Basic Algebraic & Differential Topology

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Note: This note will be updated from time to time.
If there are any mistakes, please bring it to my notice.

Contents

List of Symbols

$\emptyset$	Empty set
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integers
$\mathbb{N}$	The set of all natural numbers (i.e., positive integers)
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\subset$	Proper subset
$\subseteq$	Subset or equal to
$\subsetneq$	Subset but not equal to (c.f. proper subset)
$\exists$	There exists
$\nexists$	Does not exists
$\forall$	For all
$\in$	Belongs to
$\notin$	Does not belong to
$\sum$	Sum
$\prod$	Product
$\pm$	Plus and/or minus
∞	Infinity
$\sqrt{a}$	Square root of a
$\cup$	Union
$\coprod$	Disjoint union
$\cap$	Intersection
$A \to B$	A mapping into B
$a \mapsto b$	a maps to b
$\hookrightarrow$	Inclusion map
$A \setminus B$	A setminus B
$\cong$	Isomorphic to
$A :=$ ...	A is defined to be ...
$\square$	End of a proof

Symbol	Name	Symbol	Name
α	alpha	β	beta
γ	gamma	δ	delta
π	pi	ϕ	phi
φ	var-phi	ψ	psi
ϵ	epsilon	ε	var-epsilon
ζ	zeta	η	eta
θ	theta	ι	iota
κ	kappa	λ	lambda
μ	mu	ν	nu
υ	upsilon	ρ	rho
ϱ	var-rho	ξ	xi
σ	sigma	τ	tau
χ	chi	ω	omega
Ω	Capital omega	Γ	Capital gamma
Θ	Capital theta	Δ	Capital delta
Λ	Capital lambda	Ξ	Capital xi
Σ	Capital sigma	Π	Capital pi
Φ	Capital phi	Ψ	Capital psi

Some of the useful Greek letters

Chapter 1

Review of Point Set Topology

In this chapter, we recall some topological preliminaries from basic topology course. Star marked exercises are difficult one and are not required for this course.

1.1 Preliminaries

A topology on a set X is given by specifying which subsets of X are 'open'. Naturally those subsets should satisfies certain properties as we are familiar from basic analysis and metric space courses.

Definition 1.1.1. A *topology* is on a set X is a collection τ of subsets of X satisfying the following properties:

(i) $\emptyset$ and X are in τ,

(ii) for any collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of objects of τ, their union $\bigcup\limits_{\alpha\in\Lambda} U_\alpha \in \tau$,

(iii) for a finite collection of objects $U_1, \ldots, U_n \in \tau$, their intersection $\bigcap\limits_{i=1}^{n} U_i \in \tau$.

The pair (X, τ) is called a *topological space*, and the objects of τ are called *open subsets* of (X, τ). For notational simplicity, we suppress τ and denote a topological space (X, τ) simply by X.

Joke: An empty set may contain some air since it is open!

Remark 1.1.2. One can also define a topology on a set X by considering a collection τ_c of subsets of X such that

(a) both $\emptyset$ and X are in τ_c,

(b) τ_c is closed under arbitrary intersections, and

(c) τ_c is closed under finite unions.

This is known as the *closed set axioms for a topology*. In this settings, the objects of τ_c are called *closed subsets* of X. It is easy to switch between these two definitions by taking complements of objects of τ and τ_c in X. However, unless explicitly mentioned, we usually work with open set axioms for topology.

Example 1.1.3. (i) If $X = \emptyset$ then $\tau = \{\emptyset\}$ is the only topology on $\emptyset$.

(ii) Let X be non-empty set X. Then $(X, \mathcal{P}(X))$ is a topological space. Such a topology, where all subsets of X are open, is called the *discrete topology* on X. Note that (X, τ) with $\tau = \{\emptyset, X\}$ is a topological space; such a topology on X is called the *indiscrete topology* on X.

(iii) Let $X \neq \emptyset$, and let $\tau = \{A \in \mathcal{P}(X) : X \setminus A \text{ is finite}\}$. Then (X, τ) is a topological space; such a topology is called the *cofinite topology* on X.

(iv) The set $\mathbb{R}^n$ together with the collection of all usual open subsets of $\mathbb{R}^n$ is a topological space, known as the usual/Euclidean topology on $\mathbb{R}^n$.

(v) Any metric space (X, d) is a topological space where the topology on X is given by the collection of all open subsets of (X, d).

Exercise* **1.1.4.** This exercise is for readers interested in commutative algebra. Let A be a commutative ring with identity. Let $\mathrm{Spec}(A)$ be the set of all prime ideals of A, known as the *spectrum of* A. For each subset $E \subseteq A$, let

$$V(E) := \{\mathfrak{p} \in \mathrm{Spec}(A) : E \subseteq \mathfrak{p}\}.$$

Prove the following.

(i) $V(A) = \emptyset$ and $V(0) = \mathrm{Spec}(A)$.

(ii) $V(E) = V(\mathfrak{a})$, where $\mathfrak{a} \subseteq A$ is the ideal generated by $E \subseteq A$.

(iii) $V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$, for all ideals $\mathfrak{a}, \mathfrak{b}$ of A.

(iv) $\bigcap\limits_{i \in I} V(\mathfrak{a}_i) = V(\sum\limits_{i \in I} \mathfrak{a}_i)$, for any collection of ideals $\{\mathfrak{a}_i : i \in I\}$ of A.

(v) Conclude that the collection $\{V(\mathfrak{a}) : \mathfrak{a} \text{ is an ideal of } A\}$ satisfies axioms for closed subsets of a topological space. The resulting topology on $\mathrm{Spec}(A)$ is called the *Zariski topology* on $\mathrm{Spec}(A)$.

(vi) For any ideal $\mathfrak{a}$ of A, show that $\mathrm{Spec}(A/\mathfrak{a})$ is homeomorphic to the closed subspace $V(\mathfrak{a})$ of $\mathrm{Spec}(A)$.

(vii) Let X be a topological space. A point $\xi \in X$ is said to be a *closed point* (resp., a *generic point*) of X if $\overline{\{\xi\}} = \{\xi\}$ (resp., $\overline{\{\xi\}} = X$). If A is an integral domain, show that $\mathrm{Spec}(A)$ contains a unique generic point, which is precisely the zero ideal of A.

(viii) Show that the Zariski topology on $\mathrm{Spec}(A)$ is not even T1 let alone be it Hausdorff.

(ix) Show that, a point $\mathfrak{m} \in \mathrm{Spec}(A)$ is closed if and only if $\mathfrak{m}$ is a maximal ideal of A.

(x) Let k be an algebraically closed field, and let $A = k[x_1, \ldots, x_n]$. Use Hilbert's Nullstellensatz to show that the set of all closed points of $\mathrm{Spec}(A)$ is in bijection with the set $k^n := \{(a_1, \ldots, a_n) : a_i \in k, \ \forall\, i \in \{1, \ldots, n\}\}$.

(xi) Let k be an algebraically closed field. Fix a subset $S \subseteq A := k[x_1, \ldots, x_n]$, and let $\mathfrak{a}_S \subseteq A$ be the ideal generated by S. Show that the set

$$Z(S) := \{(a_1, \ldots, a_n) \in k^n : f(a_1, \ldots, a_n) = 0, \forall f \in S\}$$

is in bijection with the set of all closed points of $V(\mathfrak{a}_S) \subseteq \mathrm{Spec}(A)$.

The space $\mathrm{Spec}(A)$ carries rich algebro-geometric structure. They are called *affine scheme*, and are building block of all schemes in the sense that any scheme is build by suitably gluing affine schemes.

Definition 1.1.5. Let X and Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the subset $f^{-1}(V)$ is open in X.

As an immediate consequence of the above definition, we get the following.

Corollary 1.1.6. *Let X be a non-empty set together with two topologies τ_1 and τ_2. For each $j = 1, 2$, let $X_j = (X, \tau_j)$ be the topological space whose underlying set is X and the topology is τ_j. Then τ_2 is finer than τ_1 (i.e., $\tau_1 \subsetneq \tau_2$) if and only if the identity map $\mathrm{Id}_X : X_2 \to X_1$ is continuous.*

Definition 1.1.7. A topological space X is said to be *Hausdorff* or, *T2* or, *separated* if each pair of distinct points of X can be separated by a pair of disjoint open neighbourhoods of them. In other words, give $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist open subsets V_1, V_2 of X with $x_1 \in V$, $x_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$.

Exercise 1.1.8. Show that a topological space is Hausdorff if and only if the image of the *diagonal map*

$$\Delta_X : X \to X \times X, \ \ x \mapsto (x, x)$$

is closed in $X \times X$. (*Hint:* Look at the complement of $\Delta_X(X)$ in $X \times X$.)

Exercise 1.1.9. Let $f, g : X \to Y$ be continuous maps of topological spaces. If Y is Hausdorff, show that $\{x \in X : f(x) = g(x)\}$ is closed in X. (*Hint:* Look at the inverse image of $\Delta_Y(Y) \subset Y \times Y$ under the map $(f, g) : X \to Y \times Y$ given by $x \mapsto (f(x), g(x))$.)

Exercise* 1.1.10. Given continuous maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ of topological spaces, we define their *fiber product* to be a topological space $X \times_Z Y$ together with continuous maps $p_1 : X_1 \times_Y X_2 \to X_1$ and $p_2 : X_1 \times_Y X_2 \to X_2$ such that

(i) $f_1 \circ p_1 = f_2 \circ p_2$, and

(ii) given any topological space Z, and continuous maps $g_1 : Z \to X_1$ and $g_2 : Z \to X_2$ satisfying $f_1 \circ g_1 = f_2 \circ g_2$, there is a unique continuous map $g : Z \to X_1 \times_Y X_2$ such that $p_1 \circ g = g_1$ and $p_2 \circ g = g_2$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& X_1 \times_Y X_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & X_2 \arrow[d, "f_2"] \\
& X_1 \arrow[r, "f_1"] & Y
\end{tikzcd}$$

Prove the following.

(a) The fiber product $X_1 \times_Y X_2$ exists and is unique up to a unique homeomorphism. (*Hint:* Take $X_1 \times_Y X_2 := \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$.)

(b) If $X_2 \subseteq Y$ and $f_2 : X_2 \hookrightarrow Y$ is the inclusion map, show that $f_1^{-1}(X_2)$ is homeomorphic to the fiber product $X_1 \times_Y X_2$. (*Hint:* Verify properties (i) and (ii) for $f_1^{-1}(X_2)$.)

1.2 Quotient space

Let's recall the notion of quotients from algebra course. Let G be a group and H a normal subgroup of G. Then we have a relation $\sim$ on G defined by

$$g_1 \sim g_2 \quad \text{if } g_1^{-1} g_2 \in H.$$

Clearly this is an equivalence relation on G, and we have a partition of G into a disjoint union of its subsets (equivalence classes)

$$G = \bigcup_{g \in G} gH,$$

where $gH = \{g' \in G : g \sim g'\}$ is the equivalence class of g in G, for all $g \in G$.

Now question is does there exists a pair (Q, q) consisting of a group Q and a map $q : G \to Q$ such that

(QG1) $q : G \to Q$ is a surjective group homomorphism, and

(QG2) given any group G' and a group homomorphism $f : G \to G'$ with $H \subseteq \mathrm{Ker}(f)$, there should exists a unique group homomorphism $\widetilde{f} : Q \to G'$ such that $\widetilde{f} \circ q = f$?

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "q"'] & G' \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd} \tag{1.2.1}$$

Interesting point is that, without knowing existence of such a pair (Q, q), it follows immediately from the properties (QG1) and (QG2) that such a pair (Q, q), if it exists, must be unique up to a unique isomorphism of groups in the sense that, given another such pair (Q', q') satisfying the above two conditions, there is a unique group isomorphism $\phi : Q \to Q'$ such that $\phi \circ q = q'$.

Exercise 1.2.2. Prove the above mentioned uniqueness statement.

Now question is about its existence. The condition (QG1) suggests that the elements of Q should be the fibers of the map q, which are nothing but the $\sim$-equivalence classes

$$[g]_\sim = \{g' \in G : g' \sim g\} = gH, \ \forall\, g \in G.$$

This suggests us to consider $\{gH : g \in G\}$ as a possible candidate for the set Q. Now question is what should be the appropriate group structure on it? Take any group homomorphism $f : G \to G'$ such that $H \subseteq \mathrm{Ker}(f)$. This says that $f(g_1) = f(g_2)$ if $g_1 \sim g_2$ (equivalently, $g_1^{-1}g_2 \in H$). The commutativity of the diagram (1.2.1) tells us to send $gH \in Q$ to $f(g) \in G'$ to define the map $\widetilde{f} : Q \to G'$ (note that this is well-defined!), and since we want $\widetilde{f} : Q \to G'$ to be a group homomorphism, we should define a binary operation on $Q = \{gH : g \in G\}$ in such a way that $(g_1H) * (g_2H) \overset{\widetilde{f}}{\longmapsto} f(g_1)f(g_2) = f(g_1g_2)$, for all $g_1, g_2 \in G$. So the obvious choice is to define

$$(g_1H) * (g_2H) := (g_1g_2)H, \ \forall\, g_1, g_2 \in G. \tag{1.2.3}$$

Clearly this is a well-defined binary operation on $Q = \{gH : g \in G\}$.

Exercise 1.2.4. Verify that (1.2.3) makes Q a group such that the pair (Q, q) satisfies the condition (QG1) and (QG2).

Exercise 1.2.5. Verify analogous stories for the cases rings and vector spaces.

Now we are going to witness the same story in topology! Let X be a topological space.

Definition 1.2.6. Given an equivalence relation $\sim$ on X, the associated *quotient topological space* (or, *identification space*) $X/\!\sim$ is a pair (Q, q) consisting of a topological space Q and a continuous map $q : X \to Q$ such that

(QT1) q is surjective and satisfies $q(x) = q(x')$ whenever $x \sim x'$ in $(X, \sim)$; and

(QT2) given any topological space Y and a continuous map $f : X \to Y$ satisfying $f(x) = f(x')$ whenever $x \sim x'$ in $(X, \sim)$, there is a unique continuous map $\widetilde{f} : Q \to Y$ such that $\widetilde{f} \circ q = f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

The map q is called the *quotient map* (or, *identification map*) for $(X, \sim)$.

As an immediate corollary to the Definition 1.2.6, we have the following.

Corollary 1.2.7. *If (Q, q) is a quotient space for $(X, \sim)$, then the topology on Q is the largest topology on the set Q such that the map $q : X \to Q$ is continuous.*

Proof. By a topology on a set S we mean a collection τ of subsets of S that satisfies axioms for open subsets in S. Suppose on the contrary that the statement in Corollary 1.2.7 is false. Then there is a topology τ' on the set Q finer than the quotient topology on Q such that the set map $q : X \to Q' := (Q, \tau')$ is continuous. Then by property (QT2) of (Q, q) in Definition 1.2.6, there is a unique (continuous) map $f : Q \to Q'$ such that $f \circ q = q$. Since $q : X \to Q$ is surjective, it admits a right inverse (set theoretically). This forces $f : Q \to Q'$ to be the identity map. This is not possible because f is continuous and the topology on Q' is finer than that of Q by our assumption (see Corollary 1.1.6). Hence the result follows. $\square$

Remark 1.2.8. In Definition 1.2.6, the first condition suggests what should be the underlying set of points of Q and the map $q : X \to Q$, and the second condition suggests what should be the topology on the set Q.

Theorem 1.2.9. *Given a topological space X and an equivalence relation $\sim$ on X, the associated quotient space (Q, q) for $(X, \sim)$ exists, and is unique up to a unique homeomorphism in the sense that if (Q, q) and (Q', q') are two quotient spaces for $(X, \sim)$, then there is a unique homeomorphism $\varphi : Q \to Q'$ such that $\varphi \circ q = q'$.*

Proof. We first prove uniqueness of the pair (Q, q), up to a unique homeomorphism. Let (Q', q') be another quotient space for the pair $(X, \sim)$. Since q' is continuous and $q'(x) = q'(y)$ whenever $x \sim y$ in $(X, \sim)$, we have a unique continuous map $\widetilde{q} : Q' \to Q$ such that $\widetilde{q} \circ q' = q$.

$$\begin{tikzcd}
& X \arrow[dl, "q'"'] \arrow[d, "q"] \arrow[dr, "q'"] & \\
Q' \arrow[r, dashed, "\widetilde{q}"'] & Q \arrow[r, dashed, "\widetilde{q'}"'] & Q'
\end{tikzcd} \tag{1.2.10}$$

Similarly, interchanging the role of (Q, q) and (Q', q') we get a unique continuous map $\widetilde{q'} : Q \to Q'$ such that $\widetilde{q'} \circ q = q'$. Then we have $(\widetilde{q'} \circ \widetilde{q}) \circ q' = q'$. Since the identity map $\mathrm{Id}_{Q'} : Q' \to Q'$ is continuous and satisfies $\mathrm{Id}_{Q'} \circ q' = q'$, we must have $\widetilde{q'} \circ \widetilde{q} = \mathrm{Id}_{Q'}$. Similarly, we have $\widetilde{q} \circ \widetilde{q'} = \mathrm{Id}_Q$. Therefore, both $\widetilde{q}$ and $\widetilde{q'}$ are homeomorphisms. Thus the pair (Q, q) is unique, up to a unique homeomorphism.

Now (following Remark 1.2.8) we give an explicit construction of $(Q, \sim)$. For each $x \in X$, the *equivalence class of* x in $(X, \sim)$ is the subset

$$[x] := \{x' \in X : x \sim x'\} \subseteq X.$$

Let Q be the set of all distinct equivalence classes of elements of X. Consider the map $q : X \to Q$ defined by sending each point $x \in X$ to its equivalence class $[x] \in Q$. Note that, the map q is surjective. As suggested in Corollary 1.2.7, we define a topology on Q by declaring a subset $U \subseteq Q$ to be *open* if its inverse image $q^{-1}(U) \subseteq X$ is open in X. Clearly this makes $q : X \to Q$ continuous. It remains to check property (QT2) as in Definition 1.2.6. Let Y be any topological space and $f : X \to Y$ any continuous map satisfying $f(x) = f(x')$ for $x \sim x'$ in $(X, \sim)$. Define a map $\widetilde{f} : Q \to Y$ by $\widetilde{f}([x]) = f(x)$, for all $[x] \in Q$. Clearly $\widetilde{f}$ is well-defined, and by its construction it satisfies

$$\widetilde{f} \circ q = f. \tag{1.2.11}$$

Since f is continuous, for any open subset $V \subseteq Y$, the subset

$$q^{-1}(\widetilde{f}^{-1}(V)) = (\widetilde{f} \circ q)^{-1}(V) = f^{-1}(V)$$

is open in X, and hence $\widetilde{f}^{-1}(V)$ is open in Q by definition of the topology on Q. Therefore, $\widetilde{f}$ is continuous. If $g : Q \to Y$ is any continuous map satisfying $g \circ q = f$, then $g([x]) = (g \circ q)(x) = f(x)$, for all $[x] \in Q$, and hence $g = \widetilde{f}$. $\square$

Theorem 1.2.12. *Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective continuous map. Then the following are equivalent.*

(i) The pair (Y, p) is a quotient space for some equivalence relation $\sim$ on X.

(ii) A subset $U \subseteq Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

(iii) A subset $Z \subseteq Y$ is closed in Y if and only if $p^{-1}(Z)$ is closed in X.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): Follow from Corollary 1.2.7.

We show (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) together. Consider the equivalence relation $\sim$ on X defined by

$$x_1 \sim x_2 \text{ in } X, \text{ if } \; p(x_1) = p(x_2).$$

Let (Q, q) be the associated quotient space for the pair $(X, \sim)$. Note that,

$$Q = \{p^{-1}(y) \subseteq X : y \in Y\},$$

and the quotient map $q : X \to Q$ is given by

$$q(x) = p^{-1}(p(x)), \;\; \forall\, x \in X.$$

Since $p(x_1) = p(x_2)$ whenever $x_1 \sim x_2$ in $(X, \sim)$, the universal property of the quotient space (Q, q) gives a unique continuous map

$$\widetilde{p} : Q \to Y$$

such that $\widetilde{p} \circ q = p$ (see Theorem 1.2.9). In fact, in this case, the map $\widetilde{p}$ can be explicitly given as follow. Since p is surjective, given any element $p^{-1}(y) \in Q$, there exists an element $x \in p^{-1}(y) \subseteq X$ so that $p(x) = y$. Then $\widetilde{p}(p^{-1}(y)) = \widetilde{p}(q(x)) = p(x) = y$. Then the map

$$\phi : Y \to Q$$

defined by $\phi(y) := p^{-1}(y)$, for all $y \in Y$, is the set theoretic inverse of the map $\widetilde{p} : Q \to Y$.

$$\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "q"'] & Y \arrow[dl, bend left, "\phi"] \\
Q \arrow[ur, "\widetilde{p}"] &
\end{tikzcd}$$

Therefore, to show $\widetilde{p}$ a homeomorphism, it is enough to show that ϕ is continuous. Let $V \subseteq Q$ be open (resp., closed). Since $\phi^{-1}(V) = \{y \in Y : p^{-1}(y) \in V\}$, the subset

$$\begin{aligned}
p^{-1}(\phi^{-1}(V)) &= \{x \in X : p(x) \in \phi^{-1}(V)\} \\
&= \{x \in X : p^{-1}(p(x)) \in V\} \\
&= \{x \in X : q(x) \in V\} \\
&= q^{-1}(V)
\end{aligned}$$

is open (resp., closed) in X, because the quotient map q is continuous. Then $\phi^{-1}(V)$ is open (resp., closed) in Y by assumption (ii) (reps., (iii)). Therefore, ϕ is continuous as required. □

Remark 1.2.13. It follows from construction of quotient space in Theorem 1.2.9, and the proof of Theorem 1.2.12 that if $f : X \to Y$ is a quotient map then the set of all fibers $\{f^{-1}(y) : y \in Y\}$ of f gives a partition of X, and hence defines an equivalence relation $\sim$ on X such that the associated quotient space $X/\sim$ is homeomorphic to (Y, f).

Exercise 1.2.14. Let $q : X \to Q$ be a quotient map. Show that for any subset $Z \subseteq Q$, the restriction map $q|_{q^{-1}(Z)} : q^{-1}(Z) \to Z$ is a quotient map. (*Hint:* Use Theorem 1.2.12).

Definition 1.2.15. A map $f : X \to Y$ is said to be *open* (resp., *closed*) if $f(U)$ is open (resp., closed) in Y for any open (resp., closed) subset U of X.

Corollary 1.2.16. *A surjective continuous open (or, closed) map is a quotient map.*

Proof. Let $f : X \to Y$ be a surjective map. Then for any $V \subseteq Y$ we have $f(f^{-1}(V)) = V$. Suppose that f is also continuous and open (resp., closed). Then for any $V \subseteq Y$ with $f^{-1}(V)$ open (resp., closed) in X, $V = f(f^{-1}(V))$ is open (resp., closed) in Y. Hence the result follows from Theorem 1.2.12. □

Remark 1.2.17. Corollary 1.2.16 fails without continuity assumption on f. For example, take a set X with at least two elements. Let τ_0 and τ_1 be the trivial topology and the discrete topology on X, respectively. Then the identity map $\mathrm{Id}_X : (X, \tau_0) \to (X, \tau_1)$ is a surjective open map, which is not continuous let alone be a quotient map.

Corollary 1.2.18. *Let $f : X \to Y$ be a continuous surjective map. If X is compact and Y is Hausdorff, then f is a quotient map.*

Proof. Let Z be a closed subset of X. Since X is compact, Z is compact. Since f is continuous, $f(Z)$ is a compact subset of Y. Since Y is Hausdorff, $f(Z)$ is closed in Y. Hence the result follows from Corollary 1.2.16. □

Proposition 1.2.19. *Let $\sim$ be an equivalence relation on a topological space X, and let (Q, q) be the associated quotient space. Given a topological space Y, a map $\phi : Q \to Y$ is continuous if and only if the composite map $\phi \circ q : X \to Y$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\phi\circ q"] \arrow[d, "q"'] & Y \\ Q \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proof. Since the quotient map q is continuous, the composite map $\phi \circ q$ is continuous whenever ϕ is continuous. Conversely, let $\phi \circ q$ be continuous. Since for any open subset $V \subseteq Y$, we have $q^{-1}(\phi^{-1}(V)) = (\phi \circ q)^{-1}(V)$ is open in X, by construction of topology of Q, the subset $\phi^{-1}(V)$ is open in Q. Thus ϕ is continuous. □

Example 1.2.20. (i) **Cylinder**: Let $I = [0, 1] \subset \mathbb{R}$. Consider the unit square

$$I \times I = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$$

in $\mathbb{R}^2$. Define an equivalence relation $\sim_1$ on $I \times I$ by setting

$$(x,y) \sim_1 (x',y'), \text{ if } x' = x+1 = 1 \text{ and } y = y'.$$

This identifies points of two vertical sides of $I \times I$ (see Figure 1.1 below), and the associated quotient space $(I \times I)/{\sim_1}$ is homeomorphic to the cylinder

$$S^1 \times [0, 1] = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \ 0 \leq z \leq 1\}.$$

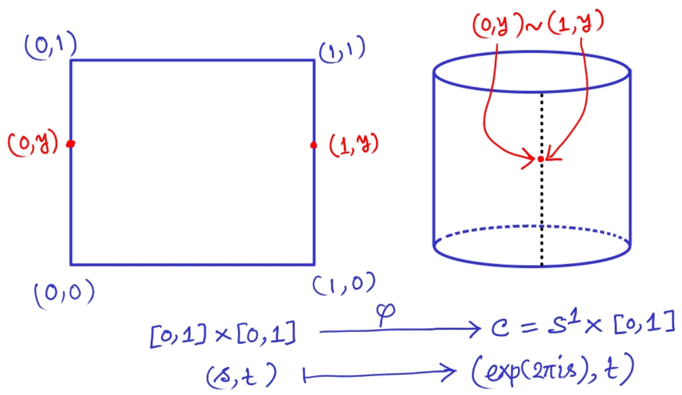


FIGURE 1.1

Indeed, we can define a (continuous) map

$$\phi : I \times I \to S^1 \times [0, 1]$$

by $\phi(s, t) = (\exp(2\pi i s), t)$, for all $(s, t) \in I \times I$. Then the set $\{\phi^{-1}(z,t) : (z, t) \in S^1 \times [0, 1]\}$ of all fibers of ϕ is precisely the partition of $I \times I$ given by the equivalence relation $\sim_1$ on $I \times I$. It follows from Corollary 1.2.18 that ϕ is a quotient map, and by Remark 1.2.13 the associated quotient space $(I \times I)/\sim$ is homeomorphic to $S^1 \times I$.

(ii) **Torus**: Consider an equivalence relation $\sim_2$ on the cylinder $S^1 \times I$ defined by

$$(z,t) \sim_2 (z',t') \text{ if } z = z', \text{ and } t' = t+1 = 1.$$

This identifies each point of the bottom circle of $S^1 \times I$ with the corresponding point of the top circle on $S^1 \times I$ (see Figure 1.2 below).

Then the associated quotient space $(S^1 \times I)/{\sim_2}$ is homeomorphic to the torus $T := S^1 \times S^1$ in $\mathbb{R}^3$. Indeed, we can define a (continuous) map

$$\psi : S^1 \times I \to S^1 \times S^1$$

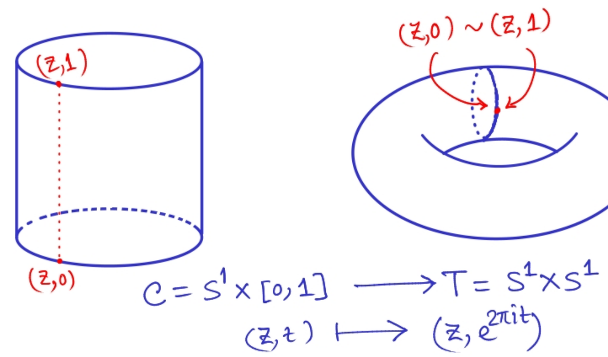


FIGURE 1.2

by $\psi(z,t) = (z, \exp(2\pi i t))$, for all $(z, t) \in S^1 \times I$. As before, it is easy to see that the set of all fibers of the map ψ is precisely the partition of $S^1 \times I$ defined by the equivalence relation $\sim_2$ on the cylinder $S^1 \times I$. As before, it follows from Corollary 1.2.18 that ψ is a quotient map, and by Remark 1.2.13 the associated quotient space $(S^1 \times I)/\sim$ is homeomorphic to $S^1 \times S^1$.

(iii) **Cone**: Let $I = [0, 1] \subseteq \mathbb{R}$. The *cone* of a topological space X is the quotient space $CX := (X \times I)/\sim$ of $X \times I$ for the equivalence relation $\sim$ on $X \times I$ defined by

$$(x,t) \sim (x',t'), \text{ if } t = t' = 1. \tag{1.2.21}$$

The associated set of all partitions of $X \times I$ is the set

$$\{X \times \{1\}, \{(x,t)\} : x \in X, 0 \le t < 1\}.$$

Thus we identify all points of $X \times \{1\} \subseteq X \times I$ into a single point, called the *vertex* of the

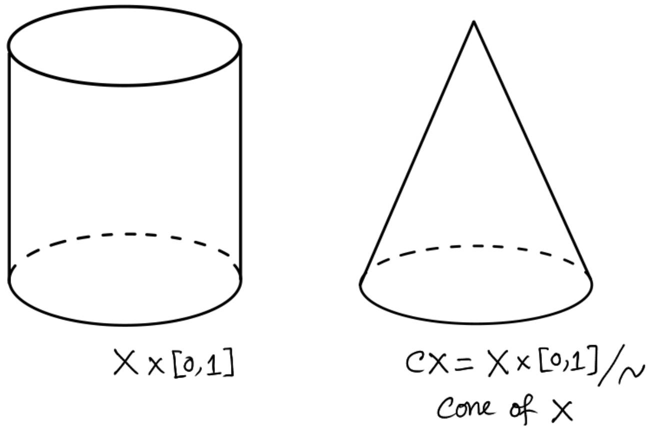


FIGURE 1.3

cone CX, and the remaining points of $X \times [0,1)$ remains as they are.

If X is a compact subset of an Euclidean space $\mathbb{R}^n$, then we can construct CX more geometrically as follow. Embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$, and fix a point $v \in \mathbb{R}^{n+1}$ which lies outside the image of this embedding; for example take $v = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Note that $\ell_{[v,x]} := \{tv + (1-t)x : 0 \leq t \leq 1\} \subseteq \mathbb{R}^{n+1}$ is the straight line segment in $\mathbb{R}^{n+1}$ joining v and $x \in X$. The subset

$$\bigcup_{x \in X} \ell_{[v,x]} \subseteq \mathbb{R}^{n+1}$$

with the subspace topology induced from $\mathbb{R}^{n+1}$ is called the *geometric cone* of X. We show that the geometric cone of X is homeomorphic to the cone of X, i.e.,

$$CX \cong \bigcup_{x \in X} \ell_{[v,x]}.$$

Define a map

$$f : X \times I \to \bigcup_{x \in X} \ell_{[v,x]}$$

by $f(x,t) = tv + (1-t)x$, for all $(x,t) \in X \times I$. Clearly f is a surjective continuous map, and $f(x,t) = f(x,t')$ if and only if either $x = x'$ and $t = t'$, or $t = t' = 1$. Since X is compact and its image is Hausdorff (being a subspace of $\mathbb{R}^{n+1}$), it follows from Corollary 1.2.18 that f is a quotient map. Since the fibers of the map f are precisely the equivalence classes for the equivalence relation on $X \times I$ defined in (1.2.21), it follows from Remark 1.2.13 that $CX = (X \times I)/\sim$ is homeomorphic to the geometric cone of X.

(iv) **The space** X/A: Let A be a subset of a topological space X. Define an equivalence relation $\sim$ on X by

$$x \sim x' \text{ if both } x \text{ and } x' \text{ are in } A.$$

We denote by X/A the associated quotient space $X/\sim$. Here we collapse the subspace A into a single point, and the remaining points of $X \setminus A$ remains as they were. For example, $CX = (X \times I)/(X \times \{1\})$.

(v) **The space** B^n/S^{n-1}: Consider the closed unit ball

$$B^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^n x_j^2 \leq 1\}$$

in $\mathbb{R}^n$, and its boundary

$$\partial B^n = \{(x_1, \ldots, x_n) \in \mathbb{B}^n : \sum_{j=1}^n x_j^2 = 1\} = S^{n-1}.$$

Then the associated quotient space is denoted by B^n/S^{n-1} is homeomorphic to S^n. This is quite easy to visualize for $n = 1$ and 2. For $n = 1$, $B^1 = [-1,1] \subseteq \mathbb{R}$, and $S^0 = \{-1,1\}$ is its boundary. If we identify all points of $S^0 = \{-1,1\}$ into a single point and keep all other points of B^1 as they were, we get a circle S^1 in $\mathbb{R}^2$; see Figure 1.4 below.

The case $n = 2$ is explained in the Figure 1.5 below.

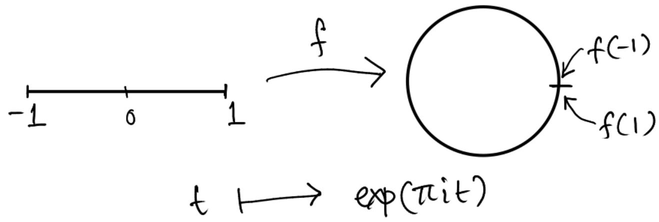


FIGURE 1.4

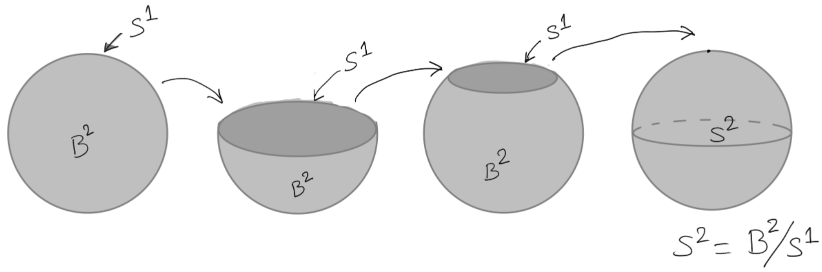


FIGURE 1.5

In general, it suffices to construct a surjective continuous map

$$f : B^n \to S^n$$

such that $f\big|_{B^n \setminus S^{n-1}}$ is injective and $f(S^{n-1})$ is a singleton subset of S^n. Then by Corollary 1.2.18, f become a quotient map producing a homeomorphism of B^n/S^{n-1} onto S^n. To construct such a map f, note that $\mathbb{R}^n$ is homeomorphic to $B^n \setminus S^{n-1}$ and $S^n \setminus \{p\}$, for any $p \in S^n$. Fix two homeomorphisms $h_1 : B^n \setminus S^{n-1} \to \mathbb{R}^n$ and $h_2 : \mathbb{R}^n \to S^n \setminus \{p\}$, and define

$$f(x) := \begin{cases} h_2(h_1(x)), & \text{if} \quad x \in B^n \setminus S^{n-1}, \\ p, & \text{if} \quad x \in S^{n-1}. \end{cases} \tag{1.2.22}$$

It is easy to check that f has desired properties (verify).

Example 1.2.23 (Attaching spaces along a map)**.** Let X and Y be two topological spaces. Suppose we wish to attach X by identifying points of a subspace $A \subseteq X$ with points of Y in a continuous way. This can be done by using a continuous map $f : A \to Y$. Indeed, we identify $x \in A$ with its image $f(x) \in Y$. This defines an equivalence relation on $X \sqcup Y$, and we denote the associated quotient space by $X \sqcup_f Y$, and call it the *space Y with X attached along A via f*. Let us discuss some examples.

(i) Let $I = [0, 1] \subset \mathbb{R}$. Given a space X, we call $X \times I$ the *cylinder over* X. Let $f : X \to Y$ be a continuous map of topological spaces. Then f induces a continuous map $\widetilde{f} : X \times \{0\} \to Y$ given by

$$\widetilde{f}(x, 0) = f(x), \ \forall\, (x, 0) \in X \times \{0\}.$$

If we attach Y with the *cylinder* $X \times I$ of X along its base $X \times \{0\} \subset X \times I$ via the map $\widetilde{f}$, by identifying $(x, 0) \sim f(x)$, then the associated quotient space $M_f = (X \times I) \sqcup_{\widetilde{f}} Y$ is called the *mapping cylinder of* f (see Figure 1.6).

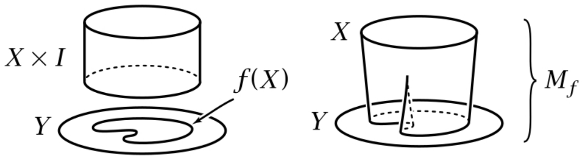


FIGURE 1.6: Mapping Cylinder

(ii) Let $CX = (X \times I)/(X \times \{1\})$ be the *cone over* X obtained by collapsing the subspace $X \times \{1\}$ of the cylinder $X \times I$ over X to a single point. Let $f : X \to Y$ be a continuous map. If we attach this cone CX with Y along its base $X \times \{0\} \subset CX$ by identifying $(x, 0) \in X \times \{0\}$ with $f(x) \in Y$, then the resulting quotient space $C_f = Y \sqcup_f CX$ is called the *mapping cone of* f (see Figure 1.7). Note that, the mapping cone C_f can also be obtained

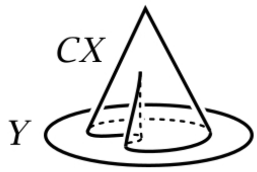


FIGURE 1.7: Mapping Cone

as a quotient space of the mapping cylinder M_f by collapsing $X \times \{1\} \subset M_f$ to a point.

