CRITERION FOR EXISTENCE OF A LOGARITHMIC CONNECTION ON A PRINCIPAL BUNDLE OVER A SMOOTH COMPLEX PROJECTIVE VARIETY

SUDARSHAN GURJAR AND ARJUN PAUL

ABSTRACT. Let X be a connected smooth complex projective variety of dimension $n \geq 1$. Let D be a simple normal crossing divisor on X. Let G be a connected complex Lie group, and E_G a holomorphic principal G-bundle on X. In this article, we give criterion for existence of a logarithmic connection on E_G singular along D.

1. INTRODUCTION

A theorem of Weil [Wei38] says that a holomorphic vector bundle E on a smooth complex projective curve X admits a holomorphic connection if and only if each indecomposable holomorphic direct summand of E has degree 0; see [Ati57]. For connected reductive linear algebraic group G over $\mathbb{C}$, this result of Weil and Atiyah is generalized to the case of holomorphic principal G-bundles on a smooth complex projective curve in [AB02]. It follows from [Ati57, Theorem 4, p. 192] that, not every holomorphic bundle on a compact Kähler manifold can admit a holomorphic connection. Therefore, one can ask for criterion for a holomorphic bundle on X to admit a meromorphic connection. Simplest case of meromorphic connection is logarithmic connection. So it natural to ask when a given holomorphic bundle on X admits a logarithmic connection singular along a given divisor with prescribed residues. When X is a smooth complex projective curve, in [BDP18], a necessary and sufficient criterion for a vector bundle on X to admit a logarithmic connection singular along a given reduced effective divisor D on X with prescribed rigid residues along D is given. This result is further generalized to the case of holomorphic principal G-bundles over smooth complex projective curve in [BDPS17] when G is a connected reductive linear algebraic group over $\mathbb{C}$. When X is a smooth complex projective variety of dimension of more than one, no such criterion for existence of logarithmic connection on a holomorphic bundle on X with prescribed residues along a given reduced effective divisor is known to the best of our knowledge. In this article, we attempt to study this problem.

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Corresponding author: Arjun Paul.

1.1. **Outline of the paper.** Unless otherwise specified, X is a connected smooth complex projective variety of dimension at least one, and D a reduced effective divisor on X. We denote by G a connected affine algebraic group over $\mathbb{C}$. In Section §2, we recall definitions of simple normal crossing divisor, logarithmic connection on holomorphic vector bundles on X, and their residues along a simple normal crossing divisor on X. In Section §3, we extend this notion of logarithmic connection for principal G-bundles E_G on X, and discuss the notion of residue of a logarithmic connection on E_G singular along a simple normal crossing divisor on X.

In Section §3.3, we study logarithmic connection on principal bundles under the extensions of structure group. Let H be a connected closed algebraic subgroup of G over $\mathbb{C}$. Let E_H be a holomorphic principal H-bundle on X. Let $E_H(G)$ be the holomorphic principal G-bundle on X obtained by extending the structure group of E_H by the inclusion map $H \subset G$. Then we have the following (see Proposition 3.3.1).

Proposition 1.1.1. *If E_H admits a logarithmic connection singular along D, then $E_H(G)$ admits a logarithmic connection singular along D. The converse holds if H is reductive.*

The case of parabolic subgroup P of a reductive affine algebraic group G over $\mathbb{C}$ is interesting. Let $L \cong P/R_u(P)$ be the Levi factor of P, where $R_u(P)$ is the unipotent radical of P. Let E_L be the corresponding holomorphic principal L-bundle on X obtained by extending the structure group from P to L. The natural action of P on the Lie algebra $\mathfrak{n} := \mathrm{Lie}(R_u(P))$ give rise to a holomorphic vector bundle $E_P(\mathfrak{n}) := E_P \times^P \mathfrak{n}$ on X. Then we have the following (see Theorem 3.3.2).

Theorem 1.1.2. *Suppose that $H^1(X, E_P(\mathfrak{n}) \otimes \Omega^1_X(\log D)) = 0$. Then E_P admits a logarithmic connection singular along D if E_L admits a logarithmic connection singular along D.*

In Section §4, we discuss how existence of logarithmic connection on E_G singular along D can be ensured from existence of logarithmic connection on $E_G\big|_{X_n}$, where X_n is some sufficiently high degree hypersurface in X intersecting D properly. More precisely, we fix an embedding $X \hookrightarrow \mathbb{CP}^N$, for some $N > 0$. By a hypersurface X_n of degree n in X, we mean $X \cap H_n$, for some hypersurface H_n in $\mathbb{CP}^N$ of degree n. In [Ati57, Proposition 21], it is shown that if $\dim_{\mathbb{C}}(X) \geq 3$, then E_G admits a holomorphic connection if and only if for some smooth hypersurface X_n in X of sufficiently large degree, the principal G-bundle $E_G\big|_{X_n}$ admits a holomorphic connection. However, it is shown in [Ati57] that this result fails if $\dim_{\mathbb{C}}(X) = 2$; see also [BG18]. Also there are no complete answers known for this problem if $\dim_{\mathbb{C}}(X) = 2$. We prove the following analogue of [Ati57, Proposition 21] in the case of logarithmic connections on E_G singular along D in X (see Theorem 4.1.1).

