Proc. Indian Acad. Sci. (Math. Sci.) Vol. XXX, No. X, August 2020, pp. 1–12
DOI xx.xxxx/xxxxxx-xxx-xxx-x

Fundamental Group Schemes of n-fold Symmetric Product of a Smooth Projective Curve

ARJUN PAUL[1,*] and RONNIE SEBASTIAN[2]

[1]Department of Mathematics, International Centre for Theoretical Sciences, Survey No. 151, Shivakote, Hesaraghatta Hobli, Bengaluru - 560 089, India.
[2]Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, Maharashtra, India.
[*]Corresponding author.
E-mail[1]: arjun.math.tifr@gmail.com;
E-mail[2]: ronnie@math.iitb.ac.in.

MS received 25 August 2020; revised

Abstract. Let k be an algebraically closed field of characteristic $p > 0$. Let X be an irreducible smooth projective curve of genus g over k. Fix an integer $n \geq 2$, and let $S^n(X)$ be the n-fold symmetric product of X. In this article we find the S-fundamental group scheme and Nori's fundamental group scheme of $S^n(X)$.

Keywords. Essentially finite vector bundle; S-fundamental group scheme; semistable bundle; Tannakian category.

2010 Mathematics Subject Classification. 14J60, 14F35, 14L15.

1. Introduction

For a connected reduced complete scheme X defined over a perfect field k and having a k-rational point x, in [8, 9], Nori introduced an affine k-group scheme $\pi^N(X, x)$ associated to the neutral Tannakian category of essentially finite vector bundles on X, known as *Nori's fundamental group scheme*. This group scheme carries more informations than the étale fundamental group scheme $\pi^{\text{ét}}(X, x)$ in positive characteristic, and is the same as $\pi^{\text{ét}}(X, x)$ when $k = \mathbb{C}$. For a connected smooth projective curve defined over an algebraically closed field k, in [2], Biswas, Parameswaran and Subramanian defined and studied the *S-fundamental group scheme* $\pi^S(X, x)$ of X. This is further generalized and extensively studied for higher dimensional smooth projective varieties over algebraically closed fields by Langer in [6, 7]. It is an interesting question to find $\pi^{\text{ét}}(X, x)$, $\pi^N(X, x)$ and $\pi^S(X, x)$ for well-known algebraic varieties.

Let X be a connected smooth projective curve defined over an algebraically closed field k of characteristic $p > 0$. Fix an integer $n \geq 2$, and let S_n be the permutation group of n symbols. Then S_n acts on X^n by permutation of its factors, and the associated quotient $S^n(X) := X^n/S_n$ is a connected smooth projective variety over k. For any affine k-group scheme G we denote by G_{ab} its abelianization. In this article we prove the following results.

Theorem 1 (Theorem 7). For any closed point $x \in X(k)$, there is an isomorphism of affine k-group schemes

$$\widetilde{\psi^S_*} : \pi^S(X, x)_{\mathrm{ab}} \longrightarrow \pi^S(S^n(X), nx).$$

Theorem 2 (Theorem 9). For any closed point $x \in X(k)$, there is an isomorphism of affine k-group schemes

$$\widetilde{\psi^N_*} : \pi^N(X, x)_{\mathrm{ab}} \longrightarrow \pi^N(S^n(X), nx).$$

As a consequence we also obtain the following result, which is already contained in [1], and proved using a different method. For any closed point $x \in X(k)$, there is an isomorphism of affine k-group schemes

$$\widetilde{\psi^{\text{ét}}_*} : \pi^{\text{ét}}(X, x)_{\mathrm{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx).$$

Note that when $n > 2g - 2$, where g is the genus of X, these isomorphisms can be easily obtained from results in [7, Section 7], since $S^n(X)$ is a projective bundle over $\mathrm{Alb}(X)$. We prove the above results without any restriction on n. Our initial strategy was to use the same method as in [10] under the assumption that $\mathrm{char}(k) > 3$. However, we observed that one can avoid using the characterization of numerically flat sheaves as strongly semistable reflexive sheaves with vanishing Chern classes; proved in [6]. Instead, we first show that $\widetilde{\psi^S_*}$ is faithfully flat and then use [7, Section 7] to conclude that it is an isomorphism.

