Think

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These are a set of problems (arranged roughly in increasing order) that require you to be creative or clever in some way. Most of the questions are intended to have little to no computation, so if a question seems easy to bash, try to find a slicker solution. Enjoy.

1. (Moscow 1954) Find the maximal value of the ratio of a three-digit number to the sum of its digits.

2. (HMMT Nov 2019) Two hexagons are attached to form a new polygon $\mathcal{P}$. Compute the minimum number of sides that $\mathcal{P}$ can have.

3. (Evan Chen) Let $ABCDEZYXWV$ be an equilateral decagon with interior angles $\angle A = \angle V = \angle E = \angle Z = \angle C = 90°, \angle W = \angle Y = 135°, \angle B = \angle D = 225°$,and $\angle X = 270°$. Determine whether or not one can dissect $ABCDEZYXWV$ into four congruent polygons.

4. (EMCC 2019) Given that
$$\begin{cases} a + 5b + 9c = 1, \\ 4a + 2b + 3c = 2, \\ 7a + 8b + 6c = 9, \end{cases}$$
find $741a + 825b + 639c$.

5. (HMMT Nov 2019) In 2019, a team, including professor Andrew Sutherland of MIT, found three cubes of integers which sum to 42:
$$42 = (-80538738812075974\square)^3 + (80435758145817515)^3 + (12602123297335631)^3$$
One of the digits, labeled by $\square$, is missing. What is that digit?

6. (PUMaC 2016) Let $f(x) = 15x - 2016$. If $f(f(f(f(f(x))))) = f(x)$, find the sum of all possible values of x.

7. (OMO Fall 2019) There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

8. (HMMT Nov 2016) Let S be a subset of the set $\{1, 2, 3, \ldots 2015\}$ such that for any two elements $a, b \in S$, the difference $a - b$ does not divide the sum $a + b$. Find the maximum possible size of S.