# MAT360 Notes

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Created: 22nd September 2022
Last updated: 28th September 2022

## 1 Determinants

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- Let $\{e_i\}$ and $\{v_i\}$ be ordered bases for $V$. Let $T : V \to V$, such that $T(e_i) = v_i$ for all $i$. Then $\{v_1, v_2, .., v_n\}$ is said to be positively oriented if $\det A > 0$ and negatively oriented if $\det A < 0$.

## 2 Diagonalization

- A linear operator $T$ on vector space $V$ is said to be diagonalizable if there exists an ordered basis $\beta$ of $V$ such that $[T]_\beta$ is a diagonal matrix. A square matrix $A$ is diagonalizable if the linear operator it represents is diagonalizable.
- Suppose $\beta = \{v_1, v_2, ..., v_n\}$ is an ordered basis for $V$ such that $D = [T]_\beta$ is a diagonal matrix. Then for each $v_j \in \beta$, $T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j$, and $[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$.
- An eigenvalue is a scalar $\lambda \in \mathbb{F}$ such that for some $v \in V$, $v \neq 0$, we have $T(v) = \lambda v$. In this case $v$ is the eigenvector corresponding to $\lambda$.
- $E_\lambda = \{v \in V : T(v) = \lambda v\}$ is the set of all eigenvectors of $T$ corresponding to $\lambda$.
- Let $A \in M_n(\mathbb{F})$. Then $f_A(x) = \det(xI - A)$ [or $\det(A - xI)$] is an $n$th degree polynomial called the characteristic polynomial of $A$. It is easy to see that the roots of this polynomial are exactly the eigenvalues of $A$.
- **Let $T$ be a linear operator on $V$, and let $c \in \mathbb{F}$. Then the following are equivalent:**

  1. **$c$ is an eigenvalue of $T$**
  2. **The operator $T - cI$ is singular (not invertible)**
  3. $\det(T - cI) = 0$

  *Proof:* Suppose there exists a nonzero vector $v \in V$ such that $Tv = cv$. Then $Tv = cIv \implies Tv - cIv = 0 \implies (T - cI)v = 0 \implies v \in \mathrm{Ker}(T - cI)$. Thus $\mathrm{Ker}(T - cI) \neq \{0\}$, so $T - cI$ is singular. Now if $T - cI$ is not invertible, then $\det(T - cI) = 0$. If $\det(T - cI) = 0$, then $c$ is a root of $T$'s characteristic polynomial and thus an eigenvalue of $T$. ■
- **Similar matrices have the same characteristic polynomial.** *Proof:* Let $A, B \in M_n(\mathbb{F})$ be similar. Then $A = P^{-1}BP$. So $\det(xI - A) = \det(xI - P^{-1}BP) = \det(xP^{-1}IP - P^{-1}BP) = \det(P^{-1}(xI - B)P) = \det(P^{-1})\det(P)\det(xI - B) = \det(xI - B)$. ■
- Thus different matrix representations of the same operator will have the same characteristic polynomial.
- The algebraic multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial. Its geometric multiplicity is the dimension of its eigenspace, $\dim E_\lambda$. GM $\leq$ AM.

- $T : V \to V$ is diagonalizable if there exists a basis $\beta$ of $V$ consisting of eigenvectors of $T$. Suppose $\beta = \{v_1, v_2, ..., v_n\}$ and $T(v_i) = c_i v_i$ for all $i$. Then clearly $[T]_\beta$ would be a diagonal matrix whose diagonal entries would be $c_1, c_2, ..., c_n$.

- $A \in M_n(\mathbb{F})$ is diagonalizable if it is similar to a diagonal matrix.

- **Let $\lambda_1, \lambda_2, ..., \lambda_r$ be $r$ distinct eigenvalues of $T$, and let $v_1, v_2, ..., v_r$ be their corresponding eigenvectors. Then $v_1, v_2, ..., v_r$ are linearly independent.** *Proof:* The case $r = 1$ is trivial as an eigenvector is nonzero, and a single nonzero vector is of course linearly independent. Suppose that the statement is true for $r - 1$, and suppose that there exists a nontrivial linear combination $\sum_{k=1}^{r} c_k v_k = 0$. Applying $A - \lambda_r I$ to both sides and since $(A - \lambda_r I)v_r = 0$, we get $\sum_{k=1}^{r-1} c_k(\lambda_k - \lambda_r)v_k = 0$. As $v_1, ..., v_{r-1}$ are linearly independent, $c_k = 0$ for $1 \le k \le r - 1$ as $\lambda_k \ne \lambda_r$ when $k \ne r$. It thus follows that $c_r = 0$, and so $v_1, ..., v_r$ are linearly independent. ■

- **Let $A \in M_n(\mathbb{F})$. Let $c_1, c_2, ..., c_k$ be distinct eigenvalues of $A$ in $\mathbb{F}$, and let $E_i$ be the eigenspace of each $c_i$. Let $\beta_i$ be an ordered basis for $E_i$. Let $P = [\beta_1, \beta_2, ..., \beta_k]$ be the matrix which has the vectors of these bases (in order) as its columns. Then, $A$ is diagonal if and only if $P$ is a square matrix, and in this case, $P$ is invertible and $P^{-1}AP = D$, a diagonal matrix.** *Proof:* If $P$ is a square matrix, then $P$ has $n$ columns, which means that $A$ has $n$ distinct eigenvectors, so $A$ is diagonalizable. As $P$'s columns are linearly independent, $\det P \ne 0$ and so $P$ is invertible.