2. Fundamental Group Schemes

Let k be an algebraically closed field. Let X be a reduced proper k-scheme, which is connected in the sense that $H^0(X, \mathcal{O}_X) \cong k$.

2.1 S-fundamental group scheme

Let $\mathrm{Coh}(X)$ be the category of coherent sheaf of $\mathcal{O}_X$-modules on X. This has a full subcategory $\mathrm{Vect}(X)$, whose objects are locally free coherent sheaves (vector bundles) on X. A vector bundle E on X is said to be *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$. An object E of $\mathrm{Coh}(X)$ is said to be *numerically flat* if E is locally free and both E and its dual $E^{\vee}$ are nef. Let $\mathscr{C}^{\mathrm{nf}}(X)$ be the full subcategory of $\mathrm{Coh}(X)$, whose objects are numerically flat vector bundles on X. It is known that, $E \in \mathrm{Ob}(\mathrm{Coh}(X))$ is an object of $\mathscr{C}^{\mathrm{nf}}(X)$ if and only if E is locally free and for any smooth projective curve C over k and any morphism $f : C \longrightarrow X$, its pullback f^*E on C is slope semistable and of degree 0 (see [6, Remark 5.2]). Note that $\mathscr{C}^{\mathrm{nf}}(X)$ is closed under finite direct sum and tensor products. Choosing a closed point $x \in X(k)$, one can define a fiber functor

$$T_x : \mathscr{C}^{\mathrm{nf}}(X) \longrightarrow \mathrm{Vect}_k$$

by sending an object E of $\mathscr{C}^{\mathrm{nf}}(X)$ to its fiber E_x at x. The quadruple $(\mathscr{C}^{\mathrm{nf}}(X), \otimes, O_X, T_x)$ is a neutral Tannakian category (see [6, Proposition 5.5]), and the affine k-group scheme $\pi^S(X, x)$ Tannaka dual to this is known as the *S-fundamental group scheme* of X with base point x.

Let X be a connected smooth projective variety of dimension d over k. Fix an ample divisor H on X. Let $\mathrm{Vect}_0^s(X)$ be the full subcategory of $\mathrm{Coh}(X)$, whose objects are reflexive coherent sheaves E on X, that are strongly H-semistable and $\mathrm{ch}_1(E) \cdot H^{d-1} = \mathrm{ch}_2(E) \cdot H^{d-2} = 0$, where $\mathrm{ch}_i(E)$ is the i-th Chern character of E. It is shown in [6, Proposition 5.1] that the objects of the category $\mathrm{Vect}_0^s(X)$ are in fact locally free coherent sheaves on X and all of their Chern classes vanishes. It follows from [6, Proposition 4.5] that the category $\mathrm{Vect}_0^s(X)$ does not depend on choice of H. For X smooth, the categories $\mathscr{C}^{\mathrm{nf}}(X)$ and $\mathrm{Vect}_0^s(X)$ are the same (see [6, Proposition 5.1], [7, Theorem 2.2]). We will not use this characterization here, however, this was crucial in [10].

It is clear from the definition of the categories $\mathrm{Vect}_0^s(X)$ and $\mathrm{EF}(X)$ that $\pi^S(X, x)$ carries more informations than $\pi^N(X, x)$. In fact, there are natural faithfully flat homomorphisms of affine k-group schemes $\pi^S(X, x) \longrightarrow \pi^N(X, x) \longrightarrow \pi^{\text{ét}}(X, x)$, (see [6, Lemma 6.2]).

2.2 Nori's fundamental group scheme

DEFINITION 1

A vector bundle E on X is said to be *finite* if there are two distinct non-zero polynomials f and g with positive integer coefficients such that $f(E) \cong g(E)$.

A vector bundle E on X is said to be *essentially finite* if there are finitely many finite vector bundles $E_1, \ldots, E_n$ and two numerically flat vector bundles V_1 and V_2 with $V_2 \subseteq V_1 \subseteq \bigoplus_{i=1}^{n} E_i$ such that $E \cong V_1/V_2$.

