Combinatorial Identities and Arguments

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Overview

In this article, I discuss some common combinatorial identities that appear in middle school and high school math competitions. I start off with some basic identities and slowly work up towards more nontrivial identities. I provide proofs of each identity. While each identity can be proven algebraically, it does not always allow us to see *why* the identity is true; hence, we will emphasize the combinatorial proof of the identity, which is far more powerful. At the end of the article, I provide selected problems along with hints. Prior experience with combinatorics, especially Pascal's triangle, is helpful, but not necessary.

Theory

Recall the following:

Definition 1

The number of ways to choose k objects out of n total objects is denoted by $\binom{n}{k}$ (pronounced "n choose k"). This is called a *binomial coefficient.*

Proposition 0.1. * For $n \geq k$, we have that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k!}$.

Proof. We have n total objects, say, lollipops, and we would like to select k lollipops. For our first lollipop, we can select any of the n total lollipops. For our second choice, we can select any of the remaining $n-1$ lollipops. For our third choice, we can select any of the remaining $n-2$ lollipops. We continue selecting lollipops in this fashion k times, where we will have $n-k+1$ lollipops remaining. Hence, the total number of ways to select k lollipops is $n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)$. However, we have counted each possible subset a total of $k!$ times. To account for this, we must divide by $k!$, which makes our final count $\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k!}$, as desired. □

Remark 0.1. Since the number of ways we can choose k objects from n total objects is an integer, $\binom{n}{k}$ must be an integer as well.

With this important fact, we can now begin to come up with some identities.

Identity 1

For integers n, k, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

While the algebraic proof for this identity is quite simple, we will see that later identities cannot be tackled as easily. Thus, I only include the combinatorial proof for this identity.

Proof. (Combinatorial) The left hand side of the equation represents the number of ways that we can choose k of these n objects. However, the number of ways to k objects is equivalent to the number of not choosing k objects (the complement), which is represented by the right hand side of the equation. Thus, the identity is true. □

This identity is fairly common and useful in simplifying binomial coefficients and highlighting symmetry. The next identity is due to French mathematician Blaise Pascal.

Identity 2 (Pascal)

For integers n, k, we have
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Proof. (Combinatorial.) Suppose we have n students, and we need to choose k of them to go on a field trip to Disneyland. There are clearly $\binom{n}{k}$ ways to choose the students. Let there be a student named Alex. We have 2 cases: either Alex is lucky and gets to go on the field trip, or he is unlucky and has to go to school.

Case 1: Alex is lucky.
If Alex is lucky, then there are $n-1$ students remaining, and we need to choose $k-1$ others to go on the field trip with him. There are $\binom{n-1}{k-1}$ ways to do this.

Case 2: Alex is *un*lucky.
If Alex is unlucky, then he can't be chosen for the field trip. There are $n-1$ remaining students, and we need to choose k of them to go on the field trip. This can be done in $\binom{n-1}{k}$ ways.

Since these are the only 2 cases, and all possibilities are counted within these cases, their sum should give us the total number of ways we can select k students from n. Hence, we have
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$
as desired. □

Note the structure of this proof: we wanted to show that the left hand side of the equation equals the right hand side. In order to do this, we showed that each quantity counted the same thing – choosing k students from n – in two different ways. This is known as *double counting,* and it's a very powerful technique in combinatorics and problem-solving in general.

I will also provide the algebraic proof so that you are familiar with it.

Proof. (Algebra.) By using the definition of a binomial coefficient, we have
$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}.$$

Recalling the fact that $a! = a \cdot (a-1)!$, we multiply the first fraction by $\frac{k}{k}$, and the second fraction by $\frac{n-k}{n-k}$ to help simplify things.

$$\frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k}{k} + \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{n-k}{n-k} \implies \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \implies \frac{n \cdot (n-1)!}{k!(n-k)!} \implies \frac{n!}{k!(n-k)!},$$

which is simply $\binom{n}{k}$, as desired. □

As you can see, the algebra proof required much more computation than actual logic. The combinatorial proof is almost always more elegant than the algebraic proof. For the rest of the handout, I will only provide combinatorial proofs. If you feel like finding an algebraic explanation of the identity, by all means, do so. Our next identity comes from the nth row of Pascal's triangle.

Identity 3

For integer n,

$$\binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = 2^n.$$

Proof. To prove this identity, we will show that the left hand side and the right hand side of the equations are counting the same thing.

Suppose we have n balls and 2 baskets, called $\mathcal{A}$ and $\mathcal{B}$. We need to place these balls in the baskets. One approach to do this is to consider each ball separately. The first ball has 2 choices: it can either be placed in basket $\mathcal{A}$ or $\mathcal{B}$. The second ball also has the same 2 choices. In fact, each of the n balls has 2 choices since each ball's choices are independent of the other balls. Thus, there are 2^n ways to distribute the balls in this fashion.

