Notes on derived category

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Abstract. In this note, we discuss basic theory of derived category following [Huy06]. After discussing some basic theories, we are interested to explore some of its applications in algebraic geometry.

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Caution! I have started writing this note (around June 2020) for myself to learn basic theories of derived category, triangulated category, Bridgeland stability conditions, their connections with mathematical physics, and more importantly their applications in various areas of algebraic geometry. The present note is unorganized, incomplete, and may contains many inaccuracies to be fixed. I may post a polished version (as an expository note) in my home page after a year. Any suggestions to improve the exposition are welcome.

Update (July 25, 2020): Since the present note become quite long, I have decided to keep its main focus on basic theories of derived category only. I am planning to write a separate note focusing on Bridgeland stability.

Current version: https://arjunpaul29.github.io/home/notes/derived.pdf.

0. Introduction

One of my preliminary motivation to start this series of discussions is to understand *stability condition* in triangulated category as introduced by Tom Bridgeland in his celebrated 2007 paper published in the Annals of Mathematics [Bri07]. I also would like to learn some of its application in some other areas, like birational geometry, mirror symmetry etc.

Bridgeland's original motivation was to mathematically formulate the concept of Π-*stability* in theoretical physics as formulated by Douglas. In physics, Π-stability is something to relate a *super-symmetric non-linear sigma model* with a *(2, 2) Super*

Conformal Field Theory (SCFT). Let's have a quick tour into an interesting intersection of geometry and physics.

0.1. **Motivation from modern physics.** I am not an expert in mathematical physics, but am interested to understand its relation with mathematics, in particular with algebraic geometry. After exploring various available sources, what I initially found and become interested in, are summarized below.

Let us start with a tailor of a largely speculating theory, known as *mirror symmetry*. A *super-symmetric non-linear sigma model* consists of a complex Calabi-Yau variety $X = (M, I)$ admitting a Ricci flat Kähler form ω and a "B-field". Let us explain the terminologies:

- M is the underlined real manifold of X and I is a complex structure on it,
- the variety X is *Calabi-Yau* means that the canonical line bundle K_X is trivial,
- the Kähler form ω is *Ricci flat* means that the curvature $F_{\det(\nabla_\omega)} = 0$, where $\det(\nabla_\omega)$ is the connection on $\det(TX)$ induced by the Chern connection ∇_ω on TX with respect to the Kähler form ω, and
- that "B-field" is something mysterious.

  In the context of SYZ mirror symmetry (an attempt to understand mathematically original version of mirror symmetry in physics), a B-field should be a class of a *unitary flat gerbe*, as suggested by Hitchin.

It is expected from physical ground that such a *super-symmetric non-linear sigma model* should give us a *(2, 2) Super Conformal Field Theory (SCFT)*. However, we don't know any precise mathematical formulation of (2, 2) SCFT, except for few cases! Roughly, a (2, 2) SCFT is some physical theory that depends on both complex and symplectic structures of varieties, and using *topological twists* one may separate its parts:

- *A-side*: depend only on symplectic structure, and
- *B-side*: depend only on complex structure.

In his famous ICM talk in 1994, Maxim Kontsevich proposed that the mathematical objects obtained from these topological twists should be in the derived category of coherent sheaves on the B-side (algebraic side), and in the derived Fukaya category of Lagrangian submanifolds on the A-side (symplectic side). Physically, objects of these categories are considered to be boundary conditions, known as *branes*. In this sense, Fukaya category is the category of A-branes and the derived category of coherent sheaves is the category of B-branes.

Conjecture 0.1.1 (Kontsevich). *If two super-symmetric non-linear sigma models (X, ω, B) and (X', ω', B'), as described above, defines mirror symmetric SCFTs, then there are equivalences of categories:*

$$D^b(X) \simeq D^b(\mathrm{Fukaya}(X', \omega')) \quad \textit{and} \quad D^b(\mathrm{Fukaya}(X, \omega)) \simeq D^b(X')\,.$$

This is mathematically quite vague because we don't have precise mathematical formulation of SCFT!

From mathematical point of view, Kontsevich's Conjecture may be considered as a definition of *homological mirror symmetry*. Two super symmetric non-linear sigma models (X, ω, B) and (X', ω', B') are said to be *homological mirror partner* to each other if there are equivalences of such derived categories.

Remark 0.1.2. I think, finding explicit examples of such homological mirror symmetric pairs of super-symmetric non-linear sigma models would be very difficult problem. There is a notion of mirror symmetric varieties in SYZ sense, which identifies X and X' as dual to each other in an appropriate sense; see e.g., works of Hitchin, Hausel-Thaddues, Donagi-Pantev etc. This notion is different from the notion of homological mirror symmetry.

We shall see from construction of $D^b(X)$ that the derived category $D^b(X)$ depends only on complex/algebraic structure of X, and so $D^b(X)$ keeps only half information of the SCFT. Douglas argued that for any Ricci flat Kähler metric ω on X, there is a subcategory of $D^b(X)$, whose objects are physical branes, and these subcategories changes as the Kähler class ω moves in the stringy Kähler moduli.

To get an intuitive idea what this mathematically means, instead of looking at whole $D^b(X)$, consider the abelian category $\mathfrak{Coh}(X)$. Then a choice of Kähler class (or polarization) singles out semistable and stable objects of $\mathfrak{Coh}(X)$, and as we change the polarization, the collection of stable/semistable objects changes. Thus, there might be some way to encode more information of SCFT purely in terms of triangulated category $D^b(X)$ together with some extra structure on it. In a series of papers, Bridgeland set out to put these ideas on a mathematical setting and introduced the notion of *stability conditions* on a triangulated category. He has shown that the space of such stability conditions forms an (infinite) dimensional manifold, and this can be thought of an approximation of the stringy Kähler moduli space.

Mathematically, interesting point is that this new theory associates to a very algebraic object, like a triangulated category, a moduli space with meaningful geometric structure.

Roughly, a *stability condition* on a triangulated category $\mathcal{A}$ is given by a heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{A}$ and an additive group homomorphism $Z : K_0(\mathcal{H}) \to \mathbb{C}$, called the "central charge", satisfying Harder-Narasimhan property.

Well, enough introduction, and we shall see these in detail! Let's first set up some languages from category theory.

1. SOME CATEGORY THEORY

Joke: Category theory is like Ramayana and Mahabharata — there are lots of arrows!

— *Nitin Nitsure*

1.1. Abelian category.

Definition 1.1.1. A *category* $\mathscr{A}$ consists of the following data:

(i) a class of objects, denoted $\mathrm{Ob}(\mathscr{A})$,
(ii) for $X, Y \in \mathrm{Ob}(\mathscr{A})$, a class of morphisms from X into Y, denoted $\mathrm{Mor}_{\mathscr{A}}(X, Y)$,
(iii) for each $X, Y, Z \in \mathrm{Ob}(\mathscr{A})$, a *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \to \mathrm{Mor}_{\mathscr{A}}(X, Z), \;\; (f, g) \mapsto g \circ f\,,$$

which satisfies associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$, $g \in \mathrm{Mor}_{\mathcal{A}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathcal{A}}(Z, W)$, for all $X, Y, Z, W \in \mathrm{Ob}(\mathscr{A})$.

A category $\mathscr{A}$ is said to be *locally small* if $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ is a set, for all $X, Y \in \mathrm{Ob}(\mathscr{A})$. A category $\mathscr{A}$ is said to be *small* if it is locally small and the class of objects $\mathrm{Ob}(\mathscr{A})$ is a set.

Example 1.1.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{A}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{A}$ such that $g \circ f = \mathrm{Id}_{A_1}$ and $f \circ g = \mathrm{Id}_{A_2}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,
(ii) for $X, Y \in \mathscr{A}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, there is $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, which are compatible with the composition maps.

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$$

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ is given by the following data: for each object $A \in \mathscr{A}$, a map $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_A"'] & \mathcal{F}(A') \arrow[d, "\varphi_{A'}"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A') \end{tikzcd} \tag{1.1.3}$$

Definition 1.1.4. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Mor}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathcal{F}un(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 1.1.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Two objects $\mathcal{F}, \mathcal{G} \in \mathcal{F}un(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 1.1.6. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Mor}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Notation. For any pre-additive category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ the abelian group $\mathrm{Mor}_{\mathscr{A}}(X, Y)$, for all $X, Y \in \mathrm{Ob}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 1.1.7 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.

(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0, 0)$ is the trivial group with one element.

(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $\mathrm{j}_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 1.1.8. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A, B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)\,, \quad (f, g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 1.1.9. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1, A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 1.1.10. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
& C \arrow[dl, dashed, "\exists ! \widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined by reversing the arrows of the above diagram.

Definition 1.1.11. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and
(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists ! \widetilde{g}"] \\ & C & \end{tikzcd}$$

Definition 1.1.12. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 1.1.13. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.
(ii) If a cokernel of f exists, then it is an epimorphism.
(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.
(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). $\square$

Exercise 1.1.14. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 1.1.15. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence $\mathrm{Coim}(f) \to Y$

factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$ (see Lemma 1.1.13 (iii)). This completes the proof.

(1.1.16)
$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\ & & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & & \end{tikzcd}$$

□

Definition 1.1.17. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 1.1.12).

Example 1.1.18. (1) For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.

(2) Let X be a scheme. Let $\mathfrak{Mod}(X)$ be the category of sheaves of $\mathcal{O}_X$-modules on X. Then $\mathfrak{Mod}(X)$ is abelian. The full subcategory $\mathfrak{QCoh}(X)$ (reps., $\mathfrak{Coh}(X)$) of $\mathfrak{Mod}(X)$ consisting of quasi-coherent (resp., coherent) sheaves of $\mathcal{O}_X$-modules on X, are also abelian. However, the full subcategory $\mathcal{V}ect(X)$ of $\mathfrak{Mod}(X)$ consisting of locally free coherent sheaves of $\mathcal{O}_X$-modules on X, is not abelian, because kernel of a morphism in $\mathcal{V}ect(X)$ may not be in $\mathcal{V}ect(X)$.

1.2. **Triangulated category.** Let $\mathscr{A}$ be an additive category. A *shift functor* is an additive functor

$$T : \mathscr{A} \longrightarrow \mathscr{A}\,, \tag{1.2.1}$$

which is an equivalence of categories. A *triangle* in $(\mathscr{A}, T)$ is given by a diagram

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,, \tag{1.2.2}$$

with objects and arrows in $\mathscr{A}$. A *morphism of triangles* in $(\mathscr{A}, T)$ is given by a commutative diagram

(1.2.3)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "{f[1]}"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

where $f[1] := T(f) \in \mathrm{Hom}_{\mathscr{A}}(A[1], A'[1])$. If, in addition, f, g, h are isomorphisms in $\mathscr{A}$, we say that (1.2.3) is an isomorphism of triangles. We denote by $A[n]$ the object $T^n(A) \in \mathscr{A}$, and denote by $f[n]$ the morphism $T^n(f) \in \mathrm{Hom}_{\mathscr{A}}(A[n], B[n])$, for $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$.

Definition 1.2.4. A *triangulated category* is an additive category $\mathscr{A}$ together with an additive equivalence (*shift functor*)

$$T : \mathscr{A} \longrightarrow \mathscr{A}\,, \tag{1.2.5}$$

and a set of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] := T(A)\,, \tag{1.2.6}$$

satisfying the following axioms (TR1) – (TR4) below.

(TR1) (i) Any triangle of the form

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is a distinguished triangle.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]\,.$$

(TR2) A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$$

is a distinguished triangle.

(TR3) Any commutative diagram of distinguished triangles with vertical arrows f and g

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dashed, "\exists\, h"'] & A[1] \arrow[d, "f[1]"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd} \tag{1.2.7}$$

can be completed to a commutative diagram (not necessarily in a unique way).

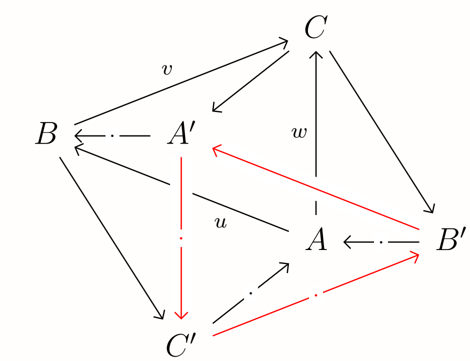
(TR4) (Octahedral axiom) Given any three distinguished triangles

$$A \xrightarrow{u} B \longrightarrow C' \longrightarrow A[1]$$
$$B \xrightarrow{v} C \longrightarrow A' \longrightarrow B[1]$$
$$A \xrightarrow{w} C \longrightarrow B' \longrightarrow A[1]$$

there is a distinguished triangle $C' \longrightarrow B' \longrightarrow A' \longrightarrow C'[1]$ such that the following diagram is commutative.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{Id}"] & B \arrow[r] \arrow[d, "v"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "w"] \arrow[d, "u"] & C \arrow[r] \arrow[d, "\mathrm{Id}"] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "u[1]"] \\
B \arrow[r, "v"] \arrow[d] & C \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\mathrm{Id}"] & B[1] \arrow[d] \\
C' \arrow[r] & B' \arrow[r] & A' \arrow[r] & C'[1]
\end{tikzcd}$$

This axiom is called *"octahedron axiom"* because of its original formulation: given composable morphisms $A \xrightarrow{u} B \xrightarrow{v} C$, with $w := v \circ u$, we have the following octahedron diagram.



where any triangle of the form

$$\begin{tikzcd}
 & Z \arrow[dl, dashed] & \\
X \arrow[rr] & & Y \arrow[ul]
\end{tikzcd}$$

are distinguished triangles, and the arrow $Z \dashrightarrow X$ stands for $Z \longrightarrow X[1]$.

Remark 1.2.8. A triangulated category need not be abelian, in general. In triangulated category, distinguished triangles play the roles of exact sequences in abelian categories. Examples of triangulated categories, we will be interested in, are derived categories of abelian categories.

Definition 1.2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulated categories. An *exact functor* of triangulated categories $\mathcal{A}$ to $\mathcal{B}$ is a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ such that for any distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ in $\mathcal{A}$, there is an isomorphism $F(A[1]) \xrightarrow{\phi} F(A)[1]$ such that

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{\phi \circ F(h)} F(A)[1]$$

is a distinguished triangle in $\mathcal{B}$. By a *morphism of triangulated categories*, we always mean an exact functor between them.

Definition 1.2.10 (Adjoint functors)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between any two categories. A functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ is said to be *right adjoint to F*, written as $F \dashv G$, if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, G(B)), \ \forall\, A \in \mathcal{A},\ B \in \mathcal{B}\,, \tag{1.2.11}$$

which is functorial in both A and B.

Similarly, a functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ is said to be *left adjoint to F*, written as $H \dashv F$, if there is an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(B, F(A)) \cong \mathrm{Hom}_{\mathcal{A}}(H(B), A), \ \forall\, A \in \mathcal{A},\ B \in \mathcal{B}\,,$$

which is functorial in both A and B.

Remark 1.2.12.

(1) Note that, G is right adjoint to F if and only if F is left adjoint to G.

(2) If $F \dashv G$, then $\mathrm{Id}_{F(A)} \in \mathrm{Hom}_{\mathcal{B}}(F(A), F(A)) \cong \mathrm{Hom}_{\mathcal{A}}(A, (G \circ F)(A))$ induces a morphism $A \longrightarrow (G \circ F)(A)$, for all $A \in \mathcal{A}$. The naturality of this morphism gives us a morphism of functors

$$\mathrm{Id}_{\mathcal{A}} \longrightarrow G \circ F\,.$$

Similarly, taking $A = G(B)$ in (1.2.11), we get a morphism of functors

$$F \circ G \longrightarrow \mathrm{Id}_{\mathcal{B}}\,.$$

In particular, if F and G are quasi-inverse to each other (in case of equivalence of categories), then one is both left and right adjoint to the other one.

(3) Using Yoneda lemma, one can check that, left (resp., right) adjoint of a functor, if it exists, is unique up to isomorphisms.

Proposition 1.2.13. *Let $F : \mathcal{D} \longrightarrow \mathcal{D}'$ be an exact functor of triangulated categories. Let $G : \mathcal{D}' \longrightarrow \mathcal{D}$ be a functor. If $F \dashv G$, or $G \dashv F$, then G is also exact.*

1.3. **Semi-orthogonal decomposition.** Let $\mathcal{D}$ be a k-linear triangulated category. An object $E \in \mathcal{D}$ is said to be *exceptional* if

$$\mathrm{Hom}_{\mathcal{D}}(E, E[\ell]) = \begin{cases} k & \text{if} \quad \ell = 0\,, \\ 0 & \text{if} \quad \ell \neq 0\,. \end{cases} \tag{1.3.1}$$

An *exceptional sequence* in $\mathcal{D}$ is a sequence of exceptional objects $E_1, E_2, \ldots, E_n$ of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = 0$, for all $i > j$ and all ℓ. In other words, if

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[\ell]) = \begin{cases} k & \text{if} \quad i = j \text{ and } \ell = 0\,, \\ 0 & \text{if} \quad i > j, \text{ or if } \ell \neq 0 \text{ and } i = j\,. \end{cases} \tag{1.3.2}$$

An exceptional sequence $\{E_i\}_{i=1}^n$ is said to be *full* if, as a triangulated category, $\mathcal{D}$ is generated by $\{E_i\}_{i=1}^n$; i.e., if $\mathscr{D}'$ is a triangulated full subcategory of $\mathcal{D}$ containing E_i, for all $i = 1, \ldots, n$, then the inclusion morphism $\mathcal{D}' \hookrightarrow \mathcal{D}$ is an equivalence of categories.

Lemma 1.3.3. *Let $\mathcal{D}$ be a k-linear triangulated category such that $\bigoplus_i \mathrm{Hom}(A, B[i])$ is a finite dimensional k-vector space, for all $A, B \in \mathcal{D}$. If $E \in \mathcal{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus n_i}$ forms an admissible triangulated subcategories.*

1.4. **t-structure and heart.** Let $(\mathscr{A}, T)$ be a triangulated category. Let $\mathscr{B}$ be a subcategory of $\mathscr{A}$. For an integer n, we denote by $\mathscr{B}[n]$ the full subcategory of $\mathscr{A}$, whose objects are of the form $X[n]$, with $X \in \mathscr{B}$. In other words, $\mathscr{B}[n] = T^n(\mathscr{B}) \subset \mathscr{A}$.

Definition 1.4.1. Let $\mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 0}$ be two full subcategories of $\mathscr{A}$. For an integer n, let $\mathscr{A}^{\leq n} := \mathscr{A}^{\leq 0}[-n]$ and $\mathscr{A}^{\geq n} := \mathscr{A}^{\geq 0}[-n]$. A *$t$-structure* on $\mathscr{A}$ is given by a pair $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ of full subcategories of $\mathscr{A}$ satisfying the following axioms.

(t1) $\mathscr{A}^{\leq -1} \subset \mathscr{A}^{\leq 0}$ and $\mathscr{A}^{\geq 1} \subset \mathscr{A}^{\geq 0}$.
(t2) For any $X \in \mathscr{A}^{\leq 0}$ and $Y \in \mathscr{A}^{\geq 1}$, we have $\mathrm{Hom}_{\mathscr{A}}(X, Y) = 0$.
(t3) For any $X \in \mathscr{A}$, there is a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]\,,$$

with $X_0 \in \mathscr{A}^{\leq 0}$ and $X_1 \in \mathscr{A}^{\geq 1}$.

In this case, the full subcategory $\mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ of $\mathscr{A}$ is called the *heart* (or, *core*) of the t-structure $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$.

Example 1.4.2 (Standard t-structure on $D^b(X)$)**.** Consider the full subcategories

$$\mathcal{A}^{\leq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i > 0\} \text{ and}$$
$$\mathcal{A}^{\geq 0} := \{E^\bullet \in D^b(X) : \mathcal{H}^i(E^\bullet) = 0,\ \forall\, i < 0\}$$

of $D^b(X)$. The axiom (T1) is easy to see. To check axiom (T2), we need some notations. For an integer $n \in \mathbb{Z}$, let $D^{\leq n}(X)$ (resp., $D^{\geq n}$) be the full subcategory of $D^b(X)$, whose objects are $E^\bullet \in D^b(X)$ satisfying $E^i = 0$, for all $i \leq n$ (resp., for all $i \geq n$). Consider the *truncation functors*

$$\tau^{\leq n} : D^b(X) \longrightarrow D^{\leq n}(X) \quad \text{and} \quad \tau^{\geq n} : D^b(X) \longrightarrow D^{\geq n}(X) \tag{1.4.3}$$

defined by

$$\tau^{\leq n}(E^\bullet) := (\cdots \to E^{n-2} \to E^{n-1} \to \mathrm{Ker}(d^n_{E^\bullet}) \to 0 \to \cdots), \text{ and}$$
$$\tau^{\geq n}(E^\bullet) := (\cdots \to 0 \to \mathrm{Ker}(d^n_{E^\bullet}) \to E^n \to E^{n+1} \to \cdots)\,.$$

where $d^n_{E^\bullet} : E^n \to E^{n+1}$, and $E^\bullet \in D^b(X)$. Now take $E^\bullet \in \mathcal{A}^{\leq 0}$ and $F^\bullet \in \mathcal{A}^{\geq 1}$. If $f \in \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet)$, then f factors as

$$\begin{tikzcd}
E^\bullet \arrow[rrr, "f"] \arrow[dr, "\cong", "\tau^{\leq 0}"'] & & & F^\bullet \\
& \tau^{\leq 0}(E^\bullet) \arrow[r, "\tau^{\leq 0}(f)"] & \tau^{\leq 0}(F^\bullet) \arrow[ur] &
\end{tikzcd}.$$

Since $\tau^{\leq 0}(F^\bullet) = 0$ in $D^b(X)$, we conclude that $f = 0$. Axiom (T3) follows from the exact triangle

$$\tau^{\leq 0}(E^\bullet) \longrightarrow E^\bullet \longrightarrow \tau^{\geq 1}(E^\bullet) \longrightarrow \tau^{\leq 0}(E^\bullet)[1], \ \ \forall\, E^\bullet \in D^b(X)\,.$$

Thus $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a t-structure on $D^b(X)$, and the associated heart $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ is isomorphic to $\mathfrak{Coh}(X)$.

The above mentioned t-structure on $D^b(X)$ is not interesting, and somehow useless. We shall be interested in some non-trivial t-structures on $D^b(X)$ giving more interesting and useful hearts different from $\mathfrak{Coh}(X)$.

The next proposition shows that, the truncation functors exists for general triangulated category admitting a t-structure.

Proposition 1.4.4. *Let $\iota : \mathscr{A}^{\geq n} \to \mathscr{A}$ (resp., $\iota' : \mathscr{A}^{\geq n} \to \mathscr{A}$) be the inclusion functor. Then there is a functor $\tau^{\geq n} : \mathscr{A} \to \mathscr{A}^{\geq n}$ (resp., $\tau^{\leq n} : \mathscr{A} \to \mathscr{A}^{\leq n}$) such that for any $X \in \mathscr{A}$ and $Y \in \mathscr{A}^{\leq n}$ (resp., $Y \in \mathscr{A}^{\geq n}$), we have an isomorphism*

$$\mathrm{Hom}_{\mathscr{A}^{\leq n}}(Y, \tau^{\leq n}(X)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{A}}(X, \iota'(Y)) \tag{1.4.5}$$

(resp.,

$$\mathrm{Hom}_{\mathscr{A}^{\geq n}}(\tau^{\geq n}(X), Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{A}}(X, \iota(Y))\,\big). \tag{1.4.6}$$

Lemma 1.4.7. *Let $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ be a bounded t-structure on a triangulated category $\mathscr{A}$. Then $\mathcal{H} := \mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ is abelian.*

Proof. Let $\mathcal{H} := \mathscr{A}^{\leq 0} \cap \mathscr{A}^{\geq 0}$ be the heart of a bounded t-structure $(\mathscr{A}^{\leq 0}, \mathscr{A}^{\geq 0})$ on $\mathscr{D}$. □

Remark 1.4.8. If $D^b(\mathcal{A}) \cong D^b(X)$ for some abelian subcategory $\mathcal{A}$ of $D^b(X)$, then $\mathcal{A}$ is a heart of a t-structure on $D^b(X)$. However, the converse is not true, in general.

1.5. **Tensor Triangulated Category.**

2. Derived category

2.1. Category of complexes. Let $\mathscr{A}$ be an abelian category. A *complex* in $\mathscr{A}$ is given by

$$A^\bullet \quad : \quad \cdots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \cdots , \tag{2.1.1}$$

where A^i are objects of $\mathscr{A}$ and d_A^i are morphisms in $\mathscr{A}$ such that $d_A^i \circ d_A^{i-1} = 0$, for all $i \in \mathbb{Z}$. A complex $A^\bullet$ in $\mathscr{A}$ is said to be *bounded above* (resp., *bounded below*) if there is an integer i_0 such that $A^i = 0$, for all $i \geq i_0$ (resp., if there is an integer j_0 such that $A^j = 0$, for all $j \leq j_0$). If $A^\bullet$ is both bounded above and bounded below, we say that $A^\bullet$ is *bounded*.

A morphism $f^\bullet : A^\bullet \to B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ of objects and morphisms from $\mathscr{A}$ is given by a collection of morphisms $\{f^i : A^i \to B^i\}_{i \in \mathbb{Z}}$ in $\mathscr{A}$ such that the following diagram commutes.

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & A^i \arrow[r, "d_B^i"] & A^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots \end{tikzcd} \tag{2.1.2}$$

Let $\mathcal{K}om(\mathscr{A})$ be the category, whose objects are complexes of objects and morphisms from $\mathscr{A}$, and morphisms are given by morphism of complexes, as defined in (2.1.2). Denote by $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ the full subcategories of $\mathcal{K}om(\mathscr{A})$, whose objects are bounded above complexes, resp., bounded below complexes, resp., bounded complexes. Then we have the following.

Proposition 2.1.3. *For any abelian category $\mathscr{A}$, the categories $\mathcal{K}om(\mathscr{A})$, $\mathcal{K}om^-(\mathscr{A})$, $\mathcal{K}om^+(\mathscr{A})$ and $\mathcal{K}om^b(\mathscr{A})$ are abelian.*

Definition 2.1.4. For any complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ and $k \in \mathbb{Z}$, we define its k^{th}-shift to be the complex $A[k]^\bullet \in \mathcal{K}om(\mathscr{A})$ satisfying

(i) $A[k]^i := A^{k+i}$, for all $i \in \mathbb{Z}$, and
(ii) $d^i_{A[k]^\bullet} := (-1)^k d^{i+k}_{A^\bullet} : A[k]^i \longrightarrow A[k]^{i+1}$, for all $i \in \mathbb{Z}$.

Proposition 2.1.5. *For any integer k, the k^{th}-shift functor*

$$\mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{A}), \quad A^\bullet \longmapsto A[k]^\bullet$$

is an equivalence of categories.

Proof. Clearly, the $(-k)^{\text{th}}$-shift functor $A^\bullet \longmapsto A[-k]^\bullet$ defines the inverse functor of the k^{th}-shift functor. $\square$

Remark 2.1.6. We shall see later that the category $\mathcal{K}om(\mathscr{A})$ together with the shift functor do not form a triangulated category, in general. However, we shall construct the derived category $D^b(\mathscr{A})$ from $\mathcal{K}om(\mathscr{A})$, which will turn out to be a triangulated category.

Given a complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, we define its i^{th} cohomology sheaf

$$\mathcal{H}^i(A^\bullet) := \frac{\mathrm{Ker}(d^i_{A^\bullet})}{\mathrm{Im}(d^{i-1}_{A^\bullet})} \in \mathscr{A}\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{2.1.7}$$

A complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$ is said to be *acyclic* if $\mathcal{H}^i(A^\bullet) = 0$, for all $i \in \mathbb{Z}$. Any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$ of complexes gives rise to natural homomorphisms

$$\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet)\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{2.1.8}$$

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories. Let

$$\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B} \tag{2.1.9}$$

be an additive functor. Then $\mathcal{F}$ induces a functor, also denoted by the same symbol,

$$\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathcal{K}om(\mathscr{B}) \tag{2.1.10}$$

defined by sending $A^\bullet \in \mathcal{K}om(\mathscr{A})$ to the complex $\mathcal{F}(A^\bullet)$, defined by

(i) $\mathcal{F}(A^\bullet)^i := \mathcal{F}(A^i)$, for all $i \in \mathbb{Z}$, and

(ii) $d^i_{\mathcal{F}(A^\bullet)} : \mathcal{F}(A^i) \xrightarrow{F(d^i_{A^\bullet})} \mathcal{F}(A^{i+1})$, for all $i \in \mathbb{Z}$,

and for any morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$, we have a natural morphism of complexes

$$\mathcal{F}(f^\bullet) : \mathcal{F}(A^\bullet) \longrightarrow \mathcal{F}(B^\bullet)$$

defined by $\mathcal{F}(f^\bullet)^i := \mathcal{F}(f^i) : \mathcal{F}(A^i) \longrightarrow \mathcal{F}(B^i)$, for all $i \in \mathbb{Z}$.

Definition 2.1.11. An additive functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *exact* if it takes exact sequence to exact sequence.

Remark 2.1.12. Note that $\mathcal{F}$ is exact if and only if for any acyclic complex $A^\bullet \in \mathcal{K}om(\mathscr{A})$, its image $\mathcal{F}(A^\bullet) \in \mathcal{K}om(\mathscr{B})$ is acyclic.

Since $\mathcal{K}om(\mathscr{A})$ is abelian for $\mathscr{A}$ abelian, we can talk about short exact sequences in $\mathcal{K}om(\mathscr{A})$. Then by standard techniques from homological algebra, any short exact sequence

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0 \tag{2.1.13}$$

gives rise to a long exact sequence of cohomologies (which are objects of $\mathscr{A}$)

$$\cdots \longrightarrow \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \longrightarrow \mathcal{H}^i(C^\bullet) \longrightarrow \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \cdots\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{2.1.14}$$

Definition 2.1.15. A morphism of complexes $f^\bullet : A^\bullet \longrightarrow B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is called *quasi-isomorphism* if the induced morphism

$$\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet) \tag{2.1.16}$$

is an isomorphism, for all $i \in \mathbb{Z}$.

Example 2.1.17. Let X be a smooth projective k-variety and let E be a coherent sheaf on X. Then we can find a finite resolution

$$0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0 \to E \to 0\,.$$

of E with E^i projective (locally free) $\mathcal{O}_X$-modules. (We can use this to study many properties of E in terms of locally free coherent sheaves.) This gives rise to a morphism of complexes

$$f^\bullet : (0 \to E^n \to E^{n-1} \to \cdots \to E^1 \to E^0) \longrightarrow (\cdots \to 0 \to E \to 0 \to \cdots),$$

which is a quasi-isomorphism.

2.2. **What is a derived category?** The main idea for definition of derived category is: *quasi-isomorphism of complexes should become isomorphism in the derived category*. Therefore, the derived category $D(\mathscr{A})$ is the localization of $\mathcal{K}om(\mathscr{A})$ by quasi-isomorphisms. This can be done by passing to the appropriate homotopy category.

Theorem 2.2.1. *Let $\mathscr{A}$ be an abelian category, and $\mathcal{K}om(\mathscr{A})$ the category of complexes in $\mathscr{A}$. Then there is a category $D(\mathscr{A})$, known as the derived category of $\mathscr{A}$, together with a functor*

$$Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A}) \tag{2.2.2}$$

such that:

(i) If $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a quasi-isomorphism, then $Q(f^\bullet)$ is an isomorphism in $D(\mathscr{A})$,
(ii) if a functor $\mathcal{F} : \mathcal{K}om(\mathscr{A}) \longrightarrow \mathscr{D}$ satisfies property (i), there is a unique functor $\widetilde{\mathcal{F}} : D(\mathscr{A}) \longrightarrow \mathscr{D}$ such that $\widetilde{\mathcal{F}} \circ Q \cong \mathcal{F}$.

$$\begin{tikzcd} \mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "\mathcal{F}"'] & & D(\mathscr{A}) \arrow[dl, dashed, "\exists\,!\,\widetilde{\mathcal{F}}"] \\ & \mathscr{D} & \end{tikzcd} \tag{2.2.3}$$

Now we go ahead for construction of the derived category $D(\mathscr{A})$ of $\mathscr{A}$. Since we want any quasi-isomorphism $C^\bullet \to A^\bullet$ of complexes in $\mathcal{K}om(\mathscr{A})$ to become isomorphism in the derived category $D(\mathscr{A})$, any morphism of complexes $C^\bullet \to B^\bullet$

in $\mathcal{K}om(\mathscr{A})$ should give rise to a morphism $A^\bullet \to B^\bullet$ in $D^b(\mathscr{A})$. This leads to the definition of morphisms in $D^b(\mathscr{A})$ as diagrams of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet, \end{tikzcd}$$

where "qis" stands for "quasi-isomorphism" of complexes. To make this more precise, we need to define when two such roofs should considered to be equal, and how to define their compositions. Then natural context for both problems is to consider the homotopy category $K(\mathscr{A})$ of complexes in $\mathcal{K}om(\mathscr{A})$, which is an intermediate step for going from $\mathcal{K}om(\mathscr{A})$ to $\mathscr{D}(\mathscr{A})$.

$$\begin{tikzcd} \mathcal{K}om(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, bend right] & & D(\mathscr{A}) \\ & K(\mathscr{A}) \arrow[ur, bend right, "Q'"'] & \end{tikzcd}$$

Definition 2.2.4. Two morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ are said to be *homotopically equivalent*, written as $f^\bullet \sim g^\bullet$, if there is a morphism of complexes $h^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

$$\begin{tikzcd} A^\bullet \arrow[d, shift right, "g^\bullet"'] \arrow[d, shift left, "f^\bullet"] & \cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, shift right, "g^{i-1}"'] \arrow[d, shift left, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[dl, "h^i"'] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "g^{i+1}"'] \arrow[d, shift left, "f^{i+1}"] & \cdots \\ B^\bullet & \cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \cdots \end{tikzcd}$$

Let $K(\mathscr{A})$ be the category, whose objects are the same as objects of $\mathcal{K}om(\mathscr{A})$ and morphisms are given by $\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, for all $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$.

Following proposition is an easy consequence of the above definition.

Proposition 2.2.5. *(i) Homotopy equivalence of morphisms $A^\bullet \to B^\bullet$ of complexes is an equivalence relation.*

(ii) Homotopically trivial morphisms of complexes form an 'ideal' in the morphisms of $\mathcal{K}om(\mathscr{A})$.

(iii) If $f^\bullet$ and $g^\bullet$ are two homotopically equivalent morphisms of complexes in $\mathcal{K}om(\mathscr{A})$, then the induced morphisms $\mathcal{H}^i(f^\bullet)$ and $\mathcal{H}^i(g^\bullet)$ from $\mathcal{H}^i(A^\bullet)$ to $\mathcal{H}^i(B^\bullet)$ coincides.

(iv) Let $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to A^\bullet$ be two morphisms of complexes. If $f^\bullet \circ g^\bullet \sim \mathrm{Id}_{B^\bullet}$ and $g^\bullet \circ f^\bullet \sim \mathrm{Id}_{A^\bullet}$, then $f^\bullet$ and $g^\bullet$ are quasi-isomorphisms, and $\mathcal{H}^i(f^\bullet)^{-1} = \mathcal{H}^i(g^\bullet)$, for all $i \in \mathbb{Z}$.

Now we complete the construction of derived category $D(\mathscr{A})$. Take $\mathrm{Ob}(D(\mathscr{A})) := \mathrm{Ob}(\mathcal{K}om(\mathscr{A}))$. As discussed before, a morphism $f : A^\bullet \to B^\bullet$ in $D(\mathscr{A})$ is given by equivalence class of roofs of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \xrightarrow{qis} A^\bullet$ is a quasi-isomorphism of complexes in $\mathcal{K}om(\mathscr{A})$, and two such roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

are considered to be equivalent if they are dominated by a third one of the same type

(2.2.6)
$$\begin{tikzcd} & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] \arrow[ddl, dashed, bend right, "qis"'] & \\ C_1^\bullet \arrow[d, "qis"'] \arrow[drr] & & C_2^\bullet \arrow[dll, "qis"] \arrow[d] \\ A^\bullet & & B^\bullet \end{tikzcd}$$

such that the above diagram commutes in the homotopy category $K(\mathscr{A})$. In particular, the compositions $C_0^\bullet \to C_1^\bullet \to A^\bullet$ and $C_0^\bullet \to C_2^\bullet \to A^\bullet$ are homotopically equivalent. To define composition of morphisms in $D(\mathscr{A})$, consider two roofs

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & B^\bullet & & C^\bullet \end{tikzcd}$$

representing two morphisms in $D(\mathscr{A})$. It is natural to guess that, one should be able to define their composition to be a morphism represented by 'the' following roof

(2.2.7)
$$\begin{tikzcd} & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet , \end{tikzcd}$$

which commutes in the homotopy category $K(\mathscr{A})$. Now we need to ensure that such a diagram exists uniquely, up to *equivalence of roofs* as defined in (2.2.6) (c.f., Proposition 2.2.12). For this, we need the concept of "mapping cone".

Definition 2.2.8. The *mapping cone* of a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathcal{K}om(\mathscr{A})$ is a complex $C(f^\bullet)$ defined as follow:

$$C(f^\bullet)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f^\bullet)} = \begin{pmatrix} -d^{i+1}_{A^\bullet} & 0 \\ f^{i+1} & d^i_{B^\bullet} \end{pmatrix}, \ \forall\, i \in \mathbb{Z}\,.$$

Note that $C(f^\bullet)$ is a complex. Moreover, there are natural morphisms of complexes

$$\tau : B^\bullet \to C(f^\bullet) \quad \text{and} \quad \pi : C(f^\bullet) \to A^\bullet[1] \tag{2.2.9}$$

given by natural injection $B^i \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively, for all i. Then we have the following.

(i) The composition $B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$ is trivial. In fact,

$$0 \longrightarrow B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1] \longrightarrow 0$$

is a short exact sequence in $\mathcal{K}om(\mathscr{A})$, and gives us a long exact sequence of cohomologies

$$\cdots \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C(f^\bullet)) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots\,.$$

(ii) The composition $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet)$ is homotopic to the trivial morphism. Indeed, take $h^\bullet : A^\bullet \longrightarrow C(f^\bullet)$ to be morphism of complexes defined by the natural injective morphism $h^i : A^i \to C(f^\bullet)^{i-1} = A^i \oplus B^{i-1}$, for all i. Then we have

$$h^{i+1} \circ d^i_{A^\bullet} + d^{i-1}_{C(f^\bullet)} \circ h^i = \tau^i \circ f^i, \quad \forall\, i \in \mathbb{Z}\,.$$

Remark 2.2.10. It follows from the above construction that any commutative diagram of complexes

$$\begin{tikzcd}
A_1^\bullet \arrow[r, "f_1^\bullet"] \arrow[d, "\phi"] & B_1^\bullet \arrow[r, "\tau_1"] \arrow[d, "\psi"] & C(f_1^\bullet) \arrow[r, "\pi_1"] \arrow[d, dashed, "\phi[1]\oplus\psi", "\exists"'] & A_1^\bullet[1] \arrow[d, "\phi[1]"] \\
A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r, "\tau_2"] & C(f_2) \arrow[r, "\pi_2"] & A_2^\bullet[1]
\end{tikzcd}$$

can be completed by a dashed arrow as above (c.f., axiom (TR3) in Definition 1.2.4).

Proposition 2.2.11. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone. Let $\tau : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$ be the natural morphisms as in*

(2.2.9). *Then there is a morphism of complexes $g : A^\bullet[1] \longrightarrow C(\tau)$ which is an isomorphism in $K(\mathscr{A})$ such that the following diagram commutes in the homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd} B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \\ & & C(\tau) \arrow[ur, "\pi_\tau"'] & \end{tikzcd}$$

Proof. Define $g : A^\bullet[1] \longrightarrow C(\tau)$ by setting

$$g^i : A^\bullet[1]^i = A^{i+1} \longrightarrow C(\tau)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$$

to be the morphism $g^i = (-f^{i+1}, \mathrm{Id}_{A^{i+1}}, 0)$, for all i. Clearly, g is a morphism of complexes, and its inverse (in $K(\mathscr{A})$) is given by the morphism of complexes g^{-1} : $C(\tau) \to A^\bullet[1]$ defined by projection onto the middle factor. Clearly, $\pi_\tau \circ g = -f$ in $\mathcal{K}om(\mathscr{A})$. However, the diagram

$$\begin{tikzcd} C(f) \arrow[r, "\pi"] \arrow[dr, "\tau_\tau"'] & A^\bullet[1] \arrow[d, "g"] \\ & C(\tau) \end{tikzcd}$$

does not commute in $\mathcal{K}om(\mathscr{A})$. We show that, $\pi \circ g \sim \tau_\tau$. For this, note that $g^{-1} \circ \tau_\tau = \pi$ in $\mathcal{K}om(\mathscr{A})$. Since $g \circ g^{-1} \sim \mathrm{Id}_{C(\tau)}$, we have $\tau_\tau \sim g \circ \pi$. This completes the proof. □

Now we use the above proposition to complete the proof of existence and uniqueness of composition of morphisms in $D(\mathscr{A})$.

Proposition 2.2.12. *Let $f : A^\bullet \to B^\bullet$ and $g : C^\bullet \to B^\bullet$ be morphism of complexes with f a quasi-isomorphism. Then there is a complex $C_0^\bullet$ together with a quasi-isomorphism $C_0^\bullet \to C^\bullet$ and a morphism $C_0^\bullet \to A^\bullet$ such that the following diagram commutes in the homotopy category $K(\mathscr{A})$.*

$$\begin{tikzcd} C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "qis"'] & B^\bullet \end{tikzcd}$$

Proof. Note that, there is a natural morphism of complexes $\phi^i : C(\tau \circ g) \to A^\bullet[1]$ given by the natural projection

$$\phi^i : C(\tau \circ g)^i = C^{i+1} \oplus C(f)^i = C^{i+1} \oplus A^{i+1} \oplus B^i \xrightarrow{pr_2} A^{i+1} = A^\bullet[1]^i$$

onto the middle factor, for each i. By Proposition 2.2.11, there is morphism of complexes $\psi : C(\tau) \xrightarrow{\sim} A^\bullet[1]$ which is an isomorphism in $K(\mathscr{A})$. Then the following diagram is commutative in $K(\mathscr{A})$.

$$\begin{tikzcd}
C_0^\bullet := C(\tau\circ g)[-1] \arrow[r] \arrow[d, dashed, "{\phi[-1]}"] & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, "g"] & C(f) \arrow[rr, "\tau_{\tau\circ g}"] \arrow[d, equal] & & C(\tau\circ g) \arrow[d, "\phi"] \\
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & C(\tau) \arrow[r, "\psi", "\sim"'] & A^\bullet[1]
\end{tikzcd}$$

Define $C_0^\bullet := C(\tau\circ g)[-1]$. Since f is a quasi-isomorphism, it follows from the commutativity of the above diagram that $C_0^\bullet \to C^\bullet$ is a quasi-isomorphism. $\square$

Remark 2.2.13. Existence and uniqueness of composition of morphisms in $D(\mathscr{A})$ follows from the above Proposition 2.2.12 (c.f., diagram (2.2.7)). This completes the construction of the derived category $D(\mathscr{A})$.

Proposition 2.2.14. *The categories $K(\mathscr{A})$ and $D(\mathscr{A})$ are additive.*

Proof. Let $A^\bullet, B^\bullet \in \mathcal{K}om(\mathscr{A})$. Since $\mathcal{K}om(\mathscr{A})$ is an abelian category, it follows from Proposition 2.2.5 that the quotient

$$\mathrm{Hom}_{K(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathcal{K}om(\mathscr{A})}(A^\bullet, B^\bullet)/\sim,$$

is an abelian group. Thus $K(\mathscr{A})$ is an additive category.

To see $D(\mathscr{A})$ is an additive category, let $f_1, f_2 \in \mathrm{Hom}_{D(\mathscr{A})}(A^\bullet, B^\bullet)$ be two morphisms in $D(\mathscr{A})$ represented by following equivalence classes of roofs

$$\begin{tikzcd}
& C_1^\bullet \arrow[dl, "\phi_1"', "qis"] \arrow[dr, "\psi_1"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& C_2^\bullet \arrow[dl, "\phi_2"', "qis"] \arrow[dr, "\psi_2"] & \\
A^\bullet & & B^\bullet
\end{tikzcd},$$

respectively. It follows from Proposition 2.2.12 that there is an object $C^\bullet \in D(\mathscr{A})$ and quasi-morphisms $\delta_i : C^\bullet \to C_i^\bullet$, for $i = 1, 2$, such that the following diagram commutes in the homotopy category.

(2.2.15)
$$\begin{tikzcd}
C^\bullet \arrow[r, "\delta_2", "qis"'] \arrow[d, "\delta_1", "qis"'] & C_2^\bullet \arrow[d, "\phi_2"] \\
C_1^\bullet \arrow[r, "\phi_1", "qis"'] & A^\bullet
\end{tikzcd}$$

Note that, both $\phi_1\circ\delta_1$ and $\phi_2\circ\delta_2$, are quasi-isomorphisms, and are equal in $K(\mathscr{A})$. Let $\delta = \phi_1\circ\delta_1 = \phi_2\circ\delta_2$ in $K(\mathscr{A})$. Then in $D(\mathscr{A})$, we can write

$$f_1 + f_2 = \psi_1\circ\phi_1^{-1} + \psi_2\circ\phi_2^{-1} = (\psi_1\circ\delta_1 + \psi_2\circ\delta_2)\circ\delta^{-1}.$$

This defines $f_1 + g_1$ in $D(\mathscr{A})$. One can check that, $f + g$ as defined above, is well-defined in $D(\mathscr{A})$. Note that, the roof $A^\bullet \xleftarrow[qis]{-\phi_1} C_1^\bullet \xrightarrow{\psi_1} B^\bullet$ is the additive inverse of f_1 in $D(\mathscr{A})$. Now one can check that $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ is an abelian group. $\square$

Definition 2.2.16. A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $K(\mathscr{A})$ (resp., in $D(\mathscr{A})$) is said to be a *distinguished triangle* if it is isomorphic in $K(\mathscr{A})$ (resp., $D(\mathscr{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1],$$

where f is a morphism of complexes with mapping cone $C(f)$, and τ and π are natural morphisms as defined in (2.2.9).

Proposition 2.2.17. *The categories $D(\mathscr{A})$ and $K(\mathscr{A})$ together with the shift functor are triangulated. Moreover, the natural functor $Q : K(\mathscr{A}) \longrightarrow D(\mathscr{A})$ is an exact functor of triangulated categories.*

Proof. Triangulated structure on both $K(\mathscr{A})$ and $D(\mathscr{A})$ are given by shift functor $A^\bullet \mapsto A^\bullet[1]$ together with the collection of 'distinguished triangles' as defined above. Verification of axioms (TR1) – (TR4) requires crucial use of mapping cone. $\square$

Example 2.2.18. Let $\mathscr{A} = \mathcal{V}ect_{fd}(k)$ be the category, whose objects are finite dimensional k-vector spaces, and morphisms between objects are k-linear homomorphisms. Then $D(\mathscr{A})$ is equivalent to the category $\prod_{i\in\mathbb{Z}} \mathscr{A}$ of graded k-vector spaces. Note that, any complex of k-vector spaces $A^\bullet \in D(\mathcal{V}ect_{fd}(k))$ is isomorphic to its cohomology complex $\bigoplus\limits_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$ with trivial differentials. More generally, this holds for any *semisimple* abelian category $\mathscr{A}$ (i.e., when $\mathscr{A}$ is abelian and any short exact sequence in $\mathscr{A}$ splits).

Remark 2.2.19. Contrary to the category $\mathcal{K}om(\mathscr{A})$ of complexes in $\mathscr{A}$, the derived category $D(\mathscr{A})$ is not abelian, in general. However, $D(\mathscr{A})$ is always triangulated. $D^b(\mathscr{A})$ is abelian if and only if $\mathscr{A}$ is semisimple
(see https://math.stackexchange.com/questions/189769).

Corollary 2.2.20. *(a) The functor $Q : \mathcal{K}om(\mathscr{A}) \longrightarrow D(\mathscr{A})$ identifies set underlying set of objects of both categories (Apply property (ii) to the identity functor $\mathcal{K}om(\mathscr{A}) \to \mathcal{K}om(\mathscr{A})$).*

(b) For any complex $A^\bullet \in D(\mathscr{A})$, its cohomology objects $\mathcal{H}^i(A^\bullet)$ are well-defined objects in $\mathscr{A}$. (This is because, quasi-isomorphisms of $\mathcal{K}om(\mathscr{A})$ turns into isomorphisms in $D(\mathscr{A})$.)

(c) Considering $A \in \mathscr{A}$ as a complex $A^\bullet \in D(\mathscr{A})$ concentrated at degree zero only, gives an equivalence between $\mathscr{A}$ and the full subcategory of objects of $D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$ for $i \neq 0$.

Proposition 2.2.21. *Let $\mathscr{A}$ be an abelian category and $K(\mathscr{A})$ its homotopy category. Let $\mathscr{C}$ be any additive category.*

(1) An additive functor $F : K(\mathscr{A}) \longrightarrow \mathscr{C}$ factors through an additive functor $\widetilde{F} : D(\mathscr{A}) \longrightarrow \mathscr{C}$ if and only if F send quasi-isomorphisms to isomorphisms.