(iii) Let X be a topological space. The *suspension* SX of X is the quotient space of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

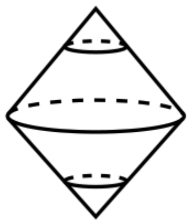


FIGURE 1.8: Suspension

For example, if we take X to be the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then $X \times I$ is a cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \leq z \leq 1\}$, and then we collapse two circular edges of C to two points to get SX, which is homeomorphic to the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

We can think of SX as a *double cone on* X: take disjoint union of two cones $C_1 := (X \times I)/(X \times \{1\})$ and $C_2 := (X \times I)/(X \times \{0\})$, and then attach C_1 with C_2 via the continuous map $f : X \times \{0\} \to C_2$ given by $f(x, 0) = (x, 0) \in C_2$, $\forall\, (x, 0) \in X \times \{0\} \subset C_1$.

The following exercise shows that a quotient of a Hausdorff space need not be Hausdorff in general.

Exercise 1.2.24. Consider the double real line $X = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$, and the equivalence relation $\sim$ on X defined by $(t,0) \sim (t,1)$, for all $t \in \mathbb{R} \setminus \{0\}$. The associated quotient space $X/\sim$ is called the *real line with double origin*. Show that $X/\sim$ is not Hausdorff.

Proposition 1.2.25. *Let $\rho \subseteq X \times X$ be an equivalence relation on a topological space X, and let $q : X \to Q := X/\rho$ be the quotient map. Then we have the following.*

(i) Q is a T1 space if and only if every ρ-equivalence class is closed in X.

(ii) If Q is Hausdorff then ρ is a closed subspace of the product space $X \times X$. The converse holds if $q : X \to Q$ is an open map.

Proof. (i) Let Q be T1. Let $x \in X$. Choose a $y \in X \setminus [x]$. Then $[x] \neq [y]$ in Q. Since Q is T1, there is an open subset $V_y \subseteq Q$ such that $[y] \in V_y$ and $[x] \notin V_y$. Then $q^{-1}(V_y)$ is an open neighbourhood of y with $q^{-1}(V_y) \cap [x] = \emptyset$. Therefore, y is an interior point of $X \setminus [x]$. Therefore, $X \setminus [x]$ is open, and hence $[x] \subseteq X$ is closed.

Conversely, suppose that $[x] \subseteq X$ is closed, for all $x \in X$. To show Q is T1, we need to show that $\{[x]\}$ is closed in Q, for all $x \in X$. Since $q^{-1}(Q \setminus \{[x]\}) = \{y \in X : q(y) \neq [x]\} = X \setminus [x]$ is open, $Q \setminus \{[x]\}$ is open in Q, for all $[x] \in Q$. Therefore, Q is a T1 space.

(ii) Consider the commutative diagram of continuous maps

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\Delta_X"'] & Q \arrow[d, "\Delta_Q"] \\ X \times X \arrow[r, "q\times q"] & Q \times Q, \end{tikzcd}$$

where $q \times q : X \times X \to Q \times Q$ is the product map given by

$$(q \times q)(x,y) = (q(x), q(y)), \ \forall\, (x,y) \in X \times X.$$

Note that, $(q \times q)^{-1}(\Delta_Q(Q)) = \{(x,y) \in X \times X : q(x) = q(y)\} = \rho$. If Q is Hausdorff, then $\Delta_Q(Q)$ is closed by Exercise 1.1.8. Since $q \times q$ is continuous, ρ is closed in $X \times X$.

Now we assume that q is an open map, and that ρ is closed in $X \times X$. Since $q \times q$ is a continuous surjective open map (verify!), it is a quotient map by Corollary 1.2.16. Since $(q \times q)^{-1}(\Delta_Q(Q)) = \rho$ is closed in $X \times X$, the diagonal $\Delta_Q(Q)$ is closed in $Q \times Q$ by Theorem 1.2.12 (iii). Therefore, Q is Hausdorff by Exercise 1.1.8.

□

Now we give an example to show that even if X is Hausdorff and the equivalence relation ρ is closed in $X \times X$, the associated quotient space $Q = X/\rho$ need not be Hausdorff without the assumption that q is an open map. For this, we first recall the following.

Definition 1.2.26. A topological space X is said to be *normal* if any two disjoint closed subsets can be separated by a pair of disjoint open subsets containing them. In other words, given two closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there are open subsets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

Most of the familiar examples of topological spaces are generally normal (e.g., $\mathbb{R}^n$), and a closed subspace of a normal space is normal. The following example shows that a Hausdorff space need not be normal.

Example 1.2.27. Let $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Consider the topology τ_K on $\mathbb{R}$ whose basis for open subsets is given by the collection

$$\mathscr{B} = \{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\} \cup \{(a, b) \setminus K : a, b \in \mathbb{R} \text{ with } a < b\}.$$

Clearly this topology on $\mathbb{R}$ is strictly finer than the Euclidean topology on $\mathbb{R}$, and hence $(\mathbb{R}, \tau_K)$ is a Hausdorff space. Note that in this topology, K and $\{0\}$ are disjoint closed subsets that cannot be separated by a pair of disjoint open subsets containing them. Therefore, $(\mathbb{R}, \tau_K)$ is not normal.

Example 1.2.28. Start with a Hausdorff space X that is not normal. Choose two disjoint closed subsets $A, B \subset X$ that cannot be separated by two disjoint open subsets containing them. Take $\rho = \Delta_X(X) \cup (A \times A) \cup (B \times B)$. Note that ρ is an equivalence relation on X, and is closed in $X \times X$. Since the ρ-equivalence classes of each elements of X are closed subsets (either singleton subset of X or A or B) of X, the quotient space X/ρ is a T1 space by Proposition 1.2.25 (i). Show that the associated quotient space X/ρ is not Hausdorff. In this example, can you think of an open subset of X whose image under the quotient map is not open?

1.3 Projective space and Grassmannian

1.3.1 Real and complex projective spaces

Fix an integer $n \geq 0$. Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$v \sim v' \text{ if } v' = \lambda \cdot v, \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}. \tag{1.3.1}$$

In other words, identify all points lying on the same straight-line in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. Then the associated quotient space

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\})/\sim$$

is called the *real projective n-space*. As a set, $\mathbb{RP}^n$ consists of all straight-lines in $\mathbb{R}^{n+1}$ passing through the origin $0 \in \mathbb{R}^{n+1}$. So an element of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \{\lambda \cdot (a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R} \setminus \{0\}\}. \tag{1.3.2}$$

Let $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map for the projective n-space. Note that the unit n-sphere $S^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} a_j^2 = 1\}$ is a compact connected subspace of $\mathbb{R}^{n+1} \setminus \{0\}$. Since the restriction map $q\big|_{S^n} : S^n \longrightarrow \mathbb{RP}^n$ is continuous and surjective, $\mathbb{RP}^n$ is compact and connected.

Exercise 1.3.3. Prove the following.

(i) Show that the map $f := q\big|_{S^n} : S^n \to \mathbb{RP}^n$ is a quotient map.

(ii) For each $\ell \in \mathbb{RP}^n$, show that $f^{-1}(\ell) = \{v, -v\}$, for some $v \in S^n$.

(iii) Show that $\mathbb{RP}^n$ is Hausdorff.

Outline of solution. Note that, the quotient map $q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is given by sending $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ to the straight line

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) : \lambda \in \mathbb{R}\} \in \mathbb{RP}^n.$$

Since $\mathbb{RP}^n$ consists of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin, given a straight-line $\ell \in \mathbb{RP}^n$, choosing any non-zero point $v := (a_0, \ldots, a_n) \in \ell$, we find an element $v/\|v\| \in S^n$ with $f(v/\|v\|) = \ell$, where $\|v\| := \big(\sum_{j=1}^{n} a_j^2\big)^{1/2}$. Thus, f is surjective.

$$f : S^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{q} \mathbb{RP}^n.$$

Note that given any subset $V \subseteq \mathbb{RP}^n$, we have $f^{-1}(V) = q^{-1}(V) \cap S^n$. So continuity of f follows from that of q. To see that f is a quotient map, suppose that $f^{-1}(V)$ is open in S^n. To show that V is open in $\mathbb{RP}^n$, fix a point $\ell \in V$. Its fiber $f^{-1}(\ell) = \{v, -v\}$ consists of the two antipodal points of S^n obtained by intersecting the line ℓ with S^n. Since the points v and $-v$ lies on two hemispheres separated by a great circle on S^n, we can find a small enough (connected) open neighbourhood $U \subset f^{-1}(V)$ of v such that $-U := \{-u : u \in U\} \subset S^n$ is an open neighbourhood of $-v$ in S^n, and $U \cap (-U) = \emptyset$. Note that, $-U \subseteq f^{-1}(V)$. Then $f\big|_U$ is a homeomorphism of U onto the open neighbourhood $f(U) \subseteq V$ of ℓ in $\mathbb{RP}^n$. Thus f is a quotient map. □

Next we show that $\mathbb{RP}^n$ can be covered by $n+1$ open subsets each homeomorphic to $\mathbb{R}^n$. Let $p_j : \mathbb{R}^{n+1} \to \mathbb{R}$ be the j-th projection map defined by

$$p_j(x_0, \ldots, x_n) = x_j, \;\; \forall\, (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}.$$

For each $j \in \{0, 1, \ldots, n\}$, consider the *hyperplane*

$$H_j := \{[a_0 : \cdots : a_n] \in \mathbb{RP}^n : a_j = 0\} \subset \mathbb{RP}^n.$$

Since q is a quotient map and

$$\begin{aligned} q^{-1}(H_j) &= \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : a_j = 0\} \\ &= p_j^{-1}(0) \cap (\mathbb{R}^{n+1} \setminus \{0\}), \end{aligned}$$

we conclude that H_j is a closed subset of $\mathbb{RP}^n$. Let $U_j := \mathbb{P}^n \setminus H_j$, $\forall\, j = 0, 1, \ldots, n$. Since any point of $\mathbb{RP}^n$ is of the form

$$[a_0 : \cdots : a_n] := \{\lambda(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\} : \lambda \in \mathbb{R}\},$$

with $a_j \neq 0$, for some j, we see that $\{U_0, U_1, \ldots, U_n\}$ is an open cover of $\mathbb{RP}^n$.

Proposition 1.3.4. *The open subset $U_j \subset \mathbb{RP}^n$ is homeomorphic to $\mathbb{R}^n$, for all j.*

Proof. Consider the map $\phi_j : U_j \longrightarrow \mathbb{R}^n$ given by

$$[a_0 : \cdots : a_n] \stackrel{\phi_j}{\longmapsto} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right).$$

Note that ϕ_j is a well-defined bijective map with its inverse $\psi_j : \mathbb{R}^n \to U_j$ given by

$$(b_0, \ldots, b_{n-1}) \mapsto [b_0 : \cdots : b_{j-1} : 1 : b_j : \cdots : b_n].$$

Note that $V_j = q^{-1}(U_j) = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} : a_j \neq 0\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$, and the map $f_j : V_j \to \mathbb{R}^n$ given by $(a_0, \ldots, a_n) \stackrel{f_j}{\longmapsto} \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_n}{a_j}\right)$ is continuous. Since $q^{-1}(\phi_j^{-1}(V)) = f_j^{-1}(V)$, $\forall\, V \subseteq \mathbb{R}^n$, and q is a quotient map, we conclude that ϕ_j is continuous, for all j (c.f. Proposition 1.2.19).

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "q"'] & V_j \arrow[l, hook'] \arrow[r, "f_j"] \arrow[d, "q_j"'] & \mathbb{R}^n \arrow[dl, dashed, bend left, "\psi_j"] \\
\mathbb{RP}^n & U_j \arrow[l, hook'] \arrow[ur, "\phi_j"] &
\end{tikzcd}$$

Since f_j is a quotient map by Corollary 1.2.16, as before we see that $\psi_j = \phi_j^{-1}$ is also continuous. This completes the proof. $\square$

Corollary 1.3.5. *$\mathbb{RP}^n$ is a compact connected Hausdorff space.*

Exercise 1.3.6. Define an equivalence relation $\sim$ on S^n by

$$v \sim v' \text{ if } v' = -v.$$

Show that the associated quotient space $S^n/\sim$ is homeomorphic to $\mathbb{RP}^n$. Conclude that $\mathbb{RP}^n$ is a compact connected Hausdorff space.

The *complex projective n-space* $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim$ defined by

$$v \sim v' \text{ if } v' = \lambda v, \text{ for some } \lambda \in \mathbb{C}.$$

So the points of $\mathbb{CP}^n$ are precisely one dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^{n+1}$ (i.e., complex lines in $\mathbb{C}^{n+1}$ passing through the origin $0 \in \mathbb{C}^{n+1}$).

Exercise 1.3.7. Show that $\mathbb{CP}^n$ is a compact connected Hausdorff space.

Remark on notations: The real projective n-space $\mathbb{RP}^n$ is also denoted by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n(\mathbb{R})$. Similar notations $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^n(\mathbb{C})$ are also used for complex projective n-space $\mathbb{CP}^n$.

1.3.2 Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$

Fix two positive integers k and n, with $k < n$. Let

$$(\mathbb{R}^n)^k := \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-times}}$$

be k-fold product of $\mathbb{R}^n$ together with the product topology. A typical element of $(\mathbb{R}^n)^k$ is of the form $(v_1, \ldots, v_k)$, where $v_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$, for all $j = 1, \ldots, k$. Note that, we can identify $(\mathbb{R}^n)^k$ with $M_{k,n}(\mathbb{R})$ using the bijective map

$$(v_1, \ldots, v_k) \longmapsto \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix}.$$

Consider the subset

$$X := \{(v_1, \ldots, v_k) \in (\mathbb{R}^n)^k \mid \{v_1, \ldots, v_k\} \text{ is } \mathbb{R}\text{-linearly independent}\}$$

with the subspace topology induced from $(\mathbb{R}^n)^k$. Given $A := (v_1, \ldots, v_k)$ and $A' := (v'_1, \ldots, v'_k)$ in X, we define $A \sim A'$ if

$$\mathrm{Span}_{\mathbb{R}}\{v_1, \ldots, v_k\} = \mathrm{Span}_{\mathbb{R}}\{v'_1, \ldots, v'_k\}.$$

Clearly $\sim$ is an equivalence relation on X. The associated quotient topological space $X/\sim$ is known as the *Grassmannian of k-dimensional $\mathbb{R}$-linear subspaces of* $\mathbb{R}^n$, and is denoted by $\mathrm{Gr}(k, \mathbb{R}^n)$. As a set, $\mathrm{Gr}(k, \mathbb{R}^n)$ consists of all $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$ of dimension k.

Corollary 1.3.8. $\mathrm{Gr}(1, \mathbb{R}^n)$ *is homeomorphic to* $\mathbb{RP}^{n-1}$.

Remark 1.3.9 (Plücker embedding)**.** Given a n-dimensional $\mathbb{R}$-vector space V, its k-th exterior power $\bigwedge^k V$ is a $\mathbb{R}$-vector space of dimension $\binom{n}{k}$. Sending $W \in \mathrm{Gr}(k, \mathbb{R}^n)$ to its k-th exterior power $\bigwedge^k W \subset \bigwedge^k \mathbb{R}^n$, we get a continuous map

$$\Phi : \mathrm{Gr}(k, \mathbb{R}^n) \longrightarrow \mathbb{RP}^N,$$

where $N = \binom{n}{k} - 1$. It turns out that Φ is a closed embedding (homeomorphism onto a closed subspace of $\mathbb{RP}^N$). From this, one can conclude that $\mathrm{Gr}(k, \mathbb{R}^n)$ is a compact Hausdorff space. We shall not go into detailed proofs of the above statements.

1.4 Topological group

Definition 1.4.1. A *topological group* is a topological space G which is also a group G such that the binary map (group operation)

$$m : G \times G \to G, \quad (x, y) \mapsto xy,$$

and the inversion map

$$\mathrm{inv} : G \to G, \;\; x \mapsto x^{-1},$$

involved in its group structure, are continuous. Here we consider $G \times G$ as the product topological space.

We recast the above definition of topological group in more formal language, without using points of G. This formalism, with appropriate type of spaces and maps between them, defines *Lie group*, *algebraic group*, *group-scheme* and more generally, a *group object* in a category (for curious readers!). Denote by $*$ the topological space whose underlying set is singleton. This space is unique up to a unique homeomorphism. Given any topological space X, any map $* \to X$ is continuous, and they are in bijection with the underlying set of points of X. On the other hand, the space $*$ is the *final object* in the category of topological spaces in the sense that, given any topological space X, there is a unique continuous map $X \to *$. Clearly the product space $X \times *$ is homeomorphic to X, and the set of all such homeomorphisms are in bijection with the set of all *automorphisms of* X (i.e., homeomorphisms of X onto itself). Unless explicitly specified, we consider the homeomorphism $X \times * \to X$ given by the identity map $\mathrm{Id}_X : X \to X$ of X.

Now the above Definition 1.4.1 essentially says that, a topological group is a pair (G, m), where G is a topological space and $m : G \times G \to G$ is a continuous map such that the following axioms holds.

(TG1) *Associativity:* The following diagram is commutative.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

(TG2) *Existence of neutral element:* There is a continuous map $e : * \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
* \times G \arrow[r, "e \times \mathrm{Id}_G"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{Id}_G \times e"'] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd}$$

(TG3) *Existence of inverse:* There is a continuous map $\mathrm{inv} : G \to G$ such that the following diagram is commutative.

$$\begin{tikzcd}
G \arrow[r, "(\mathrm{Id}_G, \mathrm{inv})"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{inv}, \mathrm{Id}_G)"'] \arrow[d] \\
* \arrow[r, "e"] & G & * \arrow[l, "e"']
\end{tikzcd}$$

Example 1.4.2. (i) Any abstract group is a topological group with respect to the discrete topology on it.

(ii) $(\mathbb{R}, +)$, the real line with usual addition of real numbers, is a topological group.

(iii) $(\mathbb{R}^*, \cdot)$, the subspace of non-zero real numbers with usual multiplication is a topological group.

(iv) $(\mathbb{Z}, +)$ is a topological group, where the topology on $\mathbb{Z}$ is discrete.

(v) For any integer $n \geq 1$, the Euclidean space $\mathbb{R}^n$ with the component wise addition, i.e.,

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) := (a_1 + b_1, \ldots, a_n + b_n), \ \forall\, a_i, b_i \in \mathbb{R},$$

is a topological group.

(vi) Given integers $m, n \geq 1$, the set of all $(m \times n)$-matrices with real entries $\mathrm{M}_{m,n}(\mathbb{R})$, considered as the Euclidean topological space $\mathbb{R}^{mn}$, is a topological group with respect to the usual matrix addition.

(vii) $\mathrm{GL}_n(\mathbb{R})$, the subspace of all invertible $(n \times n)$-matrices with real entries, is a topological group with respect to multiplication of matrices.

(viii) Circle group: The space $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$, together with the multiplication of complex numbers, is a topological group.

(ix) Any abstract subgroup of a topological group is a topological group with respect to the subspace topology.

(x) Product of two topological groups is a topological group.

Exercise 1.4.3. Let G be a topological group. If $U \subseteq G$ is an open neighbourhood of identity $e \in G$, show that there is an open neighbourhood $V \subset G$ of identity such that $V^2 := \{ab : a, b \in V\} \subseteq U$. (Hint: Use continuity of the multiplication map m.)

Exercise 1.4.4. Show that for any $a \in G$, the *right translation by a map*

$$R_a : G \to G, \ \ g \mapsto ga,$$

is a homeomorphism. Prove the same statement for the *left translation by a map* given by $L_a(g) = ga$, for all $g \in G$. (*Hint:* Note that R_a is the composite map $g \mapsto (g, a) \stackrel{m}{\mapsto} ga$ with inverse $R_{a^{-1}}$.)

Exercise 1.4.5. Show that a topological group G is Hausdorff if and only if it is a T1 space. (*Hint:* $\Delta_G(G)$ is precisely the inverse image of $\{e\} \subseteq G$ under the map $(x, y) \mapsto x^{-1}y$.)

Lemma 1.4.6. *Let G be a topological group. Let H be the connected component of G containing the neutral element $e \in G$. Then H is a closed normal subgroup of G.*

Proof. Since connected components are closed, H is closed. Since for any $a \in H$, the set $Ha^{-1} = \{ha^{-1} : h \in H\} = R_{a^{-1}}(H)$ contains e, and is homeomorphic to H, we must have $Ha^{-1} \subseteq H$. Since this holds for all $a \in H$, we see that H is a subgroup of G. To see that H is normal, note that, for any $g \in G$, the set $gHg^{-1} = L_g(R_{g^{-1}}(H))$ is a connected subset of G containing e, and hence $gHg^{-1} \subseteq H$. This completes the proof. □

Definition 1.4.7. A *right action* of a topological group G on a topological space X is a continuous map $\sigma : X \times G \to X$ such that $\sigma(x,e) = x$, and $\sigma(\sigma(x,g_1),g_2) = \sigma(x,m(g_1,g_2))$, for all $x \in X$ and $g_1, g_2 \in G$, where $m : G \times G \to G$ is the product operation (multiplication map) on G. Similarly, one can left action of G on X.

Without using points, a right G-action σ on X can be defined by commutativity of the following diagrams.

(i)

$$\begin{tikzcd} X \times * \arrow[r, "\mathrm{Id}_X \times e"] \arrow[dr, "\cong"'] & X \times G \arrow[d, "\sigma"] \\ & X \end{tikzcd}$$

(ii)

$$\begin{tikzcd} X \times G \times G \arrow[r, "\sigma \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_X \times m"'] & X \times G \arrow[d, "\sigma"] \\ X \times G \arrow[r, "\sigma"] & X \end{tikzcd}$$

A right G-action σ on X induces an equivalence relation on X, which gives a partition of X as a disjoint union of equivalence classes. A typical equivalence class is of the form

$$\mathrm{orb}_G(x) := \{x' \in X : x' = xg, \text{ for some } g \in G\} = xG,$$

and is called the *G-orbit* of $x \in X$. The associated quotient space, denoted by X/σ or X/G, consists of all G-orbits of elements of X as its points. For this reason, X/G is also called *orbit space*. If the G-action on X is *transitive* (i.e., given any $x, x' \in X$, there exists $g \in G$ such that $x' = xg$), then X is called a *homogeneous space*. In this case, the associated quotient space X/G is singleton.

Exercise 1.4.8. Given a subgroup H of a topological group G, the H-action on G defined by

$$G \times H \mapsto G, \ (g,h) \mapsto gh$$

gives a partition of G into all right cosets of H in G. Show that the orbit space G/H is a homogeneous space.

Exercise 1.4.9. Let $I = [0,1] \subset \mathbb{R}$. Define the $\mathbb{Z}_2$-action on $I \times I$ which gives identifications $(0,t) \sim (1, 1-t)$, for each $t \in I$. Convince yourself that the associated quotient space is homeomorphic to the *Möbius strip* (see Figure 1.9). Note that, Möbius strip has only one side!

Exercise 1.4.10. Define a $\mathbb{Z}_2$-action on the n-sphere $S^n \subset \mathbb{R}^{n+1}$ which identifies $v \in S^n$ with its *antipodal point* $-v \in S^n$. Show that the associated quotient space $S^n/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^n$.

Exercise 1.4.11. Show that $\mathbb{R}/\mathbb{Q}$ is a non-Hausdorff topological group. Hint: Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, and right cosets are just translates of $\mathbb{Q}$.

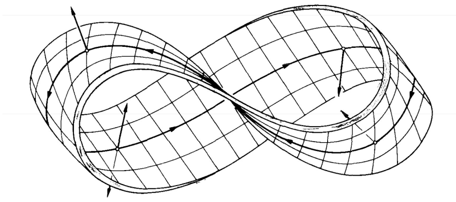


FIGURE 1.9: Möbius strip

Exercise 1.4.12. Let $\sigma : X \times G \to X$ be a right action of a topological group on a space X. For each $g \in G$, show that the induced map

$$\sigma_g : X \to X, \ \ x \mapsto xg := \sigma(x, g)$$

is a homeomorphism.

Exercise 1.4.13. Let X be a topological space together with an action of a topological group G. Show that the quotient map $q : X \to X/G$ is open. (Hint: For $V \subseteq X$ open, show that $q^{-1}(q(V)) = \bigcup_{g \in G} Vg$ is open by Exercise 1.4.12, where $Vg = \{vg : v \in V\}, \forall g \in G$.)

Proposition 1.4.14. *Let H be a subgroup of a topological group G. Then the orbit space G/H is Hausdorff if and only if H is closed in G. (Here G/H is not necessarily a group because H need not be a normal subgroup of G.)*

Proof. If G/H is Hausdorff, then it is a T1 space so that $H = \mathrm{orb}_H(e) \in G/H$ is a closed point. Since H is the inverse image of this point under the quotient map $q : G \to G/H$ (continuous), H is closed in G. Conversely, suppose that H is closed in G. Since the equivalence relation given by the H-action on G is precisely the inverse image of H under the continuous map

$$G \times G \longrightarrow G, \ \ (g_1, g_2) \mapsto g_1^{-1} g_2,$$

and the quotient map $q : G \to G/H$ is open by Exercise 1.4.13, the converse part follows from Proposition 1.2.25 because H is closed in G. $\square$

Corollary 1.4.15. *The topological group $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.*

Definition 1.4.16. A *homomorphism* of topological groups is a continuous group homomorphism. An *isomorphism* of topological groups is a bijective bi-continuous homomorphism of topological groups.

Exercise 1.4.17. If $f : G \to H$ is a homomorphism of topological groups with H Hausdorff, show that $\mathrm{Ker}(f) := \{g \in G : f(g) = e_H\}$ is a closed normal subgroup of G.

Exercise 1.4.18. If $f : G \to H$ is a homomorphism of topological groups, show that the induced map $G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is an isomorphism of topological groups.

Exercise 1.4.19. Show that $f : \mathbb{R} \to S^1$ defined by $f(t) = e^{2\pi i t}$, for all $t \in \mathbb{R}$, is a surjective homomorphism of topological groups. Use Exercise 1.4.18 to show that $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

Exercise 1.4.20. Let $f : G \to H$ be a continuous bijective homomorphism of topological groups. Show that $f^{-1} : H \to G$ is continuous (Hint: Use Exercise 1.4.13).

Corollary 1.4.21. *A bijective homomorphism $f : G \to H$ of topological groups is an isomorphism.*

Exercise 1.4.22. Consider the $\mathbb{Z}$-action on $\mathbb{R}$ given by $\sigma(t, n) = t + n$, for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Show that the associated quotient space $\mathbb{R}/\sigma$ is homeomorphic to S^1.

Exercise 1.4.23. Show that $\mathrm{GL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$ as topological groups.

Exercise 1.4.24. Show that $\mathrm{GL}_n(\mathbb{R})$ is disconnected, and has precisely two connected components, whereas $\mathrm{GL}_n(\mathbb{C})$ is path-connected. (*Hint:* For the first part, use determinant map. For the second part, given $A \in \mathrm{GL}_n(\mathbb{C})$ use left and right translation homeomorphisms to move it to an upper triangular matrix, and then use convex combination map for its entries to move it to the identity matrix in $\mathrm{GL}_n(\mathbb{C})$.)

Exercise* **1.4.25.** Show that the group

$$SO_n = \{A \in \mathrm{GL}_n(\mathbb{R}) : AA^t = A^tA = I_n \text{ and } \det(A) = 1\}$$

is compact and connected.

Chapter 2

Homotopy Theory

2.1 Homotopy of maps

Let I be the closed interval $[0,1] \subset \mathbb{R}$. Let X and Y be topological spaces.

Definition 2.1.1. Let $f_0, f_1 : X \to Y$ be continuous maps. We say that f_0 *is homotopic to* f_1, written as $f_0 \simeq f_1$, if there is a continuous map

$$F : X \times I \longrightarrow Y$$

such that $F(x,0) = f_0(x)$, $\forall\, x \in X$, and $F(x,1) = f_1(x)$, $\forall\, x \in X$. In this case, the continuous map F is called the *homotopy* from f_0 to f_1. A continuous map $f : X \to Y$ is said to be *null homotopic* if f is homotopic to a constant map from X into Y.

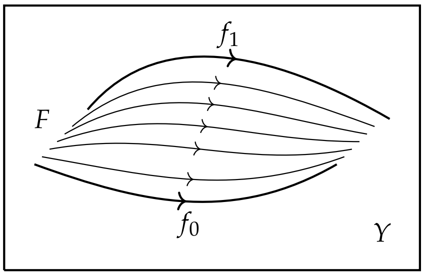


FIGURE 2.1: Homotopy

Example 2.1.2. 1. Let X be a space. Then any two continuous maps $f, g : X \to \mathbb{R}^2$ are homotopic. To see this, note that the map $F : X \times I \to \mathbb{R}^2$ defined by

$$F(x,t) = (1-t)f(x) + tg(x), \;\; \forall\, (x,t) \in X \times I,$$

is continuous and satisfies $F(x,0) = f(x)$ and $F(x,1) = g(x)$, for all $x \in X$. Thus F is a homotopy from f to g; such a homotopy is called a *straight-line homotopy*, because for each $x \in X$, it movies $f(x)$ to $g(x)$ along the straight-line segment joining them.

Before proceeding further, let us recall the following useful result from basic topology course, that we need frequently in this course.

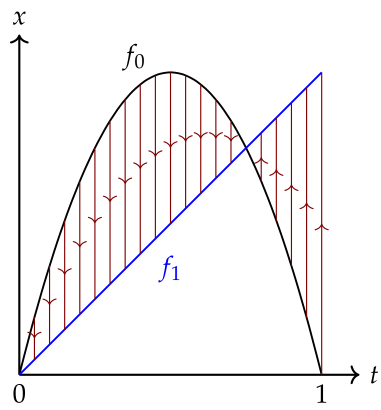


FIGURE 2.2: Example of a straight-line homotopy

Lemma 2.1.3 (Joining continuous maps)**.** *Let A and B be two closed subsets of topological space X such that $X = A \cup B$. Let Y be any topological space. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps such that $f(x) = g(x)$, for all $x \in A \cap B$. Then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & \text{if} \quad x \in A, \\ g(x), & \text{if} \quad x \in B, \end{cases}$$

is continuous.

Proof. Let $Z \subseteq Y$ be a closed subset. It is enough to check that $h^{-1}(Z)$ is closed in X. Note that

$$\begin{aligned} h^{-1}(Z) &= \left(h^{-1}(Z) \cap A\right) \bigcup \left(h^{-1}(Z) \cap B\right) \\ &= f^{-1}(Z) \bigcup g^{-1}(Z). \end{aligned}$$

Since f and g are continuous, $f^{-1}(Z)$ is closed in A and $g^{-1}(Z)$ is closed in B. Since A and B are closed in X, both $f^{-1}(Z)$ and $g^{-1}(Z)$ are closed in X, and so is their union $h^{-1}(Z)$. This completes the proof. □

Lemma 2.1.4. *The relation "being homotopic maps" is an equivalence relation on the set $C(X,Y)$ of all continuous maps from X into Y.*

Proof. For any $f \in C(X,Y)$, taking

$$F : X \times I \to Y, \ \ (x,t) \mapsto f(x)$$

we see that f is homotopic to itself, and hence "being homotopic maps" is a reflexive relation. Let $f_0, f_1 \in C(X,Y)$ be such that f_0 is homotopic to f_1 with homotopy F. Then the continuous map

$$G : X \times I \to Y, \ \ (x,t) \mapsto F(x, 1-t)$$

is a homotopy from f_1 to f_0. So "being homotopic maps" is a symmetric relation. Let $f_0, f_1, f_2 \in C(X,Y)$ be such that $f_0 \simeq f_1$ with a homotopy F, and $f_1 \simeq f_2$ with a homotopy G. Consider the map $H : X \times I \to Y$ defined by

$$H(x,t) := \begin{cases} F(x,2t), & \text{if} \quad t \in [0, \frac{1}{2}], \\ G(x,2t-1), & \text{if} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

Since at $t = \frac{1}{2}$, we have $F(x,2t) = F(x,1) = f_1(x) = G(x,0) = G(x,2t-1)$, for all $x \in X$, we see that H is a well-defined continuous map (c.f., Lemma 2.1.3). Clearly H satisfies $H(x,0) = f_0(x)$ and $H(x,1) = f_2(x)$, for all $x \in X$. Therefore, H is a homotopy from f_0 to f_2. Thus "being homotopic maps" is a transitive relation, and hence is an equivalence relation on $C(X,Y)$. $\square$

Exercise 2.1.5. Let $f, g \in C(X,Y)$, and $F : X \times I \to Y$ be a homotopy from f to g. Use F to construct a homotopy G from f to g with $G \neq F$. Therefore, homotopy between two maps need not be unique. (*Hint: take* $G(x,t) = F(x,t^2)$).

Lemma 2.1.6. *Let $f_0, f_1 : X \to Y$ be two continuous maps such that f_0 is homotopic to f_1. Then for any spaces Z and W, and continuous maps $g : Z \to X$ and $h : Y \to W$, we have $f_0 \circ g \simeq f_1 \circ g$ and $h \circ f_0 \simeq h \circ f_1$.*

$$\begin{tikzcd} Z \arrow[r, "g"] & X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, "h"] & W. \end{tikzcd}$$

Proof. Let $F : X \times I \to Y$ be a continuous map such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$, for all $x \in X$. Define $G : Z \times I \to Y$ by setting

$$G(z,t) = F(g(z),t), \ \forall\, (z,t) \in Z \times I.$$

Clearly G is a continuous function with $G(z,0) = F(g(z),0) = (f_0 \circ g)(z)$, and $G(z,1) = F(g(z),1) = (f_1 \circ g)(z)$, for all $z \in Z$. Therefore, G gives a homotopy $f_0 \circ g \simeq f_1 \circ g$. Similarly, taking

$$H : X \times I \to W, \ \ (x,t) \mapsto h(F(x,t)),$$

we see that H is a continuous map satisfying $H(x,0) = h(F(x,0)) = (h \circ f_0)(x)$ and $H(x,1) = h(F(x,0)) = (h \circ f_1)(x)$, for all $x \in X$. Therefore, H gives a homotopy $h \circ f_0 \simeq h \circ f_1$. $\square$

Definition 2.1.7. Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be continuous maps of pointed topological spaces. A *homotopy* from f_0 to f_1 is a continuous map $F : X \times I \to Y$ such that

(i) $F(x,0) = f_0(x), \ \forall\, x \in X$,

(ii) $F(x,1) = f_1(x), \ \forall\, x \in X$, and

(iii) $F(x_0,t) = y_0, \ \forall\, t \in [0,1]$.

When we talk about homotopy of continuous maps of pointed topological spaces, we always mean that the homotopy preserve the marked points in the sense of (iii) mentioned above.

Exercise 2.1.8. Let $f_0, f_1 : (X, x_0) \to (Y, y_0)$ be two continuous maps of pointed topological spaces. If f_0 is homotopic to f_1 in the sense of Definition 2.1.7, show that for any spaces Z and W, and continuous maps $g : (Z, z_0) \to (X, x_0)$ and $h : (Y, y_0) \to (W, w_0)$, we have

(i) $f_0 \circ g$ is homotopic to $f_1 \circ g$ in the sense of Definition 2.1.7, and

(ii) $h \circ f_0$ is homotopic to $h \circ f_1$ in the sense of Definition 2.1.7.

Definition 2.1.9. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is said to be a *homotopy equivalence* if there exist a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. In this case, we say that X *is homotopy equivalent to* Y (or, X *and* Y *have the same homotopy type*), and write it as $X \simeq Y$.

Lemma 2.1.10. *Being homotopy equivalent spaces is an equivalence relation.*

Proof. For any space X, we can take $f = g = \mathrm{Id}_X$ to get $f \circ g = \mathrm{Id}_X = g \circ f$ so that X is homotopy equivalent to itself (verify!). It follows from the Definition 2.1.9 that the relation "being homotopy equivalent spaces" is symmetric. Let X, Y and Z be topological spaces such that $X \simeq Y$ and $Y \simeq Z$. Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be homotopy equivalences. Then there are continuous maps $g_1 : Y \to X$ and $g_2 : Z \to Y$ such that $g_1 \circ f_1 \simeq \mathrm{Id}_X$, $f_1 \circ g_1 \simeq \mathrm{Id}_Y$, $g_2 \circ f_2 \simeq \mathrm{Id}_Y$ and $f_2 \circ g_2 \simeq \mathrm{Id}_Z$. Now using Lemma 2.1.6 we have

$$\begin{aligned}(f_2 \circ f_1) \circ (g_1 \circ g_2) &= f_2 \circ (f_1 \circ g_1) \circ g_2 \\ &\simeq f_2 \circ \mathrm{Id}_Y \circ g_2 \\ &= f_2 \circ g_2 \simeq \mathrm{Id}_Z \,.\end{aligned}$$

Similarly, we have $(g_1 \circ g_2) \circ (f_2 \circ f_1) \simeq \mathrm{Id}_X$. Therefore, $f_2 \circ f_1 : X \to Z$ is a homotopy equivalence, and hence $X \simeq Z$. Thus "being homotopy equivalent spaces" is a transitive relation, and hence is an equivalence relation. □

Definition 2.1.11. A space X is said to be *contractible* if the identity map $\mathrm{Id}_X : X \to X$ is null homotopic.