Now, we use a different approach. Instead of going one ball at a time, we now consider the possible final outcomes of the baskets. At the end of distributing, there will either be 0 balls in $\mathcal{A}$ and n balls in $\mathcal{B}$, or 1 ball in $\mathcal{A}$ and $n-1$ balls in $\mathcal{B}$, and so on and so forth, until we reach the last possibility, which is all n balls in $\mathcal{A}$ and 0 balls in $\mathcal{B}$. These are all the possible final outcomes, so the summing these outcomes will give us the total number of ways to distribute the n balls among the 2 baskets.

If we place 0 balls in $\mathcal{A}$ and n balls in $\mathcal{B}$, we are simply choosing 0 of the balls to be in $\mathcal{A}$, and placing the rest of the balls in $\mathcal{B}$. Hence, there are $\binom{n}{0}$ ways to do this. If we place 1 ball in $\mathcal{A}$ and $n-1$ balls in $\mathcal{B}$, then we are just choosing 1 ball to place in $\mathcal{A}$, and putting the rest in $\mathcal{B}$. Thus, there are $\binom{n}{1}$ ways to do this. This process continues for all n possibilities, so our final count is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}.$$

Since we have shown that each expression is just counting the same thing in 2 different ways, we have shown that they are equal. Hence, we are done.

□

The next identity is one of my favorite identities, and is named the “Hockey Stick" identity because of how it looks when represented on Pascal's triangle.

Identity 4 (Hockey Stick)

For integers n, k,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots \binom{n}{k} = \binom{n+1}{k+1}.$$

Proof. Once again, in order to prove that this identity is true, we will show that each side of the equation is counting the same thing.

Let there be $n+1$ balls, labeled $1, 2, \ldots n+1$. Then, the right hand side of the equation represents the number of ways to choose $k+1$ balls from $n+1$ total balls.

Now, we use a different approach. Instead of taking $k+1$ balls at random, we select the balls more systematically. We first choose 1 ball, and pay attention to its label. Then, we select k more balls such that the number on the label of our first ball is *greater* than the numbers on the other balls. In other words, our first ball is the "largest" ball out of the $k+1$ balls that we have selected. Let us see how this plays out by looking at different cases:

<u>If our first ball is labeled $k+1$:</u> If our first ball selected is labeled $k+1$, the labels on the remaining balls must be less than $k+1$. The only balls whose label is less than $k+1$ are balls $1, 2 \ldots k$. Hence, there are k total balls to choose from, and we need to select k of them. This can be done in $\binom{k}{k}$ ways.

<u>If our first ball is labeled $k+2$:</u> The only balls whose label is less than $k+2$ are balls $1, 2 \ldots k+1$. Hence, there are $k+1$ total balls to choose from, and we need to select k of them, which can be done in $\binom{k+1}{k}$ ways.

This process will continue in an identical manner until we reach the case where our first ball is labeled $n+1$. Since $n+1$ is greater than all of the other numbers, we have n total balls to choose from. Hence, there are $\binom{n}{k}$ ways for this final case.

Adding all of these cases together represents the total number of ways we can select $k+1$ balls from $n+1$ balls. Hence,

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots \binom{n}{k} = \binom{n+1}{k+1},$$

as desired.

□

There is another nice combinatorial proof for the Hockey Stick identity which uses a method called "stars and bars". If you know stars and bars, try to see if you can find the proof.

Our final identity is due to French mathematician Alexandre-Théophile Vandermonde.

Identity 5 (Vandermonde)

For integers n, m, k,

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\cdots\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}.$$

Proof. This proof is similar to our proof of the Hockey Stick identity. Suppose we have a committee of $n+m$ people. Furthermore, of these $n+m$ people, let n of them live in New York, and the remaining m of them to live in Massachusetts. We would like to choose k of the total $n+m$ people to move to Kentucky. Clearly, this can be done in $\binom{n+m}{k}$ ways.

Now, we will count the same thing, but in a slightly different manner. Suppose we choose 0 people from New York. This means that we have to choose all k people from Massachusetts. There are $\binom{n}{0}\binom{m}{k}$ ways to do this. Next, if we choose 1 person from New York, that means we have to choose $k-1$ of them from Massachusetts. This can be done in $\binom{n}{1}\binom{m}{k-1}$ ways. This process continues similarly until we choose all k people from New York, and no people from Massachusetts, which can be done in $\binom{n}{k}\binom{m}{0}$ ways. Summing all of these cases up gives us

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\cdots\binom{n}{k}\binom{m}{0},$$

which is the total number of ways to select k people from New York and Massachusetts. This is also precisely what the right hand side of the equation counts as well. Hence, if both quantities count the same thing, then they must be equal, as desired. $\square$

Problem Set

These are problems that were particularly helpful for me when learning this topic, are classical and will be found in almost all texts pertaining to this subject, or are problems that I have written. I have tried to cite every problem that I used, but if I am missing a citation, please let me know.

