Linear Algebra: Inner Product Spaces

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Created: 22nd July 2022
Last updated: 22nd July 2022

1 Introduction

- An inner product is a function that assigns a scalar to a pair of vectors.
- Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. The norm of a vector in $\mathbb{R}^n$ is defined as $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. The inner product of two vectors is defined as $\langle x, y \rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$.
- Let V be a vector space over $\mathbb{C}$. An inner product in V must satisfy the following properties:
  1. **Conjugate symmetry:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
  2. **Linearity:** $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{C}$.
  3. **Non-negativity:** $\langle x, x \rangle \geq 0$ for all $x \in V$.
  4. **Non-degeneracy:** $\langle x, x \rangle = 0$ if and only if $x = 0$.

  If such a function exists, then V together with its inner product is an inner product space.
- Given an inner product space, the norm of a vector x is defined as $\|x\| = \sqrt{\langle x, x \rangle}$.
- **$\langle x, 0 \rangle = 0$ for all vectors x.** *Proof:* $\langle x, 0 \rangle = \langle x, x - x \rangle = \langle x, x \rangle + \langle x, -x \rangle = \langle x, x \rangle - \langle x, x \rangle = 0$. ■
- **Let $x \in V$. Then $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in V$.** *Proof:*
- **Corollary: Let x, y be vectors in an inner product space V. $x = y$ if and only if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$.** *Proof:*
- **Suppose two operators $A, B : X \to Y$ satisfy**
- **Cauchy-Schwarz Inequality:** $|\langle x, y \rangle| \leq \|x\| \, \|y\|$. *Proof:*
- **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$. *Proof:*
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2 Orthogonality, Orthogonal and Orthonormal Bases

- Two vectors u and v are called orthogonal if $\langle u, v \rangle = 0$. This is denoted by $u \perp v$.
- **Pythagorean Identity: If $u \perp v$, then $\|u\| + \|v\| = \|u\|^2 + \|v\|^2$.** *Proof:*
- A vector v is said to be orthogonal to a subspace E if $v \perp w$ for all $w \in E$.
- Subspaces E and F are said to be orthogonal if all vectors in E are orthogonal to all vectors in F.
- **Let E be spanned by $v_1, v_2, ..., v_n$. Then $v \perp E$ if and only if $v \perp v_k$ for all k.** *Proof:*
- A system of vectors $v_1, v_2, ..., v_n$ is called orthogonal if $v_i \perp v_j$ whenever $i \neq j$. If $\|v_k\| = 1$ for all k, the system is called orthonormal.
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3 Orthogonal Projection and Gram-Schmidt Orthogonalization

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