Dimensional Analysis and Error Propagation

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Dimensional Analysis

Dimensional analysis is essentially checking the units of components of an equation/relationship and seeing if the outcome has units that you would expect. Many $F = ma$ problems do not require us to physically solve the problem, but rather simply check which answer choice has the correct units.

Example

An engineer is given a fixed volume V_m of metal with which to construct a spherical pressure vessel. Interestingly, assuming the vessel has thin walls and is always pressurized to near its bursting point, the amount of gas the vessel can contain, n (measured in moles), does not depend on the radius r of the vessel; instead it depends only on V_m (measured in m^3), the temperature T (measured in K), the ideal gas constant R (measured in $J/(K \cdot mol)$), and the tensile strength of the metal σ (measured in N/m^2). Which of the following gives n in terms of these parameters?

$$\text{(A) } n = \frac{2}{3}\frac{V_m\sigma}{RT} \qquad \text{(B) } n = \frac{2}{3}\frac{\sqrt[3]{V_m\sigma}}{RT} \qquad \text{(C) } n = \frac{2}{3}\frac{\sqrt[3]{V_m\sigma^2}}{RT}$$

$$\text{(D) } n = \frac{2}{3}\frac{\sqrt[3]{V_m^2\sigma}}{RT} \qquad \text{(E) } n = \frac{2}{3}\sqrt[3]{\frac{V_m\sigma^2}{RT}}$$

Solution: Here, note that we are solving for n, which is said to be measured in moles. This means that our answer should also have units of moles, so we need to see which combination of V_m, σ, ρ, R,and T gives units of moles. Starting with choice A, we see that $\frac{2}{3}\frac{V_m\sigma}{RT}$ has units

$$\frac{\text{m}^3 \times \frac{\text{N}}{\text{m}^2}}{\frac{\text{J}}{\text{K}\cdot\text{mol}} \times \text{K}} = \frac{\text{Nm}}{\frac{\text{J}}{\text{mol}}} = \frac{\text{Nm mol}}{\text{J}} = \text{mol},$$

which means that choice $\boxed{\text{A}}$ is our answer. The other 4 quantities can be verified to have incorrect units. ■

When doing dimensional analysis problems, it is useful to represent quantities with units that are some combination of "fundamental quantities". There are 7 fundamental quantities in physics:

Length	Mass	Time	Temperature	Electric Current	Luminous Intensity	Amount of Substance
Meter (m)	Kilogram (kg)	Second (s)	Kelvin (K)	Ampere (A)	Candela (cd)	Mole (mol)

For example, we would write the units of density as $\frac{[\text{M}]}{[\text{L}]^3}$ since we know that density is equal to mass over volume (which is length^3).

