Real Analysis I: Sequences and Series

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1 Convergent Sequences

- A sequence $\{p_n\}$ in metrix space X is said to converge if there exists $p \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq n \implies d(p_n, p) < \epsilon$. In this case, we say $\lim\limits_{n\to\infty} p_n = p$ or $p_n \to p$.
- If $\{p_n\}$ does not converge, it diverges.
- **If $p, p' \in X$ and $\{p_n\}$ converges to p and p', then $p = p'$.** *Proof:* Let $\epsilon \geq 0$ be given. Then there exist integers N and N' such that $n \geq N \implies d(p_n, p) < \frac{\epsilon}{2}$ and $n \geq N' \implies d(p_n, p') < \frac{\epsilon}{2}$. Let $N^\circ = \max(N, N')$. So if $n \geq N^\circ$ then $d(p, p') \leq d(p, p_n) + d(p', p_n) < \epsilon$. Since ϵ was arbitrary, we get $d(p, p') = 0$. $\blacksquare$
- **If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.** *Proof:* Suppose $p_n \to p$. There exists $N \in \mathbb{N}$ such that $n \geq N \implies d(p_n, p) < 1$. Let $r = \max(1, d(p_1, p), d(p_2, p), ..., d(p_N, p))$. Then $d(p_n, p) < r$ for all $n \in \mathbb{N}$. $\blacksquare$
- **If $E \subset X$ and p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $\lim\limits_{n\to\infty} p_n = p$.** *Proof:* Since p is a limit point, for each $n \in \mathbb{N}$ there exists $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. Given $\epsilon > 0$, choose N so that $N > \frac{1}{\epsilon}$. Then $n \geq N \implies n \geq \frac{1}{\epsilon} \implies \epsilon \geq \frac{1}{n} \implies d(p_n, p) < \epsilon$. So $p_n \to p$. $\blacksquare$
- **Suppose $\{s_n\}$ and $\{t_n\}$ are sequences in $\mathbb{C}$, and $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$. Then:**
  1. $\lim\limits_{n\to\infty} s_n + t_n = s + t$. *Proof:* Given $\epsilon > 0$, there exist integers N_1, N_2 such that $n \geq N_1 \implies |s_n - s| < \frac{1}{\epsilon}$ and $n \geq N_2 \implies |t_n - t| < \frac{1}{\epsilon}$. Let $N_3 = \max(N_1, N_2)$. Then $n \geq N_3 \implies |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$. $\blacksquare$
  2. $\lim\limits_{n\to\infty} cs_n = cs$, $\lim\limits_{n\to\infty} c + s_n = c + s$, **for all** $c \in \mathbb{C}$. *Proof:* Given $\epsilon > 0$, there exists N such that $n \geq N \implies |s_n - s| < \epsilon$ which implies that $|cs_n - cs| < \epsilon$ and $|(c + s_n) - (c + s)| < \epsilon$. $\blacksquare$
  3. $\lim\limits_{n\to\infty} s_n t_n = st$. *Proof:* Use the identity $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$. Given $\epsilon > 0$, there exist integers N_1 and N_2 such that $n \geq N_1 \implies |s_n - s| < \sqrt{\epsilon}$ and $n \geq N_2 \implies |t_n - t| < \sqrt{\epsilon}$. If we let $N = \max(N_1, N-2)$, then $n \geq N \implies |(s_n - s)(t_n - t)| < \epsilon$ and thus $\lim\limits_{n\to\infty} (s_n - s)(t_n - t) = 0$. By taking the limit of both sides of the identity, we get $\lim\limits_{n\to\infty} s_n t_n - st = 0$. $\blacksquare$
  4. $\lim\limits_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$**, where** $s_n \neq 0$ **for all** $n \in \mathbb{N}$. *Proof:* Choose m such that $|s_n - s| < \frac{1}{2}|s|$. Then $|s_n| > \frac{1}{2}|s|$. Given $\epsilon > 0$ there exists an integer $N > m$ such that $n \geq N \implies |s_n - s| < \frac{1}{2}|s|^2\epsilon$. So for $n \geq N$, $\left|\dfrac{1}{s_n} - \dfrac{1}{s}\right| = \left|\dfrac{s_n - s}{s_n s}\right| < \dfrac{2}{|s|^2}|s_n - s| < \epsilon$. $\blacksquare$

2 Subsequences

- Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers where $n_1 < n_2 < n_3 < ...$ and so on. Then the sequence $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$. $\{p_{n_i}\}$ converges, its limit is a subsequential limit of $\{p_n\}$.