(2) Let $\mathscr{B}$ be an abelian category, and $G : K(\mathscr{A}) \longrightarrow K(\mathscr{B})$ an additive functor which maps quasi-isomorphism to quasi-isomorphism. Then G induces an additive functor $\widetilde{G} : D(\mathscr{A}) \longrightarrow D(\mathscr{B})$ such that the following diagram commutes.

$$\begin{tikzcd} K(\mathscr{A}) \arrow[r, "G"] \arrow[d, "Q_{\mathscr{A}}"'] & K(\mathscr{B}) \arrow[d, "Q_{\mathscr{B}}"] \\ D(\mathscr{A}) \arrow[r, "\widetilde{G}"] & D(\mathscr{B}) \end{tikzcd}$$

Proof. If $F : K(\mathscr{A}) \to \mathscr{C}$ sends quasi-isomorphisms to quasi-isomorphism, we define $\widetilde{F} : D(\mathscr{A}) \longrightarrow \mathscr{C}$ by sending an object $E^\bullet \in D(\mathscr{A})$ to $F(E^\bullet) \in \mathscr{C}$, and any morphism $f/\phi : A^\bullet \xleftarrow[\sim]{\phi} C^\bullet \xrightarrow{f} B^\bullet$ in $D(\mathscr{A})$ to the morphism

$$F(f) \circ F(\phi)^{-1} : F(A^\bullet) \xrightarrow[\simeq]{F(\phi)^{-1}} F(C^\bullet) \xrightarrow{F(f)} F(B^\bullet).$$

in $\mathscr{C}$. Converse follows from the construction of $D(\mathscr{A})$ and Theorem 2.2.1.

The second assertion follows by applying the first one to the composition $F : K(\mathscr{A}) \xrightarrow{G} K(\mathscr{B}) \xrightarrow{Q_{\mathscr{B}}} D(\mathscr{B})$. $\square$

2.3. **Derived categories:** $D^-(\mathscr{A}), D^+(\mathscr{A})$**, and** $D^b(\mathscr{A})$**.**

Definition 2.3.1. Let $\mathcal{K}om^*(\mathscr{A})$, with $* = +, -$, or b, be the full subcategory of $\mathcal{K}om(\mathscr{A})$, whose objects are complexes $A^\bullet \in \mathcal{K}om(\mathscr{A})$ with $A^i = 0$ for all $i \ll 0$, $i \gg 0$, or $|i| \ll 0$, respectively.

Note that, $\mathcal{K}om^*(\mathscr{A})$ is again abelian, for $* \in \{+, -, b\}$. So following similar construction (i.e., by dividing out first by homotopy equivalence, and then by localizing with respect to quasi-isomorphisms), we can construct a category, denoted by $D^*(\mathscr{A})$. There is a natural forgetful functor

$$\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A}),$$

which just forgets boundedness condition.

Proposition 2.3.2. *The natural forgetful functor $\mathcal{F}^* : D^*(\mathscr{A}) \longrightarrow D(\mathscr{A})$, where $* = +, -$, or b, gives an equivalence of $D^*(\mathscr{A})$ with the full triangulated subcategories of all complexes $A^\bullet \in D(\mathscr{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for $i \ll 0$, for $i \gg 0$, or for $|i| \ll 0$, respectively.*

To give an idea how this works, let $A^\bullet \in \mathcal{K}om(\mathscr{A})$ be such that $\mathcal{H}^i(A^\bullet) = 0$, for $i > n$. Then the commutative diagram

$$\begin{tikzcd} B^\bullet : & \cdots \arrow[r] & A^{n-2} \arrow[r] \arrow[d, equal] & A^{n-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d_A^n) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\ A^\bullet : & \cdots \arrow[r] & A^{n-2} \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots \end{tikzcd} \tag{2.3.3}$$

defines a quasi-isomorphism between a complex $B^\bullet \in \mathcal{K}om^-(\mathscr{A})$ and $A^\bullet$. Similarly, if $\mathcal{H}^i(A^\bullet) = 0$ for $i < m$, then the commutative diagram

$$\begin{tikzcd} A^\bullet : & \cdots \arrow[r] & A^{m-1} \arrow[r] \arrow[d] & A^m \arrow[r] \arrow[d] & A^{m+1} \arrow[r] \arrow[d, equal] & \cdots \\ C^\bullet : & \cdots \arrow[r] & 0 \arrow[r] & \mathrm{Coker}(d_A^{m-1}) \arrow[r] & A^{m+1} \arrow[r] & \cdots \end{tikzcd} \tag{2.3.4}$$

defines a quasi-isomorphism of a complex $C^\bullet \in \mathcal{K}om^+(\mathscr{A})$ and $A^\bullet$. Similar idea works for $\mathcal{K}om^b(\mathscr{A})$. However, one need to pass from $\mathcal{K}om^*(\mathscr{A})$ to the derived category $D^*(\mathscr{A})$ by inverting quasi-isomorphisms. This needs some technical care.

Let $\mathscr{A} \subset \mathscr{B}$ be full abelian subcategory of an abelian category $\mathscr{B}$. Then there is an obvious functor $\iota : D(\mathscr{A}) \longrightarrow D(\mathscr{B})$. One might expect that this is an equivalence of $D(\mathscr{A})$ with the full subcategory $D_{\mathscr{A}}(\mathscr{B})$ of $D(\mathscr{B})$ consisting of objects $E^\bullet \in D(\mathscr{B})$ with $\mathcal{H}^i(E^\bullet) \in \mathscr{A}$, for all $i \in \mathbb{Z}$. However, this is not true, in general. There are several issues.

- $D_{\mathscr{A}}(\mathscr{B})$ need not be triangulated!
- The functor ι is neither faithful nor full, in general.

However, the next proposition answers when the above expectation holds true. First, we need a definition.

Definition 2.3.5. Let $\mathscr{A}$ be an abelian category. A *thick subcategory* of $\mathscr{A}$ is a full abelian subcategory $\mathscr{B} \subset \mathscr{A}$ of $\mathscr{A}$ such that for any short exact sequence (in $\mathscr{A}$)

$$0 \to A \to B \to C \to 0$$

with $A, C \in \mathscr{B}$, we have $B \in \mathscr{B}$.

Let $E, F \in \mathscr{A}$. We say that F is *embedded* in E (or, F is a *subobject* of E) if there is a monomorphism $F \to E$ in $\mathscr{A}$. An object $I \in \mathscr{A}$ is called *injective* if the functor

$$\mathrm{Hom}_{\mathscr{A}}(-, I) : \mathscr{A} \longrightarrow Ab$$

is exact.

Proposition 2.3.6. *Let $\mathscr{A} \subset \mathscr{B}$ be a thick abelian subcategory of an abelian category $\mathscr{B}$. Assume that any object $A \in \mathscr{A}$ is a subobject of an object $I_A \in \mathscr{A}$, which is injective as an object of $\mathscr{B}$. Then the natural functor $D(\mathscr{A}) \longrightarrow D(\mathscr{B})$ induces an equivalence*

$$D^*(\mathscr{A}) \longrightarrow D^*_{\mathscr{A}}(\mathscr{B}), \quad \text{where } * = + \text{ or } b,$$

of $D^*(\mathscr{A})$ and the full triangulated subcategory $D^*_{\mathscr{A}}(\mathscr{B}) \subset D^*(\mathscr{B})$ of complexes with cohomologies in $\mathscr{A}$.

P.S.: I have not seen similar statement for $* = \emptyset$ or $-$.

Next, we want to get a computable description of Hom's in the derived category. In the next section, using derived functor, we show that

$$\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]) = \mathrm{Ext}^i(A^\bullet, B^\bullet), \quad \forall\, i\,.$$

3. Derived functors

3.1. **What is it?** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. We want to know when such a functor give rise to a natural functor between derived categories. Note that, if F is not exact, then image of an acyclic complex ($\mathcal{H}^i(A^\bullet) = 0, \ \forall\, i$) may not be acyclic. So to get a induced functor at the level of derived categories, F should be exact.

Lemma 3.1.1. *Let $F : K^*(\mathcal{A}) \to K^*(\mathcal{B})$, where $* \in \{\emptyset, -, +, b\}$, be an exact functor of triangulated categories. Then F induces a functor $\widetilde{F} : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ making the following diagram commutative*

$$\begin{tikzcd}
K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d] & K^*(\mathcal{B}) \arrow[d] \\
D^*(\mathcal{A}) \arrow[r, "\widetilde{F}"] & D^*(\mathcal{B})
\end{tikzcd}$$

if and only if one of the following (equivalent) conditions holds:

(i) F sends a quasi-isomorphism to a quasi-isomorphism.
(ii) F sends any acyclic complex to an acyclic complex.

However, if the functor F is not exact or F does not satisfies one of the equivalent conditions in (i) and (ii) above, still there is a bit complicated way to induce a natural functor between derived categories. This new functor is called the derived functor of F, but they will not produce a commutative diagram as in the above lemma. However, derived functor encodes more information about objects of the abelian categories.

To ensure existence of a derived functors, we need to assume some kind of exactness of F. If F is left exact (resp., right exact), we generally get a right derived functor (resp., let derived functor)

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \quad (\text{ resp.,} \quad LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B})\,).$$

Both constructions are similar, and we only discuss the case of left exact functor.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Assume that $\mathcal{A}$ has *enough injective* (meaning that, for any $A \in \mathcal{A}$, there is an injective object I of $\mathcal{A}$ together with a monomorphism $\iota : A \hookrightarrow I$ in $\mathcal{A}$). Let $\mathcal{I}_\mathcal{A} \subset \mathcal{A}$ be the full subcategory of $\mathcal{A}$ consisting of injective objects of $\mathcal{A}$. Note that, $\mathcal{I}_\mathcal{A}$ is additive, but not necessarily abelian. However, the construction of homotopy category works for any additive category. Therefore, $K^*(\mathcal{I}_\mathcal{A})$ is defined, and is a triangulated category. Now the inclusion functor $\mathcal{I}_\mathcal{A} \hookrightarrow \mathcal{A}$ induces a natural exact functor $K^*(\mathcal{I}_\mathcal{A}) \longrightarrow K^*(\mathcal{A})$, and composing it with the exact functor $Q_\mathcal{A} : K^+(\mathcal{A}) \to D^+(\mathcal{A})$, we get a natural exact functor $K^+(\mathcal{I}_\mathcal{A}) \longrightarrow D^+(\mathcal{A})$.

Proposition 3.1.2. *The functor $Q_\mathcal{A} : K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$ induces a natural equivalence of categories $\iota : K^+(\mathcal{I}_\mathcal{A}) \longrightarrow D^+(\mathcal{A})$.*

Then we have the following diagram

$$\begin{tikzcd}
K^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] \arrow[dr, "\iota"] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_\mathcal{A}"] & K^+(\mathcal{B}) \arrow[d, "Q_\mathcal{B}"] \\
 & D^+(\mathcal{A}) \arrow[ul, bend left, "\iota^{-1}"] & D^+(\mathcal{B})\,.
\end{tikzcd}$$

where ι^{-1} is the quasi-inverse functor. Such a quasi-isomorphism (ι^{-1}) is obtained by choosing a complex of injective objects quasi-isomorphic to a given bounded below complex in $D^+(\mathcal{A})$. Note that, the functor $K(F)$ is well-defined at the level of homotopy category, because F is left exact and we are working with bounded below complexes.

Definition 3.1.3. The *right derived functor* of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ is the functor

$$RF := Q_\mathcal{B} \circ K(F) \circ \iota^{-1} : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})\,.$$

Proposition 3.1.4. *(i) There is a natural morphism of functors*

$$Q_\mathcal{B} \circ K(F) \longrightarrow RF \circ Q_\mathcal{A}\,.$$

(ii) The right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ is an exact functor of triangulated categories.

(iii) *Let $G : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ be an exact functor of triangulated categories. Then any morphism of functors*

$$Q_{\mathcal{B}} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}},$$

factorize through a unique morphism of functors $RF \longrightarrow G$.

Proof. (i) Let $A^\bullet \in K^+(\mathcal{A})$. Note that, $Q_{\mathcal{A}}(A^\bullet) = A^\bullet$. Let $I^\bullet := \iota^{-1}(A^\bullet) \in K^+(\mathcal{I}_{\mathcal{A}})$. Then the natural isomorphism of functors $\mathrm{Id}_{D^+(\mathcal{A})} \xrightarrow{\simeq} \iota \circ \iota^{-1}$ gives rise to a functorial isomorphism $A^\bullet \xrightarrow{\simeq} \iota(I^\bullet) \cong I^\bullet$. Since $I^\bullet$ is a complex of injective objects, the above isomorphism in $D^+(\mathcal{A})$ gives rise to a unique morphism $A^\bullet \longrightarrow I^\bullet$ in the homotopy category $K(\mathcal{A})$ by Proposition 3.1.13 (see below). Therefore, we have a functorial morphism

$$K(F)(A^\bullet) \longrightarrow K(F)(I^\bullet) = RF(A^\bullet),$$

which gives the required morphism of functors.

(ii) Note that, $K^+(\mathcal{I}_{\mathcal{A}})$ is a triangulated category and $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$ is an exact equivalence of categories. Then ι^{-1} being the adjoint of ι, it is also exact (c.f. Proposition 1.2.13). Now $RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1}$ being a composition of exact functors, is exact.

(iii) See [GM03, III.6.11].

□

One can rephrase the Proposition 3.1.4 as a universal property to define derived functor of a left exact functor as follow.

Definition 3.1.5 (Universal property of derived functor)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then the right derived functor of F, if it exists, is an exact functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ of triangulated categories such that

(i) there is a natural morphism of functors $Q_{\mathcal{B}} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}$, and
(ii) for any exact functor $G : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$, there is a natural bijection

$$\mathrm{Hom}(RF, G) \xrightarrow{\simeq} \mathrm{Hom}(Q_{\mathcal{B}} \circ F, G \circ Q_{\mathcal{A}}) .$$

Definition 3.1.6. Let $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$ be a right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$, we define

$$R^iF(A^\bullet) := \mathcal{H}^i(RF(A^\bullet)) \in \mathcal{B}\,, \quad \forall\, i \in \mathbb{Z}\,.$$

The induced functors $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$ given by composition

$$\mathcal{A} \hookrightarrow D^+(\mathcal{A}) \xrightarrow{R^iF} \mathcal{B}$$

are known as *higher derived functors* of F.

Given any $A \in \mathcal{A}$, choosing an injective resolution

$$A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

we see that, $R^iF(A) = \mathcal{H}^i(\cdots \to F(I^0) \to F(I^1) \to \cdots)$. In particular,

$$R^0F(A) = \mathrm{Ker}(F(I^0) \to F(I^1)) = F(A),$$

since F is left exact.

Definition 3.1.7. An object $A \in \mathcal{A}$ is called F*-acyclic* if $R^iF(A) \cong 0$, for all $i \neq 0$.

Corollary 3.1.8. *With the above assumptions, any short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ give rise to a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2(A) \to \cdots.$$

To see how it works, note that any short exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ in $\mathcal{A}$ gives rise to a distinguished triangle $A \to B \to C \to A[1]$ in $D(\mathcal{A})$. Again any distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ in $D(\mathcal{A})$ give rise to a long exact sequence of cohomologies

$$\cdots \to \mathcal{H}^{i-1}(C^\bullet) \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C^\bullet) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots.$$

Now the above corollary follows by considering the distinguished triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ in $D(\mathcal{B})$.

Example 3.1.9. Let $\mathcal{A}$ be an abelian category, and let Ab be the category of abelian groups. Consider the covariant functor

$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \longrightarrow \mathrm{Ab}.$$

Clearly, $\mathrm{Hom}(A, -)$ is left exact. If $\mathcal{A}$ contains enough injectives (for example, if $\mathcal{A}$ is $\mathfrak{Mod}(\mathcal{O}_X)$ or $\mathfrak{QCoh}(X)$ for X a noetherian scheme; c.f., [Har77, Exercise III. 3.6]), then we define

$$\mathrm{Ext}^i(A, -) := H^i(R\,\mathrm{Hom}_{\mathcal{A}}(A, -))\,. \tag{3.1.10}$$

Proposition 3.1.11. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A, B \in \mathcal{A}$. Then there are natural isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]), \quad \forall\ i,$$

where A and B are considered as complexes in $D(\mathcal{A})$ concentrated at degree 0 place.

Proof. Let
$$B \to I^0 \to I^1 \to \cdots$$
be an injective resolution of B in $\mathcal{A}$. It follows from the construction of right derived functor that $R\operatorname{Hom}(A, B)$, as an object of $D^+(\mathrm{Ab})$, is isomorphic to the complex
$$\operatorname{Hom}(A, I^\bullet): \quad \operatorname{Hom}(A, I^0) \to \operatorname{Hom}(A, I^1) \to \operatorname{Hom}(A, I^2) \to \cdots .$$
Therefore, $\operatorname{Ext}^i(A, B) = H^i(\operatorname{Hom}(A, I^\bullet))$.

Note that, a morphism $f \in \operatorname{Hom}(A, I^i)$ is a *cycle* (i.e., $f \in \operatorname{Ker}(\operatorname{Hom}(A, I^i) \to \operatorname{Hom}(A, I^{i+1}))$) if and only if f defines a morphism of complexes
$$f : A \longrightarrow I^\bullet[i].$$
This morphism of complexes f is homotopically trivial if and only if f is a *boundary* (i.e., $f \in \operatorname{image}(\operatorname{Hom}(A, I^{i-1}) \to \operatorname{Hom}(A, I^i))$). Therefore, we have
$$\operatorname{Ext}^i(A, B) \cong H^i(\operatorname{Hom}(A, I^\bullet)) \cong \operatorname{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]).$$
Since $I^\bullet$ is a complex of injective objects, by Lemma 3.1.12 (see below) we have
$$\operatorname{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]) \cong \operatorname{Hom}_{D(\mathcal{A})}(A, I^\bullet[i]).$$
Since $B \cong I^\bullet$ in $D^+(\mathcal{A})$, we have $\operatorname{Ext}^i(A, B) \cong \operatorname{Hom}_{D(\mathcal{A})}(A, B[i])$. $\square$

Lemma 3.1.12. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $A^\bullet, I^\bullet \in \mathcal{K}om^+(\mathcal{A})$ be two bounded below complexes such that I^i is injective, for all i. Then*
$$\operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) = \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet).$$

Proof. Clearly, there is a natural morphism
$$\operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet) .$$
We need to show that given any morphism $f \in \operatorname{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet)$ represented by a roof of the form
$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"', "\phi"] \arrow[dr, "\psi"] & \\ A^\bullet \arrow[rr, dashed, "\psi'"'] & & I^\bullet \end{tikzcd}$$
there is a unique morphism of complexes $\psi' : A^\bullet \to I^\bullet$ making the above diagram commutative in the homotopy category $K(\mathcal{A})$. Now the result follows from the Proposition 3.1.13 below. $\square$

Proposition 3.1.13. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for any quasi-isomorphism $\phi : B^\bullet \to A^\bullet$ in $\mathcal{K}om^+(\mathcal{A})$, the induced map*
$$\operatorname{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \operatorname{Hom}_{K(\mathcal{A})}(B^\bullet, I^\bullet),$$
obtained by precomposing with ϕ, is bijective.

Sketch of a proof. Complete the morphism $\phi : B^\bullet \to A^\bullet$ to a distinguished triangle in the triangulated
category $K^+(\mathcal{A})$. Applying the functor $\mathrm{Hom}(-, I^\bullet)$ and then taking the associated long exact $\mathrm{Hom}(-, I^\bullet)$-sequence, we see that it is enough to show that $\mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, I^\bullet) = 0$, for any acyclic complex $E^\bullet$.

Next, we take any morphism of complexes $f : E^\bullet \to I^\bullet$, and construct a homotopy between f and the zero morphism. This can be done by induction. Assume that $h^i : E^i \to I^{i-1}$ is constructed by induction. If h^j is constructed for all $j \leq i$, then the morphism

$$f^i - d^{i-1}_{I^\bullet} \circ h^i : E^i \longrightarrow I^i$$

factors through $E^i/E^{i-1} \longrightarrow I^i$. Since I^i is injective, this lifts to a morphism $h^{i+1} : E^{i+1} \longrightarrow I^i$ so that $f^i - d^{i-1}_{I^\bullet} \circ h^i = h^{i+1} \circ d^i_{E^\bullet}$. Thus the induction works! □

Remark 3.1.14. In practice, we need to deal with many important abelian categories without enough injective objects, or sometimes the functor F is defined at the level of homotopy categories only. However, one can still construct derived functors in that setup under certain assumption. Let us explain briefly.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

Case I. F is defined only at the level of homotopy category: let

$$F : K^+(\mathcal{A}) \longrightarrow K(\mathcal{B}) \tag{3.1.15}$$

be an exact functor of triangulated categories. Then the right derived functor

$$RF : D^+(\mathcal{A}) \longrightarrow D(\mathcal{B}) \tag{3.1.16}$$

of F satisfying the properties (i)–(iii) of Proposition 3.1.4 exists if there is a triangulated subcategory $K_F \subset K^+(\mathcal{A})$ *adapted to* F, meaning that K_F satisfies the following conditions:

(i) if $A^\bullet \in K_F$ is acyclic, then so is $F(A^\bullet)$, and
(ii) any $A^\bullet \in K^+(\mathcal{A})$ is quasi-isomorphic to a complex in K_F.

Roughly, with the above hypotheses (i)–(iii), we may localize the subcategory $K^+(K_F)$ with respect to the quasi-isomorphisms of objects from K_F to produce an equivalence of categories $K^+(K_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A})$. Moreover, the functor $K(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ give rise to a functor $K^+(K_F)_{qis} \longrightarrow K^+(\mathcal{B})$. Then by choosing a quasi-inverse of the above equivalence, we get the required derived functor $D^+(\mathcal{A}) \to D(\mathcal{B})$.

Case II. $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, but $\mathcal{A}$ has not enough injectives. In this situation, we may construct the right derived functor of F by looking at the *F-adapted class of objects $\mathcal{I}_F \subset \mathcal{A}$*, which is defined by the following properties.

(a) $\mathcal{I}_F$ is stable under finite sums,
(b) if $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^i \in \mathcal{I}_F$, for all i, then $F(A^\bullet)$ is acyclic, and
(c) any object of $\mathcal{A}$ can be embedded inside an object of $\mathcal{I}_F$.

Let $K^+(\mathcal{I}_F)_{qis}$ be the localization of $K^+(\mathcal{I}_F)$ by quasi-isomorphism of complexes with objects from $\mathcal{I}_F$. Then the above hypotheses (a)–(c) gives rise to an equivalence of categories

$$\iota_q : K^+(\mathcal{I}_F)_{qis} \xrightarrow{\simeq} D^+(\mathcal{A}). \tag{3.1.17}$$

Then $K(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ induces a functor $K(F)_{qis} : K^+(\mathcal{I}_F)_{qis} \to K^+(\mathcal{B})$. Now choosing a quasi-inverse ι_q^{-1} of (3.1.17) and composing with $Q_{\mathcal{B}} \circ K(F)_{qis}$, we get the required right derived functor $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$ of F as discussed before.

The definition of Ext group as given in Example (3.1.9) can be generalized for complexes as follow. Given $A^\bullet, B^\bullet \in \mathcal{K}om(\mathcal{A})$, let $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ be the complex defined by

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(A^j, B^{i+j}) \tag{3.1.18}$$

with the differential

$$d(f) := d_B \circ f - (-1)^i f \circ d_A, \quad \forall\, f \in \mathrm{Hom}(A^*, B^{i+*})\,.$$

$$\begin{tikzcd} A^j \arrow[r, "f_i^j"] \arrow[d, "d_A^j"'] & B^{i+j} \arrow[d, "d_B^{i+j}"] \\ A^{j+1} \arrow[r, "f_i^{j+1}"] & B^{i+j+1} \end{tikzcd}$$

The complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is known as the *internal hom of* $A^\bullet$ *into* $B^\bullet$.

Note that, any $A^\bullet \in \mathcal{K}om(\mathcal{A})$ gives rise to an exact functor

$$\mathrm{Hom}^\bullet(A^\bullet, -) : K^+(\mathcal{A}) \longrightarrow K(\mathbf{A}b), \quad B^\bullet \longmapsto \mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\,. \tag{3.1.19}$$

Let $\mathcal{I} \subset K^+(\mathcal{A})$ be the full triangulated subcategory, whose objects are complexes $I^\bullet$ with I^i injective object of $\mathcal{A}$, for all i. Then $\mathcal{I}$ is F-adapted, where $F = \mathrm{Hom}^\bullet(A^\bullet, -)$, as defined in Remark 3.1.14. Then the right derived functor

$$R\,\mathrm{Hom}^\bullet(A^\bullet, -) : D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.20}$$

of $\mathrm{Hom}^\bullet(A^\bullet, -)$ exists. Then we define

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := H^i\big(R\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\big)\,, \quad \forall\, i\,. \tag{3.1.21}$$

Now the proof of Proposition 3.1.11 can be modified to prove the following.

Theorem 3.1.22. *Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A^\bullet, B^\bullet \in \mathcal{K}om(\mathcal{A})$ be two bounded (or bounded below) complexes. Then there are natural isomorphisms of abelian groups*

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet[i]), \quad \forall\ i.$$

It follows from Theorem 3.1.22 that the abelian group $\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet)$ depends on the "isomorphism classes" of $A^\bullet$ and $B^\bullet$ in the derived category, not on the complexes. If $A_1^\bullet \xrightarrow{\sim} A_2^\bullet$ is a quasi-isomorphism of complexes, then the induced morphism

$$R\,\mathrm{Hom}^\bullet(A_1^\bullet, B^\bullet) \longrightarrow R\,\mathrm{Hom}^\bullet(A_2^\bullet, B^\bullet)$$

is an isomorphism in $D(\mathbf{Ab})$, because their cohomologies are isomorphic. Therefore, the functor $\mathrm{Hom}^\bullet(-, B^\bullet)$ descends to the derived category to give a bifunctor

$$D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,, \tag{3.1.23}$$

which is exact in each variable.

Definition 3.1.24. An abelian category $\mathcal{A}$ is said to have *enough projectives* if for each object $A \in \mathcal{A}$ there is a projective object P in $\mathcal{A}$ together with an epimorphism $A \to P$ in $\mathcal{A}$.

If the abelian category $\mathcal{A}$ has enough projectives, then for any complex $B^\bullet \in \mathcal{K}om(\mathcal{A})$, the left exact functor

$$\mathrm{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathbf{Ab})$$

admits a right derived functor

$$R\,\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab})\,.$$

One can check that, this depends only on $B^\bullet$ as an object of derived category. Therefore, it defines a bifunctor

$$D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.25}$$

If $\mathcal{A}$ has enough injectives and enough projectives, both bifunctors in (3.1.23) and (3.1.25) give rise to the same bifunctor

$$R\,\mathrm{Hom}^\bullet(-,-) : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.26}$$

Remark 3.1.27. If $\mathcal{A}$ has enough injectives, but not necessarily have enough projectives, using (3.1.23) we can get the derived functor

$$R\,\mathrm{Hom}^\bullet(-, B^\bullet) : D^-(\mathcal{A})^{\mathrm{op}} \longrightarrow D(\mathbf{Ab})\,. \tag{3.1.28}$$

Note that, thanks to Theorem 3.1.22, composition of morphisms in the derived category can be used to define composition for Ext groups:

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) \times \mathrm{Ext}^j_{\mathcal{A}}(B^\bullet, C^\bullet) \longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{A}}(A^\bullet, C^\bullet)\,, \tag{3.1.29}$$

for all $A^\bullet, B^\bullet, C^\bullet \in D^+(\mathcal{A})$. This follows because

$$\mathrm{Ext}^j_{\mathcal{A}}(B^\bullet, C^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(B^\bullet, C^\bullet[j]) \cong \mathrm{Hom}_{D(\mathcal{A})}(B^\bullet[i], C^\bullet[i+j]) .$$

Proposition 3.1.30 (Grothendieck's composite functor theorem)**.** *Let $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories. Suppose that there are adapted classes $\mathcal{I}_{F_1} \subset \mathcal{A}$ and $\mathcal{I}_{F_2} \subset \mathcal{B}$ for F_1 and F_2, respectively, such that $F_1(\mathcal{I}_{F_1}) \subseteq \mathcal{I}_{F_2}$. Then the derived functors $RF_1 : D^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$, $RF_2 : D^+(\mathcal{B}) \to D^+(\mathcal{C})$ and $R(F_2 \circ F_1) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ exists, and there is a natural isomorphism of functors*

$$R(F_2 \circ F_1) \xrightarrow{\simeq} RF_2 \circ RF_1 .$$

Proof. Clearly RF_1 and RF_2 exists by given assumptions (c.f., Remark 3.1.14). Since $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, we see that $\mathcal{I}_{F_1}$ is $(F_2 \circ F_1)$-adapted. Therefore, $R(F_2 \circ F_1)$ exists. Then the natural morphism of functors

$$R(F_2 \circ F_1) \longrightarrow RF_2 \circ RF_1 \tag{3.1.31}$$

follows from the universal property of derived functor $R(F_2 \circ F_1)$ (c.f., Definition 3.1.5). To see (3.1.31) is an isomorphism, given any complex $A^\bullet \in D^+(\mathcal{A})$ we choose a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$ quasi-isomorphic to $A^\bullet$. Then we have

$$R(F_2 \circ F_1)(A^\bullet) \cong (K(F_2) \circ K(F_1))(I^\bullet) \tag{3.1.32}$$

and

$$RF_2(RF_1(A^\bullet)) \cong RF_2(K(F_1)(I^\bullet)) \cong K(F_2)(K(F_1)(I^\bullet)) . \tag{3.1.33}$$

Now it follows from the natural isomorphism between (3.1.32) and (3.1.33) that the morphism of functor in (3.1.31) is an isomorphism. $\square$

Remark 3.1.34. If both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, then the hypotheses of the above Proposition 3.1.30 are satisfied if $F_1(\mathcal{I}_{\mathcal{A}}) \subset \mathcal{I}_{\mathcal{B}}$.

4. Serre functor

4.1. **Abstract Serre functor.**

Let k be a field. Let $\mathcal{A}$ be a k-linear additive category.

Definition 4.1.1. A *Serre functor* on $\mathcal{A}$ is a k-linear equivalence of categories

$$S : \mathcal{A} \longrightarrow \mathcal{A}$$

such that for any two objects $A, B \in \mathcal{A}$, there is a natural k-linear isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(B, S(A))^*,$$

which is functorial in both A and B. We write the induced k-bilinear pairing as

$$\mathrm{Hom}(B, S(A)) \times \mathrm{Hom}(A, B) \longrightarrow k , \quad (f, g) \longmapsto \langle f | g \rangle .$$

Proposition 4.1.2. *Let $\mathcal{A}$ be a k-linear additive category together with a Serre functor $S : \mathcal{A} \to \mathcal{A}$. Then for any $A, B \in \mathcal{A}$, the following diagram commutes.*

(4.1.3)
$$\begin{tikzcd} \mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] \arrow[d, "S_{A,B}"'] & \mathrm{Hom}(B,S(A))^* \\ \mathrm{Hom}(S(A),S(B)) \arrow[r, "\eta_{S(A),S(B)}"] \arrow[ur, dashed, "\exists\ \eta^*_{B,S(A)}"] & \mathrm{Hom}(S(B),S^2(A))^* \arrow[u, "S^*_{B,S(A)}"'] \end{tikzcd}$$

Proof. By abuse of notation, we denote by $\eta^*_{B,S(A)}$ the composite k-linear homomorphism

$$\eta^*_{B,S(A)} : \mathrm{Hom}(S(A),S(B)) \hookrightarrow \mathrm{Hom}(S(A),S(B))^{**} \xrightarrow{\eta^*_{B,S(A)}} \mathrm{Hom}(B,S(A))^*.$$

Therefore, it suffices to show that both upper and lower triangles in (4.1.3) commutes. Note that, commutativity of upper triangle is equivalent to

$$\langle f|g\rangle = \langle S_{A,B}(g)|f\rangle, \quad \forall\ f \in \mathrm{Hom}(B,S(A)),\ g \in \mathrm{Hom}(A,B)\,.$$

Applying functoriality of η in the second variable, we have the following commutative diagram.

(4.1.4)
$$\begin{tikzcd} \mathrm{Hom}(A,B) \arrow[r, "\eta_{A,B}"] & \mathrm{Hom}(B,S(A))^* \\ \mathrm{Hom}(B,B) \arrow[r, "\eta_{B,B}"] \arrow[u, "-\circ g"] & \mathrm{Hom}(B,S(B))^* \arrow[u, "(S(g)\circ-)^*"'] \end{tikzcd}$$

Applying commutativity of (4.1.4) to $\mathrm{Id}_B \in \mathrm{Hom}(B,B)$ we have $\langle f|g\rangle = \langle S(g)\circ f|\,\mathrm{Id}\rangle$. Applying functoriality of η in the first variable, we have the following commutative diagram

(4.1.5)
$$\begin{tikzcd} \mathrm{Hom}(B,B) \arrow[r, "\eta_{B,B}"] \arrow[d, "f\circ-"'] & \mathrm{Hom}(B,S(B))^* \arrow[d, "(-\circ f)^*"] \\ \mathrm{Hom}(B,S(A)) \arrow[r, "\eta_{B,S(A)}"] & \mathrm{Hom}(S(A),S(B))^*\,, \end{tikzcd}$$

which gives $\langle (S(g)\circ f)|\,\mathrm{Id}_B\rangle = \langle S(g)|f\rangle$. This completes the proof. □

Remark 4.1.6. In order to avoid trouble with identifying $\mathrm{Hom}(A,B)$ with its double dual $\mathrm{Hom}(A,B)^{**}$, we always assume that a k-linear additive category $\mathcal{A}$ has finite dimensional Hom's (i.e., $\dim_k \mathrm{Hom}_{\mathcal{A}}(A,B) < \infty$, for all $A, B \in \mathrm{Ob}(\mathcal{A})$).

Lemma 4.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear additive categories with finite dimensional Hom's. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively, then any k-linear equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$ commutes with Serre functors (i.e., there is an isomorphism of functors $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$).*

Proof. Since F is fully faithful, for any $A, B \in \mathcal{A}$ we have a functorial isomorphism

$$\mathrm{Hom}(A, S_{\mathcal{A}}(B)) \cong \mathrm{Hom}(F(A), F(S_{\mathcal{A}}(B))) \quad \text{and} \quad \mathrm{Hom}(B, A) \cong \mathrm{Hom}(F(B), F(A)).$$

By definition of Serre functor, we have the following functorial (in both variables) isomorphisms

$$\mathrm{Hom}(A, S_{\mathcal{A}}(B)) \cong \mathrm{Hom}(B, A)^* \quad \text{and} \quad \mathrm{Hom}(F(B), F(A)) \cong \mathrm{Hom}(F(A), S_{\mathcal{B}}(F(B)))^*.$$

These gives a functorial isomorphism

$$\mathrm{Hom}(F(A), F(S_{\mathcal{A}}(B))) \xrightarrow{\simeq} \mathrm{Hom}(F(A), S_{\mathcal{B}}(F(B))).$$

Since F is essentially surjective, any object in $\mathcal{B}$ is isomorphic to an object of the form $F(A)$, for some $A \in \mathcal{A}$. Hence the result follows from the above functorial isomorphism. □

Proposition 4.1.8. *Let $\mathcal{A}$ be a k-linear additive category. Then any two Serre functors on $\mathcal{A}$ are isomorphic.*

Proof. This follows from the definition of Serre functor and Yoneda lemma. □

4.2. **Serre duality in $D^b(X)$.** Let X be a smooth projective k-variety of dimension $n \geq 1$. Note that, for any locally free coherent sheaf E on X, the functor

$$- \otimes E : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(X)\,, \quad F \longmapsto F \otimes E$$

is exact. Let ω_X be the dualizing sheaf on X. Let $D^*(X) = D^*(\mathfrak{Coh}(X))$, where $* \in \{\emptyset, b, -, +\}$. Consider the composite functor

$$S_X : D^*(X) \xrightarrow{\omega_X \otimes -} D^*(X) \xrightarrow{[n]} D^*(X), \tag{4.2.1}$$

where $[n] : D^*(X) \to D^*(X)$ is the n-th shift functor given by sending a complex $E^\bullet$ to $E^\bullet[n]$. Since both the functors $\omega_X \otimes -$ and $[n]$ are exact, their composite functor $S_X := \omega_X \otimes (-)[n]$ is exact.

Theorem 4.2.2 (Grothendieck-Serre duality)**.** *Let X be a smooth projective variety over a field k. Then the functor $S_X : D^b(X) \longrightarrow D^b(X)$ as defined in (4.2.1) is a Serre functor in the sense of Definition 4.1.1.*

Proof. Given any two objects $E^\bullet, F^\bullet \in D^b(X)$, we need to give an isomorphism of k-vector spaces

$$\eta_{E^\bullet, F^\bullet} : \mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet))^* \tag{4.2.3}$$

which is functorial in both $E^\bullet$ and $F^\bullet$. Thanks to Theorem 3.1.22, we have

$$\mathrm{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i]) = \mathrm{Ext}^i(E^\bullet, F^\bullet), \quad \forall\, i\,.$$

Since X is smooth and projective, choosing a resolution by complex of locally free sheaves on X, we may assume that E^i is locally free, for all i. Then we have functorial isomorphisms

$$\begin{aligned}
\mathrm{Hom}^i(E^\bullet, F^\bullet) &= \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(E^j, F^{i+j}) = \bigoplus_{j\in\mathbb{Z}} H^0(X, \mathcal{H}om(E^j, F^{i+j})) \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^n(F^{i+j}, E^j\otimes\omega_X)^*, \quad \text{by classical Serre duality theorem.} \\
&\cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{D^b(X)}(F^{i+j}, E^j\otimes\omega_X[n])^*, \quad \text{by Proposition 3.1.11.} \\
&\cong \mathrm{Hom}^{n-i}(F^\bullet, E^\bullet\otimes\omega_X)^*.
\end{aligned}$$

Since for any two complexes $A^\bullet, B^\bullet$, we have

$$\mathrm{Ext}^i_{\mathcal{A}}(A^\bullet, B^\bullet) := H^i\big(R\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)\big)\,, \quad \forall\, i\,,$$

the theorem follows. □

Remark 4.2.4. Theorem 4.2.2 is a special case of Grothendieck-Verdier duality (c.f. Section §6.9). We shall see some interesting applications of the Serre functor $\omega_X\otimes(-)[n]$ on $D^b(X)$ in Section §8. For this, we need concept of local Hom complex, and spectral sequences to be explained in the next two sections.

5. Spectral sequence

5.1. **What is it?**

In this subsection, we explain how spectral sequence occur when we compose two derived functors. Let $\mathcal{A}$ be an abelian category.

Definition 5.1.1. A *spectral sequence in* $\mathcal{A}$ is given by a collection of objects

$$(E_r^{p,q}, E^n), \quad n,p,q,r\in\mathbb{Z},\ r\geq 1$$

and morphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

satisfying that the following conditions.

(i) $d_r^{p+r,q-r+1}\circ d_r^{p,q} = 0$, for all p,q,r. This yields a complex $E_r^{p+\bullet r,q-\bullet r+\bullet}$, for all $r\geq 1$.

(ii) There are isomorphisms

$$E_{r+1}^{p,q} \cong H^0(E_r^{p+\bullet r,q-\bullet r+\bullet})\,,$$

which are part of the data.

(iii) For any (p,q), there is an r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$, for all $r\geq r_0$. In particular, $E_r^{p,q}\cong E_{r_0}^{p,q}$, for all $r\geq r_0$. This object is denoted by $E_\infty^{p,q}$.

(iv) There is a decreasing filtration

$$\cdots \subset F^{p+1}E^n \subset F^pE^n \subset F^{p-1}E^n \subset \cdots \subset F^0E^n := E^n$$

such that $\bigcap\limits_{p\in\mathbb{Z}} F^pE^n = 0$ and $\bigcup\limits_{p\in\mathbb{Z}} F^pE^n = E^n$, and isomorphisms

$$E_\infty^{p,q} \cong F^pE^{p+q}/F^{p+1}E^{p+q}.$$

Remark 5.1.2. If $E_\infty^{p,q} = 0$, for all p, q, then $E^{p+q} = 0$. This follows from property (iv).

Let us try to visualize a spectral sequence. In page E_1, we have the following data.

$$\begin{tikzcd}
E_1^{p-2,q+1} \arrow[r] & E_1^{p-1,q+1} \arrow[r] & E_1^{p,q+1} \arrow[r] & E_1^{p+1,q+1} & \cdots \\
E_1^{p-2,q} \arrow[r] & E_1^{p-1,q} \arrow[r] & E_1^{p,q} \arrow[r] & E_1^{p+1,q} & \cdots \\
E_1^{p-2,q-1} \arrow[r] & E_1^{p-1,q-1} \arrow[r] & E_1^{p,q-1} \arrow[r] & E_1^{p+1,q-1} & \cdots
\end{tikzcd}$$

In page E_2, we have the following data.

$$\begin{tikzcd}
E_2^{p-2,q+1} \arrow[drr] & E_2^{p-1,q+1} \arrow[drr] & E_2^{p,q+1} \arrow[drr] & E_2^{p+1,q+1} & \cdots \\
E_2^{p-2,q} \arrow[drr] & E_2^{p-1,q} \arrow[drr] & E_2^{p,q} \arrow[drr] & E_2^{p+1,q} & \cdots \\
E_2^{p-2,q-1} & E_2^{p-1,q-1} & E_2^{p,q-1} & E_2^{p+1,q-1} & \cdots
\end{tikzcd}$$

In some sense, the condition (iv) says that the objects $E_r^{p,q}$ converges towards a subquotient of certain filtration of E^n. Usually objects of one layer, say $E_r^{p,q}$ with r fixed, are given, and objects of the next layer can be obtained using (ii). It is enough to give objects $E_r^{p,q}$ with $r \geq m$, for some m; the information is just the same. We express the spectral sequence by writing

$$E_r^{p,q} \Longrightarrow E^{p+q}.$$

In most of the applications, only $E_r^{p,q}$ are given for $r \geq 2$, and in most of the cases, we don't need to go beyond page E_2 or E_3.

Definition 5.1.3. A *double complex* $K^{\bullet,\bullet}$ is given by the following data: for each pair of integers (i, j), an object $K^{i,j} \in \mathcal{A}$ and morphisms

$$d_I^{i,j} : K^{i,j} \longrightarrow K^{i+1,j} \quad \text{and} \quad d_{II}^{i,j} : K^{i,j} \longrightarrow K^{i,j+1}$$

such that

$$d_I^2 = d_{II}^2 = d_Id_{II} + d_{II}d_I = 0.$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & K^{i-1,j-1} \arrow[r, "d_I^{i-1,j-1}"] \arrow[d, "d_{II}^{i-1,j-1}"] & K^{i,j-1} \arrow[r, "d_I^{i,j-1}"] \arrow[d, "d_{II}^{i,j-1}"] & K^{i+1,j-1} \arrow[r, "d_I^{i+1,j-1}"] \arrow[d, "d_{II}^{i+1,j-1}"] & \cdots \\
\cdots \arrow[r] & K^{i-1,j} \arrow[r, "d_I^{i-1,j}"] \arrow[d, "d_{II}^{i-1,j}"] & K^{i,j} \arrow[r, "d_I^{i,j}"] \arrow[d, "d_{II}^{i,j}"] & K^{i+1,j} \arrow[r, "d_I^{i+1,j}"] \arrow[d, "d_{II}^{i+1,j}"] & \cdots \\
\cdots \arrow[r] & K^{i-1,j+1} \arrow[r, "d_I^{i-1,j+1}"] \arrow[d, "d_{II}^{i-1,j+1}"] & K^{i,j+1} \arrow[r, "d_I^{i,j+1}"] \arrow[d, "d_{II}^{i,j+1}"] & K^{i+1,j+1} \arrow[r, "d_I^{i+1,j+1}"] \arrow[d, "d_{II}^{i+1,j+1}"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The associated *total complex* $K^\bullet := \mathrm{tot}(K^{\bullet,\bullet})$ is defined by $K^n := \bigoplus_{i+j=n} K^{i,j}$ with differentials $d = d_I + d_{II}$.

The total complex $K^\bullet = \mathrm{tot}(K^{\bullet,\bullet})$ admits a natural decreasing filtration $\{F^\ell K^n\}_\ell$ given by

$$F^\ell K^n := \bigoplus_{j \geq \ell} K^{n-j,j}\,, \tag{5.1.4}$$

which satisfies $d_I(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n. Due to symmetry of the situation, there is another such natural filtration.

Example 5.1.5. The complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ is an example of a total complex of the double complex $K^{i,j} := \mathrm{Hom}(A^{-i}, B^j)$ together with the differentials $d_I = (-1)^{j-i+1} d_A$ and $d_{II} = d_B$ (there are different sign conventions in the literature; however one can choose one sign convention, and final conclusion would be the same).

Definition 5.1.6. A *filtered complex* is a complex $K^\bullet$ together with a decreasing filtration

$$\cdots \subset F^\ell K^n \subset F^{\ell-1} K^n \subset \cdots \subset F^0 K^n := K^n, \;\; \forall\, n,$$

such that $d^n(F^\ell K^n) \subset F^\ell K^{n+1}$, for all n.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r] & K^{n} \arrow[r] & K^{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F^{\ell}(K^{n-1}) \arrow[r] \arrow[u, hook] & F^{\ell}(K^{n}) \arrow[r] \arrow[u, hook] & F^{\ell}(K^{n+1}) \arrow[r] \arrow[u, hook] & \cdots \\
\cdots \arrow[r] & F^{\ell+1}(K^{n-1}) \arrow[r] \arrow[u, hook] & F^{\ell+1}(K^{n}) \arrow[r] \arrow[u, hook] & F^{\ell+1}(K^{n+1}) \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

Consider the filtrations $\{F^{\ell}K^n\}_{\ell}$ of the total complex $K^{\bullet} = \mathrm{tot}(K^{\bullet,\bullet})$ in (5.1.4). The associated graded objects

$$\mathrm{gr}^{\ell}(K^n) := F^{\ell}(K^n)/F^{\ell+1}(K^n) = K^{n-\ell,\ell}$$

forms a complex $K^{\bullet,\ell}[-\ell]$ (up to a global sign $(-1)^{\ell}$). Hence $\mathcal{H}^k(\mathrm{gr}^{\ell}(K^{\bullet})) = \mathcal{H}^{k-\ell}(K^{\bullet,\ell})$, for all ℓ, and the cohomology of the complex $\mathcal{H}^n_I(K^{\bullet,\bullet}) := (\mathcal{H}^n(K^{\bullet,j}))_{j\in\mathbb{Z}}$, with respect to d_{II}, gives $\mathcal{H}^{\ell}_{II}(\mathcal{H}^{k-\ell}_I(K^{\bullet,\bullet}))$.

Assuming the following finiteness condition: for each n, there is $\ell_+(n)$ and $\ell_-(n)$ such that $F^{\ell}K^n = 0$, for all $\ell \geq \ell_+(n)$ and $F^{\ell}K^n = K^n$, for all $\ell \leq \ell_-(n)$, one can show that any filtered complex gives rise to a spectral sequence. In case of double complex, we have the following.

Proposition 5.1.7. *Let $K^{\bullet,\bullet}$ be a double complex such that for any n, $K^{n-\ell,\ell} = 0$, for $|\ell| \gg 0$. Then there is a spectral sequence*

$$E_2^{p,q} := \mathcal{H}^p_{II}\mathcal{H}^q_I(K^{\bullet,\bullet}) \Longrightarrow \mathcal{H}^{p+q}(K^{\bullet}).$$

Definition 5.1.8. Let $A^{\bullet} \in K^+(\mathcal{A})$. A *Cartan-Eilenberg resolution* of $A^{\bullet}$ is a double complex $C^{\bullet,\bullet}$ together with a morphism of complexes $A^{\bullet} \longrightarrow C^{\bullet,0}$ such that

(i) $C^{i,j} = 0$, for $j < 0$,
(ii) the sequences $A^n \to C^{n,0} \to C^{n,1} \to \cdots$ are injective resolutions of A^n, and the induced sequences

$$\begin{aligned}
&\mathrm{Ker}(d^n_A) \to \mathrm{Ker}(d^{n,0}_I) \to \mathrm{Ker}(d^{n,1}_I) \to \cdots \\
&\mathrm{Im}(d^n_A) \to \mathrm{Im}(d^{n,0}_I) \to \mathrm{Im}(d^{n,1}_I) \to \cdots \\
&\mathcal{H}^n(A^{\bullet}) \to \mathcal{H}^n_I(C^{\bullet,0}) \to \mathcal{H}^n_I(C^{\bullet,1}) \to \cdots
\end{aligned}$$

are injective resolutions of $\mathrm{Ker}(d^n_A)$, $\mathrm{Im}(d^n_A)$ and $\mathcal{H}^n(A^{\bullet})$, respectively, and
(iii) any short exact sequences of the form

$$0 \to \mathrm{Ker}(d^{i,j}_I) \to C^{i,j} \to \mathrm{Im}(d^{i,j}_I) \to 0$$

split.

Proposition 5.1.9. *If $\mathcal{A}$ has enough injectives, then any $A^{\bullet} \in K^+(\mathcal{A})$ admits a Cartan-Eilenberg resolution.*

Theorem 5.1.10 (Grothendieck spectral sequence)**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be abelian categories. Let $F_1 : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$ and $F_2 : K^+(\mathcal{B}) \longrightarrow K(\mathcal{C})$ be exact functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ contains enough injectives, and for any complex $I^\bullet \in K^+(\mathcal{A})$ of injective objects from $\mathcal{A}$, its image $F_1(I^\bullet)$ is inside an F_2-adapted triangulated subcategory $\mathcal{K}_{F_2}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$, there is a spectral sequence*

$$E_2^{p,q} := R^p F_2(R^q F_1(A^\bullet)) \Longrightarrow E^{p+q} := R^{p+q}(F_2 \circ F_1)(A^\bullet)\,. \tag{5.1.11}$$

Proof. Note that, if $F_1 = \mathrm{Id}$, then for a left exact functor $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$, the above spectral sequence reads

$$E_2^{p,q} := R^p F(\mathcal{H}^q(A^\bullet)) \Longrightarrow E^{p+q} := R^{p+q} F(A^\bullet)\,. \tag{5.1.12}$$

It follows from construction of derived functors that, given $A^\bullet \in D^+(\mathcal{A})$ isomorphic to a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$, we have $RF_1(A^\bullet) \cong F_1(I^\bullet)$ and

$$R^p F_2(R^q F_1(A^\bullet)) = R^p F_2(\mathcal{H}^q(F_1(I^\bullet))). \tag{5.1.13}$$

Since

$$\begin{aligned} R^n(F_2 \circ F_1)(A^\bullet) = \mathcal{H}^n(R(F_2 \circ F_1)(I^\bullet)) &\cong \mathcal{H}^n(RF_2(RF_1(A^\bullet)) \\ &\cong \mathcal{H}^n(RF_2(F_1(I^\bullet))) \cong R^n F_2(F_1(I^\bullet))\,, \end{aligned} \tag{5.1.14}$$

using (5.1.13), the general case (5.1.11) follows from the special case (5.1.12) above.

