

Real Analysis I: Sequences and Series

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1 Convergent Sequences

- A sequence $\{p_n\}$ in metric space X is said to converge if there exists $p \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(p_n, p) < \epsilon$. In this case, we say $\lim_{n \rightarrow \infty} p_n = p$ or $p_n \rightarrow p$.
- If $\{p_n\}$ does not converge, it diverges.
- **If $p, p' \in X$ and $\{p_n\}$ converges to p and p' , then $p = p'$.** *Proof:* Let $\epsilon \geq 0$ be given. Then there exist integers N and N' such that $n \geq N \implies d(p_n, p) < \frac{\epsilon}{2}$ and $n \geq N' \implies d(p_n, p') < \frac{\epsilon}{2}$. Let $N^\circ = \max(N, N')$. So if $n \geq N^\circ$ then $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$. Since ϵ was arbitrary, we get $d(p, p') = 0$. ■
- **If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.** *Proof:* Suppose $p_n \rightarrow p$. There exists $N \in \mathbb{N}$ such that $n \geq N \implies d(p_n, p) < 1$. Let $r = \max(1, d(p_1, p), d(p_2, p), \dots, d(p_N, p))$. Then $d(p_n, p) < r$ for all $n \in \mathbb{N}$. ■
- **If $E \subset X$ and p is a limit point of E , then there is a sequence $\{p_n\}$ in E such that $\lim_{n \rightarrow \infty} p_n = p$.** *Proof:* Since p is a limit point, for each $n \in \mathbb{N}$ there exists $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. Given $\epsilon > 0$, choose N so that $N > \frac{1}{\epsilon}$. Then $n \geq N \implies n \geq \frac{1}{\epsilon} \implies \epsilon \geq \frac{1}{n} \implies d(p_n, p) < \epsilon$. So $p_n \rightarrow p$. ■
- **Suppose $\{s_n\}$ and $\{t_n\}$ are sequences in \mathbb{C} , and $\lim_{n \rightarrow \infty} s_n = s$, $\lim_{n \rightarrow \infty} t_n = t$. Then:**
 1. $\lim_{n \rightarrow \infty} s_n + t_n = s + t$. *Proof:* Given $\epsilon > 0$, there exist integers N_1, N_2 such that $n \geq N_1 \implies |s_n - s| < \frac{\epsilon}{2}$ and $n \geq N_2 \implies |t_n - t| < \frac{\epsilon}{2}$. Let $N_3 = \max(N_1, N_2)$. Then $n \geq N_3 \implies |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$. ■
 2. $\lim_{n \rightarrow \infty} cs_n = cs$, $\lim_{n \rightarrow \infty} c + s_n = c + s$, **for all $c \in \mathbb{C}$.** *Proof:* Given $\epsilon > 0$, there exists N such that $n \geq N \implies |s_n - s| < \epsilon$ which implies that $|cs_n - cs| < \epsilon$ and $|(c + s_n) - (c + s)| < \epsilon$. ■
 3. $\lim_{n \rightarrow \infty} s_n t_n = st$. *Proof:* Use the identity $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$. Given $\epsilon > 0$, there exist integers N_1 and N_2 such that $n \geq N_1 \implies |s_n - s| < \sqrt{\epsilon}$ and $n \geq N_2 \implies |t_n - t| < \sqrt{\epsilon}$. If we let $N = \max(N_1, N_2)$, then $n \geq N \implies |(s_n - s)(t_n - t)| < \epsilon$ and thus $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$. By taking the limit of both sides of the identity, we get $\lim_{n \rightarrow \infty} s_n t_n - st = 0$. ■
 4. $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$, **where $s_n \neq 0$ for all $n \in \mathbb{N}$.** *Proof:* Choose m such that $|s_n - s| < \frac{1}{2}|s|$. Then $|s_n| > \frac{1}{2}|s|$. Given $\epsilon > 0$ there exists an integer $N > m$ such that $n \geq N \implies |s_n - s| < \frac{1}{2}|s|^2\epsilon$. So for $n \geq N$, $\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s_n - s}{s_n s} \right| < \frac{2}{|s|^2} |s_n - s| < \epsilon$. ■

2 Subsequences

- Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers where $n_1 < n_2 < n_3 < \dots$ and so on. Then the sequence $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$. $\{p_{n_i}\}$ converges, its limit is a subsequential limit of $\{p_n\}$.