Let $\mathrm{EF}(X)$ be the full subcategory of $\mathrm{Vect}(X)$ whose objects are essentially finite vector bundles on X. Then $\mathrm{EF}(X)$ is an abelian rigid tensor category. Let Vect_k be the category of k-vector spaces. Fixing a closed point $x \in X(k)$, we have a fiber functor

$$T_x : \mathrm{EF}(X) \longrightarrow \mathrm{Vect}_k$$

defined by sending a vector bundle $E \in \mathrm{Ob}(\mathrm{EF}(X))$ to its fiber E_x at x. This makes the quadruple $(\mathrm{EF}(X), \otimes, \mathcal{O}_X, T_x)$ a neutral Tannakian category. The affine k-group scheme $\pi^N(X, x)$ Tannaka dual to this category is called *Nori's fundamental group scheme* of X with base point x.

3. Fundamental Group Schemes of $S^n(X)$

3.1 Symmetric product of curve

Let k be an algebraically closed field of characteristic $p > 0$. Let X be an irreducible smooth projective curve over k. Fix an integer $n \geq 2$, and let us denote by S_n the permutation group of n symbols. There is a natural action of S_n on the n-fold product X^n, and the associated quotient X^n/S_n, denoted by $S^n(X)$, is a smooth projective variety of dimension n over k. Note that any closed point $q \in S^n(X)$ can be uniquely written as $\sum_{i=1}^{r} n_i x_i$, where $x_1, \ldots, x_r$ are distinct closed points of X and $n_1, \ldots, n_r$ are integers with

$$n_1 \geq \ldots \geq n_r \geq 1.$$

We call $\langle n_1, \ldots, n_r \rangle$ the *type* of q. The quotient morphism

$$\psi : X^n \longrightarrow S^n(X) \tag{3.1.1}$$

is a faithfully flat finite morphism of k-schemes.

3.2 A group scheme theoretic lemma

For an affine group scheme G, we denote by G_{ab} the group scheme quotient of G by its derived subgroup scheme $[G, G]$ (c.f., [11, §10.1]). This is the largest abelian affine quotient group scheme of G, [11, §16.3]. A proof of the following easy Lemma can be found in [10].

Lemma 2. *Let G and H be two group schemes over k. For an integer $n \geq 2$, we denote by G^n the group scheme $G \times \cdots \times G$. Then S_n acts on G^n by permuting the factors. Let f_0 be the following composite group homomorphism*

$$f_0 : G^n \xrightarrow{\alpha^n} (G_{\mathrm{ab}})^n \xrightarrow{m} G_{\mathrm{ab}}\,,$$

where $\alpha : G \to G_{\mathrm{ab}}$ denotes the abelianization homomorphism and m denotes the multiplication homomorphism. Then a homomorphism of k-group schemes $f : G^n \longrightarrow H$ is S_n-invariant if and only if there is a homomorphism $\tilde{f} : G_{\mathrm{ab}} \longrightarrow H$ of affine k-group schemes such that $\tilde{f} \circ f_0 = f$. In other words, f is S_n-invariant iff there if $\tilde{f}$ which makes the following diagram commutes.

$$\begin{tikzcd} G^n \arrow[rr, "f"] \arrow[dr, "f_0"'] & & H \\ & G_{\mathrm{ab}} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

3.3 Construction of homomorphism

The functor which sends $E \in \mathscr{C}^{\mathrm{nf}}(S^n(X))$ to $\psi^* E \in \mathscr{C}^{\mathrm{nf}}(X^n)$ defines a morphism of neutral Tannakian categories (for any closed point $p \in X^n(k)$)

$$\mathscr{F} : (\mathscr{C}^{\mathrm{nf}}(S^n(X)), \otimes, \mathcal{O}_{S^n(X)}, T_{\psi(p)}) \to (\mathscr{C}^{\mathrm{nf}}(X^n), \otimes, \mathcal{O}_{X^n}, T_p). \tag{3.3.1}$$

Thus, we get a homomorphism

$$\psi_*^S : \pi^S(X^n, p) \longrightarrow \pi^S(S^n(X), \psi(p)).$$

It follows from [7, Theorem 4.1, p. 842] that, for any closed point $x \in X(k)$, the natural homomorphism of affine k-group schemes

$$\pi^S(X^n, (x, \ldots, x)) \xrightarrow{\simeq} \pi^S(X, x)^n .$$

is an isomorphism. By abuse of notation, denote by ψ_*^S the composite of the inverse of this isomorphism and ψ_*^S. So now

$$\psi_*^S : \pi^S(X, x)^n \to \pi^S(S^n(X), nx), \tag{3.3.2}$$

where $nx = \psi(x, \ldots, x)$.

