

Polynomials

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Problems

1. (PUMaC 2016) Let $f(x) = 15x - 2016$. If $f(f(f(f(f(x))))) = f(x)$, find the sum of all possible values of x .
2. (CNCM) Given that a, b, c are the roots of $x^3 - 21x^2 + 3x + 2 = 0$, compute $(a+1)(b+1)(c+1)$.
3. (MAT) Let a, b , and c be distinct positive integers with $a + b + c = 10$. Then there exists a quadratic polynomial p satisfying $p(a) = bc$, $p(b) = ca$, and $p(c) = ab$. Find the maximum possible value of $p(10)$.
4. A polynomial $p(x)$ is called self-centered if it has integer coefficients and $p(100) = 100$. If $p(x)$ is a self-centered polynomial, what is the maximum number of integer solutions k to the equation $p(k) = k^3$?

5. (USMCA 2020) Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3 + 2} + \frac{b^2}{b^3 + 2} + \frac{c^2}{c^3 + 2} + \frac{d^2}{d^3 + 2}.$$

6. (AMC 12 2005) Let $P(x) = (x - 1)(x - 2)(x - 3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) * R(x)$?
7. (OMO 2013) Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.
8. (AIME 2015) Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

9. (AMC 12 2021) A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?
10. (HMMT 2020) Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \dots, 2020$. Compute $P(2021^2)$.