Therefore, it suffices to prove the result with $F_1 = \mathrm{Id}$ and $F := F_2$. For this we need an appropriate double complex, which is provided by a Cartan-Eilenberg resolution of $A^\bullet$. Let $C^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A^\bullet$, and set $K^{\bullet,\bullet} := F(C^{\bullet,\bullet})$. Since F is additive, it preserve direct sums, and since $C^{i,j} \cong \mathrm{Ker}(d_I^{i,j}) \oplus \mathrm{Im}(d_I^{i,j})$, we have $\mathcal{H}_I^q(K^{\bullet,p}) = F\mathcal{H}_I^q(C^{\bullet,p})$. Fixing q, and allowing p to vary, we see that $\mathcal{H}_I^q(C^{\bullet,p})$ defines an injective resolution of $\mathcal{H}^q(K^{\bullet,p}) = \mathcal{H}^q(A^\bullet)$. Then we have

$$\mathcal{H}_{II}^p \mathcal{H}_I^q(K^{\bullet,\bullet}) = R^p F(\mathcal{H}^q(A^\bullet)).$$

Now applying spectral sequence in Proposition 5.1.7 and using the fact that the natural morphism $A^\bullet \longrightarrow \mathrm{tot}(C^{\bullet,\bullet})$ is a quasi-isomorphism, we see that

$$\begin{aligned} \mathcal{H}^{p+q}(\mathrm{tot}(K^{\bullet,\bullet})) &= \mathcal{H}^{p+q}(F(\mathrm{tot}(C^{\bullet,\bullet}))) \\ &= \mathcal{H}^{p+q}(RF(A^\bullet)) \\ &= R^{p+q} F(A^\bullet). \end{aligned}$$

This completes the proof. □

Corollary 5.1.15. *Let $F : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$ be an exact functor admitting a right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$.*

(i) If $RF(A) \in D^b(\mathcal{B})$, for all $A \in \mathcal{A}$, then $RF(A^\bullet) \in D^b(\mathcal{B})$, for all $A^\bullet \in D^b(\mathcal{A})$, and hence RF induces a functor

$$RF : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}).$$

(ii) Suppose that $\mathcal{A}$ has enough injectives. If $\mathcal{C} \subset \mathcal{B}$ is a thick subcategory with $R^iF(A) \in \mathcal{C}$, for all $A \in \mathcal{A}$, and that there is an integer n such that $R^iF(A) = 0$, for all $A \in \mathcal{A}$. Then the image of RF lands inside $D^+_{\mathcal{C}}(\mathcal{B})$; i.e.,

$$RF : D^+(\mathcal{A}) \longrightarrow D^+_{\mathcal{C}}(\mathcal{B}).$$

6. Derived functors in algebraic geometry

6.1. **Cohomology.** Let X be a noetherian scheme defined over a field k. Then the global section functor

$$\Gamma : \mathfrak{QCoh}(X) \longrightarrow \mathcal{V}ect(k), \quad E \mapsto \Gamma(X, E) \tag{6.1.1}$$

is left exact. Since $\mathfrak{QCoh}(X)$ has enough injectives, the right derived functor (exact)

$$R\Gamma : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathcal{V}ect(k)) \tag{6.1.2}$$

of Γ exists, and we define

$$H^i(X, E^\bullet) := R^i\Gamma(E^\bullet) := \mathcal{H}^i(R\Gamma(E^\bullet)), \ \forall\, E^\bullet \in D^+(\mathfrak{QCoh}(X)). \tag{6.1.3}$$

Classically, this is known as the *hypercohomology* of $E^\bullet$, and is denoted by $\mathbb{H}^i(X, E^\bullet)$. For $E \in \mathfrak{QCoh}(X)$, the above definition (6.1.3) gives the usual i-th cohomology $H^i(X, E)$ of E, for all $i \geq 0$. Since any complex of k-vector spaces splits, we have an isomorphism (in $D^+(\mathcal{V}ect(k))$)

$$R\Gamma(E^\bullet) \cong \bigoplus_i H^i(X, E^\bullet)[-i], \quad \forall \ E^\bullet \in D^+(\mathfrak{QCoh}(X)). \tag{6.1.4}$$

Since for any $E \in \mathfrak{QCoh}(X)$, by Grothendieck's theorem $H^i(X, E) = 0$, for all $i > \dim(X)$ (see [Har77]), applying the Grothendieck spectral sequence

$$E_2^{p,q} := R^p\Gamma(\mathcal{H}^q(E^\bullet)) \Longrightarrow R^{p+q}\Gamma(E^\bullet),$$

one can deduce that $R\Gamma$ restrict to a functor

$$R\Gamma : D^b(\mathfrak{QCoh}(X)) \longrightarrow D^b(\mathcal{V}ect(k)) \tag{6.1.5}$$

(*Hint: If $E_\infty^{p,q} = 0$, for all p, q, then it follows from property (iv) of the spectral sequence that $E^{p+q} = 0$*; c.f., Corollary 5.1.15). The above functor (6.1.5) is exact because $R\Gamma$ in (6.1.2) is exact.

Next, we want to induce our derived functor $R\Gamma$ at the level of $D^b(X)$. Let $\mathcal{V}ect_{fd}(k)$ be the full subcategory of $\mathcal{V}ect(k)$, whose objects are finite dimensional k-vector spaces. If X is a proper k-scheme, by a theorem of Serre [Har77], for any $E \in \mathfrak{Coh}(X)$

we have $H^i(X,E) \in \mathcal{V}ect_{fd}(k)$. Since the category $\mathfrak{Coh}(X)$ doesn't have enough injectives, we cannot directly get the right derived functor $R\Gamma : D^b(X) \longrightarrow D^b(\mathcal{V}ect_{fd}(k))$ of the left exact functor $\Gamma : \mathfrak{Coh}(X) \longrightarrow \mathcal{V}ect_{fd}(k)$. Nevertheless, we can construct the right derived functor, in this case, as the composition of the exact functors

$$D^b(X) \longrightarrow D^b(\mathfrak{QCoh}(X)) \xrightarrow{R\Gamma} D^b(\mathcal{V}ect(k)). \tag{6.1.6}$$

Clearly, the image of the above composite functor lands inside $D^b(\mathcal{V}ect_{fd}(k))$.

6.2. **Derived direct image.** Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes. Then the direct image functor

$$f_* : \mathfrak{QCoh}(X) \longrightarrow \mathfrak{QCoh}(Y), \;\; E \mapsto f_*E \tag{6.2.1}$$

is left exact. Since $\mathfrak{QCoh}(X)$ has enough injectives, the right derived functor

$$Rf_* : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X)) \tag{6.2.2}$$

of f_* exists. In particular, $R^if_*(E^\bullet) := \mathcal{H}^i(Rf_*(E^\bullet)) \in \mathfrak{QCoh}(X)$, for all i. Thus, $R^if_*E \in \mathfrak{QCoh}(X)$, for all $E \in \mathfrak{QCoh}(X)$. Since $R^if_*(E) = 0$ for all $i > \dim(X)$ [Har77], by Corollary 5.1.15 (a) the functor Rf_* restricts to an exact functor

$$Rf_* : D^b(\mathfrak{QCoh}(X)) \longrightarrow D^b(\mathfrak{QCoh}(Y)). \tag{6.2.3}$$

Next, we want to get our derived functor Rf_* at the level of $D^b(X)$. Recall that, $\mathfrak{Coh}(X)$ is a thick full subcategory of $\mathfrak{QCoh}(X)$, and the inclusion functor $\mathfrak{Coh}(X) \hookrightarrow \mathfrak{QCoh}(X)$ induces a natural fully faithful exact functor

$$D^b(X) \longrightarrow D^b(\mathfrak{QCoh}(X)), \tag{6.2.4}$$

which gives an equivalence of categories

$$D^b(X) \xrightarrow{\simeq} D^b_{\mathfrak{Coh}(X)}(\mathfrak{QCoh}(X)), \tag{6.2.5}$$

where $D^b_{\mathfrak{Coh}(X)}(\mathfrak{QCoh}(X))$ is the triangulated full subcategory of $D^b(\mathfrak{QCoh}(X))$, whose objects are bounded complexes of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X with coherent cohomology sheaves.

Proposition 6.2.6. *If $f : X \longrightarrow Y$ is a proper morphism of noetherian k-schemes, then the right derived functor $Rf_* : D^b(\mathfrak{QCoh}(X)) \longrightarrow D^b(\mathfrak{QCoh}(Y))$ restricts to give an exact functor*

$$Rf_* : D^b(X) \longrightarrow D^b(Y). \tag{6.2.7}$$

Proof. Since f is proper, for any $E \in \mathfrak{Coh}(X)$, we have $R^if_*E \in \mathfrak{Coh}(Y)$, for all i. Then by Corollary 5.1.15 (b), the image of the composite functor

$$D^b(X) \longrightarrow D^b(\mathfrak{QCoh}(X)) \xrightarrow{Rf_*} D^b(\mathfrak{QCoh}(Y)). \tag{6.2.8}$$

lands inside $D^b_{\mathfrak{Coh}(Y)}(\mathfrak{QCoh}(Y))$. Then choosing a quasi-inverse

$$D^b_{\mathfrak{Coh}(Y)}(\mathfrak{QCoh}(Y)) \xrightarrow{\simeq} D^b(Y)$$

of the exact equivalence in (6.2.5), which is exact, we get the desired functor (6.2.7). This completes the proof. □

Proposition 6.2.9. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of noetherian k-schemes. The natural isomorphism of functors $(g \circ f)_* \cong g_* \circ f_*$ give rise to a natural isomorphism of the corresponding derived functors*

$$R(g \circ f)_* \cong Rg_* \circ Rf_* : D^b(\mathfrak{QCoh}(X)) \longrightarrow D^b(\mathfrak{QCoh}(Z)). \tag{6.2.10}$$

Proof. Recall that, an object $E \in \mathfrak{QCoh}(X)$ is called *flabby* if the restriction morphism $E(X) \longrightarrow E(U)$ is surjective, for any open subset U of X. Note that, any injective $\mathcal{O}_X$-module is flabby. Moreover, if $E \in \mathfrak{QCoh}(X)$ is flabby, then for any morphism of noetherian k-schemes $f : X \to Y$, we have $R^i f_*(E) = 0$, for all $i > 0$. Furthermore, $f_*(E)$ is flabby whenever E is flabby.

Let $\mathcal{I} \subset \mathfrak{QCoh}(X)$ be the full subcategory of injective $\mathcal{O}_X$-modules. Then $\mathcal{I}$ is f_*-adapted (c.f., Remark 3.1.14) and $f_*(\mathcal{I})$ is contained in the g_*-adapted full subcategory of all flabby $\mathcal{O}_Y$-modules. Hence the result follows. □

We can apply Grothendieck's spectral sequence to get what is known as *Leray spectral sequence*

$$E_2^{p,q} := R^p g_*(R^q f_*(E^\bullet)) \Longrightarrow R^{p+q}(g \circ f)_*(E^\bullet). \tag{6.2.11}$$

Taking $Y \longrightarrow \mathrm{Spec}(k)$ to be the structure morphism, we see that

$$R\Gamma(Y, -) \circ Rf_* \cong R\Gamma(X, -). \tag{6.2.12}$$

Then the above Leray spectral sequence gives its classical version

$$E_2^{p,q} := R^p g_* \mathcal{H}^q(E^\bullet) \Longrightarrow R^{p+q} g_*(E^\bullet). \tag{6.2.13}$$

Even more specially, for $f : X = Y \longrightarrow \mathrm{Spec}(k)$, we get the following Leray spectral sequence

$$E_2^{p,q} := H^p(X, \mathcal{H}^q(E^\bullet)) \Longrightarrow H^{p+q}(X, E^\bullet). \tag{6.2.14}$$

All of these are very useful computational tools in real life examples.

6.3. **Local $\mathcal{H}om^\bullet$ complex.** Let X be a noetherian scheme. For $E \in \mathfrak{QCoh}(X)$, the functor

$$\mathcal{H}om(E, -) : \mathfrak{QCoh}(X) \longrightarrow \mathfrak{QCoh}(X), \quad F \mapsto \mathcal{H}om(E, F)\,, \tag{6.3.1}$$

is left exact. Moreover, $\mathcal{H}om(E,F) \in \mathfrak{Coh}(X)$ if both E, F are coherent. Since $\mathfrak{QCoh}(X)$ has enough injectives (c.f., [Har77, Chapter III, Exercise 3.6]), its right derived functor

$$R\,\mathcal{H}om(E,-) : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X)) \tag{6.3.2}$$

exists. Then for any $E, F \in \mathfrak{QCoh}(X)$ and any integer i, we define

$$\mathcal{E}xt^i(E,F) := R^i\,\mathcal{H}om(E,F) := \mathcal{H}^i(R\,\mathcal{H}om(E,F)) \in \mathfrak{QCoh}(X)\,. \tag{6.3.3}$$

If $E \in \mathfrak{Coh}(X)$, the above definition is local in the sense that its stalk at $x \in X$ can be computed as

$$\mathcal{E}xt^i(E,F)_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(E_x, F_x)\,, \tag{6.3.4}$$

which follows from commutativity of the following diagram.

$$\begin{tikzcd} \mathfrak{QCoh}(X) \arrow[r, "\mathcal{H}om(E,-)"] \arrow[d] & \mathfrak{QCoh}(X) \arrow[d] \\ \mathrm{Mod}(\mathcal{O}_{X,x}) \arrow[r, "\mathrm{Hom}(E_x,-)"] & \mathrm{Mod}(\mathcal{O}_{X,x})\,. \end{tikzcd} \tag{6.3.5}$$

Note that, $\mathcal{E}xt^i(E,F) \in \mathfrak{Coh}(X)$ whenever both $E, F \in \mathfrak{Coh}(X)$.

When $E \in \mathfrak{Coh}(X)$, the functor (6.3.2) restricts to the bounded below derived category of coherent sheaves

$$R\,\mathcal{H}om(E,-) : D^+(X) \longrightarrow D^+(X)\,. \tag{6.3.6}$$

Since for a non-regular local ring A, the groups $\mathrm{Ext}^i_A(M,-)$ can be non-trivial even for $i \gg 0$, only for non-singular schemes X, the above functor $R\,\mathcal{H}om(E,-)$ restricts to $D^b(X)$, the bounded derived category of coherent sheaves on X.

As discussed before, the above construction easily generalizes for bounded above complexes of coherent sheaves $E^\bullet \in D^-(X)$. For this, we note that the following functor is exact.

$$\mathcal{H}om^\bullet(E^\bullet,-) : K^+(\mathfrak{QCoh}(X)) \longrightarrow K^+(\mathfrak{QCoh}(X))$$

given by sending a complex $F^\bullet \in K^+(\mathfrak{QCoh}(X))$ to the complex $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$, where

$$\mathcal{H}om^i(E^\bullet, F^\bullet) := \prod_{p\in\mathbb{Z}} \mathcal{H}om(E^p, F^{i+p})$$

and the differentials are given by $d^i = d_{E^\bullet} - (-1)^i d_{F^\bullet}$, for all $i \in \mathbb{Z}$. The following lemma follows form corresponding local statement for modules over a ring.

Lemma 6.3.7. *Let $F^\bullet \in D^-(X)$ be a complex of injective sheaves. If $F^\bullet$ or $E^\bullet \in K^+(\mathfrak{QCoh}(X))$ is acyclic, then $\mathcal{H}om^\bullet(E^\bullet, F^\bullet)$ is acyclic.*

The above Lemma 6.3.7 applied to the class

$$\mathcal{I} := \{I^\bullet \in K^+(\mathfrak{QCoh}(X)) : I^i \text{ is injective } \mathcal{O}_X\text{-module}\}$$

shows that $\mathcal{I}$ is adapted to the functor $\mathcal{H}om^\bullet(E^\bullet, -)$ (see Remark 3.1.14), and hence, the right derived functor

$$R\,\mathcal{H}om^\bullet(E^\bullet, -) : D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X)) \tag{6.3.8}$$

exists. Note that, we are working with $\mathfrak{QCoh}(X)$, because $\mathfrak{Coh}(X)$ has not enough injectives. Similarly, to see that the functor

$$\mathcal{H}om^\bullet(-, F^\bullet) : D^-(\mathfrak{QCoh}(X))^{\mathrm{op}} \longrightarrow D^+(\mathfrak{QCoh}(X))$$

descends to the derived category for any $F^\bullet \in D^+(\mathfrak{QCoh}(X))$. Therefore, we get a bifunctor

$$R\,\mathcal{H}om^\bullet(-, -) : D^-(\mathfrak{QCoh}(X))^{\mathrm{op}} \times D^+(\mathfrak{QCoh}(X)) \longrightarrow D^+(\mathfrak{QCoh}(X))\,. \tag{6.3.9}$$

This enables us to define

$$\mathcal{E}xt^i(E^\bullet, F^\bullet) := R^i\,\mathcal{H}om^\bullet(E^\bullet, F^\bullet) := \mathcal{H}^i(R\,\mathcal{H}om^\bullet(E^\bullet, F^\bullet)) \in \mathfrak{QCoh}(X)\,, \quad \forall\, i. \tag{6.3.10}$$

Assume that X is a regular noetherian k-scheme. Although the category $\mathfrak{Coh}(X)$ has not enough injectives, for the purpose of computing local Ext's (i.e., $\mathcal{E}xt$), locally free coherent sheaves are good enough. More precisely, if $E^\bullet \in D^b(X)$ is a bounded complex of locally free coherent sheaves on X, then $R\,\mathcal{H}om(E^\bullet, -)$ can be computed as $\mathcal{H}om(E^\bullet, -)$. This can be deduced from the corresponding local statement that, for any bounded complex $M^\bullet$ of free modules over a local ring A, $R\,\mathrm{Hom}(M^\bullet, -)$ can be computed as $\mathrm{Hom}(M^\bullet, -)$, which follows because free A-modules are projective.

Proposition 6.3.11. *Let X be a non-singular noetherian k-scheme. Then any bounded complex $E^\bullet \in D^b(X)$ is isomorphic to a bounded complex $\mathcal{E}^\bullet \in D^b(X)$ of locally free coherent sheaves of $\mathcal{O}_X$-modules on X.*

Proof. Since X is a noetherian non-singular k-scheme, $\mathfrak{Coh}(X)$ has enough projectives, meaning that any $E \in \mathfrak{Coh}(X)$ admits a finite resolution

$$0 \to F_i^\ell \to F_i^{\ell-1} \to \cdots \to F_i^0 \to E \to 0,$$

with F_i^j a locally free coherent sheaves of $\mathcal{O}_X$-modules on X. Moreover, one can choose such a resolution of length $\ell \leq \dim_k(X)$. Hence the result follows. □

6.4. **Trace map.** Let X be a regular noetherian k-scheme. Since any $E^\bullet \in D^b(X)$ is isomorphic to a bounded complex of locally free coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{E}^\bullet$ in $D^b(X)$, we may assume that each E^i is a locally free coherent sheaves of $\mathcal{O}_X$-modules. Then $R\,\mathcal{H}om(E^\bullet, E^\bullet) = \mathcal{H}om^\bullet(E^\bullet, E^\bullet)$. By definition, $\mathcal{H}om^0(E^\bullet, E^\bullet) =$

$\bigoplus_i \mathcal{H}om(E^i, E^i)$. Then the usual trace morphism $\mathrm{tr}_{E^i} : \mathcal{H}om(E^i, E^i) \longrightarrow \mathcal{O}_X$ for the locally free sheaves E^i give rise to the trace morphism

$$\mathrm{tr}_{E^\bullet} := \bigoplus_i (-1)^i \mathrm{tr}_{E^i} : R\mathcal{H}om(E^\bullet, E^\bullet) \longrightarrow \mathcal{O}_X. \tag{6.4.1}$$

6.5. **Derived dual.** For $E^\bullet \in D^-(\mathfrak{QCoh}(X))$, we define its *dual* (or, more precisely, its *derived dual*) to be the object

$$E^{\bullet\vee} := R\mathcal{H}om(E^\bullet, \mathcal{O}_X) \in D^+(\mathfrak{QCoh}(X)). \tag{6.5.1}$$

If $E^\bullet$ is a bounded above complex of locally free coherent sheaves on X, then we can compute its (derived) dual $E^{\bullet\vee}$ as the bounded below complex

$$\cdots \to \mathcal{H}om(E^{i+1}, \mathcal{O}_X) \to \mathcal{H}om(E^i, \mathcal{O}_X) \to \mathcal{H}om(E^{i-1}, \mathcal{O}_X) \to \cdots . \tag{6.5.2}$$

If X is regular noetherian k-scheme, then for any $E^\bullet \in D^b(X)$, its (derived) dual $E^{\bullet\vee} := R\mathcal{H}om(E^\bullet, \mathcal{O}_X) \in D^b(X)$.

Note that, even for a coherent sheaf E on X, its derived dual

$$E^\vee := R\mathcal{H}om(E, \mathcal{O}_X) \in D^b(\mathfrak{QCoh}(X))$$

need not be a sheaf on X. For example, if $E \in \mathfrak{Coh}(X)$ is a coherent sheaf on a smooth projective k-variety X with $\mathrm{codim}_X(\mathrm{Supp}(E)) \geq d$, then $R\mathcal{H}om(E, \mathcal{O}_X)$ is a complex concentrated in degree $\geq d$. (Hint: Use Serre duality and [HL10, Proposition 1.1.6]).

We shall see later, using Grothendieck-Verdier duality (Theorem 6.9.1), that for any smooth closed k-subvariety $\iota : Z \hookrightarrow X$ of codimension c in a smooth k-variety X, the derived dual of $\iota_*\mathcal{O}_Z$ can be computed as

$$(\iota_*\mathcal{O}_Z)^\vee \cong \big(\iota_*\omega_Z \otimes_{\mathcal{O}_X} \mathcal{H}om(\omega_X, \mathcal{O}_X)\big)[-c]. \tag{6.5.3}$$

As an immediate consequence of this formula, we have the following. If $D \overset{\iota}{\hookrightarrow} X$ is a divisor in X, then using the adjunction formula $\omega_D \cong (\omega_X \otimes \mathcal{O}_X(D))\big|_D$ we have, $(\iota_*\mathcal{O}_D)^\vee \cong \iota_*\mathcal{O}_D(D)[-1]$.

6.6. **Derived tensor product.** Let X be a projective k-scheme. Then any coherent sheaf $E \in \mathfrak{Coh}(X)$ admits a resolution (not necessarily finite) by locally free coherent sheaves of $\mathcal{O}_X$-modules

$$\mathcal{E}^\bullet \to E. \tag{6.6.1}$$

If X is smooth, we can choose $\mathcal{E}^\bullet$ to be a bounded complex of length $\leq \dim(X)$. Note that, for any $F \in \mathfrak{Coh}(X)$, the tensor product functor $F \otimes - : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X)$ is right exact. If $E^\bullet$ is a bounded above acyclic complex (i.e., $\mathcal{H}^i(E^\bullet) = 0$, for all i) of locally free coherent sheaves of $\mathcal{O}_X$-modules, then $F \otimes E^\bullet$ is also acyclic. Therefore, the full subcategory $\mathcal{V}ect(X)$ of locally free coherent sheaves on X is adapted to the right exact functor $F^\bullet \otimes -$.

Consider a bounded above complex of coherent sheaves of $\mathcal{O}_X$-modules $E^\bullet \in K^-(\mathfrak{Coh}(X))$. Define a functor

$$E^\bullet \otimes - : K^-(\mathfrak{Coh}(X)) \longrightarrow K^-(\mathfrak{Coh}(X)) \tag{6.6.2}$$

by sending $F^\bullet \in K^-(\mathfrak{Coh}(X))$ to the complex $E^\bullet \otimes F^\bullet$:

$$(E^\bullet \otimes F^\bullet)^i := \bigoplus_{p+q=i} E^p \otimes F^q, \text{ with } d = d_E \otimes 1 + (-1)^i 1 \otimes d_F. \tag{6.6.3}$$

So by definition, $E^\bullet \otimes F^\bullet$ is the total complex of the double complex $K^{\bullet,\bullet}$ with $K^{p,q} := E^p \otimes F^q$, and the two differentials are $d_I := d_E \otimes 1$ and $d_{II} := (-1)^{p+q} 1 \otimes d_F$. Therefore, to get the left derived functor of $E^\bullet \otimes -$, we need to check that the full subcategory $\mathcal{K}om^-(\mathcal{V}ect(X))$ of bounded above complexes of locally free coherent sheaves of $\mathcal{O}_X$-modules is adopted to $E^\bullet \otimes -$. Since any coherent sheaf $F \in \mathfrak{Coh}(X)$ admits a surjective morphism $\mathcal{F} \to F$, with $\mathcal{F}$ locally free coherent sheaf of $\mathcal{O}_X$-modules, it remains to check that, for any acyclic complex $F^\bullet \in K^-(\mathfrak{Coh}(X))$ with all F^i locally free, $E^\bullet \otimes F^\bullet$ is acyclic. For this, we use the following spectral sequence

$$\mathbb{E}_2^{p,q} := \mathcal{H}_I^p \mathcal{H}_{II}^q(K^{\bullet,\bullet}) \Longrightarrow \mathbb{E}^{p+q} := \mathcal{H}^{p+q}(E^\bullet \otimes F^\bullet). \tag{6.6.4}$$

Note that, for $F^\bullet$ acyclic with all F^i locally free, $E^p \otimes F^\bullet$ is acyclic, for all p. Therefore, $\mathcal{H}_{II}(E^p \otimes F^\bullet) = 0$, for all p, and hence, $\mathbb{E}_2^{p,q} = 0$ for all p and q. Since $\mathbb{E}_\infty^{p,q} \cong \mathrm{F}^p\mathbb{E}^{p+q}/\mathrm{F}^{p+1}\mathbb{E}^{p+q}$ and $\bigcap\limits_p \mathrm{F}^p E^{p+q} = 0$, it follows that $\mathbb{E}^{p+q} = 0$. Hence $E^\bullet \otimes F^\bullet$ is acyclic. Therefore, the left derived functor

$$E^\bullet \overset{L}{\otimes} - : D^-(X) \longrightarrow D^-(X) \tag{6.6.5}$$

exists. Similarly, one can show that for a complex of locally free sheaves $F^\bullet$ and an acyclic complex $E^\bullet$, the tensor product complex $E^\bullet \otimes F^\bullet$ is acyclic. In other words, the induced bifunctor

$$K^-(\mathfrak{Coh}(X)) \times D^-(X) \longrightarrow D^-(X) \tag{6.6.6}$$

need not be derived in the first factor, and descends to the bifunctor

$$(-\overset{L}{\otimes}-) : D^-(X) \times D^-(X) \longrightarrow D^-(X)$$

on the derived categories.

Suppose that X is a smooth projective k-scheme. Then any $E^\bullet \in D^b(X)$ is isomorphic to a bounded complex of locally free coherent sheaves of $\mathcal{O}_X$-modules. Therefore, for $E^\bullet, F^\bullet \in D^b(X)$, replacing them with isomorphic bounded complexes of locally free coherent sheaves of $\mathcal{O}_X$-modules, we can compute their (derived) tensor product $E^\bullet \overset{L}{\otimes} F^\bullet$ as the ordinary tensor product $E^\bullet \otimes F^\bullet$ of complexes. This gives us functorial isomorphisms

$$E^\bullet \overset{L}{\otimes} F^\bullet \cong F^\bullet \overset{L}{\otimes} E^\bullet \quad \text{and}$$
$$E^\bullet \overset{L}{\otimes} (F^\bullet \overset{L}{\otimes} G^\bullet) \cong (E^\bullet \overset{L}{\otimes} F^\bullet) \overset{L}{\otimes} G^\bullet .$$

For any $E^\bullet, F^\bullet \in D^b(X)$, we define

$$\mathcal{T}or_i(E^\bullet, F^\bullet) := \mathcal{H}^{-i}(E^\bullet \overset{L}{\otimes} F^\bullet); \tag{6.6.7}$$

which can be computed using the following spectral sequence.

Proposition 6.6.8. *There is a spectral sequence*

$$\mathbb{E}_2^{p,q} := \mathcal{T}or_{-p}(\mathcal{H}^q(E^\bullet), F^\bullet) \Longrightarrow \mathbb{E}^{p+q} := \mathcal{T}or_{-(p+q)}(E^\bullet, F^\bullet).$$

Remark 6.6.9. For the sake of simplicity, we only have explained how to get the left derive functor of the tensor product functor in case X is a projective k-scheme. The general case is also similar, but require more technical cares to construct it.

6.7. **Defived pullback.** Recall that, for any morphism of locally ringed spaces

$$f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y), \tag{6.7.1}$$

the pullback functor

$$f^* : \mathfrak{Mod}(\mathcal{O}_Y) \longrightarrow \mathfrak{Mod}(\mathcal{O}_X) \tag{6.7.2}$$

is defined to be the composition of the exact functor

$$f^{-1} : \mathfrak{Mod}(\mathcal{O}_Y) \longrightarrow \mathfrak{Mod}_X(f^{-1}(\mathcal{O}_Y)) \tag{6.7.3}$$

with the right exact functor

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-) : \mathfrak{Mod}_X(f^{-1}\mathcal{O}_Y) \longrightarrow \mathfrak{Mod}(\mathcal{O}_X). \tag{6.7.4}$$

Thus, f^* is right exact. Let $\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)$ be the left derived functor of $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-)$. Since f^{-1} is exact, we don't need to derive it. Then we can define the left derived functor of f^* to be the composite functor

$$Lf^* := \big(\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)\big) \circ f^{-1} : D^-(Y) \longrightarrow D^-(X). \tag{6.7.5}$$

Note that, we have discussed how to get the left derived functor $\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} (-)$ only for the case X is a projective k-scheme. However, general case being similar but little more technical in nature, we leave it to the reader to fill the gap. In most of the applications, we work with f a flat morphism, in which case, we don't need to derive f^* as it is already exact.

The following spectral sequence is useful to work with Lf^* in real life.

Proposition 6.7.6. *There is a spectral sequence*

$$\mathbb{E}_2^{p,q} := L^p f^*(\mathcal{H}^q(E^\bullet)) \Longrightarrow \mathbb{E}^{p+q} := L^{p+q} f^*(E^\bullet),$$

where $L^i f^*(E^\bullet) := \mathcal{H}^i(Lf^*(E^\bullet))$, *for all* $i \in \mathbb{Z}$.

6.8. **Compatibilities.** In this subsection, we quickly go through compatibilities among various derived functors generalizing classical ones. We only sketch their proofs, leaving the details to the readers.

(i) Let $f : X \longrightarrow Y$ be a proper morphism of projective k-schemes. Then for any $E^\bullet \in D^b(X)$ and $F^\bullet \in D^b(Y)$, we have a natural isomorphism (*projection formula*)

$$Rf_*(E^\bullet) \overset{L}{\otimes} F^\bullet \cong Rf_*(E^\bullet \overset{L}{\otimes} Lf^* F^\bullet). \tag{6.8.1}$$

This follows from the following classical projection formula [Har77]: for a coherent sheaf of $\mathcal{O}_X$-modules E on X and a locally free coherent sheaf of $\mathcal{O}_Y$-modules on Y, we have a natural isomorphism of $\mathcal{O}_Y$-modules

$$f_*(E \otimes_{\mathcal{O}_X} f^* F) \cong f_* E \otimes_{\mathcal{O}_Y} F.$$

(ii) Let $f : X \longrightarrow Y$ be a morphism of projective k-schemes. Then for any $E^\bullet, F^\bullet \in D^b(Y)$, there is a natural isomorphism

$$(Lf^* E^\bullet) \overset{L}{\otimes} (Lf^* F^\bullet) \xrightarrow{\simeq} Lf^*(E^\bullet \overset{L}{\otimes} F^\bullet). \tag{6.8.2}$$

Since Y is projective k-scheme (smoothness is not required!), we can replace $E^\bullet$ and $F^\bullet$ by bounded above complexes of locally free coherent sheaves of $\mathcal{O}_Y$-modules on Y, and use them to compute their derived tensor product as the usual tensor product of complexes. The resulting complex of locally free coherent sheaves of $\mathcal{O}_Y$-modules is again bounded above, and so we can compute its derived pullback as the ordinary pullback of complex of locally free sheaves. Thus we obtain a bounded above complex of locally free coherent sheaves on X. Now the above formula (6.8.2) can be deduced by using the classical pullback formula $f^*E \otimes f^*F \cong f^*(E \otimes F)$ for coherent sheaves.

(iii) Let $f : X \longrightarrow Y$ be a projective morphism of noetherian schemes. Then we have $Lf^* \dashv Rf_*$; i.e., there is a functorial isomorphism

$$\mathrm{Hom}(Lf^* E^\bullet, F^\bullet) \xrightarrow{\simeq} \mathrm{Hom}(E^\bullet, Rf_* F^\bullet), \tag{6.8.3}$$

for all $E^\bullet \in D^-(\mathfrak{QCoh}(Y))$ and $F^\bullet \in D^+(\mathfrak{QCoh}(X))$. To see this, replacing $E^\bullet$ with a bounded above complex of locally free sheave of $\mathcal{O}_Y$-modules quasi-isomorphic to $E^\bullet$, and $F^\bullet$ with a bounded below complex of injective $\mathcal{O}_X$-modules quasi-isomorphic to $F^\bullet$, we can compute the corresponding derived functors as the usual pullback (resp., push-forward) of complexes along f. Then the statement follows from the classical adjunction formula $\mathrm{Hom}(f^*E, F) \cong \mathrm{Hom}(E, f_*F)$ for coherent sheaves.

(iv) Assume that X is a smooth projective k-variety. Let $E^\bullet, F^\bullet, G^\bullet \in D^b(X)$ be the bounded complexes of coherent sheaves on X. Then we have the following natural isomorphisms.

$$R\mathcal{H}om(E^\bullet, F^\bullet)\overset{L}{\otimes}G^\bullet \cong R\mathcal{H}om(E^\bullet, F^\bullet\overset{L}{\otimes}G^\bullet) \tag{6.8.4}$$

6.9. **Grothendieck-Verdier duality.** In this subsection, we state a deep duality theorem known as Grothendieck-Verdier duality, and show its applications. We refer the reader to [Con00] for its proof.

Theorem 6.9.1 (Grothendieck-Verdier duality). *Let $f : X \longrightarrow Y$ be a morphism of smooth schemes over a field k of relative dimension $\dim(f) := \dim(X) - \dim(Y)$. Let*

$$\omega_f := \omega_X \otimes f^*\omega_Y^\vee \tag{6.9.2}$$

be the relative dualizing sheaf of f. Then for any $F^\bullet \in D^b(X)$ and $E^\bullet \in D^b(Y)$, there is a a functorial isomorphism

$$Rf_*R\mathcal{H}om(F^\bullet, Lf^*(E^\bullet)\overset{L}{\otimes}\omega_f[\dim(f)]) \xrightarrow{\simeq} R\mathcal{H}om(Rf_*F^\bullet, E^\bullet)\,. \tag{6.9.3}$$

7. Examples of spectral sequence

Here we give some useful examples of Grothendieck spectral sequences, which will appear in next sections.

Example 7.0.1. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $A^\bullet \in D(\mathcal{A})$ and $B^\bullet \in D^+(\mathcal{A})$. Take $F_1 = \mathrm{Id}$ so that $R^qF_1(B^\bullet) = \mathcal{H}^q(B^\bullet)$, and take $F_2 = \mathrm{Hom}^\bullet(A^\bullet, -)$. Then we have,

$$R^pF_2(R^qF_1(B^\bullet) = R^p\,\mathrm{Hom}^\bullet(A^\bullet, \mathcal{H}^q(B^\bullet)) = \mathrm{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)),$$

and

$$R^{p+q}(F_2 \circ F_1)(B^\bullet) = R^{p+q}\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet).$$

Since $\mathrm{Ext}^{p+q}(A^\bullet, B^\bullet) = \mathrm{Hom}(A^\bullet, B^\bullet[p+q])$ and $\mathrm{Ext}^p(A^\bullet, \mathcal{H}^q(B^\bullet)) \cong \mathrm{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p])$, by Theorem 5.1.10, we have a spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(A^\bullet, \mathcal{H}^q(B^\bullet)[p]) \Longrightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q]). \tag{7.0.2}$$

Similarly, if $\mathcal{A}$ has enough projectives so that we can compute $R^p\,\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ for $A^\bullet \in D^-(\mathcal{A})$ as the right derived functor of the contravariant functor $\mathrm{Hom}^\bullet(-, B^\bullet) : K^-(\mathcal{A})^{\mathrm{op}} \longrightarrow K(\mathbf{Ab})$, we have the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(A^\bullet), B^\bullet[p]) \Longrightarrow \mathrm{Hom}(A^\bullet, B^\bullet[p+q])\,. \tag{7.0.3}$$

If $B^\bullet \in D^+(\mathcal{A})$ is bounded below, and $\mathcal{A}$ has enough injectives, but may not have enough projectives, then also we have this spectral sequence.

Remark 7.0.4. It should be noted that, a spectral sequence $E_2^{p,q} \Rightarrow E^{p+q}$ given at page E_2 **does not imply** that the term $E_\infty^{p,q}$ lies in page E_2.

Example 7.0.5. Let X be a noetherian scheme so that $\mathfrak{QCoh}(X)$ has enough injectives. Then for any $E^\bullet, F^\bullet \in D^b(X)$, we have the following spectral sequences

$$E_2^{p,q} := \mathrm{Ext}^p(E^\bullet, \mathcal{H}^q(F^\bullet)) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet)\,, \tag{7.0.6}$$

and

$$E_2^{p,q} := \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), F^\bullet) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet)\,, \tag{7.0.7}$$

Remark 7.0.8. If X is a projective k-variety, $\dim_k H^i(X, E) < \infty$ for any coherent sheaf E on X. Using this, one can deduce that $\dim_k \mathrm{Ext}^i(E, F) < \infty$, for any $E, F \in \mathfrak{Coh}(X)$. Then using the spectral sequences (7.0.6) and (7.0.7) one can show that $\mathrm{Ext}^i(E^\bullet, F^\bullet)$ has finite dimension, for all $E^\bullet, F^\bullet \in D^b(X)$.

Example 7.0.9. Let $E^\bullet \in D^-(X)$. Then by definition of local Hom complex, we have $\Gamma \circ \mathcal{H}om^\bullet(E^\bullet, -) = \mathrm{Hom}^\bullet(E^\bullet, -)$. Since for a complex $I^\bullet$ of injective sheaves of $\mathcal{O}_X$-modules, the complex $\mathcal{H}om^\bullet(E^\bullet, I^\bullet)$ is Γ-acyclic (meaning that, $\mathrm{Ext}^i(E^\bullet, I^\bullet) = R^i\Gamma(\mathcal{H}om^\bullet(E^\bullet, I^\bullet)) = 0$ for all $i \neq 0$, which indeed holds), we have

$$R\Gamma \circ R\,\mathcal{H}om^\bullet(E^\bullet, -) = R\,\mathrm{Hom}^\bullet(E^\bullet, -).$$

Therefore, applying Theorem 5.1.10 we have the following spectral sequence relating local and global Ext:

$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q(E^\bullet, F^\bullet)) \Longrightarrow \mathrm{Ext}^{p+q}(E^\bullet, F^\bullet). \tag{7.0.10}$$

8. Bondal–Orlov's reconstruction theorem

8.1. **What is it?** A famous theorem of Gabriel says that two k-varieties X and Y are isomorphic if and only if there is an equivalence of categories $\mathfrak{Coh}(X)$ with $\mathfrak{Coh}(Y)$. In [Muk81], Mukai established an equivalence $D^b(A) \simeq D^b(\check{A})$, where A is an abelian variety and $\check{A}$ its dual abelian variety. Therefore, equivalence between bounded derived category of coherent sheaves fails to ensure isomorphism of varieties, in general. In their famous paper [BO01], Bondal and Orlov shows how to reconstruct a smooth projective variety X from $D^b(X)$ when ω_X or its dual is ample (see Theorem 8.1.1). More precisely,

Theorem 8.1.1 (Bondal–Orlov)**.** *Let X be a smooth projective variety over k with canonical line bundle ω_X. Assume that ω_X (resp., $\omega_X^\vee$) is ample. Let Y be any smooth projective variety over k. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k-varieties. In particular, ω_Y (resp., $\omega_Y^\vee$) is ample.*

The main idea behind the proof is to "cohomologically" characterize closed points, invertible sheaves and Zariski topology of a smooth projective k-variety. For this we need "Serre functor" as defined in Definition 4.1.1. Note that, both $D^b(X)$ and $D^b(Y)$ admits Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim(Y)]$, respectively. As a first step towards this theorem, we now establish equality of dimensions of X and Y.

8.2. **Equality of dimensions.** Let k be a field. A *k–variety* is an integral separated finite type k-scheme. For any smooth projective k-variety X, we define $D^b(X) := D^b(\mathfrak{Coh}(X))$. A rank one invertible sheaf L on X is said to have *finite order* if $L^r \cong \mathcal{O}_X$ for some integer $r > 0$. The smallest positive integer r such that $L^r \cong \mathcal{O}_X$ is called the *order* of L. If $L^r \ncong \mathcal{O}_X$, $\forall\, r > 0$, we say that L has *infinite order*. For any $x \in X$, let $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field at x. For any closed point $x \in X$, we can consider $k(x)$ as a coherent sheaf on X supported at x by taking its push-forward along the closed embedding $\iota_x : \mathrm{Spec}(k(x)) \hookrightarrow X$. This is the skyscraper sheaf supported at x given by

$$k(x)(U) = \begin{cases} k(x), & \text{if } x \in U, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 8.2.1. *Let X and Y be smooth projective varieties over k. If there is an exact equivalence $D^b(X) \xrightarrow{\sim} D^b(Y)$ of bounded derived categories, then $\dim_k(X) = \dim_k(Y)$. In this case, both ω_X and ω_Y have the same order (can be infinity too).*

Proof. Since both X and Y are smooth projective k-varieties, by Theorem 4.2.2, they admit natural Serre functors $S_X := (\omega_X \otimes -)[\dim_k(X)]$ and $S_Y := (\omega_Y \otimes -)[\dim_k(Y)]$, respectively. By Lemma 4.1.7, any k-linear equivalence $F : D^b(X) \longrightarrow D^b(Y)$ commutes with Serre functors S_X and S_Y (i.e., there is a natural isomorphism of functors $F \circ S_X \cong S_Y \circ F$).

For a closed point $x \in X$, we have $k(x) \cong k(x) \otimes \omega_X \cong S_X(k(x))[-\dim_k(X)]$. So,

$$\begin{aligned} F(k(x)) &\cong F(k(x) \otimes \omega_X) = F(S_X(k(x))[-\dim_k(X)]) \\ &\cong F(S_X(k(x)))[-\dim_k(X)], \quad \text{since } F \text{ is exact.} \\ &\cong S_Y(F(k(x)))[-\dim_k(X)], \quad \text{since } F \circ S_X \cong S_Y \circ F. \\ &\cong F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]\,. \end{aligned} \tag{8.2.2}$$

Since F is an equivalence of categories, $F(k(x))$ is a non-trivial bounded complex. Let i be the maximal (resp., minimal) integer such that $\mathcal{H}^i(F(k(x))) \neq 0$. Now from

(8.2.2) we have

$$\begin{aligned} 0 \neq \mathcal{H}^i(F(k(x))) &\cong \mathcal{H}^i(F(k(x)) \otimes \omega_Y[\dim_k(Y) - \dim_k(X)]) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x)) \otimes \omega_Y) \\ &\cong \mathcal{H}^{i+\dim_k(Y)-\dim_k(X)}(F(k(x))) \otimes \omega_Y \,. \end{aligned} \tag{8.2.3}$$

Since ω_Y is a line bundle, (8.2.3) contradicts maximality (resp., minimality) of i whenever $\dim_k(X) < \dim_k(Y)$ (resp., $\dim_k(X) > \dim_k(Y)$). Therefore, $\dim_k(X) = \dim_k(Y)$.

To see that both ω_X and ω_Y have the same order, assume that $\omega_X^k \cong \mathcal{O}_X$. Let $n = \dim_k(X) = \dim_k(Y)$. Note that, $S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)}$. Since $F \circ S_X \cong S_Y \circ F$, choosing a quasi-inverse of the equivalence F, we have

$$\begin{aligned} &F^{-1} \circ S_Y^k[-kn] \circ F \cong S_X^k[-kn] \cong \mathrm{Id}_{D^b(X)} \\ \Rightarrow &S_Y^k[-kn] \cong \mathrm{Id}_{D^b(Y)} \ . \end{aligned}$$

Applying $\mathcal{O}_Y$ to the above isomorphism of functors, we get $\omega_Y^k \cong \mathcal{O}_Y$. □

Remark 8.2.4. In the proof of above Proposition, to show both ω_X and ω_Y have the same order, under the assumption that $\dim(X) = \dim(Y)$, we don't need F to be exact.

8.3. **Point like objects.**

Definition 8.3.1. A *graded category* is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ and an equivalence functor $T_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, known as *shift functor*. A functor $F : \mathcal{D} \to \mathcal{D}'$ between graded categories is called *graded* if there is an isomorphism of functors $F \circ T_{\mathcal{D}} \xrightarrow{\simeq} T_{\mathcal{D}'} \circ F$.

Example 8.3.2. Any triangulated category is a graded category, and any morphism between two triangulated categories is a graded morphism.

Definition 8.3.3. Let $\mathcal{D}$ be a k-linear triangulated category with Serre functor S. An object $P \in \mathcal{D}$ is said to be *point like* of codimension s if

(i) $S(P) \cong P[s]$,
(ii) $\mathrm{Hom}(P, P[i]) = 0$, for $i < 0$, and
(iii) $k(P) := \mathrm{Hom}(P, P)$ is a field.

An object E of an additive category is called *simple* if $\mathrm{Hom}(E, E)$ is a field.

Example 8.3.4. Let X be a smooth projective k-variety of dimension n.

(i) For any closed point $x \in X$, we have $S_X(k(x)) = (k(x) \otimes \omega_X)[n] \cong k(x)[n]$. Therefore, $k(x) \in D^b(X)$ is a point like object of codimension d.

(ii) Let $\omega_X \cong \mathcal{O}_X$ (for example when X is an abelian variety or a K3 surface). Then any simple object $E \in \mathfrak{Coh}(X)$ defines a point like object of codimension n in $D^b(X)$.

Proposition 8.3.5. *Let $\mathcal{A}$ be an abelian category, and $A^\bullet \in D^b(\mathcal{A})$. Let*

$$i^+ := \max\{i : \mathcal{H}^i(A^\bullet) \neq 0\} \quad \text{and} \quad i^- := \min\{i : \mathcal{H}^i(A^\bullet) \neq 0\}.$$

Then in $D^b(\mathcal{A})$, there are morphisms $\phi : A^\bullet \to \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ and $\psi : \mathcal{H}^{i^-}(A^\bullet)[-i^-] \to A^\bullet$ such that $\mathcal{H}^{i^+}(\phi) = \mathrm{Id}_{\mathcal{H}^{i^+}(A^\bullet)}$ and $\mathcal{H}^{i^-}(\psi) = \mathrm{Id}_{\mathcal{H}^{i^-}(A^\bullet)}$.

Proof. There is a natural quasi-isomorphism of complexes

$$\begin{tikzcd}
A^\bullet_- : \arrow[d, "qis"'] & \cdots \arrow[r] & A^{i^+-1} \arrow[r] \arrow[d, equal] & \mathrm{Ker}(d^{i^+}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : & \cdots \arrow[r] & A^{i^+-1} \arrow[r] & A^{i^+} \arrow[r, "d^{i^+}"] & A^{i^++1} \arrow[r] & \cdots .
\end{tikzcd}$$

Since the natural morphism of complexes $A^\bullet_- \longrightarrow \mathcal{H}^{i^+}(A^\bullet)[-i^+]$ induces identity morphism at i^+-th cohomology, the first part follows. The second part is similar. □

Corollary 8.3.6. *With the above notations, for any $B \in \mathcal{A}$, we have the following natural isomorphisms*

(i) $\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[-i^+])$, *and*
(ii) $\mathrm{Hom}_{D^b(\mathcal{A})}(B, \mathcal{H}^{i^-}(A^\bullet)) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(B[-i^-], A^\bullet)$.

Proof. Send $f \in \mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{H}^{i^+}(A^\bullet), B)$ to $f[-i^+]$ and use above Proposition 8.3.5. To get the inverse map, send any $\phi \in \mathrm{Hom}_{D^b(\mathcal{A})}(A^\bullet, B[i^+])$ to $\mathcal{H}^{i^+}(\phi)[-i^+]$. The second part is similar. □

Exercise 8.3.7. Let $A^\bullet \in D(\mathcal{A})$ with $\mathcal{H}^i(A^\bullet) = 0$, for all $i < m$. Then there is a distinguished triangle

$$\mathcal{H}^m(A^\bullet)[-m] \longrightarrow A^\bullet \xrightarrow{\varphi} B^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[1-m]$$

in the derived category $D(\mathcal{A})$ such that

$$\mathcal{H}^i(B^\bullet) \cong \begin{cases} \mathcal{H}^i(A^\bullet) & \text{if } i \leq m, \text{ and} \\ 0, & \text{if } i > m. \end{cases}$$

Remark 8.3.8. Let X be a smooth projective k-variety of dimension d. Then any point like object $P \in D^b(X)$ has codimension d. This follows from assumption (i) in the Definition 8.3.3, because looking at minimal i with non-zero cohomologies, the isomorphism $P \otimes \omega_X[d] \cong P[s]$ implies

$$\mathcal{H}^i(P) \otimes \omega_X[d] \cong \mathcal{H}^i(P)[s]. \tag{8.3.9}$$

This forces $d = s$.

Lemma 8.3.10. *Let M be a finitely generated non-zero module over a noetherian ring A. Then there is a finite chain of A-submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ (as A-modules), for some $\mathfrak{p}_i \in \mathrm{Supp}(M)$.

