Linear Algebra: Determinants

Arjun Vardhan

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Created: 20th July 2022
Last updated: 21st July 2022

1 Introduction

- For square matrices A we define a measure called the determinant, denoted $\det A$, such that it satisfies the following properties:
  1. **Linearity in each column:** $\det(v_1, v_2, ..., \alpha v_k + \beta u_k, ..., v_n) = \alpha \det(v_1, v_2, ..., v_k, ..., v_n) + \beta \det(v_1, v_2, ..., u_k, ..., v_n)$. Here, $(v_1, ..., v_n)$ denotes the matrix with columns $v_1, ..., v_n$.
  2. **Antisymmetry:** If two columns of a matrix are interchanged, then its determinant changes sign.
  3. **Normalization:** $\det I = 1$.
- More properties of the determinant can be deduced from the above. Let A be a square matrix. Then:
  1. **If A has a zero column then $\det A = 0$.** *Proof:*
  2. **If A has two equal columns, then $\det A = 0$.** *Proof:*
  3. **If one column of A is a multiple of another, $\det A = 0$.** *Proof:*
  4. **If the columns of A are not linearly independent, i.e, A is not invertible, then $\det A = 0$.** *Proof:*
- **The determinant does not change if we add to a column a linear combination of the other columns, leaving the other columns intact.** *Proof:*
- A square matrix is called diagonal if all non-diagonal entries are 0. Let $A = \begin{pmatrix} a_1 & 0 & ... & 0 \\ 0 & a_2 & ... & 0 \\ ... & ... & ... & 0 \\ 0 & 0 & ... & a_n \end{pmatrix}$. Since A can be obtained from I by multiplying its kth column by a_k, we have $\det A = a_1 a_2 ... a_n$. Thus the determinant of a diagonal matrix is simply the product of all its diagonal entries.
- A square matrix A is called upper triangular if all entries below the main diagonal are 0, and lower triangular if all entries above the main diagonal are 0.
- **A square matrix A is invertible if and only if $\det A \neq 0$.**
- **Let A be a square matrix and E an elementary matrix of the same size. Then, $\det(AE) = (\det A)(\det E)$.** *Proof:*
- **For a square matrix A, $\det A = \det A^T$.** *Proof:*
- **For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.** *Proof:*
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2 Existence and Uniqueness of the Determinant

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3 Cofactor Expansion

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4 Minors and Rank

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