Problem Set 1

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1. Two altitudes of a triangle have lengths 8 and 15. How many possible integer lengths are there for the third altitude?

2. Let d be a randomly chosen divisor of 2016. Find the expected value of
$$\frac{d^2}{d^2 + 2016}$$

3. Find all ordered triples (a, b, c) of positive reals that satisfy:
$$\lfloor a \rfloor bc = 3,$$
$$a \lfloor b \rfloor c = 4,$$
$$ab \lfloor c \rfloor = 5,$$

   where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

4. Let $a_1, a_2 \ldots$ be an arithmetic sequence and $b_1, b_2 \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20, a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

5. In rectangle $ABCD$, point A is reflected over diagonal $\overline{BD}$ to a point A'. If $A'B = A'C$ and $AA' = 6$, what is the area of rectangle $ABCD$?

6. In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language in common, what is the sum of all possible values of n such that a total of n conversations are held?

7. Suppose that on a parabola with vertex V and a focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

8. Five people are in a group, including Kai and Wen. To exercise social distancing, they keep dividing their groups into two disjoint groups until no two people are in a group. Because of a long-standing grudge, Kai and Wen cannot be together in a group after the first division. If the order of divisions matters and divisions occur one at a time, how many possible processes exist?

9. An *island* is a contiguous set of at least two equal digits. Let $b(n)$ be the number of islands in the binary representation of n. For example, $202010 = 111111001002$, so $b(2020) = 3$. Compute
$$b(1) + b(2) + \ldots b(2^{2020})$$