Factors in Number Theory

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Overview

In this handout, I discuss important properties and results relating to the divisors of a number, including prime factorization, how to calculate the number of factors of a number, and how to calculate sum and product of factors. At the end of the handout, I provide challenging practice problems collected from various middle school and high school math competitions, as well as corresponding hints. If you would like to contact me, please see the information on the last page.

Theory

We begin by introducing one of the most powerful theorems in mathematics: the Fundamental Theorem of Arithmetic.

Theorem 1. (Fundamental Theorem of Arithmetic)

Every positive integer greater than 1 can be represented in exactly one way (ignoring permutations) as a product of one or more primes. In other words, for positive integer n, non-negative $a_{1,2,\cdots,k}$ and prime p_i, we have

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}.$$

This representation is called the *canonical representation,* or more popularly, the *prime factorization* of n. Often times, examining the prime factorization of a number is enough to solve a problem, or at least notice something useful. If you are not sure where to go with a number theory problem, try looking at the prime factorization of the numbers. We will leave the proof of this theorem out, but if you are curious, feel free to ask me outside of class or to look it up.

Now, we introduce another basic, yet important term that I'm sure you have all seen before when first learning about multiplication and division.

Definition 1

A *factor* (or divisor) of a number n is a number that divides evenly (leaves a remainder of 0) into n. In other words, f is a factor of n if there exists an integer k such that $n = kf$.

For example, 2 is a factor of 6 because $6 \div 2$ produces an integer. Another way to look at this is to note that since 6 is a multiple of 2, 2 must be a factor of 6. Multiples and factors go hand in hand. 3 is not a factor of 7 because $7 \div 3$ does not produce an integer.

Suppose we would like to count the number of factors of a number n. One way to do this is to test every integer until $\sqrt{n}$ and see if it is a factor of n (why do we only have to go until $\sqrt{n}$?). However, this process is very tedious and time-consuming, especially for large numbers. Fortunately, there is an easier way to count the number of factors of a number.

To motivate this idea, let us first consider an problem of this kind.

Problem 0.1. How many factors does 60 have?

Solution. First, we prime factorize 60 into $2^2 \cdot 3^1 \cdot 5^1$. From this, we can see that the divisors of 60 will be in the form $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3}$, where $a_1 \leq 2, a_2 \leq 1$, and $a_3 \leq 1$.

So, each factor can either have zero, one, or two 2s ($2^0, 2^1$, or 2^2), either zero or one 3s (3^0, or 3^1), or either zero or one 5s (5^0, or 5^1).

Thus, we have 3 options for the number of 2s in each factor, 2 options for the number of 3s in each factor, and 2 options for the number of 5s in each factor. This gives us a total of $3 \cdot 2 \cdot 2 = \boxed{12}$ total factors. When we check our work by listing the factors of 60, $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$, we see that there are indeed 12 factors. ■

We can use this same reasoning to generalize and find the total number of factors for any positive integer n.

Proposition 1

A number $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ has $(a_1 + 1) \cdot (a_2 + 1) \cdots (a_k + 1)$ factors, where $a_{1,2\cdots,k}$ is non-negative.

Proof. We see that the maximum number of p_1s that can occur in a factor is a_1. Thus, we can either have zero p_1s, one p_1, two p_1s ... or a_1 p_1s. This gives us $a_1 + 1$ possible choices for p_1. Similarly, we $a_2 + 1$ choices for p_2, $a_3 + 1$ choices for p_3, and so on and so forth until we have $a_k + 1$ choices for p_k. Hence, to count the total number of factors, we simply multiply the results together, which yields

$$(a_1 + 1) \cdot (a_2 + 1) \cdots (a_k + 1)$$

total factors, as desired. □

Since there is a lot of work regarding the number of factors of a number, mathematicians have defined a function just for this purpose: the *tau* function.

Definition 2

The number of positive divisors of n is denoted by $\tau(n)$.

Note. Here, τ is the lowercase Greek letter "tau".

Note. Don't confuse $\tau(n)$ with the constant, which is equal to $\tau = 2\pi \approx 6.283$.

From this proposition, an important fact follows.

Corollary 1

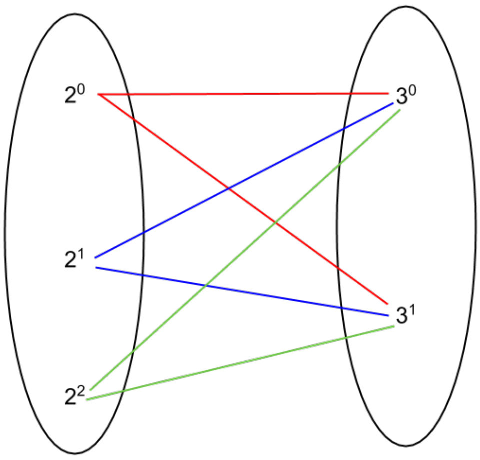
A positive integer n has an odd number of factors if and only if n is a perfect square.

The proof for this corollary is quite simple, so I leave it as an exercise.

