2020-2021
Science Quiz Bowl
Math Handouts

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Contents

1 Exponents and Logarithms

We begin with exponents and logarithms. Many questions will ask you to perform some kind of manipulation or computation involving exponents. Put simply, exponents are repeated multiplication, and logarithmic functions are the inverse of exponential functions. Let us consider exponents first.

Definition 1.1. The expression a^b represents $\underbrace{a \cdot a \ldots a}_{b \text{ times}}$. We call a the base and b the exponent.

Example 1.1. What is 3^4?

Solution 1.1. $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = \boxed{81}$

Suppose we wanted to calculate $2^4 \cdot 2^2$. Of course, we could write out the multiplication as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$, but if the exponents were larger, then this would be quite tedious. Fortunately, there are properties that can help simplify this. I claim that the following exponent properties hold true for all a, b, c:

1. $a^b \cdot a^c = a^{b+c}$
2. $\frac{a^b}{a^c} = a^{b-c}$
3. $(a^b)^c = a^{bc} = (a^c)^b$
4. $(ab)^c = a^c \cdot b^c$
5. $a^{-b} = \frac{1}{a^b}$
6. $(\frac{a}{b})^c = \frac{a^c}{b^c}$
7. $(\frac{a}{b})^{-c} = \frac{b^c}{a^c}$
8. $a^{\frac{b}{c}} = (\sqrt[c]{a})^b$
9. $a^0 = 1$
10. If $a^b = a^c$, then $b = c$

These are all very easy to prove, so I leave it as an exercise. Most of these should make sense to you. I don't want you staring at this list trying to memorize it; if you practice, then these properties will come naturally.

An extremely important thing to note is that these properties only work if the base is the same. If the base is different, then you will have to use a different method to complete the task at hand.

Now, we move on to logarithms.

Definition 1.2. If $a^b = c$, then we say that $\log_a c = b$. We say that a is the base and c is the exponent (no special name for b).

As stated before, logarithms and exponents are inverses of each other (just like how subtraction and addition are inverses and division and multiplication are inverses).

However, there is one important things to note about logarithms: if $\log_a c = b$, c must be greater than or equal to 0. If c was less than 0, then it would imply that a is non real.

Example 1.2. What is $\log_2 8$?

Solution 1.2. Since $2^3 = 8$, the answer is 8.

Logarithms and exponents are very closely related; there are a set of properties that hold true for logarithms, just like how they worked for exponents.

1. $\log_a b + \log_a c = \log_a bc$
2. $\log_a b - \log_a c = \log_a \frac{b}{c}$
3. $\log_a b^c = c \log_a b$
4. $\frac{\log_a b}{\log_a c} = \log_c b$
5. $\log_a b = \frac{1}{\log_b a}$
6. $\log_a b \cdot \log_b c = \log_a c$
7. $\log_{a^c} a^b = \frac{b}{c}$
8. $a^{\log_a n} = n$
9. If $\log_a b = \log_a c$, then $b = c$

Once again, I encourage you to not take these properties for granted. Make sure that you know why each property holds true. Proving the properties will not only help you understand the properties, but will help you understand how to solve other types of logarithm problems.

If no base is specified, then base 10 is assumed. If a problem says ln, then it means the "natural log", which is e. The logarithm rules described above apply for both of these bases. I end with a theorem that relates the logarithm to the number of digits.

Theorem 1.1. The number of digits of N is $\lfloor \log N + 1 \rfloor$.

Problem 1.1. Simplify $\dfrac{24x^4y^5z^3}{48x^{-2}y^3y^{-2}}$

Problem 1.2. Find n such that $3^n = 9^5$

Problem 1.3. Find x such that $2^{3x-1} = 2^{2x+2}$.

Problem 1.4. Evaluate $\log_5 125^{783}$.

Problem 1.5. Evaluate $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \dots \log_{127} 128$.