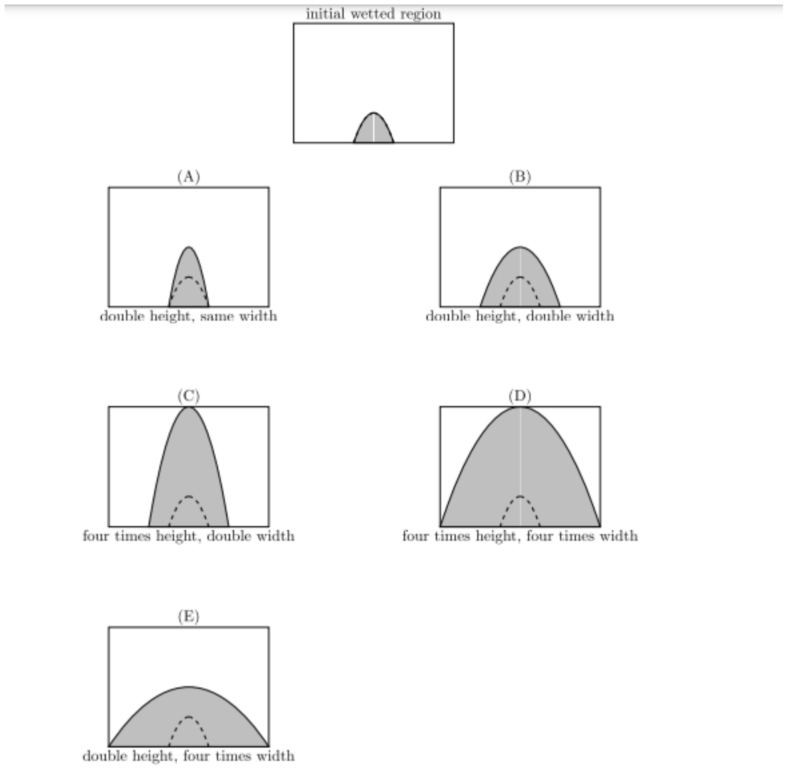
Problems

1. (2012) The softest audible sound has an intensity of $I_0 = 10^{-12} W/m^2$. In terms of the fundamental units of kilograms, meters, and seconds, this is equivalent to
   (a) 10^{-12} kg/s^3
   (b) 10^{-12} kg/s
   (c) 10^{-12} kg^{2}m/s
   (d) 10^{-12} kg^{2}m/s^2
   (e) 10^{-12} kg/m $\cdot$ s^3

2. (2015) A flywheel can rotate in order to store kinetic energy. The flywheel is a uniform disk made of a material with a density ρ and tensile strength σ (measured in Pascals), a radius r, and a thickness h. The flywheel is rotating at the maximum possible angular velocity so that it does not break. Which of the following expression correctly gives the maximum kinetic energy per kilogram that can be stored in the flywheel? Assume that α is a dimensionless constant.
   (a) $\alpha\sqrt{\rho\sigma/r}$
   (b) $\alpha h\sqrt{\rho\sigma/r}$
   (c) $\alpha\sqrt{(h/r)}(\sigma/\rho)^2$
   (d) $\alpha(h/r)(\sigma/\rho)$
   (e) $\alpha\sigma/\rho$

3. (2018) A massless beam of length L is fixed on one end. A downward force F is applied to the free end of the beam, deflecting the beam downward by a distance x. The deflection x is linear in F and is inversely proportional to the cross-section moment I, which has units m^4. The deflection is also dependent on Young's modulus E, which has units N/m^2. Then x depends on L according to
   (a) $x \propto \sqrt{L}$
   (b) $x \propto L$
   (c) $x \propto L^2$
   (d) $x \propto L^3$
   (e) $x \propto L^4$

4. (2019) A water hose is at ground level at the base of a large wall. By aiming the hose at some angle and squirting the water at speed v, one wets a region on the wall shown below. If the speed of the water is doubled, what is the new region that can be wetted? Ignore the effect of water splashing beyond the point of contact. In the answer choices, the dotted line marks the initial wetted region.



5. (2019) A very long cylinder of dust is spinning about its axis with angular velocity ω at a steady state. Let r be the distance from the axis. If the dust is only held together by gravity, the density of the dust is proportional to:
   (a) r^{-2}
   (b) r^{-1}
   (c) The density does not depend on r.
   (d) r
   (e) r^2

   Note: The units for the gravitational constant are $kg^{-1} \cdot m^3 \cdot s^{-2}$.

Error Propagation

Error analysis is method that allows scientists to give an estimate of how uncertain their final measurements were. It allows them to put error bars on their data, to express what range of values the actual value is located in. To see how this is used, consider an example of calculating how much force is exerted on an object. If I measure the mass of an object to be 1.0 ± 0.01 kg, and then I measure the acceleration of this object to be 1 ± 0.02 m/s^2, how would I find the expected range of my force measurement? Would I add the uncertainty I had with my mass to the uncertainty of the acceleration? Do I multiply uncertainties? Error analysis, and specifically error propagation, provides us with answers to these questions.

However, we first must define what the error bars mean. When we say we measured the mass of something to be 1.0 ± 0.01 kg, this means that if we take many measurements of this mass and we plot the measurement value (i.e. the number of kilograms) against the amount of times that we measure this value, we will get a Gaussian curve in which the *standard deviation* is 0.01 kg. In other words, when we measure the amount of kilograms, we will probably get a value between 0.99 kg and 1.01 kg. This does not mean we will never get a value outside of these bounds. The error in a measurement is usually associated with the instrument; depending on how accurate a digital scale is it may only be able to produce measurements accurate to 10 grams, and everything else is determined by a gaussian distribution.