Proof. Denote by $\mathrm{Ass}(M)$ the set of all associated primes of M. Recall that, $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ for any finitely generated A-module M. Since $M \neq 0$, we can choose a $\mathfrak{p}_1 \in \mathrm{Ass}(M)$ to get an A-submodule

$$M_1 := \mathrm{image}(A/\mathfrak{p}_1 \hookrightarrow M) \subset M.$$

If $M_1 \neq M$, we do the same for M/M_1 to choose a $\mathfrak{p}_2 \in \mathrm{Ass}(M/M_1)$ and apply the same to obtain a sequence $M_1 \subsetneq M_2 \subseteq M$ with $M_2/M_1 \cong A/\mathfrak{p}_2$. Since $(M/M_1)_{\mathfrak{p}_2} \neq 0$, we see that $\mathfrak{p}_2 \in \mathrm{Supp}(M)$. Since M is finitely generated, the result follows by induction. □

Corollary 8.3.11. *With the above notation, if $\mathrm{Supp}(M) = \{\mathfrak{m}\}$, for some maximal ideal $\mathfrak{m}$ of A, there is a surjective (resp., injective) A-module homomorphism $M \twoheadrightarrow A/\mathfrak{m}$ (resp., $A/\mathfrak{m} \hookrightarrow M$).*

Proof. Since $\mathrm{Ass}(M) = \{\mathfrak{m}\}$, the result follows from the above Lemma 8.3.10. □

Definition 8.3.12. Support of a complex $E^\bullet \in D^b(X)$ is the union of the supports of its cohomologies. In other words, $\mathrm{Supp}(E^\bullet)$ is the closed subset of X defined by

$$\mathrm{Supp}(E^\bullet) := \bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(\mathcal{H}^i(E^\bullet))\,.$$

Lemma 8.3.13. *Let $E^\bullet \in D^b(X)$ with $\mathrm{Supp}(E^\bullet) = Z_1 \cup Z_2$, for some disjoint closed subsets Z_1 and Z_2 in X. Then $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, for some non-zero objects $E_j^\bullet \in D^b(X)$ with $\mathrm{Supp}(E_j^\bullet) \subseteq Z_j$, for all $j = 1, 2$.*

Proof. This is clear for any $E \in \mathfrak{Coh}(X)$, and hence the result follows for $E^\bullet \cong E[n] \in D^b(X)$, for $E \in \mathfrak{Coh}(X)$ and $n \in \mathbb{Z}$. Let

$$i^+_{E^\bullet} := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\} \quad \text{and} \quad i^-_{E^\bullet} := \min\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\};$$

and we drop the subscript $E^\bullet$ when there is no confusion likely to arise. The *length* of an object $E^\bullet \in D^b(X)$ is the difference $i^+ - i^-$. For general case, we use induction on the length of a complex.

Let $E^\bullet \in D^b(X)$ be a complex of length at least 2. Let $m = i^-_{E^\bullet}$, and write $\mathcal{H} := \mathcal{H}^m(E^\bullet)$. The sheaf $\mathcal{H}$ can be decomposed as $\mathcal{H} \cong \mathcal{H}_1 \bigoplus \mathcal{H}_2$, with $\mathrm{Supp}(\mathcal{H}_j) \subset Z_j$, for

$j = 1, 2$. By Proposition 8.3.5, we have a natural morphism $\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet$ inducing identity morphism on the m-th cohomology; complete it to a distinguished triangle

$$\mathcal{H}[-m] \xrightarrow{\varphi} E^\bullet \longrightarrow F^\bullet := C(\varphi) \longrightarrow \mathcal{H}[1-m]\,.$$

Then from long exact sequence of cohomologies we have

$$\mathcal{H}^i(F^\bullet) = \begin{cases} \mathcal{H}^i(E^\bullet), & \text{if } i > m, \text{ and} \\ 0, & \text{if } i \leq m; \end{cases}$$

(c.f. Exercise 8.3.7). Since the length of $F^\bullet$ is less than the length of $E^\bullet$, induction hypothesis applied to $F^\bullet$ gives a decomposition $F^\bullet \cong F_1^\bullet \bigoplus F_2^\bullet$ with $\mathrm{Supp}(\mathcal{H}^i(F_j^\bullet)) \subset Z_j$, for all $j = 1, 2$, and $i \in \mathbb{Z}$. Since $\mathcal{H}^{-q}(F_1^\bullet)$ and $\mathcal{H}_2$ are coherent sheaves of $\mathcal{O}_X$-modules with disjoint supports, we have

$$\mathrm{Hom}_{D^b(X)}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2) = 0, \ \ \forall\, p \in \mathbb{Z},$$

which can be verified locally. Then $\mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[1-m]) = 0$ follows from the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(F_1^\bullet), \mathcal{H}_2[p]) \Longrightarrow E^{p+q} := \mathrm{Hom}(F_1^\bullet, \mathcal{H}_2[p+q]);$$

c.f., Example 7.0.1. Similarly, we have $\mathrm{Hom}(F_2^\bullet, \mathcal{H}_1[1-m]) = 0$. Choose a complex $E_j^\bullet$ to complete a distinguished triangle

$$E_j^\bullet \longrightarrow F_j^\bullet \longrightarrow \mathcal{H}_j[1-m] \longrightarrow E_j^\bullet[1], \ \ \forall\, j = 1, 2,$$

we have a decomposition $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$. Since $\mathrm{Supp}(F_j^\bullet) \subset Z_j$, it follows that $\mathrm{Supp}(E_j^\bullet) \subset Z_j$, for all $j = 1, 2$. □

Lemma 8.3.14. *Let $E^\bullet$ be a simple object in $D^b(X)$ with zero dimensional support. If $\mathrm{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, then $E^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer m.*

Proof. Since $E^\bullet$ is supported in dimension zero, $\mathrm{Supp}(E)$ is a finite subset of closed points in X. If $\mathrm{Supp}(E)$ is not a singleton set, then it has disjoint components. Then in $D^b(X)$, we have an isomorphism $E^\bullet \cong E_1^\bullet \bigoplus E_2^\bullet$, with $E_j^\bullet \not\simeq 0$, $\forall\, i = 1, 2$, which contradicts simplicity of $E^\bullet$. Therefore, $\mathrm{Supp}(E^\bullet)$ is a closed point, say $x \in X$. Let $i^+ := \max\{i : \mathcal{H}^i(E^\bullet) \neq 0\}$ and $i^- := \min\{j : \mathcal{H}^j(E^\bullet) \neq 0\}$. Since both $\mathcal{H}^{i^+}(E^\bullet)$ and $\mathcal{H}^{i^-}(E^\bullet)$ have support $\{x\}$, they are given by finite modules over the noetherian local ring $\mathcal{O}_{X,x}$ supported at $\mathfrak{m}_x$. Then applying Corollary 8.3.11, we get a non-trivial $\mathcal{O}_{X,x}$-module homomorphism $\phi : \mathcal{H}^{i^+}(E^\bullet) \longrightarrow \mathcal{H}^{i^-}(E^\bullet)$ given by the composition

$$\mathcal{H}^{i^+}(E^\bullet) \twoheadrightarrow k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x \hookrightarrow \mathcal{H}^{i^-}(E^\bullet).$$

Now it follows from Proposition 8.3.5 that the following composite morphism is non-trivial.

$$E^\bullet[i^+] \longrightarrow \mathcal{H}^{i^+}(E^\bullet) \xrightarrow{\phi} \mathcal{H}^{i^-}(E^\bullet) \longrightarrow E^\bullet[i^-]\,.$$

Since $\operatorname{Hom}(E^\bullet, E^\bullet[i]) = 0$ for all $i < 0$, we must have $i^- - i^+ \geq 0$. Hence, $i^- = i^+ =: m$ (say). Therefore, $E^\bullet \cong E[m]$, for some $E \in \mathfrak{Coh}(X)$ with $\operatorname{Supp}(E) = \{x\}$. Since $\operatorname{Hom}(E[m], E[m]) \cong \operatorname{Hom}(E, E)$, so E is simple. Then the natural surjective homomorphism $E \to k(x)$ must be isomorphism. Therefore, $E^\bullet \cong k(x)[m]$. □

Proposition 8.3.15 (Bondal–Orlov). *Let X be a smooth projective k-variety with ω_X or $\omega_X^\vee$ ample. Then any point like object in $D^b(X)$ is isomorphic to an object of the form $k(x)[m]$, for some closed point $x \in X$ and some integer m.*

Remark 8.3.16. Above result fails if neither ω_X nor $\omega_X^\vee$ is ample; c.f. Example 8.3.4.

Proof. Note that X is projective because there is an ample line bundle on X. Clearly for any closed point $x \in X$ and any integer m, the shifted skyscraper sheaf $k(x)[m] \in D^b(X)$ is a point like object of codimension $d = \dim(X)$ (c.f., Example 8.3.4).

To see the converse, let $P \in D^b(X)$ be a point like object of codimension n. It follows from $P \otimes \omega_X[d] \cong P[n]$ that $n = d$ (c.f., Remark 8.3.8). Then we have,

$$\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P), \quad \forall\, i \in \mathbb{Z}\,. \tag{8.3.17}$$

Suppose that ω_X is ample. Let

$$m \mapsto P_E(m) := \chi(E \otimes \omega_X^m)$$

be the Hilbert polynomial of $E \in \mathfrak{Coh}(X)$. Since $\deg(P_E(m)) = \dim(\operatorname{Supp}(E))$, taking tensor product with ω_X makes difference only if $\dim(\operatorname{Supp}(E)) > 0$. Therefore, form (8.3.17) we conclude that $\mathcal{H}^i(P)$ is supported in dimension zero. Since P is simple, the result follows from Lemma 8.3.14. The same argument applies for $\omega_X^\vee$ ample. □

8.4. **Invertible objects.** Now we realize line bundles on X as objects of $D^b(X)$.

Definition 8.4.1. Let $\mathcal{D}$ be a triangulated category together with a Serre functor $T_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$. An object $L \in \mathcal{D}$ is said to be *invertible* if for each point like object $P \in \mathcal{D}$, there is an integer n_P (which also depends on L) such that

$$\operatorname{Hom}_{\mathcal{D}}(L, P[i]) = \begin{cases} k(P), & \text{if } i = n_P, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Next, we characterize invertible objects in $D^b(X)$. For this, we need the following well-known result form commutative algebra.

Lemma 8.4.2. *Let M be a finitely generated module over a noetherian local ring $(A, \mathfrak{m})$. If $\operatorname{Ext}^1(M, A/\mathfrak{m}) = 0$, then M is free.*

Proof. Let $k = A/\mathfrak{m}$. Then any k-basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for the A-module M by Nakayama lemma. Thus we get a short exact sequence of

A-modules

$$0 \longrightarrow N \xrightarrow{\iota} A^n \xrightarrow{\phi} M \longrightarrow 0.$$

Note that, $N = \mathrm{Ker}(\phi)$ is finitely generated, and ι induces a trivial homomorphism $\widetilde{\iota} : N/\mathfrak{m}N \longrightarrow k^n$. Since $\mathrm{Ext}^1(M, k) = 0$, the induced homomorphism

$$\mathrm{Hom}(A^n, k) \longrightarrow \mathrm{Hom}(N, k)$$

is surjective. Since $\mathrm{Hom}_A(A^n, k) \cong \mathrm{Hom}_k(k^n, k)$ and $\mathrm{Hom}_A(N, k) \cong \mathrm{Hom}_k(N/\mathfrak{m}N, k)$, the homomorphism $\mathrm{Hom}_k(k^n, k) \longrightarrow \mathrm{Hom}_k(N/\mathfrak{m}N, k)$ induced by $\widetilde{\iota}$ is surjective. Since $\widetilde{\iota} = 0$, this forces $N/\mathfrak{m}N = 0$. Then $N = 0$ by Nakayama lemma, and hence M is a free A-module. $\square$

Proposition 8.4.3 (Bondal–Orlov). *Let X be a smooth projective k-variety. Any invertible object in $D^b(X)$ is of the form $L[m]$, for some line bundle L on X and some integer m. Conversely, if any point like object of $D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and some integer ℓ, then for any line bundle L on X and any integer m, $L[m] \in D^b(X)$ is invertible.*

Remark 8.4.4. Note that, by Proposition 8.3.15 the condition in the converse part of the above Proposition is satisfied when ω_X or $\omega_X^\vee$ is ample.

Proof of Proposition 8.4.3. **Step 1.** Let $E^\bullet \in D^b(X)$ be an invertible object. Let $m = \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then by Proposition 8.3.5, there is a morphism

$$E^\bullet \longrightarrow \mathcal{H}^m(E^\bullet)[-m]$$

in $D^b(X)$ inducing identity morphism at m-th cohomology $\mathcal{H}^m(E^\bullet)$. This gives

$$\mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,, \tag{8.4.5}$$

(c.f., Corollary 8.3.6). Fix a closed point $x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$. Then by Lemma 8.3.10, there is an associated prime ideal $\mathfrak{p} \subseteq \mathfrak{m}_{x_0}$ and a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow \mathcal{O}_{X,x_0}/\mathfrak{p}$, which gives a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x_0)$. Therefore, by (8.4.5), we have

$$0 \neq \mathrm{Hom}_{D^b(X)}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathrm{Hom}_{D^b(X)}(E^\bullet, k(x_0)[-m])\,.$$

This forces $n_{k(x_0)} = -m$ (c.f., Definition 8.4.1).

Step 2. *We show that, $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.*

Since $n_{k(x_0)} = -m$, it follows from the definition of invertible object $E^\bullet \in D^b(X)$ that

$$\mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = \mathrm{Hom}(E^\bullet, k(x_0)[1 + n_{k(x_0)}]) = 0. \tag{8.4.6}$$

Consider the spectral sequence (see Example 7.0.5)

$$
\begin{aligned}
(8.4.7)\qquad E_2^{p,q} &:= \mathrm{Hom}(\mathcal{H}^{-q}(E^\bullet), k(x_0)[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}(E^\bullet), k(x_0)) \\
&\Longrightarrow E^{p+q} := \mathrm{Hom}(E^\bullet, k(x_0)[p+q]).
\end{aligned}
$$

Since $\mathcal{H}^{m+1}(E^\bullet) = 0$, we have

$$E_2^{3,-m-1} = \mathrm{Hom}(\mathcal{H}^{m+1}(E^\bullet), k(x_0)[3]) = 0. \tag{8.4.8}$$

Also

$$E_2^{-1,-m+1} = \mathrm{Hom}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)[-1]) = \mathrm{Ext}^{-1}(\mathcal{H}^{m-1}(E^\bullet), k(x_0)) = 0. \tag{8.4.9}$$

Now using (8.4.8) and (8.4.9), and taking H^0 of the complex

$$\cdots \longrightarrow 0 = E_2^{-1,-m+1} \xrightarrow{d} E_2^{1,-m} \xrightarrow{d} E_2^{3,-m-1} = 0 \longrightarrow \cdots,$$

we see that $E_3^{1,-m} = E_2^{1,-m}$; similarly, $E_r^{1,-m} = E_2^{1,-m}$, for all $r \geq 2$. The following picture of page E_2 could be useful to understand the situation.

$$
\begin{tikzcd}
0 \arrow[drr] & E_2^{0,-m+1} \arrow[drr] & E_2^{1,-m+1} \arrow[drr] & E_2^{2,-m+1} & E_2^{3,-m+1} \\
0 & E_2^{0,-m} \arrow[drr] & E_2^{1,-m} \arrow[drr] & E_2^{2,-m} & E_2^{3,-m} \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

This shows that,

$$E_2^{1,-m} = E_\infty^{1,-m}. \tag{8.4.10}$$

Since $E_\infty^{1,-m}$ is isomorphic to a subquotient of

$$E^{1-m} = \mathrm{Hom}(E^\bullet, k(x_0)[1-m]) = 0 \tag{8.4.11}$$

(see, (8.4.6) and (8.4.7)), using (8.4.10) we conclude that $E_2^{1,-m} = 0$. Therefore,

$$\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0, \ \ \forall\, x_0 \in \mathrm{Supp}(\mathcal{H}^m(E^\bullet))\,. \tag{8.4.12}$$

Step 3. *We show that $\mathcal{H}^m(E^\bullet)$ is a locally free $\mathcal{O}_X$-module.*

For this, we consider the *local-to-global* spectral sequence (see Example 7.0.9)

$$E_2^{p,q} := H^p(X, \mathcal{E}xt^q(\mathcal{H}^m(E^\bullet), k(x_0))) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{H}^m(E^\bullet), k(x_0))\,, \tag{8.4.13}$$

which allow us to pass from the global vanishing $\mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ to the local one $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$.

Since $\mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))$ is a skyscraper sheaf supported at x_0, it is flasque, and hence is Γ-acyclic. Then form (8.4.13), we have

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{8.4.14}$$

Again,

$$E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m(E^\bullet), k(x_0))) = 0\,. \tag{8.4.15}$$

Since at page E_2, we have morphisms

$$0 = E_2^{-2,2} \xrightarrow{d} E_2^{0,1} \xrightarrow{d} E_2^{2,0} = 0,$$

we have $E_3^{0,1} = \mathcal{H}^0(\cdots \to 0 \to E_2^{0,1} \to 0 \to \cdots) = E_2^{0,1}$. Similar computations shows that $E_r^{0,1} = E_2^{0,1}$, for all $r \geq 2$. Hence we conclude that,

$$E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))) = E_\infty^{0,1}\,. \tag{8.4.16}$$

Since $E^1 = \mathrm{Ext}^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$ by Step 2, we have $E_2^{0,1} = E_\infty^{0,1} = 0$. Since $k(x_0)$ is a skyscraper sheaf supported at x_0, we see that $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))$ is supported over $\{x_0\}$, and hence is globally generated. Since

$$H^0(X, \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = E_2^{0,1} = 0,$$

we have $\mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0)) = 0$. Since $\mathcal{H}^m(E^\bullet) \in \mathfrak{Coh}(X)$, we have

$$\mathrm{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m(E^\bullet), k(x_0)) = \mathcal{E}xt^1(\mathcal{H}^m(E^\bullet), k(x_0))_{x_0} = 0. \tag{8.4.17}$$

The by Lemma 8.4.2, $\mathcal{H}^m(E^\bullet)_{x_0}$ is free $\mathcal{O}_{X,x_0}$-module. Since freeness is an open property, there is a non-empty open (dense) subset U of X containing x_0 such that $U \subseteq \mathrm{Supp}(\mathcal{H}^m(E^\bullet))$ and $\mathcal{H}^m(E^\bullet)\big|_U$ is a free $\mathcal{O}_U$-module. Since X is irreducible, $\mathcal{H}^m(E^\bullet)$ is locally free on X.

Step 4. *We show that, $\mathcal{H}^m(E^\bullet)$ is a line bundle on X.*

Since $\mathrm{Supp}(\mathcal{H}^m(E^\bullet)) = X$, there is a surjective homomorphism $\mathcal{H}^m(E^\bullet) \twoheadrightarrow k(x)$, for each $x \in X$. Then following argument of Step 1, we have

$$\mathrm{Hom}(E^\bullet, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \neq 0, \;\; \forall\, x \in X. \tag{8.4.18}$$

Now it follows from Definition 8.4.1 of invertible objects that

$$n_{k(x)} = -m, \;\forall\;\; x \in X. \tag{8.4.19}$$

If r is the rank of $\mathcal{H}^m(E^\bullet)$, we have

$$\begin{aligned} k(x) = \mathrm{Hom}(E^\bullet, k(x)[-m]) &= \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)) \\ &= \mathrm{Hom}(\mathcal{O}_{X,x}^{\oplus r}, k(x)) \cong k(x)^{\oplus r}\,. \end{aligned} \tag{8.4.20}$$

Therefore, $r = 1$, and hence $\mathcal{H}^m(E^\bullet)$ is a line bundle on X.

Step 5. *We show that, $\mathcal{H}^i(E^\bullet) = 0$, for all $i < m$.*

From the spectral sequence in (8.4.7), we have

$$\begin{aligned} E_2^{q,-m} &= \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x)[q]) \\ &= \mathrm{Ext}^q(\mathcal{H}^m(E^\bullet), k(x)) \\ &\cong H^q(X, \mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))) = 0, \quad \forall\, q > 0, \end{aligned} \tag{8.4.21}$$

because $\mathcal{H}om(\mathcal{H}^m(E^\bullet), k(x))$ is a skyscraper sheaf supported on $\{x\}$, and hence is Γ-acyclic.

Suppose that $i < m$. Then it follows from Definition 8.4.1 and (8.4.19) that

$$E^{-i} = \mathrm{Hom}(E^\bullet, k(x)[-i]) = 0, \;\; \forall\, x \in X. \tag{8.4.22}$$

Now to show $\mathcal{H}^i(E^\bullet) = 0$, it is enough to show that

$$E_2^{0,-i} = \mathrm{Hom}(\mathcal{H}^i(E^\bullet), k(x)) = 0, \forall\, x \in X. \tag{8.4.23}$$

Since $E^{-i} = 0$, if we can show that

$$E_2^{0,-i} = E_\infty^{0,-i}, \tag{8.4.24}$$

then from the spectral sequence (8.4.7) we would get $E_2^{0,-i} = 0$. We prove this by induction on i.

If $i = m-1$, then $E_2^{2,-i-1} = E_2^{2,-m} = 0$ by (8.4.21). Since negative indexed Ext groups between two coherent sheaves are zero, we have $E_2^{-2,-(m-2)} = 0$. Then (8.4.24), for the case $i = m-1$, follows from the complex

$$\cdots \to 0 = E_2^{-2,-(m-2)} \xrightarrow{d} E_2^{0,1-m} \xrightarrow{d} E_2^{2,-m} = 0 \to \cdots .$$

Therefore, $\mathcal{H}^{m-1}(E^\bullet) = 0$. Assume inductively that $\mathcal{H}^i(E^\bullet) = 0$, for all $i \in \mathbb{Z}$, with $i_0 < i \leq m-1$. Then putting $m = i_0 + 1$ in (8.4.21) and using $\mathcal{H}^{i_0+1}(E^\bullet) = 0$, we have $E_2^{2,-i_0-1} = 0$. Then (8.4.24) follows from the complex

$$\cdots \to 0 = E_2^{-2,1-i_0} \xrightarrow{d} E_2^{0,-i_0} \xrightarrow{d} E_2^{2,-i_0-1} = 0 \to \cdots .$$

This completes induction. Therefore, $\mathcal{H}^i(E^\bullet) = 0, \;\forall\, i < m$, and hence for all $i \neq m$.

Step 6. Now we prove converse part of the Proposition 8.4.3. Suppose that any point like object $P \in D^b(X)$ is of the form $k(x)[\ell]$, for some closed point $x \in X$ and $\ell \in \mathbb{Z}$. Let L be a line bundle on X, and $m \in \mathbb{Z}$. Then from Definition 8.4.1 we get

$$\begin{aligned} \mathrm{Hom}(L[m], P[i]) &\cong \mathrm{Hom}(L, k(x)[\ell + i - m]) \\ &= \mathrm{Ext}^{\ell+i-m}(\mathcal{O}_X, L^\vee \otimes k(x)) \\ &\cong H^{\ell+i-m}(X, L^\vee \otimes k(x))\,, \end{aligned} \tag{8.4.25}$$

which vanishes except for $i = m - \ell$. Then we set $n_P := m - \ell$. This completes the proof. □

Remark 8.4.26. Let $\mathcal{D}$ be a (tensor) triangulated category admitting a Serre functor S. If we naively define *Picard group* of $\mathcal{D}$ to be the set $\mathrm{Pic}(\mathcal{D})$ of all invertible objects in $\mathcal{D}$, then for a smooth projective k-variety X with ω_X or $\omega_X^\vee$ ample, we have $\mathrm{Pic}(D^b(X)) = \mathrm{Pic}(X) \times \mathbb{Z}$.

8.5. **Spanning class of $D^b(X)$.**

Definition 8.5.1. A collection Ω of objects in a triangulated category $\mathcal{D}$ is called a *spanning class of* $\mathcal{D}$ (or *spans* $\mathcal{D}$) if for all $B \in \mathcal{D}$ the following conditions hold.

(i) If $\mathrm{Hom}(A, B[i]) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
(ii) If $\mathrm{Hom}(B[i], A) = 0,\ \forall\, A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

Remark 8.5.2. If a triangulated category $\mathcal{D}$ admits a Serre functor, then the conditions (i) and (ii) in the above Definition 8.5.1 are equivalent.

Proposition 8.5.3. *Let X be a smooth projective k-variety. Then the objects of the form $k(x)$, with $x \in X$ a closed point, spans $D^b(X)$.*

Proof. It is enough to show that, for any non-zero object $E^\bullet \in D^b(X)$ there exists closed points $x_1, x_2 \in X$ and integers i_1, i_2 such that

$$\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0 \quad \text{and} \quad \mathrm{Hom}(k(x_2), E^\bullet[i_2]) \neq 0\,.$$

Since $\mathrm{Hom}(k(x_2), E^\bullet[i_2]) \cong \mathrm{Hom}(E^\bullet, k(x_2)[\dim(X){-}i_2])^*$ by Serre duality, it is enough to show that $\mathrm{Hom}(E^\bullet, k(x_1)[i_1]) \neq 0$, for some closed point $x \in X$ and some $i \in \mathbb{Z}$. Let $m := \max\{i \in \mathbb{Z} : \mathcal{H}^i(E^\bullet) \neq 0\}$. Then $\mathrm{Hom}(E^\bullet, k(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m(E^\bullet), k(x))$ by Corollary 8.3.6. Now choosing a closed point x in the support of $\mathcal{H}^m(E^\bullet)$, we see that $\mathrm{Hom}(E^\bullet, k(x)[-m]) \neq 0$. This completes the proof. □

Remark 8.5.4. Spanning class in $D^b(X)$ is not unique. For a smooth projective k-variety X, for a choice of an ample line bundle L on X, we shall see later that, $\{L^{\otimes i} : i \in \mathbb{Z}\}$ forms a spanning class in $D^b(X)$.

8.6. **Proof of the reconstruction theorem.** Now we are in a position to prove the reconstruction theorem of Bondal and Orlov in the light of the following well-known results.

Proposition 8.6.1. [Sta20, Tag01PR] *Let X be a quasi-compact scheme. Let L be an invertible sheaf of $\mathcal{O}_X$-modules on X. Consider the graded algebra $S := \bigoplus_{i\geq 0} H^0(X, L^i)$, and its ideal $S_+ = \bigoplus_{i>0} H^0(X, L^i)$. For each homogeneous element $s \in H^0(X, L^i)$, for $i > 0$, let $X_s := \{x \in X : s_x \notin \mathfrak{m}_x L^i_x\}$. Then the following are equivalent.*

(i) L is ample.
(ii) The collection of open sets X_s, with $s \in S_+$ homogeneous, covers X, and the natural morphism $X \longrightarrow \mathbf{Proj}(S)$ is an open immersion.
(iii) The collection of open sets X_s, with $s \in S_+$ homogeneous, forms a basis for the Zariski topology on X.

Proposition 8.6.2. *Let X be a smooth projective k-variety. Let L be a line bundle on X. If L or $L^\vee$ is ample, then the natural morphism of k-schemes*

$$X \longrightarrow \mathbf{Proj}\left(\bigoplus_n H^0(X, L^n)\right)$$

is an isomorphism.

Theorem 8.1.1 (Bondal–Orlov)**.** *Let X be a smooth projective variety over k with canonical line bundle ω_X. Assume that ω_X (resp., $\omega_X^\vee$) is ample. Let Y be any smooth projective variety over k. If there is an exact equivalence $F : D^b(X) \xrightarrow{\sim} D^b(Y)$, then $X \cong Y$ as k-varieties. In particular, ω_Y (resp., $\omega_Y^\vee$) is ample.*

Proof. **Step 1.** If $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or $\omega_Y^\vee$ is ample, the theorem follows.

Indeed, assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Since F is an exact equivalence of categories, $F \circ S_X \cong S_Y \circ F$ and $\dim(X) = \dim(Y) = n$ (say), (see Proposition 8.2.1). Then we have

$$F(\omega_X^k) = F(S_X^k(\mathcal{O}_X))[-kn] = S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^k, \ \forall\, k\,. \tag{8.6.3}$$

Since F is fully faithful, we have

$$H^0(X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_X, \omega_X^k) = \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^k) = H^0(Y, \omega_Y^k), \ \forall\, k\,. \tag{8.6.4}$$

The product structure on the graded k-algebra $\bigoplus_k H^0(X, \omega_X^k)$ can be expressed in terms of following composition: for $s_i \in H^0(X, \omega_X^{k_i})$, $i = 1, 2$, we have

$$s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1\,.$$

Note that, $s_1 \cdot s_2 = s_2 \cdot s_1$ follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "s_1"] \arrow[d, "s_2"'] & \omega_X^{k_1} \arrow[d, "S_X^{k_1}(s_2)[-k_1 n]"] \\ \omega_X^{k_2} \arrow[r, "S_X^{k_2}(s_1)[-k_2 n]"] & \omega_X^{k_1+k_2} \end{tikzcd} \tag{8.6.5}$$

Similarly, we have product structure on $\bigoplus_k H^0(Y, \omega_Y^k)$. Therefore, F naturally induces an isomorphism of graded k-algebras

$$\widetilde{F} : \bigoplus_k H^0(X, \omega_X^k) \longrightarrow \bigoplus_k H^0(Y, \omega_Y^k)\,, \tag{8.6.6}$$

which induces isomorphism of k-schemes

$$X \xrightarrow{\cong} \mathbf{Proj}\big(\bigoplus_k H^0(X, \omega_X^k)\big) \xrightarrow{\cong} \mathbf{Proj}\big(\bigoplus_k H^0(Y, \omega_Y^k)\big) \xrightarrow{\cong} Y, \tag{8.6.7}$$

whenever ω_Y or its dual $\omega_Y^\vee$ is ample (c.f., Proposition 8.6.2). Therefore, it is enough to show that $F(\mathcal{O}_X) = \mathcal{O}_Y$, and ω_Y or $\omega_Y^\vee$ is ample whenever ω_X or $\omega_X^\vee$ is ample.

Step 2. We can assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$.

Indeed, it follows from Definition 8.3.3 and Definition 8.4.1 that an exact equivalence $F : D^b(X) \to D^b(X)$ induce bijections

$$\begin{tikzcd} \{\text{point like objects of } D^b(X)\} \arrow[r, "F", "\simeq"'] \arrow[d, equal] & \{\text{point like objects of } D^b(Y)\} \\ \{k(x)[m] : x \in X_{closed} \text{ and } m \in \mathbb{Z}\} & \{k(y)[m] : y \in Y_{closed} \text{ and } m \in \mathbb{Z}\} \arrow[u, hook, "(*)"'] \end{tikzcd} \tag{8.6.8}$$

and

$$\begin{tikzcd} \{\text{invertible objects of } D^b(X)\} \arrow[r, "F", "\simeq"'] \arrow[d, equal] & \{\text{invertible objects of } D^b(Y)\} \\ \{L[m] : L \in \mathrm{Pic}(X) \text{ and } m \in \mathbb{Z}\} & \{M[m] : M \in \mathrm{Pic}(Y) \text{ and } m \in \mathbb{Z}\}, \arrow[u, hookleftarrow, "(**)"'] \end{tikzcd} \tag{8.6.9}$$

where X_{closed} (resp., Y_{closed}) is the set of all closed points of X (resp., Y), and the vertical inclusions and equalities are given by Proposition 8.3.15 and Proposition 8.4.3. Therefore, $F(\mathcal{O}_X) = M[m]$, for some $M \in \mathrm{Pic}(Y)$ and some $m \in \mathbb{Z}$.

If $F(\mathcal{O}_X) \neq \mathcal{O}_Y$, replacing F with the following composite functor

$$D^b(X) \xrightarrow{F} D^b(Y) \xrightarrow{(M^\vee \otimes -)[-m]} D^b(Y), \tag{8.6.10}$$

which is an exact equivalence sending $\mathcal{O}_X$ to $\mathcal{O}_Y$, we may assume that $F(\mathcal{O}_X) = \mathcal{O}_Y$. Therefore, it remains to show is that ω_Y or its dual is ample.

Step 3. We establish bijections $X_{closed} \xleftrightarrow{F} Y_{closed}$ and $\mathrm{Pic}(X) \xleftrightarrow{F} \mathrm{Pic}(Y)$.

Using the equivalence F, we first show that the vertical inclusion $(*)$ in the diagram (8.6.8) is a bijection. This immediately imply that the vertical inclusion $(**)$ in the diagram (8.6.9) is bijective by Proposition 8.4.3. Then Step 3 will follow.

By horizontal bijection in the diagram (8.6.8), for any closed point $y \in Y$ there is a closed point $x_y \in X$ and $m_y \in \mathbb{Z}$ such that $F(k(x_y)[m_y]) \cong k(y)$. Suppose on the contrary that there is a point like object $P \in D^b(Y)$, which is not of the form $k(y)[m]$, for any closed point $y \in Y$ and integer m. Because of bijection in (8.6.8), there is a unique closed point $x_P \in X$ and integer m_P such that $F(k(x_P)[m_P]) \cong P$. Then $x_P \neq x_y$, for all closed point $y \in Y$. Hence, for any closed point $y \in Y$ and any integer m, we have

$$\begin{aligned} \mathrm{Hom}(P, k(y)[m]) &= \mathrm{Hom}(F(k(x_P)[m_P]), k(y)[m]) \\ &= \mathrm{Hom}(k(x_P)[m_P], k(x_y)[m_y + m]) \\ &= \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P]) = 0, \end{aligned} \tag{8.6.11}$$

because $k(x_P)$ and $k(x_y)$ being skyscraper sheaves supported at different points, $\mathrm{Ext}^i(k(x_P), k(x_y)) = 0$, for all i. Since the objects $k(y)$, with $y \in Y$ a closed point, form a spanning class of $D^b(X)$ (c.f. Definition 8.5.1), $P \cong 0$ by Proposition 8.5.3, which contradicts our assumption that P is a point like object in $D^b(Y)$. Therefore, point like objects of $D^b(Y)$ are exactly of the form $k(y)[m]$, for $y \in Y$ a closed point and $m \in \mathbb{Z}$.

Note that, for any closed point $x \in X$, there is a closed point $y_x \in Y$ such that $F(k(x)) \cong k(y_x)[m_x]$, for some $m_x \in \mathbb{Z}$. Since F is fully faithful and $F(\mathcal{O}_X) = \mathcal{O}_Y$, we have $\mathrm{Hom}(\mathcal{O}_X, k(x)) = \mathrm{Hom}(\mathcal{O}_Y, k(y_x)[m_x]) = \mathrm{Ext}^{m_x}(\mathcal{O}_Y, k(y_x)) \neq 0$. This forces $m_x = 0$, and hence $F(k(x)) \cong k(y_x)$ (no shift!). This immediately imply that, for any $L \in \mathrm{Pic}(X)$, $F(L) \cong M$, for some $M \in \mathrm{Pic}(Y)$. Indeed, from bijections in the diagram (8.6.9), we find unique $M \in \mathrm{Pic}(Y)$ and $m_L \in \mathbb{Z}$ such that $F(L) \cong M[m_L]$. Take closed points $x \in X$ and $y_x \in Y$ such that $F(k(x)) \cong k(y_x)$. Then

$$\begin{aligned} \mathrm{Ext}^{-m_L}(M, k(y_x)) &= \mathrm{Hom}(M, k(y_x)[-m_L]) = \mathrm{Hom}(M[m_L], k(y_x)) \\ &= \mathrm{Hom}(F(L), F(k(x))) = \mathrm{Hom}(L, k(x)) \neq 0. \end{aligned}$$

This forces $m_L = 0$.

Step 4. Recovering Zariski topology from derived category to conclude ampleness.

Let Z be a quasi-compact k-scheme. Denote by Z_0 the subset of all closed points of Z. Take line bundles L_1 and L_2 on Z, and take $\alpha \in \mathrm{Hom}(L_1, L_2) = H^0(X, L_1^\vee \otimes L_2)$. For each closed point $z \in Z$, let

$$\alpha_z^* : \mathrm{Hom}(L_2, k(z)) \longrightarrow \mathrm{Hom}(L_1, k(z)) \tag{8.6.12}$$

be the homomorphism induced by α. Then $U_\alpha := \{z \in Z : \alpha_z^* \neq 0\}$ is a Zariski open subset of Z, and hence $U_\alpha \cap Z$ is open in Z_0.

Fix a line bundle $L_0 \in \mathrm{Pic}(X)$. Then it follows from Proposition 8.6.1 that the collection of all such U_α, where $\alpha \in H^0(X, L_0^n)$ and $n \in \mathbb{Z}$, forms a basis for the Zariski topology on Z if and only if either L_0 or $L_0^\vee$ is ample.

By Step 3, the exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$ sends closed points of X to closed points of Y bijectively, and sends line bundles on X to line bundles on Y bijectively. In particular, $F(\omega_X^i) \cong \omega_Y^i$, for all $i \in \mathbb{Z}$. Then the natural isomorphisms $H^0(X, \omega_X^i) \cong H^0(Y, \omega_Y^i)$, $\forall\ i \in \mathbb{Z}$, give rise to a bijection between the collection of open subsets

$$\begin{aligned} \mathcal{B}_X &:= \{U_\alpha : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}, \text{ and} \\ \mathcal{B}_Y &:= \{V_\alpha : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}. \end{aligned}$$

Since ω_X (resp., $\omega_X^\vee$) is ample, $\mathcal{B}_X$ is a basis for the Zariski topology on X, and hence $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in H^0(X, \omega_X^i) \text{ and } i > 0 \text{ (resp., } i < 0)\}$ is a basis for the Zariski topology on X_0. Therefore, $\mathcal{B}_{Y_0} := \{V_\alpha \cap Y_0 : \alpha \in H^0(Y, \omega_Y^i) \text{ and } i > 0 \,(\text{resp., } i < 0)\}$ is a basis for the Zariski topology on Y_0, and hence $\mathcal{B}_Y$ is a basis for the Zariski topology on Y (see Lemma 8.6.13 below). Therefore, ω_Y (resp., $\omega_Y^\vee$) is ample. This completes the proof. □

I thank Arideep Saha for useful discussion leading to the following Lemma.

Lemma 8.6.13. *Let X be a scheme locally of finite type over* Spec($\Bbbk$), *where $\Bbbk$ is a field or $\mathbb{Z}$. Let X_0 be a subset of X containing all closed points of X. Let $\mathcal{B}_X := \{U_\alpha : \alpha \in \Lambda\}$ be a collection of open subsets of X such that $\mathcal{B}_{X_0} := \{U_\alpha \cap X_0 : \alpha \in \Lambda\}$ is a basis for the subspace Zariski topology on X_0. Then $\mathcal{B}$ is a basis for the Zariski topology on X.*

Proof. **Step 1.** First we show that, *if an open set $U \subset X$ contains a closed point x_0, then for any point $x \in X$ which contains x_0 in its closure (i.e., $x_0 \in \overline{\{x\}}$), we have $x \in U$*. Since $\mathcal{B}_{X_0}$ is a basis, there is $\alpha \in \Lambda$ such that $x_0 \in U_\alpha \cap X_0 \subseteq U \cap X_0$. If $x \notin U_\alpha$, then x belongs to the closed set $X \setminus U_\alpha$, and hence $\overline{\{x\}} \subseteq X \setminus U_\alpha$, which contradicts the assumption that $x_0 \in \overline{\{x\}}$. Therefore, $x \in U_\alpha$. Since closure of any point in X contains a closed point, it follows that $\mathcal{B}_X$ is an open cover for X.

It remains to show that for $x \in U_\alpha \cap U_\beta$, there is $\gamma \in \Lambda$ such that $x \in U_\gamma \subseteq U_\alpha \cap U_\beta$.

Step 2. Assume that, *for any open subset U of X with $x \in U$, there is a closed point $x_0 \in \overline{\{x\}} \cap U$*. For then, taking $U = U_\alpha \cap U_\beta$, we can find a $\gamma \in \Lambda$ such that

$$x_0 \in U_\gamma \cap X_0 \subseteq U_\alpha \cap U_\beta \cap X_0.$$

Then we will have $U_\gamma \subseteq U_\alpha \cap U_\beta$. Indeed, for each $z \in U_\gamma$, by *above assumption* there is a closed point $z_0 \in \overline{\{z\}} \cap U_\alpha \cap U_\beta$. Then by Step 1, we have $z \in U_\alpha \cap U_\beta$.

Step 3. We now prove the *assumption* of Step 2. Since the statement is local, we may assume that $X = \text{Spec}(A)$, for some finitely generated $\Bbbk$-algebra A. For each $f \in A$, let $D_f := \{\mathfrak{q} \in \text{Spec}(A) : f \notin \mathfrak{q}\}$. Since $\{D_f : f \in A\}$ forms a basis for the Zariski topology on Spec(A), any point $\mathfrak{p} \in \text{Spec}(A)$ is contained in D_f, for some $f \in A \setminus \{0\}$. We claim that, there is a closed point (maximal ideal) $\mathfrak{m} \in D_f$ with $\mathfrak{p} \subset \mathfrak{m}$. If not, then all closed points (maximal ideal) $\mathfrak{m} \in \text{Max}(A/\mathfrak{p}) \subset \text{Spec}(A/\mathfrak{p})$ lies outside D_f. Since $A/\mathfrak{p}$ is a finitely generated $\Bbbk$-algebra, we have

$$\text{Jac}(A/\mathfrak{p}) = \bigcap_{\mathfrak{m} \in \text{Max}(A/\mathfrak{p})} \mathfrak{m} = \bigcap_{\mathfrak{q} \in \text{Spec}(A/\mathfrak{p})} \mathfrak{q} = \text{Nil}(A/\mathfrak{p}),$$

which is zero because $A/\mathfrak{p}$ is an integral domain. This contradicts the fact that $f \neq 0$ in $A/\mathfrak{p}$. This completes the proof. □

Although we don't need full strength of the following Lemma 8.6.14 here, let me mention it here since it can be useful in may purpose. I thank Saurav Bhaumik for explaining it to me.

Lemma 8.6.14. *Any polarized reduced projective scheme locally of finite type over a field can be reconstructed from its set of closed points.*

Proof. Let X be a reduced projective k-scheme, which is locally of finite type over Spec(k). If $\mathscr{I}_X \subset \mathcal{O}_{\mathbb{P}^n_k}$ is the ideal sheaf of a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, for some integer $n \geq 1$, then $X \cong \mathrm{Proj}(S/I)$, where $I := \bigoplus_{i \geq 0} H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ is the homogeneous ideal of the graded k-algebra $S := k[x_0, \ldots, x_n]$. Therefore, it suffices to show that, I coincides with the ideal of homogeneous polynomials in S vanishing at each closed point of X. It follows from the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n_k, \mathscr{I}_X(i)) \longrightarrow H^0(\mathbb{P}^n_k, \mathcal{O}_X(i)) \longrightarrow H^0(X, \mathcal{O}_X(i))$$

that $H^0(\mathbb{P}^n_k, \mathscr{I}_X(i))$ can be identified with the set of all homogeneous polynomials of degree i in S that vanishes at each point of X. Therefore, it suffices to show that, if X is a finite type reduced k-subscheme of a k-scheme $\widetilde{X}$, a section $s \in H^0(\widetilde{X}, L)$ of a line bundle L on $\widetilde{X}$ vanishes at every closed points of X if and only if $s|_X = 0$. This can be checked locally. Take an affine open subset $U = \mathrm{Spec}(A)$ of X such that $L\big|_U$ is trivial. Then $s\big|_U$ is given by an element $f \in A$. Since s vanishes at every closed points of X, $f \in \mathrm{Jac}(A)$. Since X is locally of finite type over Spec(k), $\mathrm{Jac}(A) = \mathrm{Nil}(A)$, which is zero because X is reduced. Therefore, $f = 0$, and hence $s\big|_X = 0$. Hence the result follows. $\square$

Remark 8.6.15. There is a more geometric proof of ampleness of ω_Y or its dual in Theorem 8.1.1 when k is algebraically closed. The idea is to use the fact that line bundle is very ample if and only if it separates points and tangent vectors.

Alternative proof of ampleness of ω_Y or its dual, for $k = \overline{k}$. In this subsection, we assume that k is algebraically closed. Let X be a projective k-scheme.

Definition 8.6.16. An invertible sheaf of $\mathcal{O}_X$-modules L on X is said to be *very ample* if there is a closed embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$, for some $n \geq 1$, such that $L \cong \iota^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$.

It should be noted that, an invertible sheaf L on X is ample if and only if L^m is very ample, for some integer $m \gg 0$; [Har77].

Definition 8.6.17. Let L be an invertible sheaf of $\mathcal{O}_X$-modules on X. We say that,

(i) *L separates points* if for any two closed points $p, q \in X$, there is a section $s \in H^0(X, L)$ such that $s_p \in \mathfrak{m}_p L_p$ and $s_q \notin \mathfrak{m}_q L_q$.
(ii) *L separates tangent vectors* if for any closed point $p \in X$ and any tangent vector $v \in T_p X = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$, there is a non-zero section $s \in H^0(X, L)$ such that $s_p \in \mathfrak{m}_p L_p$ and $v \notin T_p V$, where V is the divisor of zero locus of s.

Note that, an invertible sheaf L on X separate tangent vectors if and only if for each closed point $x \in X$, the set $\{s \in H^0(X, L) : s_x \in \mathfrak{m}_x L_x\}$ spans the k-vector space $L_x \otimes (\mathfrak{m}_x/\mathfrak{m}_x^2)$.

Theorem 8.6.18. [Har77, Proposition II.7.3] *Let X be a projective k-scheme. An invertible sheaf of $\mathcal{O}_X$-modules on X is very ample if and only if it separate points and tangent vectors.*

Continuing with above notations, it follows from the Definition 8.6.17 that $L \in \mathrm{Pic}(X)$ separates points if and only if for any two closed points $x_1, x_2 \in X$ with $x_1 \neq x_2$, the restriction homomorphism (to the fibers)

$$r_{x_1,x_2} : L \longrightarrow (L \otimes k(x_1)) \oplus (L \otimes k(x_2)) \cong k(x_1) \oplus k(x_2) \tag{8.6.19}$$

induces a surjective homomorphism

$$H^0(r_{x_1,x_2}) : H^0(X, L) \longrightarrow H^0(X, k(x_1) \oplus k(x_2)). \tag{8.6.20}$$

Let Y be a smooth projective k-variety, and $F : D^b(X) \longrightarrow D^b(Y)$ be an exact equivalence of k-linear graded categories. Then we have the following commutative diagram

$$\begin{tikzcd} H^0(X, \omega_X^i) \arrow[r, "H^0(r_{x_1,r_2})"] \arrow[d, "F"', "\cong"] & H^0(X, k(x_1) \oplus k(x_2)) \arrow[d, "\cong"', "F"] \\ H^0(Y, \omega_Y^i) \arrow[r, "H^0(r_{y_1,y_2})"] & H^0(Y, k(y_1) \oplus k(y_2)) \end{tikzcd} \tag{8.6.21}$$

where $y_j \in Y$ is the closed point such that $F(k(x_j)) = k(y_j)$, for all $j = 1, 2$. Therefore, ω_X^i separates points if and only if ω_Y^i separates points.

To see ω_Y^i separates tangent vectors if and only if ω_X^i do the same, first we need the following observation.

Lemma 8.6.22. *Let X be a scheme over any field k. To give a point $x \in X$ with residue field $k(x) = k$ and a tangent vector $v \in T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is equivalent to give a subscheme $Z_x \subset X$, supported at x, of length 2 (i.e., $\dim_k H^0(Z_x, \mathcal{O}_{Z_x}) = 2$).*

Let $Z_y \subset Y$ be a subscheme of length 2 supported at a closed point $y \in Y$. Then we have an exact sequence

$$0 \longrightarrow k(y) \longrightarrow \mathcal{O}_{Z_y} \longrightarrow k(y) \longrightarrow 0\,. \tag{8.6.23}$$

Therefore, Z_y is given by an non-trivial extension class

$$\Phi_{Z_y} \in \mathrm{Ext}^1(k(y), k(y)). \tag{8.6.24}$$

Since F is fully faithful, $\Phi_{Z_y} \in \mathrm{Ext}^1(k(y), k(y))$ corresponds to a non-trivial extension class

$$F(\Phi_{Z_y}) \in \mathrm{Ext}^1(k(x_y), k(x_y)), \tag{8.6.25}$$

where $x_y \in X$ is the closed point satisfying $F(k(x_y)) = k(y)$. Then $F(\Phi_{Z_y})$ defines a subscheme $Z_x \subset X$ of length 2 supported at $x \in X$. Therefore, $F(\mathcal{O}_{Z_x}) = \mathcal{O}_{Z_y}$. Moreover, F gives an isomorphism

$$F : \mathrm{Hom}(\omega_X^i, \mathcal{O}_{Z_x}) \xrightarrow{\simeq} \mathrm{Hom}(\omega_Y^i, \mathcal{O}_{Z_y}).$$

It follows from the Lemma 8.6.22 that a line bundle L on X separate tangent vectors if and only if the homomorphism (induced by the restriction morphism)

$$H^0(X, L) \longrightarrow H^0(X, \mathcal{O}_{Z_x}) \tag{8.6.26}$$

is surjective. Now it follows from the commutative diagram

$$\begin{tikzcd} H^0(X, \omega_X^i) \arrow[r] \arrow[d, "F"', "\simeq"] & H^0(X, \mathcal{O}_{Z_x}) \arrow[d, "\simeq"', "F"] \\ H^0(Y, \omega_Y^i) \arrow[r] & H^0(Y, \mathcal{O}_{Z_y}) \end{tikzcd} \tag{8.6.27}$$

that ω_X^i separate tangent vectors if and only if ω_Y^i separate tangent vectors. Hence, ω_X (resp., $\omega_X^\vee$) is ample if and only if ω_Y^i (resp., $\omega_Y^\vee$) is ample.