Problem 1.6. If $e^{5/x} = 30$, what is the value of x, assuming that the natural log of $2 = 0.7$, and the natural log of $15 = 2.7$.

Problem 1.7. Find all t such that $2\log_3(1-5t) = \log_3(2t+5) + 2$.

Problem 1.8 (2020 ARML Local)**.** Let $N = 2020^{20}$. Given that $10^{0.3010} < 2 < 10^{0.3011}$, compute the number of digits in the base-10 expansion of N.

2 Binomial Theorem

The Binomial Theorem is a powerful theorem in mathematics that allows us to expand a binomial raised to a certain power quite efficiently. For example, if we wanted to expand $(x+y)^3$, we could multiply all the terms together, but this is quite time-consuming and tedious. The Binomial Theorem offers a faster way to perform this calculation.

Theorem 2.1 (Binomial Theorem)**.** For positive integer n, the expansion of $(a+b)^n$ is

$$\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1++\ldots+\binom{n}{k}a^{n-k}b^k+\ldots+\binom{n}{n-2}a^2b^{n-2}+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}a^0b^n.$$

For completeness, I provide a combinatorial proof of the theorem (note that there also exists a proof by induction, but I find this proof to be more intuitive).

Proof. Let us write $(a+b)^n$ as

$$\underbrace{(a+b)\cdot(a+b)\cdot(a+b)\cdot\ldots\cdot(a+b)}_{n\text{ times}}.$$

We have to pick a total of n letters, one from each "pair" $(a+b)$. We proceed with casework. Suppose that we choose all n letters to be a's, and none of them to be b's. There are $\binom{n}{0}$ ways to do this, which gives us our first term of $\binom{n}{0}a^nb^0$. We proceed similarly for the other cases. If we choose $n-1$ of the letters to be a and 1 letter to be b, then there are $\binom{n}{1}$ ways to do this, which gives us our second term of $\binom{n}{1}a^{n-1}b^1$. This process continues for all the possible cases, which gives us a total of

$$\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{k}a^{n-k}b^k+\ldots+\binom{n}{n-2}a^2b^{n-2}+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}a^0b^n$$

ways to pick n letters, as desired. ☐

Some Science Bowl questions will be boring and ask you to just expand $(x+4)^3$. Of course, in cases like these, using the Binomial Theorem is inevitable. However, in some questions, they may ask you other questions, where your first instinct may to apply the Binomial Theorem, but there is actually a more clever thought process behind it. For instance, consider the following examples.

Example 2.1. What is the constant term of the expansion of $(2x^2+\frac{3}{x})^3$?

Solution 2.1. Yes, we could potentially use the binomial theorem to find the answer, but since there are fractions involved, things get messy pretty fast. So, instead of using the Binomial Theorem on the whole expression, we only focus on the stuff we care about.

The question is asking for the **constant** term, which means that it does not have an x term. Thus, the x's must cancel each other out. The only way for the x's to cancel out is if we have one $2x^2$ term and two $\frac{3}{x}$ terms. In other words, the we only care about the $\binom{3}{1}(2x^2)^1(\frac{3}{x})^2$ term of the expansion. This is much easier to calculate, so we receive an answer of $3 \cdot 2x^2 \cdot 3^2/x^2 = \boxed{54}$.

Example 2.2. What is the sum of the coefficients of the expansion $(2x+3)^4$?

Solution 2.2. Let $f(x) = (2x+3)^4$. Once again, we may be tempted to blindly start using the Binomial Theorem, but this is quite time-consuming. Instead, before expanding $f(x)$, let us first consider what our expansion will look like. It will probably look something like

$$ax^4 + bx^3 + cx^2 + dx + e,$$

where a, b, c, d, e are some integers. The problem is asking us for $a+b+c+d+e$. Note that if we let $x = 1$, then $ax^4 + bx^3 + cx^2 + dx + e$ becomes $1a + 1b + 1c + 1d + 1e$, which is exactly what the question is asking! So, it suffices to compute $f(1)$, which is just $(2 \cdot 1 + 3)^4 = 5^4 = \boxed{625}$.