Now, suppose that we would like to calculate the sum of the factors. Of course, one could simply write out the factors of a number and add them all up, but there is a slicker way to do this. To illustrate this, let us consider the following problem.

Problem 0.2. What is the sum of the factors of 12?

Solution. First, we prime factorize 12 into $2^2 \cdot 3^1$ From this, we can see that the divisors of 12 will be in the form $2^{a_1} \cdot 3^{a_2} \cdot$, where $0 \le a_1 \le 2$, and $0 \le a_2 \le 1$. In other words, a_1 can only be 0, 1, or 2, and a_2 can only be 0 or 1.



Thus, as shown in the image above, our possible factors are $2^0 3^0$, $2^0 3^1$, $2^1 3^0$, $2^1 3^1$, $2^2 3^0$, and $2^2 3^1$. Now, we simply add them all up to get our final answer.

$$\begin{aligned} 2^0 3^0 + 2^0 3^1 + 2^1 3^0 + 2^1 3^1 + 2^2 3^0 + 2^2 3^1 &= 2^0(3^0 + 3^1) + 2^1(3^0 + 3^1) + 2^2(3^0 + 3^1) \\ &= (2^0 + 2^1 + 2^2)(3^0 + 3^1) \\ &= 7 \cdot 4 \\ &= \boxed{28}. \end{aligned}$$

■

Note how we were able to factor the sum rather than just go straight to adding them. The way we factored this is extremely important in coming up with the general formula.

Proposition 2

The sum of the positive divisors of a number $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ is equal to $(p_1^0 + p_1^1 + \cdots p_1^{a_1}) \cdot (p_2^0 + p_2^1 + \cdots p_2^{a_2}) \cdots (p_k^0 + p_k^1 + \cdots p_k^{a_k})$.

The following proof is due to brilliant.org.

Proof. (Brilliant.) If $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then the divisors of n will be in the form

$$p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}, \text{where } 0 \leq r_i \leq a_i.$$

For each combination of $(r_2, r_3 \dots r_k)$, if we find the sum of the terms over all possible values of r_1, we will obtain

$$(p_1^1 + p_1^2 + p_1^3 + \dots p_1^{a_1}) \times p_2^{a_2} \times p_3^{a_3} \dots \times p_k^{a_k}.$$

We repeat this procedure, where we sum up over all possible values of r_2, and then r_3, and so on and so forth. The sum of all of these factors would hence be

$$(p_1^0 + p_1^1 + \cdots p_1^{a_1}) \cdot (p_2^0 + p_2^1 + \cdots p_2^{a_2}) \cdots (p_k^0 + p_k^1 + \cdots p_k^{a_k}),$$

as desired. □

Just as we have a special notation to denote the number of factors, $\tau(n)$, we also have special notation for the sum of proper divisors.

Definition 4

The sum of the positive divisors of n is denoted by $\sigma(n)$.

Note 0.1. The "σ" is the lowercase Greek letter "sigma".

There is a special type of number that can be expressed using this notation.

Definition 5

A *perfect* number is a number that is equal to twice the sum of its divisors. It is equivalent to say that a number is perfect if and only if

$$\sigma(n) = 2n.$$

The first couple of perfect numbers are 6, 28, and 496. Perfect numbers rarely appear on math competitions, but it is still useful to know what they are.

Numbers that are related to perfect numbers are *abundant* and *deficient* numbers.

Definition 6

An *abundant* number is a number such that the sum of its factors is greater than twice the original number. It is equivalent to say that a number is abundant if and only if

$$\sigma(n) > 2n.$$

Definition 7

A *deficient* number is a number such that the sum of its factors is less than twice the original number. It is equivalent to say that a number is deficient if and only if

$$\sigma(n) < 2n.$$

We will not explore the properties of perfect, abundant, or deficient numbers in this handout, but I encourage you to read up on them if you are interested.

We close the handout with an approach to quickly find the product of the factors of a number.

Problem 0.3. What is the product of the factors of 30?

Solution. Note that 30 has the following factor "pairs": $1 \times 30,\ 2 \times 15,\ 3 \times 10$, and 5×6. To find the product, we just multiply the factors together.

$$(1 \times 30) \cdot (2 \times 15) \cdot (3 \times 10) \cdot (5 \times 6) = 30 \cdot 30 \cdot 30 \cdot 30 = 30^4 = \boxed{810000}.$$

■

In general, we have:

Proposition 3

The product of the positive divisors of integer n is equal to

$$n^{\frac{\tau(n)}{2}},$$

where $\tau(n)$ denotes the number of positive divisors of n.

Proof. The structure of this proof is similar to the solution of the above problem. Note that if k is a factor of n, then so is $\frac{n}{k}$ (one could say that $(n, \frac{n}{k})$ is a factor "pair"). Additionally, observe that $k \cdot \frac{n}{k}$ always equals n, regardless of what k is. Hence, to find the product of all factors, we simply multiply all the factor pairs together. However, since we have counted the factor *pairs,* this is equal to exactly half the number of divisors. Hence, the product of all the divisors is

$$n^{\frac{\tau(n)}{2}},$$

as desired. □

As far as I know, there is not any special notation for the product of the factors. Usually, a problem will make up their own notation, like $\kappa(n)$, or $\mathcal{P}(n)$, like #16 in the problem set.