Theorem 1.1.3. *With the above notations, if $\dim_{\mathbb{C}}(X) \geq 3$ and $D \subset X$ a reduced effective divisor in X, then E_G admits a logarithmic connection singular along D if and only if for some smooth hypersurface X_n in X of sufficiently large degree n, intersecting D properly, the principal G-bundle $E_G\big|_{X_n}$ on X_n admits a logarithmic connection singular along $D \cap X_n$.*

2. Preliminaries

2.1. **Simple normal crossing divisor.** Let X be a connected smooth complex projective variety of dimension at least one. We denote by TX (respectively, Ω^1_X) the tangent bundle (respectively, cotangent bundle) of X. The ideal sheaf $\mathscr{I}_D$ of an effective divisor D on X is a line bundle on X, denoted $\mathcal{O}_X(-D)$.

Definition 2.1.1. An effective divisor D on X is said to be a *simple normal crossing divisor* if D is reduced, each irreducible components of D are smooth, and for each point $x \in X$, there is a system of regular elements (local parameters) $z_1, \ldots, z_n \in \mathfrak{m}_x$ such that the stalk $\mathcal{O}_X(-D)_x$ of the line bundle $\mathcal{O}_X(-D)$ at x is generated by the product $z_1 \cdots z_r$, for some integer r with $1 \leq r \leq n$.

In other words, a *simple normal crossing divisor* on X is a reduced effective divisor D on X, all of whose irreducible components are smooth, and locally for some choice of coordinate functions $(z_1, \ldots, z_n)$ around a point $x_0 \in U \subset X$, $D \cap U$ is given by an equation $z_1 \cdots z_r = 0$, for some integer r with $1 \leq r \leq n$. This means, the irreducible components of D passing through x_0 are given by the equations $z_i = 0$, for $i = 1, \ldots, r$, and they intersects each others transversally.

2.2. **Logarithmic connection.** Let $D \subset X$ be a reduced effective divisor on X. For an integer $p \geq 0$, a *meromorphic p-form* on X is a section of $\Omega^p_X(D) := \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. A meromorphic p-form $\alpha \in (\Omega^p_X(D))(U)$ on an open subset $U \subset X$ is said to have a *logarithmic pole along D* if α is holomorphic on $U \setminus (U \cap D)$ and α has pole of order at most one along each irreducible component of D, and the same holds for $d\alpha$, where d denotes the holomorphic exterior differential operator (see [Voi07, p. 197]). Let $\Omega^p_X(\log D)$ be the subsheaf of meromorphic p-forms on X with at most logarithmic pole along D.

Let $p : E \to X$ be a holomorphic vector bundle of rank r on X. By abuse of notation, we denote by E the sheaf of holomorphic sections of $p : E \to X$; this is a locally free coherent sheaf of $\mathcal{O}_X$-modules of rank r on X.

Definition 2.2.1. A *logarithmic connection* on E singular along D is a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = f\nabla(s) + s \otimes df,$$

for all locally defined section f of $\mathcal{O}_X$ and locally defined section s of E.

2.3. **Residue of a logarithmic connection.** We now recall the definition of reside of a logarithmic connection from [Del70, Oht82]. Let D be a simple normal crossing divisor on X. Write $D = \bigcup_{j \in J} D_j$ as a union of all of its irreducible components. Let E be a holomorphic vector bundle of rank r on X admitting a logarithmic connection

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$$

singular along D. Since each irreducible component D_j of D are smooth, using *Poincaré residue map* (see [Voi07, p. 211], [GH94, p. 147]), we have the following homomorphism

$$\mathrm{Res}_{D_j} : E \otimes \Omega^1_X(\log D) \longrightarrow E \otimes \mathcal{O}_{D_j}\,, \;\; \forall\, j.$$

Then the composite map

$$\mathrm{Res}_{D_j} \circ \nabla : E\big|_{D_j} \longrightarrow E\big|_{D_j} \tag{2.3.1}$$

is a $\mathcal{O}_{D_j}$-module homomorphism, and hence defines a section

$$\mathrm{Res}_{D_j}(\nabla) \in H^0(D_j, \mathcal{E}nd(E)\big|_{D_j})\,,$$

called the *residue of* ∇ *along* D_j. For the sake of completeness, we recall explicit description of the reside of ∇ along D_j using local coordinates; [Oht82].