Exercise 2.1.12. Show that a contractible space is path-connected.

Corollary 2.1.13. *A space X is contractible if and only if given any topological space T, any two continuous maps $f, g : T \to X$ are homotopic.*

Proof. Suppose that X is contractible. Let T be any topological space, and let $f, g : T \to X$ be any two continuous maps. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ of X is homotopic to a constant map $c_{x_0} : X \to X$ given by $c_{x_0}(x) = x_0$, $\forall\, x \in X$. Then $f = \mathrm{Id}_X \circ f$ is homotopic to the constant map $c_{x_0} \circ f : T \to X$. Similarly, g is homotopic to the constant map $c_{x_0} \circ g : T \to X$. Since $c_{x_0} \circ f = c_{x_0} \circ g$, and being homotopic maps is an equivalence relation by Lemma 2.1.4, we see that f is homotopic to g. Converse part is obvious. □

2.2 Fundamental group

2.2.1 Construction

A *path* in X is a continuous map $\gamma : I \to X$; the point $\gamma(0) \in X$ is called the *initial point* of γ, and $\gamma(1) \in X$ is called the *terminal point* or the *final point* of γ.

Definition 2.2.1. Fix two points $x_0, x_1 \in X$. Two paths $f, g : I \to X$ with the same initial point x_0 and terminal point x_1 are said to be *path homotopic* if

(i) $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, and

(ii) there is a continuous map $F : I \times I \to X$ such that for each $t \in I$, the map

$$\gamma_t : I \to X, \ \ s \mapsto F(s,t)$$

is a path in X from x_0 to x_1, and that $\gamma_0 = f$ and $\gamma_1 = g$.

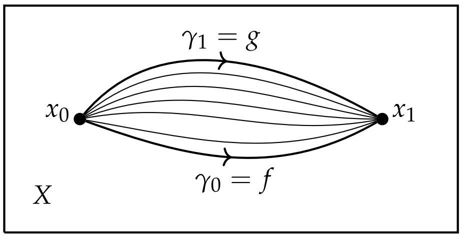


FIGURE 2.3: Path homotopy

Exercise 2.2.2. Given $x_0, x_1 \in X$, let

$$Path(X; x_0, x_1) := \{f : I \to X \,|\, f(0) = x_0,\ f(1) = x_1\}$$

be the set of all paths in X starting at x_0 and ending at x_1. Show that being path homotopic is an equivalence relation on $Path(X; x_0, x_1)$. (*Hint: Follow the proof of Lemma 2.1.4*).

Remark 2.2.3. If $\gamma, \delta : I \to X$ are two paths in X, we use the symbol $\gamma \simeq \delta$ to mean γ and δ are path-homotopic in X in the sense of Definition 2.2.1. Unless explicitly mentioned, by a *homotopy between two paths* we always mean a path-homotopy between them.

A *loop* in X is a path $\gamma : I \to X$ with the same initial and terminal point: i.e., $\gamma(0) = \gamma(1) = x_0 \in X$; the point x_0 is called the base point of the loop γ. For a loop $\gamma : I \to X$ based at $x_0 \in X$, let

$$[\gamma] := \{\delta : I \to X \,|\, \delta(0) = \delta(1) = x_0 \text{ and } \delta \simeq \gamma\},$$

the homotopy equivalence class of γ. Fix a base point $x_0 \in X$, and let

$$\pi_1(X, x_0) := \{[\gamma] \;\big|\; \gamma : I \to X \text{ with } \gamma(0) = \gamma(1) = x_0\}$$

be the set of all equivalence classes of loops in X based at x_0. Next we define a binary operation on $\pi_1(X, x_0)$ and show that it is a group.

Given any two loops $\gamma_1, \gamma_2 : I \to X$ in X with the base point $x_0 \in X$, we define the *product of* γ_1 *with* γ_2 to be the map $\gamma_1 \star \gamma_2 : I \to X$ defined by

$$(\gamma_1 \star \gamma_2)(t) := \begin{cases} \gamma_1(2t), & \text{if } \ t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1), & \text{if } \ t \in [\frac{1}{2}, 1]. \end{cases} \tag{2.2.4}$$

That is, we first travel along γ_1 with double speed from $t = 0$ to $t = \frac{1}{2}$, and then along γ_2 from $t = \frac{1}{2}$ to $t = 1$. Clearly $\gamma_1 \star \gamma_2$ is a continuous map with $(\gamma_1 \circ \gamma_2)(0) = (\gamma_1 \circ \gamma_2)(1) = x_0$, and hence $\gamma_1 \circ \gamma_2$ is a loop in X with the base point x_0. Note that $\gamma_1 \star \gamma_2 \neq \gamma_2 \star \gamma_1$, in general (Find such an example).

Remark 2.2.5. In fact, we shall see later examples of topological spaces X admitting loops $\gamma_1, \gamma_2 : I \to X$ with the same base point $x_0 \in X$ such that $\gamma_1 \star \gamma_2$ is not homotopic to $\gamma_2 \star \gamma_1$.

Proposition 2.2.6. *Let γ_1, γ_2, δ_1, δ_2 be loops in X based at x_0. If $\gamma_1 \simeq \delta_1$ and $\gamma_2 \simeq \delta_2$, then $(\gamma_1 \star \gamma_2) \simeq (\delta_1 \star \delta_2)$. Consequently, the map*

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0), \ \ ([\gamma_1], [\gamma_2]) \mapsto [\gamma_1 \star \gamma_2] \tag{2.2.7}$$

is well-defined, and hence is a binary operation on the set $\pi_1(X, x_0)$.

Proof. Let $F : I \times I \to X$ be a homotopy from $F(-, 0) = \gamma_1$ to $F(-, 1) = \delta_1$, and let $G : I \times I \to X$ be a homotopy from $G(-, 0) = \gamma_2$ to $G(-, 1) = \delta_2$. Define a map $F \star G : I \times I \to X$ by sending $(s, t) \in I \times I$ to

$$(F \star G)(s, t) := \begin{cases} F(2s, t), & \text{if} \quad 0 \leq s \leq 1/2, \\ G(2s - 1, t), & \text{if} \quad 1/2 \leq s \leq 1. \end{cases}$$

Clearly $F \star G$ is a continuous map with $(F \star G)(-, 0) = \gamma_1 \star \gamma_2$ and $(F \star G)(-, 1) = \delta_1 \star \delta_2$. Therefore, $\gamma_1 \star \gamma_2 \simeq \delta_1 \star \delta_2$. □

Theorem 2.2.8. *The set $\pi_1(X, x_0)$ together with the binary operation (2.2.7) defined in Proposition 2.2.6 is a group, known as the fundamental group of X with base point $x_0 \in X$.*

To prove this theorem, we use the following technical tool (Lemma 2.2.10).

Definition 2.2.9. A *reparametrization* of a path $\gamma : I \to X$ is defined to be a composition $\gamma \circ \varphi$, where $\varphi : I \to I$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Lemma 2.2.10. *A reparametrization of a path preserves its homotopy class.*

Proof. Let $\gamma : I \to X$ be a path in X. Let

$$\gamma \circ \varphi : I \xrightarrow{\varphi} I \xrightarrow{\gamma} X$$

be a reparametrization of γ in X, for some continuous map $\varphi : I \to I$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Consider the straight-line homotopy from φ to the identity map of I given by

$$\varphi_t(s) := (1 - t)\varphi(s) + ts, \ \ \forall\, s, t \in I.$$

Now it is easy to check that the map

$$F : I \times I \to X, \ \ (s, t) \mapsto \gamma(\varphi_t(s)),$$

is continuous and satisfies $F(s, 0) = (\gamma \circ \varphi)(s)$ and $F(s, 1) = \gamma(s)$, for all $s \in I$. Therefore, $\gamma \circ \varphi \simeq \gamma$ via the homotopy F. □

Proof of Theorem 2.2.8. We need to verify group axioms.

Associativity: Given any three loops $\gamma_1, \gamma_2, \gamma_3 : I \to X$ based at x_0, it is enough to show that $(\gamma_1 \star \gamma_2) \star \gamma_3 \simeq \gamma_1 \star (\gamma_2 \star \gamma_3)$.

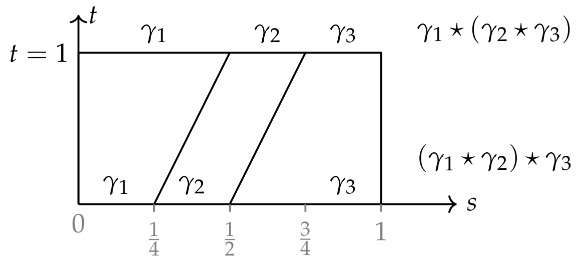


FIGURE 2.4: Homotopy for associativity

Note that

$$((\gamma_1 \star \gamma_2) \star \gamma_3)(t) = \begin{cases} \gamma_1(4t), & \text{if } 0 \le t \le 1/4, \\ \gamma_2(4t-1), & \text{if } 1/4 \le t \le 1/2, \\ \gamma_3(2t-1), & \text{if } 1/2 \le t \le 1, \end{cases}$$

and

$$(\gamma_1 \star (\gamma_2 \star \gamma_3))(t) = \begin{cases} \gamma_1(2t), & \text{if } 0 \le t \le 1/2, \\ \gamma_2(4t-2), & \text{if } 1/2 \le t \le 3/4, \\ \gamma_3(4t-3), & \text{if } 3/4 \le t \le 1. \end{cases}$$

It's an easy exercise to check that $\gamma_1 \star (\gamma_2 \star \gamma_3)$ is a reparametrization of $(\gamma_1 \star \gamma_2) \star \gamma_3$ by a piece-wise linear function (hence, continuous) $\varphi : I \to I$ defined by

$$\varphi(t) = \begin{cases} t/2, & \text{if } 0 \le t \le 1/2, \\ t - \frac{1}{4}, & \text{if } 1/2 \le t \le 3/4, \\ 2t-1, & \text{if } 3/4 \le t \le 1, \end{cases}$$

(see Figure 2.5). Then using Lemma 2.2.10 we conclude that $\gamma_1 \star (\gamma_2 \star \gamma_3) \simeq (\gamma_1 \star \gamma_2) \star \gamma_3$.

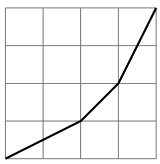


FIGURE 2.5: Graph of φ

Existence of identity: Let $e \in \pi_1(X, x_0)$ be the homotopy class of *constant loop*,

$$c_{x_0} : I \to X, \ \ t \mapsto x_0,$$

at x_0. Let $\gamma : I \to X$ be any loop in X based at x_0. Since $\gamma \star c_{x_0}$ is a reparametrization of γ via the function

$$\psi(t) := \begin{cases} 2t, & \text{if } 0 \le t \le 1/2, \\ 1, & \text{if } 1/2 \le t \le 1, \end{cases}$$

by Lemma 2.2.10 we have $\gamma \star c_{x_0} \simeq \gamma$. Similarly, $c_{x_0} \star \gamma$ is a reparametrization of γ by via the function

$$\eta(t) := \begin{cases} 0, & \text{if} \quad 0 \leq t \leq 1/2, \\ 2t-1, & \text{if} \quad 1/2 \leq t \leq 1, \end{cases}$$

by Lemma 2.2.10 we have $c_{x_0} \star \gamma \simeq \gamma$.

Existence of inverse: Given any loop γ in X based at x_0, we can define its *inverse loop* or *opposite loop* $\overline{\gamma} : I \to X$ by setting $\overline{\gamma}(t) = \gamma(1-t)$, for all $t \in I$. We need to show that $\gamma \star \overline{\gamma} \simeq c_{x_0}$ and $\overline{\gamma} \star \gamma \simeq c_{x_0}$. To show $\gamma \star \overline{\gamma} \simeq c_{x_0}$, consider the map $H : I \times I \to X$ given by

$$H(s,t) := f_t(s) \star g_t(s), \ \ \forall\, (s,t) \in I \times I,$$

where $f_t : I \to X$ is the path defined by

$$f_t(s) = \begin{cases} \gamma(s), & \text{for} \quad 0 \leq s \leq 1-t, \\ \gamma(1-t), & \text{for} \quad 1-t \leq s \leq 1, \end{cases}$$

and $g_t : I \to X$ is the inverse path of f_t, i.e., $g_t(s) = f_t(1-s)$, $\forall\, s \in I$. It is an easy exercise to check that H is a continuous map satisfying

$$H(s,0) = \gamma \star \overline{\gamma}, \ \text{ and } \ H(s,1) = c_{x_0}, \ \ \forall\, s \in I.$$

The homotopy H can be understood using the Figure 2.6. In the bottom line $t = 0$, we have

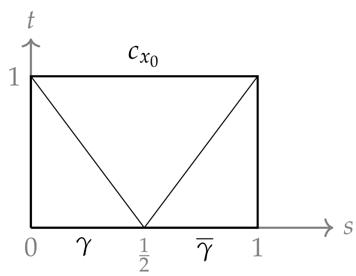


FIGURE 2.6: Homotopy H

$\gamma \star \overline{\gamma}$ while on the top line $t = 1$ we have the constant loop c_{x_0}. And below the 'V' shape we let $H(s,t)$ be independent of t while above the 'V' shape we let $H(s,t)$ be independent of s. Therefore, we have $\gamma \star \overline{\gamma} \simeq c_{x_0}$. Interchanging the roles of γ and $\overline{\gamma}$ in the above construction, we see that $\overline{\gamma} \star \gamma \simeq c_{x_0}$. Therefore, $\pi_1(X, x_0)$ is a group. □

2.2.2 Functoriality

By a *pointed topological space* we mean a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. In the above construction, given a pointed topological space (X, x_0) we attached a group $\pi_1(X, x_0)$, known as the *fundamental group of X with the base point at x_0*. Next we see how fundamental group of a pointed space behaves under continuous maps and their compositions.

Let $f : (X, x_0) \to (Y, y_0)$ be a *continuous map of pointed spaces* (this means, $f : X \to Y$ is a continuous map with $f(x_0) = y_0$). Let $\gamma : I \to X$ be a loop in X based at x_0. Then the composition $f \circ \gamma$,

$$I \xrightarrow{\gamma} X \xrightarrow{f} Y,$$

is a loop in Y based at $f(x_0) = y_0$. Let $\gamma, \delta : I \to X$ be loops in X based at x_0. If $F : I \times I \to X$ is a homotopy from γ to δ, then $f \circ F : I \times I \to Y$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$ (see Lemma 2.1.6). Thus we have a well-defined map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \ \ [\gamma] \mapsto [f \circ \gamma]. \tag{2.2.11}$$

Proposition 2.2.12. *The map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is a group homomorphism.*

Proof. Note that for any two loops $\gamma, \delta : I \to X$ based at x_0, we have

$$\begin{aligned} f_*([\gamma \star \delta]) &= [f \circ (\gamma \star \delta)] \\ &= [(f \circ \gamma) \star (f \circ \delta)] \\ &= [f \circ \gamma] \cdot [f \circ \delta] \\ &= f_*([\gamma]) \cdot f_*([\delta]). \end{aligned}$$

□

Remark 2.2.13. If $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the homomorphism of fundamental group of a pointed topological space (X, x_0) induced by the identity map of (X, x_0) onto itself, then it follows from the construction of the map f_* given in (2.2.11) that $f_* = \mathrm{Id}_{\pi_1(X, x_0)}$, the identity map of $\pi_1(X, x_0)$ onto itself.

Proposition 2.2.14. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be continuous maps of pointed spaces. Then $g_* \circ f_* = (g \circ f)_*$. In other words, the following diagram commutes.*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \pi_1(Y, y_0) \arrow[d, "g_*"] \\ & \pi_1(Z, z_0) \end{tikzcd}$$

Proof. Left as an exercise. □

Corollary 2.2.15. *If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism of pointed spaces with its inverse $g : (Y, y_0) \to (X, x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of groups with its inverse $g_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.*

Proof. Since $g \circ f = \mathrm{Id}_{(X, x_0)}$ and $f \circ g = \mathrm{Id}_{(Y, y_0)}$, applying Proposition 2.2.14 we have $g_* \circ f_* = \mathrm{Id}_{\pi_1(X, x_0)}$ and $f_* \circ g_* = \mathrm{Id}_{\pi_1(Y, y_0)}$. □

Lemma 2.2.16. *Homotopic continuous maps of pointed topological spaces induces the same homomorphism of fundamental groups.*

Proof. Let $f, g : (X, x_0) \to (Y, y_0)$ be two continuous maps of pointed space. If f is homotopic to g in the sense of Definition 2.1.7, then for any loop $\gamma : I \to X$ based at x_0, using Exercise 2.1.8 (c.f. Lemma 2.1.6) we have $f \circ \gamma$ is homotopic to $g \circ \gamma$ in the sense of Definition 2.1.7, and hence $f_*([\gamma]) = [f \circ \gamma] = [g \circ \gamma] = g_*([\gamma])$. Hence the result follows. □

Definition 2.2.17. A category $\mathscr{C}$ consists of the following data:

(i) a collection of objects $\mathrm{ob}(\mathscr{C})$,

(ii) for each ordered pair of objects (X, Y) of $\mathrm{ob}(\mathscr{C})$, there is a collection $\mathrm{Mor}_{\mathscr{C}}(X, Y)$, whose members are called *arrows* or *morphisms from* X *to* Y *in* $\mathscr{C}$; an object $\varphi \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ is usually denoted by an arrow $\varphi : X \to Y$.

(iii) for each ordered triple (X, Y, Z) of objects of $\mathscr{C}$, there is a map (called *composition map*)

$$\circ : \mathrm{Mor}_{\mathscr{C}}(X, Y) \times \mathrm{Mor}_{\mathscr{C}}(Y, Z) \to \mathrm{Mor}_{\mathscr{C}}(X, Z), \ \ (f, g) \mapsto g \circ f,$$

such that the following conditions hold.

(a) *Associativity:* Given $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, and $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \mathrm{Mor}_{\mathscr{C}}(Z, W)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(b) *Existence of identity:* For each $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathscr{C}}(X, X)$ such that given any objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ and morphism $f : Y \to Z$ we have $f \circ \mathrm{Id}_Y = f$ and $\mathrm{Id}_Z \circ f = f$.

Example 2.2.18. (i) Let (Set) be the category of sets; its objects are sets and arrows are map of sets.

(ii) Let $(\mathcal{G}rp)$ be the category of groups; its objects are groups and arrows are group homomorphisms.

(iii) Let $(\mathcal{T}op)$ be the category of topological spaces; its objects are topological spaces and arrows are continuous maps.

(iv) Let $(\mathcal{R}ing)$ be the category of rings; its objects are rings and morphisms are ring homomorphisms.

Definition 2.2.19. Let $\mathscr{C}$ be a category. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be an isomorphism if there is a morphism $g : Y \to X$ in $\mathscr{C}$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

Definition 2.2.20. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *covariant functor* (resp., a *contravariant functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a rule

$$\mathcal{F} : \mathscr{C} \to \mathscr{D}$$

which associate to each object $X \in \mathscr{C}$ an object $\mathcal{F}(X) \in \mathscr{D}$, and to each morphism $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ (resp., a morphism $\mathcal{F}(f) \in \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(Y), \mathcal{F}(X))$) such that

(i) $\mathcal{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{F}(X)}$, for all $X \in \mathscr{C}$, and

(ii) given any objects $X, Y, Z \in \mathscr{C}$ and morphisms $f \in \mathrm{Mor}_{\mathscr{C}}(X,Y)$ and $g \in \mathrm{Mor}_{\mathscr{C}}(Y,Z)$, we have $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (resp., $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$).

If $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is a functor, for each ordered pair of objects $X, Y \in \mathscr{C}$ we denote by $\mathcal{F}_{X,Y}$ the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Mor}_{\mathscr{C}}(X,Y) \to \mathrm{Mor}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y)),$$

defined by $\mathcal{F}_{X,Y}(f) := \mathcal{F}(f)$. The same notation is used for contravariant functor.

Definition 2.2.21. A functor $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is said to be

(i) *faithful* if $\mathcal{F}_{X,Y}$ is injective, $\forall\, X, Y \in \mathscr{C}$.

(ii) *full* if $\mathcal{F}_{X,Y}$ is surjective, $\forall\, X, Y \in \mathscr{C}$.

(iii) *fully faithful* if $\mathcal{F}_{X,Y}$ is bijective, $\forall\, X, Y \in \mathscr{C}$.

(iv) *essentially surjective* if given any object $Y \in \mathscr{D}$, there is an object $X \in \mathscr{C}$ and an isomorphism $\varphi : \mathcal{F}(X) \xrightarrow{\simeq} Y$ in $\mathscr{D}$.

(v) *equivalence of categories* if there is a functor $\mathcal{G} : \mathscr{D} \to \mathscr{C}$ such that $\mathcal{G} \circ \mathcal{F} \cong \mathrm{Id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathrm{Id}_{\mathscr{D}}$. This is equivalent to say that $\mathcal{F}$ is fully faithful and essentially surjective.

Remark 2.2.22. Let $\mathcal{T}op_0$ be the category of pointed topological spaces; its objects are pointed topological space, and given any two pointed topological spaces (X, x_0) and (Y, y_0), a morphism $f : (X, x_0) \to (Y, y_0)$ in $\mathcal{T}op_0$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. Then it follows from Propositions 2.2.12 and 2.2.14 and the Remark 2.2.13 that

$$\begin{aligned} \pi_1 \ : \mathcal{T}op_0 &\longrightarrow (\mathcal{G}rp) \\ (X, x_0) &\mapsto \pi_1(X, x_0) \\ f &\mapsto f_* \end{aligned}$$

is a *covariant functor* from the category of pointed topological spaces to the category of groups. It follows from Lemma 2.2.16 that the functor π_1 is not faithful. It is a non-trivial fact that π_1 is not full. (i.e., there exist pointed topological spaces (X, x_0) and (Y, y_0), and a group homomorphism $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ such that $\varphi \neq f_*$, for all continuous map $f : (X, x_0) \to (Y, y_0)$.) However, π_1 is *essentially surjective* (i.e., given any group G there is a pointed topological space (X, x_0) such that $\pi_1(X, x_0) \cong G$).

2.2.3 Dependency on base point

Now we investigate relation between fundamental groups of X for different choices of base point. Let $x_0, x_1 \in X$. Let $f : I \to X$ be a path in X joining x_0 to x_1, i.e., f is a continuous map satisfying $f(0) = x_0$ and $f(1) = x_1$. We define the *opposite path* of f to be the map

$$\overline{f} : I \to X, \;\; t \mapsto f(1-t); \tag{2.2.23}$$

note that $\overline{f}(0) = x_1$ and $\overline{f}(1) = x_0$, hence $\overline{f}$ is a path from x_1 to x_0.

Exercise 2.2.24. Show that $f \star \overline{f} \simeq c_{x_0}$ and $\overline{f} \star f \simeq c_{x_1}$, where $\simeq$ stands for path-homotopy relation (see Definition 2.2.1).

Given a loop γ in X based at x_1, we can define $\widetilde{\gamma} := f \star \gamma \star \overline{f}$. Note that $\widetilde{\gamma} : I \to X$ is a continuous map satisfying $\widetilde{\gamma}(0) = f(0) = x_0 = \overline{f}(1) = \widetilde{\gamma}(1)$, and hence is a loop in X based at x_1. Strictly speaking, we have two choices to define this product $\widetilde{\gamma}$, namely $(f \star \gamma) \star \overline{f}$ or $f \star (\gamma \star \overline{f})$, but we are interested in only homotopy classes of paths, and following the proof of associativity as in Theorem 2.2.8 one can easily verify that $(f \star \gamma) \star \overline{f} \simeq f \star (\gamma \star \overline{f})$, therefore, we just fix one ordering of taking products to define $\widetilde{\gamma}$.

If γ and γ' are two loops in X based at x_1 with $\gamma \simeq \gamma'$ via a homotopy $\{h_t\}_{t \in I}$, then $\{f \star h_t \star \overline{f}\}_{t \in I}$ is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma'}$ (*Exercise: Write down the homotopy explicitly and check details*). Thus, we have a well-defined map

$$\beta_f : \pi_1(X, x_1) \to \pi_1(X, x_0), \ \ [\gamma] \mapsto [(f \star \gamma) \star \overline{f}]. \tag{2.2.25}$$

Proposition 2.2.26. *The map β_f defined in (2.2.25) is a group isomorphism.*

Proof. Since $\overline{f} \star f \simeq c_{x_0}$ for any two loops γ and δ in X based at x_1, using Exercise 2.2.24, we have

$$\begin{aligned} f \star (\gamma \star \delta) \star \overline{f} &\simeq f \star \gamma \star c_{x_0} \star \delta \star \overline{f} \\ &\simeq (f \star \gamma \star \overline{f}) \star (f \star \delta \star \overline{f}). \end{aligned}$$

Therefore, $\beta_f([\gamma \star \delta]) = [f \star (\gamma \star \delta) \star \overline{f}] = [f \star \gamma \star \overline{f}][f \star \delta \star \overline{f}] = \beta_f([\gamma])\beta_f([\delta])$, and hence β_f is a group homomorphism. To show β_f an isomorphism of groups, it is enough to show that the group homomorphism

$$\beta_{\overline{f}} : \pi_1(X, x_0) \to \pi_1(X, x_1), \ \ [\gamma] \mapsto [\overline{f} \star \gamma \star f]$$

is the inverse of β_f. Indeed, for any $\gamma \in \pi_1(X, x_0)$ we have

$$\begin{aligned} \beta_f(\beta_{\overline{f}}([\gamma])) &= \beta_f([\overline{f} \star \gamma \star f]) \\ &= [f \star \overline{f} \star \gamma \star f \star \overline{f}] \\ &= [c_{x_0} \star \gamma \star c_{x_0}] = [\gamma], \end{aligned}$$

and similarly, for any $\delta \in \pi_1(X, x_1)$ we have

$$\begin{aligned} \beta_{\overline{f}}(\beta_f([\delta])) &= \beta_{\overline{f}}([f \star \delta \star \overline{f}]) \\ &= [\overline{f} \star f \star \delta \star \overline{f} \star f] \\ &= [c_{x_1} \star \delta \star c_{x_1}] = [\delta]. \end{aligned}$$

Therefore, $\beta_{\overline{f}}$ is the inverse homomorphism of β_f, and hence both of them are isomorphisms. $\square$

Remark 2.2.27. Thus if X is a path connected space, up to isomorphism its fundamental group is independent of choice of base point, and so we may denote it by $\pi_1(X)$ without specifying its base point.

Proposition 2.2.28. *Let $f, g : X \to Y$ be two continuous maps of topological spaces. Fix a point $x_0 \in X$, and let $y_0 = f(x_0)$ and $y_1 = g(x_0)$. Let $F : X \times I \to Y$ be a continuous map such that $F(-, 0) = f$ and $F(-, 1) = g$. Then for any loop γ in X based at x_0, the loop $f \circ \gamma$ is path-homotopic to the loop $F_0 \star (g \circ \gamma) \star \overline{F_0}$ in Y, where $F_0 : I \to Y$ is the path in Y defined by $F_0(t) = F(x_0, t)$, $\forall\, t \in I$.*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "g_*"] \arrow[dr, "f_*"'] & & \pi_1(Y, y_1) \arrow[dl, "\beta_{F_0}"] \\
& \pi_1(Y, y_0) &
\end{tikzcd}$$

Proof. Left as an exercise. □

Corollary 2.2.29. *Let $f, g : X \to Y$ be two homotopic continuous maps of topological spaces. Let $x_0 \in X$ be such that $f(x_0) = g(x_0) = y_0 \in Y$. Then the homomorphisms of fundamental groups f_* and g_*, induced by f and g, respectively, are conjugate by an element of $\pi_1(Y, y_0)$. In other words, there exists an element $[\eta] \in \pi_1(Y, y_0)$ such that $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$.*

Proof. Let $F : X \times I \to X$ be a continuous map such that

$$F(x, t) = \begin{cases} f(x), & \text{if} \quad t = 0, \\ g(x), & \text{if} \quad t = 1. \end{cases}$$

Then by Proposition 2.2.28 we have $g_*([\gamma]) = [\eta] f_*([\gamma])[\eta]^{-1}$, for all $[\gamma] \in \pi_1(X, x_0)$, where $\eta : I \to Y$ is the loop defined by $\eta(t) := F(x_0, t)$, $\forall\, t \in I$. □

Corollary 2.2.30. *If $f, g : (X, x_0) \to (Y, y_0)$ are two homotopic continuous maps of pointed topological spaces (see Definition 2.1.7), then $f_* = g_*$.*

Proof. Follows from Corollary 2.2.29. □

Definition 2.2.31. A space X is said to be *simply connected* if X is path connected and $\pi_1(X)$ is trivial.

Corollary 2.2.32. *A contractible space (see Definition 2.1.11) is simply connected.*

Proof. Let X be a contractible space. Then X is path-connected by Exercise 2.1.12. Fix a point $x_0 \in X$, and let $c_{x_0} : X \to X$ be the constant map sending all points to x_0. Since X is contractible, the identity map $\mathrm{Id}_X : X \to X$ is homotopic to the constant map c_{x_0} in X. Then by Corollary 2.2.29 the identity homomorphism $\mathrm{Id}_{\pi_1(X, x_0)} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is conjugate to the trivial homomorphism $(c_{x_0})_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by an element of $\pi_1(X, x_0)$. Therefore, the image of the identity homomorphism $\mathrm{Id}_{\pi_1(X, x_0)}$ is trivial, and hence $\pi_1(X, x_0)$ is trivial. □

Corollary 2.2.33. *A space X is simply connected if and only if there is a unique path-homotopy class of paths connecting any two points of X.*

Proof. Suppose that X is simply connected. Fix $x_0, x_1 \in X$. Since X is path-connected, there is a path in X joining x_0 to x_1. Let $f, g : I \to X$ be any two paths in X from x_0 to x_1. Let $\overline{f}$ and $\overline{g}$ be the opposite paths of f and g, respectively. Since $f \star \overline{g}$ is a loop in X based at x_0 and $\pi_1(X, x_0)$ is trivial, we have $f \star \overline{g}$ is path-homotopic to the constant loop c_{x_0} in X based at x_0. Since $\overline{g} \star g$ is path-homotopic to the constant loop c_{x_1} by Exercise 2.2.24, we have

$$f \simeq f \star c_{x_1} \simeq f \star \overline{g} \star g \simeq c_{x_0} \star g \simeq g.$$

To see the converse part, note that path connectedness of X means any two points of X can be joined by a path in X. Since there is a unique homotopy class of paths connecting any two points of X, path connectedness of X is automatic, and any loop in X based at a given point $x_0 \in X$ is homotopically trivial. Thus, X is path connected with $\pi_1(X, x_0)$ trivial, and hence is simply connected. $\square$

Proposition 2.2.34. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Note that $X \times Y$ naturally acquires product topology induced from X and Y, and the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$, for all $(x, y) \in X \times Y$, are continuous. Moreover, given any space Z and a map $f : Z \to X \times Y$, we have $f = (p_1 \circ f, p_2 \circ f)$. From this, it follows that f is continuous if and only if its components $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to Y$ are continuous. Therefore, to give a loop $\gamma : I \to X \times Y$ based at $(x_0, y_0) \in X \times Y$ is equivalent to give a pair of loops $(p_1 \circ \gamma, p_2 \circ \gamma)$ in the pointed spaces (X, x_0) and (Y, y_0), respectively. Similarly, to give a homotopy $F : I \times I \to X \times Y$ of loops $\gamma, \delta : I \to X \times Y$ based at $(x_0, y_0))$ is equivalent to give a pair of homotopies $(p_1 \circ F, p_2 \circ F)$ of the corresponding loops $p_j \circ \gamma$ with $p_j \circ \delta$, where $j \in \{1, 2\}$. Thus we have a bijection

$$\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0), \quad [\gamma] \mapsto ([p_1 \circ \gamma], [p_2 \circ \gamma]).$$

To see ϕ is an isomorphism, note that for any two loops γ and δ in $X \times Y$ based at $(x_0, y_0) \in X \times Y$, we have

$$\begin{aligned}
\phi([\gamma] \cdot [\delta]) &= \phi([\gamma \star \delta]) \\
&= ([p_1 \circ (\gamma \star \delta)], [p_2 \circ (\gamma \star \delta)]) \\
&= ([(p_1 \circ \gamma) \star (p_1 \circ \delta)], [(p_2 \circ \gamma) \star (p_2 \circ \delta)]) \\
&= ([(p_1 \circ \gamma)] \cdot [(p_1 \circ \delta)], [(p_2 \circ \gamma)] \cdot [(p_2 \circ \delta)]) \\
&= ([p_1 \circ \gamma], [p_2 \circ \gamma]) \cdot ([p_1 \circ \delta], [p_2 \circ \delta]) \\
&= \phi([\gamma]) \cdot \phi([\delta]).
\end{aligned}$$

This completes the proof. $\square$

Example 2.2.35. As an immediate application of Proposition 2.2.34 we see that the fundamental group of the 1-torus $S^1 \times S^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

We end this subsection with the following useful remark.

Remark 2.2.36. A loop in X based at x_0 can equivalently be defined as a continuous map of pointed spaces $\gamma : (S^1, 1) \to (X, x_0)$. Indeed, since a loops in X based at $x_0 \in X$ is a continuous map $\gamma : I = [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, and since S^1 is homeomorphic to the quotient space $[0, 1]/\sim$, where only the end points 0 and 1 of the interval I are identified, $\gamma : I \to X$ uniquely factors as

$$\begin{tikzcd} I \arrow[r, "\gamma"] \arrow[d, "q"'] & X \\ S^1 \arrow[ur, dashed, "\gamma'"'] & \end{tikzcd}$$

where $q : I \to S^1$ is the quotient map given by $q(t) = e^{2\pi i t}$, for all $t \in I$. Therefore, $\pi_1(X, x_0)$ is the group of all homotopy classes of continuous maps $(S^1, 1) \to (X, x_0)$.

2.2.4 Fundamental group of some spaces

Proposition 2.2.37. $\pi_1(\mathbb{R}, 0) = \{1\}$.

Proof. Consider the continuous map $F : \mathbb{R} \times I \to \mathbb{R}$ defined by

$$F(x, t) = (1 - t)x, \ \forall\, (x, t) \in \mathbb{R} \times I.$$

Note that, for all $x \in \mathbb{R}$ and $t \in I$ we have

- $F(x, 0) = x$,
- $F(x, 1) = 0$, and
- $F(0, t) = 0$.

Therefore, F "contracts" whole $\mathbb{R}$ to the point 0 leaving the point 0 intact at all times. Let $\gamma : I \to \mathbb{R}$ be a loop based at 0. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \xrightarrow{\gamma \times \mathrm{Id}_I} \mathbb{R} \times I \xrightarrow{F} \mathbb{R}$$

is a homotopy from γ to the constant loop $0 : I \to \mathbb{R}$ which sends all points of S^1 to $0 \in \mathbb{R}$. This completes the proof. □

Proposition 2.2.38. *Let*

$$D^2 := \{z \in \mathbb{C} \,:\, |z| \le 1\} = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \le 1\}$$

be the closed unit disk in the plane. Then $\pi_1(D^2, 1) = \{1\}$.

Proof. Consider the map $F : D^2 \times I \to D^2$ defined by

$$F(z, t) = (1 - t)z + t, \ \forall\, (z, t) \in D^2 \times I.$$

Note that F is continuous and for all $z \in D^2$ and $t \in I$ we have

- $F(z, 0) = z$,
- $F(z, 1) = 1$, and
- $F(1, t) = 1$.

Therefore, F contracts D^2 to the point 1 leaving 1 intact at all times. Let $\gamma : I \to D^2$ be a loop based at 1. Then the composite map

$$F \circ (\gamma \times \mathrm{Id}_I) : I \times I \xrightarrow{\gamma \times \mathrm{Id}_I} D^2 \times I \xrightarrow{F} D$$

is a homotopy from γ to the constant loop $1 : I \to D^2$ which sends all points of I to $1 \in D^2$. This completes the proof. □

2.3 Covering Space

2.3.1 Covering map

We begin this section with an aim to compute fundamental group of the unit circle in plane

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and we'll see how the idea of a 'covering map' could help us.

Let $\omega : I \to S^1$ be the map defined by $\omega(t) = e^{2\pi i t}$, $\forall\, t \in I$, where $i = \sqrt{-1}$. Then ω is a loop in S^1 based at $x_0 := 1 \in S^1$. For each integer n, let $\omega_n : I \to S^1$ be the loop based at x_0 defined by $\omega_n(t) = e^{2\pi i n t}$, $\forall\, t \in I$. So ω_n winds around the circle $|n|$-times in the anti-clockwise direction if $n > 0$, and in the clockwise direction if $n < 0$. We shall see later that $[\omega]^n = [\omega_n]$ in $\pi_1(S^1, 1)$, for all $n \in \mathbb{Z}$. The following is the main theorem of this section.

Theorem 2.3.1. *$\pi_1(S^1, x_0)$ is the infinite cyclic group $\mathbb{Z}$ generated by the loop ω.*

To prove this theorem, we compare paths in S^1 with paths in $\mathbb{R}$ via the map

$p : \mathbb{R} \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$, $\forall\, s \in \mathbb{R}$.