- **$\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.** *Proof:* Suppose every subsequence of $\{p_n\}$ converges to p. Then since $\{p_n\}$ is also a subsequence of itself, $\{p_n\}$ converges to p. Conversely, suppose $\{p_n\}$ converges to p and let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$. Given $\epsilon > 0$, there exists an integer M such that $n \geq M \implies |p_n - p| < \epsilon$. Now choose some integer $N \in \{n_k\}$ such that $N > M$. Then $n \geq N \implies |p_{n_k} - p| < \epsilon$, so $\{p_{n_k}\}$ converges to p. $\blacksquare$

- **Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.**

3 Cauchy Sequences

- A sequence $\{p_n\}$ in a metric space X is a Cauchy sequence if for every $\epsilon > 0$, there is an integer N such that $d(p_m, p_n) < \epsilon$ if $m, n \geq N$.

- **In any metric space X, every convergent sequence is a Cauchy sequence.** *Proof:* Let $\{p_n\}$ be a sequence in X. If $p_n \to p$, then for all $\epsilon > 0$ there is an integer N such that $n \geq N \implies d(p_n, p) < \epsilon$. Then $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$ whenever $n \geq N$ and $m \geq N$. So $\{p_n\}$ is a Cauchy sequence. $\blacksquare$

- **In $\mathbb{R}^k$, every Cauchy sequence converges.**

- A metric space in which every Cauchy sequence converges is said to be complete.

- A sequence $\{s_n\}$ in $\mathbb{R}$ is said to be monotonically increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.

- A sequence $\{s_n\}$ in $\mathbb{R}$ is said to be monotonically decreasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

- **If $\{s_n\}$ is monotonic, then $\{s_n\}$ converges if and only if it is bounded.** *Proof:* Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$. Since $\{s_n\}$ is bounded, let $s = \sup E$. Then $s_n \leq s$ for all $n \in \mathbb{N}$. For every $\epsilon > 0$, there exists an integer N such that $s-\epsilon < s_N \leq s$ since if it were not so, then $s - \epsilon$ would be an upper bound for E. Since $\{s_n\}$ is increasing, $n \geq N \implies s - \epsilon < s_n \leq s < s + \epsilon$, and so $\{s_n\}$ converges to s. The converse has already been proved previously, and the proof where $\{s_n\}$ is decreasing is analogous. $\blacksquare$

4 Upper and Lower Limits

- We define the extended real numbers, $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$, where $-\infty < r < \infty$ for all $r \in \mathbb{R}$. The arithmetic operations of $\mathbb{R}$ are partially extended to $\tilde{\mathbb{R}}$:

  1. $a + \infty = \infty + a = \infty$ for $a \neq -\infty$
  2. $a - \infty = -\infty + a = -\infty$ for $a \neq \infty$
  3. $\infty - \infty$ is not defined.

- Let $\{s_n\}$ be a sequence in $\mathbb{R}$ with the property that for every $M \in \mathbb{R}$ there is an integer N such that $n \geq N \implies s_n \geq M$. Then we say that $s_n \to \infty$. Similarly, if for every $M \in \mathbb{R}$ there an integer N such that $n \geq N \implies s_n \leq M$, we say that $s_n \to -\infty$.

- Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We define the limit superior and limit inferior of $\{a_n\}$ as such:

  1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
  2. If $\{a_n\}$ is bounded above, then let $M_k = \sup \{a_k, a_{k+1}, a_{k+2}, ...\}$. Then $\limsup a_n = \lim\limits_{k\to\infty} M_k$.
  3. If $a_n \to -\infty$, then $M_k \to -\infty$ and $\limsup a_n = -\infty$.
  4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
  5. If $\{a_n\}$ is bounded below, then let $m_k = \inf \{a_k, a_{k+1}, a_{k+2}, ...\}$. Then $\liminf a_n = \lim\limits_{k\to\infty} m_k$.
  6. If $a_n \to \infty$, then $m_k \to \infty$ and $\liminf a_n = \infty$.