Since D is a simple normal crossing divisor on X, we can choose an open cover $\{U_\lambda : \lambda \in \Lambda\}$ of X such that for each $\lambda \in \Lambda$,

(I) $E\big|_{U_\lambda}$ is trivial, and
(II) for each irreducible component D_j of D, with $D_j \cap U_\lambda \neq \emptyset$, we can choose a local coordinate function $f_{\lambda j} \in \mathcal{O}_X(U_\lambda)$ for a local coordinate system on U_λ, such that $f_{\lambda j}$ is a defining equation of $D_j \cap U_\lambda$. If $D_j \cap U_\lambda = \emptyset$, we take $f_{\lambda j} = 1$.

If ∇_λ is the connection matrix of ∇ with respect to a holomorphic local frame $s_\lambda = (s_{\lambda 1}, \ldots, s_{\lambda r})$ for E on U_λ, then we have

$$\nabla(s_\lambda) = \nabla_\lambda \otimes s_\lambda, \tag{2.3.2}$$

where ∇_λ is a $r \times r$ matrix whose entries are holomorphic sections of $\Omega^1_X(\log D)$ over U_λ. For each D_j, the matrix ∇_λ can be written as

$$\nabla_\lambda = R_{\lambda j}\frac{df_{\lambda j}}{f_{\lambda j}} + S_{\lambda j}\,, \tag{2.3.3}$$

where $R_{\lambda j}$ is a $r \times r$ matrix with entries in $\mathcal{O}_X(U_\lambda)$ and $S_{\lambda j}$ is a $r \times r$ matrix with entries in $(\Omega^1_X(\log D))(U_\lambda)$ with simple pole along $\bigcup_{j' \neq j} D_{j'}$. Then

$$\operatorname{Res}_{D_j}(\nabla_\lambda) := R_{\lambda j}\big|_{U_\lambda \cap D_j} \tag{2.3.4}$$

is a $r \times r$ matrix whose entries are holomorphic functions on $U_\lambda \cap D_j$; it is independent of choice of local defining equation $f_{\lambda j}$ for D_j. Then $\{\operatorname{Res}_{D_j}(\nabla_\lambda)\}_{\lambda \in \Lambda}$ defines a holomorphic global section

$$\operatorname{Res}_{D_j}(\nabla) \in H^0(D_j, \mathcal{E}nd(E\big|_{D_j}))\,, \tag{2.3.5}$$

known as the *residue of* ∇ *along* D_j.

Remark 2.3.1. If we further assume that intersections of any finite number of irreducible components of D are connected, then the Chern classes of E can be computed in terms of the residues of the logarithmic connection ∇ along the irreducible components of D, and the first Chern classes of the line bundles associated to the irreducible components of D; see [Oht82, Theorem 3, p. 16].

3. Logarithmic Connection On Principal Bundles

3.1. **Logarithmic Atiyah exact sequence.** Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$. Let

$$p : E_G \longrightarrow X \tag{3.1.1}$$

be a holomorphic principal G-bundle on X. The holomorphic G-action on E_G induces a holomorphic G-action on the holomorphic tangent bundle TE_G of E_G, and the associated quotient $\text{At}(E_G) := TE_G/G$ is a holomorphic vector bundle on X, known as the *Atiyah bundle* of E_G; the sections of $\text{At}(E_G)$ are given by G-invariant holomorphic vector fields on E_G. Let $\text{ad}(E_G) := E_G \times^G \mathfrak{g}$ be the *adjoint vector bundle* associated to the adjoint representation of G to its Lie algebra $\mathfrak{g}$. The surjective submersion p in (3.1.1) induces a short exact sequence of holomorphic vector bundles on X,

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{d'p} TX \longrightarrow 0, \tag{3.1.2}$$

called the *Atiyah exact sequence* of E_G. A *holomorphic connection* on E_G is given by a holomorphic vector bundle homomorphism $\eta : TX \to \text{At}(E_G)$ such that $d'p \circ \eta = \text{Id}_{TX}$; see [Ati57]. We now modify the exact sequence (3.1.2) to define a logarithmic Atiyah exact sequence.

Let D be a reduced effective divisor on X. Then $TX(-\log D) := (\Omega^1_X(\log D))^\vee$ is a locally free $\mathcal{O}_X$-submodule of TX. In fact, we have $TX(-D) \subseteq TX(-\log D) \subset TX$. Then we have a locally free $\mathcal{O}_X$-submodule $\mathcal{A}_D(E_G) := (d'p)^{-1}(TX(-\log D))$ of $\text{At}(E_G)$ which fits into the following short exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \longrightarrow \text{ad}(E_G) \xrightarrow{\iota_D} \mathcal{A}_D(E_G) \xrightarrow{\widetilde{d'p}} TX(-\log D) \longrightarrow 0, \tag{3.1.3}$$

called the *logarithmic Atiyah exact sequence* of E_G for the divisor D, (see also [BDP18]). Moreover, we have the following commutative diagram of $\mathcal{O}_X$-module homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r, "\iota_D"] \arrow[d, equal] & \mathcal{A}_D(E_G) \arrow[r, "\widetilde{d'p}"] \arrow[d, hook, "J"] & TX(-\log D) \arrow[r] \arrow[d, hook, "I"] & 0 \\ 0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r, "\iota"] & \mathrm{At}(E_G) \arrow[r, "d'p"] & TX \arrow[r] & 0 \end{tikzcd} \tag{3.1.4}$$