Problem Set

These are problems that were particularly helpful for me when learning this topic, are classical and will be found in almost all texts pertaining to this subject, or are problems that I have written. I have tried to cite every problem that I used, but if I am missing a citation, please let me know.

I have tried to arrange these problems in increasing difficulty, and a ★ denotes a problem which I consider to be challenging. Have fun!

1. How many odd factors does 210 have?

2. (AMC 8 2011) Let w, x, y, and z be whole numbers. If $2^w \cdot 3^x \cdot 5^y \cdot 7^z = 588$, then what does $2w + 3x + 5y + 7z$ equal?

3. (AMC 10A 2005) How many positive cubes divide $3! \cdot 5! \cdot 7!$?

4. (MathCounts) What fraction of the positive integer factors of 1000^3 are perfect squares?

5. (ARML 1984) Find all possible values of k for which $1984k$ has exactly 21 positive divisors.

6. (AMC 10B 2017) The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

7. (ARML 2014) Find the smallest positive integer n such that $214n$ and $2014n$ have the same number of divisors.

8. (AIME II 2019) Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$\begin{aligned} abc &= 70, \\ cde &= 71, \\ efg &= 72. \end{aligned}$$

9. (AMC 10B 2019) Let S be the set of all positive integer divisors of $100,000$. How many numbers are the product of two distinct elements of S?

10. (AIME I 2010) Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. (AIME II 2004) How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

12. (AHSME 1996) Suppose that n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors. How many positive divisors does $6n$ have?

13. (LMT Team 2019) Find the smallest positive integer n such that $|\tau(n+1) - \tau(n)| = 7$. Here, $\tau(n)$ denotes the number of divisors of n

14. (AMC 10A 2016) For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

15. ★ (AIME I 2005) For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

16. ★ (Mock AIME) Let $\mathcal{P}(n)$ denote the product of the factors of n, and let $\tau(n)$ denote the number of divisors of n. How many ordered pairs of integers (a, b) are there such that $1 \leq a, b \leq 100$ and
$$\mathcal{P}(a) \cdot b \cdot \tau(b) = \mathcal{P}(b) \cdot a \cdot \tau(a)?$$

Hints

1. It is easier to count the number of even numbers.

2. When in doubt, prime factorize.

3. Prime factorize $3! \cdot 5! \cdot 7!$. Now, consider what is true about the exponent of a perfect cube.

4. Prime factorize. After that, consider what is true about the exponent of a perfect cube. Also remember that
$$\text{probability} = \frac{\text{number of favorable possibilities}}{\text{total possibilities}}.$$

5. Prime factorize 1984 *and* 21. Then use the formula for $\tau(n)$ given in Proposition 1.1.

6. Refer to Hint 1.

7. Refer to Hint 5.

8. For a small hint, read (a). If you are still stuck, then read (b). Don't immediately read (b).
   a) Is there anything special about 70? What about 71? 72?
   b) What can we conclude about the values of c and e?

9. The two numbers must be in the form 2^w5^x and 2^y5^z, for $0 \leq w, x, y, z \leq 5$ and $(a, b) \neq (c, d)$. Now multiply these two numbers together, and work up from there.

10. Refer to Hint 3.

11. Prime factorize 2004. What is the form of a number that has 2004 factors?

12. Prime factorize 28 and 30. From here, try to guess the prime factorization of n. Keep in mind that in the formula for the number of divisors, *we add one* to each of the exponents.

13. If the difference of two numbers is 7 (an odd number), what does this tell us about the parities of the original two numbers? (Note: the parity of a number is whether it is odd or even). Then, use Corollary 1.

14. Similar to problem 12. Try to use the same logic.

15. List out the first few cases (I would say the first 10 cases, but you can list more or less, depending if you see the pattern or not). Notice anything about the parities of the first few $S(n)$? If you don't, keep Corollary 1 in mind.

16. Substitute $\mathcal{P}(a)$ and $\mathcal{P}(b)$ with a different expression that we know. Now, rearrange the equation, and consider the behavior of the RHS/LHS. From here, you should be able to guess a few (a, b) that work, but there is one more possibility that works – make sure you find it!

Further References

If you have any questions, noticed an error or typo in the handout, would like me to check your solution, or just want to contact me, feel free to send me an email at `kheera09@gmail.com`, or reach out on Facebook Messenger. You can also private message me on AoPS, where my username is `matharcher`.

To learn more about number theory, I suggest the following resources:

- *Competition Math for Middle School*, by Jason Batterson

- *Introduction to Number Theory*, by Mathew Crawford
- *111 Problems in Algebra and Number Theory*, by Adrian Andreescu and Vinjai Vale
- `artofproblemsolving.com`
- `brilliant.org`