Algebra II: Introduction to Module Theory

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Created: 23rd February 2022
Last updated: 27th March 2022

1 Basic Definitions

- Let R be a ring. Let M be an abelian group with respect to an operation $+$. M is a left-module on R if there exists an action of R on M, i.e, a map $R \times M \to M$, denoted rm, such that:
  1. $(r + s)m = rm + sm$, for all $r, s \in R$ and all $m \in M$.
  2. $(rs)m = r(sm)$, for all $r, s \in R$ and all $m \in M$.
  3. $r(m + n) = rm + rn$, for all $r \in R$ and all $m, n \in M$.
  4. If R has unity, then $1m = m$ for all $m \in M$.

  A right-module on R can be defined analogously.
- Modules over a field $\mathbb{F}$ and vector spaces over $\mathbb{F}$ are the same thing.
- Let R be a ring and M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of ring elements, i.e, $rn \in N$ for all $r \in R$ and all $n \in N$. A submodule of M thus just a subset of M which is itself a module with the same operations.
- If R is a field, then submodules are the same thing as subspaces.
- Every R-module M has at least two submodules: M itself, and $\{0\}$, the trivial submodule.
- Let R be any commutative ring. Then R is a module on itself, where the action is simply regular multiplication in R. In this case the submodules of R would simply be the ideals of R.
- If M is an R-module and S is a subring of R with $1_S = 1_R$, then M is also an S-module.
- Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}^+$. The affine n-space over $\mathbb{F}$ is $\mathbb{F}^n = \{(a_1, a_2, ..., a_n) : a_i \in \mathbb{F}\}$. It is a module/vector space over $\mathbb{F}$, with addition and scalar multiplication defined componentwise.
- Let R be a ring with unity and let $n \in \mathbb{Z}^+$. Define $R^n = \{(a_1, a_2, ..., a_n) : a_i \in R\}$. This is an R-module with componentwise operations. It is called the free module of rank n over R.
- If M is an R-module and if I is an ideal of R, and if $am = 0$ for all $a \in I$, and all $m \in M$, then we say that M is annihilated by I. Here, M can be made into an (R/I)-module with the operation $(r + I)m = rm$. Since $am = 0$ for all $a \in I$, this is well defined. When I is a maximal ideal, then M is a vector space over the field R/I.
- Let A be an abelian group. For any $n \in \mathbb{Z}$ and $a \in A$, define $na = a + a + ... + a$ (n times) if $n > 0$, $na = 0$ if $n = 0$, and $na = -a - a - ... - a$ (n times) if $n < 0$. This makes A into a $\mathbb{Z}$-module, and shows that every abelian group is a $\mathbb{Z}$-module. Additionally, $\mathbb{Z}$-submodules are just subgroups of A.
- **Submodule Criterion: Let R be a ring and M be an R-module. Then, $N \subseteq M$ is a submodule of M if and only if $N \neq \varnothing$ and $x + ry \in N$ for all $r \in R$ and all $x, y \in N$.** *Proof:* Suppose N is a submodule. Then $0 \in N$ so $N \neq \varnothing$. Additionally, N is closed under addition and $rn \in N$ for all $r \in R$, $n \in N$. Conversely, suppose $N \neq \varnothing$ and $x + ry \in N$ for all $x, y \in N$ and all $r \in R$. Let $r = -1$. Then by the subgroup criterion, N is an (additive) subgroup of M. So $0 \in N$. Let $x = 0$. Then N is closed under the action of ring elements and is therefore a submodule. ■
- Let M be an R-module. $m \in M$ is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. $\mathrm{Tor}(M)$ denotes the set of all torsion elements in M.

2 Quotient Modules and Module Homomorphisms

- Let R be a ring and M, N be R-modules. A map $\Phi : M \to N$ is an R-module homomorphism if $\Phi(x+y) = \Phi(x) + \Phi(y)$ for all $x, y \in M$ and $\Phi(rx) = r\Phi(x)$ for all $r \in R, x \in M$.
- $\mathrm{Hom}(M, N)$ denotes the set of all module homomorphisms from M to N.
- $\mathbb{Z}$-module homomorphisms are the same as abelian group homomorphisms.
- Let $\Phi, \Psi \in \mathrm{Hom}(M, N)$ and define $\Phi + \Psi$ as $(\Phi + \Psi)(m) = \Phi(m) + \Psi(m)$ for all $m \in M$. Then $\mathrm{Hom}(M, N)$ is an abelian group. If R is a commutative ring, then define $r\Phi$ as $(r\Phi)(m) = r\Phi(m)$ for all $m \in M$. With this action, $\mathrm{Hom}(M, N)$ is itself an R-module.
- Since $\Phi \circ \Psi \in \mathrm{Hom}(M, N)$ whenever $\Phi, \Psi \in \mathrm{Hom}(M, N)$, using function composition as the multiplication operation, $\mathrm{Hom}(M, N)$ is a ring with unity.
- The ring $\mathrm{Hom}(M, M)$ is called the endomorphism ring of M and its elements are called endomorphisms. It is also denoted $\mathrm{End}(M)$.
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3 Generation of Modules, Direct Sums, Free Modules

- Let M be an R-module and $N_1, N_2, ..., N_n$ be submodules of M.

4 Tensor Products of Modules

5 Exact Sequences; Projective, Injective and Flat Modules