The natural S_n-action on X^n gives rise to automorphisms σ_* of the affine k-group scheme $\pi^S(X^n, (x, \ldots, x)) \cong \pi^S(X, x)^n$, for all $\sigma \in S_n$. Now one can check that $\psi_*^S \circ \sigma_* = \psi_*^S$, where ψ_*^S is the homomorphism defined in (3.3.2) with $p = (x, \ldots, x)$. Consider the natural homomorphism of affine k-group schemes

$$\phi : \pi^S(X, x)^n \longrightarrow \pi^S(X, x)_{\mathrm{ab}}$$

defined as the following composite homomorphism

$$\pi^S(X, x)^n \longrightarrow (\pi^S(X, x)_{\mathrm{ab}})^n \xrightarrow{m} \pi^S(X, x)_{\mathrm{ab}} ,$$

where the first homomorphism is given by abelianization at each factor, and the second homomorphism is the multiplication. Then it follows from Lemma 2 that the homomorphism ψ^S_* in (3.3.2) factors through a homomorphism

$$\widetilde{\psi^S_*} : \pi^S(X,x)_{\mathrm{ab}} \longrightarrow \pi^S(S^n(X), nx). \tag{3.3.3}$$

We record the above discussion in the following proposition.

PROPOSITION 3

The map

$$\psi^S_* : \pi^S(X^n, (x, \dots, x)) \longrightarrow \pi^S(S^n(X), \psi(x, \dots, x))$$

factors to give a homomorphism $\widetilde{\psi^S_*} : \pi^S(X,x)_{\mathrm{ab}} \longrightarrow \pi^S(S^n(X), nx)$.

A vector bundle E on X^n is said to be S_n-*invariant* if $\sigma^*E \cong E$, for all $\sigma \in S_n$.

PROPOSITION 4

Let E be a vector bundle in $\mathscr{C}^{\mathrm{nf}}(X^n)$ associated to a representation $\rho : \pi^S(X^n, (x, \dots, x)) \cong \pi^S(X,x)^n \to \mathrm{GL}(V)$. If ρ factors through $\pi^S(X,x)_{\mathrm{ab}}$, as in Lemma 2, then E is S_n-invariant.

Proof. From the hypothesis it follows that $\rho \circ \sigma_* = \rho$. The proposition follows. □

3.4 Faithfully flatness

In this subsection we use [3, Proposition 2.21] to show that the homomorphism $\widetilde{\psi^S_*}$ in (3.3.3) is faithfully flat. We begin by recalling this result for the convenience of the reader. Let $\theta : G \longrightarrow G'$ be a homomorphism of affine group schemes over k, and let

$$\widetilde{\theta} : \mathrm{Rep}_k(G') \longrightarrow \mathrm{Rep}_k(G) \tag{3.4.1}$$

be the functor given by sending $\rho' : G' \to \mathrm{GL}(V)$ to $\rho' \circ \theta : G \to \mathrm{GL}(V)$. An object $\rho : G \to \mathrm{GL}(V)$ in $\mathrm{Rep}_k(G)$ is said to be a *subquotient* of an object $\eta : G \to \mathrm{GL}(W)$ in $\mathrm{Rep}_k(G)$ if there are two G-submodules $V_1 \subset V_2$ of W such that $V \cong V_2/V_1$ as G-modules.

PROPOSITION 5

[3, Proposition 2.21] Let $\theta : G \longrightarrow G'$ be a homomorphism of affine algebraic groups over k. Then

(a) *θ is faithfully flat if and only if the functor $\widetilde{\theta}$ in* (3.4.1) *is fully faithful and given any subobject $W \subset \widetilde{\theta}(V')$, with $V' \in \mathrm{Rep}_k(G')$, there is a subobject $W' \subset V'$ in $\mathrm{Rep}_k(G')$ such that $\widetilde{\theta}(W') \cong W$ in $\mathrm{Rep}_k(G)$.*

(b) *θ is a closed immersion if and only if every object of $\mathrm{Rep}_k(G)$ is isomorphic to a subquotient of an object of the form $\widetilde{\theta}(V')$, for some $V' \in \mathrm{Rep}_k(G')$.*