- $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p . *Proof:* Suppose every subsequence of $\{p_n\}$ converges to p . Then since $\{p_n\}$ is also a subsequence of itself, $\{p_n\}$ converges to p . Conversely, suppose $\{p_n\}$ converges to p and let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$. Given $\epsilon > 0$, there exists an integer M such that $n \geq M \implies |p_n - p| < \epsilon$. Now choose some integer $N \in \{n_k\}$ such that $N > M$. Then $n \geq N \implies |p_{n_k} - p| < \epsilon$, so $\{p_{n_k}\}$ converges to p . ■
- **Bolzano-Weierstrass Theorem:** Every bounded sequence in \mathbb{R} has a convergent subsequence.

3 Cauchy Sequences

- A sequence $\{p_n\}$ in a metric space X is a Cauchy sequence if for every $\epsilon > 0$, there is an integer N such that $d(p_m, p_n) < \epsilon$ if $m, n \geq N$.
- **In any metric space X , every convergent sequence is a Cauchy sequence.** *Proof:* Let $\{p_n\}$ be a sequence in X . If $p_n \rightarrow p$, then for all $\epsilon > 0$ there is an integer N such that $n \geq N \implies d(p_n, p) < \epsilon$. Then $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$ whenever $n \geq N$ and $m \geq N$. So $\{p_n\}$ is a Cauchy sequence. ■
- **In \mathbb{R}^k , every Cauchy sequence converges.**
- A metric space in which every Cauchy sequence converges is said to be complete.
- A sequence $\{s_n\}$ in \mathbb{R} is said to be monotonically increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.
- A sequence $\{s_n\}$ in \mathbb{R} is said to be monotonically decreasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.
- **If $\{s_n\}$ is monotonic, then $\{s_n\}$ converges if and only if it is bounded.** *Proof:* Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$. Since $\{s_n\}$ is bounded, let $s = \sup E$. Then $s_n \leq s$ for all $n \in \mathbb{N}$. For every $\epsilon > 0$, there exists an integer N such that $s - \epsilon < s_N \leq s$ since if it were not so, then $s - \epsilon$ would be an upper bound for E . Since $\{s_n\}$ is increasing, $n \geq N \implies s - \epsilon < s_n \leq s < s + \epsilon$, and so $\{s_n\}$ converges to s . The converse has already been proved previously, and the proof where $\{s_n\}$ is decreasing is analogous. ■

4 Upper and Lower Limits

- We define the extended real numbers, $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$, where $-\infty < r < \infty$ for all $r \in \mathbb{R}$. The arithmetic operations of \mathbb{R} are partially extended to $\tilde{\mathbb{R}}$:
 1. $a + \infty = \infty + a = \infty$ for $a \neq -\infty$
 2. $a - \infty = -\infty + a = -\infty$ for $a \neq \infty$
 3. $\infty - \infty$ is not defined.
- Let $\{s_n\}$ be a sequence in \mathbb{R} with the property that for every $M \in \mathbb{R}$ there is an integer N such that $n \geq N \implies s_n \geq M$. Then we say that $s_n \rightarrow \infty$. Similarly, if for every $M \in \mathbb{R}$ there an integer N such that $n \geq N \implies s_n \leq M$, we say that $s_n \rightarrow -\infty$.
- Let $\{a_n\}$ be a sequence in \mathbb{R} . We define the limit superior and limit inferior of $\{a_n\}$ as such:
 1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
 2. If $\{a_n\}$ is bounded above, then let $M_k = \sup \{a_k, a_{k+1}, a_{k+2}, \dots\}$. Then $\limsup a_n = \lim_{k \rightarrow \infty} M_k$.
 3. If $a_n \rightarrow -\infty$, then $M_k \rightarrow -\infty$ and $\limsup a_n = -\infty$.
 4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
 5. If $\{a_n\}$ is bounded below, then let $m_k = \inf \{a_k, a_{k+1}, a_{k+2}, \dots\}$. Then $\liminf a_n = \lim_{k \rightarrow \infty} m_k$.
 6. If $a_n \rightarrow \infty$, then $m_k \rightarrow \infty$ and $\liminf a_n = \infty$.
- An alternate definition follows:

1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
 2. If $\{a_n\}$ is bounded above, and there exists $u \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n < u + \epsilon$ and there exist infinitely many n where $a_n > u - \epsilon$, then $\limsup a_n = u$.
 3. Otherwise, $\limsup a_n = -\infty$.
 4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
 5. If $\{a_n\}$ is bounded below, and there exists $l \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n > l - \epsilon$ and there exist infinitely many n where $a_n < l + \epsilon$, then $\liminf a_n = l$.
 6. Otherwise, $\liminf a_n = \infty$.
- Another equivalent definition: Let \mathbb{S} be the set containing all subsequential limits of a_n , including ∞ and $-\infty$. Then $\limsup a_n = \sup \mathbb{S}$ and $\liminf a_n = \inf \mathbb{S}$. These numbers exist since \mathbb{S} is non-empty. If $\{a_n\}$ is bounded, then there exists at least one real subsequential limit. If a_n is unbounded in either direction, then there exist subsequences that diverge in either direction.
 - $\liminf a_n \leq \limsup a_n$. *Proof:* If $\limsup a_n = \infty$ or $\liminf a_n = -\infty$, we are done. So suppose $\limsup a_n = -\infty$. Then $\sup \mathbb{S} = -\infty$ and thus $\mathbb{S} = \{-\infty\}$. So $\liminf a_n = -\infty$. If $\liminf a_n = \infty$, then by similar reasoning we can show that $\limsup a_n = \infty$. So let $\limsup a_n = \alpha \in \mathbb{R}$ and let $\liminf a_n = \beta \in \mathbb{R}$. $\alpha = \sup \mathbb{S}$ and $\beta = \inf \mathbb{S}$, so $\alpha \leq \beta$. ■
 - $a_n \rightarrow \infty$ **if and only if** $\liminf a_n = \limsup a_n = \infty$. *Proof:* Suppose $a_n \rightarrow \infty$. Then for all $\alpha \in \mathbb{R}$, there exists K such that $n \geq K \implies a_n > \alpha$. So a_n is bounded below.
 - $a_n \rightarrow -\infty$ **if and only if** $\liminf a_n = \limsup a_n = -\infty$. *Proof:*
 - **If there exists $v \in \mathbb{R}$ such that given $\epsilon > 0$, there exists M such that $n \geq M \implies a_n < v + \epsilon$, then $v \geq \limsup a_n$.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n < v + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n > \alpha - \frac{1}{2}\delta = v + \frac{1}{2}\delta$, so we have a contradiction. Thus, $v \geq \limsup a_n$. ■
 - **If there exists $v \in \mathbb{R}$ such that given $\epsilon > 0$, there exist infinitely many n such that $a_n > v - \epsilon$, then $v \leq \limsup a_n$.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n < \alpha + \frac{1}{2}\delta$, and so we have a contradiction. Thus $v \leq \limsup a_n$. ■
 - **If there exists $v \in \mathbb{R}$ such that given $\epsilon > 0$, there exists M such that $n \geq M \implies a_n > v - \epsilon$, then $v \leq \liminf a_n$.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n < \alpha + \frac{1}{2}\delta$, so we have a contradiction. Thus $v \leq \liminf a_n$. ■
 - **If there exists $v \in \mathbb{R}$ such that given $\epsilon > 0$, there exist infinitely many n such that $a_n < v + \epsilon$, then $v \geq \liminf a_n$.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n < v + \frac{1}{2}\delta = \alpha - \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n > \alpha - \frac{1}{2}\delta$, so we have a contradiction. Thus $v \geq \liminf a_n$. ■
 - **For a sequence $\{a_n\}$ in \mathbb{R} , $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$ if and only if $\limsup a_n = \liminf a_n = a$.** *Proof:* Let $\limsup a_n = \liminf a_n = a$. Then for all $\epsilon > 0$, there exist integers M, N such that $n \geq M \implies a_n < a + \epsilon$ and $n \geq N \implies a_n > a - \epsilon$. Let $P = \max(M, N)$. Then $n \geq P \implies |a_n - a| < \epsilon$. Conversely, suppose $a_n \rightarrow a$. For all $\epsilon > 0$, there exists K such that $n \geq K \implies a - \epsilon < a_n < a + \epsilon$. Thus $a \leq \liminf a_n$ and $a \geq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$, we have $\limsup a_n = \liminf a_n = a$. ■
 - $\liminf(-a_n) = -\limsup a_n$. *Proof:* Let $\limsup a_n = \alpha$. Then, for every $\epsilon > 0$ there exists M such that $n \geq M \implies a_n < \alpha + \epsilon$ and there exist infinitely many n such that $a_n > \alpha - \epsilon$. So for every $\epsilon > 0$, there exists M such that $n \geq M \implies -a_n > -\alpha - \epsilon$ and there exist infinitely many n such that $-a_n < -\alpha + \epsilon$. So $\liminf(-a_n) = -\alpha$. ■
 - **Let $a_n \leq b_n$ for all $n \geq K$. Then $\limsup a_n \leq \limsup b_n$.** *Proof:* If $\limsup b_n = \infty$ then we are done.

