Real Analysis I: Differentiation

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1 Derivative of a Real Function

- Let $f : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. f is said to be differentiable at c if and only if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists. In this case, this we say that f is differentiable at c, the limit is called the derivative of f at c and denoted $f'(c)$. Thus $f'(x)$ is called the derivative of f. Its domain is the set of points at which the limit exists.
- If $f'(x)$ is differentiable at every point in $E \subseteq [a, b]$, then we say that f is differentiable on E.
- **Let f be defined on $[a, b]$. If f is differentiable at $x \in [a, b]$, then f is continuous at x.** *Proof:* $f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c)$. Taking the limit as $x \to c$ on both sides, we get $\lim_{x \to c} f(x) - f(c) = 0 \implies \lim_{x \to c} f(x) = f(c)$. ■
- **Let f, g be defined on $[a, b]$ and differentiable at $x \in [a, b]$. Then, $f + g$, fg, and $\frac{f}{g}$ (if $g(x) \neq 0$) are differentiable at x and:**
  1. $(f + g)'(x) = f'(x) + g'(x)$. *Proof:* $(f + g)'(x) = \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c} \right) = f'(x) + g'(x)$. ■
  2. $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. *Proof:* $(fg)'(x) = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} = \lim_{x \to c} \frac{f(x)(g(x) - g(c)) + g(c)(f(x) - f(c))}{x - c} = f(x) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} + g(x) \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f(x)g'(x) + g(x)f'(x)$. ■
  3. $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$. *Proof:* $\left(\frac{f}{g}\right)'(x) = \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \lim_{x \to c} \frac{1}{g(c)g(x)} \left(\frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{x - c} \right) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$. ■
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2 Mean Value Theorems

3 Continuity of Derivatives

4 L'Hopital's Rule

5 Taylor's Theorem

6 Differentiation of Vector-Valued Functions