One final note: if we measure a variable, we denote what we get as x, and our error as Δx. When we report our measurements, we write $x \pm \Delta x$. The relative error, which is just the percent error divided by 100, is denoted as $RE(x) = \frac{\Delta x}{x}$.

With that out of the way, here are a couple of the most useful rules for error propagation:

1. Scalar multiplication: $c \cdot (x \pm \Delta x) = c \cdot x \pm c \cdot \Delta x \implies RE(x) = RE(c \cdot x)$.
2. $(x \pm \Delta x)^2 = x^2 \pm 2x\Delta x \implies RE(x^2) = 2RE(x)$.
3. $RE(x) \approx RE(\frac{1}{x})$ for small percent errors.
4. $(x \pm \Delta x) + (y \pm \Delta y) = (x + y) \pm \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Errors add in quadrature when you add or subtract two quantities.
5. $RE(x \cdot y) = \sqrt{RE(x)^2 + RE(y)^2}$. When you multiply two quantities, the relative errors add in quadrature.

Example

Let's say you measure the mass of an object to be 1 ± 0.01 kg and its acceleration to be $1 \pm 0.02 \frac{\text{m}}{\text{s}^2}$. What is the uncertainty of the net force?

To solve this problem we first compute the relative error of each measurement. $RE(m) = \frac{0.01}{1} = 1\%$. $RE(a) = \frac{0.02}{1} = 2\%$. Adding these in quadrature gives us that the relative error of the net force: $RE(F_{net}) = RE(m \cdot a) = \sqrt{0.01^2 + 0.02^2} = 0.0224$, which is a 2.24% error. ■

Problems

1. (2019) To test the speed of a model car, you time the car with a stopwatch as it travels a distance of 100 m. You record a time of 5.0 s, and your measurement has an uncertainty of 0.2 s. What is the uncertainty in your estimate of the car's speed? Assume that the car travels at a constant speed and the distance of 100 m is known very precisely.
   (a) $v = 20 \pm 0.16$ m/s
   (b) $v = 20 \pm 0.8$ m/s
   (c) $v = 20 \pm 1.0$ m/s
   (d) $v = 20 \pm 1.25$ m/s
   (e) $v = 20 \pm 4$ m/s

2. (2018) A group of students wish to measure the acceleration of gravity with a simple pendulum. They take one length measurement of the pendulum to be $L = 1.00 \pm 0.05$ m. They then measure the period of a single swing to be $T = 2.00 \pm 0.10$ s. Assume that all uncertainties are Gaussian. The computed acceleration of gravity from this experiment illustrating the range of possible values should be recorded as
   (a) 9.87 ± 0.10 m/s^2
   (b) 9.87 ± 0.15 m/s^2
   (c) 9.87 ± 0.25 m/s^2

(d) 9.87 ± 1.1 m/s^2

(e) 9.87 ± 1.5 m/s^2

Note: $T = 2\pi\sqrt{\frac{l}{g}}$

3. (2018) Alice and Bob are working on a lab report. Alice measures the period of a pendulum to be 1.013 ± 0.008 s, while Bob independently measures the period to be 0.997 ± 0.016 s. Alice and Bob can combine their measurements in several ways.

   1: Keep Alice's result and ignore Bob's

   2: Average Alice's and Bob's results

   3: Perform a weighted average of Alice's and Bob's results with Alice's result weighted 4 times more than Bob's

   How are the uncertainties of these results related?

   (a) Method 1 has the lowest uncertainty, and method 2 has the highest

   (b) Method 3 has the lowest uncertainty, and method 2 has the highest

   (c) Method 2 has the lowest uncertainty, and method 1 has the highest

   (d) Method 3 has the lowest uncertainty, and method 1 has the highest

   (e) Method 1 has the lowest uncertainty, and method 3 has the highest

Answers

Dimensional Analysis:

1. A
2. E
3. D
4. D
5. C

Error Propagation:

1. B
2. D
3. B