Overall, majority of the Science Bowl questions will just be a simple application of the Binomial Theorem. However, if you think something is taking too long/requires too much computation, keep an eye out for a simpler method.

Problem 2.1. Expand $(x+4)^3$.

Problem 2.2. Expand $(2x-1)^4$.

Problem 2.3. What is the coefficient of y^3 in $(y+6)^6$?

Problem 2.4. How many terms are there in the expansion of $(x+1)^{30}$?

Problem 2.5. Find the constant term in the expansion of $\left(x + \frac{2}{x^2}\right)^9$

Problem 2.6. What is the integer part of $(\sqrt{3}+1)^4$ when expanded?

Problem 2.7. The coefficient of the x^5y^3 term in the expansion of $(x+ny)^8$ is 1512. Compute n.

Problem 2.8 (AIME)**.** ★ Find the sum of all the roots, real and nonreal, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots. **Hint:** You will also need to know Vieta's formulas. If you don't know Vieta's, you can skip this problem.

3 Sequences and Series

Today, we discuss sequences and series. Specifically, we will cover arithmetic sequences and series, geometric sequences and series, and infinite geometric series. Before we begin, let us establish the distinction between *sequences* and *series*.

Definition 3.1. A *sequence* is a list of numbers written in some particular order.

Example 3.1. $2, 4, 6, 8$ is a sequence.

Definition 3.2. A *series* is the sum of numbers that are written in a particular order.

Example 3.2. $1 + 0.5 + 0.25 + \ldots$ is a series.

Definition 3.3. An *arithmetic sequence* is a sequence such that the difference between two consecutive terms is constant. In other words, in order to get from one term to the next term, we add the "common difference".

Definition 3.4. An *arithmetic series* is the sum of the terms of a arithmetic sequence.

Suppose we have an arithmetic sequence $a, a + d, a + 2d \ldots$, where a represents the first term, and d represents the common difference. The formula to find the nth term of the sequence is given by

$$\boxed{a_n = a + d(n - 1).}$$

Instead of memorizing this formula, let us see why it is true. To find the second term, we add the common difference *once* to the first term. To find the third term, we add the common difference *twice* to the first term. So, to find the nth term, we would add the common difference $n - 1$ times to the first term, which is exactly what the formula states.

Now, suppose we wanted to find the sum of a arithmetic series

$$a, a + d, a + 2d \ldots a + d(n - 1).$$

Let this sum be S. There is a slick way we can calculate the sum: let us write this series backwards, and add it to the original series. So, we get

$$\begin{array}{rcccccccc}
S = & a & + & a+d & + & a+2d & + & \ldots & + & a+d(n-1) \\
S = & a+d(n-1) & + & a+d(n-2) & + & a+d(n-3) & + & \ldots & + & a \\
\hline
2S = & 2a+d(n-1) & + & 2a+d(n-1) & + & 2a+d(n-1) & + & \ldots & + & 2a+d(n-1)
\end{array}$$

Dividing both sides by 2, we get that

$$\boxed{S = \frac{1}{2}n[2a + d(n - 1)]}.$$

Once again, I wouldn't spend time trying to memorize this formula. The key takeaway of this approach is to see how we "flipped" the series backwards, which made it much easier to calculate the sum. Now, we move on to geometric sequences and series.

Definition 3.5. A *geometric sequence* is a sequence such that the ratio between two consecutive terms is constant. In other words, to get from one term to the next term, we multiply the term by the "common ratio".

Definition 3.6. A *geometric series* is the sum of the terms of a geometric sequence.

