MAT422 Notes

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1 Basic Definitions

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2 Convergence of Sequences

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3 Open Sets and Interior Points

- Let $x \in X$, $r > 0$. The open sphere/ball/disc centered at x with radius r, denoted $S(x,r)$ is $\{y \in X : d(y,x) < r\}$.
- Alternate definition of convergence:
  $(x_n) \to x \Leftrightarrow \forall \epsilon > 0, \exists M : n \geq M \implies d(x_n, x) < \epsilon$
  $\implies \forall \epsilon > 0, \exists M : n \geq M \implies x_n \in S(x, \epsilon)$.
- Let $a \in X$, $A \subseteq X$. a is said to be an interior point of A if and only if $\exists r > 0 : S(a,r) \subseteq A$.
- The set of all interior points of A is called the interior of A, denoted $\mathrm{Int}(A)$ or A°.
- Let $b \in X$. $N \subseteq X$ is said to be a neighborhood of b if and only if $b \in N^\circ$.
- **$A^\circ \subseteq A$.** *Proof:* Let $a \in A^\circ$. Then $\exists r > 0 : S(a,r) \subseteq A$. As $a \in S(a,r)$, $a \in A$. ■
- **$A \subseteq B \implies A^\circ \subseteq B^\circ$.** *Proof:* Let $a \in A^\circ$. Then $\exists r > 0 : S(a,r) \subseteq A \subseteq B$. So $a \in B^\circ$. ■
- **Let $A \subseteq B$, and A be a neighborhood of a. Then B is a neighborhood of a.** *Proof:* $a \in A^\circ \implies a \in B^\circ \implies B$ is a neighborhood of A. ■
- **$(A^\circ)^\circ = A^\circ$.** *Proof:* We know that $(A^\circ)^\circ \subseteq A^\circ$. So let $a \in A^\circ$. Then $\exists r > 0 : S(a,r) \subseteq A$.
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4 Closed Sets and Limit Points

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5 Countability and Uncountability

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6 Functions

- Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$, $a \in X$, $l \in Y$. Then we say $\lim_{x \to a} f(x) = l$ if and only if given $\epsilon > 0$, $\exists \delta > 0 : \forall x \in X$, $0 < d(x, a) < \delta \implies \rho(f(x), l) < \epsilon$.
- $S'(x, r) = S(x, r) \setminus \{x\}$ is called the deleted open sphere of x with radius r. Similarly, $S'[x, r] = S[x, r] \setminus \{x\}$ is the deleted closed sphere of x with radius r.
- Thus, $\lim_{x \to a} f(x) = l$ if and only if given $\epsilon > 0$, $\exists \delta > 0 : x \in S'(x, \delta) \implies f(x) \in S(l, \epsilon)$.
- $\lim_{x \to a} f(x) \neq l$ when $\exists \epsilon_0 > 0$, $\forall \delta > 0 : \exists x_\delta : 0 < d(a, x_\delta) < \delta$ but $\rho(f(x), l) \geq \epsilon_0$.
- **If a is an isolated point of X, then every member of Y is a limit of f as $x \to a$.** *Proof:*
- **If $a \in X^l$, then $\lim_{x \to a} f(x)$ is unique.** *Proof:*
- f is said to be continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.
- Thus every function on X is continuous at every isolated point in X.
- Alternatively, f is continuous at a when given $\epsilon > 0$, $\exists \delta > 0 : d(a, x) < \delta \implies \rho(f(a), f(x)) < \epsilon$. Or when given $\epsilon > 0$, $\exists \delta > 0 : x \in S(a, \delta) \implies f(x) \in S(f(a), \epsilon)$. Or when given $\epsilon > 0$, $\exists \delta > 0 : S(a, \delta) \subseteq f^{-1}(S(f(a), \epsilon))$.
- **f is continuous if and only if $f^{-1}(G)$ is open in X for all open sets G in Y.** *Proof:*
- **f is continuous if and only if $f^{-1}(G)$ is closed in X for all closed sets G in Y.** *Proof:*
- f is an open function if $f(G)$ is open in Y when G is open in X.
- f is a closed function if $f(G)$ is closed in Y when G is closed in X.
- **Sequential Criterion: $\lim_{x \to a} f(x) = l$ if and only if for all sequences $(x_n) \in X$ such that $x_n \neq a$ and $x_n \to a$, $f(x_n) \to l$.** *Proof:*
- Let (X, d) and (Y, ρ) be metric spaces. Then $f : (X, d) \to (Y, \rho)$ is a homeomorphism if and only if f is bijective, f is continuous and f^{-1} is continuous.
- X is said to be homeomorphic to Y or $X \sim Y$ if a homeomorphism exists between the two spaces. It can be shown that $\sim$ here is an equivalence relation.
- **Let d_1, d_2 be metrics on X. Then d_1 and d_2 are equivalent if and only if the identity map from (X, d_1) to (X, d_2) is a homeomorphism.** *Proof:*
- **If d is a discrete metric on X, then every function from (X, d) to any metric space is continuous.** *Proof:*
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- Let (X, d) and (Y, ρ) be metric spaces. Then $f : (X, d) \to (Y, \rho)$ is an isometry if and only if $d(x_1, x_2) = \rho(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.
- **Isometries are injective.** *Proof:*
- **Isometries are continuous.** *Proof:*
- (X, d) is said to be isometric, or isometrically isomorphic to (Y, ρ) if and only if $\exists \phi : X \to Y$ such that ϕ is an isometry and ϕ is surjective.
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- Let $f : (X, d) \to (Y, \rho)$. f is said to be uniformly continuous on $A \subseteq X$ if and only if: given $\epsilon > 0$, $\exists \delta > 0$ such that $d(a_1, a_2) < \delta \implies \rho(f(a_1), f(a_2)) < \epsilon$, $\forall a_1, a_2 \in A$.
- f is NOT uniformly continuous on A if $\exists \epsilon_0 > 0, \forall \delta > 0, \exists a_\delta, b_\delta \in A : d(a_\delta, b_\delta) < \delta$ and $\rho(f(a_\delta), f(b_\delta)) \geq \epsilon_0$.

- Alternatively, f is not uniformly continuous if there exist sequences $(a_n), (b_n)$ in A such that $d(a_n, b_n) \to 0$ but $\rho(f(a_n), f(b_n)) \not\to 0$.
- $f : X \to Y$ is said to be a Lipschitz function if $\exists K : \rho(f(x), f(y)) \leq K d(x, y)$. If $K < 1$, then f is called a contraction.
- $(X_d \times Y_\rho, d_\infty)$, where $d_\infty((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), \rho(y_1, y_2))$, is called the product space of X and Y.
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