Let E be a holomorphic vector bundle E of rank n on X. Let $p : E_{\mathrm{GL}_n(\mathbb{C})} \longrightarrow X$ be the holomorphic frame bundle of E; this is a principal $\mathrm{GL}_n(\mathbb{C})$-bundle on X. Note that, $\mathrm{ad}(E_{\mathrm{GL}_n(\mathbb{C})})$ is naturally isomorphic to $\mathcal{E}nd(E)$.

Proposition 3.1.1. *E admits a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ singular along D if and only if the exact sequence in (3.1.3) associated to $E_{\mathrm{GL}_n(\mathbb{C})}$ splits holomorphically.*

Proof. Let $G = \mathrm{GL}_n(\mathbb{C})$. Let $\mathcal{D}er_{\mathbb{C}}(E_G)$ be the sheaf of $\mathbb{C}$-linear derivations of $\mathcal{O}_{E_G}$. Then there is a natural $\mathcal{O}_{E_G}$-module isomorphism $\mathcal{D}er_{\mathbb{C}}(E_G) \xrightarrow{\simeq} \mathcal{H}om(\Omega^1_{E_G}, \mathcal{O}_{E_G}) = TE_G$ defined by sending a locally defined $\mathbb{C}$-linear derivation ξ of $\mathcal{O}_{E_G}$ to the unique $\mathcal{O}_{E_G}$-module homomorphism $\widetilde{\xi} : \Omega^1_{E_G} \to \mathcal{O}_{E_G}$ such that $\widetilde{\xi} \circ d = \xi$, where $d : \mathcal{O}_{E_G} \to \Omega^1_{E_G}$ is the Kähler differential operator on E_G. Then the G-invariant sections of $\mathcal{D}er_{\mathbb{C}}(E_G)$ descend to sections of $\mathrm{At}(E_G)$.

Now it is clear that given a $\mathcal{O}_X$-module homomorphism $\eta : TX(-\log D) \to \mathcal{A}_D(E)$ with $\eta \circ \widetilde{d'p} = \mathrm{Id}_{TX(-\log D)}$, for each locally defined section ξ of $TX(-\log D)$, its image $\eta(\xi)$ defines a G-invariant $\mathbb{C}$-linear derivation of E. Thus we have a logarithmic connection on E singular along D. Conversely, given a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$ singular along D, for each locally defined section ξ of $TX(-\log D) = \mathcal{H}om(\Omega^1_X(\log D), \mathcal{O}_X)$, we get a $\mathrm{GL}_n(\mathbb{C})$-invariant $\mathbb{C}$-linear derivation $\nabla_\xi := (\mathrm{Id}_E \otimes \xi) \circ \nabla$ of E. This defines a splitting of the short exact sequence (3.1.3). $\square$

The above Proposition 3.1.1 motivates us to define the following (see also [BDPS17, §2.2]).

Definition 3.1.2. Let $p : E_G \to X$ be a holomorphic principal G-bundle on X. A *logarithmic connection* on E_G singular along D is a holomorphic vector bundle homomorphism $\eta : TX(-\log D) \to \mathcal{A}_D(E_G)$ such that $\widetilde{d'p} \circ \eta = \mathrm{Id}_{TX(-\log D)}$, where $\widetilde{d'p}$ is the homomorphism in (3.1.3).

We refer the exact sequence (3.1.3) as the *logarithmic Atiyah exact sequence* of E_G associated to the divisor D. The exact sequence (3.1.3) defines a cohomology class

$$\Phi_D(E) \in H^1(X, \mathrm{ad}(E_G) \otimes \Omega^1_X(\log D)), \tag{3.1.5}$$

which we call the *logarithmic Atiyah class* of E along D, such that the exact sequence (3.1.3) splits holomorphically if and only if $\Phi_D(E) = 0$.

