Pascal's Triangle

Arjun Rastogi

Sharon High School Math Team

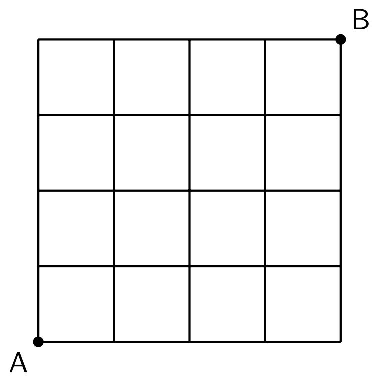
Overview

In this article, I discuss Pascal's triangle, an extremely important figure in combinatorics. Before delving into the topic right away, I explain a classic combinatorial problem, the "block-walking" problem. Then, I explain how to derive Pascal's triangle, and explain its relevance to combinatorics. I also discuss the binomial theorem. At the end of the article, I provide selected problems along with hints.

Theory

Before talking about Pascal's triangle, it will help to understand a different problem first. These problems are known as "block-walking" problems.

Problem 0.1. Consider the 4×4 grid below. Let there be an grasshopper starting on point A. The grasshopper can only jump 1 unit right or 1 unit up at a time. How many different ways can the grasshopper reach point B if it can only jump exactly 8 times?



Solution. If the grasshopper can only jump 8 times and must reach point B, then it must jump 4 units to the right and 4 units up, in some order. In other words, of the 8 total jumps the grasshopper makes, 4 of them must be rightwards and 4 of them must be upwards. This can be done in $\binom{8}{4}$ ways, so our answer is $\boxed{\text{70 ways.}}$ ■

In this problem, we first counted how many *total* jumps the grasshopper had to take. Then, we noted that the grasshopper had to make exactly 4 jumps to the right, which allowed us to calculate the total number of ways. This kind of reasoning will help us later on.

Now, we are ready to introduce Pascal's Triangle.

Definition 1

Pascal's Triangle is a triangular array of integers where each entry is generated by summing the two numbers directly above it.

Definiton 2

A *row* of Pascal's triangle is the horizontal set of numbers in the triangle. We start our row count at 0.

Definition 3

An *entry* in Pascal's triangle is one particular number in the triangle.

The first few rows, specifically, rows 0 to 5, of Pascal's triangle are shown below. The triangle was generated using the procedure described in Definition 1.1.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Note that Pascal's triangle will continue infinitely.

Remark 0.1. If we color all the odd numbers in Pascal's triangle black, and color the even numbers white, then we get a nice fractal. This result is known as *Sierpinski's Triangle.* I highly recommend you search up an image of this triangle.

Now, we look at another way to generate the entries of Pascal's triangle by viewing it as a block-walking problem. Let us replace the entries of the first 6 rows of Pascal's triangle with letters, as shown below.

$$
\begin{array}{ccccccccccc}
 & & & & & \text{A} & & & & & \\
 & & & & \text{B} & & \text{C} & & & & \\
 & & & \text{D} & & \text{E} & & \text{F} & & & \\
 & & \text{G} & & \text{H} & & \text{I} & & \text{J} & & \\
 & \text{K} & & \text{L} & & \text{M} & & \text{N} & & \text{O} & \\
\text{P} & & \text{Q} & & \text{R} & & \text{S} & & \text{T} & & \text{U}
\end{array}
$$

Our starting point is A, and the only valid moves are those one "unit" downwards to the left, or one unit downwards to the right (abbreviated to left and right from this point on). Now, we try to see how many different ways we can move from A to another letter. To do this, we employ the same reasoning we used in our block-walking problem:

- First, count the *total* number of moves needed.
- Then, count how many of those moves need to be to the right (say, k moves). Then, clearly, the remaining moves ($n-k$ moves) will be to the left.
- Thus, the total number of ways to get from A to that point will be $\binom{n}{k}$ or $\binom{n}{n-k}$ (they are both equal). I will just use $\binom{n}{k}$ to be consistent.

To get from A to A, we don't need any moves at all. We don't need any moves to the left or right, so there are $\binom{0}{0}$ ways to get from A to A.

To get from A to B, we need 1 move: 0 to the right, and 1 to the left. Thus, there are $\binom{1}{0}$ ways to do this.