PROPOSITION 6

The homomorphism

$$\widetilde{\psi_*^S} : \pi^S(X, x)_{\mathrm{ab}} \longrightarrow \pi^S(S^n(X), nx)$$

defined in (3.3.3) *is faithfully flat.*

Proof. We will apply Proposition 5 (a). Let E_1 be an object in the category $\mathscr{C}^{\mathrm{nf}}(S^n(X))$. Clearly $\psi^* E_1$ has the same rank as that of E_1. If $\mathcal{E}_2 \subset \mathcal{E}_1 := \psi^* E_1$ is a subbundle corresponding to a representation of $\pi^S(X, x)_{\mathrm{ab}}$, we need to show that there is a subbundle $E_2 \subset E_1$ such that $\psi^* E_2 = \mathcal{E}_2$. We will prove this by induction on rank of E_1. If $\mathrm{rank}(E_1) = 1$, there is nothing to prove. Assume that $\mathrm{rank}(E_1) \geq 2$.

The vector bundles $\mathcal{E}_i$ corresponds to a representation

$$\pi^S(X^n, (x, \dots, x)) \xrightarrow{f_0} \pi^S(X, x)_{\mathrm{ab}} \xrightarrow{\rho_i} \mathrm{GL}(V_i), \quad \forall\, i = 1, 2.$$

It follows from Proposition 4 that $\mathcal{E}_2$ is a S_n-invariant numerically flat vector bundle on X^n. Since $\pi^S(X, x)_{\mathrm{ab}}$ is an abelian k-group scheme, it follows from [11, Theorem 9.4, p. 70], that we can find a surjective $\pi^S(X, x)_{\mathrm{ab}}$-module homomorphism $V_1 \to L_1$, where L_1 is one dimensional and V_2 is a $\pi^S(X, x)_{\mathrm{ab}}$-submodule of the kernel of this homomorphism. Let $\mathcal{L}$ be the line bundle on X^n corresponding to the representation L_1. Then it is clear that $\mathcal{L}$ is S_n-invariant (see Proposition 4) and there is an S_n-equivariant exact sequence of vector bundles

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{L} \longrightarrow 0$$

on X^n such that $\mathcal{E}_2 \subset \mathcal{K}$.

Every S_n-invariant line bundle on X^n is the pullback of a line bundle from $S^n(X)$ (see [4, Proposition 3.6], also [10, Proposition 5.1.1]). Therefore, it follows that $L := (\psi_* \mathcal{L})^{S_n}$ is a line bundle on all of $S^n(X)$. We now show that L is numerically flat on $S^n(X)$. Given a morphism $C \longrightarrow S^n(X)$ from a smooth projective curve C into $S^n(X)$, we can find a curve $\widetilde{C}$ and a morphism $\widetilde{C} \longrightarrow C$ making the following diagram commutative.

$$\begin{tikzcd} \widetilde{C} \arrow[r] \arrow[d] & X^n \arrow[d, "\psi"] \\ C \arrow[r] & S^n(X) \end{tikzcd}$$

Since $\mathcal{L}$ is numerically flat on X^n and $\mathcal{L} \cong \psi^* L$, it follows that L is numerically flat.

We claim that

$$0 \to (\psi_* \mathcal{K})^{S_n} \longrightarrow (\psi_* \mathcal{E}_1)^{S_n} \cong E_1 \xrightarrow{q} (\psi_* \mathcal{L})^{S_n} = L \longrightarrow 0 \tag{3.4.2}$$

is exact. The sequence (3.4.2) can fail to be exact only on the right. Since both E_1 and L are numerically flat and L is a line bundle, q is surjective since it is nonzero. This proves the exactness of (3.4.2). It follows that $K := (\psi_* \mathcal{K})^{S_n}$ is locally free and numerically flat on $S^n(X)$. It is clear that $\psi^* K = \mathcal{K}$ on X^n. Since $\mathcal{E}_2 \subset \mathcal{K}$ the assertion that there is $E_2 \subset E_1$ such that $\mathcal{E}_2 = \psi^* E_2$ on X^n follows by induction on rank.

