pi_1^S(X, x_0)$ carries more geometric information about X than that of $\pi_1^N(X, x_0)$ and $\pi_1^{\text{ét}}(X, x_0)$. There are faithfully flat homomorphisms

$$\pi_1^S(X, x_0) \longrightarrow \pi_1^N(X, x_0) \longrightarrow \pi_1^{\text{ét}}(X, x_0).$$

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$\mathcal{H}ilb^n_X$, the Hilbert scheme of n points on X

- Let k be an algebraically closed field of characteristic $p \geq 0$.
- Let X be an irreducible smooth projective k-variety of dimension $d \in \{1, 2\}$.
- Fix an integer $n \geq 2$, and let $\mathcal{H}ilb^n_X$ be the Hilbert scheme parametrizing 0-cycles of length n on X.
- $\mathcal{H}ilb^n_X$ is a smooth projective k-variety of dimension nd, for $d = 1, 2$.
- $\mathcal{H}ilb^n_X$ is very important and studied by many authors; see e.g., [FGA05].
- It is an interesting problem to find the S-fundamental group scheme and Nori's fundamental group schemes of $\mathcal{H}ilb^n_X$.
- Let $S^n(X) := X^n/S_n$ be the n-fold symmetric product of X. This is a normal projective k-variety of dimension nd.

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Hilbert-Chow morphism

- Let X be an irreducible smooth projective surface over $k = \overline{k}$.
- Let $n \geq 2$. Then there is a morphism of k-schemes $\varphi : \mathcal{H}ilb_X^n \longrightarrow S^n(X)$, known as the Hilbert-Chow morphism, which sends

$$Z \in \mathcal{H}ilb_X^n \longmapsto \sum_{x \in \mathrm{Supp}(Z)} \ell(\mathcal{O}_{Z,x})[x] \in S^n(X),$$

where $\mathrm{Supp}(Z) = \{x \in X : \mathcal{O}_{Z,x} \neq 0\}$ and $\ell(\mathcal{O}_{Z,x})$ is the length of $\mathcal{O}_{Z,x}$.

- It is known that φ is a proper morphism.
- Fix a closed point $x_0 \in X$, and let $\widetilde{nx_0} \in \mathcal{H}ilb_X^n$ be such that $\varphi(\widetilde{nx_0}) = nx_0 \in S^n(X)$, where $\varphi : \mathcal{H}ilb_X^n \to S^n(X)$ is the Hilbert-Chow morphism.

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Fundamental group schemes of $\mathcal{H}ilb^n_X$

- **Notation:** For an affine k-group scheme G, denote by G_{ab} the abelianization of G. This is the largest abelian affine quotient k-group scheme of G.

Theorem (Paul, Sebastian [PS20])

Assume that $\mathrm{char}(k) = p > 3$. Let X be an irreducible smooth projective surface over k. Then there is an isomorphism of affine k-group schemes

$$f^{?} : \pi_1^{?}(X, x_0)_{\mathrm{ab}} \xrightarrow{\simeq} \pi_1^{?}(\mathcal{H}ilb^n_X, \widetilde{nx_0}),$$

where $? = S$, N, ét. In particular, $\pi_1^{?}(\mathcal{H}ilb^n_X, \widetilde{nx_0})$ is abelian.

- In [PS21], we have proved similar results for the case $\dim_k(X) = 1$ and $\mathrm{char}(k) = p > 0$.
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Outline of proof

- A point of $S^n(X)$ is of the form $\overline{x} = r_1x_1 + \cdots + r_px_p$, with $x_j \in X$ and $r_1 \geq \cdots \geq r_p$ positive integers with $\sum_{j=1}^{p} r_j = n$. We call $\langle r_1, \ldots, r_p\rangle$ the type of $\overline{x}$.
- Let $W \subset S^n(X)$ be the open subscheme consisting of points of types $\langle 1, 1, \ldots, 1\rangle$ and $\langle 2, 1, 1, \ldots, 1\rangle$, and let $V = \varphi^{-1}(W) \subset \mathcal{H}ilb^n_X$.
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- Let $\theta : G \to G'$ be a homomorphism of affine algebraic groups over k. Then
  (i) θ is faithfully flat if and only if the functor $\widetilde{\theta} : \mathcal{R}ep_k(G') \to \mathcal{R}ep_k(G)$ is fully faithful and given any subobject $W \subset \widetilde{\theta}(V')$, with $V' \in \mathcal{R}ep_k(G')$, there is a subobject $W' \subset V'$ in $\mathcal{R}ep_k(G')$ such that $\widetilde{\theta}(W') \cong W$ in $\mathcal{R}ep_k(G)$.
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Thank you!