To get from A to C, we need 1 move: 1 to the right and 0 to the left. There are $\binom{1}{1}$ ways to get there.

To get from A to D, we need 2 moves: 2 to the left and 0 to the right. There are $\binom{2}{0}$ ways to get there.

To get from A to E, we need 2 moves: 1 to the left and 1 to the right. There are $\binom{2}{1}$ ways do this.

To get from A to F, we need 2 moves: 0 to the left and 2 to the right. There are $\binom{2}{2}$ ways to get there.

This process continues for all of the infinite letters in Pascal's triangles. Now, we replace each letter with the number of ways to get there from A.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Thus, we found a different way to represent Pascal's triangle! Let us recap what we did. In order to derive the combinatorial representation of Pascal's triangle, we used a different problem, the block-walking problem, to help us. We forced our current, unfamiliar problem into a problem that we already knew how to do. In other words, we formed a *bijection* between the two problems.

Definition 4

A *bijection* is a one-to-one correspondence between two functions.

Forming a bijection between two problems is an extremely powerful strategy in math. As mentioned above, forming a bijection allows us to transform a harder problem into one that we know how to solve. We end the theory section of this handout with a nice theorem, the Binomial Theorem. We will explore how it relates to combinatorics and Pascal's triangle in the problem section.

Theorem 1. (Binomial Theorem)

For positive integer n, the expansion of $(a+b)^n$ is

$$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n.$$

There are many different proofs for the binomial theorem. If you are curious, I recommend that you look them up. If you are familiar with induction, then feel free to prove the Binomial Theorem using induction. Here, I provide a combinatorial proof.

Proof. Let us write $(a+b)^n$ as

$$\underbrace{(a+b)\cdot(a+b)\cdot(a+b)\cdot\ldots\cdot(a+b)}_{n \text{ times}}.$$

We have to pick a total of n letters, one from each "pair": $(a+b)$. We proceed with casework. Suppose that we choose all n letters to be a's, and none of them to be b's. There are $\binom{n}{0}$ ways to do this, which gives us our first term of $\binom{n}{0}a^n b^0$. We proceed similarly for the other cases. If we choose $n-1$ of the letters to be a and 1 letter to be b, then there are $\binom{n}{1}$ ways to do this, which gives us our second term of $\binom{n}{1}a^{n-1}b^1$. This process continues for all the possible cases, which gives us a total of

$$\binom{n}{0}a^n b^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{k}a^{n-k}b^k+\ldots+\binom{n}{n-2}a^2 b^{n-2}+\binom{n}{n-1}a^1 b^{n-1}+\binom{n}{n}a^0 b^n$$

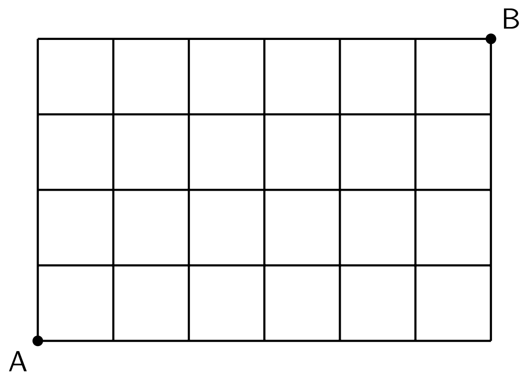
ways to pick n letters, as desired. $\square$

Problem Set

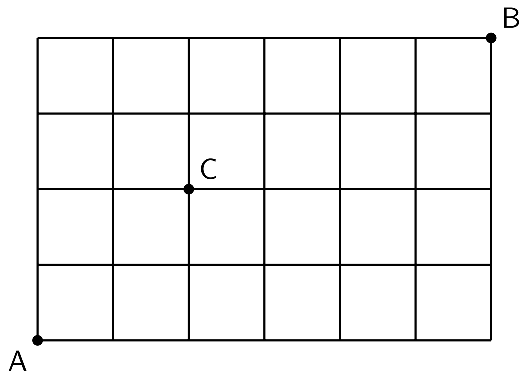
These are problems that were particularly helpful for me when learning this topic, are classical and will be found in almost all texts pertaining to this subject, or are problems that I have written. I have tried to cite every problem that I used, but if I am missing a citation, please let me know.