To complete the proof of the Proposition, we need to show that if E_1 and E_2 are numerically flat vector bundles on $S^n(X)$ then the natural map

$$\mathrm{Hom}_{S^n(X)}(E_1, E_2) \longrightarrow \mathrm{Hom}_{X^n}(\psi^* E_1, \psi^* E_2)$$

is bijective. It is clear that this natural map is injective (faithful). Therefore, it suffices to show that given any numerically flat vector bundle E on $S^n(X)$, any nonzero homomorphism $\phi : O_{X^n} \longrightarrow \psi^* E$ comes from a nonzero homomorphism $\widetilde{\phi} : O_{S^n(X)} \longrightarrow E$. Since the homomorphism $\pi^S(X^n, x) \longrightarrow \pi^S(X, x)_{\mathrm{ab}}$ is faithfully flat, and $\psi^* E$ corresponds to a representation of $\pi^S(X, x)_{\mathrm{ab}}$, it follows that ϕ is a morphism between two representations of $\pi^S(X, x)_{\mathrm{ab}}$. This shows that ϕ is S_n-equivariant on X^n. Now from the preceding discussion it follows that ϕ arises from a morphism $O_{S^n(X)} \longrightarrow E$. □

3.5 Proofs of Theorems

Let X be a connected smooth projective variety over k and $f : \mathbb{P}(E) \longrightarrow X$ be a projective bundle over X. It is easy to see, using $\pi^S(\mathbb{P}^n, s) = \{1\}$ and [5, Corollary 12.9, Chapter III], that for a numerically flat sheaf F on $\mathbb{P}(E)$, the sheaf $f_* F$ is locally free and the natural map $f^* f_* F \to F$ is an isomorphism. From this it easily follows that the homomorphism of S-fundamental group schemes

$$f_*^S : \pi^S(\mathbb{P}(E), y) \longrightarrow \pi^S(X, f(y))$$

is an isomorphism, for all $y \in \mathbb{P}(E)$.

Theorem 7. *The homomorphism of affine k-group schemes*

$$\widetilde{\psi^S_*} : \pi^S(X,x)_{\mathrm{ab}} \longrightarrow \pi^S(S^n(X), nx)$$

is an isomorphism, for all $x \in X(k)$.

Proof. Let $\mathrm{Alb}(X)$ be the Albanese variety of X. Let g be the genus of the curve X. Fix a closed point $x \in X(k)$. If $n \geq 2g-1$, then the morphism $\eta : S^n(X) \to \mathrm{Alb}(X)$ given by

$$\sum_{i=1}^n x_i \mapsto \sum_{i=1}^n x_i - nx,$$

makes $S^n(X)$ into a projective bundle over $\mathrm{Alb}(X)$. It follows that the induced homomorphism of affine k-group schemes is an isomorphism,

$$\eta_* : \pi^S(S^n(X), nx) \xrightarrow{\sim} \pi^S(\mathrm{Alb}(X), 0)\,.$$

From [7, Section 7] it follows that the Albanese morphism $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ given by $t \mapsto t - x$ induces maps

$$\mathrm{alb}_{X*} : \pi^S(X,x) \xrightarrow{a_0} \pi^S(X,x)_{\mathrm{ab}} \xrightarrow{\widetilde{\mathrm{alb}_{X*}}} \pi^S(\mathrm{Alb}(X), 0),$$

where $\widetilde{\mathrm{alb}_{X*}}$ is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[dr, "\psi\circ a"'] & X^n \arrow[d, "\psi"] & \\
& S^n(X) \arrow[r, "\eta"] & \mathrm{Alb}(X)\,,
\end{tikzcd}$$

where $a(t) = (t, x, x, \ldots, x)$. At the level of S-fundamental group schemes we get the commutative diagram

$$\begin{tikzcd}
\pi^S(X,x) \arrow[r, "a_*"] \arrow[dr, "a_0"'] & \pi^S(X,x)^n \arrow[r] \arrow[d, "\psi_0"] & \pi^S(X,x)^n_{\mathrm{ab}} \arrow[dl, "m"'] & \\
& \pi^S(X,x)_{\mathrm{ab}} \arrow[r, "\widetilde{\psi^S_*}"] & \pi^S(S^n(X), nx) \arrow[r, "\eta_*"] & \pi^S(\mathrm{Alb}(X), 0)\,,
\end{tikzcd}$$

We have $\eta \circ \psi \circ a = \mathrm{alb}_X$. It is easy to check that $(\psi \circ a)_* = \widetilde{\psi^S_*} \circ a_0$. Since a_0 is faithfully flat and

$$\widetilde{\psi^S_*} \circ a_0 = \eta_*^{-1} \circ \widetilde{\mathrm{alb}_{X*}} \circ a_0\,,$$

it follows that $\widetilde{\psi^S_*} = \eta_*^{-1} \circ \widetilde{\mathrm{alb}_{X*}}$ and so the theorem is true is $n \geq 2g-1$.