3.2. **Residue of logarithmic connection on a principal bundle.** Let D be a simple normal crossing divisor on X, locally defined by $z_1 \cdots z_r = 0$. Let us denote by D_j the irreducible component of D locally defined by $z_j = 0$, for each $j = 1, \ldots, r$. Let $TX(-\log D)$ be the dual of $\Omega^1_X(\log D)$; this is a locally free coherent sheaf of $\mathcal{O}_X$-modules of rank $d = \dim_{\mathbb{C}}(X)$, with local frame fields given by $\left(z_1 \frac{\partial}{\partial z_1}, \ldots, z_r \frac{\partial}{\partial z_r}, \frac{\partial}{\partial z_{r+1}}, \ldots, \frac{\partial}{\partial z_d}\right)$. For each $j = 1, \ldots, r$, over D_j, we can identify $z_j \frac{\partial}{\partial z_j}$ with 1; this identification is independent of choice of local coordinate system $(z_1, \ldots, z_d)$ on X such that D_i is locally given by vanishing locus of z_i, for all $i = 1, \ldots, r$. Thus, $TX(-\log D)\big|_{D_j}$ is locally free $\mathcal{O}_{D_j}$-module generated by

$$\left(z_1 \frac{\partial}{\partial z_1}, \ldots, z_{j-1} \frac{\partial}{\partial z_{j-1}}, 1, z_{j+1} \frac{\partial}{\partial z_{j+1}}, \ldots, z_r \frac{\partial}{\partial z_r}, \frac{\partial}{\partial z_{r+1}}, \ldots, \frac{\partial}{\partial z_d}\right).$$

Therefore, we have an injective homomorphism $\mathcal{O}_{D_j} \longrightarrow TX(-\log D)\big|_{D_j}$. Let

$$\eta : TX(-\log D) \longrightarrow \mathcal{A}_D(E_G) \tag{3.2.1}$$

be a logarithmic connection on E_G singular along D; that means, η is an $\mathcal{O}_X$-module homomorphism such that $\widetilde{d'p} \circ \eta = \mathrm{Id}_{TX(-\log D)}$ (see (3.1.3)). Note that the image of $\eta\big|_{\mathcal{O}_{D_j}}$ lands inside $\mathrm{ad}(E_G)\big|_{D_j} \subset \mathcal{A}_{D_j}(E_G)\big|_{D_j}$. This gives a section

$$\mathrm{Res}_{D_j}(\eta) \in H^0(D_j, \mathrm{ad}(E_G)\big|_{D_j}), \tag{3.2.2}$$

called the *residue* of η along D_j, for all $j = 1, \ldots, r$. Then we have the following.

Proposition 3.2.1. *Let E be a holomorphic vector bundle of rank n on X, and let $E_{\mathrm{GL}_n(\mathbb{C})}$ be the holomorphic frame bundle of E. If η in (3.2.1) is the logarithmic connection on $E_{\mathrm{GL}_n(\mathbb{C})}$ associated to a logarithmic connection ∇ on E as defined in (2.2.1), then for each irreducible component D_j of D, we have*

$$\mathrm{Res}_{D_j}(\nabla) = \mathrm{Res}_{D_j}(\eta)\,, \tag{3.2.3}$$

where $\mathrm{Res}_{D_j}(\nabla)$ is as defined in (2.3.5) and $\mathrm{Res}_{D_j}(\eta)$ is as defined in (3.2.2).

Proof. Follows from the proof of Proposition 3.1.1 and the definition of residue in (2.3.1). □

3.3. **Extension of structure group.** Let G and H be two connected complex Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $f : H \longrightarrow G$ be a homomorphism of complex Lie groups, and $df : \mathfrak{h} \longrightarrow \mathfrak{g}$ the Lie algebra homomorphism induced by f. Let $p : E_H \to X$ be a holomorphic principal H-bundle on X. Then we have a holomorphic principal G-bundle $p' : E_G := E_H(G) \to X$ on X obtained by extending the structure group of E_H by the homomorphism f. Then there is a natural vector bundle homomorphisms $\alpha : \mathrm{ad}(E_H) \longrightarrow$

$\text{ad}(E_G)$ and $\beta : \text{At}(E_H) \longrightarrow \text{At}(E_G)$ induced by f. Then we have the following commutative diagram of vector bundle homomorphisms with two rows exact (see [Ati57]).

$$\begin{tikzcd} 0 \arrow[r] & \text{ad}(E_H) \arrow[r] \arrow[d, "\alpha"] & \text{At}(E_H) \arrow[r, "d'p"] \arrow[d, "\beta"] & TX \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \text{ad}(E_G) \arrow[r] & \text{At}(E_G) \arrow[r, "d'p'"] & TX \arrow[r] & 0 \end{tikzcd} \tag{3.3.1}$$

Let D be a reduced effective divisor on X. Then the commutative diagram (3.1.4) and (3.3.1) gives the following commutative diagram of vector bundle homomorphisms with two rows exact.

$$\begin{tikzcd} 0 \arrow[r] & \text{ad}(E_H) \arrow[r] \arrow[d, "\alpha"] & \mathcal{A}_D(E_H) \arrow[r, "\widetilde{d'p}"] \arrow[d, "\beta"] & TX(-\log D) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \text{ad}(E_G) \arrow[r] & \mathcal{A}_D(E_G) \arrow[r, "\widetilde{d'p'}"] & TX(-\log D) \arrow[r] & 0 \end{tikzcd} \tag{3.3.2}$$