You might note that some of these problems are not just direct applications of the theorems and definitions in the text - instead, more insight is required to solve these problems. I have tried to arrange these problems in increasing difficulty. A ★ denotes a problem which I consider to be challenging. Have fun!

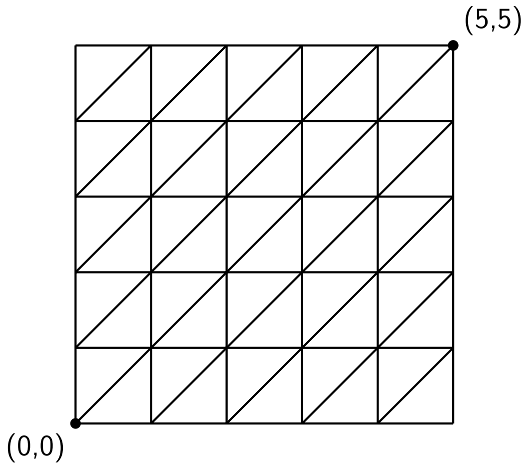
1. How many ways can a grasshopper get from A to B on the grid below if it can only jump 1 unit right or one unit up at a time?



2. How many ways can a grasshopper get from A to B on the grid on the next page if it can only jump 1 unit right or one unit up at a time, and can not land on C?



3. (ABMC 2018) Consider the following grid below. Stefan the bug starts from the origin, and can move either to the right, diagonally in the positive direction, or upwards. In how many ways can he reach (5, 5)?



4. Evaluate
$$\binom{6}{2} + \binom{6}{3} + \binom{7}{4}.$$

5. Prove that
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$
(this result is known as Pascal's identity).

6. (AoPS) One entry from the first 7 rows of Pascal's triangle is chosen at random. What is the probability that this entry is 1?

7. Find the sum of the entries in the 0th row, 1st row, 2nd row, 3rd row, 4th row, and 5th row of Pascals triangle.
   a) What is special about these numbers?
   b) Prove that this pattern will hold for the nth row of Pascal's triangle.

8. (AoPS) There is a row of Pascal's triangle that has three successive positive entries, a, b and c such that is b double c and is a triple c. If this row begins $1, n, \ldots$ then find n.

9. (AIME 1992) In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3 : 4 : 5$?

10. What is the coefficient of the x^4y^5 term in the expansion $(2x + y)^9$?

11. ★ (AMC 10B 2011) What is the hundreds digit of 2011^{2011}?

12. ★ (MathCounts State 2019) If a and b are real numbers with $a - b = 1$, what is the least possible value of $a^5 - b^5$? Express your answer as a common fraction.

13. ★ (AoPS) What is the smallest whole number greater than $(\sqrt{3} + \sqrt{6})^6$?

Hints

1. How many total moves are there? How many of them need to be up?
2. Find the number of "bad" paths, and subtract them from the total paths.
3. Use casework.
4. Draw out Pascal's Triangle using the binomial coefficient representation. Where is $\binom{6}{2}+\binom{6}{3}$ on Pascal's triangle? Then, look at where $\binom{7}{3}$ is on Pascal's triangle.
5. Use the block-walking problem to guide you.
6. How many 1's are there? How many total numbers are there? Then, calculate the probability.
7. a) Do these numbers have a common difference? What about a common ratio? Ask yourself more questions like these ones in order to see what is special about them.

   b) Use the binomial theorem
8. Express a, b, c using binomial coefficients, then create and solve the equation. The algebra is not as scary as it seems to be.
9. Express the terms using binomial coefficients, then create and solve the equation.
10. Direct application of binomial theorem.
11. Try to make this problem a binomial theorem question.
12. Since $a-b=1$, let $a=c+\frac{1}{2}$, and $b=c-\frac{1}{2}$. Then, use the binomial theorem.
13. What makes this problem hard? Then, try to *eliminate* it.

Further References

If you have any questions, noticed an error or typo in the handout, would like me to check your solution, or just want to contact me, feel free to send me an email at `kheera09@gmail.com`. You can also private message me on AoPS, where my username is `matharcher`.

To learn more about introductory combinatorics, I suggest the following resources:

- *Introduction to Counting and Probability*, by David Patrick
- *102 Combinatorial Problems*, by Titu Andreescu and Zuming Feng
- `brilliant.org`