Assume that $n < 2g-1$. Consider the maps

$$X \xrightarrow{a} X^n \xrightarrow{\psi} S^n(X) \xrightarrow{c} S^{2g-1}(X) \xrightarrow{\eta} \mathrm{Alb}(X),$$

where $a(t) = (t, x, x, \ldots, x)$ and $c(\sum_{i=1}^n x_i) = \sum_{i=1}^n x_i + (2g-1-n)x$. It is clear that the composite morphism is alb_X. As above, one easily checks that

$$\eta_* \circ c_* \circ \widetilde{\psi_*^S} = \widetilde{\mathrm{alb}_{X*}}\,.$$

Thus, we get homomorphisms of affine k-group schemes

$$\pi^S(X,x)_{\mathrm{ab}} \xrightarrow{\widetilde{\psi_*^S}} \pi^S(S^n(X), nx) \xrightarrow{\eta_*\circ c_*} \pi^S(\mathrm{Alb}(X), 0),$$

such that the composite homomorphism is an isomorphism. This forces that the first map is a closed immersion. Since we know from Proposition 6 that the first map is faithfully flat, the theorem follows. □

Remark 8. That $\widetilde{\psi_*^S}$ is a closed immersion could have been proved using the same method in [10] under the assumption that $\mathrm{char}(k) > 3$.

Let X be a reduced proper k-scheme with $H^0(X, \mathcal{O}_X) = k$. Let E be an essentially finite vector bundle on X. Then there is a finite k-group scheme G, a principal G-bundle $p : P \to X$ and a finite dimensional k-linear representation $\rho : G \to \mathrm{GL}(W)$ such that $E \cong P \times^\rho W$, the vector bundle associated to the representation ρ. It follows from the proof of [8, Proposition 3.8] that there is a finite vector bundle $\mathcal{V}$ on X such that E is a subbundle of $\mathcal{V}$.

As before, let X be a connected smooth projective curve over k and $S^n(X)$ the n-fold symmetric product of X. It is clear that ψ^* takes a finite vector bundle to a finite vector bundle. Thus, $\psi^* E \subset \psi^* \mathcal{V}$, which shows that $\mathscr{F}$ takes essentially finite vector bundles to essentially finite vector bundles. Note that there is a commutative diagram of homomorphisms of affine k-group schemes

$$\begin{tikzcd}
\pi^S(X,x)_{\mathrm{ab}} \arrow[r, "\simeq"] \arrow[d] & \pi^S(S^n(X), nx) \arrow[d] \\
\pi^N(X,x)_{\mathrm{ab}} \arrow[r, "\widetilde{\psi_*^N}"] & \pi^N(S^n(X), nx)
\end{tikzcd}$$

where the vertical arrows are faithfully flat by [6, Lemma 6.2]. It follows that $\widetilde{\psi_*^N}$ is faithfully flat.

Now let $\mathcal{E}$ be an essentially finite S_n-invariant vector bundle on X^n. It is easy to find a finite and S_n-invariant bundle $\mathcal{V}$ on X^n and an S_n-equivariant inclusion $\mathcal{E} \subset \mathcal{V}$. Define $E = (\psi_*\mathcal{E})^{S_n}$ and $V := (\psi_*\mathcal{V})^{S_n}$. It is clear that V is a finite vector bundle and $E \subset V$. So E is essentially finite and $\mathscr{F}(E) = \mathcal{E}$. This shows that $\widetilde{\psi_*^N}$ is a closed immersion. Thus, we have the following.