We usually represent a general geometric sequences as $a, ar, ar^2, ar^3 \dots$ where a is the first term and r is the common ratio. Suppose we wanted to calculate the nth term. We take a similar (in fact, almost identical!) approach as we did for arithmetic sequences. The second term is given by $a \cdot r$, the third term as $a \cdot r \cdot r$, and so on. So, the nth term is just

$$\boxed{a_n = ar^{n-1}}.$$

Now suppose that we wanted to find the sum of a geometric series $S = a + ar + ar^2 + ar^3 + \dots ar^n$. To find this sum, we first multiply S by r to get

$$rS = ar + ar^2 + ar^3 + \dots ar^{n+1}.$$

Subracting gives

$$rS - S = S(r-1) = ar + ar^2 + ar^3 + \dots ar^{n+1} - (a + ar + ar^2 + ar^3 + \dots ar^n) = ar^n - a = a(r^n - 1),$$

and finally dividing by $r - 1$ gives

$$\boxed{S = \frac{a(r^n - 1)}{r - 1}}.$$

Finally, we consider how to find the sum if the series is *infinite*. We take an almost identical approach as above, and find that the sum is

$$\boxed{S = \frac{a}{1 - r}}.$$

Note that this only holds if $|r| < 1$, because if $|r|$ were greater than one, then we would not be able to find the sum, as it would just keep on getting bigger and bigger (converge).

Problem 3.1. The first term of an arithmetic sequence is 3 and the sixth term is 38. Find the 492^{nd} term of the sequence.

Problem 3.2. What is the sum of the first 50 terms of the arithmetic sequence with second term 35 and fourth term 55?

Problem 3.3 (2014 AMC 12A)**.** The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

Problem 3.4 (AoPS)**.** Find all integers k such that $46 + 44 + 42 + \cdots + k = 510$.

Problem 3.5 (David Altizio)**.** Let p and q be real numbers with $|p|, |q| < 1$, such that

$$p + pq + pq^2 + pq^3 + \ldots = 2 \quad \text{and} \quad q + qp + qp^2 + qp^3 + \ldots = 3.$$

Compute $100pq$.

Problem 3.6 (2014 AMC 12A)**.** Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

4 Factors

Today, we discuss the factors of a number, and interesting functions relating to factors. We begin with the definition of a factor, which hopefully everyone knows.

Definition 4.1. A *factor* (or divisor) of a number n is a number that divides evenly (leaves a remainder of 0) into n. A *proper* factor is a factor that is less than n.

A number with only 2 factors, 1 and itself, is known as a prime number, and numbers with more than 2 factors are known as composite numbers. Let's say we wanted to quickly find the number of factors of n. If n has a prime factorization of $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then the number of factors is given by

$$\boxed{(a_1+1)\cdot(a_2+1)\cdots(a_k+1)}.$$

Proof. We see that the maximum number of p_1s that can occur in a factor is a_1. Thus, we can either have zero p_1s, one p_1, two p_1s $\ldots$ or a_1 p_1s. This gives us a_1+1 possible choices for p_1. Similarly, we a_2+1 choices for p_2, a_3+1 choices for p_3, and so on and so forth until we have a_k+1 choices for p_k. Hence, to count the total number of factors, we simply multiply the results together, which yields

$$(a_1+1)\cdot(a_2+1)\cdots(a_k+1)$$

total factors, as desired. □

Please make sure that you follow this proof because this reasoning can easily be modified to solve many more problems.

Example 4.1. How many factors does 45 have?

Solution 4.1. Write 45 as $3^2 \cdot 5$, so the number of factors is $(2+1)(1+1) = 3 \cdot 2 = \boxed{6}$.

Now, suppose we wanted to find the sum of the factors of a number. Sure, we could list out all the factors and sum them all up, but that takes a lot of time. There is an interesting and slick way to calculate this quantity. If we have the same number n as before, then the sum of its factors is given by

$$\boxed{(p_1^0+p_1^1+\ldots p_1^{a_1})\cdot(p_2^0+p_2^1+\ldots p_2^{a_2})\ldots(p_k^0+p_k^1+\ldots p_k^{a_k})}.$$

I omit the proof of this proposition.