You might note that some of these problems are not direct applications of the identities - instead, more insight is required to solve these problems.

I have tried to arrange these problems in increasing difficulty, and a ★ denotes a problem which I consider to be challenging. Have fun!

1. If $n, m \in \mathbb{N}$ and $n < m$, then what is $\binom{n}{m}$?
2. For $n \in \mathbb{N}$, what is $\binom{0}{n}$?
3. A club's executive board has 7 openings: 2 secretaries, 2 treasurers, and 3 historians. If 5 people run for secretary, 6 people run for treasurer, and 7 people run for historian, how many possible executive boards can be formed?
4. (Feb HMMT 2007) A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

5. (AIME II 2005) A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.

6. (AoPS) Evaluate

$$9\binom{4}{4}+8\binom{5}{4}+7\binom{6}{4}+6\binom{7}{4}+5\binom{8}{4}+4\binom{9}{4}+3\binom{10}{4}+2\binom{11}{4}+1\binom{12}{4}$$

7. Prove that $\binom{n}{r}\binom{r}{m} = \binom{n}{m}\binom{n-m}{r-m}$.

8. Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

9. (MathStackExchange) Prove that $\binom{2n}{2} = 2\binom{n}{2} + n^2$

10. (AIME I 2020) A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N

11. Show that

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$$

using

   a) Vandermonde's identity.

   b) ★ a combinatorial argument.

12. ★ Fifteen students join a summer course. Every day, three students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

13. ★ (MMATHS 2014) Compute

$$\frac{\binom{54}{23}+6\binom{54}{24}+15\binom{54}{25}+15\binom{54}{27}+6\binom{54}{28}+\binom{54}{29}-\binom{60}{29}}{\binom{54}{26}}$$

14. ★ (AIME I 2015) Consider all 1000-element subsets of the set $1, 2, 3, ..., 2015$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$

15. ★ (Feb HMMT 2015) 2015 people sit down at a restaurant. Each person orders a soup with probability $\frac{1}{2}$. Independently, each person orders a salad with probability $\frac{1}{2}$. What is the probability that the number of people who ordered a soup is exactly one more than the number of people who ordered a salad?

Hints

1. Think back to the definition of a binomial coefficient. What does it actually represent, in words?
2. Refer to Hint 1.
3. Consider each position separately.
4. Use cases.
5. Try to translate the words into equations, using binomial coefficients. Then, when simplifying, recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.
6. Write out $9\binom{4}{4}$ as $\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}+\binom{4}{4}$. Do the same for the other terms, and then try to collapse them using an identity from the text.
7. Use a committee argument (or something along those lines) to show that both sides are counting the same thing, so they are equal.
8. Refer to Hint 7.
9. Refer to Hint 7.
10. Translate the words into math and then try to simplify using one of the identities in the text.
11. a) Plug in different values for n, m and see what you get. There is a nice relationship between n and m.
    b) Write $\binom{n}{0}^2$ as $\binom{n}{0} \cdot \binom{n}{0}$. What else is $\binom{n}{0}$ equal to? Do the same for the other terms, then, refer to Hint 7.
12. Try to use each piece of information that is given to you. Then, you should be able to set two quantities equal to each other, and then solve for the number of days.
13. What is special about the numbers $1, 6, 15, 15, 6, 1$? If you are unsure, look at the first few rows of Pascal's triangle. Then, let the desired sum be x, and rearrange the equation to $\binom{54}{23}+6\binom{54}{24}+15\binom{54}{25}+x\binom{54}{26}+15\binom{54}{27}+6\binom{54}{28}+\binom{54}{29}=\binom{60}{29}$. Finally, use an identity from the text to solve for x.
14. Look back to the proof of the Hockey Stick Identity, specifically, how we labeled the balls. Try to do something similar here. Then, when you make progress, actually use the Hockey Stick identity to simplify the computations.
15. This problem is similar to Problem $\#10$. Use Vandermonde's Identity.

Further References

If you have any questions, noticed an error in the handout, would like me to check your solution, or want to contact me for any reason, feel free to send me an email at kheera09@gmail.com. You can also private message me on AoPS, where my username is matharcher.
To learn more about combinatorics, I suggest the following resources:

- *Introduction to Counting and Probability*, by David Patrick
- *Intermediate Counting and Probability*, by David Patrick
- *102 Combinatorial Problems*, by Titu Andreescu and Zuming Feng

- *112 Combinatorial Problems*, by Elizabeth Reiland and Vlad Matei
- artofproblemsolving.com
- brilliant.org