8.7. **Auto equivalence of derived category.**

9. Fourier-Mukai transforms

9.1. **Integral functor.** Let X and Y be smooth projective schemes defined over a field k. Consider the two projections

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y\,. \tag{9.1.1}$$

Definition 9.1.2. An *integral functor* with *kernel* $P \in D^b(X \times Y)$ is a functor

$$\Phi_P^{X \to Y} : D^b(X) \longrightarrow D^b(Y) \tag{9.1.3}$$

defined by

$$\Phi_P^{X \to Y}(E) := p_{Y*}(p_X^* E \otimes P), \;\; \forall\, E \in D^b(X);$$

where p_{Y*}, p_X^* and $\otimes$ are derived functors.

When there is no confusion regarding the direction of the functor likely to arise, we just drop the superscript $X \to Y$ from $\Phi_P^{X \to Y}$, and simply denote it by Φ_P. An integral functor Φ_P, which is an equivalence of categories, is called a *Fourier-Mukai functor* with kernel P. We say that X and Y are *Fourier-Mukai partner* if there is a Fourier-Mukai transform $\Phi_P : D^b(X) \to D^b(Y)$.

Remark 9.1.4. Since the derived functors p_X^*, p_{Y*} and $\otimes$ are exact, Φ_P is exact.

Example 9.1.5 (Examples of integral functors)**.** Let X be a proper k-scheme.

(i) The identity functor $\mathrm{Id} : D^b(X) \longrightarrow D^b(X)$ is naturally isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_{\Delta_X}$, where $\Delta_X \subset X \times X$ is the diagonal.

Let $p_i : X \times X \to X$ be the projection onto the i-th factor, for $i = 1, 2$. Let $\iota : X \xrightarrow{\simeq} \Delta \subset X \times X$ be the embedding of the diagonal into $X \times X$. Let $\mathcal{O}_\Delta$ be the structure sheaf of the diagonal Δ. Then for any $E^\bullet \in D^b(X)$, we have

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(E^\bullet) &= p_{1*}(p_2^*E^\bullet \otimes \mathcal{O}_\Delta) = p_{1*}(p_2^*E^\bullet \otimes \iota_*\mathcal{O}_X)\\ &\cong p_{1*}(\iota_*(\iota^*(p_2^*E^\bullet) \otimes \mathcal{O}_X)), \quad \text{by projection formula;}\\ &\cong (p_1 \circ \iota)_*(p_2 \circ \iota)^*(E^\bullet) \cong E^\bullet, \ \text{ since } \ p_1 \circ \iota = \mathrm{Id}_X = p_2 \circ \iota.\end{aligned}$$

(ii) (Pullback and direct image functors) Let $f : X \to Y$ be a morphism of smooth projective k-schemes. Then $Rf_* : D^b(X) \longrightarrow D^b(Y)$ is isomorphic to the integral functor $\Phi_{\mathcal{O}_{\Gamma_f}}^{X \to Y}$, where $\Gamma_f \subset X \times Y$ is the graph of f. Indeed, take any $E^\bullet \in D^b(X) := D^b(\mathfrak{Coh}(X))$. Then

$$\Phi_{\mathcal{O}_{\Gamma_f}}^{X \to Y}(E^\bullet) = Rp_{2*}((Rp_1^*E^\bullet) \overset{L}{\otimes} \mathcal{O}_{\Gamma_f}) \cong Rf_*E^\bullet.$$

On the other hand, taking integral functor on the reverse direction with the same kernel gives a natural isomorphism of functors $\Phi_{\mathcal{O}_{\Gamma_f}}^{Y \to X} \cong Lf^*$.

(iii) The cohomology functor $H^*(X, -) : D^b(X) \longrightarrow D^b(\mathcal{V}ect(k))$ is isomorphic to the integral functor $\Phi_{\mathcal{O}_{\Gamma_f}}^{X \to \mathrm{Spec}(k)}$, where $\Gamma_f \subset X \times \mathrm{Spec}(k)$ is the graph of the structure morphism $f : X \to \mathrm{Spec}(k)$ of X.

(iv) Let $\mathcal{L}$ be an invertible sheaf on a proper k-scheme X. Let $\iota : X \overset{\simeq}{\hookrightarrow} \Delta_X \subset X \times X$ be the diagonal immersion. Then the integral functor $\Phi_{\iota_*\mathcal{L}}$ is isomorphic to the functor $\mathcal{L} \overset{L}{\otimes} -$. Indeed, for any $E^\bullet \in D^b(X)$ we have

$$\Phi_{\iota_*\mathcal{L}}(E^\bullet) = p_{2*}((p_1^*E^\bullet) \otimes \iota_*(\mathcal{L})) \cong E^\bullet \overset{L}{\otimes} \mathcal{L}.$$

(v) The shift functor $T : D^b(X) \to D^b(X)$ given by $T(E^\bullet) = E^\bullet[1]$, for all $E^\bullet \in D^b(X)$, is isomorphic to the integral functor $\Phi_{\mathcal{O}_\Delta[1]}$.

(vi) Let X be a smooth projective k-variety of dimension n. Recall that the Serre functor $S_X : D^b(X) \to D^b(X)$ is defined by $E^\bullet \mapsto (E^\bullet \otimes \omega_X)[n]$. Since the integral functor $\Phi_{\iota_*\omega_X^i}$ is isomorphic to $S_X^i[-ni]$ for all $i \in \mathbb{Z}$, as a corollary to Proposition 9.1.21 below, we conclude that S_X is an integral functor with kernel $(\iota_*\omega_X) * (\mathcal{O}_\Delta[n])$.

(vii) (Kodaira-Spencer morphism) Let X and T be smooth projective k-varieties. Let $\mathcal{P}$ be a coherent sheaf on $X \times T$ flat over T, and consider the integral functor $\Phi_{\mathcal{P}}^{T \to X} : D^b(T) \longrightarrow D^b(X)$ with kernel $\mathcal{P}$. Fix a k-rational point $t \in T$ (i.e., a closed point $t \in T$ with $k(t) \cong k$), we have

$$\Phi_{\mathcal{P}}^{T\to X}(k(t)) = Rp_{X*}\big((Lp_T^* k(t)) \overset{L}{\otimes} \mathcal{P}\big) \cong \mathcal{P}_t,$$

where $\mathcal{P}_t := \mathcal{P}\big|_{X\times\{t\}}$ considered as a coherent sheaf on X.

Note that, a tangent vector $v \in \mathrm{T}_t T$ at t is uniquely defined by a subscheme $Z_v \subset T$ of length 2 (i.e., $\dim_k H^0(Z_v, \mathcal{O}_{Z_v}) = 2$) concentrated at t. Then we have a short exact sequence

$$0 \longrightarrow k(t) \longrightarrow \mathcal{O}_{Z_v} \longrightarrow k(t) \longrightarrow 0. \tag{9.1.6}$$

Pulling back this exact sequence by p_T and tensoring with $\mathcal{P}$ (note that, $\mathcal{P}$ is flat over T by assumption), we get a short exact sequence

$$0 \longrightarrow \mathcal{P}_t \longrightarrow \mathcal{P}\big|_{X\times Z_v} \longrightarrow \mathcal{P}_t \longrightarrow 0. \tag{9.1.7}$$

Considering this as a sequence on X, we get an extension class in $\mathrm{Ext}^1_X(\mathcal{P}_t, \mathcal{P}_t)$. Now one can check that, this gives a k-linear map

$$\mathrm{KS}_t : \mathrm{T}_t T \longrightarrow \mathrm{Ext}^1_X(\mathcal{P}_t, \mathcal{P}_t), \tag{9.1.8}$$

known as the *Kodaira-Spencer map*. It follows from the above construction that the following diagram is commutative.

$$\begin{tikzcd} \mathrm{T}_t T = \mathrm{Ext}^1_T(k(t), k(t)) \arrow[r, "\mathrm{KS}_t"] \arrow[d, "\simeq"'] & \mathrm{Ext}^1_X(\mathcal{P}_t, \mathcal{P}_t) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{D^b(T)}(k(t), k(t)[1]) \arrow[r, "\Phi_{\mathcal{P}}^{X\to T}"] & \mathrm{Hom}_{D^b(X)}(\mathcal{P}_t, \mathcal{P}_t[1]) \end{tikzcd} \tag{9.1.9}$$

In other words, the Kodaira-Spencer morphism KS_t is compatible with the integral functor $\Phi_{\mathcal{P}}^{T\to X}$.

Remark 9.1.10. Note that, an integral functor need not be compatible with Serre functors (c.f., Section §4). For example, let X be a smooth projective k-scheme of dimension $n \geq 1$. Let $f : X \longrightarrow \mathrm{Spec}(k)$ be the structure morphism. Then the right derived functor $Rf_* : D^b(X) \longrightarrow D^b(\mathcal{V}ect(k))$ sends a coherent sheaf E on X to the graded k-vector space $H^*(X, E) := \bigoplus\limits_{i\geq 0} H^i(X, E)$. Note that $S_{\mathrm{Spec}(k)}(H^0(X, E)) = H^0(X, E)$, for all $i \geq 0$. Again $R^0 f_*(S_X(E)) \cong \mathrm{Ext}^0(\mathcal{O}_X, S_X(E)) = \mathrm{Ext}^0(\mathcal{O}_X, E \otimes \omega_X[n]) \cong \mathrm{Ext}^n(\mathcal{O}_X, E\otimes\omega_X) \cong H^n(X, E\otimes\omega_X)$. Since $H^n(X, E\otimes\omega_X) \ncong H^0(X, E)$, in general, we conclude that $Rf_* \circ S_X \ncong S_{\mathrm{Spec}(k)} \circ Rf_*$.

We are interested to know when an integral functor $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ is a Fourier-Mukai transform (i.e., an equivalence of categories). For $\Phi_{\mathcal{P}}^{X\to Y}$ to be an equivalence of categories, it must admit both left adjoint and right adjoint. As a first step towards this, we show that an integral functor $\Phi_{\mathcal{P}}^{X\to Y}$ always admit both left adjoint and right adjoint, which are again integral functors, and their kernels can be explicitly described.

Let X and Y be smooth projective k-schemes, and p_X and p_Y denote the projection morphisms from $X \times Y$ onto X and Y, respectively.

Definition 9.1.11. For any object $\mathcal{P} \in D^b(X \times Y)$, we define

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes p_Y^*\omega_Y[\dim Y], \quad \text{and} \quad \mathcal{P}_R := \mathcal{P}^\vee \otimes p_X^*\omega_X[\dim X].$$

Note that, both $\mathcal{P}_L$ and $\mathcal{P}_R$ are objects of $D^b(X \times Y)$.

Proposition 9.1.12. *There is a natural isomorphism of functors*

$$\Phi_{\mathcal{P}_R}^{Y \to X} \cong S_X \circ \Phi_{\mathcal{P}_L} \circ S_Y^{-1}.$$

Proof. It suffices to show that there are natural isomorphism of functors

$$\Phi_{\mathcal{P}_L}^{Y \to X} \cong \Phi_{\mathcal{P}^\vee}^{Y \to X} \circ S_Y \quad \text{and} \quad \Phi_{\mathcal{P}_R}^{Y \to X}(E^\bullet) \cong S_X \circ \Phi_{\mathcal{P}^\vee}^{Y \to X}. \tag{9.1.13}$$

For any $E^\bullet \in D^b(Y)$, we have

$$\begin{aligned}
\Phi_{\mathcal{P}_L}^{Y \to X}(E^\bullet) &= Rp_{X*}\big((Lp_Y^*E^\bullet)\overset{L}{\otimes}\mathcal{P}_L\big) \\
&\cong Rp_{X*}\big((Lp_Y^*E^\bullet)\overset{L}{\otimes}(\mathcal{P}^\vee \otimes p_Y^*\omega_Y[\dim Y])\big) \\
&\cong Rp_{X*}\big((Lp_Y^*E^\bullet)\overset{L}{\otimes}(p_Y^*\omega_Y[\dim Y])\overset{L}{\otimes}\mathcal{P}^\vee\big) \\
&\cong Rp_{X*}\big(Lp_Y^*(E^\bullet\overset{L}{\otimes}\omega_Y[\dim Y])\overset{L}{\otimes}\mathcal{P}^\vee\big) \\
&\cong Rp_{X*}\big(Lp_Y^*(S_Y(E^\bullet))\overset{L}{\otimes}\mathcal{P}^\vee\big) = (\Phi_{\mathcal{P}^\vee}^{Y \to X} \circ S_Y)(E^\bullet)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Phi_{\mathcal{P}_R}^{Y \to X}(E^\bullet) &= Rp_{X*}\big((Lp_Y^*E^\bullet)\overset{L}{\otimes}(\mathcal{P}^\vee \otimes p_X^*\omega_X[\dim X])\big) \\
&\cong Rp_{X*}\big((Lp_Y^*E^\bullet)\overset{L}{\otimes}\mathcal{P}^\vee\big)\overset{L}{\otimes}\omega_X[\dim X], \quad \text{by projection formula}. \\
&= \Phi_{\mathcal{P}^\vee}(E^\bullet)\overset{L}{\otimes}\omega_X[\dim X] \\
&= (S_X \circ \Phi_{\mathcal{P}^\vee}^{Y \to X})(E^\bullet).
\end{aligned}$$

Hence the result follow. □

Theorem 9.1.14 (Grothendieck-Verdier duality)**.** *Let $f : X \longrightarrow Y$ be a morphism of smooth schemes over a field k of relative dimension* $\dim(f) := \dim(X) - \dim(Y)$*. Let*

$$\omega_f := \omega_X \otimes f^*\omega_Y^\vee \tag{9.1.15}$$

be the relative dualizing sheaf of f. Then for any $F^\bullet \in D^b(X)$ and $E^\bullet \in D^b(Y)$, there is a a functorial isomorphism

$$Rf_*R\mathcal{H}om(F^\bullet, Lf^*(E^\bullet)\overset{L}{\otimes}\omega_f[\dim(f)]) \xrightarrow{\simeq} R\mathcal{H}om(Rf_*F^\bullet, E^\bullet)\,. \tag{9.1.16}$$

Proposition 9.1.17 (Mukai). *Let X and Y be smooth projective k-schemes. Let $F = \Phi_{\mathcal{P}} : D^b(X) \longrightarrow D^b(Y)$ be an integral functor with kernel $\mathcal{P} \in D^b(X \times Y)$. Then $\Phi_{\mathcal{P}_L}$ (resp., $\Phi_{\mathcal{P}_R}$) is the left adjoint (resp., right adjoint) of F.*

Proof. This is an application of Grothendieck-Verdier duality (Theorem 6.9.1), projection formula and compatibilities among derived functors. The relative dualizing sheaf for the projection morphism $p_X : X \times Y \to X$ is

$$\omega_{p_X} := \omega_{X\times Y} \otimes p_X^*\omega_X^\vee \cong p_Y^*\omega_Y \otimes p_X^*\omega_X \otimes p_X^*\omega_X^\vee \cong p_Y^*\omega_Y,$$

and the relative dimension of p_X is $\dim(p_X) := \dim(X \times Y) - \dim(X) = \dim(Y)$. For any $E^\bullet \in D^b(Y)$ and $F^\bullet \in D^b(X)$, we have

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(\Phi_{\mathcal{P}_L}^{Y\to X}(E^\bullet), F^\bullet) &= \mathrm{Hom}_{D^b(X)}\left(Rp_{X*}(Lp_Y^*E^\bullet \overset{L}{\otimes} \mathcal{P}_L), F^\bullet\right) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}\left(Lp_Y^*E^\bullet \overset{L}{\otimes} \mathcal{P}_L, p_X^*F^\bullet \overset{L}{\otimes} \omega_{p_X}[\dim(p_X)]\right), \text{ by Theorem 6.9.1.} \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}\left(Lp_Y^*E^\bullet \overset{L}{\otimes} \mathcal{P}_L, p_X^*F^\bullet \overset{L}{\otimes} p_Y^*\omega_Y[\dim(Y)]\right) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}\left(\mathcal{P}^\vee \overset{L}{\otimes} p_Y^*\omega_Y[\dim(Y)] \overset{L}{\otimes} p_Y^*E^\bullet, Lp_X^*F^\bullet \overset{L}{\otimes} p_Y^*\omega_Y[\dim Y]\right) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}\left(\mathcal{P}^\vee \overset{L}{\otimes} p_Y^*E^\bullet, p_X^*F^\bullet\right) \\
&\cong \mathrm{Hom}_{D^b(X\times Y)}\left(p_Y^*E^\bullet, \mathcal{P} \overset{L}{\otimes} p_X^*F^\bullet\right) \\
&\cong \mathrm{Hom}_{D^b(Y)}\left(E^\bullet, Rp_{Y*}(\mathcal{P} \overset{L}{\otimes} p_X^*F^\bullet)\right) \\
&= \mathrm{Hom}_{D^b(Y)}\left(E^\bullet, \Phi_{\mathcal{P}}^{X\to Y}(F^\bullet)\right).
\end{aligned}$$

Therefore, $\Phi_{\mathcal{P}_L}^{Y\to X}$ is the left adjoint of $\Phi_{\mathcal{P}}^{X\to Y}$. To show $\Phi_{\mathcal{P}_R}^{Y\to X}$ is the right adjoint of $\Phi_{\mathcal{P}}^{X\to Y}$, one can again do similar calculations as above, or alternatively use Proposition 9.1.12 and Lemma 9.1.18 (below) to complete the proof. □

Lemma 9.1.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be two k-linear categories with finite dimensional Homs. Assume that $\mathcal{A}$ and $\mathcal{B}$ admits Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$, respectively. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be k-linear functors. If G is the left adjoint of F, then $S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}$ is the right adjoint of F.*

Proof. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have the following sequence of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}\left(A, (S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1})B\right) &\cong \mathrm{Hom}\left((G \circ S_{\mathcal{B}}^{-1})(B), A\right)^* \\
&\cong \mathrm{Hom}\left(S_{\mathcal{B}}^{-1}(B), F(A)\right) \\
&\cong \mathrm{Hom}\left(F(A), S_{\mathcal{B}}(S_{\mathcal{B}}^{-1}(B))\right) \\
&\cong \mathrm{Hom}\left(F(A), B\right).
\end{aligned}$$

Hence the result follow. □

An important property of integral functors is that composition of two integral functors is again integral, and its kernel is given by *convolution product*, which we explain below. Let X, Y and Z be proper k-schemes. Consider the diagram

(9.1.19)
$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{XZ}"] \arrow[dr, "p_{YZ}"] & \\
X \times Y & X \times Z & Y \times Z\,,
\end{tikzcd}$$

where p_{XY}, p_{YZ} and p_{XZ} are projection morphisms. For $P \in D^-(X \times Y)$ and $Q \in D^-(Y \times Z)$, we define their *convolution product* to be the object

(9.1.20)
$$Q * P := p_{XZ*}\big((p_{XY}^* P) \otimes (p_{YZ}^* Q)\big) \in D^-(X \times Z).$$

Clearly, $Q * P$ is naturally isomorphic to $P * Q$. Proof of the following result is an easy consequence of projection formula.

Proposition 9.1.21 (Mukai)**.** *For any $\mathcal{P} \in D^-(X \times Y)$ and $\mathcal{Q} \in D^-(Y \times Z)$, there is a natural isomorphism of functors*

$$\Phi_{\mathcal{Q}}^{Y \to Z} \circ \Phi_{\mathcal{P}}^{X \to Y} \cong \Phi_{\mathcal{Q} * \mathcal{P}}^{X \to Z}\,.$$

Proof. Proof is very simple, but what makes it difficult is to work with 11 projection morphisms. The following commutative diagram could be useful to keep track of the morphisms.

$$\begin{tikzcd}
 & & & \overset{p_{XY}^*\mathcal{P} \otimes p_{YZ}^*\mathcal{Q}}{X \times Y \times Z} \arrow[dll, "p_{XY}"'] \arrow[drr, "p_{YZ}"] \arrow[dd, "p_{XZ}"] \arrow[dddlll, bend right, "p_X"'] \arrow[dddrrr, bend left, "p_Z"] & & & \\
 & \overset{\mathcal{P}}{X \times Y} \arrow[dl, "q"'] \arrow[dr, "p"] & & & & \overset{\mathcal{Q}}{Y \times Z} \arrow[dl, "u"'] \arrow[dr, "t"] & \\
X \arrow[drr, equal] & & Y \arrow[rr, bend left, equal] & \underset{\mathcal{R}}{X \times Z} \arrow[dl, "s"'] \arrow[dr, "r"] & Y & & Z \arrow[dll, equal] \\
 & & X & & Z & &
\end{tikzcd}$$

Let $\mathcal{R} := \mathcal{Q} * \mathcal{P} = p_{XZ*}\big(p_{XY}^*\mathcal{P} \otimes p_{YZ}^*\mathcal{Q}\big) \in D^b(X \times Z)$ be the convolution product of P and Q. Then for any $E^\bullet \in D^-(X)$, we have

$$\begin{aligned}
\Phi_{\mathcal{R}}^{X\to Z}(E^\bullet) &= r_*(Ls^*E^\bullet \otimes \mathcal{R}) \\
&= r_*\big(s^*E^\bullet \otimes p_{XZ*}\big(p_{XY}^*\mathcal{P} \otimes p_{YZ}^*\mathcal{Q}\big)\big) \\
&\cong r_*\big(p_{XZ*}(p_X^*E^\bullet \otimes p_{XY}^*\mathcal{P} \otimes p_{YZ}^*\mathcal{Q})\big), \text{ by projection formula.} \\
&\cong p_{Z*}\big(p_{XY}^*(q^*E^\bullet \otimes \mathcal{P}) \otimes p_{YZ}^*\mathcal{Q}\big), \text{ since } r \circ p_{XZ} = p_Z. \\
&\cong t_*p_{YZ*}\big(p_{XY}^*(q^*E^\bullet \otimes \mathcal{P}) \otimes p_{YZ}^*\mathcal{Q}\big), \text{ since } t \circ p_{YZ} = p_Z. \\
&\cong t_*\big(p_{YZ*}p_{XY}^*(q^*E^\bullet \otimes \mathcal{P}) \otimes \mathcal{Q}\big), \text{ by projection formula.} \\
&\cong t_*\big(u^*p_*(q^*E^\bullet \otimes \mathcal{P}) \otimes \mathcal{Q}\big), \text{ by flat base change } p_{YZ*} \circ p_{XY}^* = u^* \circ p_*. \\
&= t_*\big(u^*\Phi_{\mathcal{P}}^{X\to Y}(E^\bullet) \otimes \mathcal{Q}\big) = (\Phi_{\mathcal{Q}}^{Y\to Z} \circ \Phi_{\mathcal{P}}^{X\to Y})(E^\bullet)
\end{aligned}$$

This completes the proof. □

Remark 9.1.22. If the composite functor $\Phi_{\mathcal{Q}}^{Y\to Z} \circ \Phi_{\mathcal{P}}^{X\to Y}$ is not an equivalence of categories, then the kernel $\mathcal{R} := \mathcal{P} * \mathcal{Q}$ is not necessarily unique. However this choice of $\mathcal{R}$ is a natural one in the sense that it is compatible with taking left adjoint and right adjoint. More precisely, we have natural isomorphism of functors

(9.1.23) $$\Phi_{\mathcal{R}_L} \cong p_{XZ*}\big(p_{XY}^*\mathcal{P}_L \otimes p_{YZ}^*\mathcal{Q}_L\big) \quad \text{and} \quad \Phi_{\mathcal{R}_R} \cong p_{XZ*}\big(p_{XY}^*\mathcal{P}_R \otimes p_{YZ}^*\mathcal{Q}_R\big).$$

This can easily be checked using Grothendieck-Verdier duality as in Proposition 9.1.17.

Remark 9.1.24. Let $\mathcal{P}, \mathcal{Q} \in D^b(X \times Y)$. Then any morphism $\varphi \in \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}, \mathcal{Q})$ induces a morphism of the associated integral functors: $\Phi_\varphi : \Phi_{\mathcal{P}} \longrightarrow \Phi_{\mathcal{Q}}$. One might wonder if this induced morphism Φ_φ is non-trivial if φ is non-trivial. In general, the answer is no! For example, let C be an elliptic curve over a field k, and denote by Δ the image of the diagonal embedding $C \hookrightarrow C \times C$. Consider $\mathcal{O}_\Delta$ as an object of $D^b(C \times C)$. Using Serre duality on the product $C \times C$, one can conclude that $\dim_k \mathrm{Ext}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = 1$. So there is a non-trivial morphism

(9.1.25) $$f : \mathcal{O}_\Delta \longrightarrow \mathcal{O}_\Delta[2]$$

in $D^b(C \times C)$. This morphism f induce a morphism of associated integral functors

(9.1.26) $$\Phi_f : \Phi_{\mathcal{O}_\Delta} \longrightarrow \Phi_{\mathcal{O}_\Delta[2]}.$$

Note that, $\Phi_{\mathcal{O}_\Delta} \cong \mathrm{Id}_{D^b(C)}$, and $\Phi_{\mathcal{O}_\Delta[2]}$ is isomorphic to the 2-shift functor $E^\bullet \mapsto E^\bullet[2]$. However, $\Phi_f = 0$. To see this, note that C being one dimensional, $\mathrm{Ext}^2(E, E) = 0$ for any coherent sheaf E on C. So the induced morphism $\Phi_f(E) : \Phi_{\mathcal{O}_\Delta}(E) \longrightarrow \Phi_{\mathcal{O}_\Delta[2]}(E)$ is trivial. Since any object of $D^b(C)$ is isomorphic to a finite direct sum of shifted coherent sheaves on C, we conclude that $\Phi_f(E^\bullet) : E^\bullet \longrightarrow E^\bullet[2]$ is zero, for any $E^\bullet \in D^b(C)$.

The following useful results are easy to verify.

Proposition 9.1.27. *Let X, Y and Z be smooth projective k-schemes. Let $\mathcal{P} \in D^b(X \times Y)$.*

(i) For any morphism $f : Y \longrightarrow Z$ of k-schemes, we have an isomorphism of functors $Rf_* \circ \Phi_{\mathcal{P}}^{X\to Y} \cong \Phi_{(\mathrm{Id}_X \times f)_*\mathcal{P}}^{X\to Z}$.
(ii) For any morphism $f' : Z \longrightarrow Y$ of k-schemes, we have an isomorphism of functors $Lf^* \circ \Phi_{\mathcal{P}}^{X\to Y} \cong \Phi_{(\mathrm{Id}_X \times f')^*\mathcal{P}}^{X\to Z}$.
(iii) For any morphism $g : Z \to X$ of k-schemes, we have an isomorphism of functors $\Phi_{\mathcal{P}}^{X\to Y} \circ Rg_* \cong \Phi_{(g\times \mathrm{Id}_Y)^*\mathcal{P}}^{Z\to Y}$.
(iv) For any morphism $g' : X \to Z$ of k-schemes, we have an isomorphism of functors $\Phi_{\mathcal{P}}^{X\to Y} \circ Lg^* \cong \Phi_{(g'\times \mathrm{Id}_Y)^*\mathcal{P}}^{Z\to Y}$.

It is natural to ask which functors are isomorphic to an integral functor? The answer is given by the celebrated representability theorem due to Orlov [Orl03]. Orlov's representability theorem says that, *if X and Y are smooth projective varieties defined over a field k, then any exact fully faithful functor $F : D^b(X) \longrightarrow D^b(Y)$ admitting both left and right adjoints is isomorphic to an integral functor $\Phi_{P_F}^{X\to Y}$, where $P_F \in D^b(X \times Y)$ is unique up to isomorphism*.

In their celebrated paper [BvdB03], Bondal and Van den Bergh proved a deep result which ensures that any exact functor $F : D^b(X) \to D^b(Y)$ admits a right adjoint. Since both $D^b(X)$ and $D^b(Y)$ admit Serre functors, it follows from Lemma 9.1.18 that F admits a left adjoint too. Therefore, the assumption of existence of both left and right adjoints of F is redundant, and we get the following stronger version of Orlov's representability theorem (whose proof will be given later).

Theorem 9.1.28. *Let X and Y be two smooth projective k-varieties. Let*

$$F : D^b(X) \longrightarrow D^b(Y)$$

be an exact fully faithful functor (resp., exact equivalence of categories). Then there is an object $P_F \in D^b(X\times Y)$, unique up to isomorphism, such that F is isomorphic to the integral functor $\Phi_{P_F}^{X\to Y}$ with kernel P_F.

As an immediate corollary to this, we get the following.

Corollary 9.1.29. *Let X and Y be smooth projective k-varieties with an exact equivalence of derived categories $F : D^b(X) \longrightarrow D^b(Y)$. Then $\dim_k(X) = \dim_k(Y)$.*

Proof. By Orlov's representability theorem, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}$. By Proposition 9.1.17, due to Mukai, F admits both left adjoint and right adjoint, which are also Fourier-Mukai transformations with kernels

$$\mathcal{P}_L := \mathcal{P}^\vee \overset{L}{\otimes} p_Y^*\omega_Y[\dim Y] \quad \text{and} \quad \mathcal{P}_R := \mathcal{P}^\vee \overset{L}{\otimes} p_X^*\omega_X[\dim X], \tag{9.1.30}$$

respectively. Since F is an equivalence of categories, $\Phi_{\mathcal{P}_L}^{Y\to X} \cong \Phi_{\mathcal{P}_R}^{Y\to X}$. Since quasi-inverse of F is also an exact equivalence from $D^b(Y)$ to $D^b(X)$, using uniqueness (up to isomorphism) of kernel of a Fourier-Mukai transformation (c.f., Theorem 9.1.28), we conclude that $\mathcal{P}_L \cong \mathcal{P}_R$ in $D^b(X \times Y)$. Therefore,

$$\mathcal{P}^\vee \cong \mathcal{P}^\vee \overset{L}{\otimes} \big(p_X^*\omega_X \overset{L}{\otimes} p_Y^*\omega_Y^\vee[\dim X - \dim Y]\big). \tag{9.1.31}$$

From this, it follows that $\dim X = \dim Y$. $\square$

Let X and Y be smooth projective k-varieties. We show that if an exact equivalence $F : D^b(X) \to D^b(Y)$ sends skyscraper sheaves $k(x)$ to skyscraper sheaves $k(y_x)$, where $x \in X$ and $y_x \in Y$ are closed points, then there is an isomorphism of k-schemes $f : X \to Y$ sending x to y_x, and that F is isomorphic to $(- \otimes L) \circ f_*$, for some $L \in \mathrm{Pic}(Y)$. In particular, F is a Fourier-Mukai transform. First, we need the following.

Lemma 9.1.32. *Let $\phi : X \longrightarrow S$ be a morphism of k-schemes. For a closed point $s \in S$, we denote by $\iota_s : X_s \hookrightarrow X$ the closed embedding of the scheme theoretic fiber $X_s := X \times_S \mathrm{Spec}(k(s))$ over s into X. Let $\mathcal{P} \in D^b(X)$ be such that for each closed point $s \in S$, the derived pullback $L\iota_s^*\mathcal{P} \in D^b(X_s)$ is a complex concentrated at degree 0. Then $\mathcal{P}$ is isomorphic to a coherent sheaf on X flat over S.*

Proof. Let

$$m := \max\{i \in \mathbb{Z} : \mathcal{H}^i(\mathcal{P}) \neq 0\}.$$

First, we show that $m = 0$. Then we show that

$$\mathcal{H}^{-1}(L\iota_s^*\mathcal{H}^0(\mathcal{P})) = 0, \ \ \forall\, s \in S,$$

which implies $\mathrm{Tor}_1(\mathcal{H}^0(\mathcal{P}), k(s)) = 0$, for all $s \in S$, and hence $\mathcal{H}^0(\mathcal{P})$ is flat over S. Finally to complete the proof, we show that $\mathcal{H}^q(\mathcal{P}) = 0$, for all $q < 0$.

Let $s \in S$ be a closed point. Consider the spectral sequence

$$E_2^{p,q} := \mathcal{H}^p(L\iota_s^*(\mathcal{H}^q(\mathcal{P}))) \Longrightarrow E^{p+q} := \mathcal{H}^{p+q}(L\iota_s^*(\mathcal{P})), \ \ \forall\, \mathcal{P} \in D^b(X). \tag{9.1.33}$$

Then there is a closed point $s \in S$ such that

$$E_2^{0,m} = \mathcal{H}^0(L\iota_s^*(\mathcal{H}^m(\mathcal{P}))) \neq 0;$$

note that, this is just the ordinary pullback of $\mathcal{H}^m(\mathcal{P}) \in \mathfrak{Coh}(X)$ over X_s.

$$\begin{tikzcd}
E_2^{-2,m+1} = 0 \arrow[drr] & E_2^{-1,m+1} = 0 \arrow[drr] & E_2^{0,m+1} = 0 \arrow[drr] & E_2^{1,m+1} = 0 & E_2^{2,m+1} \\
E_2^{-2,m} \arrow[drr] & E_2^{-1,m} \arrow[drr] & E_2^{0,m} \arrow[drr] & E_2^{1,m} = 0 & E_2^{2,m} \\
E_2^{-2,m-1} & E_2^{-1,m-1} & E_2^{0,m-1} & E_2^{1,m-1} = 0 & E_2^{2,m-1} = 0
\end{tikzcd}$$

Since, by assumption, $E^m = \mathcal{H}^m(L\iota_s^*(\mathcal{P})) = 0$ except possibly for $m = 0$, we conclude that $m = 0$. Similarly, since $\mathcal{H}^{-1}(L\iota_s^*\mathcal{P}) = 0$ by assumption, we have $E_2^{-1,0} = \mathcal{H}^{-1}(L\iota_s^*\mathcal{H}^0(\mathcal{P})) = 0$, for all $s \in S$. Then $\mathrm{Tor}_1(\mathcal{H}^0(\mathcal{P}), k(s)) = 0$, for all closed points $s \in S$, and hence $\mathcal{H}^0(\mathcal{P})$ is flat over S. Now $\mathcal{H}^0(\mathcal{P})$ being flat over S, its higher derived pullbacks $E_2^{-p,0} = \mathcal{H}^{-p}(L\iota_s^*\mathcal{H}^0(\mathcal{P})) = \mathcal{T}or_p(\mathcal{H}^0(\mathcal{P}), k(s))$ are trivial for $p > 0$.

Now it remains to show that $\mathcal{H}^q(\mathcal{P}) = 0$, for all $q < 0$. If not, then let n be the largest integer such that $n < 0$ and $\mathcal{H}^n(\mathcal{P}) \neq 0$. Choose a closed point $s \in S$ such that $X_s \cap \mathrm{Supp}(\mathcal{H}^n(\mathcal{P})) \neq \emptyset$. Since $E_2^{-p,q} = \mathcal{H}^{-p}(L\iota_s^*\mathcal{H}^q(\mathcal{P})) = 0$, for all $p < 0$ and $q > m = 0$, it follows that $E_\infty^{0,n} = E_2^{0,n} = \mathcal{H}^0(L\iota_s^*\mathcal{H}^n(\mathcal{P})) \neq \emptyset$. This is a contradiction because $E^n = \mathcal{H}^n(L\iota_s^*\mathcal{P}) = 0$, since $n < 0$. This completes the proof. $\square$

Proposition 9.1.34. *Let k be an algebraically closed field. Let X and Y be smooth projective k-varieties with an exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$. Suppose that for each closed point $x \in X$ there is a (unique) closed point $y_x \in Y$ such that $F(k(x)) \cong k(y_x)$. Then f gives rise to an isomorphism of k-schemes, also denoted by $f : X \longrightarrow Y$, such that $F \cong (L \otimes -) \circ f_*$, for some $L \in \mathrm{Pic}(Y)$.*

Proof. We only sketch a proof leaving the details to the readers. By Orlov's representability theorem 9.1.28, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{X\to Y}$. By assumption,

$$k(y_x) \cong F(k(x)) \cong \Phi_{\mathcal{P}}^{X\to Y}(k(x)) = p_{Y*}\big(\mathcal{P} \overset{L}{\otimes} p_X^* k(x)\big) \cong \mathcal{P}\big|_{\{x\}\times Y}. \tag{9.1.35}$$

Then by above Lemma 9.1.32, we may assume that $\mathcal{P}$ is a coherent sheaf on $X \times Y$ flat over X. Then choosing local sections of $\mathcal{P}$, using (9.1.35) we find a morphism of k-schemes $f : X \longrightarrow Y$ such that $f(x) = y_x$, for all closed points $x \in X$.

Since $\{k(x) \in D^b(X) : x \text{ is a closed point of } X\}$ spans $D^b(X)$, the exact equivalence $F : D^b(X) \to D^b(Y)$ ensures that the objects $F(k(x)) \cong k(f(x))$ spans $D^b(Y)$. Therefore, for a closed point $y \in Y$, there is a closed point $x_y \in X$ such that

$$\mathrm{Hom}_{D^b(Y)}(F(k(x_y)), k(y)[m_y]) \neq 0,$$

for some integer m_y. This implies, $k(y)$ is of the form $k(f(x))$ in $D^b(Y)$. Therefore, f is surjective over the set of closed points. Similarly, one can show that f is injective at the level of closed points. Since the set of all closed points is dense in a finite type k-scheme, f is birational. Then one can use Zariski's main theorem and Stein factorization to deduce that f is an isomorphism of k-schemes in characteristic 0. In positive characteristic, one need to use exact quasi-inverse $F^{-1} : D^b(Y) \to D^b(X)$ of F to produce a morphism of k-schemes $g : Y \longrightarrow X$, which gives the inverse of f in the category of k-schemes.

Eventually, $\mathcal{P}$ considered as a sheaf on its support, which is the graph of f in $X \times Y$, is a sheaf of constant fiber dimension 1, and hence is a line bundle. Since the projection p_Y induces an isomorphism $\mathrm{Supp}(\mathcal{P}) \xrightarrow{\simeq} Y$, we can consider $\mathcal{P}$ as a line bundle over Y. Then the result follows. □

Corollary 9.1.36 (Gabriel)**.** *Let X and Y be smooth projective k-varieties. If there is an exact equivalence of abelian categories $F : \mathfrak{Coh}(X) \longrightarrow \mathfrak{Coh}(Y)$, then X is isomorphic to Y.*

Proof. Clearly F give rise to an exact equivalence of derived categories $\widetilde{F} : D^b(X) \to D^b(Y)$. Then by Orlov's theorem 9.1.28, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $\widetilde{F} \cong \Phi_{\mathcal{P}}$. An object $E \in \mathfrak{Coh}(X)$ is called *indecomposable* if any non-trivial epimorphism $E \twoheadrightarrow E''$ in $\mathfrak{Coh}(X)$ is an isomorphism. One can check that, $E \in \mathfrak{Coh}(X)$ is indecomposable if and only if $E \cong k(x)$, for some closed point $x \in X$. Since an exact equivalence $F : \mathfrak{Coh}(X) \to \mathfrak{Coh}(Y)$ sends indecomposable object to indecomposable object, for each closed point $x \in X$, we find a closed point $f(x) \in Y$ with $\Phi_{\mathcal{P}}(k(x)) \cong k(f(x))$. Then by above Proposition, f give rise to a morphism of k-schemes $f : X \to Y$ such that $\Phi_{\mathcal{P}} \cong (L \otimes -) \circ f_*$, for some $L \in \mathrm{Pic}(Y)$. □

9.2. K-theoretic integral transformation.

Definition 9.2.1. Let $\mathcal{A}$ be an abelian category. Let $\mathbf{F}(\mathcal{A})$ be the free abelian group generated by the set of all isomorphism classes of objects of $\mathcal{A}$. Let $\mathbf{N}(\mathcal{A})$ be the normal subgroup of $\mathbf{F}(\mathcal{A})$ generated by the elements $[E'] - [E] + [E''] \in \mathbf{F}(\mathcal{A})$, where $0 \to E' \to E \to E'' \to 0$ is an exact sequence in $\mathcal{A}$. Then the *Grothendieck group* of $\mathcal{A}$ is defined to be the quotient group

$$K_0(\mathcal{A}) := \mathbf{F}(\mathcal{A})/\mathbf{N}(\mathcal{A})\,.$$

Remark 9.2.2. The above definition of Grothendieck group perfectly make sense for exact categories.

Let X be a smooth projective k-variety. Let $\mathcal{V}ect(X)$ be the full subcategory of $\mathfrak{Coh}(X)$, whose objects are locally free coherent sheaves on X. The category $\mathcal{V}ect(X)$ is exact, but not abelian.

Lemma 9.2.3. *There is a natural isomorphism $K_0(\mathcal{V}ect(X)) \cong K_0(\mathfrak{Coh}(X))$.*

Proof. The homomorphism $\iota_* : K_0(\mathcal{V}ect(X)) \to K_0(\mathfrak{Coh}(X))$ induced by the fully faithful (inclusion) functor $\iota : \mathcal{V}ect(X) \hookrightarrow \mathfrak{Coh}(X)$ is injective. Since X is a smooth projective algebraic k-variety, any $E \in \mathfrak{Coh}(X)$ admits a finite resolution

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to E \to 0,$$

with $E_i \in \mathcal{V}ect(X)$, for all i. The map given by sending $[E] \in K_0(\mathfrak{Coh}(X))$ to $\sum\limits_{i=0}^{n}(-1)^i[E_i] \in K_0(\mathcal{V}ect(X))$ is independent of choice of the resolution of E, and is a group homomorphism. This gives the inverse of ι_*. $\square$

Define $K_0(X) := K_0(\mathfrak{Coh}(X))$. For $E^\bullet \in D^b(X)$, we associate an element

$$[E^\bullet] := \sum_j (-1)^j [E^j] \in K_0(X). \tag{9.2.4}$$

Since any object of $\mathfrak{Coh}(X)$ admits a finite resolution by locally free coherent sheaves on X, any element of $K_0(X)$ can be written as a finite $\mathbb{Z}$-linear combination $\sum\limits_i \alpha_i[E_i]$, with E_i locally free coherent sheaves on X. One can use this to define a ring structure on $K_0(X)$ by setting

$$[E] \cdot [F] := [E \otimes F], \quad \forall\, E, F \in \mathfrak{Coh}(X), \tag{9.2.5}$$

and then extending this operation $\mathbb{Z}$-linearly over $K_0(X)$. Define a map

$$D^b(X) \longrightarrow K_0(X) \tag{9.2.6}$$

by sending $E^\bullet \in D^b(X)$ to $[E^\bullet] := \sum\limits_j (-1)^j [E^j] \in K_0(X)$. One can check that,

$$[E^\bullet] = \sum_j (-1)^j [\mathcal{H}^j(E^\bullet)] \tag{9.2.7}$$

in $K_0(X)$, and hence $[E^\bullet] = [F^\bullet]$ in $K_0(X)$ whenever $E^\bullet \cong F^\bullet$ in $D^b(X)$. Note that,

$$[E^\bullet[i]] = \sum_j (-1)^j E^{i+j} = (-1)^i [E^\bullet] \quad \text{and} \quad [E^\bullet \oplus F^\bullet] = [E^\bullet] + [F^\bullet].$$

Since X is a smooth projective k-variety, derived tensor product of two complexes in $D^b(X)$ can be computed as a ordinary tensor product of bounded complexes of locally free coherent sheaves on X isomorphic to them, $[E^\bullet \overset{L}{\otimes} F^\bullet] = [E^\bullet] \cdot [F^\bullet]$ in $K_0(X)$. Therefore, the map $D^b(X) \to K_0(X)$ given by $E^\bullet \longmapsto [E^\bullet]$ is compatible with the natural additive and multiplicative structures on both sides.

Lemma 9.2.8. *Let $f : X \longrightarrow Y$ be a morphism of smooth projective k-varieties. Then f induces a homomorphism of their Grothendieck groups $f^* : K_0(Y) \to K_0(X)$.*

Proof. Let $E \in \mathfrak{Coh}(Y)$. Then $\widetilde{f}(E) := \sum_{i\geq 0}(-1)^i[L^i f^*E]$ is an element of the free abelian group generated by the isomorphism classes of objects from $\mathfrak{Coh}(X)$. Since any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

in $\mathfrak{Coh}(Y)$ induces a (bounded) long exact sequence of $\mathcal{O}_X$-modules

$$\cdots \to L^{i+1}f^*E'' \to L^i f^*E' \to L^i f^*E \to L^i f^*E'' \to L^{i-1}f^*E' \to \cdots,$$

we see that $\widetilde{f}(E) = \widetilde{f}(E') + \widetilde{f}(E'')$ in $K_0(X)$. Thus $\widetilde{f}$ induces a well-defined group homomorphism $f^* : K_0(Y) \to K_0(X)$. □

Lemma 9.2.9. *Let $f : X \to Y$ be a proper morphism of projective k-schemes. Then f induces a homomorphism of Grothendieck groups $f_! : K_0(X) \to K_0(Y)$.*

Proof. Since f is proper, $R^i f_*(E) \in \mathfrak{Coh}(Y)$, for all $E \in \mathfrak{Coh}(X)$. Then following the proof of the above Lemma 9.2.8, we see that $[E] \mapsto \sum_{i\geq 0}(-1)^i[R^i f_*(E)]$ gives the required group homomorphism. □

Remark 9.2.10. Both $f^* : K_0(Y) \to K_0(X)$ and $f_! : K_0(X) \to K_0(Y)$ are compatible with derived pullback and derived direct image functors in the sense that the following diagrams are commutative.

$$\begin{tikzcd} D^b(Y) \arrow[r, "Lf^*"] \arrow[d, "{[\,]}"'] & D^b(X) \arrow[d, "{[\,]}"] \\ K_0(Y) \arrow[r, "f^*"] & K_0(X) \end{tikzcd} \qquad \begin{tikzcd} D^b(X) \arrow[r, "Rf_*"] \arrow[d, "{[\,]}"'] & D^b(Y) \arrow[d, "{[\,]}"] \\ K_0(X) \arrow[r, "f_!"] & K_0(Y) \end{tikzcd} \tag{9.2.11}$$

Commutativity of the square on the left hand side is easy to check. To see the commutativity of the square on the right hand side, we need to show that $[Rf_*E^\bullet] = \sum_j(-1)^j[R^jf_*E^\bullet]$ is equal to

$$f_![E^\bullet] = \sum_j(-1)^j f_![\mathcal{H}^j(E^\bullet)] = \sum_j(-1)^j\sum_i(-1)^i[R^if_*\mathcal{H}^j(E^\bullet)],$$

which can be checked by using the Leray spectral sequence

$$E_2^{p,q} := R^pf_*\mathcal{H}^q(E^\bullet) \Longrightarrow E^{p+q} := R^{p+q}f_*(E^\bullet). \tag{9.2.12}$$

Definition 9.2.13. We define the *K-theoretic integral transform*

$$\Phi_\xi^{K_0, X\to Y} : K_0(X) \longrightarrow K_0(Y)$$

with kernel $\xi \in K_0(X \times Y)$ by sending $\alpha \in K_0(X)$ to $\Phi_\xi^{K_0,X\to Y}(\alpha) := p_{Y!}(\xi \otimes p_X^*\alpha)$.

It follows from the above compatibility relations in (9.2.11) that the integral transform is compatible with the corresponding K-theoretic integral transform in the sense that the following diagram commutes.

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_{\mathcal{P}}^{X\to Y}"] \arrow[d, "[\,]"'] & D^b(Y) \arrow[d, "[\,]"] \\ K_0(X) \arrow[r, "\Phi_{[\mathcal{P}]}^{K_0, X\to Y}"] & K_0(Y) \end{tikzcd} \tag{9.2.14}$$

Remark 9.2.15. The above compatibility between $\Phi_{\mathcal{P}}^{X\to Y}$ and $\Phi_{[\mathcal{P}]}^{K_0,X\to Y}$ can also be seen from the following more general fact: any exact functor $F : D^b(X) \to D^b(Y)$ induces a group homomorphism $F^{K_0} : K_0(X) \longrightarrow K_0(Y)$ such that $F^{K_0}([E^\bullet]) = [F(E^\bullet)]$, for all $E^\bullet \in D^b(X)$. In other words, the above diagram commutes.

Proposition 9.2.16. *Let $\mathcal{P} \in D^b(X \times Y)$. If the integral functor $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ is an equivalence of categories, then the induced K-theoretic integral transform $\Phi_{[\mathcal{P}]}^{K,X\to Y}$: $K_0(X) \to K_0(Y)$ is an isomorphism of abelian groups.*

Proof. Note that, for $Y = X$ and $\mathcal{P} = \mathcal{O}_{\Delta_X}$, the induced K-theoretic integral transform

$$\Phi_{\mathcal{O}_{\Delta_X}}^{K,X\to X} : K_0(X) \longrightarrow K_0(X)$$

is just the identity map $K_0(X)$. Since $\Phi_{\mathcal{P}}^{X\to Y}$ is an equivalence of categories, its left adjoint and right adjoint functors are isomorphic, and they are quasi-inverse to $\Phi_{\mathcal{P}}^{X\to Y}$. Note that, the left adjoint and right adjoint of $\Phi_{\mathcal{P}}^{X\to Y}$ are again Fourier-Mukai functors whose kernels are explicitly given by Proposition 9.1.17. Since the composite Fourier-Mukai functor $\Phi_{\mathcal{P}_R}^{Y\to X} \circ \Phi_{\mathcal{P}}^{X\to Y} \cong \Phi_{\mathcal{O}_{\Delta_X}}$ (resp., $\Phi_{\mathcal{P}}^{X\to Y} \circ \Phi_{\mathcal{P}_R}^{Y\to X} \cong \Phi_{\mathcal{O}_{\Delta_Y}}$) is isomorphic to the identity functor on $D^b(X)$ (resp., $D^b(Y)$), we have $\Phi_{[\mathcal{P}]}^{K,X\to Y} \circ \Phi_{[\mathcal{P}_R]}^{K,Y\to X} \in \mathrm{Aut}(K_0(Y))$ and $\Phi_{[\mathcal{P}_R]}^{K,Y\to X} \circ \Phi_{[\mathcal{P}]}^{K,X\to Y} \in \mathrm{Aut}(K_0(X))$. Hence $\Phi_{[\mathcal{P}]}^{K,X\to Y}$ is an isomorphism. $\square$

Remark 9.2.17. For X and Y smooth projective k-varieties, following the similar procedure, one can also define integral transformation at the level of Chow groups $\Phi_Z^{\mathrm{Chow},X\to Y} : \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(Y)$ with kernel $Z \in \mathrm{CH}^*(X \times Y)$. However, since Chow group and K_0-group coincides after tensoring with $\mathbb{Q}$, we don't gain much.