Example 4.2. Compute the sum of the factors of 45.

Solution 4.2. Write 45 as $3^2 \cdot 5$, so the sum of factors is $(3^0+3^1+3^2)(5^0+5^1) = 13 \cdot 6 = \boxed{78}$.

Finally, suppose we wanted to compute the product of the factors of n. This quantity is given by the expression

$$\boxed{n^{\frac{\tau(n)}{2}}},$$

where $\tau(n)$ denotes the number of positive divisors of n.

I omit the proof of this proposition.

Example 4.3. Compute the product of the factors of 45.

Solution 4.3. As calculated in Example 4.1, the number of divisors of n is 6, so the product of the factors is $45^{6/2} = \boxed{45^3}$.

These formulas can be simplified greatly if n is prime. Try to see if you can derive these yourself.

Problem 4.1. How many odd factors does 210 have?

Problem 4.2 (2011 AMC 8)**.** Let w, x, y, and z be whole numbers. If $2^w \cdot 3^x \cdot 5^y \cdot 7^z = 588$, then what does $2w + 3x + 5y + 7z$ equal?

Problem 4.3 (2005 AMC 10A)**.** How many positive cubes divide $3! \cdot 5! \cdot 7!$?

Problem 4.4 (MathCounts)**.** What fraction of the positive integer factors of 1000^3 are perfect squares?

Problem 4.5 (2017 AMC 10B)**.** The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

Problem 4.6 (1996 AHSME)**.** Suppose that n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors. How many positive divisors does $6n$ have?

5 Graph Theory

Today, I will discuss the basics of Graph Theory. We begin with some definitions, and work our way to the famous Handshaking Lemma and Euler's characteristic formula.

Definition 5.1. An *undirected graph* consists of a set of vertices V and a set of edges E between the vertices.

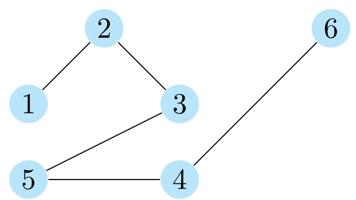
Definition 5.2. Two vertices u and v in an undirected graph G are *adjacent* if there is an edge e between them. Alternatively, two edges are *adjacent* if they share a common vertex.

Definition 5.3. An edge e is called *incident* to vertices u and v if it has u and v as endpoints.

Definition 5.4. An graph G is *simple* if between every two vertices, there is at most one edge, and there are no *loops* (a loop is an edge in the form (v, v)).

Definition 5.5. The *degree* of a vertex v is the number of edges incident to v. The degree of a vertex v is denoted by $\deg(v)$.

An example of an undirected graph G is shown below.



In this graph, we see that there are 6 vertices labeled $1, 2, 3, 4, 5, 6$, and there are a total of 5 edges. We also can find the degree of each vertex: $\deg(1) = 1, \deg(2) = 2, \deg(3) = 2, \deg(4) = 2, \deg(5) = 2$, and $\deg(6) = 1$. Now, with these definitions and example in mind, we state a very useful lemma.

Handshaking Lemma. In every undirected graph G, we have

$$\sum_{v \in V} \deg(v) = 2|E|,$$

where $|E|$ represents the number of edges.

To prove this lemma, we use a type of argument called "double counting", where we show that each side of the equation is counting the same quantity, but in different ways.

Proof. Each edge contributes twice to the total sum of the degrees of the vertices. Since both sides of the equation count the same thing, they must be equal, as desired. $\square$

You might be wondering why is this called the "handshaking" lemma. To show why it is called this, we consider an example.

Example 5.1. At a party, everyone shook hands with everybody else exactly once. If there were 10 people at the party, how many handshakes took place?

