# Linear Algebra: Systems of Linear Equations

Arjun Vardhan

†

Created: 11th July 2022
Last updated: 11th July 2022

## 1 Introduction

- Consider a system of $m$ linear equations with $n$ unknowns: $\sum_{k=1}^{n} a_{1,k}x_k = b_1$, $\sum_{k=1}^{n} a_{2,k}x_k = b_2$, ..., $\sum_{k=1}^{n} a_{m,k}x_k = b_m$. Taking the coefficients from the equations, we can form a matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,n} \\ a_{2,1} & a_{2,2} & ... & a_{2,n} \\ ... & ... & ... & ... \\ a_{m,1} & a_{m,2} & ... & a_{m,n} \end{pmatrix}$. Then the linear system can be written in the form of a matrix equation $Ax = b$, where $x = (x_1, x_2, ..., x_n)^T$ and $b = (b_1, b_2, ..., b_m)^T$.
- $A$ here is the coefficient matrix of the system. If we join the coefficient matrix to the vector $b$, we get the augmented matrix $\begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & ... & a_{2,n} & b_2 \\ ... & ... & ... & ... & ... \\ a_{m,1} & a_{m,2} & ... & a_{m,n} & b_m \end{pmatrix}$, which contains all the information necessary to solve the system.

## 2 Echelon Form and Reduced Echelon Form

- Linear systems can be solved by Gauss-Jordan elimination, also known as row reduction. By performing row operations on the augmented matrix, we can bring it into echelon form.
- There are three types of legal row operations:
  1. Row exchange: Interchanging two rows of the matrix.
  2. Scaling: Multiplying a row with a scalar.
  3. Row replacement: Replacing a row by its sum with a constant multiple of another row.

  None of these operations alter the solution set of the linear system.
- For each row in a matrix, the leftmost nonzero entry is called the pivot entry or just pivot.
- A matrix is said to be in echelon form or row echelon form when all zero rows are below all nonzero entries and the pivot entry of each nonzero row is strictly to the right of the pivot of the row above it.
- A matrix is said to be in reduced echelon form or reduced row echelon form when it is in echelon form, all pivot entries equal 1 and all entries above pivots are 0.
- When in echelon form, the variables corresponding to columns without pivots are called free variables.

## 3 Analyzing the Pivots

- A linear system is said to be inconsistent if it has no solutions.
- A linear system is inconsistent if and only if there is a pivot in the last column of an echelon form of the augmented matrix. I.e, if the echelon form has a row of the type (0 0 0 ... 0 $b$), with $b \neq 0$. In this case one of the equations ends up being $0x_1 + 0x_2 + ... + 0x_n = b$, which obviously has no solution in $\mathbb{C}$.
- If a linear system has a solution, the solution is unique if and only the system has no free variables, i.e, when the echelon form of the coefficient matrix has a pivot in every column.
- The equation $Ax = b$ has a solution for any $b \in \mathbb{F}^m$ if and only if the echelon form of $A$ has a pivot in every row. So a linear system is consistent only when its coefficient matrix has a pivot in every row when in echelon form.
- The equation $Ax = b$ has a unique solution for any $b \in \mathbb{F}^m$ if and only if the echelon form of $A$ has a pivot in every row and every column. So a linear system has a unique solution only when its coefficient matrix has a pivot in every row and every column when in echelon form.
- **Let $v_1, v_2, ..., v_m \in \mathbb{F}^n$. Let $A = [v_1, v_2, ..., v_n]$ be the matrix with these vectors as its columns. Then:**
  1. **The system $v_1, v_2, ..., v_m$ is linearly independent if and only if the echelon form of $A$ has a pivot in every column.** *Proof:*
  2. **The system $v_1, v_2, ..., v_m$ spans $\mathbb{F}^n$ if and only if the echelon form of $A$ has a pivot in every row.** *Proof:*
  3. **The system $v_1, v_2, ..., v_m$ is a basis in $\mathbb{F}^n$ if and only if the echelon form of $A$ has a pivot in every row and every column.** *Proof:*
- **Any linearly independent system of vectors in $\mathbb{F}^n$ cannot have more than $n$ vectors in it.** *Proof:*
- **Any two bases in a vector space $V$ have the same number of vectors in them.** *Proof:*
- **Any basis in $\mathbb{F}^n$ must have exactly $n$ vectors in it.** *Proof:*
- **Any spanning set in $\mathbb{F}^n$ must have at least $n$ vectors in it.** *Proof:*
- **A matrix $A$ is invertible if and only if the echelon form of $A$ has a pivot in every column and every row.** *Proof:*
- **Corollary: An invertible matrix must be a square matrix.**
- **If a square matrix is either left invertible or right invertible, then it is invertible.** *Proof:*

## 4 Finding $A^{-1}$ by Row Reduction

-

## 5 Dimension

-

## 6 Solution Set of a Linear System

-

## 7 Fundamental Subspaces of a Matrix, Rank

-

## 8 Represenation of a Linear Transformation in Arbitrary Bases

-