If $\eta : TX(-\log D) \to \mathcal{A}_D(E_H)$ is a holomorphic vector bundle homomorphism with $\widetilde{d'p} \circ \eta = \text{Id}_{TX(-\log D)}$, then $f_*(\eta) := \beta \circ \eta$ satisfies $\widetilde{d'p'} \circ (f_*\eta) = \text{Id}_{TX(-\log D)}$. Consequently, if D is a simple normal crossing divisor D in X, for each irreducible component D_j of D, we have $\text{Res}_{D_j}(f_*\eta) = \alpha \circ \text{Res}_{D_j}(\eta)$; (see also [BDPS17, §2.4]).

In fact, it follows from commutativity of the diagram (3.3.2) that there is a natural homomorphism of cohomologies

$$f_* : H^1(X, \text{ad}(E_H) \otimes \Omega^1_X(\log D)) \longrightarrow H^1(X, \text{ad}(E_G) \otimes \Omega^1_X(\log D)), \tag{3.3.3}$$

induced by f, which sends the cohomology class $\Phi_D(E_H)$ to $\Phi_D(E_G)$; see (3.1.5). Since the homomorphism (3.3.3) is not necessarily injective, in general, existence of a logarithmic connection on E_G singular along D may not ensure existence of a logarithmic connection on E_H singular along D. However, if $f : H \longrightarrow G$ is an injective homomorphism of connected affine algebraic groups over $\mathbb{C}$ with H reductive, then the above homomorphism (3.3.3) can be shown to be injective (see the proof of [BDPS17, Lemma 3.3] for more details). Therefore, from the above discussions, we have the following.

Proposition 3.3.1. *With the above notations, E_G admits a logarithmic connection singular along D if E_H admits a logarithmic connection singular along D. Converse holds if $f : H \to G$ is an injective homomorphism of connected affine algebraic groups over $\mathbb{C}$ with H reductive.*

Let G be a connected reductive affine algebraic group over $\mathbb{C}$. Let P be a parabolic subgroup of G. Let $R_u(P)$ be the unipotent radical of P. Then there is a closed connected algebraic subgroup $L \subset P$ such that the restriction to L of the quotient homomorphism

$$q : P \longrightarrow P/R_u(P),$$

is an isomorphism of algebraic groups over $\mathbb{C}$. Clearly, L is reductive; and it is known as the *Levi factor* of P (see e.g., [Mil17, p. 559]). Consider the homomorphism

$$q' := \left(q\big|_L\right)^{-1} \circ q : P \longrightarrow L\,. \tag{3.3.4}$$

Let E_P be a homomorphic principal P-bundle on X. Let $E_L := E_P(L)$ be the holomorphic principal L-bundle on X obtained by extending the structure group of E_P by the homomorphism q' in (3.3.4). The Lie algebra $\mathfrak{n} := \mathrm{Lie}(R_u(P))$ of $R_u(P)$ is the nilpotent radical of the Lie algebra $\mathfrak{p} := \mathrm{Lie}(P)$ of P. The action of P on $\mathfrak{n}$ gives rise to a holomorphic vector bundle $E_P(\mathfrak{n}) := E_P \times^P \mathfrak{n}$ on X. Note that, $E_P(\mathfrak{n})$ is a subbundle of $\mathrm{ad}(E_P) = E_P(\mathfrak{p})$, and the associated quotient vector bundle $\mathrm{ad}(E_P)/E_P(\mathfrak{n})$ is isomorphic to $E_P(\mathfrak{l}) \cong \mathrm{ad}(E_L)$, where $\mathfrak{l} = \mathrm{Lie}(L)$. Then we have the following.

Theorem 3.3.2. *With the above notations, if $H^1(X, E_P(\mathfrak{n}) \otimes \Omega^1_X(\log D)) = 0$, then E_P admits a logarithmic connection singular along D whenever E_L admits a logarithmic connection singular along D.*

Proof. Replacing H by P and G by L in the commutative diagram (3.3.2), we have the following commutative diagram of holomorphic vector bundle homomorphisms, with all rows and columns exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E_P(\mathfrak{n}) \arrow[r, equal] \arrow[d] & E_P(\mathfrak{n}) \arrow[d] & & \\
0 \arrow[r] & \mathrm{ad}(E_P) \arrow[r] \arrow[d, "\alpha"] & \mathcal{A}_D(E_P) \arrow[r, "\sigma_P"] \arrow[d, "\beta"] & TX(-\log D) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{ad}(E_L) \arrow[r] \arrow[d] & \mathcal{A}_D(E_L) \arrow[r, "\sigma_L"] \arrow[d] & TX(-\log D) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.3.5}$$