Solution 5.1. We interpret the problem in terms of graph theory: let a person be a vertex and a handshake be an edge. Since each person shook hands with everyone else once, each person shook hands 9 times, so the degree of each vertex is 9. Thus, the total sum of the degrees is $10 \cdot 9 = 90$. We want to find the total number of handshakes, which is equivalent to finding the total number of edges. We apply the handshaking lemma to find that

$$90 = 2|E| \implies |E| = \boxed{45}.$$

In general, if n people shake hands with everybody else exactly one time, then the total number of handshakes is given by $\frac{n(n-1)}{2}$ (or equivalently, $\binom{n}{2}$). However, if the question is modified and this general formula can't be used, then we should use the Handshaking Lemma.

We discuss one more important result of graph theory: Euler's polyhedron formula. Its proof is a bit too difficult for this lecture, so I leave it out.

Theorem 5.1 (Euler's Polyhedron Formula)**.** In a polyhedron with F faces, E edges, and V vertices, we have

$$F + V - E = 2.$$

If you are skeptical of this result, try it out for yourself! Consider a tetrahedron, which has 4 faces, 4 vertices, and 6 edges. Since $4+4-6$ does indeed equal 2, it checks out. Euler's formula works for all other polyhedra as well.

Problem 5.1. Given that a icosahedron has 20 faces and 30 faces, how many vertices does it have?

Problem 5.2 (Brilliant.org)**.** At a party with n people, everyone shook hands with everybody else exactly once. If 666 handshakes took place, compute n.

Problem 5.3 (2019 MathCounts)**.** After a hockey game, each member of the losing team shook hands with each member of the winning team. Afterwards, each member of the winning team gave a fist-bump to each of their teammates. Each team has 20 players. If n handshakes occurred and m fist-bumps occurred, what is the value of $n + m$?

Problem 5.4 (2017 AMC 10A)**.** At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?

Problem 5.5 (2013 AMC 10A)**.** A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have?

Problem 5.6 (2020 ARML Local)**.** In a Taekwondo class of 21 students, the students compete in "sparring teams" of three students. Suppose that there are 56 different sparring teams that have ever been formed by the students in the class (here, two teams are the same if they consist of the same three students and different otherwise). Given that each student has been, on average, a part of N different sparring teams, compute N.

6 Angles

Today, we discuss how to find the measure of angles. We measure angles in a unit called degrees ($\circ$). All non-degenerate angles are greater than 0° and less than 360°. If we have an angle $\angle ABC$, there are 5 different classifications for $\angle ABC$ depending on its measure:

Name	Constraint
Acute	$0^\circ < \angle ABC < 90^\circ$
Right	$\angle ABC = 90^\circ$
Obtuse	$90^\circ < \angle ABC < 180^\circ$
Straight	$\angle ABC = 180^\circ$
Reflex	$180^\circ < \angle ABC < 360^\circ$
Full	$\angle ABC = 360^\circ$

Now, we define some important relationships between angles.

Definition 6.1. Two angles are *complementary* if their sum is equal to 90 degrees.

Definition 6.2. Two angles are *supplementary* if their sum is equal to 180 degrees.

Example 6.1. The complement of 63° is 27°. The supplement of 63° is 117°.

Now, we shift our focus from individual angles to angles in polygons (**Note:** when we talk about the angles in polygons, we are talking about *interior* angles, which are the angles inside the polygon. We are not talking about the *exterior* angles). We first look at an important fact about the angles in a triangle.

Theorem 6.1 (Sum of Angles in Triangles)**.** The sum of angles in any triangle is equal to 180°.

The proof of this theorem is quite easy, so I leave it out (Hint: draw two parallel lines). Despite its simple statement, this fact is very useful as it allows us easily generalize to shapes with more sides.