- An alternate definition follows:

1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
2. If $\{a_n\}$ is bounded above, and there exists $u \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n < u + \epsilon$ and there exist infinitely many n where $a_n > u - \epsilon$, then $\limsup a_n = u$.
3. Otherwise, $\limsup a_n = -\infty$.
4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
5. If $\{a_n\}$ is bounded below, and there exists $l \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n > l - \epsilon$ and there exist infinitely many n where $a_n < l + \epsilon$, then $\liminf a_n = l$.
6. Otherwise, $\liminf a_n = \infty$.

- Another equivalent definition: Let $\mathbb{S}$ be the set containing all subsequential limits of a_n, including ∞ and $-\infty$. Then $\limsup a_n = \sup \mathbb{S}$ and $\liminf a_n = \inf \mathbb{S}$. These numbers exist since $\mathbb{S}$ is non-empty. If $\{a_n\}$ is bounded, then there exists at least one real subsequential limit. If a_n is unbounded in either direction, then there exist subsequences that diverge in either direction.

- $\liminf a_n \leq \limsup a_n$**.** *Proof:* If $\limsup a_n = \infty$ or $\liminf a_n = -\infty$, we are done. So suppose $\limsup a_n = -\infty$. Then $\sup \mathbb{S} = -\infty$ and thus $\mathbb{S} = \{-\infty\}$. So $\liminf a_n = -\infty$. If $\liminf a_n = \infty$, then by similar reasoning we can show that $\limsup a_n = \infty$. So let $\limsup a_n = \alpha \in \mathbb{R}$ and let $\liminf a_n = \beta \in \mathbb{R}$. $\alpha = \sup \mathbb{S}$ and $\beta = \inf \mathbb{S}$, so $\alpha \leq \beta$. ∎

- $a_n \to \infty$ **if and only if** $\liminf a_n = \limsup a_n = \infty$**.** *Proof:* Suppose $a_n \to \infty$. Then for all $\alpha \in \mathbb{R}$, there exists K such that $n \geq K \implies a_n > \alpha$. So a_n is bounded below.

- $a_n \to -\infty$ **if and only if** $\liminf a_n = \limsup a_n = -\infty$**.** *Proof:*

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exists** M **such that** $n \geq M \implies a_n < v+\epsilon$**, then** $v \geq \limsup a_n$**.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n < v + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n > \alpha - \frac{1}{2}\delta = v + \frac{1}{2}\delta$, so we have a contradiction. Thus, $v \geq \limsup a_n$. ∎

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exist infinitely many** n **such that** $a_n > v - \epsilon$**, then** $v \leq \limsup a_n$**.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n < \alpha + \frac{1}{2}\delta$, and so we have a contradiction. Thus $v \leq \limsup a_n$. ∎

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exists** M **such that** $n \geq M \implies a_n > v-\epsilon$**, then** $v \leq \liminf a_n$**.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n < \alpha + \frac{1}{2}\delta$, so we have a contradiction. Thus $v \leq \liminf a_n$. ∎

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exist infinitely many** n **such that** $a_n < v + \epsilon$**, then** $v \geq \liminf a_n$**.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n < v + \frac{1}{2}\delta = \alpha - \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n > \alpha - \frac{1}{2}\delta$, so we have a contradiction. Thus $v \geq \liminf a_n$. ∎

- **For a sequence** $\{a_n\}$ **in** $\mathbb{R}$**,** $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$ **if and only if** $\limsup a_n = \liminf a_n = a$**.** *Proof:* Let $\limsup a_n = \liminf a_n = a$. Then for all $\epsilon > 0$, there exist integers M, N such that $n \geq M \implies a_n < a + \epsilon$ and $n \geq N \implies a_n > a - \epsilon$. Let $P = \max(M, N)$. Then $n \geq P \implies |a_n - a| < \epsilon$. Conversely, suppose $a_n \to a$. For all $\epsilon > 0$, there exists K such that $n \geq K \implies a - \epsilon < a_n < a + \epsilon$. Thus $a \leq \liminf a_n$ and $a \geq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$, we have $\limsup a_n = \liminf a_n = a$. ∎

- $\liminf(-a_n) = -\limsup a_n$. *Proof:* Let $\limsup a_n = \alpha$. Then, for every $\epsilon > 0$ there exists M such that $n \geq M \implies a_n < \alpha + \epsilon$ and there exist infinitely many n such that $a_n > \alpha - \epsilon$. So for every $\epsilon > 0$, there exists M such that $n \geq M \implies -a_n > -\alpha - \epsilon$ and there exist infinitely many n such that $-a_n < -\alpha + \epsilon$. So $\liminf(-a_n) = -\alpha$. ∎

- **Let** $a_n \leq b_n$ **for all** $n \geq K$**. Then** $\limsup a_n \leq \limsup b_n$**.** *Proof:* If $\limsup b_n = \infty$ then we are done.

