

Linear Algebra: Inner Product Spaces

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1 Introduction

- An inner product is a function that assigns a scalar to a pair of vectors.
- Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. The norm of a vector in \mathbb{R}^n is defined as $\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. The inner product of two vectors is defined as $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.
- Let V be a vector space over \mathbb{C} . An inner product in V must satisfy the following properties:
 1. **Conjugate symmetry:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
 2. **Linearity:** $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{C}$.
 3. **Non-negativity:** $\langle x, x \rangle \geq 0$ for all $x \in V$.
 4. **Non-degeneracy:** $\langle x, x \rangle = 0$ if and only if $x = 0$.

If such a function exists, then V together with its inner product is an inner product space.

- Given an inner product space, the norm of a vector x is defined as $\|x\| = \sqrt{\langle x, x \rangle}$.
- $\langle x, 0 \rangle = 0$ for all vectors x . *Proof:* $\langle x, 0 \rangle = \langle x, x - x \rangle = \langle x, x \rangle + \langle x, -x \rangle = \langle x, x \rangle - \langle x, x \rangle = 0$. ■
- Let $x \in V$. Then $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in V$. *Proof:*
- **Corollary:** Let x, y be vectors in an inner product space V . $x = y$ if and only if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$. *Proof:*
- Suppose two operators $A, B : X \rightarrow Y$ satisfy
- **Cauchy-Schwarz Inequality:** $|\langle x, y \rangle| \leq \|x\| \|y\|$. *Proof:*
- **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$. *Proof:*
- **Polarization Identities:** Let $x, y \in V$. If V is a real inner product space, then $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$. If V is a complex inner product space, then $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$. *Proof:*
- **Parallelogram Identity:** $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$. *Proof:*
- It is easy to show that the norm satisfies the following properties:
 1. **Homogeneity:** $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all vectors v , all scalars α .
 2. **Non-negativity:** $\|v\| \geq 0$ for all vectors v .
 3. **Non-degeneracy:** $\|v\| = 0$ if and only if $v = 0$.

A vector space equipped with a function satisfying the above properties, along with the triangle inequality, is called a normed space.

2 Orthogonality, Orthogonal and Orthonormal Bases

- Two vectors u and v are called orthogonal if $\langle u, v \rangle = 0$. This is denoted by $u \perp v$.
- **Pythagorean Identity:** If $u \perp v$, then $\|u\|^2 + \|v\|^2 = \|u + v\|^2$. *Proof:*
- A vector v is said to be orthogonal to a subspace E if $v \perp w$ for all $w \in E$.
- Subspaces E and F are said to be orthogonal if all vectors in E are orthogonal to all vectors in F .
- **Let E be spanned by v_1, v_2, \dots, v_n . Then $v \perp E$ if and only if $v \perp v_k$ for all k .** *Proof:*
- A system of vectors v_1, v_2, \dots, v_n is called orthogonal if $v_i \perp v_j$ whenever $i \neq j$. If $\|v_k\| = 1$ for all k , the system is called orthonormal.

- **Generalized Pythagorean Identity:** Let v_1, v_2, \dots, v_n be orthogonal. Then, $\left\| \sum_{k=1}^n \alpha_k v_k \right\|^2 = \sum_{k=1}^n |\alpha_k|^2 \|v_k\|^2$. *Proof:*

- **Any orthogonal system v_1, v_2, \dots, v_n of nonzero vectors is linearly independent.** *Proof:*
- If an orthogonal or orthonormal system v_1, v_2, \dots, v_n is also a basis, it is called an orthogonal or orthonormal basis.
- Let v_1, v_2, \dots, v_n be an orthogonal basis for V , and let $x \in V$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$. Taking the inner product of both sides with v_1 , we get $\langle x, v_1 \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, v_1 \rangle = \alpha_1 \langle v_1, v_1 \rangle$ (as $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$). Thus $\alpha_1 = \frac{\langle x, v_1 \rangle}{\|v_1\|^2}$. The same process can be used to find all other coordinates for any vector in this basis, i.e., $\alpha_k = \frac{\langle x, v_k \rangle}{\|v_k\|^2}$.

3 Orthogonal Projection and Gram-Schmidt Orthogonalization

- Let E be a subspace of inner product space V . For a vector $v \in V$, its orthogonal projection onto E , denoted $P_E v$, is a vector $w \in E$ such that $v - w \perp E$.
- **Let $w = P_E v$. Then for all $x \in E$, $\|v - w\| \leq \|v - x\|$. If $\|v - x\| = \|v - w\|$ for some $x \in E$, then $x = w$.** *Proof:*
- **Let v_1, v_2, \dots, v_n be an orthogonal basis in E . Then the orthogonal projection of v in E is given by $P_E v = \sum_{k=1}^n \alpha_k v_k$, where $\alpha_k = \frac{\langle v, v_k \rangle}{\|v_k\|^2}$.** *Proof:*
- **Gram-Schmidt Orthogonalization Algorithm:** Let x_1, x_2, \dots, x_n be a linearly independent system in inner product space V . Then we can find an orthogonal system v_1, v_2, \dots, v_n in V such that $\text{span}(x_1, x_2, \dots, x_n) = \text{span}(v_1, v_2, \dots, v_n)$. Additionally, for all $r \leq n$, we get $\text{span}(v_1, v_2, \dots, v_r) = \text{span}(x_1, x_2, \dots, x_r)$. First, let $v_1 = x_1$. Let $E_1 = \text{span}(v_1) = \text{span}(x_1)$. Define $v_2 = x_2 - P_{E_1} x_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1$. Let $E_2 = \text{span}(v_1, v_2)$. Clearly $\text{span}(x_1, x_2) = E_2$. Define $v_3 = x_3 - P_{E_2} x_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle x_3, v_2 \rangle}{\|v_2\|^2} v_2$. Let $E_3 = \text{span}(v_1, v_2, v_3)$. Again, it is clear that $\text{span}(x_1, x_2, x_3) = E_3$. As $x_3 \notin E_2$, $v_3 \neq 0$. (If x_3 had been in E_2 , so would v_3 , which would imply that $\langle v_3, v_3 \rangle = 0$ and thus $v_3 = 0$). Now suppose that we have made r steps in this process and obtained v_1, v_2, \dots, v_r such that $E_r = \text{span}(v_1, v_2, \dots, v_r) = \text{span}(x_1, x_2, \dots, x_r)$. Then let $v_{r+1} = x_{r+1} - P_{E_r} x_{r+1} = x_{r+1} - \sum_{k=1}^r \frac{\langle x_{r+1}, v_k \rangle}{\|v_k\|^2} v_k$. Since $x_{r+1} \notin E_r$, $v_{r+1} \neq 0$. Continuing this algorithm to the n th step we get an orthogonal system v_1, v_2, \dots, v_n .
- For a subspace E , its orthogonal complement, denoted E^\perp , is the set of all vectors orthogonal to E .

- For a subspace E , $(E^\perp)^\perp = E$. *Proof:*