We can visualize this map geometrically by embedding $\mathbb{R}$ inside $\mathbb{R}^3$ as the helix parametrized as

$$s \mapsto (\cos 2\pi s, \sin 2\pi s, s),$$

and then p is the restriction of the projection map

$$\mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y)$$

from this helix onto $S^1 \subset \mathbb{R}^2$, as shown in the Figure 2.7.

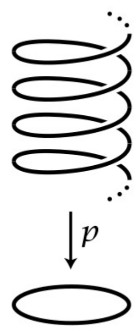


FIGURE 2.7

In this setup, the loop

$$\omega_n : I \to S^1, \ \ s \mapsto (\cos 2n\pi s, \sin 2n\pi s)$$

is the composition $p \circ \widetilde{\omega}_n$, where

$$\widetilde{\omega}_n : I \to \mathbb{R}, \ \ s \mapsto ns$$

is the path in $\mathbb{R}$ starting at 0 and ending at n, winding around the helix $|n|$-times, upward direction if $n > 0$, and downward direction if $n < 0$.

Before proceeding further, we introduce notion of a *covering map*, and discuss some of its useful properties.

Definition 2.3.2. Let $f : X \to Y$ be a continuous map. An open subset $V \subseteq Y$ is said to be *evenly covered by* f if $f^{-1}(V)$ is a union of pairwise disjoint open subsets of X each of which are homeomorphic to V by f (meaning that, $f^{-1}(V) = \bigcup\limits_{i\in I} U_i$, where $U_i \subseteq X$ is an open subset of X with $U_i \cap U_j = \emptyset$, for all $i \neq j$ in I, and $f\big|_{U_i} : U_i \to V$ is a homeomorphism, for all $i \in I$).

Example 2.3.3. (i) Let $f : \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ be the map defined by $f(t) = e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$, for all $t \in \mathbb{R}$. For $a, b \in \mathbb{R}$ with $a < b$, we define an open subset

$$V_{a,b} := \{f(t) : a \le t \le b\} \subseteq S^1.$$

If $b - a < 1$, then $V_{a,b}$ is evenly covered by f. In fact, in this case, we have $f^{-1}(V_{a,b}) =$

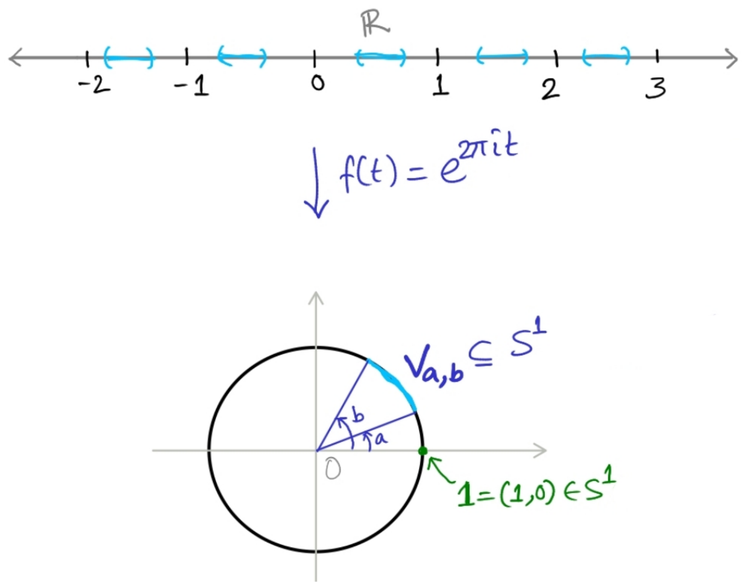


FIGURE 2.8

$\bigsqcup\limits_{n\in\mathbb{Z}} (a+n, b+n)$, and $f : (a+n, b+n) \xrightarrow{\simeq} V_{a,b}$ is a homeomorphism, $\forall\, n \in \mathbb{Z}$. See Figure 2.8.

If $b - a \geq 1$, then $V_{a,b} = S^1$, and hence $f^{-1}(V_{a,b}) = \mathbb{R}$. In this case, $V_{a,b}$ is not evenly covered by f, for otherwise we would have $\mathbb{R} = \bigsqcup_{i \in I} U_i$ with each U_i open subset of $\mathbb{R}$ and $f|_{U_i} : U_i \to S^1$ is a homeomorphism, which is not possible because S^1 is compact, whereas an open subset of $\mathbb{R}$ cannot be compact.

(ii) Let $\mathbb{R}_{>0} := \{t \in \mathbb{R} : t > 0\}$ be the positive part of the real line. Let

$$f : \mathbb{R}_{>0} \to S^1, \quad t \mapsto e^{2\pi i t}. \tag{2.3.4}$$

For any point $x \in S^1$ with $x \neq \mathbb{1} := (1, 0) \in S^1$, we can choose a small enough open neigbourhood V of x in S^1 with $\mathbb{1} \notin V$. Then it is easy to see that V is evenly covered by f. However, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$. To see this, note that

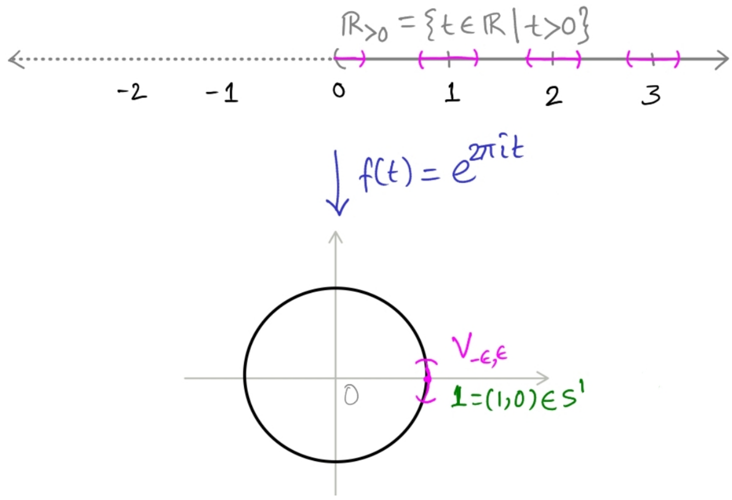


FIGURE 2.9

if $U \subseteq V$ is an open subset of an evenly covered neighbourhood V, then U is also evenly covered. Thus, if there is a neighbourhood V of $\mathbb{1}$ which is evenly covered, then we may find $\epsilon \in (0, 1/2)$ small enough such that $V_{-\epsilon,\epsilon} \subseteq V$, and hence $V_{-\epsilon,\epsilon}$ is evenly covered. Then we must have $f^{-1}(V_{-\epsilon,\epsilon}) = \bigsqcup_{i \in I} U_i$, with $f|_{U_i} : U_i \to V_{-\epsilon,\epsilon}$ homeomorphism, for all $i \in I$. In particular, each U_i is connected and are path components of $f^{-1}(V_{-\epsilon,\epsilon})$. Let U_0 be the path component of $\epsilon/2 \in \mathbb{R}_{>0}$. Since

$$f^{-1}(V_{-\epsilon,\epsilon}) = (0, \epsilon) \bigcup \left(\bigcup_{n \geq 1} (n - \epsilon, n + \epsilon) \right),$$

we must have $U_0 = (0, \epsilon)$. But $f|_{(0,\epsilon)} : (0, \epsilon) \to V_{-\epsilon,\epsilon}$ cannot be surjective because only possible preimage of $\mathbb{1} \in V_{-\epsilon,\epsilon}$ in $\mathbb{R}^+$ could be positive integers, and none of which are in the domain of $f|_{(0,\epsilon)}$. Thus we get a contradiction. See Figure 2.9. Therefore, there is no evenly covered neighbourhood of $\mathbb{1} \in S^1$ for the map f in (2.3.4).

Definition 2.3.5. A continuous map $f : X \to Y$ is called a *covering map* if each point $y \in Y$ has an open neighbourhood $V_y \subseteq Y$ that is evenly covered by f.

Note that, a covering map is always surjective. This follows immediately from the Definition 2.3.5.

Example 2.3.6. (i) Let F be a non-empty discrete topological space, and let X be any topological space. Give $X \times F$ the product topology. Then the projection map $pr_1 : X \times F \to X$ defined by $pr_1(x, v) = x$, $\forall\ (x, v) \in X \times F$, is a covering map. Such a covering map is called a *trivial cover* of X.

(ii) The continuous map
$$f : \mathbb{R} \to S^1,\ \ t \mapsto e^{2\pi i t},$$
as discussed in Example 2.3.3 (i), is a covering map, while its restriction $f\big|_{\mathbb{R}^+} : \mathbb{R}^+ \to S^1$, in Example 2.3.3 (ii), is not a covering map.

(iii) The map $f : \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by $f(z) = e^z$, for all $z \in \mathbb{C}$, is a covering map.

(iv) Fix an integer $n \geq 1$. Then the map $f : \mathbb{C}^* \to \mathbb{C}^*$ defined by $f(z) = z^n$, for all $z \in \mathbb{C}$, is a covering map, known as the *n-sheeted covering map* of $\mathbb{C}^*$.

Exercise 2.3.7. If $f_i : X_i \to Y_i$ is a covering map, for $i = 1, 2$, show that the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by sending $(x_1, x_2) \in X_1 \times X_2$ to $(f_1(x_1), f_2(x_2)) \in Y_1 \times Y_2$, is a covering map.

Exercise 2.3.8. If $f : X \to Y$ is a covering map, for any subspace $Z \subseteq Y$, the restriction of f on $f^{-1}(Z) \subseteq X$ is a covering map.

Definition 2.3.9. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covering maps. A *morphism of covering maps* from p_1 to p_2 is a continuous map $\phi : Y_1 \to Y_2$ such that $p_2 \circ \phi = p_1$. In other words, the following diagram commutes.

$$\begin{tikzcd} Y_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A morphism of covering maps $\phi : Y_1 \to Y_2$ is said to be an *isomorphism of covering maps* if there is a covering map $\psi : Y_2 \to Y_1$ such that $\phi \circ \psi = \mathrm{Id}_{Y_2}$ and $\psi \circ \phi = \mathrm{Id}_{Y_1}$. In other words, an isomorphism of covering spaces is a homeomorphism of the covers compatible with the base. An isomorphism of a covering map $p : Y \to X$ to itself is called a *Deck transformation* or a *covering transformation*.

Exercise 2.3.10. Show that any covering map $p : Y \to X$ is locally trivial (i.e., each point $x \in X$ has an open neighbourhood $U_x \subseteq X$ such that the restriction map $p : p^{-1}(U_x) \to U_x$ is isomorphic to a trivial covering map over U_x).

A continuous map $f : X \to Y$ is said to be an *open map* if for any open subset U of X, $f(U)$ is open in Y.

Proposition 2.3.11. *If $f : X \to Y$ is a covering map, then f is an open map.*

Proof. Let $U \subseteq X$ be an open subset of X, and let $y \in f(U)$. Then there is $x_0 \in U$ such that $f(x_0) = y$. Since f is a covering map, there is an open neighbourhood $V \subseteq Y$ of y such that $f^{-1}(V) = \bigcup\limits_{j\in J} W_i$ is a union of pairwise disjoint open subsets $W_j \subseteq X$, and that $f\big|_{W_j} : W_j \to V$ is a homeomorphism, for all $j \in J$. Then $x_0 \in U \cap W_{j_0}$, for some unique $i_0 \in I$. Since $f\big|_{W_i}$ is a homeomorphism, $f(U \cap W_{j_0}) \subseteq V$ is an open neighbourhood of $f(x_0) = y$. Since V is open in Y, $f(U \cap W_{j_0})$ is open in Y. Thus $f(U)$ is open in Y, and hence f is an open map. $\square$

Theorem 2.3.12 (Lifting path to a cover)**.** *Let $f : X \to Y$ be a covering map. Let $\gamma : [0,1] \to Y$ be a path in Y. Fix a point $x_0 \in X$ such that $f(x_0) = y_0 := \gamma(0)$. Then there is a unique path $\widetilde{\gamma} : [0,1] \to X$ with $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$.*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ {[0,1]} \arrow[ur, dashed, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & Y \end{tikzcd}$$

The path $\widetilde{\gamma}$ is called a lift of γ in X starting at x_0.

Proof. We first prove uniqueness of lift of γ, if it exists. Let $\eta_1, \eta_2 : [0,1] \to X$ be any two continuous maps such that $\eta_1(0) = x_0 = \eta_2(0)$ and $f \circ \eta_1 = \gamma = f \circ \eta_2$. We need to show that $\eta_1 = \eta_2$ on $[0,1]$. Let

$$S = \{t \in [0,1] \, : \, \eta_1(t) = \eta_2(t)\}.$$

Since both η_1 and η_2 are continuous, S is a closed subset of $[0, 1]$. Note that $S \neq \varnothing$ since $0 \in S$. Since $[0, 1]$ is connected, it is enough to show that S is both open and closed in $[0,1]$, so that S is a connected component of $[0, 1]$, and hence $S = [0, 1]$.

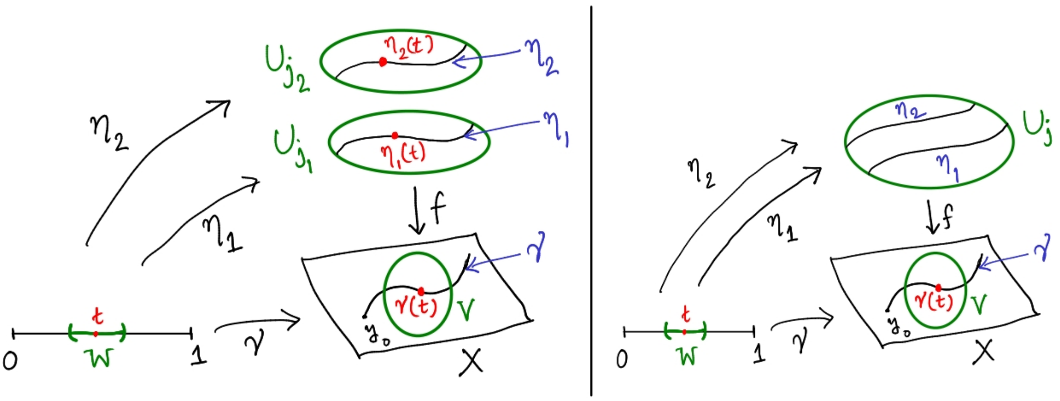


FIGURE 2.10

Fix a $t \in S$, and let $V \subseteq Y$ be an open neighbourhood of $y := \gamma(t)$ that is evenly covered by f. So $f^{-1}(V) = \bigcup\limits_{j\in J} U_j$, where $\{U_j\}_{j\in J}$ is a collection of pairwise disjoint open subsets of X each of which gets mapped homeomorphically onto V by f. Then there are $j_1, j_2 \in J$ such that $\eta_1(t) \in U_{j_1}$ and $\eta_2(t) \in U_{j_2}$. Since η_1 and η_2 are continuous at $t \in [0, 1]$, there is an open neighbourhood $W \subseteq [0,1]$ of t such that $\eta_1(W) \subseteq U_{j_1}$ and $\eta_2(W) \subseteq U_{j_2}$. Since $U_{j_1} \cap U_{j_2} = \varnothing$

for $j_1 \neq j_2$, and since $\eta_1(t) = \eta_2(t)$ by assumption, we must have $j_1 = j_2$ and $U_{j_1} = U_{j_2}$. Since $f|_{U_j} : U_j \to V$ is injective (in fact, homeomorphism), for all $j \in J$, and $f \circ \eta_1 = f \circ \eta_2$, we must have $\eta_1|_W = \eta_2|_W$. Therefore, $W \subseteq S$. Thus S is both open and closed in $[0, 1]$, and hence is the connected component of $[0, 1]$. Therefore, $S = [0, 1]$, and hence $\eta_1 = \eta_2$ on $[0, 1]$.

Remark 2.3.13. Note that, by replacing $[0, 1]$ with any connected topological space T in the above proof of uniqueness of lift of γ, we get the following result:

Lemma 2.3.13. *Let $f : X \to Y$ be a covering map. Let $\eta_1, \eta_2 : T \to X$ be any continuous maps such that $f \circ \eta_1 = f \circ \eta_2$. If T is connected and $\eta_1(t) = \eta_2(t)$, for some $t \in T$, then $\eta_1 = \eta_2$ on whole T.*

To complete the proof of Theorem 2.3.12, it remains to construct an explicit lift of γ to the cover $f : X \to Y$ starting at x_0. For this we use a result from basic topology course, called *Lebesgue number lemma* (to keep the note self-contained, we include its proof later).

Lemma 2.3.14 (Lebesgue number lemma)**.** *Let $\{U_j\}_{j \in J}$ be an open cover of a compact metric space (X, d). Then there is a $\delta > 0$ such that for each $x_0 \in X$, the open ball $B_\delta(x_0)$ is contained in U_{j_0}, for some $j_0 \in J$.*

Since $f : X \to Y$ is a covering map, we can write $Y = \bigcup\limits_{y \in Y} V_y$, where $V_y \subseteq Y$ is an open neighbourhood of y that is evenly covered by f, for all $y \in Y$. Since $[0, 1] = \bigcup\limits_{y \in Y} \gamma^{-1}(V_y)$, by Lebesgue covering lemma (c.f. Lemma 2.3.14) we can find a $\delta > 0$ such that for each $t \in (0, 1)$ there is a $y_t \in Y$ such that $\gamma\big([t - \frac{\delta}{2}, t + \frac{\delta}{2}] \cap [0, 1]\big) \subseteq V_{y_t}$. Choose $n \gg 0$ such that $\frac{1}{n} < \delta$, and write

$$[0, 1] = \bigcup_{k=0}^{n-1} \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Now $\gamma([0, 1/n]) \subseteq V_0$, for some open subset $V_0 \subset Y$ evenly covered by f, and $y_0 = \gamma(0) \in V_0$. Write

$$f^{-1}(V_0) = \bigsqcup_{j \in J} U_{0,j}\,,$$

where $\{U_{0,j}\}_{j \in J}$ is a collection of pair-wise disjoint open subsets of X each of which are homeomorphic to V_0 via the restriction of f onto them. Since $x_0 \in f^{-1}(V_0)$, there is a unique $j_0 \in J$ such that $x_0 \in U_{0,j_0}$. Let $s_0 : V_0 \to U_{0,j_0}$ be the inverse of the homeomorphism $f|_{U_{0,j_0}}$. Clearly $s_0(y_0) = x_0$. Consider the map $\widetilde{\gamma}_0 : [0, \frac{1}{n}] \to U_{0,j_0}$ defined by

$$\widetilde{\gamma}_0(t) := s_0(\gamma(t)), \;\; \forall \;\; t \in [0, 1/n]\,.$$

Then $\widetilde{\gamma}_0$ satisfies $\widetilde{\gamma}_0(0) = x_0$ and $f \circ \widetilde{\gamma}_0 = \gamma$ on $[0, \frac{1}{n}]$.

Let $x_1 = \widetilde{\gamma}_0(\frac{1}{n})$ and $y_1 = \gamma(\frac{1}{n}) = (f \circ \widetilde{\gamma}_0)(\frac{1}{n})$. Then there is an open subset $V_1 \subseteq Y$ which is evenly covered by f and $\gamma([\frac{1}{n}, \frac{2}{n}]) \subseteq V_1$. Proceeding in the same way as above, we can write

$$f^{-1}(V_1) = \bigsqcup_{j \in J} U_{1,j},$$

where $U_{1,j}$ are pairwise disjoint open subsets of X each of which are homeomorphic to V_1 by the restriction of f onto them. Since $x_1 = \widetilde{\gamma}_0(\frac{1}{n}) \in f^{-1}(V_1)$, there is a $j_1 \in J$ such that $x_1 \in U_{1,j_1}$. Let $s_1 : V_1 \to U_{1,j_1}$ be the inverse of the homeomorphism $f : U_{1,j_1} \to V_1$. Clearly $s_1(y_1) = x_1$. Then the continuous map $\widetilde{\gamma}_1 : [\frac{1}{n}, \frac{2}{n}] \to U_{1,j_1}$ defined by

$$\widetilde{\gamma}_1(t) = s_1(\gamma(t)), \ \forall\, t \in [1/n, 2/n]$$

satisfies $\widetilde{\gamma}_1(\frac{1}{n}) = x_1$ and $f \circ \widetilde{\gamma}_1 = \gamma$ on $[\frac{1}{n}, \frac{2}{n}]$. Since the maps $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ agrees on $[0, \frac{1}{n}] \cap [\frac{1}{n}, \frac{2}{n}] = \{\frac{1}{n}\}$, by Lemma 2.1.3 we can join them to get a continuous map $\widetilde{\gamma} : [0, \frac{2}{n}] \to X$ such that $\widetilde{\gamma}(0) = x_0$ and $f \circ \widetilde{\gamma} = \gamma$ on $[0, \frac{2}{n}]$. Proceeding in this way we can construct a lift $\widetilde{\gamma}$ of γ to the whole $[0, 1]$ as required. □

We now give a proof of Lemma 2.3.14. If you have already learned it from your basic topology course, you may skip it. Let's begin with the following observation from metric space.

Proposition 2.3.15. *Let X be a metric space with the metric d, and let Y be a subspace of X. Then the function $\phi_Y : X \to \mathbb{R}$ defined by*

$$\phi_Y(x) = \inf_{y \in Y} d(x, y), \ \forall\, x \in X,$$

is continuous.

Proof. Let $x_0 \in X$. Fix a $\delta > 0$, and let $z \in B_\delta(x_0) := \{x \in X : d(x_0, x) < \delta\}$. Then for any $y \in X$, using the triangle inequality we have

$$\phi_Y(z) = \inf_{y' \in Y} d(z, y') \leq d(z, y) \leq d(z, x_0) + d(x_0, y).$$

Now taking infimum over $y \in Y$ and subtracting $\phi_Y(x_0)$, we have

$$\phi_Y(z) - \phi_Y(x_0) \leq d(z, x_0).$$

Since $z \in B_\delta(x_0)$ implies $x_0 \in B_\delta(z)$, repeating the above construction we have $\phi_Y(x_0) - \phi_Y(z) \leq d(z, x_0)$, and hence

$$|\phi_Y(x_0) - \phi_Y(z)| \leq d(z, x_0).$$

Therefore, given any $\epsilon > 0$, taking $\delta = \epsilon$, we see that

$$|\phi_Y(x_0) - \phi_Y(z)| < \epsilon, \ \text{ whenever } \ d(x_0, z) < \delta.$$

Since $x_0 \in X$ is arbitrary, ϕ_Y is continuous on X. □

Now we give a proof of Lebesgue number lemma.

Proof of Lemma 2.3.14. Since X is compact, we can cover it by finitely many U_j's, say $X = \bigcup_{j=1}^{n} U_j$. Consider the function $\phi : X \to \mathbb{R}$ defined by

$$\phi(x) = \sum_{j=1}^{n} \phi_{Y_j}(x),$$

where $\phi_{Y_j} : X \to \mathbb{R}$ is the map as defined in Proposition 2.3.15 with $Y_j = X \setminus U_j$, for $j = 1, \ldots, n$. Since each ϕ_{Y_j} is continuous, ϕ is continuous. Since (X, d) is compact, its image $\phi(X)$ is a compact subset of $\mathbb{R}$. Since each ϕ_{Y_j} takes non-negative values, for $x \in X$ with $\phi(x) = 0$ we must have $\phi_{Y_j}(x) = 0$, for all $j = 1, \ldots, n$, and hence $x \in \bigcap_{j=1}^{n} Y_j$, since each Y_j is closed in X. But this is not possible because $\bigcap_{j=1}^{n} Y_j = X \setminus (\bigcup_{j=1}^{n} U_j) = \emptyset$. Therefore, $0 \notin \phi(X)$. Since ϕ is a non-negative valued function, there is a $\delta > 0$ such that $n\delta < \phi(x)$, for all $x \in X$. Then for each $x \in X$, there is at least one $j_x \in \{1, \ldots, n\}$ such that $\phi_{Y_{j_x}}(x) > \frac{n\delta}{n} = \delta$, and hence $B_\delta(x) \subseteq X \setminus Y_{j_x} = U_{j_x}$. □

Next we lift homotopy from a base to its cover.

Lemma 2.3.16 (Glueing continuous maps)**.** *Let X and Y be two topological spaces. Let $\{U_j\}_{j\in J}$ be an open covering of X. Then given a family of continuous maps $\{f_j : U_j \to Y\}_{j\in J}$ satisfying $f_j|_{U_j\cap U_k} = f_k|_{U_j\cap U_k}$, for all $j, k \in J$, there is a unique continuous map $f : X \to Y$ such that $f|_{U_j} = f_j$, for all $j \in J$.*

Proof. Left as an exercise. □

Theorem 2.3.17 (Lifting homotopy to covers)**.** *Let $I := [0, 1] \subset \mathbb{R}$. Let $f : X \to Y$ be a covering map. Let $F : I \times I \to Y$ be a continuous map. Let $y_0 := F(0, 0)$ and fix a point $x_0 \in f^{-1}(y_0)$. Then there is a unique continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F$.*

Proof. Since $I \times I$ is connected, uniqueness of $\widetilde{F}$, if it exists, follows from Remark 2.3.13. We only show a construction of such a lift $\widetilde{F}$.

It is enough to show that, for each $s \in I$ there is a connected open neighbourhood $U_s \subseteq I$ of $s \in I$ such that $\widetilde{F}$ can be constructed on $U_s \times I$. Indeed, since $\{U_s \times I : s \in I\}$ is a connected open covering of $I \times I$ and those $\widetilde{F}$'s agree on their intersections $(U_s \times I) \cap (U_{s'} \times I) = (U_s \cap U_{s'}) \times I$, which are connected (because U_s' are open intervals), uniqueness of liftings $\widetilde{F}$'s defined on connected domains ensures that they can be glued together to get a well-defined continuous map $\widetilde{F} : I \times I \to X$ such that $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F$ on $I \times I$.

Now we construct such a lift $\widetilde{F} : U \times I \to X$, for some open neighbourhood $U \subseteq I$ of a given point $s_0 \in I$. Since F is continuous, each point $(s_0, t) \in I \times I$ has an open neighbourhood $U_t \times (a_t, b_t) \subset I \times I$ such that $F(U_t \times (a_t, b_t))$ is contained in some open neighbourhood of $F((s_0, t)) \in Y$ that is evenly covered by f. Since $\{s_0\} \times I$ is compact, finitely many such open subsets $U_t \times (a_t, b_t)$ cover $\{s_0\} \times I$. Taking intersection of those finitely many open subsets $U_t \subseteq I$, we can find a single open neighbourhood $U \subset I$ of s_0 and a partition $0 = t_0 < t_1 <$

$\cdots < t_m = 1$ of $I = [0,1]$ such that for each $i \in \{0,1,\ldots,m\}$, $F(U \times [t_i, t_{i+1}]) \subseteq V_i$, for some open subset $V_i \subset Y$ that is evenly covered by f.

By Theorem 2.3.12 (Lifting paths to a cover), we can find a unique continuous function $\widetilde{F} : I \times \{0\} \to X$ with $\widetilde{F}(0, 0) = x_0$ and $f \circ \widetilde{F} = F\big|_{I\times\{0\}}$. Assume inductively that $\widetilde{F}$ has been constructed on $U \times [0, t_i]$, starting with the given $\widetilde{F}$ on $U \times \{0\} \subseteq I \times \{0\}$. Since $F(U \times [t_i, t_{i+1}]) \subseteq V_i$, and V_i is evenly covered by f, there is an open subset $W_i \subseteq X$ such that $\widetilde{F}(s_0, t_i) \in W_i$ and $f\big|_{W_i} : W_i \to V_i$ is a homeomorphism. Replacing U by a smaller open neighbourhood of $s_0 \in I$, if required, we may assume that $\widetilde{F}(U \times \{t_i\}) \subseteq W_i$; for instance, it is enough to replace $U \times \{t_i\}$ with $\big(U \times \{t_i\}\big) \bigcap \big(\widetilde{F}\big|_{U\times\{t_i\}}\big)^{-1}(W_i)$. Then we can define $\widetilde{F}$ on $U \times [t_i, t_{i+1}]$ to be the composition $\varphi \circ F$, where $\varphi : V_i \to W_i$ is the inverse of the homeomorphism $f\big|_{W_i} : W_i \to V_i$. Continuing in this way, after a finite number of steps, we get a continuous map $\widetilde{F} : U \times I \to X$ with $\widetilde{F}(0,0) = x_0$ and $f \circ \widetilde{F} = F\big|_{U\times I}$, as required. □

Lemma 2.3.18. *Let $f : X \to Y$ be a covering map, and let $\gamma : I \to X$ be a continuous map. If $f \circ \gamma$ is a constant map, so is γ.*

Proof. Suppose that $(f \circ \gamma)(t) = y_0$, for all $t \in I$. Let $V \subseteq Y$ be an open neighbourhood of y_0 that is evenly covered by f. Then $f^{-1}(V) = \bigsqcup\limits_{\alpha\in\Lambda} U_\alpha$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is a family of pairwise disjoint open subsets of X with $f\big|_{U_\alpha} : U_\alpha \to V$ a homeomorphism, for all $\alpha \in \Lambda$. Since $\gamma(t) \in f^{-1}(V)$, for all $t \in I$, and I is connected, there is a unique $\alpha_0 \in \Lambda$ such that $\gamma(t) \in U_{\alpha_0}$, for all $t \in I$. Since $f\big|_{U_{\alpha_0}}$ is a homeomorphism, its restriction on the image of γ must be a homeomorphism; this is not possible since $f \circ \gamma$ is a constant map. □

Corollary 2.3.19 (Lifting of path-homotopy)**.** *Let $f : X \to Y$ be a covering map. Let $\gamma_0, \gamma_1 : I \to Y$ be two paths in Y with $\gamma_0(0) = \gamma_1(0) = y_0$ and $\gamma_0(1) = \gamma_1(1) = y_1$. Let $F : I \times I \to Y$ be a path-homotopy from γ_0 to γ_1 in X. If $\widetilde{F} : I \times I \to X$ is a lifting of F on X, then $\widetilde{F}$ is a path-homotopy.*

Proof. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{F} : I \times I \to X$ be the lifting of F on X with $\widetilde{F}(0,0) = x_0$. Then by Theorem 2.3.17, $\widetilde{F}$ is a homotopy of maps from $\widetilde{\gamma}_0 := \widetilde{F}(-,0)$ to $\widetilde{\gamma}_1 := \widetilde{F}(-,1)$. Let $x_1 := \widetilde{\gamma}_0(1) = \widetilde{F}(1,0)$. To show $\widetilde{F}$ is a path-homotopy, we need to ensure that $\widetilde{F}(0,t) = x_0$ and $\widetilde{F}(1,t) = x_1$, for all $t \in I$. This follows from the Lemma 2.3.18 applied to the paths $t \mapsto \widetilde{F}(0,t)$ and $t \mapsto \widetilde{F}(1,t)$. □

Corollary 2.3.20. *Let $f : X \to Y$ be a covering map. Let $y_0 \in Y$ and fix a point $x_0 \in f^{-1}(y_0)$. Then the group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ induced by f is injective. The image subgroup $f_*\big(\pi_1(X, x_0)\big)$ in $\pi_1(Y, y_0)$ consists of the homotopy classes of loops in Y based at y_0 whose lifts to X starting at x_0 are loops.*

Proof. Let $[\gamma], [\delta] \in \pi_1(X, x_0)$ be such that $f_*([\gamma]) = f_*([\delta])$. Then $f \circ \gamma$ is homotopic to $f \circ \delta$. Let $F : I \times I \to Y$ be a path homotopy from $f \circ \gamma$ to $f \circ \delta$. Then by Theorem 2.3.17 and its Corollary 2.3.19, we can lift F to a path-homotopy $\widetilde{F} : I \times I \to X$ with $\widetilde{F}(0,0) = x_0$. By uniqueness of path-lifting (see Theorem 2.3.12), $\widetilde{F}$ must be a path-homotopy from γ to δ (verify!). Therefore, f_* is injective.

To see the second part, note that any element of $f_*\big(\pi_1(X, x_0)\big)$ is of the form $[f \circ \gamma]$, for some loop $\gamma : I \to X$ in X based at x_0. By Theorem 2.3.12 (Path-lifting) we can lift $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ starting at $\gamma(0)$. Then by uniqueness of path-lifting, we have $\widetilde{f \circ \gamma} = \gamma$. Conversely, if δ is a loop in Y based at x_0 such that its lift $\widetilde{\delta}$ in X is a loop in X based at x_0, then $f_*([\widetilde{\delta}]) = [f \circ \widetilde{\delta}] = [\delta]$. This completes the proof. $\square$

Exercise 2.3.21 (Lifting of opposite path)**.** Let $f : X \to Y$ be a covering map. Let $\gamma : I \to Y$ be a path in Y from y_0 to y_1. Fix a point $x_0 \in f^{-1}(y_0)$, and let $\widetilde{\gamma}$ be the lift of γ in X starting at x_0. Let $\overline{\gamma}$ be the opposite path of γ. If $\widetilde{\overline{\gamma}}$ is the lift of $\overline{\gamma}$ in X starting at $\widetilde{\gamma}(1)$, then show that $\overline{\widetilde{\gamma}} = \widetilde{\overline{\gamma}}$.

Exercise 2.3.22 (Lifting of product of paths)**.** Let $f : X \to Y$ be a covering map. Let $\gamma, \delta : I \to Y$ be two paths in Y such that $\gamma(1) = \delta(0)$. Fix a point $x_0 \in f^{-1}(\gamma(0))$, and let $\widetilde{\gamma}$ and $\widetilde{\gamma \star \delta}$ be the liftings of the paths γ and $\gamma \star \delta$, respectively, in X starting at x_0. If $\widetilde{\delta}$ is the lifting of δ in X starting at $x_1 := \widetilde{\gamma}(1)$, show that $\widetilde{\gamma} \star \widetilde{\delta} = \widetilde{\gamma \star \delta}$.

Lemma 2.3.23. *Let $f : X \to Y$ be a covering space. If both X and Y are path-connected, then the cardinality of the fiber $f^{-1}(y)$ is independent of $y \in Y$.*

Proof. Fix a point $y_0 \in Y$, and a point $x_0 \in f^{-1}(y_0) \subseteq X$. Let $G = \pi_1(Y, y_0)$ and $H = f_*\big(\pi_1(X, x_0)\big)$. Let $H\backslash G := \{Hg : g \in G\}$ be the set of all right cosets of H in G. Since both X and Y are path-connected, the cardinality of the set $H\backslash G$ is independent of choices of $y_0 \in Y$ and $x_0 \in f^{-1}(y_0)$. Therefore, to show the cardinality of the fibers $f^{-1}(y)$ is independent of $y \in Y$, it is enough to construct a bijective map

$$\Phi : H\backslash G \longrightarrow f^{-1}(y_0). \tag{2.3.24}$$

Given a loop γ in Y based at y_0, let $\widetilde{\gamma}$ be the lifting of γ in X starting at x_0. Note that, $x_1 := \widetilde{\gamma}(1) \in f^{-1}(y_0)$. Then we define

$$\Phi(H[\gamma]) := \widetilde{\gamma}(1). \tag{2.3.25}$$

We need to show that x_1 is independent of choice of γ. Let δ be a loop in Y based at y_0 with $H[\gamma] = H[\delta]$. Then $[\gamma \star \overline{\delta}] = [\gamma][\delta]^{-1} \in H = f_*\big(\pi_1(X, x_0)\big)$, where $\overline{\delta}$ is the opposite path of δ. Then by Corollary 2.3.20 the loop $\gamma \star \overline{\delta}$ lifts to a unique loop $\widetilde{(\gamma \star \overline{\delta})}$ in X based at x_0. Let $\widetilde{\overline{\delta}}$ be the lifting of $\overline{\delta}$ in X starting at $x_1 := \widetilde{\gamma}(1)$. Then by Exercises 2.3.22 we have $\widetilde{\gamma \star \overline{\delta}} = \widetilde{\gamma} \star \widetilde{\overline{\delta}}$. Since $\widetilde{\gamma \star \overline{\delta}}$ is a loop in X based at x_0, we have $\widetilde{\overline{\delta}}(1) = x_0$. Let η be the opposite path of $\widetilde{\overline{\delta}}$ in X. Since

$$\begin{aligned}(f \circ \eta)(t) = f(\eta(t)) &= f\big(\widetilde{\overline{\delta}}(1-t)\big) \\ &= \overline{\delta}(1-t) = \delta(t), \quad \forall\, t \in I,\end{aligned}$$

η is a lift of δ in X starting at $\eta(0) = \widetilde{\overline{\delta}}(1) = x_0$. Then by uniqueness of path-lifting (Theorem 2.3.12) we have $\eta = \widetilde{\delta}$. Then $\widetilde{\delta}(1) = \eta(1) = \widetilde{\overline{\delta}}(0) = x_1$. Therefore, the map Φ in (2.3.25) is well-defined. Since X is path connected, given any $x_1 \in f^{-1}(y_0)$, there is a path φ in X from x_0 to x_1. Then $f \circ \varphi$ is a loop in Y based at y_0 whose lift $\widetilde{f \circ \varphi}$ starting at x_0 is the unique path φ ending at $x_1 = \varphi(1)$. Therefore, Φ is surjective. Let $[\gamma], [\delta] \in \pi_1(Y, y_0) = G$ be such that $\Phi(H[\gamma]) = \Phi(H[\delta])$. Let $\widetilde{\gamma}$ and $\widetilde{\delta}$ be the lifts of γ and δ, respectively, in X starting at x_0. Let $\overline{\widetilde{\delta}}$ be the opposite path of $\widetilde{\delta}$ in X. Since $\Phi(H[\gamma]) = \Phi(H[\delta])$, we have $\widetilde{\gamma}(1) = \widetilde{\delta}(1)$, and hence $\widetilde{\gamma} \star \overline{\widetilde{\delta}}$ is a loop in X based at x_0. Since $f \circ (\widetilde{\gamma} \star \overline{\widetilde{\delta}}) = \gamma \circ \overline{\delta}$, by uniqueness of path-lifting and Corollary 2.3.20, we conclude that $[\gamma \star \overline{\delta}] \in f_*\big(\pi_1(X, x_0)\big) = H$. Therefore, $H[\gamma] = H[\delta]$, and hence Φ is injective. Therefore, $\Phi : H \backslash G \to f^{-1}(y_0)$ is a bijection. □

Exercise 2.3.26. Give an example to show that the Lemma 2.3.23 fails if X and Y are not path-connected.