Theorem 9. *There is a natural isomorphism of affine k-group schemes*

$$\widetilde{\psi_*^N} : \pi^N(X,x)_{\mathrm{ab}} \longrightarrow \pi^N(S^n(X), nx).$$

3.6 Étale Fundamental Group Scheme of $S^n(X)$

In this subsection we sketch how to deduce from Theorem 9 the same assertion for étale fundamental group schemes. This result is a special case of [1, Theorem 1.2]. Note that there is a commutative diagram

$$\begin{tikzcd}
\pi^N(X,x) \arrow[r, two heads] \arrow[d] & \pi^N(X,x)_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d] & \pi^N(S^n(X), nx) \arrow[d, "d"] \\
\pi^{\text{ét}}(X,x) \arrow[r, two heads] & \pi^{\text{ét}}(X,x)_{\mathrm{ab}} \arrow[r] & \pi^{\text{ét}}(S^n(X), nx) .
\end{tikzcd}$$

From this it follows that $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx)$ is faithfully flat. Consider a homomorphism $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \to \mathrm{GL}(V)$. It follows using [8, Proposition 3.10] that this homomorphism factors through a finite and reduced k-group scheme G. Now consider the diagram

$$\begin{tikzcd}
\pi^N(X,x)_{\mathrm{ab}} \arrow[r, "\sim"] \arrow[d] & \pi^N(S^n(X), nx) \arrow[r, two heads, "d"] \arrow[d, dashed] & \pi^{\text{ét}}(S^n(X), nx) \arrow[dl, dashed] \\
\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \arrow[r] & G \arrow[r] & \mathrm{GL}(V) .
\end{tikzcd}$$

The right vertical arrow is the unique map which makes the square commute. It factors through d since G is finite and reduced. Now it follows from Proposition 5 (b) that $\pi^{\text{ét}}(X,x)_{\mathrm{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx)$ is a closed immersion. This proves the following.

Theorem 10. *For any closed point $x \in X(k)$, there is an isomorphism of affine k-group schemes*

$$\widetilde{\psi_*^{\text{ét}}} : \pi^{\text{ét}}(X,x)_{\mathrm{ab}} \longrightarrow \pi^{\text{ét}}(S^n(X), nx).$$

References

[1] Biswas I and Hogadi A, On the fundamental group of a variety with quotient singularities, *Int. Math. Res. Not. IMRN*, **5** (2015) 1421–1444. doi:10.1093/imrn/rnt261.

[2] Biswas I, Parameswaran A J, and Subramanian S, Monodromy group for a strongly semistable principal bundle over a curve, *Duke Math. J.*, **132 (1)** (2006) 1–48. doi:10.1215/S0012-7094-06-13211-8.

[3] Deligne P and Milne J S, Tannakian Categories. Chapter 2, pages 101–228. In: *Hodge Cycles, Motives, and Shimura Varieties*. Lecture Notes in Mathematics, vol. 900. *Springer-Verlag, Berlin-New York, 1982*. doi:10.1007/978-3-540-38955-2_4.

[4] Fogarty J, Line bundles on quasi-symmetric powers of varieties, *J. Algebra*, **44 (1)** (1977) 169–180. doi:10.1016/0021-8693(77)90171-5.

[5] Hartshorne R, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. *Springer-Verlag, New York-Heidelberg, 1977*. xvi+496 pp. doi:10.1007/978-1-4757-3849-0.

[6] Langer A, On the S-fundamental group scheme, *Ann. Inst. Fourier (Grenoble)*, **61 (5)** (2011) 2077–2119 (2012). doi:10.5802/aif.2667.

[7] Langer A, On the S-fundamental group scheme. II, *J. Inst. Math. Jussieu*, **11 (4)** (2012) 835–854. doi:10.1017/S1474748012000011.

[8] Nori M V, On the representations of the fundamental group, *Compositio Math.*, **33 (1)** (1976) 29–41. URL `http://www.numdam.org/item?id=CM_1976__33_1_29_0`.

[9] Nori M V, The fundamental group-scheme, *Proc. Indian Acad. Sci. Math. Sci.*, **91 (2)** (1982) 73–122. doi:10.1007/BF02967978.

[10] Paul A and Sebastian R, Fundamental group schemes of Hilbert scheme of n points on a smooth projective surface, *Bull. Sci. Math.*, **164** (2020) 102898, 25 pp. doi:10.1016/j.bulsci.2020.102898; arXiv:1907.04290.

[11] Waterhouse W C, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66. *Springer-Verlag, New York-Berlin, 1979*. xi+164 pp. doi:10.1007/978-1-4612-6217-6.