9.3. **Cohomological integral transformation.** Let X be a smooth projective variety over $\mathbb{C}$. We denote by $H^*(X,\mathbb{Q})$ the cohomology of the constant sheaf $\mathbb{Q}$ over the underlying complex manifold of X. Note that, $H^*(X,\mathbb{Q})$ has a natural ring structure. Moreover, any continuous map of compact connected complex manifolds $f : X \to Y$ induces a ring homomorphism

$$f^* : H^*(Y,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q}). \tag{9.3.1}$$

Let $n = \dim(X)$ and $m = \dim(Y)$. Then by Poincaré duality

$$H^i(X, \mathbb{Q}) \cong H^{2n-i}(X, \mathbb{Q})^* \quad \text{and} \quad H^i(Y, \mathbb{Q}) \cong H^{2m-i}(Y, \mathbb{Q})^*$$

to define

$$f_* : H^*(X, \mathbb{Q}) \longrightarrow H^{*+2m-2n}(Y, \mathbb{Q}) \tag{9.3.2}$$

as the dual of f_* in (9.3.1). Then we have the following projection formula:

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta). \tag{9.3.3}$$

Definition 9.3.4. Let X and Y be smooth projective varieties over $\mathbb{C}$. Denote by p_X and p_Y the projection morphisms from $X \times Y$ onto X and Y, respectively. Given a cohomology class $\alpha \in H^*(X \times Y, \mathbb{Q})$, we define the *cohomological integral transform*

$$\Phi_\alpha^{H,\, X \to Y} : H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$$

by $\Phi_\alpha^{H,\, X\to Y}(\beta) := p_{Y*}\big(\alpha \cdot p_X^* \beta\big)$, for all $\beta \in H^*(X, \mathbb{Q})$. Note that, $\Phi_\alpha^{H,\, X\to Y}$ is $\mathbb{Q}$-linear.

Now one can use the *Chern character map*

$$\mathrm{ch} : K_0(X) \longrightarrow H^*(X, \mathbb{Q}) \tag{9.3.5}$$

to pass from Grothendieck's K_0-group to cohomology. Unfortunately, the Chern character map (9.3.5) does not commute with integral transforms at the level of K_0-group and cohomology (i.e., the following diagram is not commutative), in general.

$$\begin{tikzcd} K_0(X) \arrow[r, "\Phi_\alpha^{K_0, X\to Y}"] \arrow[d, "\mathrm{ch}"'] & K_0(Y) \arrow[d, "\mathrm{ch}"] \\ H^*(X,\mathbb{Q}) \arrow[r, "\Phi_{\mathrm{ch}(\alpha)}^{H, X\to Y}"] & H^*(Y,\mathbb{Q}) \end{tikzcd}$$

To remedy the situation, we need to consider the Todd class. By definition, Todd class is multiplicative, i.e.,

$$\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1) \cdot \mathrm{td}(E_2), \quad \forall\, E_1, E_2 \in \mathcal{V}ect(X), \tag{9.3.6}$$

and for a line bundle L on X, we have

$$\mathrm{td}(L) := \frac{c_1(L)}{1 - \exp(-c_1(L))}. \tag{9.3.7}$$

For X a smooth variety over $\mathbb{C}$, we denote $\mathrm{td}(X) := \mathrm{td}(TX)$. The key ingredient for the compatibility relation is the Grothendieck-Riemann-Roch formula.

Theorem 9.3.8 (Grothendieck-Riemann-Roch)**.** *Let $f : X \longrightarrow Y$ be a projective morphism of smooth projective k-varieties. Then for any $\alpha \in K_0(X)$, we have*

$$\mathrm{ch}(f_!(\alpha)) \cdot \mathrm{td}(Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(X)).$$

Taking $f : X \to \mathrm{Spec}(k)$, the structure morphism of X, we get the following.

Corollary 9.3.9 (Hirzebruch-Riemann-Roch formula). *For any $\alpha \in K_0(X)$, we have*

$$\chi(\alpha) = \int_X \mathrm{ch}(\alpha) \cdot \mathrm{td}(X).$$

Remark 9.3.10. We define $\chi(E^\bullet) := \sum_i (-1)^i \chi(E^i)$, for all $E^\bullet \in D^b(X)$. By definition of the map $[\ ] : D^b(X) \to K_0(X)$ we have $[E^\bullet[j]] = \sum_i (-1)^i [E^{i+j}] = (-1)^j [E^\bullet]$ (see (9.2.4)). Since the Chern character map

$$\mathrm{ch} : K_0(X) \to H^*(X, \mathbb{Q})$$

is additive, we have $\mathrm{ch}(E^\bullet) := \sum_i (-1)^i \mathrm{ch}([E^i])$. As a result, for any $E^\bullet \in D^b(X)$, from Corollary 9.3.9 we have

$$\chi(E^\bullet) = \sum_j (-1)^j \chi(E^j) = \sum_j (-1)^j \int_X \mathrm{ch}([E^j]) \cdot \mathrm{td}(X) = \int_X \mathrm{ch}([E^\bullet]) \cdot \mathrm{td}(X).$$

Definition 9.3.11 (Mukai vector). *Mukai vector* of a class $\alpha \in K_0(X)$ (resp., an object $E^\bullet \in D^b(X)$) is defined to be the cohomology class

$$v(\alpha) := \mathrm{ch}(\alpha) \cdot \sqrt{\mathrm{td}(X)} \quad \left(\text{resp.,} \quad v(E^\bullet) := v([E^\bullet]) = \mathrm{ch}([E^\bullet]) \cdot \sqrt{\mathrm{td}(X)} \right). \tag{9.3.12}$$

Here $\sqrt{\mathrm{td}(X)}$ is the cohomology class whose square is $\mathrm{td}(X)$, and its existence can be shown by explicit computation with the formal (but finite) power series calculation. It follows from the above definition that the *Mukai vector map*

$$v : K_0(X) \longrightarrow H^*(X, \mathbb{Q}) \tag{9.3.13}$$

is additive.

Corollary 9.3.14. *Let X and Y be smooth projective $\mathbb{C}$-varieties, and let $\alpha \in K_0(X \times Y)$. Then for any $\beta \in K_0(X)$, we have*

$$\Phi^{H, X\to Y}_{v(\alpha)} \left(\mathrm{ch}(\beta) \cdot \sqrt{\mathrm{td}(X)} \right) = \mathrm{ch}\left(\Phi^{K, X \to Y}_{\alpha}(\beta) \right) \cdot \sqrt{\mathrm{td}(Y)}. \tag{9.3.15}$$

In other words, the following diagram is commutative.

$$\begin{tikzcd} K_0(X) \arrow[r, "\Phi^{K,X\to Y}_{\alpha}"] \arrow[d, "v"'] & K_0(Y) \arrow[d, "v"] \\ H^*(X, \mathbb{Q}) \arrow[r, "\Phi^{H,X\to Y}_{v(\alpha)}"] & H^*(Y, \mathbb{Q}). \end{tikzcd} \tag{9.3.16}$$

Proof. It suffices to show that the following diagram is commutative.

$$\begin{tikzcd}
K_0(X) \arrow[r, "p_X^*"] \arrow[d, "v"'] & K_0(X\times Y) \arrow[r, "\cdot\alpha"] \arrow[d, "v(-)\cdot\left(\sqrt{\mathrm{td}(Y)}\right)^{-1}"] & K_0(X\times Y) \arrow[r, "p_{Y!}"] \arrow[d, "v(-)\cdot\sqrt{\mathrm{td}(X)}"] & K_0(Y) \arrow[d, "v"] \\
H^*(X,\mathbb{Q}) \arrow[r, "p_X^*"'] & H^*(X\times Y) \arrow[r, "\cdot v(\alpha)"'] & H^*(X\times Y) \arrow[r, "p_{Y*}"'] & H^*(Y,\mathbb{Q}).
\end{tikzcd}$$

Commutativity of the first two squares follows from projection formula and the commutativity of the last square follows from Grothendieck-Riemann-Roch formula. □

Remark 9.3.17. In general, cohomological integral transform neither preserve grading nor the multiplicative structure of $H^*(-,\mathbb{Q})$, even for the Mukai vector $\alpha = v(\beta)$. However, it indeed preserve parity. More precisely, let $\mathcal{P} \in D^b(X \times Y)$, and consider the associated integral functor

$$\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \longrightarrow D^b(Y).$$

Denote by

$$\Phi_{v(\mathcal{P})}^{H,\,X\to Y} : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q}), \quad \beta \longmapsto p_{Y*}\big(v(\mathcal{P}) \cdot p_X^*\beta\big) \tag{9.3.18}$$

the cohomological integral transform with the Kernel $v(\mathcal{P}) = \mathrm{ch}([\mathcal{P}]) \cdot \sqrt{\mathrm{td}(X \times Y)} \in H^*(X \times Y, \mathbb{Q})$. Since the characteristic classes ch and td takes values in even pieces of $H^*(-,\mathbb{Q})$, it follows that

$$\Phi_{v(\mathcal{P})}^{H,X\to Y}(H^{\text{even}}(X,\mathbb{Q})) \subset H^{\text{even}}(Y,\mathbb{Q}) \quad \text{and} \quad \Phi_{v(\mathcal{P})}^{H,X\to Y}(H^{\text{odd}}(X,\mathbb{Q})) \subset H^{\text{odd}}(Y,\mathbb{Q}).$$

Actually the singular cohomology theory $H^*(X,\mathbb{Q})$ is not the right target for the Mukai vector to consider. A. Căldăraru argued that it is the Hochschild cohomology theory one need to consider for studying the map $\Phi_{v(\mathcal{P})}^{H,X\to Y}$ to get the graded and multiplicative structure of the corresponding source and target of $\Phi_{v(\mathcal{P})}^{H,X\to Y}$ be preserved. We shall discuss it later.

Remark 9.3.19. In general, given an exact equivalence of categories $F : D^b(X) \longrightarrow D^b(Y)$, we don't know how to associate a cohomological integral transform $\widetilde{F} : H^*(X,\mathbb{Q}) \longrightarrow H^*(Y,\mathbb{Q})$ with F without using the existence of kernel $\mathcal{P}$ of F (coming from the Orlov's representability theorem 9.1.28).

Proposition 9.3.20. *Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ and $\Phi_{\mathcal{Q}} : D^b(Y) \to D^b(Z)$ be two integral functors with kernels $\mathcal{P} \in D^b(X \times Y)$ and $\mathcal{Q} \in D^b(Y \times Z)$, respectively. Let $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$ be their composite integral functor with $\mathcal{R} = \mathcal{Q} * \mathcal{P} \in D^b(X \times Z)$. Then the induced cohomological integral transforms $\Phi_{v(\mathcal{P})}^{H,\,X\to Y}$, $\Phi_{v(\mathcal{Q})}^{H,\,Y\to Z}$ and $\Phi_{v(\mathcal{R})}^{H,\,X\to Z}$ satisfies $\Phi_{v(\mathcal{Q})}^{H,\,Y\to Z} \circ \Phi_{v(\mathcal{P})}^{H,\,X\to Y} = \Phi_{v(\mathcal{R})}^{H,\,X\to Z}$.*

Proof. Similar to proof of Proposition 9.1.21. □

Remark 9.3.21. The analogous statement for K-theoretic integral transforms follows from Proposition 9.1.21 and the fact that the map $D^b(X) \to K_0(X)$, given by $E^\bullet \longmapsto [E^\bullet] := \sum_j (-1)^j [E^j]$, is surjective. However, the map $K_0(X) \to H^*(X, \mathbb{Q})$ is not surjective, in general. In fact, the image of the Mukai vector map $v : K_0(X) \to H^*(X, \mathbb{Q})$ could be very small. Nevertheless, surprisingly we have the following.

Proposition 9.3.22. *If $\mathcal{P} \in D^b(X \times Y)$ defines an equivalence of categories*

$$\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y),$$

then the induced cohomological integral transform

$$\Phi_{v(\mathcal{P})}^{H,\, X\to Y} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}) \tag{9.3.23}$$

is an isomorphism of $\mathbb{Q}$-vector spaces.

Proof. Since $\Phi_{\mathcal{P}}^{X\to Y}$ is an exact equivalence, we have

$$\Phi_{\mathcal{P}_R}^{Y\to X} \circ \Phi_{\mathcal{P}}^{X\to Y} \cong \mathrm{Id}_{D^b(X)} \cong \Phi_{\mathcal{O}_{\Delta_X}}^{X\to X} \quad \text{and} \quad \Phi_{\mathcal{P}}^{X\to Y} \circ \Phi_{\mathcal{P}_R}^{Y\to X} \cong \mathrm{Id}_{D^b(Y)} \cong \Phi_{\mathcal{O}_{\Delta_Y}}^{Y\to Y}.$$

Then by above Proposition 9.3.20, we have

$$\begin{aligned} \Phi_{v(\mathcal{P}_R)}^{H,\, Y\to X} \circ \Phi_{v(\mathcal{P})}^{H,\, X\to Y} &= \Phi_{v(\mathcal{O}_{\Delta_X})}^{H,\, X\to X}, \quad \text{and} \\ \Phi_{v(\mathcal{P})}^{H,\, X\to Y} \circ \Phi_{v(\mathcal{P}_R)}^{H,\, Y\to X} &= \Phi_{v(\mathcal{O}_{\Delta_Y})}^{H,\, Y\to Y}. \end{aligned}$$

Therefore, it is enough to show that $\Phi_{v(\mathcal{O}_{\Delta_X})}^{H,\, X\to X} = \mathrm{Id}_{H^*(X,\mathbb{Q})}$ for any smooth projective $\mathbb{C}$-variety X. For this, using Grothendieck-Riemann-Roch theorem 9.3.8 for the diagonal embedding $\iota : X \xrightarrow{\cong} \Delta_X \hookrightarrow X \times X$ and $[\mathcal{O}_X] \in K_0(X)$, we have

$$\mathrm{ch}(\mathcal{O}_{\Delta_X}) \cdot \mathrm{td}(X \times X) = \iota_*\big(\mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{td}(X)\big) = \iota_* \, \mathrm{td}(X). \tag{9.3.24}$$

Since $\iota^* \sqrt{\mathrm{td}(X \times X)} = \mathrm{td}(X)$, dividing both sides of (9.3.24) by $\sqrt{\mathrm{td}(X \times X)}$, we have

$$v(\mathcal{O}_{\Delta_X}) = \mathrm{ch}(\mathcal{O}_{\Delta_X}) \cdot \sqrt{\mathrm{td}(X \times X)} = \iota_*(1). \tag{9.3.25}$$

Therefore, for any $\beta \in H^*(X, \mathbb{Q})$, we have

$$\begin{aligned} \Phi_{v(\mathcal{O}_{\Delta_X})}^{H,\, X\to X}(\beta) &= p_{2*}\big(v(\mathcal{O}_{\Delta_X}) \cdot p_1^*(\beta)\big) \\ &= p_{2*}\big(\iota_*(1) \cdot p_1^*\beta\big) \\ &= p_{2*}\big(\iota_*(1 \cdot \iota^* p_1^*\beta)\big) = \beta. \end{aligned}$$

This completes the proof. □

We shall show that the above $\mathbb{Q}$-linear isomorphism $\Phi_{v(\mathcal{P})}^{H, X\to Y}$ in (9.3.23) is, in fact, an isometry with respect to a natural quadratic form on $H^*(-, \mathbb{Q})$, known as the

Mukai pairing:

$$\langle v, v' \rangle_X := \int_X \exp\left(\frac{1}{2} c_1(X)\right) \cdot (v^\vee \cdot v'), \tag{9.3.26}$$

where for $v = \sum_j v_j \in \bigoplus_j H^j(X, \mathbb{C})$ we define its dual $v^\vee := \sum_j \left(\sqrt{-1}\right)^j v_j \in \bigoplus_j H^j(X, \mathbb{C})$, and $c_1(X) := c_1(TX)$. More precisely, we shall show that

$$\langle \Phi^{H, X \to Y}_{v(\mathcal{P})}(\alpha),\, \Phi^{H, X \to Y}_{v(\mathcal{P})}(\beta) \rangle_Y = \langle \alpha,\, \beta \rangle_X\,, \;\; \forall\, \alpha,\, \beta \,\in\, H^*(X,\, \mathbb{Q})\,. \tag{9.3.27}$$

Definition 9.3.28. For $E^\bullet, F^\bullet \in D^b(X)$, we define

$$\chi(E^\bullet, F^\bullet) := \sum_j (-1)^j \dim_k \mathrm{Ext}^j(E^\bullet, F^\bullet)\,.$$

Let $\Phi^{X \to Y}_{\mathcal{P}} : D^b(X) \to D^b(Y)$ be an equivalence of categories. Then the naturally induced isomorphism of k-vector spaces

$$\mathrm{Ext}^i_X(E^\bullet, F^\bullet) \cong \mathrm{Ext}^i_Y(\Phi^{X \to Y}_{\mathcal{P}}(E^\bullet), \Phi^{X \to Y}_{\mathcal{P}}(F^\bullet)) \tag{9.3.29}$$

gives

$$\chi(E^\bullet, F^\bullet) = \chi(\Phi^{X \to Y}_{\mathcal{P}}(E^\bullet),\, \Phi^{X \to Y}_{\mathcal{P}}(F^\bullet)). \tag{9.3.30}$$

We shall see that both sides of (9.3.30) can be interpreted as natural bilinear pairing of Mukai vectors in $H^*(-, \mathbb{Q})$, known as the Mukai pairing.

Since X is a smooth projective k-variety, replacing $E^\bullet$ and $F^\bullet$ with bounded complexes of locally free coherent sheaves of $\mathcal{O}_X$-modules isomorphic to them in $D^b(X)$, one can show that

$$\chi(E^\bullet, F^\bullet) = \chi(X, (E^\bullet)^\vee \otimes F^\bullet). \tag{9.3.31}$$

Then by Hirzebruch-Riemann-Roch formula, we have

$$\begin{aligned} \chi(E^\bullet, F^\bullet) = \chi(X, (E^\bullet)^\vee \otimes F^\bullet) &= \int_X \mathrm{ch}(E^{\bullet\vee}) \cdot \mathrm{ch}(F^\bullet) \cdot \mathrm{td}(X) \\ &= \int_X \left(\mathrm{ch}(E^{\bullet\vee}) \cdot \sqrt{\mathrm{td}(X)}\right) \cdot \left(\mathrm{ch}(F^\bullet) \cdot \sqrt{\mathrm{td}(X)}\right) \\ &= \int_X v(E^{\bullet\vee}) \cdot v(F^\bullet)\,. \end{aligned} \tag{9.3.32}$$

Now we need to determine $v(E^{\bullet\vee})$ in terms of $v(E^\bullet)$. For this, we need the following notion of dual vector.

Definition 9.3.33. For $\alpha = \sum_i \alpha_i \in \bigoplus_i H^{2i}(X, \mathbb{Q})$, we define its *dual* vector

$$\alpha^\vee := \sum_i (-1)^i \alpha_i \in \bigoplus_i H^{2i}(X, \mathbb{Q}).$$

Clearly, for any $\alpha = \sum_i \alpha_i, \beta = \sum_i \beta_i \in \bigoplus_i H^{2i}(X, \mathbb{Q})$, we have

$$\begin{aligned} & \big(\alpha + \beta\big)^\vee = \alpha^\vee + \beta^\vee, \quad \text{and} \\ & \alpha^\vee \cdot \beta^\vee = \sum_i \sum_j (-1)^{i+j} \alpha_i \cdot \beta_j = \big(\alpha \cdot \beta\big)^\vee. \end{aligned} \tag{9.3.34}$$

Lemma 9.3.35. *With the above notations, we have*

$$v(E^{\bullet\vee}) = \mathrm{ch}(E^{\bullet\vee}) \cdot \sqrt{\mathrm{td}(X)} = v(E^\bullet)^\vee \cdot \exp\left(\frac{1}{2} c_1(X)\right),$$

where $c_1(X) := c_1(TX)$.

Proof. Recall that, for any locally free coherent sheaf of $\mathcal{O}_X$-modules E on X, we have $c_i(E^\vee) = (-1)^i c_i(E)$ and $\mathrm{ch}_i(E^\vee) = (-1)^i \mathrm{ch}_i(E)$, for all $i \geq 0$. Therefore, we have

$$\begin{aligned} \mathrm{ch}(E^\bullet)^\vee = \left(\sum_i (-1)^i \mathrm{ch}(E^i)\right)^\vee &= \left(\sum_{j\geq 0} \big(\sum_i (-1)^i \mathrm{ch}_j(E^i)\big)\right)^\vee \\ &= \sum_{j \geq 0} (-1)^j \big(\sum_i (-1)^i \mathrm{ch}_j(E^i)\big) \\ &= \sum_i (-1)^i \sum_{j\geq 0} (-1)^j \mathrm{ch}_j(E^i) \\ &= \sum_i (-1)^i \sum_{j \geq 0} \mathrm{ch}_j((E^i)^\vee) = \mathrm{ch}(E^{\bullet\vee}). \end{aligned}$$

Then we have,

$$\begin{aligned} & v(E^\bullet)^\vee = \big(\mathrm{ch}(E^\bullet) \cdot \sqrt{\mathrm{td}(X)}\big)^\vee = \mathrm{ch}(E^\bullet)^\vee \cdot \sqrt{\mathrm{td}(X)}^\vee \\ \Rightarrow \ & v(E^\bullet)^\vee \frac{\sqrt{\mathrm{td}(X)}}{\sqrt{\mathrm{td}(X)}^\vee} = \mathrm{ch}(E^{\bullet\vee}) \sqrt{\mathrm{td}(X)} = v(E^{\bullet\vee}). \end{aligned}$$

Therefore, it suffices to show that $\sqrt{\mathrm{td}(X)} = \sqrt{\mathrm{td}(X)}^\vee \cdot \exp(c_1(X)/2)$ or, equivalently, $\mathrm{td}(X) = \mathrm{td}(X)^\vee \cdot \exp(c_1(X))$. Since Todd class is multiplicative, using splitting principal, we can write it as $\mathrm{td}(X) = \prod_i \frac{\gamma_i}{1 - \exp(-\gamma_i)}$. Using multiplicative property of dual as in (9.3.34), we have $\mathrm{td}(X)^\vee = \prod_i \frac{(-\gamma_i)}{1-\exp(\gamma_i)}$. Then using additivity of c_1 we have,

$$\frac{\mathrm{td}(X)}{\mathrm{td}(X)^\vee} = \prod_i \frac{\gamma_i}{1 - \exp(-\gamma_i)} \prod_j \frac{1 - \exp(\gamma_j)}{-\gamma_j} = \prod_i \exp(\gamma_i) = \exp(c_1(TX)).$$

Hence the lemma follows. □

With the above observations in mind, it is natural to extend the Definition 9.3.33 to the following.

Definition 9.3.36. For $\alpha = \sum_j \alpha_j \in \bigoplus_j H^j(X, \mathbb{C})$, we define its dual

$$\alpha^\vee := \sum_j (\sqrt{-1})^j \alpha_j \in \bigoplus_j H^j(X, \mathbb{C}).$$

Clearly, the above definition of dual vector is compatible with Definition 9.3.33.

Definition 9.3.37. The *Mukai pairing* on $H^*(X, \mathbb{C})$ is a quadratic form defined by

$$\langle \alpha, \beta \rangle_X := \int_X \exp\left(\frac{1}{2} c_1(X)\right) \cdot (\alpha^\vee \cdot \beta). \tag{9.3.38}$$

Here $(\alpha^\vee \cdot \beta)$ is the intersection product. Now it follows from (9.3.32) and Lemma 9.3.35 that

$$\begin{aligned} \chi(E^\bullet, F^\bullet) &= \int_X v(E^{\bullet\vee}) \cdot v(F^\bullet) \\ &= \int_X v(E^\bullet)^\vee \cdot \exp\left(\frac{1}{2} c_1(X)\right) \cdot v(F^\bullet) = \langle v(E^\bullet), v(F^\bullet) \rangle . \end{aligned} \tag{9.3.39}$$

Note that, the Mukai pairing is non-degenerate $\mathbb{C}$-bilinear pairing on $H^*(X, \mathbb{C})$.

Remark 9.3.40. (i) It is clear from the above Definition 9.3.37 that if $c_1(X) = 0$, then $\langle \, , \, \rangle_X$ is symmetric if $\dim(X)$ is even, and alternating if $\dim(X)$ is odd.

(ii) If $p_Y : X \times Y \to Y$ is the projection morphism, then for any $\alpha \in H^*(X \times Y, \mathbb{C})$ we have,

$$\left(p_{Y*}(\alpha)\right)^\vee = (-1)^{\dim(X)} p_{Y*}(\alpha^\vee).$$

Proposition 9.3.41 (Andrei Căldăraru)**.** *Let $\Phi_{\mathcal{P}}^{X \to Y} : D^b(X) \to D^b(Y)$ be an equivalence of categories. Then the induced cohomological Fourier-Mukai transform*

$$\Phi_{v(\mathcal{P})}^{H, X \to Y} : H^*(X, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q})$$

is isometric with respect to the Mukai pairing; i.e., for all $\alpha, \beta \in H^*(X, \mathbb{Q})$, we have

$$\langle \Phi_{v(\mathcal{P})}^{H, X \to Y}(\alpha), \Phi_{v(\mathcal{P})}^{H, X \to Y}(\beta) \rangle_Y = \langle \alpha, \beta \rangle_X.$$

Proof. Since $\Phi_{\mathcal{P}}^{X \to Y}$ is an exact equivalence of categories, $\dim(X) = \dim(Y) = n$, say. Since $\Phi_{v(\mathcal{P})}^{H, X \to Y}$ is a $\mathbb{Q}$-linear isomorphism (see Proposition 9.3.22), it is enough to show that,

$$\langle \Phi_{v(\mathcal{P})}^{H, X \to Y}(\alpha), \beta \rangle_Y = \langle \alpha, \left(\Phi_{v(\mathcal{P})}^{H, X \to Y}\right)^{-1}(\beta) \rangle_X , \tag{9.3.42}$$

for all $\alpha \in H^*(X, \mathbb{Q})$ and $\beta \in H^*(Y, \mathbb{Q})$. Since $\Phi_{\mathcal{P}}^{X \to Y}$ is an equivalence of categories, by Proposition 9.1.17 $\Phi_{\mathcal{P}_L}^{X \to Y}$ is a quasi-inverse of $\Phi_{\mathcal{P}}^{X \to Y}$, where $\mathcal{P}_L = \mathcal{P}^\vee \otimes p_Y^*(\omega_Y)[n]$

and $n = \dim(X) = \dim(Y)$. Then by Proposition 9.3.20, $\left(\Phi_{v(\mathcal{P})}^{H,\,X\to Y}\right)^{-1} = \Phi_{v(\mathcal{P}_L)}^{H,\,X\to Y}$. Note that,

$$\begin{aligned} v(\mathcal{P}_L) = v(\mathcal{P}^\vee \otimes p_Y^*\omega_Y[n]) &= (-1)^n v(\mathcal{P}^\vee)\cdot \mathrm{ch}(p_Y^*\omega_Y) \\ &= (-1)^n v(\mathcal{P}^\vee)\cdot p_Y^* \exp(-c_1(Y)). \end{aligned} \tag{9.3.43}$$

Now using multiplicative property of dual vectors (see (9.3.34)), Remark 9.3.40 (ii), Lemma 9.3.35 and equation (9.3.43), we have

$$\begin{aligned} &\langle \Phi_{v(\mathcal{P})}^{H,X\to Y}(\alpha),\, \beta\rangle_Y \\ &= \int_Y \exp(c_1(Y)/2)\cdot \big(p_{Y*}\big(v(\mathcal{P})\cdot p_X^*\alpha\big)\big)^\vee \cdot \beta \\ &= (-1)^n \int_Y \exp(c_1(Y)/2)\cdot p_{Y*}\left(\big(v(\mathcal{P})\cdot p_X^*\alpha\big)^\vee\right)\cdot\beta \\ &= (-1)^n \int_{X\times Y} p_Y^*(\exp(c_1(Y)/2))\cdot v(\mathcal{P})^\vee \cdot \big(p_X^*\alpha\big)^\vee p_Y^*\beta \\ &= (-1)^n \int_{X\times Y} p_Y^*(\exp(c_1(Y)/2))\cdot v(\mathcal{P}^\vee)\cdot \exp(-c_1(X\times Y)/2)\cdot \big(p_X^*\alpha\big)^\vee p_Y^*\beta \\ &= \int_{X\times Y} p_X^*\exp(c_1(X)/2)\cdot p_X^*\alpha^\vee \cdot v(\mathcal{P}_L)\cdot p_Y^*\beta \\ &= \int_X \exp(c_1(X)/2)\cdot \alpha^\vee p_{X*}\big(v(\mathcal{P}_L)\cdot p_Y^*\beta\big) \\ &= \langle \alpha,\, \Phi_{v(\mathcal{P}_L)}^{H,Y\to X}(\beta)\rangle_X \end{aligned}$$

This completes the proof. □

9.4. **Derived Torelli theorem for complex elliptic curves.** Let C be a complex elliptic curve. We would like to know if we can reconstruct C from its bounded derived category $D^b(C)$. Since the canonical line bundle ω_C is trivial, Bondal-Orlov's reconstruction theorem 8.1.1 does not applies here. However, by analyzing Hodge structure under cohomological Fourier-Mukai transforms, we can recover C from $D^b(C)$ as follow.

Let X be a connected smooth projective variety over $\mathbb{C}$. By Hodge theory, for each $i = 0, 1, \ldots, 2\dim_{\mathbb{C}}(X)$, we have a direct sum decomposition

$$H^i(X,\mathbb{C}) = H^i(X,\mathbb{Q})\otimes\mathbb{C} = \bigoplus_{p+q=i} H^{p,q}(X), \tag{9.4.1}$$

with $\overline{H^{p,q}(X)} = H^{q,p}(X)$. Moreover, $H^{p,q}(X) \cong H^q(X,\Omega_X^p)$. Since the Chern classes, and hence all characteristic classes, are algebraic (i.e., of the type (p,p)), the Mukai vector map factors through the algebraic part of the cohomology

$$v(-) := \mathrm{ch}(-)\cdot\sqrt{\mathrm{td}(X)} : K_0(X) \longrightarrow \bigoplus_p H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q}). \tag{9.4.2}$$

Then we have the following.

Proposition 9.4.3. *Let X and Y be connected smooth complex projective varieties. Let $\mathcal{P} \in D^b(X \times Y)$. If $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ is an equivalence of categories, then the induced cohomological Fourier-Mukai transform $\Phi_{v(\mathcal{P})}^{H,\, X\to Y} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ gives isomorphisms*

$$\bigoplus_{p-q=i} H^{p,q}(X) \xrightarrow{\simeq} \bigoplus_{p-q=i} H^{p,q}(Y),\ \ \forall\, i = 0, \pm 1, \ldots, \pm \dim_{\mathbb{C}}(X). \tag{9.4.4}$$

Proof. Since $\Phi_{\mathcal{P}}^{X\to Y}$ is an equivalence of categories, the induced $\mathbb{Q}$-linear homomorphism $\Phi_{v(\mathcal{P})}^{H,X\to Y} : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ of rational cohomologies is an isomorphism by Proposition 9.3.22. Therefore, it is enough to show that its $\mathbb{C}$-linear extension (obtained by applying $(-) \otimes_{\mathbb{Q}} \mathbb{C}$)

$$\widetilde{\Phi}_{v(\mathcal{P})}^{H,X\to Y} : H^*(X,\mathbb{C}) \longrightarrow H^*(Y,\mathbb{C})$$

satisfies

$$\widetilde{\Phi}_{v(\mathcal{P})}^{H,X\to Y}(H^{p,q}(X)) \subseteq \bigoplus_{r-s=p-q} H^{r,s}(Y).$$

For this, let

$$\sum \alpha^{p',q'} \boxtimes \beta^{r,s}, \tag{9.4.5}$$

with $\alpha^{p',q'} \in H^{p',q'}(X)$ and $\beta^{r,s} \in H^{r,s}(Y)$, be the Künneth decomposition of $v(\mathcal{P}) = \mathrm{ch}(\mathcal{P}) \cdot \sqrt{\mathrm{td}(X \times Y)}$. Since the cohomology class $v(\mathcal{P})$ is algebraic (i.e., of type (t,t)), only terms with $p' + r = q' + s$ contributes in (9.4.5).

Let $\alpha \in H^{p,q}(X)$ be such that $\Phi_{v(\mathcal{P})}^{H,X\to Y}(\alpha) = p_{Y*}\big(v(\mathcal{P}) \cdot p_X^*\alpha\big) \in H^{r,s}(Y)$. We need to show that (r,s) satisfies $r - s = p - q$. Note that, by above assumption, only terms in (9.4.5) with

$$(p,q) + (p',q') = (\dim(X), \dim(X))$$

contribute to $\Phi_{v(\mathcal{P})}^{H,X\to Y}(\alpha)$. Hence $p - q = q' - p' = r - s$. In fact, we have

$$\Phi_{v(\mathcal{P})}^{H,X\to Y}(\alpha) = \sum \left(\int_X \alpha \wedge \alpha^{p',q'}\right) \beta^{r,s} \in \bigoplus_{r-s=p-q} H^{r,s}(Y)\,.$$

This completes the proof. $\square$

Theorem 9.4.6. *Let C be an elliptic curve over $\mathbb{C}$, and let C' be any connected smooth projective variety over $\mathbb{C}$. If there is an exact equivalence of categories $F : D^b(C) \to D^b(C')$, then $C \cong C'$ as complex varieties.*

Proof. Since F is an exact equivalence, we have $\dim(C') = \dim(C) = 1$. Denote by $g(C')$ the genus of the curve C'. If $g(C') \neq 1$, then $\omega_{C'}$ is either ample or anti-ample, and hence $C \cong C'$ by Bondal-Orlov's reconstruction theorem 8.1.1, which is not possible since C is an elliptic curve. Therefore, $g(C') = 1$, i.e., C' is an elliptic curve.

Now by Orlov's representability theorem 9.1.28, there is an object $\mathcal{P} \in D^b(C \times C')$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{C \to C'}$. Since $v(\mathcal{P}) = \mathrm{ch}(\mathcal{P}) \cdot \sqrt{\mathrm{td}(C \times C')}$ is algebraic (i.e., of type (t,t)), the induced cohomological Fourier-Mukai transform

$$\Phi_{v(\mathcal{P})}^{H,\, C \to C'} : H^*(C, \mathbb{Q}) \longrightarrow H^*(C', \mathbb{Q}) \tag{9.4.7}$$

is a $\mathbb{Q}$-linear isomorphism satisfying

$$\Phi_{v(\mathcal{P})}^{H,\, C \to C'}\big(H^1(C, \mathbb{Q})\big) = H^1(C', \mathbb{Q})\,, \quad \text{and}$$

$$\Phi_{v(\mathcal{P})}^{H,\, C \to C'}\big(H^0(C, \mathbb{Q}) \oplus H^2(C, \mathbb{Q})\big) = H^0(C', \mathbb{Q}) \oplus H^2(C', \mathbb{Q}). \tag{9.4.8}$$

Recall that, the weight 1 Hodge structure determines the elliptic curve completely. More precisely, we have $C \cong H^{1,0}(C)^*/H_1(C, \mathbb{Z}) \cong H^{0,1}(C)/H^1(C, \mathbb{Z})$.

Therefore, it suffices to show that the induced cohomological Fourier-Mukai transform in (9.4.7) descends to

$$\Phi_{v(\mathcal{P})}^{H,\, C \to C'} : H^*(C, \mathbb{Z}) \longrightarrow H^*(C', \mathbb{Z})\,. \tag{9.4.9}$$

Since C and C' are elliptic curves over $\mathbb{C}$, we have $\mathrm{td}(C \times C') = 1$ and $\mathrm{ch}(\mathcal{P}) = r + c_1(\mathcal{P}) + \frac{1}{2}(c_1^2(\mathcal{P}) - 2c_2(\mathcal{P}))$. Note that, the degree 4 term $\frac{1}{2}(c_1^2(\mathcal{P}) - 2c_2(\mathcal{P}))$ could a priori be non-integral. However, this term does not contribute to $H^1(C, \mathbb{Q}) \to H^1(C', \mathbb{Q})$. Hence the result follows. $\square$

Remark 9.4.10. It turns out that, $\mathrm{ch}_2(\mathcal{P}) = \frac{1}{2}(c_1^2(\mathcal{P}) - 2c_2(\mathcal{P}))$ is also integral.

9.5. **Canonical ring and Kodaira dimension.** In this subsection, we briefly recall the notions of canonical ring and Kodaira dimension of a smooth projective k-variety, and use Fourier-Mukai functor to show derived equivalence implies isomorphism of canonical rings, and hence equality of Kodaira dimensions.

Definition 9.5.1. Let X be a smooth projective k-variety and L a line bundle on X. The *Kodaira dimension of* L on (X) is the integer $\mathrm{kod}(X, L) := m$ such that the function

$$\mathbb{Z} \to \mathbb{Z}, \;\; \ell \mapsto h^0(L^\ell) := \dim_k H^0(X, L^\ell) \tag{9.5.2}$$

grows like a polynomial of degree m, for $\ell \gg 0$. If $h^0(L^\ell) = 0$, for all $\ell > 0$, we define $\mathrm{kod}(X, L) = -\infty$. The integer $\mathrm{kod}(X) := \mathrm{kod}(X, \omega_X)$ is called the it Kodaira dimension of X.

For a line bundle L on X, the linear system $|L|$ defines a rational morphism $\varphi_L : X \dashrightarrow \mathbb{P}_k^{h^0(L)-1}$. The associated graded k-algebra $R(X, L) := \bigoplus\limits_{i \geq 0} H^0(X, L^i)$ is called the canonical ring of L. If $\operatorname{kod}(X, L) \geq 0$, it turns out that

$$\begin{aligned}\operatorname{kod}(X, L) &= \max\{\dim(\operatorname{Im}(\varphi_{L^i})) : i \geq 0\} \\ &= \operatorname{trdeg}_k Q(R(X, L)) - 1\,,\end{aligned}$$

where $Q(R(X, L))$ is the field of fractions of $R(X, L)$. Moreover, we have $\operatorname{kod}(X, L) \leq \operatorname{kod}(X)$, for all $L \in \operatorname{Pic}(X)$. It is a well-known fact that Kodaira dimension is birational invariant, i.e., if X and Y are two birational smooth projective k-varieties, then $\operatorname{kod}(X) = \operatorname{kod}(Y)$.

For a smooth projective k-variety X, we define $R(X) := R(X, \omega_X) = \bigoplus_{i \geq 0} H^0(X, \omega_X^i)$.

Proposition 9.5.3 (Orlov). *Let X and Y be smooth projective k-varieties. If there is an exact equivalence $F : D^b(X) \to D^b(Y)$, then $R(X) \cong R(Y)$ as graded k-algebras. In particular,* $\operatorname{kod}(X) = \operatorname{kod}(Y)$.

Remark 9.5.4. Since we are not assuming ω_X is ample or anti-ample, we cannot conclude if $\mathcal{O}_X$ is an invertible object in $D^b(X)$. Therefore, one cannot apply the arguments given in Step 1 and 2 of the proof of Bondal-Orlov's reconstruction Theorem 8.1.1 to obtain the above result! Here we need Orlov's representability theorem and Fourier-Mukai functors to prove this result.

To prove Proposition 9.5.3, we need the following result, which is easy to check.

Proposition 9.5.5. *Let X_1, X_2, Y_1 and Y_2 be smooth projective k-schemes. For each $i = 1, 2$, consider the objects $\mathcal{P}_i \in D^b(X_i \times Y_i)$, and denote by $\mathcal{P}_1 \boxtimes \mathcal{P}_2 \in D^b((X_1 \times Y_1) \times (X_2 \times Y_2))$ their external derived tensor product.*

(i) Consider the induced integral functors $\Phi_{\mathcal{P}_i} : D^b(X_i) \to D^b(Y_i)$, for $i = 1, 2$, and $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2} : D^b(X_1 \times X_2) \longrightarrow D^b(Y_1 \times Y_2)$. Then there is an isomorphism

$$\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(E_1^\bullet \boxtimes E_2^\bullet) \cong \Phi_{\mathcal{P}_1}(E_1^\bullet) \boxtimes \Phi_{\mathcal{P}_2}(E_2^\bullet)\,, \tag{9.5.6}$$

which is functorial in $E_i^\bullet \in D^b(X_i)$, for all $i = 1, 2$.

(ii) If $\Phi_{\mathcal{P}_i} : D^b(X_i) \longrightarrow D^b(Y_i)$ is an equivalence of categories, for $i = 1, 2$, then

$$\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2} : D^b(X_1 \times X_2) \longrightarrow D^b(Y_1 \times Y_2)$$

is also an equivalence of categories.

(iii) For $\mathcal{R} \in D^b(X_1 \times X_2)$, let $\mathcal{S} = \Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{R}) \in D^b(Y_1 \times Y_2)$. Then the following diagram commutes.

$$\begin{tikzcd} D^b(X_1) \arrow[d, "\Phi_{\mathcal{R}}"'] & D^b(Y_1) \arrow[l, "\Phi_{\mathcal{P}_1}^{Y_1 \to X_1}"'] \arrow[d, "\Phi_{\mathcal{S}}"] \\ D^b(X_2) \arrow[r, "\Phi_{\mathcal{P}_2}"] & D^b(Y_2) \end{tikzcd} \tag{9.5.7}$$

(It should be noted that, $\mathcal{P}_1$ is used in the above diagram to define integral functor in the opposite direction).

We use the above Proposition with $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$, $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}_2 = \mathcal{Q}$ in the proof of Proposition 9.5.3 below.

Proof of Proposition 9.5.3. By Orlov's representability theorem 9.1.28, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{X \to Y}$. In particular, the left adjoint and the right adjoint functors of $\Phi_{\mathcal{P}}^{X \to Y}$,

$$\Phi_{\mathcal{P}_L}^{Y \to X} : D^b(Y) \to D^b(X) \quad \text{and} \quad \Phi_{\mathcal{P}_R}^{Y \to X} : D^b(Y) \to D^b(X), \tag{9.5.8}$$

respectively, are isomorphic, and hence by uniqueness (up to isomorphism) of kernel in Orlov's representability theorem 9.1.28, we have

$$\mathcal{P}^\vee \otimes p_Y^* \omega_Y[n] =: \mathcal{P}_L \cong \mathcal{P}_R := \mathcal{P}^\vee \otimes p_X^* \omega_X[n], \tag{9.5.9}$$

where $n = \dim(X) = \dim(Y)$ (c.f., Proposition 8.2.1).

Now we show that, the functor (note the change of direction from (9.5.8))

$$\Phi_{\mathcal{P}_R}^{X \to Y} : D^b(X) \to D^b(Y) \tag{9.5.10}$$

is also an equivalence. Since the composite functor

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{P}_R}} D^b(X) \tag{9.5.11}$$

is isomorphic to the identity functor on $D^b(X)$, again by uniqueness (up to isomorphism) of kernel, we have $\mathcal{P} * \mathcal{P}_R := p_{13*}\big(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}_R\big) \cong \mathcal{O}_{\Delta_X}$.

$$\begin{tikzcd}
 & Y \times X \arrow[dl, "\simeq", "\sigma_{12}"'] & & \\
X \times Y & X \times Y \times X \arrow[l, "p_{12}"'] \arrow[dl, "p_{23}"'] \arrow[d, "p_{32}"] \arrow[dr, "p_{13}"'] & X \times Y \times X \arrow[l, "\tau_{13}", "\simeq"'] \arrow[ul, "p_{23}"'] \arrow[dr, "p_{13}"] & \\
Y \times X \arrow[r, "\sigma_{12}", "\simeq"'] & X \times Y & X \times X & X \times X \arrow[l, "\tau_{12}"', "\simeq"]
\end{tikzcd}$$

Applying the automorphism $\tau_{12} : X \times X \to X \times X$, which interchanges two factors, we have

$$\begin{aligned}
\mathcal{O}_{\Delta_X} \cong \tau_{12}^*\mathcal{O}_{\Delta_X} &\cong \tau_{12}^* p_{13*}\big(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}_R\big) \\
&\cong p_{13*}\tau_{13}^*\big(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}_R\big) \\
&\cong p_{13*}\big(p_{32}^*\mathcal{P} \otimes p_{21}^*\mathcal{P}_R\big) \\
&\cong p_{13*}\big(p_{12}^*\mathcal{P}_R \otimes p_{23}^*\mathcal{P}\big) .
\end{aligned} \tag{9.5.12}$$

Therefore, the composite functor

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}_R}^{X \to Y}} D^b(Y) \xrightarrow{\Phi_{\mathcal{P}}^{Y \to X}} D^b(X) \tag{9.5.13}$$

is also isomorphic to the identity functor on $D^b(X)$. Since $\Phi_{\mathcal{P}}^{Y\to X}$ is adjoint to $\Phi_{\mathcal{P}_R}^{X\to Y}$, we conclude that $\Phi_{\mathcal{P}_R}^{X\to Y}$ is fully faithful.

Now interchanging the role of $\mathcal{P}$ and $\mathcal{Q} := \mathcal{P}_R \cong \mathcal{P}_L$, and using the fact that $\Phi_{\mathcal{Q}}^{Y\to X}$ is the quasi-inverse of $\Phi_{\mathcal{P}}^{X\to Y}$ and hence $\Phi_{\mathcal{Q}}^{Y\to X}$ is fully faithful, the same argument shows that the composite functor

$$D^b(Y) \xrightarrow{\Phi_{\mathcal{P}}} D^b(X) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(Y) \tag{9.5.14}$$

is also isomorphic to the identity functor on $D^b(Y)$. Therefore, $\Phi_{\mathcal{Q}}^{X\to Y} : D^b(X) \to D^b(Y)$ is also an equivalence of categories.

By Proposition 9.5.5 the external derived tensor product $\mathcal{Q} \boxtimes \mathcal{P} \in D^b((Y \times X) \times (X \times Y)) \cong D^b\big((X \times X) \times (Y \times Y)\big)$ defines a Fourier-Mukai equivalence functor $\Phi_{\mathcal{Q}\boxtimes\mathcal{P}} : D^b(X \times X) \longrightarrow D^b(Y \times Y)$, and if we define

$$\mathcal{S} := \Phi_{\mathcal{Q}\boxtimes\mathcal{P}}(\iota_* \omega_X^i) \in D^b(Y \times Y)\,, \tag{9.5.15}$$

where $\iota : X \hookrightarrow \Delta_X \subset X \times X$ is the diagonal embedding, then $\Phi_{\mathcal{S}} : D^b(Y) \longrightarrow D^b(Y)$ is an equivalence of categories, which can be computed as the composite functor

$$D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(X) \xrightarrow{\Phi_{\iota_*\omega_X^i}} D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y)\,. \tag{9.5.16}$$

Since $\Phi_{\iota_*\omega_X^i}^{X\to X} \cong S_X^i[-in]$, where S_X is the Serre functor on X and $n = \dim_k(X)$, and since any equivalence of derived categories $D^b(X) \to D^b(Y)$ commutes with Serre functors S_X and S_Y, we conclude that

$$\begin{aligned} \Phi_{\mathcal{S}} &\cong \Phi_{\mathcal{P}} \circ S_X^i[-in] \circ \Phi_{\mathcal{Q}} \\ &\cong \Phi_{\mathcal{P}} \circ \Phi_{\mathcal{Q}} \circ S_Y^i[-in] \\ &\cong S_Y^i[-in] \cong \Phi_{\mathrm{j}_*\omega_Y^i}^{Y\to Y}\,, \end{aligned} \tag{9.5.17}$$

where $\mathrm{j} : Y \hookrightarrow \Delta_Y \subset Y \times Y$ is the diagonal embedding of Y. Then by uniqueness (up to isomorphism) of kernel in Orlov's representability theorem (Theorem 9.1.28), we conclude that $\mathcal{S} \cong \mathrm{j}_*\omega_Y^i$. Then from (9.5.15) we have

$$\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}(\iota_*\omega_X^i) = \mathrm{j}_*\omega_Y^i, \quad \forall\, i \in \mathbb{Z}\,. \tag{9.5.18}$$

Since $\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}$ is an exact equivalence of categories, we have

$$\mathrm{Hom}_{D^b(X\times X)}(\iota_*\omega_X^p, \iota_*\omega_X^q) \cong \mathrm{Hom}_{D^b(Y\times Y)}(\mathrm{j}_*\omega_Y^p, \mathrm{j}_*\omega_Y^q)\,, \quad \forall\, p, q \in \mathbb{Z}\,. \tag{9.5.19}$$

Putting $p = 0$ and q arbitrary, the above isomorphism gives a k-linear isomorphism

$$\begin{aligned} H^0(X, \omega_X^q) &\cong \mathrm{Hom}_{D^b(X\times X)}(\iota_*\mathcal{O}_X, \iota_*\omega_X^q) \\ &\cong \mathrm{Hom}_{D^b(Y\times Y)}(\mathrm{j}_*\mathcal{O}_Y, \mathrm{j}_*\omega_Y^q) \\ &\cong H^0(Y, \omega_Y^q)\,. \end{aligned} \tag{9.5.20}$$

As we have already seen in the Step 1 of the proof of the Theorem 8.1.1, the multiplicative structure of the canonical graded ring $R(X) := \bigoplus_{i\geq 0} H^0(X, \omega_X^i)$ can be given by compositions, and hence is compatible with any exact functor. Therefore, the k-linear isomorphisms in (9.5.20) gives an isomorphism of graded k-algebras

(9.5.21) $$R(X) := \bigoplus_{i\geq 0} H^0(X, \omega_X^i) \xrightarrow{\simeq} R(Y) := \bigoplus_{i\geq 0} H^0(Y, \omega_Y^i)\,.$$

This completes the proof. □

Remark 9.5.22. An immediate consequence of the above Proposition 9.5.3 is that $\mathrm{kod}(X) = \mathrm{kod}(Y)$. It is clear from the above proof that F gives rise to an isomorphism of graded anti-canonical k-algebras $R(X, \omega_X^\vee) \cong R(Y, \omega_Y^\vee)$, and hence $\mathrm{kod}(X, \omega_X^\vee) = \mathrm{kod}(Y, \omega_Y^\vee)$. The above Proposition 9.5.3 also provides an alternative proof of Bondal-Orlov's reconstruction theorem (Theorem 8.1.1) when both ω_X and ω_Y are ample or anti-ample.