Theorem 2.3.27 (General Lifting Criterion)**.** *Let $f : (X, x_0) \to (Y, y_0)$ be a covering map. Let T be a path-connected and locally path-connected space. A continuous map $g : (T, t_0) \to (Y, y_0)$ lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ if and only if $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Note that, such a lift $\widetilde{g}$ of g, if it exists, is unique by Lemma 2.3.13.*

Proof. If g lifts to a continuous map $\widetilde{g} : (T, t_0) \to (X, x_0)$ such that $f \circ \widetilde{g} = g$, then $g_*\big(\pi_1(T, t_0)\big) = f_*\big(\widetilde{g}_*(\pi_1(T, t_0))\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$.

To see the converse, suppose that $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$. Since T is path-connected, given a point $t_1 \in T$, there is a path $\gamma : I \to T$ with $\gamma(0) = t_0$ and $\gamma(1) = t_1$. Then $g \circ \gamma : I \to Y$ is a path in Y from $g(t_0) = y_0$ to $y_1 := g(t_1) = (g \circ \gamma)(1)$. Since $f : (X, x_0) \to (Y, y_0)$ is a covering map, by Theorem 2.3.12 (Path-lifting) the path $g \circ \gamma$ lifts to a unique path $\widetilde{g \circ \gamma}$ in X starting at x_0. Define a map

$$\widetilde{g} : T \to X \tag{2.3.28}$$

by sending t_1 to $x_1 := \widetilde{g \circ \gamma}(1) \in X$. To show the map $\widetilde{g}$ is independent of choice of a path γ in T from t_0 to t_1, note that given any path $\delta : I \to T$ from t_0 to t_1, the product path $\gamma \star \overline{\delta}$ is a loop in T based at t_0. Since $g_*\big(\pi_1(T, t_0)\big) \subseteq f_*\big(\pi_1(X, x_0)\big)$, by the second part of the Corollary 2.3.20 the loop $g \circ (\gamma \star \overline{\delta}) = (g \circ \gamma) \star (g \circ \overline{\delta})$ lifts to a unique loop, say φ, in X based at x_0. Let $\widetilde{(g \circ \overline{\delta})}$ be the lifting of $g \circ \overline{\delta}$ in X starting at $x_1 := \widetilde{g \circ \gamma}(1)$. Then by Exercises 2.3.22 we have $\varphi = \widetilde{(g \circ \gamma)} \star \widetilde{(g \circ \overline{\delta})}$. Since φ is a loop in X based at x_0, we have $\widetilde{(g \circ \overline{\delta})}(1) = x_0$. Let η be the opposite path of $\widetilde{(g \circ \overline{\delta})}$. Since

$$\begin{aligned}
(f \circ \eta)(t) &= f(\widetilde{(g \circ \overline{\delta})}(1 - t)) \\
&= (g \circ \overline{\delta})(1 - t) \\
&= (g \circ \delta)(t), \ \forall\, t \in I,
\end{aligned}$$

and $\eta(0) = \widetilde{(g \circ \overline{\delta})}(1) = x_0$, by uniqueness of path-lifting, we have $\eta = \widetilde{(g \circ \delta)}$. Then $\widetilde{(g \circ \delta)}(1) = \eta(1) = \widetilde{(g \circ \overline{\delta})}(0) = \widetilde{(g \circ \gamma)}(1) = x_1$. Therefore, the map $\widetilde{g}$ in (2.3.28) is well-defined. It follows from the construction of $\widetilde{g}$ that $f \circ \widetilde{g} = g$. It remains to show that $\widetilde{g}$ is continuous. Here we need to use local path-connectedness of T.

Fix a point $t_1 \in T$ and let $y_1 = g(t_1) \in Y$ and $x_1 := \widetilde{g}(t_1) \in f^{-1}(y_1)$. Since f is a covering map, there is an open neighbourhood $U \subseteq Y$ of y_1 and an open neighbourhood $\widetilde{U} \subseteq X$ of x_1 such that

$$f\big|_{\widetilde{U}} : \widetilde{U} \to U \tag{2.3.29}$$

is a homeomorphism. Since T is locally path-connected and g is continuous, there is a path-connected neighbourhood $V \subseteq T$ of t_1 such that $g(V) \subseteq U$. To show $\widetilde{g} : T \to X$ continuous, it is enough to show that $\widetilde{g}(V) \subseteq \widetilde{U}$. Given $t' \in V$, choose a path α inside V joining t_1 to t'. Then $\gamma \star \alpha$ is a path in T joining t_0 to t', and its image $g \circ (\gamma \star \alpha)$ has a lifting, say β, in X starting at x_0. Let $\widetilde{\alpha} := s \circ (g \circ \alpha)$, where $s : U \to \widetilde{U}$ is the inverse of the homeomorphism $f\big|_{\widetilde{U}}$ given in (2.3.29). Since $\widetilde{\gamma}(1) = (s \circ g \circ \alpha)(0)$, by uniqueness of path-lifting, β coincides with $\widetilde{\gamma} \star (s \circ g \circ \alpha)$. Then $\widetilde{g}(t') = \beta(1) = (\widetilde{\gamma} \star (s \circ g \circ \alpha))(1) = (s \circ g \circ \alpha)(1) \in \widetilde{U}$. Therefore, $\widetilde{g}(V) \subseteq \widetilde{U}$, and hence $\widetilde{g}$ is continuous. □

2.3.2 Fundamental group of S^1

Now we are in a position to compute fundamental group of the unit circle

$$S^1 := \{z \in \mathbb{C} \, : \, |z| = 1\} = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = 1\}.$$

Assuming that the reader has forgotten the statement of Theorem 2.3.1 by now, let's recall it once again.

Theorem 2.3.1. *The fundamental group $\pi_1(S^1, 1)$ of the unit circle S^1 with the base point $1 \in S^1$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ generated by the loop $\omega : I \to S^1$ defined by $\omega(t) = e^{2\pi i t}$, for all $t \in I = [0, 1]$.*

Proof. Let $\gamma : I \to S^1$ be a loop based at $x_0 = 1 \in S^1$. Since

$$p : \mathbb{R} \to S^1, \;\; t \mapsto e^{2\pi i t}$$

is a covering map (c.f. Example 2.3.6 (i)), there is a unique continuous map $\widetilde{\gamma} : I \to \mathbb{R}$ such that $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} = \gamma$. Since $p^{-1}(\gamma(1)) = \mathbb{Z}$, the path $\widetilde{\gamma}$ ends at some integer, say n. Note that, we have a path

$$\widetilde{\omega}_n : I \to \mathbb{R}, \;\; s \mapsto ns,$$

starting at 0 and ending at n. Clearly the path $\widetilde{\gamma}$ is homotopic to $\widetilde{\omega}_n$ by the linear homotopy

$$F : I \times I \to \mathbb{R}, \;\; (s, t) \mapsto (1 - t)\widetilde{\gamma}(s) + t\,\widetilde{\omega}_n(s).$$

Then the composition $p \circ F : I \times I \to S^1$ is a homotopy from γ to ω_n, where $\omega_n : I \to S^1$ is the loop based at $1 \in S^1$ defined by

$$\omega_n(s) = e^{2\pi i n s}, \;\; \forall\, s \in I.$$

Therefore, $[\gamma] = [\omega_n]$ in $\pi_1(S^1, 1)$.

Define a map

$$\varphi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1), \;\; n \mapsto [\omega_n].$$

It follows from the above construction that φ is surjective. To show that φ is a group homomorphism, we need to show that $\omega_m \star \omega_n \simeq \omega_{m+n}$, for all $m, n \in \mathbb{Z}$. To see this, consider the *"translation by m"* map

$$\tau_m : \mathbb{R} \to \mathbb{R}, \;\; x \mapsto x + m.$$

Note that $\tau_m \circ \widetilde{\omega}_n$ is a path in $\mathbb{R}$ starting at m and ending at $m+n$, and hence the path $\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)$ in $\mathbb{R}$ starts at 0 and ends at $m+n$. Then it follows from the first paragraph that $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n))$ is homotopic to ω_{m+n}. Since $p \circ (\widetilde{\omega}_m \star (\tau_m \circ \widetilde{\omega}_n)) = \omega_m \star \omega_n$, we conclude that φ is a group homomorphism.

To show that φ is injective, it is enough to show if a loop $\gamma : I \to S^1$ based at 1 is homotopic to both ω_n and ω_m, for some $m, n \in \mathbb{Z}$, then $m = n$. Indeed, if $\gamma \simeq \omega_m$ and $\gamma \simeq \omega_n$, then $\omega_m \simeq \omega_n$ by Lemma 2.1.4. Let $G : I \times I \to S^1$ be a homotopy from ω_m to ω_n in S^1. By Theorem 2.3.17 there is a unique continuous map $\widetilde{G} : I \times I \to \mathbb{R}$ such that $p \circ \widetilde{G} = G$ and $\widetilde{G}(0,0) = 0$. Then by uniqueness of path lifting (c.f. Theorem 2.3.12) we have $\widetilde{G}\big|_{\{0\}\times I} = \widetilde{\omega}_n$ and $\widetilde{G}\big|_{\{1\}\times I} = \widetilde{\omega}_m$. Since $\{\widetilde{G}\big|_{\{t\}\times I} : I \to \mathbb{R}\}_{t\in I}$ is a homotopy of paths, the end points $\widetilde{G}\big|_{\{t\}\times I}(1)$ are independent of t. Thus, $m = \widetilde{G}\big|_{\{0\}\times I}(1) = \widetilde{G}\big|_{\{1\}\times I}(1) = n$, and hence φ is injective. This completes the proof. □

2.3.3 Fundamental group of S^n, for $n \geq 2$

In this subsection we show that S^n is simply connected, for $n \geq 2$. First we need the following.

Lemma 2.3.30. *Let (X, x_0) be a pointed topological space. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X such that*

1. *each U_α is path-connected,*
2. *$x_0 \in U_\alpha$, for all $\alpha \in \Lambda$,*
3. *$U_\alpha \cap U_\beta$ is path-connected, for all $\alpha, \beta \in \Lambda$.*

Then any loop in X based at x_0 is homotopic to a finite product of loops each of which is contained in a single U_α, for finitely many α's.

Proof. Let $\gamma : I \to X$ be a loop based at x_0. Since γ is continuous, each $s \in I$ is contained in an open neighbourhood $V_s := (s - \delta_s, s + \delta_s) \subseteq I$ of s such that $\gamma(\overline{V}_s) \subseteq U_{\alpha_s}$, for some $\alpha_s \in \Lambda$. Since I is compact, we can choose finitely many such open neighbourhoods V_s's to cover I. Thus we get a finite partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $I = [0, 1]$ such that $\gamma([s_{j-1}, s_j]) \subseteq U_{\alpha_j}$, for some $\alpha_j \in \Lambda$, for all $j = 1, \ldots, m$. Therefore, the restriction

$$\gamma_j := \gamma\big|_{[s_{j-1}, s_j]} : [s_{j-1}, s_j] \to U_{\alpha_j} \subseteq X$$

is a path in U_{α_j}, for each $j = 1, \ldots, m$, and that $\gamma = \gamma_1 \star \cdots \star \gamma_m$. Since $U_j \cap U_{j+1}$ is path-connected, we may choose a path η_i in $U_{\alpha_j} \cap U_{\alpha_{j+1}}$ from the base point x_0 to the point $\gamma(s_j) \in U_{\alpha_j} \cap U_{\alpha_{j+1}}$, for all j (see Figure 2.11). Denote by $\overline{\eta}_j$ the opposite path of η_j, for all j (see definition (2.2.23) in §2.2.3). Then the product loop

$$(\gamma_1 \star \overline{\eta_1}) \star (\eta_1 \star \gamma_2 \star \overline{\eta_2}) \star (\eta_2 \star \gamma_3 \star \overline{\eta_3}) \star \cdots \star (\eta_{m-1} \star \gamma_m) \tag{2.3.31}$$

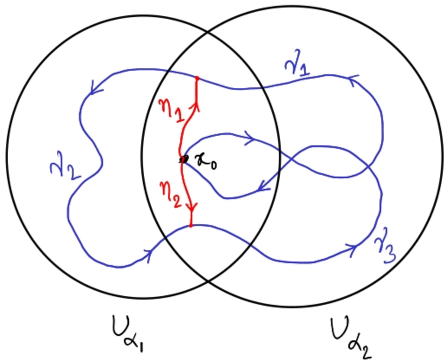


FIGURE 2.11

is homotopic to γ (see Exercise 2.2.24). Clearly this loop is a composition of the loops $\gamma_1 \star \overline{\eta_1}$, $\eta_1 \star \gamma_2 \star \overline{\eta_2}$, $\eta_2 \star \gamma_3 \star \overline{\eta_3}, \cdots, \eta_{m-1} \star \gamma_m$ based at x_0, each lying inside a single U_{α_j}, for all $j = 1, \ldots, m$. This completes the proof. $\square$

Exercise 2.3.32. Fix an integer $n \geq 1$.

(i) For any $x_0 \in S^n$, show that $S^n \setminus \{x_0\}$ is homeomorphic to $\mathbb{R}^n$.

(ii) For a pair of antipodal points $x_1, x_2 \in S^n$, let $U_j := S^n \setminus \{x_j\}$, for $j = 1, 2$. Show that $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Proposition 2.3.33. *For an integer $n \geq 2$, we have $\pi_1(S^n) = \{1\}$.*

Proof. Fix a pair of antipodal points x_1, x_2 in S^n. Then we have two open subsets $U_1 = S^n \setminus \{x_1\}$ and $U_2 = S^n \setminus \{x_2\}$ each homeomorphic to $\mathbb{R}^n$. Clearly $S^n = U_1 \cup U_2$ and $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. Then by Exercise 2.3.32 we have $U_1 \cap U_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, which is path-connected because $n \geq 2$. Fix a base point $x_0 \in U_1 \cap U_2$. Let γ be a loop in S^n based at x_0. Then by Lemma 2.3.30 γ is homotopic to a product of finitely many loops in S^n based at x_0 each of which are contained in either U_1 or U_2. Since both U_1 and U_2 are homeomorphic to $\mathbb{R}^n$ by Exercise 2.3.32, we have $\pi_1(U_j) = \pi_1(\mathbb{R}^n) = \{1\}$, for $j = 1, 2$. Therefore, γ is homotopic to a finite product of loops based at x_0 each of which are null-homotopic, and hence γ is null-homotopic. $\square$

Corollary 2.3.34. *S^n is simply connected, for $n \geq 2$.*

Exercise 2.3.35. For a point $x_0 \in \mathbb{R}^n$, show that the space $\mathbb{R}^n \setminus \{x_0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

Corollary 2.3.36. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for $n \neq 2$.*

Proof. If possible let $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a homeomorphism. For $n = 1$, since $\mathbb{R}^2 \setminus \{0\}$ is path-connected while $\mathbb{R} \setminus \{f(0)\}$ is disconnected, there is no such homeomorphism in this case. Suppose that $n > 2$. In this case, we cannot distinguish $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{R}^n \setminus \{f(0)\}$ in terms of number of path-components; but we can distinguish them by their fundamental groups.

Since for any point $x \in \mathbb{R}^n$ the space $\mathbb{R}^n \setminus \{x\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$ by Exercise 2.3.35, we have

$$\begin{aligned}\pi_1(\mathbb{R}^n \setminus \{x\}) &\cong \pi_1(S^{n-1} \times \mathbb{R}) \\ &\cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \\ &\cong \pi_1(S^{n-1}),\end{aligned}$$

because $\pi_1(\mathbb{R})$ is trivial. Since $\pi_1(S^1) \cong \mathbb{Z}$ by Theorem 2.3.1 while $\pi_1(S^{n-1}) \cong \{1\}$, for $n > 2$, by Proposition 2.3.33, such a homeomorphism cannot exists. $\square$

Remark 2.3.37. A more general result that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$ can be proved in a similar fashion using higher homotopy groups or homology groups. In fact, using homology groups one can show that *non-empty open subsets of* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *can be homeomorphic if and only if* $m = n$.

2.3.4 Some applications

Theorem 2.3.38 (Fundamental theorem of algebra)**.** *Every non-constant polynomial with coefficients from* $\mathbb{C}$ *has a root in* $\mathbb{C}$.

Proof. Take a non-constant polynomial $p(z) \in \mathbb{C}[z]$. Diving $p(z)$ by its leading coefficient, if required, we may assume that

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z].$$

If $p(z)$ has no roots in $\mathbb{C}$, then for each real number $r \geq 0$, the map $\gamma_r : I \to S^1 \subset \mathbb{C}$ defined by

$$\gamma_r(s) := \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \quad \forall\, s \in I, \tag{2.3.39}$$

is a loop in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ with the base point $1 \in \mathbb{C}$. As r varies, the collection $\{\gamma_r\}_{r \geq 0}$ defines a homotopy of loops in S^1 based at 1. Since γ_0 is the constant loop 1 in S^1, we see that the homotopy class $[\gamma_r] \in \pi_1(S^1, 1)$ is trivial, for all $r \geq 0$.

Choose any $r \in \mathbb{R}$ with $r > \max\{1, |a_1| + \cdots + |a_n|\}$. Then for $|z| = r$ we have

$$\begin{aligned}|z^n| = r^n = r \cdot r^{n-1} &> \big(|a_1| + \cdots + |a_n|\big)|z^{n-1}| \\ &\geq |a_1 z^{n-1} + \cdots + a_n|\end{aligned}$$

From this inequality, it follows that for each $t \in [0, 1]$, the polynomial

$$p_t(z) := z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = r$. Replacing $p(z)$ with $p_t(z)$ in the expression of γ_r in (2.3.39) and letting t vary from 1 to 0, we get a homotopy from the loop γ_r to the loop

$$\omega_n : I \to S^1, \;\; s \mapsto e^{2\pi ins}.$$

Since the loop ω_n represents n times a generator of the infinite cyclic group $\pi_1(S^1, 1) \cong \mathbb{Z}$, and that $[\omega_n] = [\gamma_t] = 0$, we must have $n = 0$. Thus the only polynomials without roots in $\mathbb{C}$ are constants. □

Definition 2.3.40. A *deformation retraction* of X onto its subspace A is a continuous map $F : X \times I \to X$ such that the associated family of continuous maps

$$\left\{ f_t := F\big|_{X\times\{t\}} : X \to X \right\}_{t\in I}$$

obtained by restricting F on the slices $X \times \{t\} \hookrightarrow X \times I$, for each $t \in I$, satisfies $f_0 = \mathrm{Id}_X$, $f_1(X) = A$, and $f_t\big|_A = \mathrm{Id}_A$, $\forall\, t \in I$. In this case, we say that A is a deformation retract of X.

Example 2.3.41. (i) Let $D = \{re^{i\theta} \in \mathbb{C} : 0 < r \leq 1,\; 0 \leq \theta < 2\pi\}$ be the punctured disk of radius 1 in the plane $\mathbb{C}$, and let $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subseteq X$ be the unit circle. For each $t \in I = [0, 1]$, we define a map

$$f_t : D \longrightarrow D$$

by sending $re^{i\theta} \in D$ to $\big(t + (1-t)r\big)e^{i\theta} \in D$. It is easy to verify that $\{f_t\}_{t\in I}$ is a family of continuous maps from D into itself, and satisfies $f_0 = \mathrm{Id}_D$, $f_1(D) = S^1$ and $f_t\big|_{S^1} = \mathrm{Id}_{S^1}$. Therefore, $\{f_t\}_{t\in I}$ is a deformation retraction of D onto S^1.

(ii) Let X be the Möbius strip (see Figure 2.12) and $A \subset X$ be the central simple loop of X. Then there is a deformation retraction of X onto A.

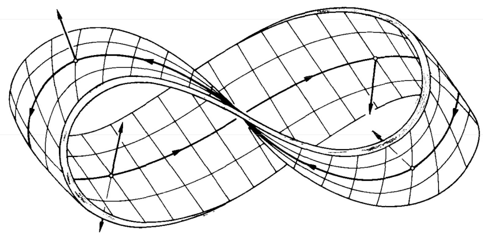


FIGURE 2.12: Möbius strip

Definition 2.3.42. A *retraction* of X onto a subspace $A \subset X$ is a continuous map $f : X \to X$ such that $f(X) = A$ and $f\big|_A = \mathrm{Id}_A$. A subspace $A \subseteq X$ is said to be a *retract* of X if there is a retraction of X onto A.

Note that a retraction $f : X \to X$ of X onto a subspace $A \subseteq X$ can be characterized by its property $f \circ f = f$, and hence we can think of it as a topological analogue of a *projection operator* in algebra.

Lemma 2.3.43. *If $A \subseteq X$ is a retract of X, for any $a_0 \in A$ the homomorphism of fundamental groups*

$$\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0),$$

induced by the inclusion map $\iota : A \hookrightarrow X$, is injective.

Proof. Let $f : X \to X$ be a retraction of X onto A. Then $f \circ \iota = \mathrm{Id}_A$, the identity map of A. Then by Proposition 2.2.12 and Remark 2.2.13 we have $f_* \circ \iota_* = \mathrm{Id}_{\pi_1(A,a_0)}$. Thus ι_* admits a left inverse, and hence is injective. □

Proposition 2.3.44. *If $A \subseteq X$ is a deformation retract of X, then X is homotopically equivalent to A (see Definition 2.1.9).*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto its subspace A. Since

$$f_0 : X \to X, \ x \mapsto F(x, 0)$$

is the identity map $\mathrm{Id}_X : X \to X$, and

$$f_1 : X \to X, \ x \mapsto F(x, 1)$$

is a retraction of X onto A, we conclude that F is a homotopy from Id_X to a retraction of X onto A. Since $f_1 \circ \iota = \mathrm{Id}_A$ and $\iota \circ f_1$ is homotopic to the identity map of X, we conclude that X and A are homotopically equivalent. □

Corollary 2.3.45. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$ we have an isomorphism of fundamental groups $\pi_1(A, a_0) \cong \pi_1(X, a_0)$.*

Proof. Follows from Lemma 2.2.16. □

Remark 2.3.46. Note that the constant map $X \to \{x_0\} \subseteq X$ being continuous, every space X admits a retraction onto a point of it. However, the next Proposition 2.3.47 and Lemma 2.3.48 produce examples of topological spaces that do not admit any deformation retract onto a point of it.

Proposition 2.3.47. *If there is a deformation retract of X onto a point $x_0 \in X$, then X is path connected.*

Proof. Let $F : X \times I \to X$ be a deformation retract of X onto a point $x_0 \in X$. Since for any point $x \in X$, the continuous map

$$\phi_x : I \to X, \ t \mapsto F(x, t)$$

is a path joining $F(x, 0) = x$ and $F(x, 1) = x_0$, X is path connected. □

Lemma 2.3.48. *If $A \subseteq X$ is a deformation retract of X, then for any $a_0 \in A$, the homomorphism of fundamental groups $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ induced by the inclusion map $\iota : A \hookrightarrow X$ is an isomorphism.*

Proof. Let $F : X \times I \to X$ be a deformation retraction of X onto X. Then $f_1 := F|_{X\times\{1\}} : X \to X$ is a retraction of X onto A. Then by Lemma 2.3.43 the homomorphism $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective. To show ι_* is an isomorphism, it enough to show that it is surjective. Note that, given any loop $\gamma : I \to X$ in X based at a_0, the composite map

$$G : I \times I \xrightarrow{\gamma\times \mathrm{Id}_I} X \times I \xrightarrow{F} X$$

is a path-homotopy from $G|_{I\times\{0\}} = \gamma$ to a loop $g := G|_{I\times\{1\}} : I \to A$ based at a_0. Thus, $\iota_*([g]) = [g] = [\gamma]$, and hence ι_* is surjective. □

Remark 2.3.49. The notion of deformation retraction of a space X onto a subspace $A \subseteq X$ is a way to continuously deform X onto A in a very strong sense, while the notion of homotopy equivalence seems to be a weaker notion of being able to deform a space into another space. However, if two spaces X and Y are homotopically equivalent, then there is a space Z such that both X and Y are deformation retracts of Z. Such a space Z can be constructed as a mapping cylinder

$$M_f := \big((X \times I) \sqcup Y\big)/(x, 1) \sim f(x)$$

of a homotopy equivalence $f : X \to Y$. We shall not go into details for its proof in this course.

Exercise 2.3.50. Show that the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ do not admit any deformation retraction onto a point of it.

Exercise 2.3.51. Show that $\pi_1(\mathbb{R}^2 \setminus \{(0, 0)\}) \cong \mathbb{Z}$.

For an integer $n \geq 1$, let

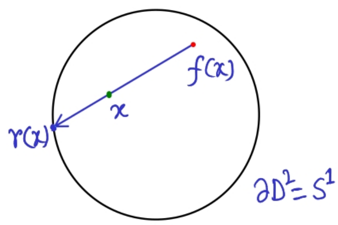
$$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1\}$$

be the *closed unit disk* in $\mathbb{R}^n$. Its boundary ∂D^n is the *unit sphere* in $\mathbb{R}^n$ given by

$$S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 = 1\}.$$

Theorem 2.3.52 (Brouwer's fixed point theorem)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Suppose on the contrary that $f : D^2 \to D^2$ has no fixed point, i.e., $f(x) \neq x$, $\forall\, x \in D^2$. Then for each $x \in D^2$, the ray in $\mathbb{R}^2$ starting at $f(x)$ and passing through x hits a unique point, say $r(x) \in S^1$. This defines a map $r : D^2 \to S^1$. Since f is continuous, small perturbations of



x produce small perturbations of $f(x)$, and hence small perturbations of the ray starting from $f(x)$ and passing through x, it follows that the function $x \mapsto r(x)$ is continuous. Explicit proof of continuity could be given by writing down the explicit expression for $r(x)$ in terms of $f(x)$. Note that $r(x) = x$, for all $x \in S^1$. Therefore, $r : D^2 \to S^1$ is a retraction of D^2 onto its subspace $S^1 = \partial D^2$. Then by Lemma 2.3.43 the homomorphism of fundamental groups

$$\iota_* : \pi_1(S^1, (1,0)) \longrightarrow \pi_1(D^2, (1,0))$$

induced by the inclusion map $\iota : S^1 \hookrightarrow D^2$, is injective. Since $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1,0))$ is trivial, we get a contradiction. □

Remark 2.3.53. The corresponding statement for Brouwer's fixed point theorem holds, more generally, for a closed unit disk $D^n \subset \mathbb{R}^n$, for all $n \geq 2$. If time permits, we shall give a proof of it using homology. However, the original proof of it, due to Brouwer, neither uses homology nor uses homotopy groups, which was not invented at that time. Instead, Brouwer's proof uses the notion of degree of maps $S^n \to S^n$, which could be defined later using homology, but Brouwer defined it more directly in a geometric way.

Definition 2.3.54. For $x = (x_1, \ldots, x_{n+1}) \in S^n$, we define its *antipodal point* to be the point $-x := (-x_1, \ldots, -x_{n+1}) \in S^n$.

Theorem 2.3.55 (Borsuk-Ulam)**.** *Let $n \in \{1, 2\}$. Then for every continuous map $f : S^n \to \mathbb{R}^n$, there is a pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$.*

Proof. The case $n = 1$ is easy. Indeed, since the function

$$g : S^1 \to \mathbb{R}, \quad x \mapsto f(x) - f(-x)$$

changes its sign after the point $x \in S^1$ moves half way along the circle S^1, there must be a pint $x \in S^1$ such that $f(x) = f(-x)$.

Assume that $n = 2$. We use the same technique used to compute the fundamental group of S^1. Suppose on the contrary that there is a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$, for all $x \in S^2$. Then we can define a map $g : S^2 \to \mathbb{R}^2$ by

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}, \ \forall\, x \in S^2,$$

where $\|(y_1, y_2)\| := \sqrt{y_1^2 + y_2^2}$ is the *norm* of $(y_1, y_2) \in \mathbb{R}^2$. Since $\|g(x)\| = 1$, the image of the map g lands inside $S^1 \subset \mathbb{R}^2$. Note that the map $g : S^2 \to S^1$ is continuous. Define a loop $\eta : I = [0, 1] \to S^2$ by

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0), \ \forall\, s \in I. \tag{2.3.56}$$

Then η circles around the equator of the sphere $S^2 \subset \mathbb{R}^3$. Let $h : I \to S^1$ be the composite map $h := g \circ \eta$.

$$h : I \xrightarrow{\eta} S^2 \xrightarrow{g} S^1.$$

Since $g(x) = -g(-x)$, we have

$$h(s + \frac{1}{2}) = -h(s), \ \forall\, s \in [0, 1/2]. \tag{2.3.57}$$

Now consider the covering map

$$p : \mathbb{R} \to S^1, \ \ s \mapsto e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s).$$

Lift the loop $h : I \to S^1$ to this cover to get a unique path $\widetilde{h} : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$ (see Theorem 2.3.12). Then it follows from the relation (2.3.57) that

$$\widetilde{h}(s + \frac{1}{2}) = \widetilde{h}(s) + \frac{q(s)}{2}, \tag{2.3.58}$$

for some odd integer $q(s)$ depending on $s \in [0, \frac{1}{2}]$. Since $\widetilde{h}$ is continuous, it follows from the equation (2.3.58) that the map

$$I \to \mathbb{R}, \ \ s \mapsto q(s),$$

is continuous on $[0, \frac{1}{2}]$. Since q is a discrete function taking values in odd integers, we must have $q(s) = q$, for some odd integer q, for all $s \in [0, \frac{1}{2}]$. In particular, putting $s = 1/2$ and 0 in (2.3.58) we have

$$\widetilde{h}(1) = \widetilde{h}(1/2) + \frac{q}{2} = \widetilde{h}(0) + q.$$

This means that the loop h represents q times a generator of $\pi_1(S^1)$. Since q is an odd integer, h cannot be null homotopic. But this cannot happen because the loop $\eta : I \to S^2$ being null-homotopic, the loop $h := g \circ \eta : I \to S^2 \to S^1$ should be null-homotopic. Thus we get a contradiction. This completes the proof. □

Remark 2.3.59. (i) Borsuk-Ulam theorem (Theorem 2.3.55) holds for all integer $n \geq 1$. A general proof could be given using homology theory later.

(ii) Theorem 2.3.55 says that there is no one-to-one continuous map from S^n into $\mathbb{R}^n$. As a result, S^n cannot be homeomorphic to a subspace of $\mathbb{R}^n$.

2.4 Galois theory for covering spaces

2.4.1 Universal cover

Since we shall work with paths in X, and a locally path-connected space is connected if and only if it is path-connected, and path-connected components of X are the same as connected components of X, there is no harm in assuming that X is connected or equivalently path-connected. Unless explicitly mentioned, in this section, we always assume that X is path-connected and locally path-connected.

Proposition 2.4.1. *Let X be a connected and locally path-connected topological space. Fix a point $x_0 \in X$. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a simply connected covering. Then for any connected covering*

$f : (Y, y_0) \to (X, x_0)$, there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $p \circ F = f$.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x_0}) \arrow[dl, dashed, "F"'] \arrow[dd, "p"] \\ (Y, y_0) \arrow[dr, "f"'] & \\ & (X, x_0) \end{tikzcd}$$

Proof. Since X is locally path-connected and $\widetilde{X}$ is a simply connected covering of X, $\widetilde{X}$ is path-connected and locally path-connected. Since $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial and $f : (Y, y_0) \to (X, x_0)$ is a covering map, by general lifting criterion (see Theorem 2.3.27) there is a unique continuous map $F : (\widetilde{X}, \widetilde{x_0}) \to (Y, y_0)$ such that $f \circ F = p$. □

Proposition 2.4.2. *Let (X, x_0) be a locally path-connected and path-connected topological space. Let $p_1 : (\widetilde{X_1}, \widetilde{x_1}) \to (X, x_0)$ and $p_2 : (\widetilde{X_2}, \widetilde{x_2}) \to (X, x_0)$ be two simply connected covering spaces of (X, x_0). Then there is a unique homeomorphism of pointed topological spaces $F : (\widetilde{X_1}, \widetilde{x_1}) \to (\widetilde{X_2}, \widetilde{x_2})$ such that $p_2 \circ F = p_1$.*

$$\begin{tikzcd} (\widetilde{X_1}, \widetilde{x_1}) \arrow[rr, "F"] \arrow[dr, "p_1"'] & & (\widetilde{X_2}, \widetilde{x_2}) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

Proof. Follows from Proposition 2.4.1. □

Definition 2.4.3. A simply connected covering space of a path-connected locally path-connected topological space (X, x_0) is called the *universal cover* of (X, x_0). This name is due to its universal property (c.f. Proposition 2.4.1) and uniqueness upto a unique homeomorphism (c.f. Proposition 2.4.2).

It is not yet clear if universal cover of a path-connected locally path-connected topological space exists or not, however if it exists, it is unique up to a unique homeomorphism of pointed topological space by Proposition 2.4.2. The following Lemma 2.4.4 gives a necessary condition on (X, x_0) for existence of a universal covering space.

Lemma 2.4.4. *Let $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be the universal cover of (X, x_0). Then each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.*

Proof. Fix $x \in X$. Then there is a path-connected open neighbourhood $U \subseteq X$ which is evenly covered by the covering map p. Let $\widetilde{U} \subseteq \widetilde{X}$ be the path-connected open subset such that $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Let γ be a loop in U based at x. Using the homeomorphism $p|_{\widetilde{U}}$, we can lift it to a loop $\widetilde{\gamma}$ in $\widetilde{X}$ based at the point $\widetilde{x} \in \widetilde{U} \cap p^{-1}(x)$. Since $\widetilde{X}$ is simply-connected, we have a path-homotopy $F : I \times I \to \widetilde{X}$ from $\widetilde{\gamma}$ to the constant loop $c_{\widetilde{x}}$ at $\widetilde{x}$ in $\widetilde{X}$. Composing F with p we get a path-homotopy $p \circ F$ from γ to the constant loop c_x at x in X.

This shows that the homomorphism $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion map $\iota : U \hookrightarrow X$ is trivial. $\square$

Definition 2.4.5. A path-connected and locally path-connected topological space X is said to be *semi-locally simply connected* if each point $x \in X$ has a path-connected open neighbourhood $U \subseteq X$ such that the homomorphism of fundamental groups $\iota_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion map $\iota : U \hookrightarrow X$, is trivial.

2.4.2 Construction of universal cover

The following theorem shows that the condition on (X, x_0) for existence of its universal covering space given in Lemma 2.4.4 is, in fact, sufficient.

Theorem 2.4.6. *Let X be a path-connected, locally path-connected topological space. Fix a point $x_0 \in X$. Then a simply connected covering space of (X, x_0) exists if and only if X is semi-locally simply connected.*

Proof. If a simply connected covering space for X exists, then X is semi-locally simply connected by Lemma 2.4.4.

Suppose that X is semi-locally simply connected. We give an explicit construction of a simply connected covering space of X. Note that, if $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a simply connected covering space for (X, x_0), then for each $\widetilde{x} \in \widetilde{X}$, there is a unique path-homotopy class of paths in $\widetilde{X}$ from $\widetilde{x_0}$ to $\widetilde{x}$ (see Corollary 2.2.33). Thus, points of $\widetilde{X}$ can be thought of as homotopy classes of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$, and hence can be thought of as the homotopy classes of paths in X starting at x_0 thanks to the homotopy lifting property. This motivates us to construct the underlined set of points of $\widetilde{X}$ as

$$\widetilde{X} := \{[\gamma] : \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

where $[\gamma]$ denotes the path-homotopy class of a path γ in X. Define

$$p : \widetilde{X} \to X \tag{2.4.7}$$

by sending a $[\gamma] \in \widetilde{X}$ to the end point $\gamma(1) \in X$ of γ; this map is well-defined because of the definition of path-homotopy (see Definition 2.2.1). Since X is path-connected, given any $x_1 \in X$ there is a path γ in X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then $[\gamma] \in \widetilde{X}$ with $p([\gamma]) = x_1$. Thus, p is surjective. If we set $\widetilde{x_0} \in \widetilde{X}$ to be the path-homotopy class of the constant path $c_{x_0} : I \to X$ given by $c_{x_0}(t) = x_0$, $\forall\, t \in I$, then $p(\widetilde{x_0}) = x_0$.

