Linear Algebra: Eigenvalues and Eigenvectors

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1 Introduction

- A scalar λ is called an eigenvalue of operator (linear transformation) $A : V \to V$ if there exists $v \in V$, $v \neq 0$ such that $A(v) = \lambda v$. The vector v is called an eigenvector of A.
- Given an eigenvalue λ for A, we can find all corresponding eigenvectors by finding all vectors v such that $Av = \lambda v$, i.e, one must check the nullspace of $A - \lambda I$, which would contain all eigenvectors along with the 0 vector.
- The nullspace $\text{Ker}(A - \lambda I)$ is called the eigenspace of λ.
- The set of all eigenvalues of A is called the spectrum of A, denoted $\sigma(A)$.
- λ is an eigenvalue if and only if the nullspace $\text{Ker}(A - \lambda I)$ is non-trivial.
- Let $A : \mathbb{F}^n \to \mathbb{F}^n$. Then as $A - \lambda I$ has a non-trivial nullspace if and only if it is not invertible, so λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.
- If A is an $n \times n$ matrix, $\det(A - \lambda I) = 0$ is a polynomial of degree n with the variable λ. This is called the characteristic polynomial of A. The roots of this polynomial are precisely the eigenvalues of A.
- **Characteristic polynomials of similar matrices are equal.** *Proof:*
- **If $T : V \to V$ is a linear transformation, and A and B are two bases in V, then $[T]_A$ and $[T]_B$ are similar.** *Proof:*
- The (algebraic) multiplicity of eigenvalue λ of operator A is the multiplicity of λ as a root of $p(z) = \det(A - zI)$, $z \in \mathbb{C}$. This is the largest integer k such that $(z - \lambda)^k$ divides $p(z)$.
- The geometric multiplicity of the eigenvalue λ is the dimension of the eigenspace $\text{Ker}(A - \lambda I)$.
- **The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.** *Proof:*
- **Let A be an $n \times n$ matrix, and let $\lambda_1, \lambda_2, ..., \lambda_n$ be all of its eigenvalues (including multiplicities). Then,** $\text{trace}\, A = \lambda_1 + \lambda_2 + ... + \lambda_n$ **and** $\det A = \lambda_1 \lambda_2 ... \lambda_n$**.** *Proof:*
- **Eigenvalues of a triangular matrix (counting multiplicities) are precisely its diagonal entries.** *Proof:*
- **Eigenvalues of a diagonal matrix (counting multiplicities) are precisely its diagonal entries.** *Proof:*

2 Diagonalization

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