Algebra I: Subgroups

Arjun Vardhan

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1 Definition

- Let G be a group. $H \subseteq G$ is a subgroup of G if $H \neq \varnothing$ and if $x, y \in H \implies x^{-1}, xy \in H$. We denote this relation by $H \leq G$, or $H < G$ if the containment is proper.
- Subgroups are just subsets of a group that are also groups themselves with the same operations.
- **Subgroup Criterion: $H \subseteq G$ is a subgroup if and only if $H \neq \varnothing$ and for all $x, y \in H$, $xy^{-1} \in H$.** *Proof:* If $H \leq G$, then $H \neq \varnothing$ and $x, y \in H \implies xy^{-1} \in H$. Conversely, suppose that H satisfies the two conditions. Then $x \in H \implies xx^{-1} = e \in H$. And thus $e, x \in H \implies ex^{-1} = x^{-1} \in H$. Suppose $x, y \in H$. Then, $y^{-1} \in H \implies xy \in H$. ■

2 Centralizers, Normalizers, Stabilizers and Kernels

- Let $A \subseteq G$, $A \neq \varnothing$. Let $C_G(A) = \{g \in G : gag^{-1} = a, \forall a \in A\}$. $C_G(A)$ is called the centralizer of A in G. Since $gag^{-1} = a$ if and only if $ga = ag$, $C_G(A)$ is the set of all elements in G that commute with all elements in A.
- $C_G(A) \leq G$. *Proof:* Let $a \in A$. $ea = ae$ so $e \in C_G(A)$ and thus $C_G(A) \neq \varnothing$. Suppose $x, y \in C_G(A)$. Then $xax^{-1} = y^{-1}ay = a$ for all $a \in C_G(A) \implies xy^{-1}ayx^{-1} = a \implies xy^{-1} \in C_G(A)$. ■
- The center of G, denoted $Z(G)$ is the set of all elements that commute with all elements of G. So $Z(G) = C_G(G)$. $Z(G) = G$ if and only if G is abelian.
- Let $A \subseteq G$, $A \neq \varnothing$. Let $gAg^{-1} = \{gag^{-1} : a \in A\}$. The normalizer of A in G, is the set $N_G(A) = \{g \in G : gAg^{-1} = A\}$. If $g \in C_G(A)$, then $gag^{-1} = a$ for all $a \in A$, so $C_G(A) \leq N_G(A)$.
- $N_G(A) \leq G$. *Proof:* Clearly, $e \in N_G(A)$ so $N_G(A) \neq \varnothing$. Suppose $x, y \in N_G(A)$. Then $xAx^{-1} = yAy^{-1} = A$.
- If G is a group acting on a set S, and $s \in S$, then the stabilizer of s in G is the set $G_s = \{g \in G : g \cdot s = s\}$.
- $G_s \leq G$. *Proof:* Since $e \in G_s$, $G_s \neq \varnothing$. Suppose $x, y \in G_s$. Then, $s = e \cdot s = y^{-1}y \cdot s = y^{-1}(y \cdot s) = y^{-1} \cdot s$, so $y^{-1} \in G_s$. Also, $(xy) \cdot s = x(y \cdot s) = x \cdot s = s$, so $xy \in G_s$. ■
- It can similarly be shown that the kernel of a group action is also a subgroup.

3 Cyclic Groups and Subgroups

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4 Subgroups Generated by Subsets

5 Lattice of Subgroups