It remains to give a suitable topology on $\widetilde{X}$ to make $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ a simply connected covering space of (X, x_0). Let

$$\begin{aligned} \mathscr{U} := \{V \overset{\iota}{\hookrightarrow} X \mid\ & V \text{ is a path-connected open subset of } X \text{ such that} \\ & \text{the homomorphism } \iota_* : \pi_1(V) \to \pi_1(X) \text{ is trivial } \}. \end{aligned}$$

Note that, if the homomorphism $\iota_* : \pi_1(V,x) \to \pi_1(X,x)$, induced by the inclusion map $\iota : V \hookrightarrow X$, is trivial for some $x \in V$, then it is trivial for all points of V, whenever V is path-connected. Moreover, if U and V are two path-connected open subsets of X with $V \subseteq U$ and $U \in \mathscr{U}$, then it follows from the following commutative diagram

$$\begin{tikzcd} \pi_1(V) \arrow[rr, "\iota_{V,U*}"] \arrow[dr, "\iota_{V*}"'] & & \pi_1(U) \arrow[dl, "\iota_{U*}"] \\ & \pi_1(X) & \end{tikzcd}$$

that $V \in \mathscr{U}$, where $\iota_U : U \hookrightarrow X$, $\iota_V : V \hookrightarrow X$ and $\iota_{V,U} : V \hookrightarrow U$ are inclusion maps. Since X is locally-path-connected, path-connected and semi-locally simply connected, now it follows that $\mathscr{U}$ is a basis for the topology on X (verify!).

We now use the collection $\mathscr{U}$ to construct a collection $\mathscr{B}$ of subsets of $\widetilde{X}$ which forms a basis for the desired topology on $\widetilde{X}$. Given $U \in \mathscr{U}$ and a path γ in X starting at x_0 and ending at a point in U, consider the subset

$$U_{[\gamma]} := \{[\gamma \star \eta] : \eta \text{ is a path in } U \text{ starting at } \gamma(1)\} \subseteq \widetilde{X}.$$

Note that, if γ is path-homotopic to γ' in X, then $\gamma(1) = \gamma'(1)$, and hence for any path η in U starting at $\gamma(1) = \gamma'(1)$, we have $[\gamma \star \eta] = [\gamma' \star \eta]$. Therefore, the subset $U_{[\gamma]} \subseteq \widetilde{X}$ depends only on U and the path-homotopy class of γ in X.

Observation 1: The restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U \tag{2.4.8}$$

is bijective. Indeed, it is surjective because U is path-connected. To see it is injective, note that if $p([\gamma \star \eta]) = p([\gamma \star \eta'])$, then $\eta(1) = \eta'(1)$ and so the loop $\eta \star \overline{\eta'}$ is path-homotopic to the constant path $c_{\eta(0)}$ inside X, because the homomorphism $\iota_* : \pi_1(U) \to \pi_1(X)$ is trivial. Then it follows that $[\gamma \star \eta] = [\gamma \star \eta']$. Therefore, the restriction of p on $U_{[\gamma]}$ (see (2.4.8)) is injective, and hence is bijective.

Observation 2: Given $U \in \mathscr{U}$ and any two paths γ and δ in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1), \delta(1) \in U$, if $[\delta] \in U_{[\gamma]}$, then we must have $U_{[\gamma]} = U_{[\delta]}$. Indeed, if $[\delta] \in U_{[\gamma]}$, then $[\delta] = [\gamma \star \eta]$, for some path η in U with $\eta(0) = \gamma(1)$. Then for any path α in U with $\alpha(0) = \delta(1)$, we have $[\delta \star \alpha] = [(\gamma \star \eta) \star \alpha] = [\gamma \star (\eta \star \alpha)] \in U_{[\gamma]}$. Thus $U_{[\delta]} \subseteq U_{[\gamma]}$. Conversely, given any $[\gamma \star \alpha] \in U_{[\gamma]}$ we have $[\gamma \star \alpha] = [\gamma \star \eta \star \overline{\eta} \star \alpha] = [\delta \star (\overline{\eta} \star \alpha)] \in U_{[\delta]}$, which shows that $U_{[\gamma]} \subseteq U_{[\delta]}$. Therefore, we conclude that $U_{[\gamma]} = U_{[\delta]}$ if $[\delta] \in U_{[\gamma]}$.

Now we use the above two observations to show that the collection

$$\mathscr{B} := \{U_{[\gamma]} : U \in \mathscr{U} \text{ and } \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

forms a basis for a topology on $\widetilde{X}$. Note that, X being path-connected, we have $\widetilde{X} = \bigcup\limits_{U_{[\gamma]} \in \mathscr{B}} U_{[\gamma]}$. To check the second property for $\mathscr{B}$ to be a basis for a topology on $\widetilde{X}$, suppose that we are given two objects $U_{[\gamma]}, V_{[\delta]} \in \mathscr{B}$ and an element

$$[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}. \tag{2.4.9}$$

Now $U, V \in \mathscr{U}$, and γ and δ are paths in X with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) \in U$, $\delta(1) \in V$. We claim that

$$U_{[\gamma]} = U_{[\alpha]} \quad \text{and} \quad V_{[\delta]} = V_{[\alpha]}. \tag{2.4.10}$$

Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, we have $[\alpha] = [\gamma \star \eta] = [\delta \star \eta']$, for some paths η and η' in U and V respectively, with $\eta(0) = \gamma(1)$ and $\eta'(0) = \delta(1)$. Since $\gamma \star \eta$ is path-homotopic to $\delta \star \eta'$, both of them have the same end point, and hence $\alpha(1) = \eta(1) = \eta'(1) \in U \cap V$. Then the claim in (2.4.10) follows from the Observation 2. Since $\mathscr{U}$ is a basis for the topology on X, and $\alpha(1) \in U \cap V$, there is an object $W \in \mathscr{U}$ such that $\alpha(1) \in W$ and $W \subseteq U \cap V$. Since $[\alpha] \in U_{[\gamma]} \cap V_{[\delta]}$, the argument given in Observation 2 shows that

$$W_{[\alpha]} \subseteq U_{[\alpha]} \cap V_{[\alpha]} = U_{[\gamma]} \cap V_{[\delta]},$$

where the equality of sets on the right side is by (2.4.10). Clearly $[\alpha] \in W_{[\alpha]}$. Therefore, $\mathscr{B}$ is a basis for a topology on $\widetilde{X}$. Give $\widetilde{X}$ the topology generated by this basis $\mathscr{B}$.

Now it remains to show that $p : \widetilde{X} \to X$ in (2.4.7) is a covering map and that $\widetilde{X}$ is simply connected. We first show that, for each $U_{[\gamma]} \in \mathscr{B}$, the restriction map

$$p\big|_{U_{[\gamma]}} : U_{[\gamma]} \to U$$

is a homeomorphism. We already have shown that $p\big|_{U_{[\gamma]}}$ is bijective. Note that, for any $V'_{[\delta]} \in \mathscr{B}$ with $V'_{[\delta]} \subseteq U_{[\gamma]}$ we have $p(V'_{[\delta]}) = V' \subseteq U$. Since both $\mathscr{U}$ and $\mathscr{B}$ are basis for the topologies of X and $\widetilde{X}$, respectively, this shows that the restriction map $p\big|_{U_{[\gamma]}}$ is open. To show that $p\big|_{U_{[\gamma]}}$ is continuous, it suffices to show that for any $V \in \mathscr{U}$ with $V \subseteq U$, we have $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma]}$. Indeed, for any $[\alpha] \in p^{-1}(V) \cap U_{[\gamma]}$, we have $\alpha(1) \in V \cap U$, and so $V_{[\alpha]} \subseteq U_{[\alpha]} = U_{[\gamma]}$ by Observation 2. Since $p(V_{[\alpha]}) = V$, it follows that $p^{-1}(V) \cap U_{[\gamma]} = V_{[\alpha]}$.

Since $\mathscr{B}$ is a basis for the topologies on $\widetilde{X}$, it follows that $p^{-1}(V)$ is open in $\widetilde{X}$, for all $V \in \mathscr{U}$. Since $\mathscr{U}$ is a basis for the topology on X, it follows that $p : \widetilde{X} \to X$ is continuous. Given a point $x \in X$, choose an object $U \in \mathscr{U}$ with $x \in U$. We claim that the collection

$$\mathscr{C}_U := \{U_{[\gamma]} : \gamma \text{ is a path in } X \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) \in U\}$$

is a partition of $p^{-1}(U)$. Since $p^{-1}(U) = \bigcup_{U_{[\gamma]} \in \mathscr{C}_U} U_{[\gamma]}$, it suffices to show that objects of the collection $\mathscr{C}_U$ are either disjoint or identical. If $[\alpha] \in U_{[\gamma]} \cap U_{[\delta]}$, then α is a path in X with $\alpha(0) = x_0$ and $\alpha(1) \in U$, and hence by Observation 2 we have $U_{[\gamma]} = U_{[\alpha]} = U_{[\delta]}$. Since the restriction of p on each of $U_{[\gamma]}$ is a homeomorphism, $p : \widetilde{X} \to X$ is a covering map.

It remains to show that $\widetilde{X}$ is simply connected. Given a point $[\gamma] \in \widetilde{X}$ and $t \in I$, consider the map $\gamma_t : I \to X$ defined by

$$\gamma_t(s) := \begin{cases} \gamma(s), & \text{if } \quad 0 \leq s \leq t, \text{ and} \\ \gamma(t), & \text{if } \quad t \leq s \leq 1. \end{cases} \tag{2.4.11}$$

Note that, each γ_t is a path in X starting at x_0, and hence its path-homotopy class is an element of $\widetilde{X}$. Then the map $\phi_{[\gamma]} : I \to \widetilde{X}$ defined by

$$\phi_{[\gamma]}(t) = [\gamma_t], \;\; \forall\, t \in I,$$

is a path (why it is continuous?) in $\widetilde{X}$ starting at $\widetilde{x_0} = [c_{x_0}] \in \widetilde{X}$ and ending at $[\gamma] \in \widetilde{X}$. Therefore, $\widetilde{X}$ is path-connected. Since $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ is a covering map, the homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x_0}) \to \pi_1(X, x_0)$ induced by the map p is injective by Corollary 2.3.20. Therefore, to show $\pi_1(\widetilde{X}, \widetilde{x_0})$ is trivial it suffices to show that $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big)$ is the trivial subgroup of $\pi_1(X, x_0)$. By Corollary 2.3.20 elements of $p_*\big(\pi_1(\widetilde{X}, \widetilde{x_0})\big) \subseteq \pi_1(X, x_0)$ are given by loops γ in X based at x_0 whose lift to the cover $p : \widetilde{X} \to X$ starting at $\widetilde{x_0}$ is a loop in $\widetilde{X}$ based at $\widetilde{x_0}$. Since $\phi_{[\gamma]}$ is a path in $\widetilde{X}$ starting at $\widetilde{x_0}$ and $p \circ \phi_{[\gamma]} = \gamma$, we must have $[\gamma] = \phi_{[\gamma]}(1) = \widetilde{x_0} = [c_{x_0}]$. In other words, γ is path-homotopic to the constant loop c_{x_0} in X. This completes the proof. □

We now go towards establishing Galois correspondence for covering spaces. Whenever we talk about simply connected covering space of X, we assume that X is semi-locally simply connected in addition to be it path-connected and locally path-connected.

2.4.3 Group action and covering map

Before proceeding further, let's recall some standard terminologies related to group action. Let G be a group, and let $\sigma : G \times X \to X$ be a left G-action on X. For notational simplicity, we denote by $g \cdot x$ the element $\sigma(g, x) \in X$, for all $(g, x) \in G \times X$. Given $x \in X$, the subset

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\} \subseteq G$$

is a subgroup of G, known as the *stabilizer of* x or the *isotropy subgroup* for x. The G-action σ is said to be *free* if $\mathrm{Stab}_G(x) = \{e\}$, for all $x \in X$. This means that, for each $x \in X$, given $g_1, g_2 \in G$, we have $g_1 \cdot x = g_2 \cdot x$ if and only if $g_1 = g_2$. Note that the G-action σ on X defines an equivalence relation on X; for $x \in X$, its equivalence class is the subset

$$\mathrm{Orb}_G(x) := \{g \cdot x : g \in G\} \subseteq X,$$

called the *G-orbit* of x in X. The G-action σ is said to be *transitive* if there is exactly one G-orbit in X. In other words, given any two points $x_1, x_2 \in X$, there exists $g \in G$ such that $x_2 = g \cdot x_1$.

Definition 2.4.12. Let G be a group. A G-action $\sigma : G \times X \to X$ on X is said to be *even* (or, *properly discontinuous* according to old texts) if the G-action map σ is continuous, and each point $x_0 \in X$ has an open neighbourhood $V \subseteq X$ such that $(g \cdot V) \cap V = \emptyset$, for all $g \neq e$ in G, where $g \cdot V := \{g \cdot x : x \in V\} \subseteq X$.

Remark on old notation: Most of the old texts uses the term *properly discontinuous G-action* to mean an even G-action. This terminology is awkward because the G-action on X itself is a continuous map.

Proposition 2.4.13. *If a group G is acting evenly on a path-connected and locally path-connected topological space Y, then the associated quotient map $q : Y \to Y/G$ is a covering map.*

Proof. Clearly the quotient map $q : Y \to Y/G$ is continuous. Note that, for any subset $V \subseteq Y$ we have

$$q^{-1}(q(V)) = \bigcup_{g \in G} g \cdot V, \tag{2.4.14}$$

where $g \cdot V = \{g \cdot v : v \in V\} \subseteq X$, for all $g \in G$. Since the left translation map $L_g : Y \to Y$ given by

$$L_g(y) = g \cdot y := \sigma(g, y), \ \ \forall\, y \in Y$$

is a homeomorphism, V is open in Y if and only if $g \cdot V = L_g(V)$ is open in Y, for all $g \in G$. Since q is a quotient map, it follows that $q(V)$ is open in Y/G if V is open in Y. Therefore, q is an open map.

To see $q : Y \to Y/G$ is a covering map, let's fix a point $v \in Y/G$, and a point $y \in q^{-1}(v)$. Since the G-action on Y is even, y has an open neighbourhood $U_y \subseteq Y$ such that $(g \cdot U_y) \cap U_y = \emptyset$, for all $g \neq e$ in G. Take $V_y := q(U_y)$. Then it follows that

$$q^{-1}(V_y) = \bigsqcup_{g \in G} g \cdot U_y.$$

It remains to show that the restriction map

$$q\big|_{g \cdot U_y} : g \cdot U_y \to V_y = q(U_y)$$

is a homeomorphism, for all $g \in G$. Since q is continuous and open, it suffices to show that $q\big|_{g \cdot U_y}$ is bijective, for all $g \in G$.

If $q\big|_{g \cdot U_y}$ were not injective, then there exist $y_1, y_2 \in g \cdot U_y$ with $y_1 \neq y_2$ such that $q(y_1) = q(y_2)$. Then there exists $h \in G$ such that $y_2 = h \cdot y_1$. Then $y_2 = h \cdot y_1 \in U_y \cap (h \cdot U_1)$ implies $h = e$ because the G-action on Y is even. This contradicts our assumption that $y_1 \neq y_2 = h \cdot y_1$. Therefore, $q\big|_{g \cdot U_y}$ must be injective. To show $q\big|_{g \cdot U_y}$ is surjective, note that a typical element of $V_y = q(U_y)$ is of the form $q(y_1)$, for some $y_1 \in U_y$. Since $q(y_1) = \mathrm{Orb}_G(y_1) = \{a \cdot y_1 : a \in G\}$, we see that $g \cdot y_1 \in g \cdot U_y$ satisfies $q\big|_{g \cdot U_y}(g \cdot y_1) = q(y_1)$. Therefore, $q\big|_{g \cdot U_y}$ is surjective. □

Proposition 2.4.13 allow us to construct a lot of examples of covering maps.

2.4.4 Group of Deck transformations

Let $f : Y \to X$ be a covering map. An *automorphism of* $f : Y \to X$ is a homeomorphisms $\phi : Y \to Y$ satisfying $f \circ \phi = f$. The set

$$\mathrm{Aut}(Y/X) := \{\phi : Y \to Y \mid \phi \text{ is a homeomorphism satisfying } f \circ \phi = f\}$$

of all automorphisms of $f : Y \to X$ forms a group with respect to the binary operation on $\mathrm{Aut}(Y/X)$ given by composition of homeomorphisms. The group $\mathrm{Aut}(Y/X)$ is also known as the group of *Deck transformations* or *covering transformations* of $f : Y \to X$. Note that, $\mathrm{Aut}(Y/X)$ acts on Y from the left by automorphisms:

$$a : \mathrm{Aut}(Y/X) \times Y \to Y, \quad (\phi, y) \mapsto \phi(y). \tag{2.4.15}$$

We shall show in Proposition 2.4.19 that if we equip $\mathrm{Aut}(Y/X)$ with discrete topology, then the action map in (2.4.15) become continuous.

Proposition 2.4.16. *Fix a point $x_0 \in X$, and a path-connected covering space $f : Y \to X$ of X. Then the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to give a free $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If Y is simply connected, then the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive.*

Proof. Let $y_0 \in f^{-1}(x_0)$ be given. Since $\phi \in \mathrm{Aut}(Y/X)$ satisfies $f \circ \phi = f$, we have $f(\phi(y_0)) = f(y_0) = x_0$, and hence $\phi(y_0) \in f^{-1}(x_0)$. Therefore, the natural $\mathrm{Aut}(Y/X)$-action on Y restricts to an $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$. If $\phi(y_0) = y_0$, for some $\phi \in \mathrm{Aut}(Y/X)$, then by uniqueness of lifting of maps (see Theorem 2.3.27 or Lemma 2.3.13) we must have $\phi = \mathrm{Id}_Y$. Therefore, the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is free.

Now assume that Y is simply connected. To show that $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive, choose two points $y_0, y_1 \in f^{-1}(x_0)$. Since X is locally path-connected and $f : Y \to X$ is a covering map, Y is locally path-connected. Since by assumption Y is path-connected and locally path-connected (since X is so) with $\pi_1(Y)$ trivial, by general lifting criterion (Theorem 2.3.27) there is a unique continuous map $\phi : (Y, y_0) \to (Y, y_1)$ such that $f \circ \phi = f$. Similarly, there is a unique continuous map $\psi : (Y, y_1) \to (Y, y_0)$ such that $f \circ \psi = f$. Then by uniqueness of lifting (see Theorem 2.3.27), we must have $\phi \circ \psi = \mathrm{Id}_{(Y, y_1)}$ and $\psi \circ \phi = \mathrm{Id}_{(Y, y_0)}$. Therefore, both ϕ and ψ are homeomorphisms, and that $\phi(y_0) = y_1$. Thus, the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive. □

Let $f : Y \to X$ be a covering map. Fix a point $x_0 \in X$. Since X is locally path-connected, there is a path-connected open neighbourhood $U \subset X$ of x_0 which is evenly covered by f. Then we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y, \tag{2.4.17}$$

where $V_y \subset Y$ is the path-connected open neighbourhood of $y \in f^{-1}(y_0)$ such that $f\big|_{V_y} : V_y \to U$ is a homeomorphism. Note that, $\{V_y : y \in f^{-1}(x_0)\}$ is precisely the set of all path-components of $f^{-1}(U)$.

Proposition 2.4.18. *With the above notations,* $\mathrm{Aut}(Y/X)$ *acts freely on the set of all path-components* $\{V_y : y \in f^{-1}(x_0)\}$ *of* $f^{-1}(U)$. *Moreover, this action is transitive when* Y *is simply connected.*

Proof. Since $f : Y \to X$ is a covering map, the restricted map

$$f_U := f\big|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$$

is a covering map. Since for any $\phi \in \mathrm{Aut}(Y/X)$ we have $f \circ \phi = f$, image of the restriction map $\phi\big|_{f^{-1}(U)} : f^{-1}(U) \to Y$ lands inside $f^{-1}(U)$, and hence gives rise to an automorphism of the covering space $f_U : f^{-1}(U) \to U$, i.e., $\phi\big|_{f^{-1}(U)} \in \mathrm{Aut}(f^{-1}(U)/U)$. Clearly $\phi \in \mathrm{Aut}(Y/X)$ takes path-components of $f^{-1}(U)$ to path-components of $f^{-1}(U)$. In particular, for each $y \in f^{-1}(x_0)$, the induced map

$$\phi : V_y \to V_{\phi(y)}$$

is a homeomorphism. Since $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is free by Proposition 2.4.16, if for some $y \in f^{-1}(x_0)$, the automorphism $\phi \in \mathrm{Aut}(Y/X)$ takes V_y to itself, then we must have $\Phi = \mathrm{Id}_Y$.

Now assume that Y is simply connected. Since the $\mathrm{Aut}(Y/X)$-action on $f^{-1}(x_0)$ is transitive by Proposition 2.4.16, and the path-components of $f^{-1}(U)$ are uniquely determined by the conditions that $V_y \cap f^{-1}(x_0) = \{y\}$ and $V_{y_1} \cap V_{y_2} = \emptyset$ for $y_1 \neq y_2$ in $f^{-1}(x_0)$, given any two path-components V_{y_1}, V_{y_2} of $f^{-1}(U)$, there exists $\phi \in \mathrm{Aut}(Y/X)$ such that $\phi(y_1) = y_2$, and hence $\phi(V_{y_1}) = V_{y_2}$. Thus, the $\mathrm{Aut}(Y/X)$-action on the set of all path-components of $f^{-1}(U)$ is transitive. $\square$

Proposition 2.4.19. *Let* $f : Y \to X$ *be a path-connected covering space of* X. *Equip* $\mathrm{Aut}(Y/X)$ *with discrete topology. Then there is a continuous map (action map)*

$$a : \mathrm{Aut}(Y/X) \times Y \to Y, \tag{2.4.20}$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}(Y/X) \times Y \arrow[r, "a"] \arrow[d, "pr_2"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X, \end{tikzcd} \tag{2.4.21}$$

where $pr_2 : \mathrm{Aut}(Y/X) \times Y \to Y$ *is the projection map onto the second factor.*

Proof. Clearly $a : \mathrm{Aut}(Y/X) \times Y \to X$ is defined by

$$a(\phi, y) = \phi(y), \ \forall\ (\phi, y) \in \mathrm{Aut}(Y/X) \times Y,$$

makes the above diagram commutative. We only need to show that the action map a is continuous.

Let

$$\mathcal{B} := \{V \subseteq Y : V \text{ is path-connected, open and } f(V) \text{ is evenly covered by } f\}.$$

Since Y is path-connected and locally path-connected covering space for X, it is easy to check that $\mathcal{B}$ is a basis for the topology on Y. Therefore, to show the action map a is continuous, it is enough to show that $a^{-1}(V)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $V \in \mathcal{B}$. Fix $V \in \mathcal{B}$. Since V is path-connected and $f : Y \to X$ is a covering map, $U := f(V)$ is path-connected and open in X. Fix a point $x_0 \in U$. Since $U = f(V)$ is evenly covered by f, we can write

$$f^{-1}(U) = \bigsqcup_{y \in f^{-1}(x_0)} V_y,$$

where $V_y \subseteq Y$ is an open neighbourhood of $y \in f^{-1}(x_0)$ such that $f\big|_{V_y} : V_y \to U$ is a homeomorphism. Since V is path-connected and $p(V) = U$, we have $V \subseteq V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Since $f\big|_{V_{y_0}} : V_{y_0} \to U$ is a homeomorphism, we must have $V = V_{y_0}$, for some $y_0 \in f^{-1}(x_0)$. Therefore, it is enough to show that $a^{-1}(V_y)$ is open in $\mathrm{Aut}(Y/X) \times Y$, for all $y \in f^{-1}(x_0)$.

Let $(\phi, y) \in a^{-1}(V_{y_0}) = \{(\psi, y') \in \mathrm{Aut}(Y/X) \times Y : \psi(y') \in V_{y_0}\}$ be arbitrary. Then $\phi(y) \in V_{y_0}$. Since ϕ is an automorphism of Y, there is a unique $y_1 \in Y$ such that $\phi(y_1) = y_0$. Then $\phi : V_{y_1} \to V_{y_0}$ is a homeomorphism. Since $\phi(y) \in V_{y_0}$, we must have $y \in V_{y_1}$. Then $\{\phi\} \times V_{y_1}$ is an open neighbourhood of (ϕ, y) in $\mathrm{Aut}(Y/X) \times Y$ such that $a(\{\phi\} \times V_{y_1}) \subseteq V_{y_0}$. Therefore, $a^{-1}(V_{y_0})$ is open in $\mathrm{Aut}(Y/X) \times Y$. This completes the proof. □

Corollary 2.4.22. *If $f : Y \to X$ is a connected cover of X, the action of* $\mathrm{Aut}(Y/X)$ *on Y is even (see Definition 2.4.12).*

Proof. Follows from Proposition 2.4.19 and 2.4.18. □

Proposition 2.4.23. *If a group G acts evenly on a connected topological space Y, then the automorphism group* $\mathrm{Aut}(Y/X)$ *of the covering map $q : Y \to X := Y/G$ is naturally isomorphic to G.*

Proof. Let $\sigma : G \times Y \to Y$ be the left G-action which is even. Since σ is continuous, for each $g \in G$, the induced map

$$\sigma_g : Y \to Y, \; y \mapsto g \cdot y := \sigma(g, y)$$

is a homeomorphism of Y onto itself. Since the quotient map $q : Y \to X := Y/G$ sends a point $y \in Y$ to its G-orbit $\mathrm{Orb}_G(y) \in Y/G$, it follows that $q(\sigma_g(y)) = q(y)$, for all $g \in G$. Therefore, $\sigma_g \in \mathrm{Aut}(Y/X)$. Thus we have a natural map

$$\Phi : G \longrightarrow \mathrm{Aut}(Y/X), \; g \longmapsto \sigma_g. \tag{2.4.24}$$

Note that, for any $g, h \in G$ we have

$$\sigma_{gh}(y) = (gh) \cdot y = g \cdot (h \cdot y) = \sigma_g(\sigma_h(y)), \; \forall\, y \in Y.$$

Therefore, Φ is a group homomorphism. Since the G-action on Y is even (see Definition 2.4.12), it follows that $\mathrm{Ker}(\Phi)$ is trivial, and hence Φ is injective. Let $\varphi \in \mathrm{Aut}(Y/X)$ be arbitrary. Fix a pint $y \in Y$, and let $x := q(y) \in X$. Since $\varphi(y) \in q^{-1}(x) = \mathrm{Orb}_G(y)$, we have $\varphi(y) = g \cdot y = \sigma_g(y)$, for some $g \in G$. Since both $\varphi, \sigma_g \in \mathrm{Aut}(Y/X)$ and they agree at a point of Y and Y is connected, by uniqueness of lifting (see Lemma 2.3.13) we have $\varphi = \sigma_g$. Therefore, Φ is surjective, and hence is an isomorphism. $\square$

2.4.5 Galois covers

Let $f : Y \to X$ be a path-connected covering space of X. Then the natural left $\mathrm{Aut}(Y/X)$-action on Y gives rise to an equivalence relation on Y, where the equivalence classes are $\mathrm{Aut}(Y/X)$-orbits of points of Y. Given $y \in Y$, its $\mathrm{Aut}(Y/X)$-orbit is the subset

$$\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y) = \{\phi(y) : \phi \in \mathrm{Aut}(Y/X)\} \subseteq Y.$$

Fix $y_0 \in Y$, and let $x_0 = f(y_0)$. Clearly, $\mathrm{Orb}_{\mathrm{Aut}(Y/X)}(y_0) \subseteq f^{-1}(x_0)$, and equality holds if and only if the $\mathrm{Aut}(Y/X)$-action on the fiber $f^{-1}(x_0)$ is transitive. By the universal property of quotient space, there is a unique continuous map

$$\widetilde{f} : Y/\,\mathrm{Aut}(Y/X) \to X \tag{2.4.25}$$

such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "q"'] & & X \\ & Y/\,\mathrm{Aut}(Y/X) \arrow[ur, dashed, "\exists!\,\widetilde{f}"'] & \end{tikzcd} \tag{2.4.26}$$

where $q : Y \to Y/\,\mathrm{Aut}(Y/X)$ is the quotient map.

Definition 2.4.27 (Galois cover)**.** A covering map $f : Y \to X$ is said to be a *Galois cover* of X if Y is path-connected and the continuous map $\widetilde{f} : Y/\,\mathrm{Aut}(Y/X) \to X$ in (2.4.25), induced by f, is a homeomorphism (see the diagram (2.4.26)).

Proposition 2.4.28. *A connected covering map $p : Y \to X$ is Galois if and only if* $\mathrm{Aut}(Y/X)$ *acts transitively on each fiber of the covering map p.*

Proof. Suppose that $p : Y \to X$ is Galois cover. Consider the commutative diagram.

$$\begin{tikzcd} Y \arrow[rr, "p"] \arrow[dr, "q"'] & & X \\ & Y/\,\mathrm{Aut}(Y/X) \arrow[ur, "\widetilde{p}"'] & \end{tikzcd}$$

Since the induced map $\widetilde{p} : Y/\,\mathrm{Aut}(Y/X) \to X$ is a homeomorphism (by definition), for each $x \in X$, the fiber $p^{-1}(x)$ coincides with the $\mathrm{Aut}(Y/X)$-orbit of a point of the fiber $p^{-1}(x)$. In other words, the $\mathrm{Aut}(Y/X)$-action on each of the fibers of p is transitive.

Conversely, if the $\operatorname{Aut}(Y/X)$-action on each of the fibers of p is transitive, then the induced continuous map $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is bijective. Therefore, to show that $p : Y \to X$ a Galois cover, it suffices to show that $\widetilde{p}$ is an open map. Let $U \subseteq Y/\operatorname{Aut}(Y/X)$ be an open subset. Since the quotient map $q : Y \to Y/\operatorname{Aut}(Y/X)$ is continuous, $q^{-1}(U)$ is open in Y. Since the covering map $p : Y \to X$ is an open map, $p(q^{-1}(U))$ is open in X. Since q is surjective, we have $q(q^{-1}(U)) = U$. Since $p = \widetilde{p} \circ q$, we have

$$\widetilde{p}(U) = \widetilde{p}(q(q^{-1}(U))) = p(q^{-1}(U)).$$

Therefore, $\widetilde{p}(U)$ is open in X. This completes the proof. □

If Y is simply connected, as remarked above, the $\operatorname{Aut}(Y/X)$-orbit of y_0 is precisely the fiber $f^{-1}(x_0)$, for all $y_0 \in f^{-1}(x_0)$. Therefore, in that case, the map $\widetilde{f}$ is bijective. This leads to the following.

Corollary 2.4.29. *A simply-connected covering map $p : \widetilde{X} \to X$ is Galois cover.*

Proof. $\widetilde{X}$ being simply connected, $\operatorname{Aut}(\widetilde{X}/X)$ acts transitively on each fiber of p by Proposition 2.4.18. Therefore, the result follows from Proposition 2.4.28. □

Remark 2.4.30. If $p : Y \to X$ is a covering map with Y connected, then to show $p : Y \to X$ is a Galois cover it suffices to show that $\operatorname{Aut}(Y/X)$ acts transitively on one fibre. Indeed, since in this case $Y/\operatorname{Aut}(Y/X)$ is a connected cover of X where one of the fibres is singleton, it follows that $\widetilde{p} : Y/\operatorname{Aut}(Y/X) \to X$ is a homeomorphism.

2.4.6 Galois correspondence for covering spaces

Theorem 2.4.31. *Let $p : Y \to X$ be a Galois cover. For each subgroup H of the Galois group $G := \operatorname{Aut}(Y/X)$, the projection map p induces a natural continuous map $\widetilde{p}_H : Y/H \to X$ which is a covering map. Conversely, if $f : Z \to X$ is a connected cover of X fitting into a commutative diagram*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "p"'] & & Z \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

then $\phi : Y \to Z$ is a Galois cover and Z is homeomorphic to Y/H. The maps $H \mapsto Y/H$ and $Z \mapsto \operatorname{Aut}(Y/Z)$ induces a natural one-to-one correspondence between the collection of subgroups of G and the intermediate covers of $p : Y \to X$ as above. Moreover, the cover $f : Z := Y/H \to X$ is Galois if and only if H is a normal subgroup of G; and in this case we have $\operatorname{Aut}(Z/X) \cong G/H$.

2.4.7 Monodromy action

Chapter 3

Differential Topology

3.1 Linear algebraic preliminaries

Fix an integer $n \geq 1$. Recall that the set $\mathbb{R}^n$ together with component-wise addition and scalar multiplication

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) := (x_1 + y_1, \dots, x_n + y_n)$$
$$\lambda \cdot (x_1, \dots, x_n) := (\lambda x_1, \dots, \lambda x_n)$$

is a $\mathbb{R}$-vector space. For each $i \in \{1, \dots, n\}$, let

$$p_i : \mathbb{R}^n \to \mathbb{R}, \ \ (x_1, \dots, x_n) \mapsto x_i, \tag{3.1.1}$$

be the projection map onto the i-th factor, and let $e_i \in \mathbb{R}^n$ be the point whose i-th component is 1, and all other components are 0. More formally, $e_i \in \mathbb{R}^n$ can be defined by the formula

$$p_j(e_i) = \delta_{i,j} := \begin{cases} 1, & \text{if} \quad j = i, \\ 0, & \text{if} \quad j \neq i, \end{cases} \quad \forall\, j \in \{1, \dots, n\}. \tag{3.1.2}$$

Then the subset $\{e_1, \dots, e_n\} \subseteq \mathbb{R}^n$ forms a basis for the $\mathbb{R}$-vector space $\mathbb{R}^n$, known as the *standard ordered basis* for $\mathbb{R}^n$. Thus we see that $\dim_{\mathbb{R}}(\mathbb{R}^n) = n$.

Given two integers $m, n \geq 1$, let

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) := \{f : \mathbb{R}^n \to \mathbb{R}^m \mid f \text{ is a } \mathbb{R}\text{-linear map}\}$$

be the set of all $\mathbb{R}$-linear maps of $\mathbb{R}^n$ into $\mathbb{R}^m$. Given $f, g \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ and $\alpha \in \mathbb{R}$, we define two maps

$$f + g,\ \alpha \cdot f : \mathbb{R}^n \to \mathbb{R}^m$$

by the formulae: $\forall\, x \in \mathbb{R}^n$,

$$(f + g)(x) := f(x) + g(x),$$
$$\text{and } (\alpha \cdot f)(x) := \alpha \cdot f(x).$$

It is easy to see that $f+g$ and $\alpha \cdot f$ are $\mathbb{R}$-linear maps, and hence are elements of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, the above two operations makes $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ a $\mathbb{R}$-vector space. Let $\mathrm{M}_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with entries from $\mathbb{R}$. Note that, $\mathrm{M}_{m\times n}(\mathbb{R})$ is a $\mathbb{R}$-vector space with respect to the addition of matrices and scalar multiplication of matrices with real numbers. Then we have a natural isomorphism of $\mathbb{R}$-vector spaces

$$mat : \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \to \mathrm{M}_{m\times n}(\mathbb{R}) \tag{3.1.3}$$

defined by sending a $\mathbb{R}$-linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ to its matrix representation with respect to the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A \in \mathrm{M}_{m\times n}(\mathbb{R})$ be the matrix whose (i,j)-th entry is $a_{ij} \in \mathbb{R}$. If $A_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$ denotes the i-th row vector of A, sending A to the mn-tuple of real numbers

$$(A_1, \ldots, A_m) := (a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn}) \in \mathbb{R}^{mn}$$

we have a natural bijective map $\mathrm{M}_{m\times n}(\mathbb{R}) \to \mathbb{R}^{mn}$, which is an isomorphism of $\mathbb{R}$-vector spaces. Thus, we see that the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ gives us natural isomorphisms of $\mathbb{R}$-vector spaces

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \to \mathrm{M}_{m\times n}(\mathbb{R}) \to \mathbb{R}^{mn}. \tag{3.1.4}$$

In particular, $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ is a $\mathbb{R}$-vector space of dimension mn.

3.2 Review of $\mathbb{R}^n$ Calculus

3.2.1 Topological preliminaries

Recall that the *Euclidean norm* of a vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is the non-negative real number $\|a\|$ defined by

$$\|a\| := \sqrt{a_1^2 + \cdots + a_n^2}. \tag{3.2.1}$$

The Euclidean norm induces a metric on $\mathbb{R}^n$, known as the *Euclidean metric*, which is defined as follow: given two points (vectors) $x, y \in \mathbb{R}^n$, the Euclidean distance between them is the non-negative real number

$$d(x,y) := \|x - y\|,$$

the Euclidean norm of the vector $x - y \in \mathbb{R}^n$. More precisely, if $x := (x_1, \ldots, x_n)$ and $y := (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, then $d(x,y)$ is the non-negative real number

$$\|(x_1, \ldots, x_n) - (y_1, \ldots, y_n)\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}. \tag{3.2.2}$$

It is easy to check that the Euclidean distance between points, as defined above, is a metric on $\mathbb{R}^n$. Thus, $\mathbb{R}^n$ has a structure of a normed linear space and hence of a metric space. Given a point $a \in \mathbb{R}^n$ and a real number $r > 0$, we define the *open ball* in $\mathbb{R}^n$ with centre at a and radius

r to be the subset

$$B(a,r) := \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

The collection $\{B(a,r) : a \in \mathbb{R}^n, r > 0\}$ forms a basis for a topology on $\mathbb{R}^n$, known as the *Euclidean topology on* $\mathbb{R}^n$. One can easily check that the Euclidean topology on $\mathbb{R}^n$ coincides with the product topology on it induced from the Euclidean topology on $\mathbb{R}$.