9.6. **Derived Torelli theorem for K3 surface.** Let k be a field. A *k-variety* is a separated geometrically integral finite type k-scheme.

Definition 9.6.1. An *algebraic K3 surface* over k is a proper smooth k-variety X of dimension 2 such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

It is a well-known fact that, any smooth proper algebraic surface over an algebraically closed field is projective. Therefore, an algebraic K3 surface defined over an algebraically closed field is always projective.

In complex geometry, *a K3 surface is a compact complex manifold X of dimension 2 with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$*. This definition includes non-projective K3 surfaces. However, it turns out that any complex K3 surface is Kähler (not easy to see).

The complex analytic manifold X_{an} associated to a complex algebraic K3 surface X is again a K3 surface over $\mathbb{C}$. Moreover, the natural functor sending X to X_{an} (by Serre's GAGA principal) gives a full embedding of the category of complex algebraic K3 surfaces into the category of complex K3 surfaces.

Proposition 9.6.2. *Let X be an algebraic K3 surface over $\mathbb{C}$. Let Y be a smooth projective variety over $\mathbb{C}$. If there is an exact equivalence of categories $F : D^b(X) \to D^b(Y)$, then Y is an algebraic K3 surface.*

Proof. Hodge theory for a smooth complex projective surface S gives a direct sum decomposition (for each $i \geq 0$), $H^i(S, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(S)$, with

(9.6.3) $$\overline{H^{p,q}(S)} \cong H^{q,p}(S) \quad \text{and} \quad H^{p,q}(S) \cong H^q(S, \Omega_X^p)\,,$$

for all $p, q \geq 0$. Since F is an exact equivalence of categories, Y is a surface with $\omega_Y \cong \mathcal{O}_Y$ by Proposition 8.2.1. Since $\Omega_S^2 \cong \mathcal{O}_S$, for $S \in \{X, Y\}$, using (9.6.3), we have

$$h^{1,2}(X) = h^{2,1}(X) = h^{0,1}(X) \quad \text{and} \quad h^{1,2}(Y) = h^{2,1}(Y) = h^{0,1}(Y)\,. \tag{9.6.4}$$

By Orlov's representability theorem 9.1.28, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{X \to Y}$. For $i = -1$, the functor $F \cong \Phi_{\mathcal{P}}^{X \to Y}$ induces an isomorphism

$$H^{0,1}(X) \oplus H^{1,2}(X) \xrightarrow{\simeq} H^{0,1}(Y) \oplus H^{1,2}(Y)\,, \tag{9.6.5}$$

(see Proposition 9.4.3). Since X is a K3 surface, using (9.6.5) and (9.6.4) we have

$$\begin{aligned} h^1(Y, \mathcal{O}_Y) = h^{0,1}(Y) &= \frac{1}{2}(h^{0,1}(Y) + h^{1,2}(Y)) \\ &= \frac{1}{2}(h^{0,1}(X) + h^{1,2}(X)) = h^{0,1}(X) = h^1(X, \mathcal{O}_X) = 0. \end{aligned}$$

Hence the result follows. □

Let X be a smooth projective surface over $\mathbb{C}$. Let E be a locally free coherent sheaf of $\mathcal{O}_X$-modules on X. Then by Hirzebruch-Riemann-Roch formula (Corollary 9.3.9), we have

$$\begin{aligned} \chi(E) &= \int_X \mathrm{ch}(E) \cdot \mathrm{td}(X) \\ &= \frac{1}{12}\big(c_1^2(X) + c_2(X)\big) + \frac{1}{2}c_1(E)c_1(X) + \frac{1}{2}\big(c_1^2(E) - 2c_2(E)\big)\,. \end{aligned} \tag{9.6.6}$$

Putting $E = \mathcal{O}_X$ in (9.6.6) we get Max Noether's formula (c.f., [Har77, p. 433])

$$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X))\,. \tag{9.6.7}$$

Now assume that X is an algebraic K3 surface over $\mathbb{C}$. Then $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. Therefore, by Serre duality, $h^2(X, \mathcal{O}_X) = h^0(X, \Omega_X^2) = 0$, and hence $\chi(X, \mathcal{O}_X) = 2$. Since ω_X is trivial, $c_1(X) = 0$. Therefore, interpreting $c_2(X)$ as the *topological Euler number* $e(X) := \sum\limits_{i \geq 0} (-1)^i b_i(X)$, we have $e(X) = 24$. Since $H^{0,1}(X) \cong H^1(X, \mathcal{O}_X)$, Hodge decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ gives $b_1(X) = 0$, and hence by Poincaré duality, $b_3(X) = 0$. Then we have

$$b_0(X) = b_4(X) = 1, \;\; b_1(X) = b_3(X) = 0, \;\; \text{and} \;\; b_2(X) = 22\,. \tag{9.6.8}$$

Moreover, the Hodge diamond for a K3 surface looks like

$$
\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{2,1} & & h^{1,2} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\tag{9.6.9}
$$

If $L \in \mathrm{Pic}(X)$, then for $\alpha = c_1(L) \in H^2(X, \mathbb{Z})$, it follows from Hirzebruch-Riemann-Roch formula (9.6.6) that the self intersection number of α is even:

$$\alpha^2 = 2\chi(L) - 4 \in 2\mathbb{Z}\,. \tag{9.6.10}$$

Remark 9.6.11. More generally, for compact Kähler manifold with $c_1(X) = 0$, one can show that the self intersection pairing

$$(\ ,\ ) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z} \tag{9.6.12}$$

is even (i.e., $\alpha^2 := (\alpha, \alpha) \in 2\mathbb{Z}$, for all $\alpha \in H^2(X, \mathbb{Z})$). From topological point of view, the evenness of the intersection pairing also follows from the vanishing of second Stiefel-Whitney class.

Since for any smooth compact complex surface X, the the Hodge-Frölicher spectral sequence

$$H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C}) \tag{9.6.13}$$

degenerates at page E_1, we have an isomorphism of complex vector spaces

$$H^1(X, \mathbb{C}) \cong H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X^1)\,. \tag{9.6.14}$$

The (exponential) short exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0 \tag{9.6.15}$$

gives a long exact sequence of cohomologies

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \tag{9.6.16}$$

$$\longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^\times) \longrightarrow H^3(X, \mathbb{Z}) \longrightarrow 0\,, \tag{9.6.17}$$

which, for X a K3 surface, gives $H^1(X, \mathbb{Z}) = 0$ since $H^1(X, \mathcal{O}_X) = 0$. Therefore, $H^1(X, \mathbb{C}) = 0$, and we get $H^0(X, \Omega_X^1) = 0$ for free! In other words, a complex K3 surface has no non-zero global vector fields. Since $H^1(X, \mathbb{Z}) = 0$, by Poincaré duality $H^3(X, \mathbb{Z}) = 0$ up to torsion. Also $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$.

Remark 9.6.18. It is a non-trivial fact that any K3 surface is simply connected, and hence $H^2(X, \mathbb{Z})$ is torsion free.

The most interesting structure on cohomology of a K3 surface X is its weight 2 Hodge structure

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)\,. \tag{9.6.19}$$

Since $H^{2,0}(X) \cong H^0(X,\Omega_X^2) = H^0(X,\mathcal{O}_X) = \mathbb{C}$, we have $h^{0,2}(X) = h^{2,0} = 1$. Since $H^1(X,\mathcal{O}_X) = 0$, the first Chern class map

$$c_1 : \text{Pic}(X) \longrightarrow H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \tag{9.6.20}$$

is injective. Since $b_2(X) = 22$, this gives an upper bound for the Picard number $\rho(X)$ of X (i.e., the rank of Pic(X)):

$$\rho(X) := \text{rk}(\text{Pic}(X)) \leq 20\,. \tag{9.6.21}$$

Remark 9.6.22. Since the Hodge decomposition is orthogonal with respect to the intersection pairing, it is completely determined by the complex line $H^{2,0}(X) \subset H^2(X,\mathbb{C})$.

Definition 9.6.23. A *Hodge isometry* between two complex K3 surfaces X and Y is a group isomorphism

$$\varphi : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z})$$

such that

(i) (intersection product is preserved): $(\varphi(\alpha),\varphi(\beta)) = (\alpha,\beta),\ \forall\ \alpha,\beta \in H^2(X,\mathbb{Z})$, and

(ii) $\varphi(H^{2,0}(X)) \subseteq H^{2,0}(Y)$.

Since $H^{2,0}(-) \cong H^0(-,\omega_-)$, the second condition says that φ sends holomorphic global sections of ω_X to that of ω_Y.

One of the most important theorem for a K3 surface is the global Torelli theorem.

Theorem 9.6.24 (Global Torelli). *Let X and Y be two complex K3 surfaces. Then X and Y are isomorphic as complex varieties if and only if there is a Hodge isometry*

$$\varphi : H^2(X,\mathbb{Z}) \longrightarrow H^2(Y,\mathbb{Z}).$$

Moreover, if φ maps at least one Kähler class of X to a Kähler class of Y, then there is a unique isomorphism $f : X \xrightarrow{\simeq} Y$ such that $f_* = \varphi$.

A natural question to ask at this point if there is a cohomological criterion to decide equivalence of bounded derived categories of K3 surfaces? This is given by derived Torelli theorem for K3 surface. For this, we need some preliminary results.

Recall that, for $E^\bullet \in D^b(X)$, its Mukai vector

$$v(E^\bullet) := \text{ch}(E^\bullet) \cdot \sqrt{\text{td}(X)} \tag{9.6.25}$$

is algebraic, and hence lies in $\widetilde{H}^{1,1}(X) \subset \widetilde{H}(X,\mathbb{Z})$. Since $c_1(X) = 0$ for X a K3 surface, we have $\mathrm{td}(X) = 1+\frac{1}{2}c_1(X)+\frac{1}{12}\big(c_1^2(X)+c_2(X)\big) = c_2(X)/12 = \chi(X,\mathcal{O}_X) = 2$ (see Max Noether's formula (9.6.7)). Therefore, $\mathrm{td}(X) = (1,0,2)$ and hence $\sqrt{\mathrm{td}(X)} = (1,0,1)$. If we write $v(E^\bullet) = (v_0(E^\bullet), v_1(E^\bullet), v_2(E^\bullet)) \in H^*(X,\mathbb{Z})$, then we have

$$(v_0(E^\bullet), v_1(E^\bullet), v_2(E^\bullet)) = \big(\mathrm{rk}(E^\bullet), c_1(E^\bullet), \mathrm{rk}(E^\bullet) + \frac{1}{2}c_1^2(E^\bullet) - c_2(E^\bullet)\big)\,. \tag{9.6.26}$$

Since the intersection pairing on $H^2(X,\mathbb{Z})$ is even for X a K3 surface, the Mukai vector $v(E^\bullet)$ is an integral cohomology class.

Lemma 9.6.27 (Mukai). *Let X and Y be two complex K3 surfaces. Then for any $E^\bullet \in D^b(X\times Y)$, it Mukai vector $v(E^\bullet)$ is an integral cohomology class (i.e., $v(E^\bullet) \in H^*(X\times Y,\mathbb{Z})$).*

Proof. Recall that, $v(E^\bullet) := \mathrm{ch}(E^\bullet)\cdot\sqrt{\mathrm{td}(X\times Y)}$. If we write $\sqrt{\mathrm{td}(X)} = (r,c,s) \in H^*(X,\mathbb{Q})$, then $\mathrm{td}(X) = (r^2, 2rc, 2rs)$. Since $c_1(X) = 0$ and $c_2(X) = 24$, for X a K3 surface, we have $\mathrm{td}(X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}\big(c_1^2(X) + c_2(X)\big) = (1,0,2)$. Then $\sqrt{\mathrm{td}(X)} = (r,c,s) = (1,0,1)$, and we can compute $\sqrt{\mathrm{td}(X\times Y)}$ as

$$\sqrt{\mathrm{td}(X\times Y)} = \pi_X^*\sqrt{\mathrm{td}(X)}\cdot\pi_Y^*\sqrt{\mathrm{td}(Y)} = \pi_X^*(1,0,1)\cdot\pi_Y^*(1,0,1)\,, \tag{9.6.28}$$

where $\pi_X : X\times Y \to X$ and $\pi_Y : X\times Y\to Y$ are projection morphisms. Therefore, it is enough to show that $\mathrm{ch}(E^\bullet) \in H^*(X\times Y,\mathbb{Z})$, for all $E^\bullet \in D^b(X\times Y)$. Note that,

$$\mathrm{ch}(E^\bullet) = \big(\mathrm{rk}(E^\bullet), c_1(E^\bullet), \frac{1}{2}(c_1^2(E^\bullet) - 2c_2(E^\bullet)), \mathrm{ch}_3(E^\bullet), \mathrm{ch}_4(E^\bullet)\big)\,, \tag{9.6.29}$$

where $\mathrm{rk}(E^\bullet)$ and $c_1(E^\bullet)$ are certainly integral classes. The Künneth formula gives

$$H^2(X\times Y,\mathbb{Z}) \cong \bigoplus_{p+q=2} H^p(X,\mathbb{Z})\otimes H^q(Y,\mathbb{Z}) \cong H^2(X,\mathbb{Z})\oplus H^2(Y,\mathbb{Z})\,,$$

since $H^0(-,\mathbb{Z})\cong\mathbb{Z}$ and $H^1(-,\mathbb{Z}) = 0$ for complex K3 surfaces. Therefore, $c_1(E^\bullet) \in H^2(X\times Y,\mathbb{Z})$ can be written as

$$c_1(E^\bullet) = \pi_X^*\alpha \oplus \pi_Y^*\beta,$$

for some $\alpha \in H^2(X,\mathbb{Z})$ and $\beta \in H^2(Y,\mathbb{Z})$. Then

$$c_1^2(E^\bullet) = \pi_X^*\alpha^2 + 2\pi_X^*\alpha\cdot\pi_Y^*\beta + \pi_Y^*\beta^2,$$

which is even because the self intersection product on $H^2(-,\mathbb{Z})$ is even for K3 surfaces. Therefore, $\mathrm{ch}_1(E^\bullet) = \frac{1}{2}\big(c_1^2(E^\bullet) - 2c_2(E^\bullet)\big)$ is integral.

Now it remains to show that $\mathrm{ch}_3(E^\bullet)$ and $\mathrm{ch}_4(E^\bullet)$ are integral. Since Todd class is multiplicative, using Grothendieck-Riemann-Roch theorem (Theorem 9.3.8) for the projection map $\pi_Y : X\times Y\to Y$, we have

$$\mathrm{ch}(\pi_{Y!}E^\bullet) = \pi_{Y*}\big(\mathrm{ch}(E^\bullet)\cdot\pi_X^*\,\mathrm{td}(X)\big)\,. \tag{9.6.30}$$

Note that the self intersection pairing on $H^2(-,\mathbb{Z})$ being even for K3 surfaces,

$$\mathrm{ch}(-) = \mathrm{rk}(-) + c_1(-) + \frac{1}{2}\big(c_1^2(-) + 2c_2(-)\big) \tag{9.6.31}$$

is integral. Consider the Künneth decomposition

$$\sum_{i=0}^{4} \mathrm{ch}_i(E^\bullet) = \mathrm{ch}(E^\bullet) = \sum_{p,q\le 4} \gamma_p^q\,, \tag{9.6.32}$$

where $\gamma_p^q := \pi_X^*\alpha_p \otimes \pi_Y^*\beta_q$ with $\alpha_p \in H^p(X,\mathbb{Q})$ and $\beta_q \in H^q(Y,\mathbb{Q})$, for all $p, q \in \{0,1,2,3,4\}$. We have seen in the above computation that γ_p^q is integral for $p+q \le 4$. Since $\mathrm{td}(X) = (1,0,2)$, from (9.6.30) we have $c_1\big(\pi_{Y!}(E^\bullet)\big) = \int_X \gamma_4^2 + 2\gamma_0^2$, which implies that γ_4^2 is integral. Similarly using Grothendieck-Riemann-Roch theorem for the projection morphism $\pi_X : X \times Y \to X$, interchanging the roles of X and Y, we see that γ_2^4 is integral. Since γ_2^4 and γ_4^2 are the only terms contributing in $\mathrm{ch}_3(E^\bullet)$, we conclude that $\mathrm{ch}_3(E^\bullet)$ is integral. Similarly, from $\mathrm{ch}_2(\pi_{Y!}(E^\bullet)) = \int_X \gamma_4^4 + 2\gamma_0^4$, using integrality of second Chern character (c.f. (9.6.31)), we conclude that γ_4^4 is integral, and hence $\mathrm{ch}_4(E^\bullet)$ is integral. This completes the proof. □

Let X be a K3 surface over $\mathbb{C}$. Since $c_1(X) = 0$, for any $\alpha = (\alpha_0, \alpha_1, \alpha_2), \beta = (\beta_0, \beta_1, \beta_2) \in H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$, the Mukai pairing on X is given by

$$\langle \alpha, \beta\rangle_X = \int_X \exp\big(\frac{1}{2}c_1(X)\big)(\alpha^\vee \cdot \beta) = \alpha_0 \cdot \beta_2 + \alpha_2 \cdot \beta_0 - \alpha_1 \cdot \beta_1\,, \tag{9.6.33}$$

(c.f. Definition 9.3.37).

Remark 9.6.34. It should be noted that, classically Mukai pairing on X is defined by

$$\langle (\alpha_0, \alpha_1, \alpha_2), (\beta_0, \beta_1, \beta_2)\rangle_X = \alpha_1 \cdot \beta_1 - \alpha_0 \cdot \beta_2 - \alpha_2 \cdot \beta_0\,, \tag{9.6.35}$$

which differs from the above definition (9.6.33) by a minus sign.

Mukai introduced a weight 2 Hodge structure on $H^*(X,\mathbb{Z})$ by declaring $H^0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C})$ to be of type $(1,1)$ and by keeping the standard Hodge structure on $H^2(X,\mathbb{C})$. More precisely,

$$\begin{aligned} \widetilde{H}^{1,1}(X) &= H^0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C}) \oplus H^{1,1}(X), \\ \widetilde{H}^{2,0}(X) &= H^{2,0}(X) \quad\text{and}\quad \widetilde{H}^{0,2}(X) = H^{0,2}(X)\,. \end{aligned} \tag{9.6.36}$$

We denote by $\widetilde{H}(X,\mathbb{Z})$ to mean $H^*(X,\mathbb{Z})$ together with the Mukai pairing and this weight 2 Hodge structure.

Definition 9.6.37. Let X and Y be K3 surfaces over $\mathbb{C}$. With the above weight 2 Hodge structure structure on $\widetilde{H}(-,\mathbb{Z})$, a *Hodge isometry* of two K3 surfaces X and Y is a group isomorphism

$$\varphi : \widetilde{H}(X,\mathbb{Z}) \longrightarrow \widetilde{H}(Y,\mathbb{Z})$$

such that

(i) (Mukai pairing is preserved): $\langle \varphi(\alpha), \varphi(\beta)\rangle_X = \langle \alpha, \beta\rangle_X$, $\forall\, \alpha, \beta \in H^2(X,\mathbb{Z})$, and
(ii) $\varphi(H^{2,0}(X)) \subseteq H^{2,0}(Y)$.

Remark 9.6.38. Since $H^4(X,\mathbb{C}) \cong \mathbb{C}$, we have $\widetilde{H}^{1,1}(X) = H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^{2,2}(X)$; see the Hodge diamond in (9.6.9). Therefore, vanishing of odd cohomologies together with the condition (ii) in the Definition 9.6.37 ensures that φ preserves that new weight 2 Hodge structure (see (9.6.36)).

Corollary 9.6.39 (Mukai)**.** [Muk87] *Let X be a complex K3 surface and Y is a smooth complex projective variety. Let $E^\bullet \in D^b(X \times Y)$. If $\Phi^{X\to Y}_{E^\bullet} : D^b(X) \to D^b(Y)$ is an equivalence of categories, then the induced cohomological Fourier-Mukai transform $\Phi^{H,\,X\to Y}_{v(E^\bullet)}$ defines a Hodge isometry (in the sense of Definition 9.6.37)*

$$\Phi^{H,\,X\to Y}_{v(E^\bullet)} : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\simeq} \widetilde{H}(Y,\mathbb{Z}).$$

Proof. By Proposition 9.6.2, Y is also a K3 surface over $\mathbb{C}$. Recall that the Fourier-Mukai functor $\Phi^{X\to Y}_{E^\bullet}$ induces a cohomological Fourier-Mukai transform

$$\Phi^{H,\,X\to Y}_{v(E^\bullet)} : H^*(X,\mathbb{Q}) \xrightarrow{\simeq} H^*(Y,\mathbb{Q}), \quad \alpha \mapsto \pi_{Y!}\big(v(E^\bullet)\cdot \pi_X^*\alpha\big)\,,$$

which is an isometry with respect to the Mukai pairing (see Proposition 9.3.41). Now by Lemma 9.6.27, we have $\Phi^{H,\,X\to Y}_{v(E^\bullet)}(\alpha) \in H^*(Y,\mathbb{Z})$, for all $\alpha \in H^*(X,\mathbb{Z})$. Since the quasi-inverse of $\Phi^{X\to Y}_{E^\bullet}$ is also a Fourier-Mukai functor $\Phi^{Y\to X}_{E^\bullet_L}$, applying the above argument to the corresponding induced cohomological Fourier-Mukai transform $\Phi^{H,\,Y\to X}_{v(E^\bullet_L)}$, we can conclude that the induced map

$$\Phi^{H,\,X\to Y}_{v(E^\bullet)} : H^*(X,\mathbb{Z}) \longrightarrow H^*(Y,\mathbb{Z}) \tag{9.6.40}$$

is an isomorphism, which is also an isometry with respect to the Mukai pairing. Therefore, to conclude that $\Phi^{H,\,X\to Y}_{v(E^\bullet)}$ in (9.6.40) is a Hodge isometry, it is enough to show that its $\mathbb{C}$-linear extension sends $H^{2,0}(X)$ to $H^{2,0}(Y)$, which follows from Proposition 9.4.3. This completes the proof. □

Theorem 9.6.41 (Derived Torelli theorem for K3 surfaces)**.** *Let X and Y be two K3 surfaces over $\mathbb{C}$. Then there is an exact equivalence of derived categories $D^b(X) \to D^b(Y)$ if and only if there is a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \xrightarrow{\simeq} \widetilde{H}(Y,\mathbb{Z})$.*

Since any exact equivalence $D^b(X) \to D^b(Y)$ is isomorphic to a Fourier-Mukai functor by Orlov's representability theorem (Theorem 9.1.28), thanks to the above

Corollary 9.6.39 of Mukai, it remains to show that existence of a Hodge isometry $\widetilde{H}(X,\mathbb{Z}) \xrightarrow{\simeq} \widetilde{H}(Y,\mathbb{Z})$ implies an exact equivalence of derived categories $D^b(X) \xrightarrow{\simeq} D^b(Y)$. This part, due to Orlov [Orl97], is quite involved. It requires some theories from moduli space of semistable bundles on K3 surfaces, and some technical tools on integral functors. Here we give an outline of the proof, and the details would be filled up later after discussing required technologies.

Proof. Let $\varphi : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\simeq} \widetilde{H}(Y,\mathbb{Z})$ be a Hodge isometry.

Case 1. If $\varphi(0,0,1) = \pm(0,0,1)$, then one can show that φ respect intersection products on $H^2(-,\mathbb{Z})$ and sends $H^{2,0}(X)$ to $H^{2,0}(Y)$, thus producing a Hodge isometry $H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(Y,\mathbb{Z})$. Then by Global Torelli Theorem 9.6.24 we have an isomorphism $f : X \xrightarrow{\simeq} Y$ such that $f_* = \varphi$. This gives us an exact equivalence of categories $D^b(X) \xrightarrow{\simeq} D^b(Y)$ in this case.

Case 2. If $\varphi(0,0,1) = (r,\ell,s) =: v$ with $r \neq 0$, then replacing φ with $-\varphi$, if required, we may assume that $r < 0$ (if we want to work with classical Mukai pairing), or $r > 0$ (if we want to work with the general definition of Mukai pairing). If $v' := \varphi(-1,0,0)$, then $\langle v, v\rangle = 0$ and $\langle v, v'\rangle = 1$. Then one can apply the following general fact from the moduli space of bundles over a K3 surface:

If Y is a K3 surface and $v, v' \in \widetilde{H}^{1,1}(Y,\mathbb{Z})$ with $\langle v, v\rangle = 0$ and $\langle v, v'\rangle = 1$, then there is another K3 surface M and a sheaf $\mathcal{P}$ on $Y \times M$ such that for each closed point $m \in M$, the Mukai vector of the sheaf $\mathcal{P}\big|_{Y\times\{m\}}$ on $Y \times \{m\} \cong Y$ is v, and the integral functor (with kernel $\mathcal{P}$)

$$\Phi_{\mathcal{P}}^{Y\to M} : D^b(Y) \longrightarrow D^b(M) \tag{9.6.42}$$

is an equivalence of categories (this could be checked by using a result of Bondal and Orlov's, which would be discussed later).

Now in the situation of Case 2, one would use the induced cohomological Fourier-Mukai transform $\Phi_{v(\mathcal{P})}^{H,Y\to M} : \widetilde{H}(Y,\mathbb{Z}) \longrightarrow \widetilde{H}(M,\mathbb{Z})$ to show that the composite homomorphism

$$\psi : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\varphi} \widetilde{H}(Y,\mathbb{Z}) \xrightarrow{\Phi_{v(\mathcal{P})}^{H,Y\to M}} \widetilde{H}(M,\mathbb{Z}) \tag{9.6.43}$$

satisfies $\psi(0,0,1) = (0,0,1)$. Then using Global Torelli theorem, as argued in Case 1, one finds an isomorphism of X with M, which gives an exact equivalence $D^b(X) \xrightarrow{\simeq} D^b(M)$. Since $\Phi_{\mathcal{P}_R}^{M\to L} : D^b(M) \to D^b(Y)$ is an exact equivalence (c.f., (9.6.42)), the result follows.

Case 3. Suppose that $\varphi(0,0,1) = (0,\ell,s) =: v$, with $\ell \neq 0$. Then use a Hodge isometry

$$\widetilde{H}(Y,\mathbb{Z}) \xrightarrow{\exp(c_1(L))\cdot -} \widetilde{H}(Y,\mathbb{Z}), \tag{9.6.44}$$

for some $L \in \mathrm{Pic}(Y)$, to see that

$$\exp(c_1(L))(0, \ell, s) = (0, \ell, s + (c_1(L), \ell)) \tag{9.6.45}$$

[To be completed...] ☐

Remark 9.6.46. In general, the cohomological Fourier-Mukai transform $\Phi^{H,\, X\to Y}_{v(E^\bullet)}$ need not preserve cohomological degree. In fact, it need not give a Hodge isometry $H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(Y,\mathbb{Z})$ in the sense of Definition 9.6.23; for otherwise by Global Torelli theorem (Theorem 9.6.24) it would give an isomorphism of X with Y, which is not true in general.

In fact, for each K3 surface X there are only finitely many non-isomorphic K3 surfaces Y with exact equivalence $D^b(X) \xrightarrow{\simeq} D^b(Y)$. More surprisingly, for each positive integer $n > 1$, there is a K3 surface X with at least n non-isomorphic Fourier-Mukai partner Y (proof of this result, due to Oguiso [Ogu02], depends on a result on "almost primes" from analytic number theory).

9.7. **Geometric aspects of kernels of Fourier-Mukai functors.** In this subsection, we discuss a series of technical but useful results that shed light on the geometry of the support of the kernel of an integral functor

$$\Phi^{X\to Y}_{\mathcal{P}} : D^b(X) \longrightarrow D^b(Y). \tag{9.7.1}$$

When $\mathcal{P}$ is a locally free coherent sheaf on $X \times Y$ (e.g., when $\mathcal{P}$ is the Poincaré sheaf on the product of abelian variety and its dual), noting interesting can be said about the geometry of its support $\mathrm{Supp}(\mathcal{P})$, which is just $X \times Y$. However, when the kernel $\mathcal{P}$ is supported on a smaller subvariety of $X \times Y$ (e.g., a graph of a morphism or a correspondence), then it encodes interesting geometric relation between X and Y. This usually happens when the canonical bundles of the variety has some kind of positivity property.

Let X be smooth projective k-variety. Recall that the support of an object $E^\bullet \in D^b(X)$, which we denote and define by

$$\mathrm{Supp}(E^\bullet) := \bigcup_i \mathrm{Supp}\left(\mathcal{H}^i(E^\bullet)\right), \tag{9.7.2}$$

is a closed subset of X with possibly many irreducible components. Note that, for any line bundle L on X, we have $\mathrm{Supp}(E^\bullet \otimes L) = \mathrm{Supp}(E^\bullet)$.

Lemma 9.7.3. *For any $E^\bullet \in D^b(X)$, we have $\mathrm{Supp}(E^\bullet) = \mathrm{Supp}(E^{\bullet\vee})$.*

Proof. Consider the spectral sequence

$$E_2^{p,q} := \mathcal{E}xt^p(\mathcal{H}^{-q}(E^\bullet), \mathcal{O}_X) \Longrightarrow E^{p+q} := \mathcal{E}xt^{p+q}(E^\bullet, \mathcal{O}_X) \cong \mathcal{H}^{p+q}(E^{\bullet\vee}) . \tag{9.7.4}$$

It follows from the above spectral sequence that $\mathrm{Supp}(E^{\bullet\vee}) \subseteq \mathrm{Supp}(E^\bullet)$. Since $E^{\bullet\vee\vee} \cong E^\bullet$, we get the reverse inclusion, which completes the proof. $\square$

Let X and Y be smooth projective k-varieties. Let $\mathcal{P} \in D^b(X \times Y)$, and consider the integral functor

$$\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \longrightarrow D^b(Y),\;\; E^\bullet \longmapsto p_{Y*}\big(\mathcal{P} \otimes p_X^* E^\bullet\big) \tag{9.7.5}$$

with kernel $\mathcal{P}$. Recall that the left and the right adjoint of $\Phi_{\mathcal{P}}^{X\to Y}$ are integral functors with kernels $\mathcal{P}_L := \mathcal{P}^\vee \otimes p_Y^*\omega_Y[\dim(Y)]$ and $\mathcal{P}_R := \mathcal{P}^\vee \otimes p_X^*\omega_X[\dim(X)]$, respectively. Then from the above Lemma 9.7.3 we have,

$$\mathrm{Supp}(\mathcal{P}) = \mathrm{Supp}(\mathcal{P}^\vee) = \mathrm{Supp}(\mathcal{P}_L) = \mathrm{Supp}(\mathcal{P}_R)\,. \tag{9.7.6}$$

When $\Phi_{\mathcal{P}}$ is fully faithful, by uniqueness of kernel (up to isomorphism) ensured by Orlov's representability theorem (Theorem 9.1.28) we have

$$\mathcal{P} \otimes p_X^*\omega_X[\dim(X)] = \mathcal{P} \otimes p_Y^*\omega_Y[\dim(Y)]\,. \tag{9.7.7}$$

Moreover, when $\dim(X) = \dim(Y)$ (e.g., if $\Phi_{\mathcal{P}}$ is an equivalence of categories), we can further deduce that

$$\mathcal{H}^i(\mathcal{P}) \otimes p_X^*\omega_X \cong \mathcal{H}^i(\mathcal{P}) \otimes p_Y^*\omega_Y\,, \quad \forall\, i \in \mathbb{Z}\,. \tag{9.7.8}$$

Lemma 9.7.9. *With the above notations, if $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ is faithful, then the natural projection morphism*

$$p_X\big|_{\mathrm{Supp}(\mathcal{P})} : \mathrm{Supp}(\mathcal{P}) \longrightarrow X \tag{9.7.10}$$

is surjective. Moreover, there is an integer i and an irreducible component Z of $\mathcal{H}^i(\mathcal{P})$ which projects onto X.

Proof. Consider the spectral sequence

$$E_2^{p,q} := \mathcal{T}or_{-p}\left(\mathcal{H}^q(\mathcal{P}), p_X^*k(x)\right) \Longrightarrow E^{p+q} := \mathcal{T}or_{-(p+q)}\left(\mathcal{P}, p_X^*k(x)\right). \tag{9.7.11}$$

If the projection map $\mathrm{Supp}(\mathcal{P}) \to X$ were not surjective, there would be a closed point $x \in X$ with $x \notin p_X(\mathrm{Supp}(\mathcal{P}))$. Then the above spectral sequence could be used to show that the derived tensor product $\mathcal{P} \otimes p_X^* k(x)$ is trivial, and hence $\Phi_{\mathcal{P}}^{X\to Y}(k(x)) = p_{Y*}\big(\mathcal{P} \otimes p_X^* k(x)\big) \cong 0$ in $D^b(Y)$, which is absurd since $\Phi_{\mathcal{P}}^{X\to Y}$ is faithful by assumption. Since X is irreducible, the last assertion follows. $\square$

Remark 9.7.12. Since $\mathrm{Supp}(\mathcal{P}) = \mathrm{Supp}(\mathcal{P}_R) = \mathrm{Supp}(\mathcal{P}_L)$, when $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ is an equivalence of categories, one can conclude that the natural projection morphism $\mathrm{Supp}(\mathcal{P}) \longrightarrow Y$ is also surjective. However, the integer i and the irreducible component Z of $\mathcal{H}^i(\mathcal{P})$ could be different for two different projections.

Definition 9.7.13. Let Z be a proper k-scheme.

(i) A line bundle L on Z is called *nef* if for any complete reduced curve C over k and any morphism of k-schemes $\varphi : C \longrightarrow Z$, we have $\deg(\varphi^* L) \geq 0$.
(ii) A line bundle L on Z is said to be *numerically trivial* if both L and its dual $L^\vee$ are nef.
(iii) Two line bundles L and L' over Z are said to be *numerically equivalent* if for any morphism of k-schemes $\varphi : C \longrightarrow Z$, with C a proper reduced curve over k, we have $\deg(\varphi^* L) = \deg(\varphi^* L')$.

Remark 9.7.14. Replacing $\varphi : C \longrightarrow Z$ with its image $C' = \varphi(C) \subseteq Z$, it suffices to check the above criterion with proper reduced curves over k that are embedded inside Z. Furthermore, considering the normalization of C, it suffices to check the above criterion for smooth proper curves C embedded in Z.

Lemma 9.7.15. *Let $p : Z \to W$ be a projective morphism of proper k-schemes. Let $L \in$* Pic*(W).*

(i) If L is nef, then $p^* L$ is nef.
(ii) If p is surjective, then L is nef if and only if $p^* L$ is nef.

Proof. Since for any morphism of k-schemes $\varphi : C \to Z$ with C a reduced proper curve over k, $(p \circ \varphi)^* L = \varphi^*(p^* L)$, the first assertion follows.

To see the second assertion, for any morphism of k-schemes $\psi : C \to W$, with C an integral proper curve over k, we construct a ramified cover $\eta : \widetilde{C} \to C$ of C by an irreducible curve $\widetilde{C}$ such that $\psi \circ \eta : \widetilde{C} \to W$ factors through Z.

$$\begin{tikzcd} \widetilde{C} \arrow[r, dashed, "\varphi"] \arrow[d, "\eta"'] & Z \arrow[d, "p"] \\ C \arrow[r, "\psi"] & W \end{tikzcd}$$

Since a dominant projective morphism of k-schemes $C \times_W Z \to C$ admits a (multi) section (this can be seen by embedding $C \times_W Z$ into some $C \times \mathbb{P}^N_k$ and intersecting it with a generic linear subspace of $\mathbb{P}^N_k$ of appropriate degree), such a ramified cover $\eta : \widetilde{C} \to C$ exists. Since

$$\deg(\varphi^*(p^* L)) = \deg(\eta^*(\psi^* L)) = \deg(\eta) \cdot \deg(\psi^* L),$$

and $p^* L$ is nef by assumption, the result follows. $\square$

Lemma 9.7.16. *Let C be a complete reduced curve over k and let $\varphi : C \longrightarrow X \times Y$ be a morphism with image inside* $\mathrm{Supp}(\mathcal{P})$*. If there is an exact equivalence of categories $F : D^b(X) \to D^b(Y)$, then*

$$\deg\left(\varphi^* p_X^* \omega_X\right) = \deg\left(\varphi^* p_Y^* \omega_Y\right). \tag{9.7.17}$$

In other words, the pullbacks $p_X^* \omega_X$ and $p_Y^* \omega_Y$ are numerically equivalent.

Proof. By Orlov's representability theorem, there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{X \to Y}$. As remarked above, we may assume that C is integral and smooth so that $\varphi(C) \subset \operatorname{Supp}\big(\mathcal{H}^i(\mathcal{P})\big)$ for some i. Then the pullback $\varphi^*\mathcal{H}^i(\mathcal{P})$ is a coherent sheaf on C, which is locally free out side a finite subset of C. Going modulo the torsion part $T(\varphi^*\mathcal{H}^i(\mathcal{P}))$ of $\varphi^*\mathcal{H}^i(\mathcal{P})$, we find a non-zero locally free coherent sheaf $F := \varphi^*\mathcal{H}^i(\mathcal{P})/T(\varphi^*\mathcal{H}^i(\mathcal{P}))$ of rank $r > 0$ on C. Since $\Phi_{\mathcal{P}}^{X \to Y}$ is an equivalence of categories, using Orlov's theorem one find that

(9.7.18)
$$\mathcal{P} \otimes p_X^*\omega_X \cong \mathcal{P} \otimes p_Y^*\omega_Y,$$

and hence $\mathcal{H}^i(\mathcal{P}) \otimes p_X^*\omega_X \cong \mathcal{H}^i(\mathcal{P}) \otimes p_Y^*\omega_Y$. Pulling back to C, we have $F \otimes \varphi^*p_X^*\omega_X \cong F \otimes \varphi^*p_Y^*\omega_Y$. Then taking r-th exterior power both sides, we have $\varphi^*p_X^*\omega_X^r \cong \varphi^*p_Y^*\omega_Y^r$. Comparing degrees both sides, the lemma follows. □

Remark 9.7.19. Tensoring (9.7.18) with $\omega_X^\vee \otimes \omega_Y$, we have $\mathcal{P} \otimes \omega_X^\vee \cong \mathcal{P} \otimes \omega_Y^\vee$. Then the same argument as in the above proof shows that $\deg(\varphi^*p_X^*\omega_X^\vee) = \deg(\varphi^*p_Y^*\omega_Y^\vee)$.

Corollary 9.7.20. *If there is an exact equivalence of categories $F : D^b(X) \xrightarrow{\simeq} D^b(Y)$, then ω_X is numerically trivial if and only if ω_Y is so.*

Proof. By Orlov's representability theorem, $F \cong \Phi_{\mathcal{P}}^{X \to Y}$, for some $\mathcal{P} \in D^b(X \times Y)$. Suppose that ω_X is numerically trivial. Then for any reduced proper curve C over k and any morphism of k-schemes $\varphi : C \to X \times Y$, we have $\deg\big(\varphi^*p_X^*\omega_X\big) = 0$. Then $p_Y^*\omega_Y\big|_{\operatorname{Supp}(\mathcal{P})}$ is numerically trivial by Lemma 9.7.16. Since the natural projection morphism $p_Y : \operatorname{Supp}(\mathcal{P}) \to Y$ is surjective (see Lemma 9.7.9), ω_Y is numerically trivial by Lemma 9.7.15. □

Corollary 9.7.21. *Let $\mathcal{P} \in D^b(X \times Y)$ and $\Phi_{\mathcal{P}}^{X \to Y} : D^b(X) \to D^b(Y)$ a Fourier-Mukai equivalence. If $Z \subset \operatorname{Supp}(\mathcal{P})$ is a closed subvariety such that the restriction of ω_X (or $\omega_X^\vee$) to the image of $p_X : \operatorname{Supp}(\mathcal{P}) \to X$ is ample, then $p_Y : Z \to Y$ is a finite morphism.*

Proof. If $p_Y : Z \to Y$ were not finite, there would exists a non-trivial irreducible curve $\varphi : C \hookrightarrow Z$ with $p_Y \circ \varphi : C \to Y$ is constant. Then $\varphi^*p_Y^*\omega_Y$ is a trivial line bundle on C, and so

(9.7.22)
$$\deg(\varphi^*p_X^*\omega_X) = \deg(\varphi^*p_Y^*\omega_Y) = 0$$

by Lemma 9.7.16. Since $p_Y \circ \varphi$ is constant, $p_X \circ \varphi : C \to X$ must be non-constant. Since ω_X (or $\omega_X^\vee$) is ample on $p_X(Z)$ and hence on $p_X(\varphi(C))$, we get a contradiction with (9.7.22). Hence the result follows. □

Lemma 9.7.23. *Let Z be a normal k-variety and $E \in \mathfrak{Coh}(Z)$ which is generically of rank r. If $L_1, L_2 \in \operatorname{Pic}(X)$ such that $E \otimes L_1 \cong E \otimes L_2$, then $L_1^r \cong L_2^r$.*

Proof. Taking quotient of E by its torsion part, we may assume that E is torsion free of rank r on X. Since Z is normal, there is an open subscheme U of Z with $\mathrm{codim}_Z(Z \setminus U) \geq 2$ such that $E\big|_U$ is locally free. Since $\det\big((E \otimes L_i)\big|_U\big) \cong \big(\det(E) \otimes L_i^r\big)\big|_U$, we have $L_1^r\big|_U \cong L_2^r\big|_U$. Since Z is normal, we have $L_1^r \cong L_2^r$. □

Lemma 9.7.24. *With the above notations, let $Z \subset \mathrm{Supp}(\mathcal{P})$ be a closed irreducible subvariety with normalization $\mu : \widetilde{Z} \to Z$. Then there is an integer $r > 0$ such that*

$$\mu^*\big((p_X^*\omega_X^r)\big|_Z\big) \cong \mu^*\big((p_Y^*\omega_Y^r)\big|_Z\big). \tag{9.7.25}$$

Proof. Let $\mu : \widetilde{Z} \longrightarrow Z$ be the normalization of a closed irreducible subvariety $Z \subset \mathrm{Supp}(\mathcal{P})$. Then there is an integer i such that $Z \subset \mathrm{Supp}(\mathcal{H}^i(\mathcal{P}))$. Then $\mu^*\big(\mathcal{H}^i(\mathcal{P})\big)$ is a coherent sheaf on $\widetilde{Z}$ generically of positive rank, say $r > 0$. Since $\Phi_{\mathcal{P}}^{X \to Y} : D^b(X) \to D^b(Y)$ is an equivalence of categories, using Orlov's representability theorem, we have

$$\mathcal{H}^i(\mathcal{P}) \otimes p_X^*\omega_X \cong \mathcal{H}^i(\mathcal{P}) \otimes p_Y^*\omega_Y\,. \tag{9.7.26}$$

Pulling back the above isomorphism by μ over the normal variety $\widetilde{Z}$, the result follows form Lemma 9.7.23. □

Lemma 9.7.27. *Let $\iota : T \hookrightarrow X$ be a closed embedding. Then for any $E^\bullet \in D^b(X)$ we have*

$$\mathrm{Supp}(E^\bullet) \cap T = \mathrm{Supp}(\iota^* E^\bullet). \tag{9.7.28}$$

Lemma 9.7.29. *If $\Phi_{\mathcal{P}}^{X \to Y} : D^b(X) \to D^b(Y)$ is a Fourier-Mukai equivalence, the fibers of the projection morphism $p_X : \mathrm{Supp}(\mathcal{P}) \longrightarrow X$ are connected.*

Proof. Suppose on the contrary that the fiber over a point $x \in X$ is disconnected. Then we can write $\mathrm{Supp}(\mathcal{P}) \cap p_X^{-1}(x) = \mathrm{Supp}(\mathcal{P}) \cap (\{x\} \times Y) = Y_1 \sqcup Y_2$, for some non-empty distinct closed subsets $Y_1, Y_2 \subset Y$. Then by above Lemma 9.7.27, we have $\mathrm{Supp}(\mathcal{P}) \cap \big(\{x\} \times Y\big) = \mathrm{Supp}(\mathcal{P}\big|_{\{x\}\times Y})$. Thus $\Phi_{\mathcal{P}}^{X \to Y}(k(x))$ has disconnected support $Y_1 \sqcup Y_2$, and hence we con write it as $\Phi_{\mathcal{P}}^{X \to Y}(k(x)) \cong E_1^\bullet \oplus E_2^\bullet$, with $\mathrm{Supp}(E_i^\bullet) = Y_i$, for $i = 1, 2$. Then $\mathrm{End}(E_1^\bullet \oplus E_2^\bullet)$ is not a field. But $\Phi_{\mathcal{P}}^{X \to Y}$ being an exact equivalence, we have $\mathrm{End}(E_1^\bullet \oplus E_2^\bullet) = \mathrm{Hom}\big(\Phi_{\mathcal{P}}^{X \to Y}(k(x)), \Phi_{\mathcal{P}}^{X \to Y}(k(x))\big) = \mathrm{Hom}(k(x), k(x)) = k(x)$, which is a contradiction. □

Corollary 9.7.30. *Let $\Phi_{\mathcal{P}}^{X \to Y} : D^b(X) \to D^b(Y)$ is a Fourier-Mukai equivalence. Let $Z \subset \mathrm{Supp}(\mathcal{P})$ be an irreducible component which surjects onto X. If $\dim(Z) = \dim(X)$, then the restriction morphism $p_X : Z \longrightarrow X$ is birational. Moreover, if such a component exists, no other component of $\mathrm{Supp}(\mathcal{P})$ dominates X.*

Proof. By Lemma 9.7.29, the fibers of $p_X : \mathrm{Supp}(\mathcal{P}) \to X$ are connected. Let $Z \subset \mathrm{Supp}(\mathcal{P})$ be an irreducible component with $\dim(Z) = \dim(X)$ and $p_X : Z \to X$ surjective. Note that, the fibers of $Z \to X$ are zero dimensional. If $Z \neq \mathrm{Supp}(\mathcal{P})$, consider the union $\bigcup_i Z_i$ of all irreducible components Z_i of $\mathrm{Supp}(\mathcal{P})$ with $Z_i \neq Z$. If the generic fiber of the projection map $p_X : \bigcup_i Z_i \longrightarrow X$ is non-empty, it contains the (zero dimensional) fiber of $Z \to X$, and then $Z \subseteq \bigcup_i Z_i$ — which is absurd. Therefore, Z is the only irreducible component of $\mathrm{Supp}(\mathcal{P})$ that dominates X.

To show the restriction morphism $p_X : Z \to X$ birational, choosing a generic point $\xi \in X$, we see that the generic fiber $p_X^{-1}(\xi) \cap Z = Z \cap \big(\{\xi\} \times Y\big)$ is a finite set (since $\dim(Z) = \dim(X)$) disjoint from other irreducible components of $\mathrm{Supp}(\mathcal{P})$. Since the fiber of the projection map $p_X : \mathrm{Supp}(\mathcal{P}) \to X$ is connected, we conclude that $p_X^{-1}(\xi) \cap Z$ is singleton. Therefore, $Z \to X$ is birational. □

Remark 9.7.31. Birationality of Z with X in the above Corollary holds even if Z is a priori non-reduced.

Corollary 9.7.32. *Let $\Phi_{\mathcal{P}}^{X\to Y} : D^b(X) \to D^b(Y)$ be a Fourier-Mukai equivalence. Let $x_0 \in X$ be a closed point such that*

$$\Phi_{\mathcal{P}}^{X\to Y}(k(x_0)) = k(y_0), \tag{9.7.33}$$

for some closed point $y_0 \in Y$. Then there is an open neighbourhood $U \subseteq X$ of x_0 and a morphism $f : U \longrightarrow Y$ with $f(x_0) = y_0$ such that

$$\Phi_{\mathcal{P}}^{X\to Y}\big(k(x)\big) = f\big(k(f(x))\big), \tag{9.7.34}$$

for all closed point $x \in U$.

Proof. Since $\Phi_{\mathcal{P}}^{X\to Y}\big(k(x_0)\big) \cong k(y_0)$, the fiber $p_X^{-1}(x_0) \cap \mathrm{Supp}(\mathcal{P})$ over x_0 of the projection morphism $p_X : \mathrm{Supp}(\mathcal{P}) \to X$ is zero dimensional. By semicontinuity, there is an open neighbourhood $U \subseteq X$ of x_0 such that $p_X^{-1}(x) \cap \mathrm{Supp}(\mathcal{P})$ is zero dimensional, for all $x \in U$. Then for each $x \in U$, the complex $\Phi_{\mathcal{P}}^{X\to Y}\big(k(x)\big)$ is concentrated in dimension 0. Then

$$\mathrm{Hom}\left(\Phi_{\mathcal{P}}^{X\to Y}\big(k(x)\big), \Phi_{\mathcal{P}}^{X\to Y}\big(k(x)\big)[i]\right) = 0, \ \ \forall\, i < 0.$$

Then by Lemma 8.3.14, for each $x \in U$ there is a closed point $y_x \in Y$ and an integer m_x such that $\Phi_{\mathcal{P}}^{X\to Y}\big(k(x)\big) \cong k(y_x)[m_x]$. Using semicontinuity, one can check that $m_x = m$ are locally constant, around $x_0 \in U$. Thus shrinking U further, if required, we may assume that $m_x = 1$, for all $x \in U$ (because of given condition (9.7.33)).

Using Lemma 9.1.32 as argued in Proposition 9.1.34, one concludes that $\mathcal{P}|_{p_X^{-1}(U)}$ is a coherent sheaf, and can choose local sections to construct a morphism of k-schemes $f : U \longrightarrow Y$ such that $\Phi_{\mathcal{P}}^{U\to Y} \cong (L \otimes -) \circ f_*$, for some line bundle L on Y. This completes the proof. □

Then next result, due to Kawamata, shows nefness of the (anti)-canonical line bundle under exact equivalence of derived categories.