Let $\eta : TX(-\log D) \to \mathcal{A}_D(E_L)$ be an $\mathcal{O}_X$-module homomorphism such that $\sigma_L \circ \eta = \mathrm{Id}_{TX(-\log D)}$, where σ_L is the homomorphism in (3.3.5). Let $\mathcal{F} := \beta^{-1}(\eta(TX(-\log D))) \subset \mathcal{A}_D(E_P)$. This fits into the following short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow E_P(\mathfrak{n}) \longrightarrow \mathcal{F} \longrightarrow TX(-\log D) \longrightarrow 0\,. \tag{3.3.6}$$

Then the logarithmic Atiyah exact sequence for E_P in (3.3.5) splits $\mathcal{O}_X$-linearly if the exact sequence (3.3.6) splits $\mathcal{O}_X$-linearly. Since the obstruction for splitting of the exact sequence (3.3.6) lies in $H^1(X, E_P(\mathfrak{n}) \otimes \Omega^1_X(\log D))$, the result follows. $\square$

4. Existence of Logarithmic Connection

4.1. **Restriction theorem for logarithmic connection.** Let X be a smooth complex projective variety of dimension $d \geq 1$. Fix an embedding of X into a complex projective space $\mathbb{CP}^N$, for some positive integer N. A hypersurface X_n of degree n in X is given by $X \cap H_n$, where H_n is a hypersurface of degree n in $\mathbb{CP}^N$. For general hypersurfaces H_n, we get $X_n = X \cap H_n$ smooth [Har77]. Let $\mathrm{Div}(X)$ be the group of all divisors in X. For $D_1, D_2 \in \mathrm{Div}(X)$, we say that D_1 and D_2 *meets properly* if for each prime divisor V (respectively, W) appearing with non-zero coefficient in D_1 (respectively, D_2), we have $\dim(V \cap W) = d - 2$. It is clear that if two reduced effective divisors $D_1, D_2 \in \mathrm{Div}(X)$ meets properly, then $D_1 \cap D_2$ is a divisor in both D_1 and D_2.

Let G be a connected complex Lie group, and E_G a holomorphic principal G-bundle on X. Then we have the following result.

Theorem 4.1.1. *Assume that $\dim_{\mathbb{C}}(X) \geq 3$ and $D \subset X$ a reduced effective divisor in X. Then E_G admits a logarithmic connection singular along D if and only if for some smooth hypersurface X_n of sufficiently large degree n, which intersects D properly, the principal G-bundle $E_G\big|_{X_n}$ on X_n admits a logarithmic connection singular along $D \cap X_n$.*

Proof. For any divisor H on X, we denote by $\mathcal{O}_X(H)$ the line bundle on X associated to H. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_X(-X_n) \longrightarrow \mathcal{F} \longrightarrow \iota_{n*}(\mathcal{F}\big|_{X_n}) \longrightarrow 0\,, \tag{4.1.1}$$

where $\iota_n : X_n \hookrightarrow X$ is the inclusion morphism. Since $\dim_{\mathbb{C}}(X) \geq 3$, it follows from Serre's theorem [Har77, p. 228] that for $n \gg 0$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(-X_n)) = 0\,, \quad \forall\, i = 1, 2. \tag{4.1.2}$$

Then the long exact sequence of cohomologies associated to the short exact sequence (4.1.1) gives an isomorphism.

$$H^1(X, \mathcal{F}) \xrightarrow{\cong} H^1(X_n, \mathcal{F}\big|_{X_n})\,. \tag{4.1.3}$$

Since X_n intersects D properly by assumption, $D_n := X_n \cap D$ is an effective divisor in X_n, and we have a natural isomorphism $\mathcal{O}_X(D)\big|_{X_n} \cong \mathcal{O}_{X_n}(D_n)$. Then from [Har77, Chapter II, Theorem 8.17], we have an exact sequence of $\mathcal{O}_{X_n}$-modules

$$0 \longrightarrow \left(\mathscr{I}_{X_n}/\mathscr{I}_{X_n}^2\right) \otimes \mathcal{O}_{X_n}(D_n) \longrightarrow \Omega^1_X(D)\big|_{X_n} \xrightarrow{\xi} \Omega^1_{X_n}(D_n) \longrightarrow 0\,, \tag{4.1.4}$$

where $\mathscr{I}_{X_n}$ is the ideal sheaf of the hypersurface X_n in X. Note that there is a natural $\mathcal{O}_{X_n}$-module isomorphism

$$\Omega^1_X(\log D)\big|_{X_n} \xrightarrow{\cong} \xi^{-1}(\Omega^1_{X_n}(\log D_n))\,. \tag{4.1.5}$$