Recall that the *closure* of a subset A in a topological space X is the smallest closed subset $\overline{A}$ containing A; this can be constructed as the intersection of all closed subsets of X containing the given subset A, i.e.,

$$\overline{A} = \bigcap_{Z \in C_A} Z,$$

where C_A is the collection of all closed subsets of X that contains A. With this definition, check that the closure of $B(a,r)$ in $\mathbb{R}^n$ is the subset

$$\overline{B(a,r)} = \{x \in \mathbb{R}^n : \|x - a\| \leq r\},$$

known as the *closed ball in* $\mathbb{R}^n$ *with centre at a and radius r*. Let A be a non-empty subset of $\mathbb{R}^n$.

Definition 3.2.3. A function $f : A \to \mathbb{R}^m$ is said to be *continuous* at $a \in A$ if for given a real number $\epsilon > 0$, there exists a real number $\delta > 0$ (depending on both a and ϵ) such that for any $x \in A$ with $\|x - a\| < \delta$, we have $\|f(x) - f(a)\| < \epsilon$. f is said to be continuous on A if f is continuous at each point of A.

Let A be a non-empty subset of $\mathbb{R}^n$, and let $a \in \overline{A}$. Let $f : A \to \mathbb{R}^m$ be a map. A vector $v \in \mathbb{R}^m$ is said to be a *limit* of f at a, denoted by the symbol $\lim\limits_{x \to a} f(x)$, if for each real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that

$$f\left(\left(B(a,\delta) \setminus \{a\}\right) \cap A\right) \subseteq B(v,\epsilon).$$

Exercise 3.2.4. If $a \in A$, show that f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Proposition 3.2.5. *A function $f : A \to \mathbb{R}^m$ is continuous if and only if the inverse image $f^{-1}(U)$ of any open subset $U \subseteq \mathbb{R}^n$ is open in A.*

Proof. Suppose that f is continuous on A. Let $U \subseteq \mathbb{R}^n$ be an open subset. To show $f^{-1}(U)$ open in A, let $a \in f^{-1}(U)$ be arbitrary but fixed after choice. Since U is open subset of $\mathbb{R}^m$ containing $f(a)$, there is a real number $\epsilon > 0$ such that the open ball $B(f(a),\epsilon) \subseteq U$. Since f is continuous at a, there is a real number $\delta > 0$ such that for any point $x \in B(a,\delta) \cap A$, we have $f(x) \in B(f(a),\epsilon)$. In other words, $f(B(a,\delta) \cap A) \subseteq B(f(a),\epsilon)$. Thus $B(a,\delta) \subseteq f^{-1}(U)$ is an open subset containing a. Since $a \in f^{-1}(U)$ is arbitrary, $f^{-1}(U)$ is open in A.

Conversely, suppose that $f^{-1}(U)$ is open in A for any open subset $U \subseteq \mathbb{R}^n$. Fix $a \in A$. Let $\epsilon > 0$ be given. Since $U := B(f(a),\epsilon)$ is open subset of $\mathbb{R}^m$ containing $f(a)$, its inverse image $f^{-1}(U)$ is an open subset of A containing a. Then there is a real number $\delta > 0$ such that $B(a,\delta) \cap A \subseteq f^{-1}(U)$. Consequently, $f(B(a,\delta) \cap A) \subseteq B(f(a),\epsilon)$. Therefore, fa is continuous at a. Since $a \in A$ is arbitrary, f is continuous on A. ☐

Fix two positive integers m, n. For each $j \in \{1, \dots, n\}$, the *projection map onto the j-th factor* is the map $p_j : \mathbb{R}^n \to \mathbb{R}$ defined by $p_j(x_1, \dots, x_n) := x_j$, for all $(x_1, \dots, x_n) \in \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^m$. Given a map $f : A \to \mathbb{R}^n$, its j-th component map is the composition $f_j := p_j \circ f$,

$$f_j : A \xrightarrow{f} \mathbb{R}^n \xrightarrow{p_j} \mathbb{R}, \ \forall\, j = 1, \dots, n.$$

Exercise 3.2.6. Show that $f : A \to \mathbb{R}^m$ is continuous if and only if its j-th component function $f_j := p_j \circ f : A \to \mathbb{R}$ is continuous, for all $j = 1, \dots, m$.

3.2.2 Differentiation of functions in higher dimensions

Let $A \subseteq \mathbb{R}$, and $a \in A$ be an interior point of A. A map $f : A \to \mathbb{R}$ is said to be *differentiable at $a \in A$* if there is a real number $Df(a) \in \mathbb{R}$ such that

$$Df(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}; \tag{3.2.7}$$

in this case $Df(a)$ is called the *derivative of f at a*. In many texts, $Df(a)$ is also denoted by $df(a)$ or $f'(a)$. Note that, the above equation is equivalent to

$$\lim_{h \to 0} \frac{f(a+h) - [f(a) + Df(a)h]}{h} = 0. \tag{3.2.8}$$

Note that, the map

$$h \mapsto Df(a)h$$

is a $\mathbb{R}$-linear map from $\mathbb{R}$ to $\mathbb{R}$, whose matrix representation is the 1×1 matrix $(Df(a))$. Therefore, the above equation says that, in a sufficiently small open neighbourhood of a in A, the map $f : A \to \mathbb{R}$ can be approximated by the unique *affine transformation** defined by

$$h \mapsto f(a) + Df(a)(h),$$

where $Df(a) : \mathbb{R} \to \mathbb{R}$ is the $\mathbb{R}$-linear map defined by

$$h \mapsto Df(a) \cdot h.$$

This gives a way to extend the notion of differentiability of a function of in higher dimensional Euclidean spaces. Here is a formal definition.

Definition 3.2.9. Let $A \subseteq \mathbb{R}^n$ and $a \in A$ an interior point of A. A map $f : A \to \mathbb{R}^m$ is said to be *differentiable at a* if there is a $\mathbb{R}$-linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - Df(a)(h)\|}{\|h\|} = 0, \tag{3.2.10}$$

where the norms in the numerator and the denominator are the Euclidean norms in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

*Given two vector spaces V and W, an *affine transformation* from V to W is a map of the form $w_0 + T$, where $T : V \to W$ is a linear map and $w_0 \in W$.

The next Proposition 3.2.12 says that the $\mathbb{R}$-linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ in Definition 3.2.9 is unique. We use the following.

Exercise 3.2.11. For any real number $\delta > 0$, show that $B(0, \delta) \subset \mathbb{R}^n$ contains a basis for the $\mathbb{R}$-vector space $\mathbb{R}^n$.

Proposition 3.2.12. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there is a unique $\mathbb{R}$-linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - Df(a)(h)\|}{\|h\|} = 0.$$

Proof. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any $\mathbb{R}$-linear map such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0.$$

Since $a \in A$ is an interior point, there is a real number $\delta > 0$ such that $B(a, \delta) \subseteq A$. Given a real number $\epsilon > 0$, $\exists$ positive real numbers $\delta_1, \delta_2 > 0$ such that

$$\frac{\|f(a+h) - f(a) - Df(a)(h)\|}{\|h\|} < \epsilon/2, \tag{3.2.13}$$

$$\text{and} \quad \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} < \epsilon/2. \tag{3.2.14}$$

Take $\delta' := \min\{\delta_1, \delta_2, \delta\}$. Then for any $h \in B(0, \delta') \setminus \{0\} \subset \mathbb{R}^n$ we have

$$\begin{aligned}
0 &\leq \frac{\|T(h) - Df(a)(h)\|}{\|h\|} \\
&= \frac{\|\,[f(a+h) - f(a) - Df(a)(h)] - [f(a+h) - f(a) - T(h)]\,\|}{\|h\|} \\
&\leq \frac{\|f(a+h) - f(a) - Df(a)(h)\|}{\|h\|} + \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{h \to 0} \frac{\|T(h) - Df(a)(h)\|}{\|h\|} = 0.$$

Fix a non-zero element $h_0 \in B(0, \delta') \setminus \{0\}$. Then we have

$$\begin{aligned}
0 &= \lim_{t \to 0} \frac{\|T(th_0) - Df(a)(th_0)\|}{\|th_0\|} \\
&= \lim_{t \to 0} \frac{|t| \cdot \|T(h_0) - Df(a)(h_0)\|}{|t| \cdot \|h_0\|} \\
&= \lim_{t \to 0} \frac{\|T(h_0) - Df(a)(h_0)\|}{\|h_0\|} \\
&= \frac{\|T(h_0) - Df(a)(h_0)\|}{\|h_0\|}.
\end{aligned}$$

Since $h_0 \neq 0$, we conclude that $T(h) = Df(a)(h),\ \forall\, h \in B(0,\delta') \setminus \{0\}$. Then by Exercise 3.2.11 we have

$$T(h) = Df(a)(h),\ \ \forall\, h \in \mathbb{R}^n.$$

This completes the proof. □

Note that, the unique $\mathbb{R}$-linear map $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ in Definition 3.2.9 is the *best* $\mathbb{R}$*-linear approximation of* f *at* $a \in A$, and is called the *derivative of* f *at* a. The matrix representation of $Df(a)$ with respect to the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ is a $m \times n$ matrix with coefficients from $\mathbb{R}$, called the *Jacobian of* f *at* a. If $A \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, a map $f : A \to \mathbb{R}^m$ is said to be *differentiable on* A if it is differentiable at each point of A. When $A \subseteq \mathbb{R}^n$ is not necessarily open, $f : A \to \mathbb{R}^m$ is said to be *differentiable on* A if it can be extended to a differentiable function on an open neighbourhood of A in $\mathbb{R}^n$.

Proposition 3.2.15. *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a* $\mathbb{R}$*-linear map. Then there exists a positive real number* $M \in \mathbb{R}$ *such that* $\|T(x)\| \leq M \cdot \|x\|$, $\forall\, x \in \mathbb{R}^n$.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard ordered basis for $\mathbb{R}^n$, and for each $i \in \{1, \ldots, n\}$, let $w_i := T(e_i) \in \mathbb{R}^m$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $\|x\| = 1$. Since $T(x) = x_1 w_1 + \cdots + x_n w_n$, by Cauchy-Schwarz inequality we have

$$\|T(x)\| \leq \sum_{j=1}^{n} |x_j| \cdot \|w_j\| \leq \|x\|^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n} \|w_j\|\right)^{1/2} = \left(\sum_{j=1}^{n} \|w_j\|\right)^{1/2}.$$

Therefore, the subset $S := \{\|T(x)\| \,:\, x \in \mathbb{R}^n \text{ with } \|x\| = 1\} \subseteq \mathbb{R}$ is bounded above, and hence $\|T\| := \sup(S)$ exists in $\mathbb{R}$. Since for any non-zero $x \in \mathbb{R}^n$ we have $\|(x/\|x\|)\| = 1$, it follows that $\|T(x)\| \leq \|T\| \cdot \|x\|$, for all $x \in \mathbb{R}^n$. □

Exercise 3.2.16. Let $A \subseteq \mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be differentiable at an interior point $a \in A$. Show that f is continuous at a.

Theorem 3.2.17 (Chain rule)**.** *Let* $A \subseteq \mathbb{R}^n$ *and* $B \subseteq \mathbb{R}^m$. *If* $f : A \to \mathbb{R}^n$ *is differentiable at* $a \in A$, *with* $f(A) \subseteq B$, *and if* $g : B \to \mathbb{R}^p$ *is differentiable at* $f(a)$, *then* $g \circ f : A \to \mathbb{R}^p$ *is differentiable at* a, *and* $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$.

Proof. For notational simplicity, let's introduce the following.

$$\varphi(x) = f(x) - f(a) - Df(a)(x-a), \tag{3.2.18}$$

$$\psi(y) = g(y) - g(b) - Dg(b)(y-b), \tag{3.2.19}$$

$$\text{and}\quad \rho(x) = (g \circ f)(x) - (g \circ f)(a) - \big(Dg(b) \circ Df(a)\big)(x-a). \tag{3.2.20}$$

Since f and g are differentiable at a and $b = f(a)$, respectively, we have

$$\lim_{x \to a} \frac{\|\varphi(x)\|}{\|x-a\|} = 0, \tag{3.2.21}$$

$$\text{and}\quad \lim_{y \to b} \frac{\|\psi(y)\|}{\|y-b\|} = 0. \tag{3.2.22}$$

Since

$$\begin{aligned}\rho(x) =&(g\circ f)(x) - (g\circ f)(a) - \big(Dg(b)\circ Df(a)\big)(x-a)\\ =& [g(y)-g(b)-Dg(b)(y-b)] + Dg(b)\big(y-b-Df(a)(x-a)\big)\\ =&\psi(y) + Dg(b)(f(x)-f(a)-Df(a)(x-a))\\ =&\psi(f(x)) + Dg(b)(\varphi(x)),\end{aligned}$$

it suffices to show that

$$\lim_{x\to a}\frac{\|\psi(f(x))\|}{\|x-a|} = 0, \tag{3.2.23}$$

$$\text{and} \quad \lim_{x\to a}\frac{\|Dg(b)(\varphi(x))\|}{\|x-a|} = 0. \tag{3.2.24}$$

Now it follows from (3.2.22) that given a real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that if $\|f(x)-b\| < \delta$, we have

$$\begin{aligned}\|\psi(f(x))\|\epsilon \le \|f(x)-f(a)\|&\\ &= \epsilon\|\varphi(x) + Df(a)(x-a)\|, \text{ by equation 3.2.21.}\\ &\le \epsilon\|\varphi(x)\| + \epsilon\|Df(a)\|\cdot\|x-a\|, \text{ by Proposition 3.2.15.}\\ &\le \epsilon^2\|x-a\| + \epsilon\|Df(a)\|\cdot\|x-a\|\\ &= \epsilon\big(\epsilon + \|Df(a)\|\big)\|x-a\|.\end{aligned}$$

From this, the equation (3.2.23) follows. And the equation (3.2.24) follows from the inequality

$$\|Dg(b)(\varphi(x))\| \le \|Dg(b)\|\cdot|\varphi(x)\|, \ \forall\, x,$$

and the equation (3.2.21). □

Theorem 3.2.25. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$.*

(i) If f is a constant map, then it is differentiable on $\mathbb{R}^n$ with $Df(a) = 0$, for all $a \in \mathbb{R}^n$.

(ii) If f is a $\mathbb{R}$-linear map, then f is differentiable on $\mathbb{R}^n$ with $Df(a) = f$, for all $a \in \mathbb{R}^n$.

(iii) For each $j \in \{1,\ldots,m\}$, let $f_j := p_j\circ f : \mathbb{R}^n \to \mathbb{R}$ be the j-th component map of f. Then f is differentiable at $a \in \mathbb{R}^n$ if and only if f_j is differentiable at a, for all $j \in \{1,\ldots,m\}$. And in this case, we have $Df(a) = (Df_1(a),\ldots,Df_m(a))$, for all $a \in \mathbb{R}^n$.

(iv) If $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x_1,\ldots,x_n) = \sum\limits_{j=1}^{n} x_j$, for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$, then f is differentiable on $\mathbb{R}^n$ with $Df(a) = f$, for all $a \in \mathbb{R}^n$.

(v) If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = xy$, for all $(x,y) \in \mathbb{R}^2$, then f is differentiable on $\mathbb{R}^2$, and its derivative $Df(a,b) : \mathbb{R}^2 \to \mathbb{R}$ at $(a,b) \in \mathbb{R}^2$ is given by

$$Df(a,b)(x,y) = bx + ay, \ \forall\,(x,y) \in \mathbb{R}^2.$$

Proof. (i) Note that, if f is a constant function, then

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-0\|}{\|h\|}=0.$$

(ii) If $f:\mathbb{R}^n\to\mathbb{R}^m$ is a $\mathbb{R}$-linear map, then

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-f(h)\|}{\|h\|}=\lim_{h\to 0}\frac{\|f(a)+f(h)-f(a)-f(h)\|}{\|h\|}=0.$$

(iii) Since the projection maps $p_i:\mathbb{R}^m\to\mathbb{R}$ are $\mathbb{R}$-linear, they are differentiable with derivative $Dp_i(b)=p_i$, for all $b\in\mathbb{R}^m$. If f is differentiable at $a\in\mathbb{R}^n$, by chain rule (Theorem 3.2.17) the component functions $f_i:=p_i\circ f$ are differentiable at a with

$$Df_i(a)=Dp_i(f(a))\circ Df(a)=p_i\circ Df(a),\ \ \forall\, i\in\{1,\ldots,m\}.$$

Conversely suppose that the component maps $f_i:\mathbb{R}^n\to\mathbb{R}$ are differentiable at $a\in\mathbb{R}^n$. Let $T_a:\mathbb{R}^n\to\mathbb{R}^m$ be the $\mathbb{R}$-linear map defined by

$$T_a(x)=(Df_1(a)(x),\ldots,Df_m(a)(x)),\ \ \forall\, x\in\mathbb{R}^n.$$

Since $f(x)=(f_1(x),\ldots,f_m(x)),\ \forall\, x\in\mathbb{R}^n$, and

$$\begin{aligned}&f(a+h)-f(a)-T_a(h)\\ &=(f_1(a+h)-f_1(h)-T_a(h),\ldots,f_m(a+h)-f_m(a)-T_a(h)),\end{aligned}\tag{3.2.26}$$

we have

$$\begin{aligned}&\lim_{h\to 0}\frac{\|f(a+h)-f(a)-T_a(h)\|}{\|h\|}\\ &\le\lim_{h\to 0}\sum_{i=1}^{m}\frac{|f_i(a+h)-f_i(a)-Df_i(a)(h)|}{\|h\|}=0.\end{aligned}$$

Therefore, f is differentiable at a with $Df(a)=(Df_1(a),\ldots,Df_m(a))$.

(iv) Since $f(x_1,\ldots,x_n):=\sum\limits_{i=1}^{n}x_i$ is a $\mathbb{R}$-linear map (verify!), it follows that f is differentiable on $\mathbb{R}^n$ with $Df(a)=f$, for all $a\in\mathbb{R}^n$.

(v) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=xy$, for all $(x,y)\in\mathbb{R}^2$. Fix $(a,b)\in\mathbb{R}^2$, and consider the map $T_{a,b}:\mathbb{R}^2\to\mathbb{R}$ defined by

$$T_{a,b}(x,y)=bx+ay,\ \ \forall\,(x,y)\in\mathbb{R}^2.$$

Then we have,

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{\|f(a+h,\, b+k) - f(a,\, b) - T_{a,b}(h,\, k)\|}{\|(h,\, k)\|} \\
&= \lim_{(h,k)\to(0,0)} \frac{\|(a+h)(b+k) - ab - bh - ak\|}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}}.
\end{aligned}$$

Since

$$|hk| \leq \begin{cases} |h^2|, & \text{if} \quad |k| \leq |h|, \\ |k^2|, & \text{if} \quad |h| \leq |k|, \end{cases}$$

we have $\displaystyle\lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} \leq \lim_{(h,k)\to(0,0)} \frac{h^2+k^2}{\sqrt{h^2+k^2}} = 0$. Hence the result follows.

□

Given $f, g : \mathbb{R}^n \to \mathbb{R}$, we define $f + g, f \cdot g : \mathbb{R}^n \to \mathbb{R}$ by the formulae

$$\begin{aligned}
(f+g)(x) &:= f(x) + g(x), \;\; \forall\, x \in \mathbb{R}^n, \\
(f \cdot g)(x) &:= f(x) \cdot g(x), \;\; \forall\, x \in \mathbb{R}^n.
\end{aligned}$$

If g is continuous at $a \in \mathbb{R}^n$ with $g(a) \neq 0$, then there is an open neighbourhood $V_a \subseteq \mathbb{R}^n$ of a with $g(x) \neq 0$, for all $x \in V_a$ (verify!). Therefore, we can define a function $f/g : V_a \to \mathbb{R}$ by $(f/g)(x) = \frac{f(x)}{g(x)}$, for all $x \in V_a$. Then we have the following.

Corollary 3.2.27. *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $a \in \mathbb{R}^n$, so are $f + g$ and $f \cdot g$; and in this case*

$$\begin{aligned}
D(f+g)(a) &= Df(a) + Dg(a), \\
D(f \cdot g)(a) &= g(a)Df(a) + f(a)Dg(a).
\end{aligned}$$

Moreover, if $g(a) \neq 0$, then f/g is differentiable at a with

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}.$$

3.2.3 Partial derivatives

Let $A \subseteq \mathbb{R}^n$, and let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be an interior point of A. For each $j \in \{1, \ldots, n\}$, consider the embedding $\varphi_{a,j} : \mathbb{R} \hookrightarrow \mathbb{R}^n$ defined by

$$p_i(\varphi_{a,j}(x)) = \begin{cases} x, & \text{if} \quad i = j, \\ a_i, & \text{if} \quad i \neq j, \end{cases}$$

where $p_i : \mathbb{R}^n \to \mathbb{R}$ is the projection map onto the i-th factor, for all $i \in \{1, \ldots, n\}$. In other words,

$$\varphi_{a,j}(x) = (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n), \;\; \forall\, x \in \mathbb{R}. \tag{3.2.28}$$

Let $A_j := \varphi_{a,j}^{-1}(A) \subseteq \mathbb{R}$. Given a map $f : A \to \mathbb{R}$, note that the composite map

$$f \circ \varphi_{a,j} : A_j (\subseteq \mathbb{R}) \to \mathbb{R}$$

is a function from a subset of $\mathbb{R}$ into $\mathbb{R}$. So we can talk about its differentiability at a_j as in one dimensional case. If

$$\lim_{h \to 0} \frac{(f \circ \varphi_{a,j})(a_j + h) - (f \circ \varphi_{a,j})(a_j)}{h} \tag{3.2.29}$$

exists in $\mathbb{R}$, we denote it by $D_j f(a)$, and call it the *j-th partial derivative of f at a*. Therefore, $D_j f(a) = (f \circ \varphi_{a,j})'(a_j)$, the usual derivative of $f \circ \varphi_{a,j}$ at a_j.

Proposition 3.2.30. *If $f : A \to \mathbb{R}$ is differentiable at an interior point a of A, then its j-th partial derivative $D_j f(a)$ exists, and is equal to the j-th factor of $Df(a)$, for all $j \in \{1, \ldots, n\}$.*

Proof. Since $\varphi_{a,j}$ is differentiable on $A_j := \varphi_{a,j}^{-1}(A) \subseteq \mathbb{R}$ by Theorem 3.2.25 (iii) and $\varphi_{a,j}(a_j) = a$, by chain rule of differentiation (see Theorem 3.2.17) we see that the composite map $f \circ \varphi_{a,j}$ is differentiable at a_j whenever f is differentiable at a, and in this case, $D(f \circ \varphi_{a,j})(a_j) = Df(a) \circ D\varphi_{a,j}(a_j)$. Since $D\varphi_{a,j}(a_j) = (0, \ldots, 0, \mathrm{Id}_{\mathbb{R}}, 0, \ldots, 0)$, where the identity map $\mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ appears at the j-th component of $D\varphi_{a,j}(a_j)$, and all other components of $D\varphi_{a,j}(a_j)$ are the zero maps, it follows that $D_j f(a)$ is the j-th component of $Df(a)$. $\square$

Let $A \subseteq \mathbb{R}^n$ be an open subset, and $f : A \to \mathbb{R}$. If $D_i f(x)$ exists, for all $x \in A$, we have a function

$$D_i f : A \to \mathbb{R},\ x \mapsto D_i f(x). \tag{3.2.31}$$

If the j-th partial derivative of this map $D_i f$ in (3.2.31) exists at $x \in A$, we denote it by $D_{j,i} f(x) := D_j(D_i f)(x)$, and call it a *second order mixed partial derivative of f at x*. Similarly, we can define *higher order mixed partial derivatives of f at a point of A*. The following example shows that, in general, $D_{j,i} f(x)$ need not be equal to $D_{i,j} f(x)$, for $i \neq j$.

Example 3.2.32. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} xy \cdot \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } \ (x,y) \neq (0,0), \text{ and} \\ 0, & \text{if } \ (x,y) = (0,0). \end{cases}$$

Note that, $D_2 f(x, 0) = x$, $\forall\, x \in \mathbb{R}$, and $D_1 f(0, y) = -y$, $\forall\, y \in \mathbb{R}$ (verify!). Consequently, $D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$.

However, the following theorem says that we could have $D_{i,j} f(a) = D_{j,i} f(a)$ under certain assumptions.

Theorem 3.2.33. *Fix an interior point $a \in A$, and two integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. If $D_{i,j} f$ and $D_{j,i} f$ exist and are continuous in an open subset of A containing a, then $D_{i,j} f(a) = D_{j,i} f(a)$.*

Definition 3.2.34. Let $A \subseteq \mathbb{R}$, and $a \in A$ is an interior point. We say that a map $f : A \to \mathbb{R}$ admits a *local maximum* (or, *local minimum*) at a if there is an open subset $V_a \subseteq A$ with $a \in V_a$ such that $f(x) \leq f(a)$, $\forall\, x \in V_a$ (resp., $f(x) \geq f(a)$, $\forall\, x \in V_a$).

Proposition 3.2.35. *Let $A \subseteq \mathbb{R}^n$, and $a \in A$ an interior point of A. If $f : A \to \mathbb{R}$ admits a local maximum (or, local minimum) at a in A, and if $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. Let $g_i(x) = (f \circ \varphi_{i,a})(x)$, where $\varphi_{a,i}$ is the map defined in (3.2.28). If f admits a local maximum (resp., local minimum) at a, then g_i admits a local maximum (resp., local minimum) at a. Then we have $Dg_i(a) = 0$. Since $D_i f(a) = Dg_i(a)$ by definition, the result follows. □

Definition 3.2.36. A function $f : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$ is said to be a C^∞ *function* if f has continuous partial derivatives of all orders.

Theorem 3.2.37. *Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Let $a \in A$ be an interior point. If f is differentiable at a, then all partial derivatives $D_j f^i(a)$ of the component functions $f_i := p_i \circ f$ exists, and the Jacobian matrix of f at a is given by*

$$Df(a) = \begin{pmatrix} D_1 f_1(a) & \cdots & D_n f_m(a) \\ \vdots & \ddots & \vdots \\ D_1 f_m(a) & \cdots & D_n f_m(a) \end{pmatrix}. \tag{3.2.38}$$

Proof. □

Definition 3.2.39. Let $A \subseteq \mathbb{R}^n$ and $a \in A$ an interior point of A. A map $f : A \to \mathbb{R}^m$ is said to be *continuously differentiable at a* if there is an open neighbourhood $V_a \subseteq A$ of a such that $Df(x)$ exists for each $x \in V_a$ and the map $Df : V_a \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ defined by sending $x \in V_a$ to $Df(x) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

Theorem 3.2.40. *Let $A \subseteq \mathbb{R}^n$ and $a \in A$ an interior point of A. A map $f : A \to \mathbb{R}^m$ is continuously differentiable at $a \in A$ if and only if there is an open subset $V_a \subseteq A$ containing a such that for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the partial derivatives $D_j f_i(x)$ of the component functions f_i exists at each point $x \in V_a$, and $D_j f_i : V_a \to \mathbb{R}$ is continuous at a.*

Proof. If f is continuously differentiable at a, it follows from Theorem 3.2.37 all partial derivatives $D_j f_i$ of its component functions f_i of f exists in an open neighbourhood $V_a \subseteq A$ of a, and the functions $D_j f_i : V_a \to \mathbb{R}$ are continuous. Conversely, suppose that there is an open neighbourhood $V_a \subseteq A$ of a such that for all i and j, the partial derivatives $D_j f_i(x)$ exists, for all $x \in V_a$, and the functions $D_j f_i : V_a \to \mathbb{R}$ are continuous.

□

3.2.4 Directional derivatives

3.2.5 Sheaf of C^k functions

Let X be a topological space. Let τ_X be the category whose objects are open subsets of X, and morphisms are given by inclusion maps: let $U, V \subseteq X$ be open subsets of X; if $V \subseteq U$ then we define $\mathrm{Mor}_{\tau_X}(V, U) := \{\iota_{V,U}\}$, the singleton set containing the inclusion map $\iota_{V,U} : V \hookrightarrow U$, otherwise we define $\mathrm{Mor}_{\tau_X}(V, U) = \emptyset$. In other words,

$$\mathrm{Mor}_{\tau_X}(U, V) = \begin{cases} \{\iota_{V,U}\}, & \text{if} \quad \iota_{V,U} : V \subseteq U, \\ \emptyset, & \text{if} \quad V \nsubseteq U. \end{cases}$$

Consider the Euclidean space $\mathbb{R}^n$. Given an open subset $U \subseteq \mathbb{R}^n$, let

$$C^k_{\mathbb{R}^n}(U) := \{f : U \to \mathbb{R} \,|\, f \text{ is a } C^k \text{ function}\}.$$

Given any two elements $f, g \in C^k_{\mathbb{R}^n}$, consider the functions

$$f + g, f \cdot g : U \to \mathbb{R}$$

defined by

$$(f + g)(x) := f(x) + g(x), \;\; \forall\, x \in U,$$
$$(f \cdot g)(x) := f(x)g(x), \;\; \forall\, x \in U.$$

Clearly $f + g, f \cdot g \in C^k_{\mathbb{R}^n}(U)$. It is easy to check that, with these two operations, $C^k_{\mathbb{R}^n}$ is a commutative ring with identity 1 (the constant map $c_1 : U \to \mathbb{R}$ which sends all points of U to $1 \in \mathbb{R}$). Given any $\alpha \in \mathbb{R}$, and $f \in C^k_{\mathbb{R}^n}(U)$, we define

$$\alpha \cdot f : U \to \mathbb{R}, \;\; x \mapsto \alpha \cdot f(x). \tag{3.2.41}$$

Clearly $\alpha \cdot f \in C^k_{\mathbb{R}^n}(U)$.

Moreover, if $c_\alpha : U \to \mathbb{R}$ is the constant map that sends all points of U to $\alpha \in \mathbb{R}$, then

$$\alpha \cdot f = c_\alpha \cdot f, \;\; \forall\, f \in C^k_{\mathbb{R}^n}, \; \alpha \in \mathbb{R}. \tag{3.2.42}$$

Thus, $C^k_{\mathbb{R}^n}(U)$ has a structure of an $\mathbb{R}$-vector space compatible with its ring structure. Therefore, $C^k_{\mathbb{R}^n}(U)$ is a commutative $\mathbb{R}$-algebra with identity.

Given two open subsets $U, V \in \tau_{\mathbb{R}^n}$ with $V \subseteq U$, we have a restriction map

$$\mathrm{res}_{U,V} : C^k_{\mathbb{R}^n}(U) \to C^k_{\mathbb{R}^n}(V), \;\; f \mapsto f\big|_V, \tag{3.2.43}$$

where $f\big|_V : V \to \mathbb{R}$ is the restriction of $f : U \to \mathbb{R}$ on V. Clearly, $\mathrm{res}_{U,V}$ is a homomorphism of $\mathbb{R}$-algebras. Note that

(i) $\mathrm{res}_{U,U} = \mathrm{Id}_{C^k_{\mathbb{R}^n}(U)}$, for all $U \subseteq \tau_{\mathbb{R}^n}$, and

(ii) $\mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W}$, for all $U, V, W \in \tau_{\mathbb{R}^n}$ with $W \subseteq V \subseteq U$.

Therefore, if we denote by $\mathrm{Alg}_{\mathbb{R}}$ the category of all commutative $\mathbb{R}$-algebras, then the association

$$C^k_{\mathbb{R}^n} : \tau_{\mathbb{R}^n} \to \mathrm{Alg}_{\mathbb{R}}, \;\; U \mapsto C^k_{\mathbb{R}^n}(U), \tag{3.2.44}$$

is a contravariant functor.

Let X be a topological space. Let τ_X be the category whose objects are open subsets of X and morphisms between two objects are given by inclusion maps. Let $\mathrm{Alg}_{\mathbb{R}}$ be the category of $\mathbb{R}$-algebras. Its objects are $\mathbb{R}$-algebras and morphisms between two objects are $\mathbb{R}$-algebra homomorphisms.

Definition 3.2.45. A *presheaf* of $\mathbb{R}$-algebras on X is a contravariant functor

$$\mathcal{F} : \tau_X \to \mathrm{Alg}_{\mathbb{R}}. \tag{3.2.46}$$

A *sheaf* of $\mathbb{R}$-algebras on X is a presheaf of $\mathbb{R}$-algebras $\mathcal{F} : \tau_X \to \mathrm{Alg}_{\mathbb{R}}$ satisfying the following properties: If $\{U_\alpha\}_{\alpha\in\Lambda}$ is a collection of objects from τ_X with $U = \bigcup\limits_{\alpha\in\Lambda} U_\alpha$, then

(i) given any $f, g \in \mathcal{F}(U)$ with $f\big|_{U_\alpha} = g\big|_{U_\alpha}$, for all $\alpha \in \Lambda$, we have $f = g$.

(ii) given $f_\alpha \in \mathcal{F}(U_\alpha)$, for each $\alpha \in \Lambda$, if $f_\alpha\big|_{U_\alpha\cap U_\beta} = f_\beta\big|_{U_\alpha\cap U_\beta}$, for all $\alpha, \beta \in \Lambda$, there exists $f \in \mathcal{F}(U)$ such that $f\big|_{U_\alpha} = f_\alpha$, for all $\alpha \in \Lambda$.

Exercise 3.2.47. Show that $C^k_{\mathbb{R}^n}$ is a sheaf of $\mathbb{R}$-algebras on $\mathbb{R}^n$.

Exercise 3.2.48. Let $\mathcal{F} : \tau_X \to \mathrm{Alg}_{\mathbb{R}}$ be a sheaf of $\mathbb{R}$-algebras on X. Fix a point $x \in X$, and let $\mathcal{N}_x$ be the set of all open subsets of X containing x. Then the direct limit

$$\mathcal{F}_x := \varinjlim_{U\in\mathcal{N}_x} \mathcal{F}(U)$$

is a local $\mathbb{R}$-algebra, called the *stalk* of $\mathcal{F}$ at x.

We now give a description of its stalk at a point. Fix a point $a \in \mathbb{R}^n$. Consider the collection of all pairs (U, f), where $U \subseteq \mathbb{R}^n$ is an open subset containing a and $f : U \to \mathbb{R}$ is a differentiable function (resp., a C^k function). Two such pairs (U, f) and (V, g) are said to be equivalent if there is an open subset $W \subseteq U \cap V$ containing a such that $f\big|_W = g\big|_W$. Note that, this is an equivalence relation on the collection of all such pairs (U, f). The equivalence class of (U, f) is denoted by

$$\begin{aligned}\langle (U, f)\rangle := \{(U, f) :\ & U \text{ is an open neighbourhood of } a \\ & \text{and } f : U \to \mathbb{R} \text{ is a } C^k \text{ function}\}.\end{aligned}$$

Let C^k_a be the set of all equivalence classes of pairs (U, f). Define addition and multiplication operations on $C^k_{\mathbb{R}^n,a}$, and show that $C^k_{\mathbb{R}^n,a}$ is a local $\mathbb{R}$-algebra. Verify that $C^k_{\mathbb{R}^n,a} \cong \varinjlim_{a\in U} C^k_{\mathbb{R}^n}(U)$ as $\mathbb{R}$-algebras.

3.2.6 Inverse and Implicit function theorems

Lemma 3.2.49. *Let $A \subseteq \mathbb{R}^n$ be a convex open subset (for example, an open rectangle), and let $f : A \to \mathbb{R}^n$ be continuously differentiable on A. If there is a real number $M \geq 0$ such that $|D_j f^i(x)| \leq M$, for all $x \in A$, then*

$$\|f(x) - f(y)\| \leq n^2 M \|x - y\|, \ \ \forall\, x, y \in A.$$

Proof. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in A$ be arbitrary. Note that, for each $i \in \{1, \ldots, n\}$ we have

$$f_i(x) - f_i(y) = \sum_{j=1}^{n} \left[f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n) - f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) \right].$$

Then by mean value theorem for real valued differentiable functions of one variable, we have

$$f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n) - f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) = (y_j - x_j) \cdot D_j f_i(z_{ij}),$$

for some z_{ij} in the interior of A. Since $|D_j f^i(x)| \leq M$, for all i, j and $x \in \mathrm{Int}(A)$ by assumption, we have

$$\|f_i(y) - f_i(x)\| \leq \sum_{j=1}^{n} |y_j - x_j| \cdot M \leq nM\|y - x\|,$$

since $|y_j - x_j| \leq \|y - x\|$, for all $j = 1, \ldots, n$. Therefore, we have

$$\|f(y) - f(x)\| \leq \sum_{i=1}^{n} \|f_i(y) - f_i(x)\| \leq n^2 M \|y - x\|.$$

□

Theorem 3.2.50 (Inverse function theorem)**.** *Let $A \subseteq \mathbb{R}^n$, and $f : A \to \mathbb{R}^n$. Let $a \in A$ be an interior point of A. If f is continuously differentiable in an open neighbourhood of a in A and $\det(Df(a)) \neq 0$, then there is an open subset $V \subseteq A$ containing a and an open subset $W \subseteq \mathbb{R}^n$ containing $f(a)$ such that $f\big|_V : V \to W$ is bijective with differentiable inverse $f^{-1} : W \to V$, and*

$$D(f^{-1})(y) = \left[Df(f^{-1}(y)) \right]^{-1}, \ \forall\, y \in W.$$

Proof. Claim 1: *It is enough to prove this theorem assuming $Df(a) = \mathrm{Id}_{\mathbb{R}^n}$.* To see this, let $T_a := Df(a)$. Since $\det(T_a) \neq 0$, we have a $\mathbb{R}$-linear map $T_a^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. Note that, T_a^{-1} is differentiable on $\mathbb{R}^n$ by Theorem 3.2.25 (ii). Since f is continuously differentiable on an open subset $U_a \subseteq A$ containing a, by chain rule (Theorem 3.2.17) the composite map $T_a^{-1} \circ f : A \to \mathbb{R}^n$ is differentiable on U_a with $D(T_a^{-1} \circ f)(a) = DT_a^{-1}(f(a)) \circ Df(a) = T_a^{-1} \circ T_a = \mathrm{Id}_{\mathbb{R}^n}$. Since T_a is injective and differentiable with $DT_a(x) = T_a$, for all $x \in \mathbb{R}^n$, it is easy to check that if the theorem is true for $T_a^{-1} \circ f$, then it is true for $T_a \circ (T_a^{-1} \circ f) = f$ (verify!). Therefore, we assume that $Df(a) = \mathrm{Id}_{\mathbb{R}^n}$.