Proposition 9.7.35 (Kawamata)**.** *Let X and Y be smooth projective k-varieties with an exact equivalence of their derived categories $F : D^b(X) \longrightarrow D^b(Y)$. Then ω_X (resp., $\omega_X^\vee$) is nef if and only if ω_Y (resp., $\omega_X^\vee$) is nef.*

Proof. By Orlov's representability theorem (Theorem 9.1.28), there is an object $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism, such that $F \cong \Phi_{\mathcal{P}}^{X \to Y}$. Then the projection morphisms $p_X : \mathrm{Supp}(\mathcal{P}) \longrightarrow X$ and $p_Y : \mathrm{Supp}(\mathcal{P}) \longrightarrow Y$ are surjective by Lemma 9.7.9. Then by Lemma 9.7.15, a line bundle $L \in \mathrm{Pic}(X)$ (resp., $L' \in \mathrm{Pic}(Y)$) is nef if and only if $(p_X^*L)\big|_{\mathrm{Supp}(\mathcal{P})}$ (resp., $(p_Y^*L')\big|_{\mathrm{Supp}(\mathcal{P})}$) is a nef line bundle on $\mathrm{Supp}(\mathcal{P})$. Since for any complete reduced curve C and any morphism of k-schemes $\varphi : C \to \mathrm{Supp}(\mathcal{P}) \subset X \times Y$, we have $\deg(\varphi^* p_X^* \omega_X) = \deg(\varphi^* p_Y^* \omega_Y)$ and $\deg(\varphi^* p_X^* \omega_X^\vee) = \deg(\varphi^* p_Y^* \omega_Y^\vee)$ by Lemma 9.7.16 (see also Remark 9.7.19), the proposition follows from Definition 9.7.13. □

Corollary 9.7.36. *Let X and Y be smooth projective k-varieties with an exact equivalence of their derived categories $F : D^b(X) \longrightarrow D^b(Y)$. Then ω_X is numerically trivial if and only if ω_Y is numerically trivial.*

Proof. Since a line bundle L is numerically trivial if and only if both L and $L^\vee$ are nef, the result follows from the above Proposition 9.7.35. □

Recall that the intersection number $([L]^n \cdot Z)$ of a line bundle L on a proper k-scheme Z of dimension n is the degree n coefficient of the polynomial $\chi(Z, L^\ell) \in \mathbb{Q}[\ell]$.

Definition 9.7.37. The *numerical Kodaira dimension* of a line bundle L on a projective k-scheme X is the maximal non-negative integer $\nu(X, L)$ such that there is a proper morphism of k-schemes $\varphi : Z \longrightarrow X$ with $\dim(W) = \nu(X, L)$ such that

$$\big([\varphi^* L]^{\nu(X,L)} \cdot Z\big) \neq 0.$$

Denote by $\nu(X) := \nu(X, \omega_X)$, the numerical Kodaira dimension of ω_X.

Remark 9.7.38. (i) It suffices to check the conditions in the above definition with closed subschemes $Z \subseteq X$.

(ii) In general, there is no relation between $\nu(X, L)$ and $\mathrm{kod}(X, L)$. However, if L is a nef line bundle on X, then one can show that $\mathrm{kod}(X, L) \leq \nu(X, L)$.

Lemma 9.7.39. *Let $p : X \longrightarrow Y$ be a projective morphism of projective k-schemes and let $L \in \mathrm{Pic}(Y)$. Then $\nu(X, p^*L) \leq \nu(Y, L)$, and equality holds if $p : X \to Y$ is surjective.*

Sketch of a proof. The inequality $\nu(X, p^*L) \leq \nu(Y, L)$ follows from the definition by taking any $\varphi : Z \to X$ and pulling back L by the composite morphism $\varphi \circ p : Z \to Y$.

Suppose that $p : X \to Y$ is surjective. To show the equality of numerical Kodaira dimensions, given a proper morphism of k-schemes $\varphi : Z \to Y$, one constructs a generically finite surjective morphism of k-schemes $\psi : \widetilde{Z} \to Z$ and a morphism of k-schemes $\widetilde{\varphi} : \widetilde{Z} \to X$ such that the following diagram commutes.

$$\begin{tikzcd} \widetilde{Z} \arrow[r, dashed, "\widetilde{\varphi}"] \arrow[d, dashed, "\psi"'] & X \arrow[d, "p"] \\ Z \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Roughly, this could be done by embedding the fiber product $Z \times_Y X$ into some $Z \times \mathbb{P}^N_k$ and taking appropriate intersection with some cycle, and choosing its section. Since for any line bundle L on Y we have

$$\big([\widetilde{\varphi}^* p^* L]^m \cdot \widetilde{Z}\big) = \deg(\psi) \cdot \big([\varphi^* L]^m \cdot Z\big),$$

we conclude that $\nu(X, p^*L) \leq \nu(Y, L)$. $\square$

Proposition 9.7.40 (Kawamata)**.** *Let X and Y be smooth projective k-varieties with an exact equivalence of derived categories $F : D^b(X) \longrightarrow D^b(Y)$. Then we have $\nu(X) = \nu(Y)$.*

Proof. As before, $F \cong \Phi_{\mathcal{P}}^{X \to Y}$, for some $\mathcal{P} \in D^b(X \times Y)$, unique up to isomorphism. By Lemma 9.7.9, there is an integer i and an irreducible component Z of $\mathrm{Supp}(\mathcal{H}^i(\mathcal{P}))$ which surjects onto X. Let $\mu : \widetilde{Z} \to Z$ be the normalization of Z. Then by Lemma 9.7.24, there is an integer $r > 0$ such that $\mu^*\big((p_X^* \omega_X^r)\big|_Z\big) \cong \mu^*\big((p_Y^* \omega_Y^r)\big|_Z\big)$. Since $\mu \circ p_X : \widetilde{Z} \to X$ is surjective, by Lemma 9.7.39 we have

$$\nu\big(\widetilde{Z}, \mu^*\big((p_X^* \omega_X^r)\big|_Z\big)\big) = \nu(X, \omega_X^r). \tag{9.7.41}$$

Since $\nu(L) = \nu(L^r)$, for any line bundle L and integer $r \neq 0$, we have

$$\begin{aligned} \nu(X, \omega_X) = \nu(X, \omega_X^r) &= \nu\left(\widetilde{Z}, \mu^*\big((p_X^* \omega_X^r)\big|_Z\big)\right) \\ &= \nu\left(\widetilde{Z}, \mu^*\big((p_Y^* \omega_Y^r)\big|_Z\big)\right) \\ &\leq \nu(Y, \omega_Y^r) = \nu(Y, \omega_Y). \end{aligned}$$

Due to symmetry of the situation, we have $\nu(Y, \omega_Y) \leq \nu(X, \omega_X)$, and hence $\nu(Y, \omega_Y) = \nu(X, \omega_X)$. $\square$

Definition 9.7.42. (i) Let X and Y be two k-schemes. A *birational correspondence* between X and Y is given by a k-scheme Z together with birational morphisms of k-schemes $\pi_X : Z \longrightarrow X$ and $\pi_Y : Z \longrightarrow Y$. In particular, X and Y are birationally equivalent.

(ii) Two projective k-schemes X and Y are said to be *K-equivalent* if there is a birational correspondence $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ such that $\pi_X^* \omega_X$ is linearly equivalent to $\pi_Y^* \omega_Y$ (i.e., if $\pi_X^* \omega_X \cong \pi_Y^* \omega_Y$).

(iii) Two smooth projective k-varieties are said to be *D-equivalent* if there is an exact equivalence of their bounded derived categories $F : D^b(X) \to D^b(Y)$.

We just have seen above that, D-equivalent smooth projective k-varieties X and Y have the same Kodaira dimension: $\operatorname{kod}(X) = \operatorname{kod}(Y)$ (also $\operatorname{kod}(X, \omega_X^\vee) = \operatorname{kod}(Y, \omega_Y^\vee)$), and also have the same numerical Kodaira dimension: $\nu(X) = \nu(Y)$ (also $\nu(X, \omega_X^\vee) = \nu(Y, \omega_Y^\vee)$).

Proposition 9.7.43 (Kawamata). *Let k be an algebraically closed field. Let X and Y be smooth projective k-varieties. If there is an exact equivalence $F : D^b(X) \longrightarrow D^b(Y)$ and if $\operatorname{kod}(X) = \dim(X)$ or $\operatorname{kod}(X, \omega_X^\vee) = \dim(X)$, then X and Y are birational. More precisely, there is a birational correspondence*

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

with $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$.

Proof. We only work with the case $\operatorname{kod}(X) = \dim(X)$; the case $\operatorname{kod}(X, \omega_X^\vee) = \dim(X)$ is similar. Let $H \subset X$ be a smooth ample hypersurface. The exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_H \longrightarrow 0$$

induces an exact sequence of k-vector spaces

$$0 \longrightarrow H^0(X, \omega_X^\ell(-H)) \longrightarrow H^0(X, \omega_X^\ell) \longrightarrow H^0(H, \omega_X^\ell\big|_H), \ \ \forall\, \ell \in \mathbb{Z}\,.$$

Recall that the *Kodaira dimension of* $L \in \operatorname{Pic}(X)$ is an integer m such that the function $\ell \longmapsto \dim H^0(X, L^\ell)$ grows like a polynomial of degree m. Since $\operatorname{kod}(X) := \operatorname{kod}(X, \omega_X) = \dim(X)$ by assumption, the function $h^0(\omega_X^\ell)$ grows like $\ell^{\dim(X)}$. Since $\dim(H) < \dim(X)$, the function $h^0(\omega_X^\ell\big|_H)$ has smaller growth than that of $h^0(\omega_X^\ell)$. Therefore, for $\ell \gg 0$ large enough, $\omega_X^\ell(-H)$ has a non-zero global section. Then there is an effective divisor $D \subset X$ such that $\omega_X^\ell(-H) \cong \mathcal{O}_X(D)$, and hence

$$\omega_X^\ell \cong \mathcal{O}_X(H + D)\,, \tag{9.7.44}$$

with H smooth ample and D effective divisor. This is also known as *Kodaira lemma*. As discussed before, by Lemma 9.7.9 there is an integer i and an irreducible component $Z \subset \operatorname{Supp}(\mathcal{H}^i(\mathcal{P}))$ that surjects onto X. Let $\mu : \widetilde{Z} \longrightarrow Z$ be the normalization. Then by Lemma 9.7.24, there is an integer $r > 0$ such that

$$\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r\,, \tag{9.7.45}$$

where $\pi_X := p_X\big|_Z \circ \mu : \widetilde{Z} \to X$ and $\pi_Y := p_Y\big|_Z \circ \mu : \widetilde{Z} \to Y$.

We claim that, the morphism π_Y restricted to $\widetilde{Z} \setminus \pi_X^{-1}(D)$ is quasi-finite, i.e.,

$$\pi_{Y,D} := \pi_Y\big|_{\widetilde{Z} \setminus \pi_X^{-1}(D)} : \widetilde{Z} \setminus \pi_X^{-1}(D) \longrightarrow Y \tag{9.7.46}$$

is quasi-finite (i.e., fibers are finite sets of points). If not, then there is a point $y \in Y$ and an irreducible curve $C \subset \widetilde{Z}$ such that $C \not\subseteq \backslash \pi_X^{-1}(D)$ and $\pi_Y(C) = \{y\}$. Then

$$\deg\left(\pi_Y^*\omega_Y\big|_C\right) = 0\,. \tag{9.7.47}$$

On the other hand, since the intersection $\pi_X(C) \cap D$ is at most a finite set of points, using (9.7.44) we have

$$\ell \cdot \deg\left(\pi_X^*\omega_X\big|_C\right) = \deg\left(\pi_X^*\omega_X^\ell\big|_C\right) \geq \deg\left(\pi_X^*\mathcal{O}_X(H)\big|_C\right) > 0\,, \tag{9.7.48}$$

where the last inequality holds because H is ample. Then combining (9.7.48), (9.7.47) and (9.7.45), we get a contradiction. This proves our claim.

Then the projection morphism $p_Y : Z \longrightarrow Y$ is generically finite, and hence $\dim(Z) \leq \dim(Y)$. On the other hand, since $p_X : Z \longrightarrow X$ is surjective, $\dim(X) \leq \dim(Z)$. Since $D^b(X) \simeq D^b(Y)$, $\dim(X) = \dim(Y)$, and hence $\dim(Z) = \dim(X)$. Therefore, the correspondence morphisms $\pi_X : \widetilde{Z} \longrightarrow X$ and $\pi_Y : \widetilde{Z} \longrightarrow Y$ are generically finite and generically surjective. Then Corollary 9.7.30 ensures that, in fact, we have constructed a birational correspondence

$$\begin{tikzcd} & \widetilde{Z} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y. \end{tikzcd} \tag{9.7.49}$$

Now it remains to show that, $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$. Let's give a sketch how to achieve this.

We already have an isomorphism $\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$, for some integer $r > 0$ (see (9.7.45)). Since $\widetilde{Z}$ is normal, its singular locus has codimension at least 2. Replacing $\widetilde{Z}$ with a desingularization $p : \widehat{Z} \to \widetilde{Z}$, if required, from the above birational correspondence in (9.7.49), we have

$$p^*\pi_X^*\omega_X \otimes \mathcal{O}_{\widehat{Z}}\big(\sum_i a_iE_i\big) \cong p^*\pi_Y^*\omega_Y \otimes \mathcal{O}_{\widehat{Z}}\big(\sum_i b_iE_i\big)\,, \tag{9.7.50}$$

where E_i are exceptional divisors with respect to $\pi_X \circ p$ and $\pi_Y \circ p$, and $a_i, b_i \in \mathbb{Z}$, for all i (see Lemma 9.7.52 below). Taking r-th tensor power in (9.7.49) and using the isomorphism $p^*\pi_X^*\omega_X^r \cong p^*\pi_Y^*\omega_Y^r$, we have

$$\mathcal{O}_{\widehat{Z}}\big(\sum_i r(a_i - b_i)E_i\big) \cong \mathcal{O}_{\widehat{Z}}\,. \tag{9.7.51}$$

Therefore, it suffices to show that if $\sum\limits_i \alpha_iE_i$ is linearly equivalent to the zero divisor in $\widehat{Z}$, then $\alpha_i = 0$, for all i. In our setup, $\alpha_i = r(a_i - b_i)$, and their vanishing would give $a_i = b_i$, for all i, and hence $p^*\pi_X^*\omega_X \cong p^*\pi_Y^*\omega_Y$ by (9.7.50), completing the proof.

Note that, outside the union of pairwise intersections of distinct exceptional divisors, they can be contracted at once. For the sake of simplicity, we assume that there is a single contraction $\widehat{Z} \longrightarrow X$ which contracts all E_i's together. Suppose that,

$\sum_i \alpha_i E_i$ is linearly equivalent to the zero divisor; i.e., $\mathcal{O}_{\widehat{Z}}(\sum_i \alpha_i E_i) \cong \mathcal{O}_{\widehat{Z}}$. Assume that $\alpha_i < 0$ for $i \le m$ and $\alpha_i \ge 0$ for all $i > m$. We may assume that $m > 0$, otherwise change the global sign by dualizing. Let $s \in H^0\big(\widehat{Z}, \mathcal{O}_{\widehat{Z}}\big(-\sum_{i=1}^m \alpha_i E_i\big)\big)$ be the (unique) section vanishing to order $-\alpha_i$ along the divisors E_i, for all $i = 1, \dots, m$. Then for a trivializing section t of $\mathcal{O}_{\widehat{Z}}\big(\sum_i \alpha_i E_i\big)$, st is a section of $\mathcal{O}_{\widehat{Z}}\big(\sum_{i \ge m+1} \alpha_i E_i\big)$ vanishing along E_i for all $i \le m$.

Now by contracting the exceptional divisors E_i, for $i \ge m+1$, we see that any two sections of $\mathcal{O}_{\widehat{Z}}\big(\sum_{i \ge m+1} \alpha_i E_i\big)$ give rise to two functions on the complement of a closed subset of X of codimension ≥ 2, which by Hartong's theorem, differs by a scalar multiplication. Therefore, $\mathcal{O}_{\widehat{Z}}\big(\sum_{i \ge m+1} \alpha_i E_i\big)$ admits only one global section, up to scalar multiplication, namely the one which vanishes only along E_i of order α_i, for all $i \ge m+1$. Since the section st, as constructed above, is different from this section, we get a contradiction. Similarly, for positive α_i's, taking dual, we can make them negative, and similarly get a contradiction. This shows that all α_i's are zero, and completes the proof. □

Lemma 9.7.52. *If we have a morphism of smooth projective k-varieties $f : X \to Y$ of the same dimension, which is birational (isomorphism over an open subset with a rational inverse), then $\omega_X \cong f^*\omega_Y \otimes \mathcal{O}_X(D)$, for some effective divisor D on X.*

Proof. Since f is unramified over an open (and hence dense) subset of Y, there is a short exact sequence

$$0 \to TX \xrightarrow{df} f^*TY \to N_f \to 0,$$

where N_f is the normal sheaf supported on the ramification divisor of f. Dualizing it, we get

$$0 \to f^*\Omega_Y^1 \to \Omega_X^1 \to \mathcal{E}xt^1(N_f, \mathcal{O}_X) \to 0,$$

where $\mathcal{E}xt^1(N_f, \mathcal{O}_X)$ is again supported on the ramification divisor of f. Taking the n-th exterior power, where $n = \dim(X) = \dim(Y)$, we get the required identity. □

9.8. **Relative integral functor and base change formula.** Let S be a proper k-scheme, to be used as a parameter scheme. For any k-scheme Z, define $Z_S := Z \times S$. Let X and Y be smooth projective k-schemes, and consider the projection morphisms

$$\pi_{12} : (X \times Y)_S := X \times Y \times S \longrightarrow X \times Y$$
$$\pi_{13} : (X \times Y)_S := X \times Y \times S \longrightarrow X_S := X \times S$$
$$\pi_{23} : (X \times Y)_S := X \times Y \times S \longrightarrow Y_S := Y \times S.$$

For $\mathcal{P} \in D^b(X \times Y)$, let $\mathcal{P}_S := L\pi_{12}^*\mathcal{P} \in D^b\big((X \times Y)_S\big)$. Define a *relative integral functor*

$$\Phi_{\mathcal{P}_S}^{X_S \to Y_S} : D^b(X_S) \longrightarrow D^b(Y_S) \tag{9.8.1}$$

by sending $E^\bullet \in D^b(X_S)$ to

$$\Phi_{\mathcal{P}_S}^{X_S \to Y_S}(E^\bullet) := R\pi_{23*}\big(\iota_*\mathcal{P}_S \overset{L}{\otimes} L\pi_{13}^* E^\bullet\big) \in D^b(Y_S), \tag{9.8.2}$$

where $\iota : X_S \times_S Y_S \hookrightarrow X_S \times Y_S$.

Proposition 9.8.3. *Let $f : T \longrightarrow S$ be a morphism of proper k-schemes. For any k-scheme Z, we denote by f_Z the induced morphism $f_Z : Z_T \longrightarrow Z_S$. Let X, Y be smooth projective k-schemes. Denote by $\pi_{ij}^{XY?}$ the projection morphism from $X \times Y \times ?$ onto the (i,j)-th factor, where $? \in \{S, T\}$. For $\mathcal{P} \in D^b(X \times Y)$, let $\mathcal{P}_? := (\pi_{12}^{XY?})^*\mathcal{P}$, where $? \in \{S, T\}$. Then there is a functorial isomorphism of functors*

$$Lf_Y^* \circ \Phi_{\mathcal{P}_S}^{X_S \to Y_S} \cong \Phi_{\mathcal{P}_T}^{X_T \to Y_T} \circ Lf_X^*. \tag{9.8.4}$$

Proof. For any $E^\bullet \in D^b(X_S)$, we have

$$\begin{aligned} Lf_Y^* \circ \Phi_{\mathcal{P}_S}^{X_S \to Y_S}(E^\bullet) &\cong Lf_Y^*\big(R(\pi_{23}^{XYS})_*(\mathcal{P}_T \overset{L}{\otimes} (\pi_{13}^{XYS})^* E^\bullet\big) \\ &\cong R(\pi_{23}^{XYT})_*\big(Lf_{X\times Y}^*((\pi_{13}^{XYS})^* E^\bullet \overset{L}{\otimes} \mathcal{P}_S\big), \text{ by base change.} \\ &\cong R(\pi_{13}^{XYT})_*\big((Lf_X^* E^\bullet) \overset{L}{\otimes} \mathcal{P}_T\big) \\ &= (\Phi_{\mathcal{P}_T}^{X_T \to Y_T} \circ Lf_X^*)(E^\bullet). \end{aligned}$$

This completes the proof. □

The following proposition can be considered as the first step towards recovering the kernel of an integral transform, uniquely up to isomorphism.

Proposition 9.8.5. *Let X and Y be smooth projective k-schemes. Denote by π_{ij} the projection morphism from $X \times X \times Y$ onto the (i,j)-th factor. Let $\mathcal{P} \in D^b(X \times Y)$, and $\mathcal{P}_X := \pi_{23}^*\mathcal{P} \in D^b(X \times X \times Y)$. Then*

$$\mathcal{P} \cong \Phi_{\mathcal{P}_X}^{X \times X \to X \times Y}(\mathcal{O}_\Delta),$$

where $\Delta \subset X \times X$ is the image of the diagonal embedding $X \to X \times X$.

Proof. Let $\delta : X \hookrightarrow X \times X$ be the diagonal embedding and $\delta_Y : X \times Y \xrightarrow{\delta \times \mathrm{Id}_Y} X \times X \times Y$ the induced embedding. Note that, $\mathcal{O}_\Delta \cong \delta_*\mathcal{O}_X$. Let $\pi_X : X \times Y \to X$ be the projection morphism. Then we have

$$\begin{aligned}\Phi_{\mathcal{P}_X}^{X\times X\to X\times Y}(\mathcal{O}_\Delta) &= R\pi_{23*}\big(\mathcal{P}_X\overset{L}{\otimes}\pi_{12}^*(\delta_*\mathcal{O}_X)\big)\\ &\cong R\pi_{23*}\big(\pi_{23}^*\mathcal{P}\overset{L}{\otimes}\delta_{Y*}\pi_X^*\mathcal{O}_X\big)\\ &\cong R\pi_{23*}\big(\delta_{Y*}\delta_Y^*\pi_{23}^*\mathcal{P}\big)\\ &\cong R\pi_{23*}(\delta_{Y*}\mathcal{P})\cong\mathcal{P}.\end{aligned}$$

□

9.9. **Bondal-Orlov's reconstruction theorem revisited.**

9.10. **Equivalence of triangulated categories.**

9.11. **Equivalence criterion for integral functors.** Let X and Y be smooth projective k-varieties. Fix an object $\mathcal{P}\in D^b(X\times Y)$, and consider the integral functor (with kernel $\mathcal{P}$)

(9.11.1) $$\Phi_{\mathcal{P}}^{X\to Y}: D^b(X)\longrightarrow D^b(Y).$$

The following criterion

Proposition 9.11.2 (Bondal-Orlov)**.** *The integral functor $\Phi_{\mathcal{P}}: D^b(X)\longrightarrow D^b(Y)$, with kernel $\mathcal{P}\in D^b(X\times Y)$, is fully faithful if and only if for any closed points $x_1,x_2\in X$ and any integer i, we have*

$$\mathrm{Hom}(\Phi_{\mathcal{P}}(k(x_1)),\Phi_{\mathcal{P}}(k(x_2))[i]) = \begin{cases} k, & \textit{if } x_1=x_2 \textit{ and } i=0\,,\\ 0, & \textit{if } x_1=x_2 \textit{ and } i\notin\{0,1,\dots,\dim_k(X)\}\,,\\ 0, & \textit{if } x_1\neq x_2 \textit{ and } i\in\mathbb{Z}\,.\end{cases}$$

10. Grothendieck group

Definition 10.0.1. Let $\mathscr{A}$ be a small abelian category. The *Grothendieck group* of $\mathscr{A}$, denoted by $K_0(\mathscr{A})$, is the quotient of the free abelian group generated by the set of all isomorphism classes of objects of $\mathscr{A}$ by its normal subgroup generated by all elements $[B]-[A]-[C]$, where $0\to A\to B\to C\to 0$ is an exact sequence in $\mathscr{A}$.

Let $\mathscr{A}$ be an abelian category. For an object $A\in\mathscr{A}$, we denote by $A^\bullet\in\mathcal{K}om(\mathscr{A})$ the complex defined by

$$A^i=\begin{cases} A, & \text{if } i=0,\\ 0, & \text{if } i\neq 0\,.\end{cases}$$

We also denote by $A^\bullet\in D^b(\mathscr{A})$, the object in $D^b(\mathscr{A})$ represented by the complex $A^\bullet$.

Definition 10.0.2. Let $\mathscr{T}$ be a small triangulated category. For example, $\mathscr{T}$ can be the bounded derived category $D^b(\mathscr{A})$ of a small abelian category $\mathscr{A}$. Let $F(\mathscr{T})$ be the free abelian group generated by the set of all objects of $\mathscr{T}$. Let $\mathcal{R}(\mathscr{T})$ be the normal subgroup of $F(\mathscr{T})$ generated by the elements $[A]+[C]-[B] \in F(\mathscr{A})$, whenever there is a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathscr{T}$. The quotient abelian group

$$K_0(\mathscr{T}) := F(\mathscr{T})/\mathcal{R}(\mathscr{T}) \tag{10.0.3}$$

is called the *Grothendieck group* of $\mathscr{T}$.

Remark 10.0.4. It follows from the axiom (TR3) in Definition 1.2.4 that $A[1] = 0$ in $K_0(\mathscr{T})$.

The following proposition establishes relation between $K_0(\mathscr{A})$ and $K_0(D^b(\mathscr{A}))$.

Proposition 10.0.5. *Let $\mathscr{A}$ be a small abelian category, and let $D^b(\mathscr{A})$ be the bounded derived category of $\mathscr{A}$. Then the natural functor $\iota : \mathscr{A} \longrightarrow D^b(\mathscr{A})$ defined by sending an object $A \in \mathscr{A}$ to $A^\bullet \in D^b(\mathscr{A})$ induces an isomorphism of their Grothendieck groups $K_0(\mathscr{A}) \xrightarrow{\simeq} K_0(D^b(\mathscr{A}))$.*

11. Bridgeland Stability

The definition of stability condition generalized in different ways from curve to higher dimensional varieties. We follow [MS17, BBHR09, Huy06].

11.1. **Stability condition in an abelian category.**

Let $\mathcal{A}$ be an abelian category.

Definition 11.1.1. A *subobject* of an object $E \in \mathcal{A}$ is a monomorphism $\iota : F \hookrightarrow E$.

Let $\iota_1 : F_1 \hookrightarrow E$ and $\iota_2 : F_2 \hookrightarrow E$ be two subobjects of $E \in \mathcal{A}$. Since direct sum of two objects in $\mathcal{A}$ is both product and coproduct in $\mathcal{A}$, by the universal property of coproduct there is a unique morphism $\varphi : F_1 \oplus F_2 \to E$ such that the following diagram commutes.

$$\begin{tikzcd} & E & \\ F_1 \arrow[ur, "\iota_1"] \arrow[r] & F_1 \oplus F_2 \arrow[u, dashed, "\varphi"] & F_2 \arrow[ul, "\iota_2"'] \arrow[l] \end{tikzcd}$$

Since the image of $\varphi : F_1 \oplus F_2 \to E$ is the kernel of the cokernel $E \to \mathrm{Coker}(\varphi)$, $\mathrm{im}(\varphi) \hookrightarrow E$ is a monomorphism, and hence is a *subobject* of E, denoted by $F_1 + F_2$.

We denote by $F_1 \cap F_2$ the kernel of the epimorphism $F_1 \oplus F_2 \longrightarrow \mathrm{im}(\varphi)$ in $\mathcal{A}$. Thus, we have an exact sequence

$$0 \longrightarrow F_1 \cap F_2 \longrightarrow F_1 \oplus F_2 \longrightarrow F_1 + F_2 \longrightarrow 0. \tag{11.1.2}$$

Note that, $F_1 \cap F_2$ is the kernel of both composite morphisms

$$F_1 \xrightarrow{\iota_1} E \longrightarrow \mathrm{Coker}(\iota_2) \quad \text{and} \quad F_2 \xrightarrow{\iota_2} E \longrightarrow \mathrm{Coker}(\iota_1). \tag{11.1.3}$$

Therefore, $F_1 \cap F_2$ is a subobject of both F_1 and F_2, and hence of E. Note that, for any two morphisms $f_i : A_i \to B$, $i = 1, 2$, their fiber product $A_1 \times_{f_1,B,f_2} A_2$ exists in $\mathcal{A}$, and can be described as $\mathrm{Ker}((f_1, -f_2) : A_1 \oplus A_2 \to B)$. In particular, the *preimage* (fiber product) of a subobject $C \subset B$ along a morphism $f : A \to B$ exists uniquely as a subobject of A.

Remark 11.1.4. Note that, "being a subobject" is a transitive relation on $\mathrm{Ob}(\mathscr{C})$. (This may fails to hold if we consider equivalence class of monomorphisms instead of just monomorphism).

Let $\mathcal{A}$ be an abelian category. Denote by $K_0(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$ (c.f., Definition 10.0.1). For any complex number z, we denote by $\mathrm{Im}(z)$ (resp., $\mathrm{Re}(z)$) the *imaginary part* (resp., the *real part*) of z.

Definition 11.1.5. A *stability function* on $\mathcal{A}$ is an additive group homomorphism

$$Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

such that for any non-zero object $A \in \mathcal{A}$, we have $\mathrm{Im}(Z(A)) \geq 0$, and if $\mathrm{Im}(Z(A)) = 0$, then $\mathrm{Re}(Z(A)) < 0$.

Note that, "$\mathrm{Im}(Z(A)) \geq 0, \ \forall\ A \in \mathcal{A}$" does not imply that "$\mathrm{Im}(Z(B)) \geq 0, \ \forall\ B \in K_0(\mathcal{A})$".

Given a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$, we may think of

$$\deg_Z(A) := -\mathrm{Re}(Z(A)) \quad \text{and} \quad \mathrm{rk}_Z(A) := \mathrm{Im}(Z(A))$$

to be the *degree* and the *rank* of A with respect to the stability function Z, respectively. We may define the *slope* of $A \in \mathcal{A} \setminus \{0\}$ with respect to Z by

$$\mu_Z(A) := \begin{cases} \dfrac{\deg_Z(A)}{\mathrm{rk}_Z(A)}, & \text{if } \mathrm{rk}_Z(A) \neq 0, \text{ and} \\ +\infty, & \text{otherwise.} \end{cases} \tag{11.1.6}$$

Example 11.1.7. Let X be a smooth projective curve defined over an algebraically closed field k. Let $\mathfrak{Coh}(X)$ be the category of coherent sheaves of $\mathcal{O}_X$-modules on X. Let $Z : K_0(\mathfrak{Coh}(X)) \longrightarrow \mathbb{C}$ be the additive group homomorphism defined by sending a non-zero object $E \in \mathfrak{Coh}(X)$ to

$$Z(E) := -\deg(E) + \sqrt{-1} \cdot \mathrm{rk}(E) \in \mathbb{C}\,.$$

Clearly, Z is a stability function on $\mathfrak{Coh}(X)$. Note that, $\mu_Z(E)$ coincides with the usual slope $\mu(E) := \deg(E)/\operatorname{rk}(E)$ of E, and hence in this case, Z-(semi)stability coincides with the usual slope (semi)stability of coherent sheaves on X.

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k. Fix an ample class $\omega \in \operatorname{Amp}(X) \subseteq N^1(X) := \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and a divisor class $B \in N^1(X)$.

Definition 11.1.8 (K. Matsuki, R. Wentworth)**.** The *B-twisted Chern character* of $E \in \mathfrak{Coh}(X)$ is defined by

$$\operatorname{ch}^B(E) := \operatorname{ch}(E) \cdot e^{-B} = \sum_{i \geq 0} \operatorname{ch}_i(E) \cdot \sum_{j \geq 0} \frac{(-1)^j}{j!} B^j .$$

Thus, for $i \geq 0$, the *B-twisted i-th Chern character of E*, denoted $\operatorname{ch}_i^B(E)$, are given by

$$\begin{aligned}
\operatorname{ch}_0^B(E) &= \operatorname{ch}_0(E) = \operatorname{rk}(E), \\
\operatorname{ch}_1^B(E) &= \operatorname{ch}_1(E) - \operatorname{ch}_0(E) \cdot B, \\
\operatorname{ch}_2^B(E) &= \operatorname{ch}_2(E) - \operatorname{ch}_1(E) \cdot B + \frac{1}{2} \operatorname{rk}(E) \cdot B^2 ,
\end{aligned}$$

and so on. Note that, taking $B = 0$, we get back the usual Chern characters.

Define an additive group homomorphism

$$Z_{\omega,B} : K_0(\mathfrak{Coh}(X)) \longrightarrow \mathbb{C}$$

by sending a non-zero object E of $\mathfrak{Coh}(X)$ to the complex number

$$Z_{\omega,B}(E) := -\omega^{n-1} \cdot \operatorname{ch}_1^B(E) + \sqrt{-1} \cdot \omega^n \cdot \operatorname{ch}_0^B(E). \tag{11.1.9}$$

If T is a torsion coherent sheaf on X supported in dimension $\leq n-2$, then $\operatorname{rk}(T) = 0$, and the line bundle $\det(E)$ admits a nowhere vanishing global section (c.f., [Kob87, Proposition 5.6.14]). Then $\det(T) \cong \mathcal{O}_X$, and hence $\operatorname{ch}_1(T) = 0$. Therefore, $Z_{\omega,B}(T) = 0$. This shows that, $Z_{\omega,B}$ is not a stability function.

Remark 11.1.10. Let $\mathfrak{Coh}_{\leq n-2}(X)$ be the full subcategory of coherent sheaves on X whose supports have dimension $\leq n-2$, and let $\mathcal{A}$ be the localized category

$$\mathfrak{Coh}_{n,n-2}(X) = \mathfrak{Coh}(X)/\,\mathfrak{Coh}_{\leq n-2}(X) .$$

Then the function

$$Z_{\omega,B} : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$$

as defined in (11.1.9) above, is a stability function.

Definition 11.1.11. A *stability condition* is a pair $(\mathcal{A}, Z)$, where $\mathcal{A}$ is an abelian category and $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a stability function such that for any non-zero object E of $\mathcal{A}$, there is a filtration of E by its subobjects

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = E\,, \tag{11.1.12}$$

such that all E_i/E_{i-1} are Z-semistable, and their Z-slopes satisfies

$$\mu_Z(E_1) > \mu_Z(E_2/E_1) > \cdots > \mu_Z(E_\ell/E_{\ell-1}).$$

Such a filtration (11.1.12) is known as *Harder-Narasimhan filtration* of E.

Proposition 11.1.13. *Let $\mathcal{A}$ be an abelian category. Given a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$, Harder-Narasimhan filtration of an object $E \in \mathcal{A}$, if it exists, is unique up to isomorphism in $\mathcal{A}$.*

Existence of Harder-Narasimhan filtration requires some additional assumption on the category $\mathscr{A}$ and the stability function Z. For this we need some definitions.

Definition 11.1.14. An additive category $\mathcal{A}$ is said to be *noetherian* if for any object $E \in \mathcal{A}$, and any any ascending chain of subobjects

$$E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots\cdots \subseteq E,$$

of E, there is an integer $i_0 \geq 0$ such that $E_i = E_{i+1}$, for all $i \geq i_0$.

Lemma 11.1.15. *Let $\mathcal{A}$ be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function. If the image of the imaginary part of Z,*

$$\mathrm{Im}(Z) : K_0(\mathcal{A}) \to \mathbb{R},$$

is discrete in $\mathbb{R}$, then for any object $E \in \mathcal{A}$, there is a number $D_E \in \mathbb{R}$ such that for any subobject F of E in $\mathcal{A}$, we have $D(F) \leq D_E$.

Proof. Since the image of R is discrete in $\mathbb{R}$, we can do induction on $R(E)$. Our induction hypothesis would be the following: *if $E' \in \mathcal{A}$ with $R(E') < R(E)$, then there is $D_{E'} \in \mathbb{R}$ such that for any subobject $F' \subset E'$ we have $D(F') \leq D_{E'}$.*

If $R(E) = 0$, then $D(E) > 0$. Then for any subobject $F \subset E$, from the exact sequence $0 \to F \to E \to E/F \to 0$, we have $R(E) = R(F) + R(E/F)$ and $D(E) = D(F) + D(E/F)$ (since both $D := -\mathrm{Re}\, Z$ and $R := \mathrm{Im}\, Z$ are homomorphisms of additive groups). Since Z is a stability function, we have $R(F), R(E/F) \geq 0$. Then $R(F) = 0$, and hence $D(F) > 0$. Thus, $0 < D(F) \leq D(E)$; note that, $D(E/F) \geq 0$, and the inequality is strict if $E/F \neq 0$.

Assume that $R(E) > 0$. Suppose on the contrary that there is an infinite sequence of subobjects $\{F_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} D(F_n) = +\infty. \tag{11.1.16}$$

If for some $n \in \mathbb{N}$, $R(F_n) = R(E)$, then $R(E/F_n) = 0$ implies $D(E/F_n) \geq 0$, and so $D(F_n) \leq D(E)$. Therefore, we may assume that

$$R(F_n) < R(E), \;\; \forall\, n \in \mathbb{N}\,. \tag{11.1.17}$$

Note that, (by induction) it suffices to construct *an increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that*

$$D\big(\sum_{i=1}^{k} F_{n_i}\big) \geq k \quad \textit{and} \quad R\big(\sum_{i=1}^{k} F_{n_i}\big) < R(E). \tag{11.1.18}$$

Because then, $\{\sum\limits_{i=1}^{k} F_{n_i}\}_{k\in\mathbb{N}}$ would form an increasing sequence of proper subobjects of E, contradicting the fact that $\mathcal{A}$ is noetherian.

By our assumption (11.1.16), there is $n_1 \in \mathbb{N}$ such that $D(F_{n_1}) \geq 1$. Suppose that we have constructed $n_1, \ldots, n_{k-1}$. Then we have an exact sequence

$$0 \longrightarrow F_n \cap \sum_{i=1}^{k-1} F_{n_i} \longrightarrow F_n \oplus \sum_{i=1}^{k-1} F_{n_i} \longrightarrow F_n + \sum_{i=1}^{k-1} F_{n_i} \longrightarrow 0\,, \tag{11.1.19}$$

where the sum and intersection are taken inside of E in $\mathcal{A}$ (see (11.1.2)). This gives

$$D(F_n + \sum_{i=1}^{k-1} F_{n_i}) = D(F_n) + D(\sum_{i=1}^{k-1} F_{n_i}) - D(F_n \cap \sum_{i=1}^{k-1} F_{n_i}). \tag{11.1.20}$$

Since $R(\sum\limits_{i=1}^{k-1} F_{n_i}) < R(E)$ by induction hypothesis (11.1.18) and $F_n \cap \sum\limits_{i=1}^{k-1} F_{n_i}$ is a subobject of $\sum\limits_{i=1}^{k-1} F_{n_i}$, by induction hypothesis $D(F_n \cap \sum\limits_{i=1}^{k-1} F_{n_i}) \leq D_0$, for some $D_0 \in \mathbb{R}$, which depends only on $\sum\limits_{i=1}^{k-1} F_{n_i}$. Then from (11.1.20) and (11.1.18), we have

$$D(F_n + \sum_{i=1}^{k-1} F_{n_i}) \geq D(F_n) + (k-1) - D_0. \tag{11.1.21}$$

Taking limit as $n \to +\infty$ in (11.1.21) and using (11.1.16), we have

$$\lim_{n\to+\infty} D(F_n + \sum_{i=1}^{k-1} F_{n_i}) = +\infty. \tag{11.1.22}$$

As before, it follows from the exact sequence

$$0 \longrightarrow F_n + \sum_{i=1}^{k-1} F_{n_i} \longrightarrow E \longrightarrow E/(F_n + \sum_{i=1}^{k-1} F_{n_i}) \longrightarrow 0$$

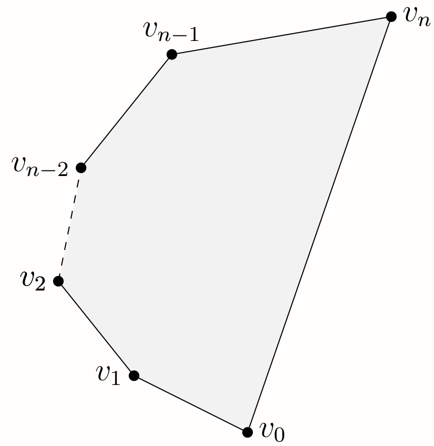


FIGURE 1. The polygon $\mathcal{P}(E)$

that, if $R(F_n + \sum_{i=1}^{k-1} F_{n_i}) = R(E)$ then $D(F_n + \sum_{i=1}^{k-1} F_{n_k}) \leq D(E)$. Therefore, in view of the limit in (11.1.22), we must have

$$R(F_n + \sum_{i=1}^{k-1} F_{n_i}) < R(E), \quad \forall\, n \gg 0. \tag{11.1.23}$$

Therefore, we can choose n_k to be some integer $n > n_{k-1}$ for which (11.1.23) holds, as claimed in (11.1.18). This completes the proof. □

Theorem 11.1.24. *Let $\mathcal{A}$ be a noetherian abelian category. Let $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ be a stability function. Assume that $R := \mathrm{Im}(Z) : K_0(\mathcal{A}) \to \mathbb{R}$ has discrete image in $\mathbb{R}$. Then any non-zero object $E \in \mathcal{A}$ admits a unique Harder-Narasimhan filtration.*

Proof. Since we have nothing to prove in case E is Z-semistable, we assume that E is not Z-semistable. Suppose that the image of $D := -\mathrm{Re}(Z)$ is discrete in $\mathbb{R}$ (and hence the image of Z is discrete in $\mathbb{C}$). Let $\mathcal{H}(E)$ be the convex hull in $\mathbb{C}$ of the (discrete) subset $\{Z(F) \in \mathbb{C} : F \text{ is a subobject of } E\}$. Then by Lemma 11.1.15, $\mathcal{H}(E)$ is bounded from the left side in $\mathbb{C}$. Let $\mathcal{H}_\ell$ be the half plane to the left of the straight line passing through $Z(E)$ and 0 in $\mathbb{C}$. Since the image of Z is discrete, $\mathcal{P}(E) := \mathcal{H}_\ell \cap \mathcal{H}(E)$ is a convex polygon in $\mathbb{C}$. Let $v_0 = 0, v_1, \ldots, v_{n-1}, v_n := Z(E)$ be the extremal vertices of $\mathcal{P}(E)$ in the ascending order of their imaginary parts. For each $i \in \{1, \ldots, n-1\}$, choose a subobject $F_i \subset E$ with $Z(F_i) = v_i$. We claim that,

(i) $F_{i-1} \subset F_i$, for all $i = 1, \ldots, n$, with $F_0 := 0$ and $F_n := E$,
(ii) $Q_i := F_i/F_{i-1}$ is Z-semistable, for all $i = 1, \ldots, n$, and
(iii) $\mu_Z(Q_1) > \cdots > \mu_Z(Q_{n-1})$.

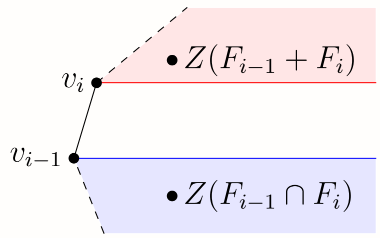


FIGURE 2. Locations of $Z(F_{i-1}\cap F_i)$ and $Z(F_{i-1}+F_i)$

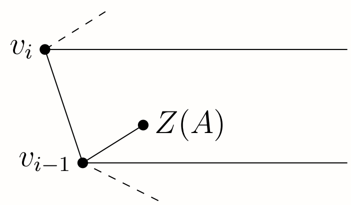


FIGURE 3. Slope of subobjects of F_i/F_{i-1}

Since $F_{i-1}\cap F_i$ and $F_{i-1}+F_i$ are subobjects of E, we have

$$Z(F_{i-1}\cap F_i), Z(F_{i-1}+F_i) \in \mathcal{H}(E). \tag{11.1.25}$$

Moreover, we have

$$R(F_{i-1}\cap F_i) \le R(F_{i-1}) < R(F_i) \le R(F_{i-1}+F_i). \tag{11.1.26}$$

Therefore, $Z(F_{i-1}\cap F_i) \in \mathcal{P}(E)$ lies on or below the line $\operatorname{Im}(z) = \operatorname{Im}(v_{i-1})$, and $Z(F_{i-1}+F_i) \in \mathcal{P}(E)$ lies on or above the line $\operatorname{Im}(z) = \operatorname{Im}(v_i)$; see Figure 2. From the exact sequence $0 \to F_{i-1}\cap F_i \to F_{i-1}\oplus F_i \to F_{i-1}+F_i \to 0$, we have

$$Z(F_{i-1}\cap Z_i) + Z(F_{i-1}+F_i) = v_{i-1}+v_i\,; \tag{11.1.27}$$

which gives

$$Z(F_{i-1}+F_i) - Z(F_{i-1}\cap F_i) = (v_{i-1} - Z(F_{i-1}\cap F_i)) + (v_i - Z(F_{i-1}\cap F_i)). \tag{11.1.28}$$

Comparing the real parts, in view of the Figure 2, we conclude that

$$Z(F_{i-1}\cap F_i) = v_{i-1} \quad \text{and} \quad Z(F_{i-1}+F_i) = v_i. \tag{11.1.29}$$

Therefore, $Z(F_{i-1}/(F_{i-1}\cap F_i)) = 0$. Since Z is a stability function, $F_{i-1}/(F_{i-1}\cap F_i) = 0$, and hence $F_{i-1}\cap F_i = F_{i-1}$. This proves our claim (i).

Note that, the Z-slope $\mu_Z(F_i/F_{i-1})$ is given by the slope of the line segment joining v_{i-1} to v_i. Hence the convexity of the polygon $\mathcal{P}(E)$ proves claim (iii); see Figure 1. Let $\overline{A}$ be a non-zero subobject of $Q_i := F_i/F_{i-1}$. Let $A \subset F_i$ be the preimage of $\overline{A}$ in F_i along the epimorphism $F_i \to F_i/F_{i-1}$. Then $Z(A) \in \mathcal{H}(E)$ and $R(F_{i-1}) \le R(A) \le R(F_i)$, since F_{i-1} is a subobject of A. Then $Z(A) - Z(F_{i-1}) = Z(A) - v_{i-1}$ has smaller of equal slope than that of $Z(F_i) - Z(F_{i-1}) = v_i - v_{i-1}$. In other words, $\mu_Z(\overline{A}) \leq \mu_Z(Q_i)$, which proves claim (ii); c.f. Figure 3.

It remains to prove the theorem without assuming that the image of $D := -\mathrm{Re}(Z)$ is discrete in $\mathbb{R}$. Since the image of Z need not be discrete in $\mathbb{C}$ anymore, we cannot directly conclude if the extremal vertices of $\mathcal{P}(E)$ are of the form $Z(F_i)$, for some subobject F_i of E. Suppose on the contrary that, there is an integer $i \in \{1, \ldots, n-1\}$ for which there is no subobject $F_i \subset E$ with $Z(F_i) = v_i$. By definition of $\mathcal{P}(E)$, there is a sequence of subobjects $\{F_j\}_{j\in\mathbb{N}}$ of E such that $\lim\limits_{j\to+\infty} Z(F_j) = v_i$. Since the image of $R := \mathrm{Im}(Z)$ is discrete in $\mathbb{R}$, there is an integer $n_0(i) \geq 1$ such that $\mathrm{Im}(Z(F_j)) = \mathrm{Im}(v_i)$, for all $j \geq n_0(i)$. Therefore, we can find a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that

$$R(v_i) = R(F_{n_k}) \quad \text{and} \quad D(v_i) - D(F_{n_k}) < \frac{1}{k}, \ \ \forall\, k \in \mathbb{N}. \tag{11.1.30}$$

Therefore, we may assume that all terms in the sequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ are distinct. From the exact sequence

$$0 \longrightarrow F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j} \longrightarrow F_{n_k} \oplus \sum_{j=1}^{k-1} F_{n_j} \longrightarrow \sum_{j=1}^{k} F_{n_j} \longrightarrow 0 \tag{11.1.31}$$

we have

$$D(\sum_{j=1}^{k} F_{n_k}) - D(\sum_{j=1}^{k-1} F_{n_k}) = D(F_{n_k}) - D(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) = D(F_{n_k}/F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) \tag{11.1.32}$$

and

$$R(\sum_{j=1}^{k} F_{n_k}) - R(\sum_{j=1}^{k-1} F_{n_k}) = R(F_{n_k}) - R(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) = R(F_{n_k}/F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}). \tag{11.1.33}$$

Since $F_{n_k} \cap \sum\limits_{j=1}^{k-1} F_{n_j}$ is a subobject of F_{n_k}, and Z is a stability condition,

$$R(F_{n_k}) - R(F_{n_k} \cap \sum_{j=1}^{k-1} F_{n_j}) \geq 0, \tag{11.1.34}$$

and if equality holds in (11.1.34) then $D(F_{n_k}) - D(F_{n_k} \cap \sum\limits_{j=1}^{k-1} F_{n_j}) > 0$. Since $\sum\limits_{j=1}^{k-1} F_{n_j}$ is a subobject of $\sum\limits_{j=1}^{k} F_{n_j}$, now it follows from (11.1.32) and (11.1.33) that $\{\sum\limits_{j=1}^{k} F_{n_j}\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of subobjects of E, which contradicts our assumption that $\mathcal{A}$ is noetherian. This completes the proof. $\square$

11.2. **Stability conditions on $D^b(X)$.** Let k be a field. Let X be a connected smooth projective k-variety. For any two objects $E^\bullet, F^\bullet \in D^b(X)$, we define

$$\chi(E^\bullet, F^\bullet) := \sum_i (-1)^i \dim_k \operatorname{Ext}^i(E^\bullet, F^\bullet)\,.$$

Let

$$K_0(X)^\perp := \{\alpha \in K_0(X) : \chi(\alpha, \beta) = 0,\ \forall\, \beta \in K_0(X)\}\,. \tag{11.2.1}$$

Definition 11.2.2. Let

12. Birational Geometry

13. Mirror Symmetry

13.1. **Fukaya category.**

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