Since $\mathscr{I}_{X_n}/\mathscr{I}_{X_n}^2 \cong \mathcal{O}_X(-X_n)\big|_{X_n}$, from (4.1.4) using (4.1.5) we have the following short exact sequence of $\mathcal{O}_{X_n}$-modules

$$0 \longrightarrow \mathcal{O}_X(D - X_n)\big|_{X_n} \longrightarrow \Omega^1_X(\log D)\big|_{X_n} \longrightarrow \Omega^1_{X_n}(\log D_n) \longrightarrow 0\,. \tag{4.1.6}$$

Now tensoring the exact sequence (4.1.6) with $\mathrm{ad}(E_G)\big|_{X_n}$, we get the following short exact sequence of $\mathcal{O}_{X_n}$-modules

$$\begin{aligned} 0 \longrightarrow (\mathrm{ad}(E_G) \otimes \mathcal{O}_X(D - X_n))\,\big|_{X_n} &\longrightarrow \big(\mathrm{ad}(E_G) \otimes \Omega^1_X(\log D)\big)\,\big|_{X_n} \\ &\longrightarrow \mathrm{ad}(E_G)\big|_{X_n} \otimes \Omega^1_{X_n}(\log D_n) \longrightarrow 0\,. \end{aligned} \tag{4.1.7}$$

Now taking $\mathcal{F} = \mathrm{ad}(E_G) \otimes \mathcal{O}_X(D)$ and $\mathcal{F} = \mathrm{ad}(E_G) \otimes \mathcal{O}_X(D - X_n)$ in (4.1.2), we get

$$H^1(X, \mathrm{ad}(E_G) \otimes \mathcal{O}_X(D - X_n)) = 0 = H^2(X, \mathrm{ad}(E_G) \otimes \mathcal{O}_X(D - 2X_n))\,, \tag{4.1.8}$$

for n large enough. Fix one such $n \gg 0$. Then applying (4.1.3) for $\mathcal{F} = \mathrm{ad}(E_G)\otimes\mathcal{O}_X(D-X_n)$, using (4.1.8) we get

$$H^1(X_n, (\mathrm{ad}(E_G) \otimes \mathcal{O}_X(D - X_n))\,\big|_{X_n}) = 0\,. \tag{4.1.9}$$

Now from the long exact sequence of cohomologies associated to (4.1.7), using (4.1.9) we get an exact sequence of cohomologies

$$0 \longrightarrow H^1(X_n, \big(\mathrm{ad}(E_G) \otimes \Omega^1_X(\log D)\big)\,\big|_{X_n}) \longrightarrow H^1(X_n, \mathrm{ad}(E_G\big|_{X_n}) \otimes \Omega^1_{X_n}(\log D_n))\,. \tag{4.1.10}$$

Now taking $\mathcal{F} = \mathrm{ad}(E_G) \otimes \Omega^1_X(\log D)$ in (4.1.3), from (4.1.10) we see that the inclusion map $\iota_n : X_n \hookrightarrow X$ induces an injective homomorphism

$$\widetilde{\iota_n} : H^1(X, \mathrm{ad}(E_G) \otimes \Omega^1_X(\log D)) \longrightarrow H^1(X_n, \mathrm{ad}(E_G\big|_{X_n}) \otimes \Omega^1_{X_n}(\log D_n))\,. \tag{4.1.11}$$

The inclusion morphism $\iota_n : X_n \hookrightarrow X$ induces the following commutative diagram of homomorphisms of sheaves of $\mathcal{O}_X$-modules on X with two rows exact.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(E_G) \arrow[r] \arrow[d] & \mathcal{A}_D(E_G) \arrow[r] \arrow[d] & TX(-\log D) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \iota_{n*}(\mathrm{ad}(E_G\big|_{X_n})) \arrow[r] & \iota_{n*}(\mathcal{A}_{D_n}(E_G\big|_{X_n})) \arrow[r] & \iota_{n*}(TX_n(-\log D_n)) \arrow[r] & 0 \end{tikzcd}$$

Now one can check that the homomorphism (4.1.11) sends the cohomology class $\Phi_D(E_G) \in H^1(X, \mathrm{ad}(E_G)\otimes\Omega^1_X(\log D))$, as defined in (3.1.5), to the cohomology class $\Phi_{D_n}(E_G\big|_{X_n})$. Thus $\Phi_D(E_G) = 0$ if and only if $\Phi_{D_n}(E_G\big|_{X_n}) = 0$. This completes the proof. □

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SUDARSHAN GURJAR
DEPARTMENT OF MATHEMATICS,
INDIAN INSTITUTE OF TECHNOLOGY BOMBAY,
POWAI, MUMBAI 400076, MAHARASHTRA, INDIA.
EMAIL: sgurjar@math.iitb.ac.in

ARJUN PAUL
DEPARTMENT OF MATHEMATICS,
INDIAN INSTITUTE OF TECHNOLOGY BOMBAY,
POWAI, MUMBAI 400076, MAHARASHTRA, INDIA.
EMAIL: arjunp@math.iitb.ac.in