Real Analysis II: Sequences and Series of Functions

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1 Introduction

- Let $\{f_n\}$, $n \in \mathbb{N}$, be a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. Then, we can define a function $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in E$. In this case we say that $\{f_n\}$ converges on E, that f is the limit function of $\{f_n\}$, or that $\{f_n\}$ converges to f pointwise on E.
- Alternatively, we can state that $\{f_n\}$ converges to f pointwise on E if and only if given $\epsilon > 0, e \in E$, there exists M such that $n \geq M \implies |f_n(e) - f(e)| < \epsilon$.

2 Uniform Convergence

- A sequence of functions $\{f_n\}$ converges uniformly on E to a function f if and only if for every $\epsilon > 0$, there exists N such that $n \geq N \implies |f_n(x) - f(x)| < \epsilon$ for all $x \in E$.
- Let $\{a_n\}$ be a sequence of complex valued functions defined on $D \subseteq \mathbb{C}$. Let a be a complex valued function defined on $C \subseteq D$. Then a_n does not converge uniformly to a on C if and only if there exists $\epsilon_0 > 0$, a subsequence $\{a_{n_k}\}$ and a sequence x_k in C such that $|a_{n_k}(x_k) - a(x_k)| \geq \epsilon_0$ for all $k \in \mathbb{N}$.
- If f_n converges uniformly to f on E, then f_n converges pointwise to f on E.
- Let $\{a_n\}$ be a sequence of complex valued functions defined on D. We say $\{a_n\}$ is uniformly bounded on D if and only if there exists K such that $|a_n(z)| \leq K$ for all $n \in \mathbb{N}$, $z \in D$. If $\{a_n\}$ is a uniformly bounded pointwise convergent sequence, then it is said to be boundedly convergent.
- **If a_n is boundedly convergent to a on C, then a is bounded on C.** *Proof:* Let $c \in C$. Then there exists M such that $n \geq M \implies |a_n(c) - a(c)| < 1$. Then $|a(c)| \leq |a(c) - a_M(c)| + |a_M(c)| \leq K + 1$. ∎
- **If $\{a_n\}$ is a sequence of bounded functions that converges uniformly to a on C, then a is bounded on C.** *Proof:* There exists M such that $n \geq M \implies |a(c) - a_n(c)| < 1$ for all $c \in C$. Then $|a(c)| \leq |a(c) - a_M(c)| + |a_M(c)| \leq K_M + 1$. ∎
- **Cauchy Criterion for Uniform Convergence: $\{f_n\}$, a sequence of functions defined on E, converges uniformly on D if and only if for every $\epsilon > 0$, there exists an integer N such that $m, n \geq N$, $x \in E \implies |f_n(x) - f_m(x)| < \epsilon$.** *Proof:* Suppose $\{f_n\}$ converges uniformly to f on D. Then, there exists M such that $n \geq M \implies |f_n(c) - f(c)| < \frac{\epsilon}{2}$, for all $c \in D$. Thus, for all $m, n \geq M$, $|f_m(c) - f_n(c)| \leq |f_m(c) - f(c)| + |f(c) - f_n(c)| < \epsilon$. Conversely, suppose $\{f_n\}$ satisfies Cauchy's criterion. Let $c \in D$. Then $\{f_n(c)\}$ is a Cauchy sequence and thus converges. So $\{f_n\}$ has a pointwise limit on D, say f. Let $\epsilon > 0$. Then there exists M such that $m, n \geq M \implies |f_m(c) - f_n(c)| < \frac{\epsilon}{2}$. As $f_n(c) \to f(c)$, $f_m(c) - f_n(c) \to f_m(c) - f(c)$ as $n \to \infty$. But since $|f_m(c) - f_n(c)| < \frac{\epsilon}{2}$, $|f_m(c) - f(c)| \leq \frac{\epsilon}{2}$. Thus $m \geq M \implies |f_m(c) - f(c)| < \epsilon$ for all $c \in D$. So $\{f_n\}$ converges uniformly to f on D. ∎
- **Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ $(x \in E)$. Let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$.** *Proof:*

- **Suppose $\{f_n\}$ is a sequence of functions defined on E, and $|f_n(x)| \leq M_n$ for $x \in E$, $n \in \mathbb{N}$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.** *Proof:*

3 Uniform Convergence and Continuity

- **Let a_n converge uniformly to a on D. Let c be a limit point of D, and suppose $\lim_{z \to c} a_n(z) = \gamma_n$. Then γ_n converges and $\lim_{z \to c} a(z) = \lim_{n \to \infty} \gamma_n$.** *Proof:*

- **Preservation of Continuity: Let $\{f_n\}$ converge uniformly to f on D. Let each f_n be continuous at $c \in D$. Then f is continuous at c.** *Proof:*

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4 Uniform Convergence and Integration

5 Uniform Convergence and Differentiation

6 Equicontinuous Families of Functions

7 Stone-Weierstrass Theorem