5 Special Sequences

6 Series

- Given a sequence $\{a_n\}$, let $S_n = \sum_{k=0}^{n} a_k$. Then, $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$. We say that $\sum_{n=0}^{\infty} a_n$ converges if and only if S_n converges. If S_n properly diverges to $\pm\infty$, then $\sum_{n=0}^{\infty} a_n$ properly diverges.
- S_n is called the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$.
- The Cauchy criterion can be restated in terms of series. S_n converges if and only if for all $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies |S_n - S_m| < \epsilon \implies \left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right| < \epsilon \implies \left|\sum_{k=n}^{m} a_k\right| < \epsilon$.
- If we let $m = n$, then we get $|a_n| < \epsilon$. Thus, if $\sum a_n$ converges, then $a_n \to 0$. The converse is not necessarily true.
- $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.
- $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ diverges but $\sum a_n$ converges.
- Let $x \in R$. Then, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. So $x = x^+ - x^-$ and $|x| = x^+ + x^-$.
- **Let a_n be a sequence that is ultimately non-negative, and let A_n be the sequence of its partial sums. Then $\sum a_n$ converges if and only if A_n is bounded above.** *Proof:* Suppose $\sum a_n$ converges. Then A_n converges and is thus bounded. Conversely, suppose A_n is bounded above. Since a_n is ultimately non-negative, A_n is ultimately monotonically increasing. Thus A_n and $\sum a_n$ converge. ■
- **Basic Comparison Test: If $|a_n| \leq b_n$ for $n \geq N_1$, and if $\sum b_n$ converges, then $\sum a_n$ converges. If $c_n \geq d_n \geq 0$ for $n \geq N_2$, and if d_n diverges, then c_n diverges. Here, N_1, N_2 are fixed integers.** *Proof:* Suppose $\sum b_n$ converges. Given $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies \left|\sum_{k=n}^{m} b_k\right| < \epsilon$. Thus, $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} b_k \leq \left|\sum_{k=n}^{m} b_k\right| < \epsilon$. So $\sum a_n$ converges. Now, suppose $\sum d_n$ diverges. If $\sum c_n$ converges, then $\sum d_n$ must also converge. So $\sum c_n$ diverges. ■
- **Comparison Test V1: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$, $\alpha, \beta > 0$ such that $n > M \implies \alpha a_n \leq b_n \leq \beta a_n$, then $\sum b_n$ converges if and only if $\sum a_n$ converges.** *Proof:*
- **Comparison Test V2: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$ such that $n > M \implies 0 \leq \frac{b_n}{b_{n+1}} \leq \frac{a_n}{a_{n+1}}$, then $\sum a_n$ converges if $\sum b_n$ converges.** *Proof:*
- **Comparison Test V3: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.** *Proof:*
- **Comparison Test V4: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 = \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if $\sum b_n$ converges.** *Proof:*
- **Comparison Test V5: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} = \infty$, then $\sum b_n$ converges if $\sum a_n$ converges.** *Proof:*

7 Series of Non-negative Terms

- **If** $0 \leq x < 1$**, then** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$**. If** $x \geq 1$**, this series diverges.** *Proof:* If $x \neq 1$, then $X_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n-1}}{1-x}$. If $0 \leq x < 1$, then $\lim_{n\to\infty} \frac{1-x^{n-1}}{1-x} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x = 1$, then the sum is $1 + 1 + 1 + ...$ which diverges. If $x > 1$ then $\frac{1-x^{n-1}}{1-x}$ diverges. ■
- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges if and only if** $p > 1$**.** *Proof:*

8 Euler's Number

9 Root and Ratio Tests

10 Power Series

11 Summation by Parts

12 Absolute Convergence

13 Addition and Multiplication of Series

14 Rearrangements