- **Let $c_1, c_2, ..., c_k$ be distinct eigenvalues of $T : V \to V$. Let $E_i = \text{Ker}(T - c_i I)$. Then the following are equivalent:**

  1. **T is diagonalizable**
  2. **The characteristic polynomial for $T$ is $(x - c_1)^{d_1}(x - c_2)^{d_2}...(x - c_k)^{d_k}$, and $\dim E_i = d_i$ for all $1 \le i \le k$**
  3. $\dim E_1 + \dim E_2 + ... + \dim E_k = \dim V$

  *Proof:*

- Let $T : V \to V$ over $\mathbb{F}$. A polynomial $p(x)$ is said to be a minimal polynomial of $T$ if $p(x)$ is monic and is of least positive degree such that $p(T) = 0$, i.e, $p(T) = T^n + a_{n-1}T^{n-1} + ... + a_1 T + a_0 I = 0$.

- **Let $p(x)$ be a minimal polynomial for $T$. Then, for any polynomial $g(x)$ such that $g(T) = 0$, $p(x) \mid g(x)$. Also, $p(x)$ is unique.** *Proof:* Let $g(T) = 0$. By the division algorithm, there exist polynomials $q(x), r(x)$ such that $g(x) = q(x)p(x) + r(x)$, where $\deg r < \deg p$ or $r = 0$. Now, $g(T) = q(T)p(T) + r(T) \implies 0 = 0 + r(T) \implies r(T) = 0 \implies r(x) = 0$ as $p$ is the polynomial of least degree that sends $T$ to 0. Thus $g(x) = q(x)p(x) \implies p(x) \mid g(x)$. Now, suppose $p_1$ and $p_2$ are both minimal polynomials for $T$. Then as $p_1 \mid p_2$ and $p_2 \mid p_1$, $p_1 = cp_2$, where $c$ is some polynomial. As $\deg p_1 = \deg p_2$, $c$ is a constant. But as $p_2$ is monic, we must have $c = 1$. So $p_1 = p_2$. ■

- The characteristic and minimal polynomials of $T$ have the same roots (not up to multiplicity).

- **Let $p(x)$ be the minimal polynomial of $T$. Then $p(c) = 0$ if and only if $c$ is an eigenvalue of $T$.** *Proof:* Let $p(c) = 0$. Then $p(x) = (x - c)q(x)$, $\deg q < \deg p$. Thus $0 = (T - cI)q(T)$, where $q(T) \ne 0$. So there exists some $\beta \in V$ such that $q(T)\beta = \alpha \ne 0$. So $(T - cI)q(T)\beta = 0 \implies (T - cI)\alpha = 0 \implies \alpha$ is an eigenvector and $c$ is an eigenvalue of $T$. ■

- Let $T : V \to V$. Let $W$ be a subspace of $V$. Then $W$ is called a $T$-invariant subspace of $V$ if and only if $T(W) \subseteq W$. That is, for all $w \in W$, $T(w) \in W$. Easy examples: $\{0\}$, $V$ and $\text{Ran}(T)$.

- $\text{Ker}(T)$ **is $T$-invariant.** *Proof:* Let $x \in T(\text{Ker}(T))$. Then there exists some $y \in \text{Ker}(T)$ such that $x = T(y) = 0$. As $0 \in \text{Ker}(T)$, we are done. ■

- $E_\lambda$ **is $T$-invariant.** *Proof:* Let $x \in T(E_\lambda)$. Then there exists some $y \in E_\lambda$ such that $x = T(y) = \lambda y$. So $T(x) = \lambda T(y) = \lambda x$. Thus $x \in E_\lambda$. ■

- **Let $T, U$ be linear operators on $V$ such that $T \circ U = U \circ T$. Then** $\text{Ran}(U)$ **and** $\text{Ker}(U)$ **are $T$-invariant.** *Proof:* Let $x \in T(\text{Ran}(U))$. Then there exists some $y \in \text{Ran}(U)$ such that $T(y) = x$. There exists some $z \in V$ such that $U(z) = y$. So $x = T(U(z)) = U(T(z))$. Thus $x \in \text{Ran}(U)$. Now let $x \in T(\text{Ker}(U))$. There exists some $y \in Ker(U)$ such that $x = T(y)$. Then $0 = T(U(y)) = U(T(y)) = U(x)$. So $x \in \text{Ker}(U)$. ■

- If $W$ is a $T$-invariant subspace of dimension 1, then there exists some nonzero $\alpha \in W$ such that $W = \text{span}(\alpha)$. As $T(W) \subseteq W$, $T(\alpha) \in W \implies T(\alpha) = c\alpha$ for some scalar $c$. Thus $\alpha$ is an eigenvector of $T$.