Suppose we wanted to find the sum of a quadrilateral. It seems like our parallel lines approach won't work here like it did for triangles. However, we can split the quadrilateral into 2 triangles, each with sum of angles as 180°. Since we have 2 triangles, the sum of angles in a quadrilateral is $2 \cdot 180 = 360^\circ$.

But what if we wanted to find the sum of the angles in a pentagon? Hexagon? $n-$gon? Fortunately, we can use the same approach that we used for quadrilaterals: split the polygon into triangles. For a pentagon, we would split it into 3 triangles, for a hexagon it would be 4 triangles, and for a $n-$gon, it would be $n-2$ triangles. Thus, the sum of the angles in a pentagon is $3 \cdot 180$, hexagon is $4 \cdot 180$, and polygon is

$$\boxed{180(n-2),}$$

where n is the number of sides.

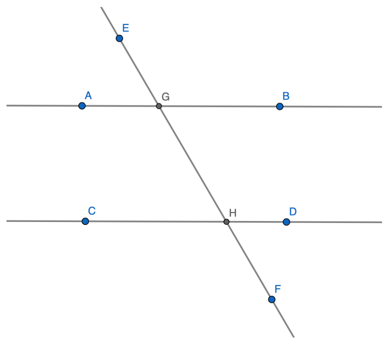
Corollary 6.1. If a polygon is regular, then all the angles are congruent to each other (and the sides are all congruent to each other). So, the measure of a single interior angle of the polygon would be the total sum of the measures divided by the number of sides, or $\boxed{\frac{180(n-2)}{180}}$.

It is definitely worth it to remember these formulas, but it is also important to remember the method that we used to get the formulas. Finally, I state some other important theorems regarding angles (without proof) because it is important to know them as well.

Theorem 6.2 (Base Angles)**.** In an isosceles triangle, the two angles opposite of the congruent sides are congruent.

Theorem 6.3 (Exterior Angles)**.** The sum of the exterior angles of a polygon equals 360°.

Theorem 6.4 (Angles formed by Tranversals)**.** In the figure below, lines AB and CD are parallel, and EF is a transversal. The alternate interior angles are congruent, the alternate exterior are congruent, same-side interior angles are supplementary, same-side exterior angles are supplementary, and corresponding angles are congruent (an example of corresponding angles are $\angle AGE$ and $\angle CHG$.

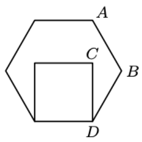


Problem 6.1. The supplement of the complement of an angle is 47°. Find the measure of the angle.

Problem 6.2. The angles in a heptagon are $110°, 120°, 130°, 140°, 150°, 160°, x°$. Find x.

Problem 6.3. Compute the difference between the measure of an interior and exterior angle of a 100−gon?

Problem 6.4 (Mathcounts 2011)**.** A square is located in the interior of a regular hexagon, and certain vertices are labeled as shown below. What is the degree measure of $\angle ABC$?



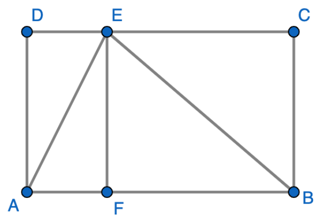
Problem 6.5 (2020 AIME I)**.** In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7 Triangle Area

Today we discuss many ways to compute the area of a triangle. This information may seem a bit contrived, but is one of the most important topics that appear in contest. We begin with the most basic method.

Theorem 7.1. The area of a triangle with base b and height h is $\frac{1}{2}bh$.

Proof. Every triangle ABE can be inscribed in a rectangle ABCD with base b and height h. It is clear that the area of $\triangle ABE$ is equal to the sum of the areas of $\triangle AFE$ and $\triangle BFE$. However, the areas of $\triangle AFE$ and $\triangle BFE$ are half of the area of their respective rectangle that they are inscribed in. Thus, the area of $\triangle ABE$ is half the area of the rectangle, or $\frac{1}{2}bh$, as desired.



