Linear Algebra: Vector Spaces

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1 Vector Spaces

- A vector space V is a set, along with two operations, vector addition and scalar multiplication, satisfying the following properties:
  1. $v + w = w + v$ for all $v, w \in V$.
  2. $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
  3. There exists a zero vector $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
  4. For every $v \in V$ there exists $w \in V$ such that $v + w = 0$. Usually denoted $-v$.
  5. $1 \cdot v = v$ for all $v \in V$.
  6. $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$, for all $v \in V$, for all scalars α, β.
  7. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$, for all $u, v \in V$, for all scalars α.
  8. $(\alpha + \beta) \cdot v = \alpha \cdot v + \alpha \cdot v$, for all $v \in V$, for all scalars α, β.
- **The zero vector is unique.** *Proof:* Let $0, 0' \in V$ such that $0 + v = v$ and $0' + v = v$ for all $v \in V$. Then $0 = 0 + 0' = 0'$. ■
- **Additive inverses are unique.** *Proof:* Let $v \in V$. Let $u, w \in V$ such that $v + w = v + u = 0$. Then $v + w + w = v + u + w \implies (v + w) + w = (v + w) + u \implies w = u$. ■
- $0 = 0 \cdot v$ **for all** $v \in V$**.** *Proof:* $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 = 0 \cdot v$. ■
- $-v = (-1) \cdot v$ **for all** $v \in V$**.** *Proof:* $(-1) \cdot v + v = (-1) \cdot v + 1 \cdot v = (-1 + 1) \cdot v = 0$. Thus $(-1) \cdot v = -v$. ■
- The scalars are always from a field, usually $\mathbb{R}$ or $\mathbb{C}$.
- An $m \times n$ matrix is a rectangular array with m rows and n columns. Entries of a matrix are denoted a_{ij} or $(A)_{i,j}$, where i is the row and j is the column.
- Given a matrix A, its transpose A^T is the matrix formed by transforming the rows of A into columns. Formally, $(A)_{i,j} = (A^T)_{j,i}$.

2 Linear Combinations and Bases

- Let V be a vector space, and let $v_1, v_2, ..., v_p \in V$ be a collection of vectors. A linear combination of $v_1, v_2, ..., v_p$ is a sum of the form $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p = \displaystyle\sum_{k=1}^{p} \alpha_k v_k$.
- A system of vectors $v_1, v_2, ..., v_p \in V$ is called a basis for V if any vector $v \in V$ admits a unique representation as a linear combination of $v_1, v_2, ..., v_p$, i.e, $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p = \displaystyle\sum_{k=1}^{p} \alpha_k v_k$. The scalars $\alpha_1, \alpha_2, ..., \alpha_p$ are called the coordinates of v with respect to this basis.

- Consider the vector space $\mathbb{F}^n$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $e_1, e_2, ..., e_n \in \mathbb{F}^n$, where e_k is the vector whose entries are all 0 except the kth entry, which is 1. Clearly, any vector in $\mathbb{F}^n$ can be expressed uniquely as a linear combination of $e_1, e_2, ..., e_n$. This system is called the standard basis in $\mathbb{F}^n$.
- A system of vectors $v_1, v_2, ..., v_p \in V$ is called a generating system or spanning system if any vector in V can be represented as a linear combination of them.
- A linear combination $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p$ is called trivial if $a_k = 0$ for all k.
- A system of vectors $v_1, v_2, ..., v_p \in V$ is called linearly independent if the trivial linear combination is the only linear combination of $v_1, v_2, ..., v_p$ that equals 0.
- If a system is not linearly independent, it is said to be linearly dependent. I.e, if there exists a nontrivial linear combination of $v_1, v_2, ..., v_p$ that equals 0.
- **A system of vectors $v_1, v_2, ..., v_p \in V$ is linearly dependent if and only if one of the vectors v_k can be represented as a linear combination of the others.** *Proof:* Suppose the system $v_1, v_2, ..., v_p$ is linearly dependent. Then there exist scalars $\alpha_1, \alpha_2, ..., \alpha_p$, such that $\sum_{m=1}^{p} \alpha_m v_m = 0$ with $\sum_{m=1}^{p} |\alpha_m| \neq 0$. Let k be an index such that $\alpha_k \neq 0$. Then $\alpha_k v_k = -\sum_{m=1, m\neq k}^{p} \alpha_m v_m \implies v_k = \frac{1}{\alpha_k} \sum_{m=1, m\neq k}^{p} \alpha_m v_m$. Conversely, suppose $v_k = \sum_{m=1, m\neq k}^{p} \beta_m v_m$. Then $0 = v_k - \sum_{m=1, m\neq k}^{p} \beta_m v_m$, which is a nontrivial linear combination. ■
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3 Linear Transformations and Matrix-Vector Multiplication

4 Vector Space of Linear Transformations

5 Composition of Linear Transformations and Matrix Multiplication

6 Invertible Transformations and Isomorphisms

7 Subspaces