<u>Claim 2</u>: *There is a closed rectangle $U \subseteq A$ containing a in its interior such that*

$$f(x) \neq f(a), \ \ \forall\, x \in U.$$

If this were not true, then there would exist a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in A converging to a (with respect to the Euclidean metric on $\mathbb{R}^n$) such that $f(x_n) = f(a)$. Since $Df(a) = \mathrm{Id}_{\mathbb{R}^n}$ by assumption,

$$\frac{\|f(x_n) - f(a) - Df(a)(x_n - a)\|}{\|x_n - a\|} = \frac{\|x_n - a\|}{\|x_n - a\|} = 1, \ \forall\, n \in \mathbb{N}, \tag{3.2.51}$$

which would contradict the fact that

$$\lim_{x \to a} \frac{\|f(x) - f(a) - Df(a)(x-a)\|}{\|x-a\|} = 0,$$

since f is differentiable at a, (see Definition 3.2.9). This proves Claim 2.

Since f is continuously differentiable in an open neighbourhood of a in A, replacing U with a possibly smaller closed rectangle in A containing a in its interior, if required, we may assume that

(A1) $\det(Df(x)) \neq 0$, $\forall\, x \in U$, and

(A2) $|D_j f_i(x) - D_j f_i(a)| < \frac{1}{2n^2}$, $\ \forall\, i, j$, and $\forall\, x \in U$;

(see Theorem 3.2.40 for a justification of the assumption (A2)). Let

$$g(x) := f(x) - x, \ \ \forall\, x \in A.$$

Since $D_j g_i(x) = D_j f_i(x) - 1$, for all $x \in A$, applying Lemma 3.2.49 to g and using the above assumption (A2), we see that

$$\|\big(f(x_1) - x_1\big) - \big(f(x_2) - x_2\big)\| \leq \frac{1}{2} \cdot \|x_1 - x_2\|, \ \ \forall\, x_1, x_2 \in U. \tag{3.2.52}$$

Then for all $x_1, x_2 \in U$ we have

$$\begin{aligned}
\|x_1 - x_2\| &= \|\,\big(\big(f(x_1) - x_1\big) - \big(f(x_2) - x_2\big)\big) - \big(f(x_1) - f(x_2)\big)\| \\
&\leq \|\big(f(x_1) - x_1\big) - \big(f(x_2) - x_2\big)\| + \|f(x_1) - f(x_2)\| \\
\Rightarrow \ \|x_1 - x_2\| - \|f(x_1) - f(x_2)\| &\leq \|\big(f(x_1) - x_1\big) - \big(f(x_2) - x_2\big)\| \\
&\leq \frac{1}{2} \cdot \|x_1 - x_2\|, \ \text{ by above inequality.} \\
\Rightarrow \ \frac{1}{2} \cdot \|x_1 - x_2\| &\leq \|f(x_1) - f(x_2)\| \\
\Rightarrow \ \|x_1 - x_2\| &\leq 2\|f(x_1) - f(x_2)\|.
\end{aligned} \tag{3.2.53}$$

Let ∂U be the boundary of the closed rectangle $U \subseteq A$. Since ∂U is compact, its image $f(\partial U)$ is a compact subset of $\mathbb{R}^n$ such that $f(a) \notin f(\partial U)$ by Claim 2. Then there is a real number $d > 0$ such that

$$\|f(a) - f(x)\| \geq d, \ \ \forall\, x \in f(\partial U).$$

Let $W := \{y \in \mathbb{R}^n : \|f(a) - y\| < d/2\}$. Then for any $y \in W$ we have

$$\|y - f(a)\| < \|y - f(x)\|, \ \forall\, x \in \partial U. \tag{3.2.54}$$

<u>Claim 3</u>: *For each $y \in W$, there is a unique $x \in \mathrm{Int}(U)$ such that $f(x) = y$.* To see this, consider the map $\phi : U \to \mathbb{R}^2$ defined by

$$\psi(x) = \|y - f(x)\|^2 = \sum_{i=1}^{n} (y_i - f_i(x))^2,$$

where y_i (resp., f_i) denotes the i-th component of $y \in W \subseteq \mathbb{R}^n$ (resp., of f). Since U is a closed rectangle (with non-empty interior) and ψ is continuous, ψ admits a minima at some point $x \in U$. It follows from (3.2.54) that x must be an interior point of U. Then by Proposition 3.2.35 we have $D_j\psi(x) = 0$, for all $j = 1, \ldots, n$. Then we have

$$\sum_{i=1}^{n} 2(y_i - f_i(x)) \cdot D_j f_i(x) = 0, \ \forall\, j \in \{1, \ldots, n\}. \tag{3.2.55}$$

Since for each $x' \in U$, the matrix $\big(D_j f_i(x')\big)_{1 \leq i,j \leq n}$ is invertible by assumption (A1), the set of all column vectors of $\big(D_j f_i(x)\big)_{1 \leq i,j \leq n}$ must be $\mathbb{R}$-linearly independent. Therefore, it follows from equation (3.2.55) that

$$y_i = f_i(x), \ \ \forall\, i \in \{1, \ldots, n\}. \tag{3.2.56}$$

In other words, $y = f(x)$. Uniqueness of such a point $x \in U$ follows from (3.2.53).

If $V = \mathrm{Int}(U) \cap f^{-1}(W)$, then we just have proved above that the map $f\big|_V : V \to W$ has is bijective, and has an inverse $f^{-1} : W \to C$. Then we can rewrite equation (3.2.53) as

$$\|f^{-1}(y_1) - f^{-1}(y_2)\| \leq 2 \cdot \|y_1 - y_2\|, \ \ \forall\, y_1, y_2 \in W. \tag{3.2.57}$$

This proves that $f^{-1} : W \to V$ is continuous. Now it remains to show that it is differentiable on W.

We now show that $f^{-1} : W \to V$ is differentiable at each $y = f(x) \in W$ with its derivative $[Df(x)]^{-1}$. Since f is differentiable on V, for $x_1, x \in V$ if we write

$$\varphi(x_1 - x) := f(x_1) - f(x) - Df(x)(x_1 - x), \tag{3.2.58}$$

then we have

$$\lim_{x_1 \to x} \frac{\|\varphi(x_1 - x)\|}{\|x_1 - x\|} = 0. \tag{3.2.59}$$

Since $Df(x)$ is an invertible $\mathbb{R}$-linear map for $x \in V$, applying its inverse to the above equation we have

$$[Df(x)]^{-1}\left(f(x_1) - f(x)\right) = x_1 - x + [Df(x)]^{-1}\left(\varphi(x_1 - x)\right). \tag{3.2.60}$$

Since each $y_1 \in W$ is of the form $f(x_1)$ for a unique $x_1 \in V$, the above equation can be rewritten as

$$f^{-1}(y_1) = f^{-1}(y) + [Df(x)]^{-1}(y_1 - y) - [Df(x)]^{-1}\left(\varphi(f^{-1}(y_1) - f^{-1}(y))\right). \tag{3.2.61}$$

Therefore, it suffices to show that

$$\lim_{y_1 \to y} \frac{\|[Df(x)]^{-1}\left(\varphi(f^{-1}(y_1) - f^{-1}(y))\right)\|}{\|y_1 - y\|} = 0. \tag{3.2.62}$$

Since $\|[Df(x)]^{-1}\|$ is a finite non-negative real number (see Proposition 3.2.15), it suffices to show that

$$\lim_{y_1 \to y} \frac{\|\varphi(f^{-1}(y_1) - f^{-1}(y))\|}{\|y_1 - y\|} = 0. \tag{3.2.63}$$

Note that,

$$\frac{\|\varphi(f^{-1}(y_1) - f^{-1}(y))\|}{\|y_1 - y\|} = \frac{\|\varphi(f^{-1}(y_1) - f^{-1}(y))\|}{\|f^{-1}(y_1) - f^{-1}(y)\|} \cdot \frac{\|f^{-1}(y_1) - f^{-1}(y)\|}{\|y_1 - y\|}. \tag{3.2.64}$$

Since $f^{-1} : W \to V$ is continuous, we have $\lim\limits_{y_1 \to y} f^{-1}(y_1) = f(y)$. Since $y_1 \to y$, using continuity of f^{-1} we have $x_1 = f^{-1}(y_1) \to x = f^{-1}(y)$. Therefore, by (3.2.59) the limit of first factor on the right side of (3.2.64) is 0 as $y_1 \to y$. Since the second factor on the right side of (3.2.64) is bounded by 2 (see inequality (3.2.57)), the limit as $y_1 \to y$ of the product term on the right side of the equation (3.2.64) is 0. This completes the proof. $\square$

Remark 3.2.65. An inverse function of f may exists even if $\det(f'(a)) = 0$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, for all $x \in \mathbb{R}$, then $Df(0) = 0$ but f has an inverse function defined by $f^{-1}(y) = \sqrt[3]{y}$, for all $y \in \mathbb{R}$. However, one thing is certain that if $\det(Df(a)) = 0$, then f^{-1} cannot be differentiable at $f(a)$. To see this, note that if f^{-1} were differentiable at $f(a)$, then applying chain rule (Theorem 3.2.17) to $f \circ f^{-1}(x) = x$ we would have

$$\begin{aligned}
&Df(f^{-1}(f(a))) \circ D(f^{-1})(f(a)) = \mathrm{Id}_{\mathbb{R}^n} \\
\Rightarrow\ &Df(a) \circ D(f^{-1})(f(a)) = \mathrm{Id}_{\mathbb{R}^n} \\
\Rightarrow\ &\det(Df(a)) \det(D(f^{-1})(f(a))) = 1,
\end{aligned}$$

which is not possible since $\det(Df(a)) = 0$ by assumption.

Theorem 3.2.66 (Implicit function theorem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable in an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ containing (a, b) and $f(a, b) = 0$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Let M be the $m \times m$ matrix*

$$M = \left(D_{n+j} f_i(a, b)\right) = \begin{pmatrix} D_{n+1} f_1(a, b) & \cdots & D_{n+m} f_1(a, b) \\ \vdots & \ddots & \vdots \\ D_{n+1} f_m(a, b) & \cdots & D_{n+m} f_m(a, b) \end{pmatrix}.$$

If $\det(M) \neq 0$, then there is an open subset $A \subseteq \mathbb{R}^n$ containing a and an open subset $B \subseteq \mathbb{R}^m$ containing b such that for each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$, and the *function*

$$g : A \to B, \ \ x \mapsto g(x)$$

is differentiable.

Proof. Define a map $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$F(x,y) = (x, f(x,y)), \ \ \forall\, (x,y) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{3.2.67}$$

Note that, $\det(DF(a,b)) = \det(M) \neq 0$. Since f is continuously differentiable, so is F. Then by inverse function theorem 3.2.50 there is an open subset $V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing (a,b) and an open subset $W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a,b) = (a,0)$ such that $F\big|_V : V \to W$ has a differentiable inverse, say $h : W \to V$. We may choose $V \subseteq \mathbb{R}^n \times \mathbb{R}^m$ of the form $A \times B$, for some open subsets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Since $h \circ F = \mathrm{Id}$, we see that h is of the form $h(x,y) = (x, k(x,y))$, for some differentiable function k. Let $pr_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projection map onto the second factor (i.e., $pr_2(x,y) = y$, for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$). Then $pr_2 \circ F = f$. Since $F \circ h = \mathrm{Id}$, we have

$$\begin{aligned} f(x, k(x,y)) &= f \circ h(x,y) \\ &= ((pr_2 \circ F) \circ h)(x,y) \\ &= (pr_2 \circ (F \circ h))(x,y) \\ &= pr_2(x,y) = y. \end{aligned}$$

Therefore, $f(x, k(x,y)) = 0$, and so we can define $g : A \to B$ by $g(x) = k(x,0)$, for all $x \in A$. This completes the proof. □

Theorem 3.2.68. *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuously differentiable in an open subset of $\mathbb{R}^n$ containing a, and let $p \leq n$. If $f(a) = 0$ and the $p \times n$ matrix $(D_j f_i(a))$ has rank p, then there is an open subset $A \subseteq \mathbb{R}^n$ containing a and a differentiable map $h : A \to \mathbb{R}^n$ with differentiable inverse map such that*

$$(f \circ h)(x_1, \ldots, x_n) = (x_{n-p+1}, \ldots, x_n), \ \forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Proof. We can consider f as a function $f : \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^p$. If $\det(M) \neq 0$, □

3.3 Differentiable and C^k Manifolds

A *topological manifold* is a Hausdorff second countable topological space X such that each point $x \in X$ has an open neighbourhood $U_x \subseteq X$ homeomorphic to an open subset of $\mathbb{R}^{n_x}$, for some integer $n_x \geq 1$. It is a non-trivial fact that if X is connected, that positive integer n_x does not depend on x. In other words, each point of X admits an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, for a fixed $n \in \mathbb{N}$. This integer n is called the *topological dimension* of X.

Let X be a topological manifold. A *coordinate chart* (or, a *chart*) on X is a pair (U, f), where $U \subseteq X$ is an open subset and $f : U \to \widetilde{U}$ is a homeomorphism of U onto an open subset

$\widetilde{U} \subseteq \mathbb{R}^n$. We say that a coordinate chart (U, f) on X is *centred at (or, around)* $x \in X$ if $x \in U$ and $f(x) = 0$ in $\mathbb{R}^n$.

A map $f : U(\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ is said to be a C^ω map if each of the component functions $f_j := p_j \circ f : U \to \mathbb{R}$ are real analytic. Note that, any C^ω map is a C^∞ map, but the converse is not true. Let $k \in \mathbb{N} \cup \{\infty, \omega\}$.

Two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) on X are said to be *differentiable compatible* (respectively, C^k*-compatible*) if the induced *transition maps*

$$\begin{aligned} \varphi_{\alpha\beta} &:= \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \\ \text{and} \quad \varphi_{\beta\alpha} &:= \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \end{aligned}$$

are differentiable maps (respectively, C^k maps).

Definition 3.3.1. A *differentiable atlas* (respectively, a C^k*-atlas*) on X is an indexed family $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of pairwise differentiable compatible (respectively, C^k-compatible) charts $(U_\alpha, \varphi_\alpha)$ such that $\bigcup\limits_{\alpha \in \Lambda} U_\alpha = X$.

As before, let $k \in \mathbb{N} \cup \{\infty\}$. Two differentiable (respectively, C^k) atlases $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ and $\mathscr{C}' = \{(W_\beta, \psi_\beta)\}_{\beta \in \Lambda'}$ on X are said to be *compatible* if every objects of $\mathscr{C}$ are pairwise differentiable compatible (respectively, C^k-compatible) with each object of $\mathscr{C}'$. In other words, given any $\alpha \in \Lambda$ and $\beta \in \Lambda'$, the induced *transition* maps

$$\begin{aligned} \psi_\beta \circ \varphi_\alpha^{-1} &: \varphi_\alpha(U_\alpha \cap W_\beta) \to \varphi_\beta(U_\alpha \cap W_\beta) \\ \text{and} \quad \varphi_\alpha \circ \psi_\beta^{-1} &: \psi_\beta(U_\alpha \cap W_\beta) \to \varphi_\alpha(U_\alpha \cap W_\beta) \end{aligned}$$

are differentiable maps (respectively, C^k maps).

Remark 3.3.2. If $(U_\alpha, \varphi_\alpha)$ is compatible with (U_β, φ_β), and if (U_β, φ_β) is compatible with $(U_\gamma, \varphi_\gamma)$, then we have

$$\begin{aligned} \varphi_{\alpha\beta} &:= \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \\ \text{and} \quad \varphi_{\beta\gamma} &:= \varphi_\gamma \circ \varphi_\beta^{-1} : \varphi_\beta(U_\beta \cap U_\gamma) \to \varphi_\gamma(U_\beta \cap U_\gamma) \end{aligned}$$

are differentiable (resp., C^k) maps. Therefore, being differentiable compatible (resp., C^k compatible) charts is reflexive and symmetric relation. However, in general this is not a transitive relation because from the above two equations we can say at most that the restriction of the map

$$\varphi_{\alpha\gamma} := \varphi_\gamma \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\gamma) \to \varphi_\gamma(U_\alpha \cap U_\gamma) \tag{3.3.3}$$

on $\varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$ is differentiable (resp., C^k), i.e.,

$$\varphi_{\alpha\gamma}\big|_{\varphi_\alpha(U_\alpha \cap U_\gamma)} := \varphi_\gamma \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma) \to \varphi_\gamma(U_\alpha \cap U_\beta \cap U_\gamma) \tag{3.3.4}$$

is differentiable (resp., C^k) map.

Exercise 3.3.5. Show that the transition maps satisfy the following properties.

(i) $\varphi_{\alpha\alpha}$ is the identity map, for all $\alpha \in \Lambda$, and

(ii) on each triple intersection $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, the transition maps $\varphi_{\alpha\beta}, \varphi_{\beta\gamma}$ and $\varphi_{\alpha\gamma}$ satisfy the following *cocycle condition*: $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.

Lemma 3.3.6. *Let $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ be an atlas on X, and let (V_1, ψ_1) and (V_2, ψ_2) be two charts on X each of which are pairwise compatible with every charts in $\mathscr{C}$. Then (V_1, ψ_1) is compatible with (V_2, ψ_2).*

Proof. Let $X_\alpha := V_1 \cap V_2 \cap U_\alpha$, for all $\alpha \in \Lambda$. Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X, $\{X_\alpha\}_{\alpha\in\Lambda}$ is an open cover of $V_1 \cap V_2$. Since (V_j, ψ_j) is pairwise compatible with each $(U_\alpha, \varphi_\alpha)$, $\forall\ \alpha \in \Lambda$, the maps

$$\begin{aligned} &\psi_j \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_j) \to \psi_j(U_\alpha \cap V_j) \\ \text{and}\quad &\varphi_\alpha \circ \psi_j^{-1} : \psi_j(U_\alpha \cap V_j) \to \varphi_\alpha(U_\alpha \cap V_j) \end{aligned}$$

are differentiable (resp., C^k), for all $j \in \{1, 2\}$. Then the composite maps

$$\begin{aligned} &\psi_1(U_\alpha \cap V_1 \cap V_2) \xrightarrow{\varphi_\alpha \circ \psi_1^{-1}} \varphi_\alpha(U_\alpha \cap V_1 \cap V_2) \xrightarrow{\psi_2 \circ \varphi_\alpha^{-1}} \psi_2(U_\alpha \cap V_2 \cap V_2) \\ \text{and}\quad &\psi_2(U_\alpha \cap V_1 \cap V_2) \xrightarrow{\varphi_\alpha \circ \psi_2^{-1}} \varphi_\alpha(U_\alpha \cap V_1 \cap V_2) \xrightarrow{\psi_1 \circ \varphi_\alpha^{-1}} \psi_1(U_\alpha \cap V_2 \cap V_2) \end{aligned}$$

are differentiable (resp., C^k) maps. But these two composite maps coincides with the restrictions of the transition maps

$$\begin{aligned} &\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2) \\ \text{and}\quad &\psi_1 \circ \psi_2^{-1} : \psi_2(V_1 \cap V_2) \to \psi_1(V_1 \cap V_2) \end{aligned}$$

on $\psi_1(X_\alpha)$ and $\psi_2(X_\alpha)$, respectively, where $X_\alpha = U_\alpha \cap V_1 \cap V_2$, for all $\alpha \in \Lambda$. Since $\{X_\alpha\}_{\alpha\in\Lambda}$ is an open cover of $V_1 \cap V_2$, the transition maps $\psi_2 \circ \psi_1^{-1}$ and $\psi_1 \circ \psi_2^{-1}$ are differentiable (resp., C^k). Therefore, (V_1, ψ_1) and (V_2, ψ_2) are pairwise compatible charts. $\square$

A differentiable (resp., C^k) atlas $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ on X is said to be *maximal* if it is not properly contained in any other differentiable (resp., C^k) atlas on X.

Lemma 3.3.7. *Any differentiable (resp., C^k) atlas on X is contained in a unique maximal differentiable (resp., C^k) atlas on X.*

Proof. Let $\mathscr{C} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ be a differentiable (resp., C^k) atlas on X. Let $\widehat{\mathscr{C}}$ be the collection of all charts (U, ϕ) on X that are pairwise differentiable compatible (resp., C^k compatible) with all charts in $\mathscr{C}$. By above Lemma 3.3.6, $\widehat{\mathscr{C}}$ is a differentiable (resp., C^k) atlas containing $\mathscr{C}$.

Let $\mathscr{D}$ be any differentiable (resp., C^k) atlas on X containing $\mathscr{C}$. Let $(U, \varphi) \in \mathscr{D}$ be arbitrary. Since (U, φ) is pairwise compatible with all charts of $\mathscr{C}$, by construction of $\widehat{\mathscr{C}}$, we conclude that $(U, \varphi) \in \widehat{\mathscr{C}}$. Therefore, $\mathscr{D} \subseteq \widehat{\mathscr{C}}$. This proves that $\widehat{\mathscr{C}}$ is a maximal differentiable (resp., C^k) atlas on X containing $\mathscr{C}$. If $\mathscr{D}$ is any maximal differentiable (resp., C^k) atlas on X, the above argument shows that $\mathscr{D} \subseteq \widehat{\mathscr{C}}$, and so by maximality of $\mathscr{D}$, we have $\mathscr{D} = \widehat{\mathscr{C}}$. This proves uniqueness of maximal atlas. $\square$

Definition 3.3.8. A *differentiable* (resp., C^k) *manifold* is a topological manifold X together with a maximal differentiable (resp., C^k) atlas on it. A C^∞ manifold is called a *smooth manifold*.

Remark 3.3.9. Similarly one can define a *holomorphic atlas* on a topological manifold X to be an indexed collection of pairs (U_α, f_α), where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open covering of X and $f_\alpha : U_\alpha \to \widetilde{U}_\alpha \subseteq \mathbb{C}^n$ is a homeomorphism of U_α onto an open subset $\widetilde{U}_\alpha$ of $\mathbb{C}^n$ such that given any $\alpha, \beta \in \Lambda$, the transition map

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap W_\beta) \to \psi_\beta(U_\alpha \cap W_\beta)$$

is a holomorphic map† Then we define a *complex manifold* to be a topological manifold X together with a maximal holomorphic atlas on it.

Example 3.3.10. Given an open subset $U \subseteq \mathbb{R}^n$, an a diffeomorphism $\varphi_U : U \to$ consider the pair (U, Id_U), where $\mathrm{Id}_U : U \to U$ is the identity map from U onto U. Note that, given any two open subsets $U, V \subseteq \mathbb{R}^n$, the map

$$\varphi_{U,V} = \mathrm{Id}_V \circ \mathrm{Id}_U^{-1} : U \cap V \to U \cap V$$

is precisely the identity map from $U \cap V$ into itself.

The Euclidean space $\mathbb{R}^n$ together with a maximal atlas containing all pairs (U, Id_U), where U is an open subset of $\mathbb{R}^n$ and

3.3.1 Submanifold

3.3.2 Tangent bundle

†A continuously differentiable map $f : U(\subseteq \mathbb{C}^n) \to \mathbb{C}^m$ is said to be *homomorphic* if each point $z_0 \in U$ has an open neighbourhood $V_0 \subseteq U$ such that f admits a power series expansion of the form $f(z_0 + z) = \sum_I \alpha_I z_I$, where I runs through the set of n-tuples of integers $(i_1, \ldots, i_n)$ with $i_k \geq 0$, and $z_I = z_1^{i_1} \cdots z_n^{i_n}$, and the coefficients $\alpha_I \in \mathbb{C}$ satisfies the property that there exists positive real numbers $R_1, \ldots, R_n$ such that the power series $\sum_I |\alpha_I| r_1^{i_1} \cdots r_n^{i_n}$ converges for $r_1 < R_1, \ldots, r_n < R_n$.

Chapter 4

Appendix

4.1 Category and Functor

Definition 4.1.1. * A *category* $\mathscr{C}$ consists of the following data:

(i) a class of objects, denoted $\text{ob}(\mathscr{C})$,

(ii) for $X, Y \in \text{ob}(\mathscr{C})$, a class of morphisms from X into Y, denoted $\text{Mor}_{\mathscr{C}}(X, Y)$,

(iii) for each $X, Y, Z \in \text{ob}(\mathscr{C})$, a *composition map*

$$\text{Mor}_{\mathscr{C}}(X, Y) \times \text{Mor}_{\mathscr{C}}(Y, Z) \to \text{Mor}_{\mathscr{C}}(X, Z), \ \ (f, g) \mapsto g \circ f\,,$$

which satisfies associative property: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \text{Mor}_{\mathcal{C}}(X, Y)$, $g \in \text{Mor}_{\mathscr{C}}(Y, Z)$ and $h \in \text{Mor}_{\mathscr{C}}(Z, W)$, for all $X, Y, Z, W \in \text{ob}(\mathscr{C})$.

A category $\mathscr{C}$ is said to be *locally small* if $\text{Mor}_{\mathscr{C}}(X, Y)$ is a set, for all $X, Y \in \text{ob}(\mathscr{C})$. A category $\mathscr{C}$ is said to be *small* if it is locally small and the class of objects $\text{ob}(\mathscr{C})$ is a set.

Example 4.1.2. The category (Set), whose objects are sets and morphisms are given by set maps, is a locally small, but not small. However, the category (FinSet), whose objects are finite sets and morphisms are given by set maps, is a small category.

Two objects $A_1, A_2 \in \mathscr{C}$ are said to be *isomorphic* if there are morphisms (arrows) $f : A_1 \to A_2$ and $g : A_2 \to A_1$ in $\mathscr{C}$ such that $g \circ f = \text{Id}_{A_1}$ and $f \circ g = \text{Id}_{A_2}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is given by the following data:

(i) for each $X \in \mathscr{A}$ there is an object $\mathcal{F}(X) \in \mathscr{B}$,

(ii) for $X, Y \in \mathscr{A}$ and $f \in \text{Hom}_{\mathscr{A}}(X, Y)$, there is $\mathcal{F}(f) \in \text{Mor}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$, which are compatible with the composition maps.

A functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is said to be *faithful* (resp., *full*) if for any two objects $A_1, A_2 \in \mathscr{A}$, the induced map

$$\mathcal{F} : \text{Mor}_{\mathscr{A}}(A_1, A_2) \longrightarrow \text{Mor}_{\mathscr{B}}(F(A_1), F(A_2))$$

* *Joke: Category theory is like Ramayana & Mahabharata — there are lots of arrows!*

is injective (resp., surjective). We say that $\mathcal{F}$ is *fully faithful* if it is both full and faithful.

Let $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ be two functors. A morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ is given by the following data: for each object $A \in \mathscr{A}$, a map $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ which is *functorial*; that means, for any arrow $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\varphi_A"'] & \mathcal{F}(A') \arrow[d, "\varphi_{A'}"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(A') \end{tikzcd} \tag{4.1.3}$$

Definition 4.1.4. A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *monomorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Hom}_{\mathscr{A}}(T, A)$ with $f \circ g = f \circ h$, we have $g = h$.

A morphism $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is said to be a *epimorphism* if for any object $T \in \mathscr{A}$ and two morphisms $g, h \in \mathrm{Mor}_{\mathscr{A}}(B, T)$ with $g \circ f = h \circ f$, we have $g = h$.

Given any two categories $\mathscr{A}$ and $\mathscr{B}$, we can define a category $\mathrm{Func}(\mathscr{A}, \mathscr{B})$, whose objects are functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, and for any two such objects $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$, there is a morphism set $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ consisting of all morphisms of functors $\varphi_A : \mathcal{F} \to \mathcal{G}$, as defined above.

Proposition 4.1.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories. Then $\mathcal{F}, \mathcal{G} \in \mathrm{Func}(\mathscr{A}, \mathscr{B})$ are isomorphic if there exists a morphism of functors $\varphi : \mathcal{F} \to \mathcal{G}$ such that for any object $A \in \mathscr{A}$, the induced morphism $\varphi_A : \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism in $\mathscr{B}$.*

Definition 4.1.6. A category $\mathscr{A}$ is said to be *pre-additive* if for any two objects $X, Y \in \mathscr{A}$, the set $\mathrm{Mor}_{\mathscr{A}}(X, Y)$ has a structure of an abelian group such that the *composition map*

$$\mathrm{Mor}_{\mathscr{A}}(X, Y) \times \mathrm{Mor}_{\mathscr{A}}(Y, Z) \longrightarrow \mathrm{Mor}_{\mathscr{A}}(X, Z),$$

written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $X, Y, Z \in \mathscr{A}$.

Notation. For any pre-additive category $\mathscr{A}$, we denote by $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ the abelian group $\mathrm{Mor}_{\mathscr{A}}(X, Y)$, for all $X, Y \in \mathrm{ob}(\mathscr{A})$.

Let $\mathscr{A}$ and $\mathscr{B}$ be pre-additive categories. A functor $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ is said to be *additive* if for all objects $X, Y \in \mathscr{A}$, the induced map

$$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a group homomorphism.

Definition 4.1.7 (Additive category)**.** A category $\mathscr{A}$ is said to be *additive* if for any two objects $A, B \in \mathscr{A}$, the set $\mathrm{Hom}_{\mathscr{A}}(A, B)$ has a structure of an abelian group such that the following conditions holds.

(i) The composition map $\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)$, written as $(f, g) \mapsto g \circ f$, is $\mathbb{Z}$-bilinear, for all $A, B, C \in \mathscr{A}$.

(ii) There is a zero object 0 in $\mathscr{A}$, i.e., $\mathrm{Hom}_{\mathscr{A}}(0, 0)$ is the trivial group with one element.

(iii) For any two objects $A_1, A_2 \in \mathscr{A}$, there is an object $B \in \mathscr{A}$ together with morphisms $\mathrm{j}_i : A_i \to B$ and $p_i : B \to A_i$, for $i = 1, 2$, which makes B the direct sum and the direct product of A_1 and A_2 in $\mathscr{A}$.

Definition 4.1.8. Let k be a field. A *k-linear category* is an additive category $\mathscr{A}$ such that for any $A, B \in \mathscr{A}$, the abelian groups $\mathrm{Hom}_{\mathscr{A}}(A, B)$ are k-vector spaces such that the composition morphisms

$$\mathrm{Hom}_{\mathscr{A}}(A, B) \times \mathrm{Hom}_{\mathscr{A}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, C)\,, \quad (f, g) \mapsto g \circ f$$

are k-bilinear, for all $A, B, C \in \mathscr{A}$.

Remark 4.1.9. Additive functors $\mathcal{F} : \mathscr{A} \longrightarrow \mathscr{B}$ between two k-linear additive categories $\mathscr{A}$ and $\mathscr{B}$ over the same base field k are assumed to be k-linear, i.e., for any two objects $A_1, A_2 \in \mathscr{A}$, the map $\mathcal{F}_{A_1, A_2} : \mathrm{Hom}_{\mathscr{A}}(A_1, A_2) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is k-linear.

Let $\mathscr{A}$ be an additive category. Then there is a unique object $0 \in \mathscr{A}$, called the *zero object* such that for any object $A \in \mathscr{A}$, there are unique morphisms $0 \to A$ and $A \to 0$ in $\mathscr{A}$. For any two objects $A, B \in \mathscr{A}$, the *zero morphism* $0 \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined to be the composite morphism

$$A \longrightarrow 0 \longrightarrow B\,.$$

In particular, taking $A = 0$, we see that, the set $\mathrm{Hom}_{\mathscr{A}}(0, B)$ is the trivial group consisting of one element, which is, in fact, the zero morphism of 0 into B in $\mathscr{A}$.

Definition 4.1.10. Let $f : A \to B$ be a morphism in $\mathscr{A}$. Then *kernel* of f is a pair $(\iota, \mathrm{Ker}(f))$, where $\mathrm{Ker}(f) \in \mathscr{A}$ and $\iota \in \mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), A)$ such that

(i) $f \circ \iota = 0$ in $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Ker}(f), B)$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : C \to A$ with $f \circ g = 0$, there is a unique morphism $\widetilde{g} : C \to \mathrm{Ker}(f)$ such that $\iota \circ \widetilde{g} = g$.

$$\begin{tikzcd}
& C \arrow[dl, dashed, "\exists ! \widetilde{g}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The *cokernel* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ is defined by reversing the arrows of the above diagram.

Definition 4.1.11. The *cokernel* of $f : A \to B$ is a pair $(\pi, \mathrm{Coker}(f))$, where $\mathrm{Coker}(f)$ is an object of $\mathscr{A}$ together with a morphism $\pi : B \to \mathrm{Coker}(f)$ in $\mathscr{A}$ such that

(i) $\pi \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Coker}(f))$, and

(ii) given any object $C \in \mathscr{A}$ and a morphism $g : B \to C$ with $g \circ f = 0$ in $\mathrm{Hom}_{\mathscr{A}}(A, C)$, there is a unique morphism $\widetilde{g} : \mathrm{Coker}(f) \to C$ such that $\widetilde{g} \circ \pi = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[dl, dashed, "\exists!\widetilde{g}"] \\ & C & \end{tikzcd}$$

Definition 4.1.12. The *coimage* of $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, denoted by $\mathrm{Coim}(f)$, is the cokernel of $\iota : \mathrm{Ker}(f) \longrightarrow A$ of f, and the *image* of f, denoted $\mathrm{Im}(f)$, is the kernel of the cokernel $\pi : B \longrightarrow \mathrm{Coker}(f)$ of f.

Lemma 4.1.13. *Let $\mathscr{C}$ be a preadditive category, and $f : X \to Y$ a morphism in $\mathscr{C}$.*

(i) If a kernel of f exists, then it is a monomorphism.

(ii) If a cokernel of f exists, then it is an epimorphism.

(iii) If a kernel and coimage of f exist, then the coimage is an epimorphism.

(iv) If a cokernel and image of f exist, then the image is a monomorphism.

Proof. Assume that a kernel $\iota : \mathrm{Ker}(f) \to X$ of f exists. Let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(Z, \mathrm{Ker}(f))$ be such that $\iota \circ \alpha = \iota \circ \beta$. Since $f \circ (\iota \circ \alpha) = f \circ (\iota \circ \beta) = 0$, by definition of $\mathrm{Ker}(f) \xrightarrow{\iota} X$ there is a unique morphism $g \in \mathrm{Hom}(Z, \mathrm{Ker}(f))$ such that $\iota \circ \alpha = \iota \circ g = \iota \circ \beta$. Therefore, $\alpha = g = \beta$.

The proof of (ii) is dual.

(iii) follows from (ii), since the coimage is a cokernel. Similarly, (iv) follows from (i). □

Exercise 4.1.14. Let $\mathscr{A}$ be an additive category. Let $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ be such that $\mathrm{Ker}(f) \xrightarrow{\iota} X$ exists in $\mathscr{A}$. Then the kernel of $\iota : \mathrm{Ker}(f) \to X$ is the unique morphism $0 \to \mathrm{Ker}(f)$ in $\mathscr{A}$.

Lemma 4.1.15. *Let $f : X \to Y$ be a morphism in a preadditive category $\mathscr{C}$ such that the kernel, cokernel, image and coimage all exist in $\mathscr{C}$. Then f uniquely factors as $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ in $\mathscr{C}$.*

Proof. Since $\mathrm{Ker}(f) \to X \to Y$ is zero, there is a canonical morphism $\mathrm{Coim}(f) \to Y$ such that the composite morphism $X \to \mathrm{Coim}(f) \to Y$ is f. The composition $\mathrm{Coim}(f) \to Y \to \mathrm{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $X \to Y \to \mathrm{Coker}(f)$, which is zero. Hence by Lemma 4.1.13 (iii), $\mathrm{Coim}(f) \to Y$ factors uniquely through $\mathrm{Im}(f) = \mathrm{Ker}(\pi_f)$. This completes the proof.

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & X \arrow[rrr, "f"] \arrow[dr, "\pi_\iota"'] & & & Y \arrow[r, "\pi_f"] & \mathrm{Coker}(f) \\ & & \mathrm{Coim}(f) \arrow[r] & \mathrm{Im}(f) \arrow[ur, "j"'] & & \end{tikzcd} \tag{4.1.16}$$

□

Definition 4.1.17. An *abelian category* $\mathscr{A}$ is an additive category such that for any morphism $f : A \to B$ in $\mathscr{A}$, its kernel $\iota : \mathrm{Ker}(f) \to A$ and cokernel $p : B \to \mathrm{Coker}(f)$ exists in $\mathscr{A}$, and the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism in $\mathscr{A}$ (c.f. Definition 4.1.12).

Example 4.1.18. For any commutative ring A with identity, the category Mod_A of A-modules is an abelian category.