□

This is by far the most common and simplest area formula for triangles. Whenever a question asks for the area of a triangle (or any figure), you should first try to use this method. We now move on to a formula that results from this formula.

Theorem 7.2. The area of a triangle with sides a, b, c is $\dfrac{ab \sin C}{2}$, where $\angle C$ is opposite side c.

Proof. Use the same diagram as above. Let b be the base and a one of the other sides. The length of the altitude is simply $a \sin C$. By using formula $bh/2$, we conclude that the area of the triangle is indeed $\frac{ab \sin C}{2}$. □

This formula is not used as often, but can be useful if you are given a nice angle like 30° or 60°. We now move on to an extremely important area formula that only uses the sides of a triangle.

Theorem 7.3 (Heron's Formula)**.** The area of $\triangle ABC$ with sides a, b, c and semiperimeter $s = \frac{a+b+c}{2}$ is $\sqrt{s(s-a)(s-b)(s-c)}$.

This result is known as Heron's Formula. Its proof is a bit bashy, so I leave it out. Although this is an extremely useful formula, I would use this only in emergencies as it involves a bit more computation. Now, we move on to some lesser-known area formulas.

Theorem 7.4. The area of $\triangle ABC$ with sides a, b, c, semiperimeter s and inradius r is rs.

Proof. Call the incenter I .Recall that the incenter is equidistant from the sides of the triangle, and that this distance is equal to r. If we draw AI, BI, CI, then we have divided the triangle into three smaller triangles with bases a, b, c and height r,. The area of $\triangle ABC$ is simply the sum of these triangles, which is equal to $\frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{(a+b+c)r}{2} = rs$, as desired. □

Now that we have a formula for area involving the inradius, we wonder if there is a formula for area involving the circumradius. Fortunately, there is one.

Theorem 7.5. The area of $\triangle ABC$ with sides a, b, c, and circumradius R is $\dfrac{abc}{4R}$.

Proof. By using an earlier formula, the area of $\triangle ABC$ is $\frac{ab\sin C}{2}$. By the Extended Law of Sines, we have $\frac{\sin C}{c} = \frac{1}{2R}$, so $\sin C = \frac{c}{2R}$. Substituting this into our equation yields $\frac{abc}{4R}$, as desired. $\square$

Next, I present a technique to calculate the area of *any* polygon given its coordinates.

Theorem 7.6 (Shoelace Theorem)**.** The area of a triangle with coordinates $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ is

$$\frac{x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2}{2}.$$

More generally, for any polygon with coordinates (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), the area given by is

$$\frac{1}{2}|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)|.$$

This formula is known as the Shoelace Formula. When using this formula, you must be careful to label the coordinates (a_i, b_i) in clockwise order. Its proof is too difficult for this lecture, so I leave it out. Finally, I include the formula for an equilateral triangle because it shows up enough that it is worth memorizing (even though it is just a result of the other formulas).

Theorem 7.7. The area of an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$.

Problem 7.1 (2017 Mathcounts)**.** A triangle has three vertices given by coordinates $(2, 2), (2, -6)$ and $(-5, -9)$. What is the area of the triangle?

Problem 7.2. Compute the inradius and circumradius of a $13, 14, 15$ triangle.

Problem 7.3 (2019 Mathcounts)**.** What is the area of an isosceles triangle that has a base of length 12 units and base angles measuring 30 degrees? Express your answer in simplest radical form.

Problem 7.4 (2019 Mathcounts)**.** The length of the base of a particular triangle is 2 cm more than its height. If the triangle has area 12 cm squared, what is its height?

Problem 7.5. A regular hexagon and square share one side in common. If the hexagon has area 27, compute the area of the square.

Problem 7.6 (2007 AMC 10B)**.** A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle?

Problem 7.7 (Evan Chen)**.** The side lengths of a triangle are 30, 40 and 50. What is the length of the shortest altitude?

Problem 7.8 (2013 AMC 10A)**.** Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?