5 Special Sequences

6 Series

- Given a sequence $\{a_n\}$, let $S_n = \sum_{k=0}^n a_k$. Then, $\sum_{n=0}^{\infty} a_n = \lim_{n \rightarrow \infty} S_n$. We say that $\sum_{n=0}^{\infty} a_n$ converges if and only if S_n converges. If S_n properly diverges to $\pm\infty$, then $\sum_{n=0}^{\infty} a_n$ properly diverges.
- S_n is called the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$.
- The Cauchy criterion can be restated in terms of series. S_n converges if and only if for all $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies |S_n - S_m| < \epsilon \implies \left| \sum_{k=0}^n a_k - \sum_{k=0}^m a_k \right| < \epsilon \implies \left| \sum_{k=n}^m a_k \right| < \epsilon$.
- If we let $m = n$, then we get $|a_n| < \epsilon$. Thus, if $\sum a_n$ converges, then $a_n \rightarrow 0$. The converse is not necessarily true.
- $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.
- $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ diverges but $\sum a_n$ converges.
- Let $x \in \mathbb{R}$. Then, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. So $x = x^+ - x^-$ and $|x| = x^+ + x^-$.
- **Let a_n be a sequence that is ultimately non-negative, and let A_n be the sequence of its partial sums. Then $\sum a_n$ converges if and only if A_n is bounded above. Proof:** Suppose $\sum a_n$ converges. Then A_n converges and is thus bounded. Conversely, suppose A_n is bounded above. Since a_n is ultimately non-negative, A_n is ultimately monotonically increasing. Thus A_n and $\sum a_n$ converge. ■
- **Basic Comparison Test: If $|a_n| \leq b_n$ for $n \geq N_1$, and if $\sum b_n$ converges, then $\sum a_n$ converges. If $c_n \geq d_n \geq 0$ for $n \geq N_2$, and if d_n diverges, then c_n diverges. Here, N_1, N_2 are fixed integers. Proof:** Suppose $\sum b_n$ converges. Given $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies \left| \sum_{k=n}^m b_k \right| < \epsilon$. Thus, $\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m b_k \leq \left| \sum_{k=n}^m b_k \right| < \epsilon$. So $\sum a_n$ converges. Now, suppose $\sum d_n$ diverges. If $\sum c_n$ converges, then $\sum d_n$ must also converge. So $\sum c_n$ diverges. ■
- **Comparison Test V1: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$, $\alpha, \beta > 0$ such that $n > M \implies \alpha a_n \leq b_n \leq \beta a_n$, then $\sum b_n$ converges if and only if $\sum a_n$ converges. Proof:**
- **Comparison Test V2: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$ such that $n > M \implies 0 \leq \frac{b_n}{b_{n+1}} \leq \frac{a_n}{a_{n+1}}$, then $\sum a_n$ converges if $\sum b_n$ converges. Proof:**
- **Comparison Test V3: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges. Proof:**
- **Comparison Test V4: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 = \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if $\sum b_n$ converges. Proof:**
- **Comparison Test V5: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} = \infty$, then $\sum b_n$ converges if $\sum a_n$ converges. Proof:**

7 Series of Non-negative Terms

- If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, this series diverges. *Proof:* If $x \neq 1$, then $X_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$. If $0 \leq x < 1$, then $\lim_{n \rightarrow \infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}$. If $x = 1$, then the sum is $1 + 1 + 1 + \dots$ which diverges. If $x > 1$ then $\frac{1-x^{n+1}}{1-x}$ diverges. ■
- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. *Proof:* If $p \leq 0$, then $\frac{1}{n^p}$ does not tend to 0, and thus the series diverges.

8 Euler's Number

9 Root and Ratio Tests

10 Power Series

11 Summation by Parts

12 Absolute Convergence

13 Addition and Multiplication of Series

14 Rearrangements