Linear Algebra: Inner Product Spaces

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1 Introduction

- An inner product is a function that assigns a scalar to a pair of vectors.
- Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. The norm of a vector in $\mathbb{R}^n$ is defined as $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. The inner product of two vectors is defined as $\langle x, y \rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$.
- Let V be a vector space over $\mathbb{C}$. An inner product in V must satisfy the following properties:
  1. **Conjugate symmetry:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
  2. **Linearity:** $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{C}$.
  3. **Non-negativity:** $\langle x, x \rangle \geq 0$ for all $x \in V$.
  4. **Non-degeneracy:** $\langle x, x \rangle = 0$ if and only if $x = 0$.

  If such a function exists, then V together with its inner product is an inner product space.
- Given an inner product space, the norm of a vector x is defined as $\|x\| = \sqrt{\langle x, x \rangle}$.
- **Let $x \in V$. Then $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in V$.** *Proof:*
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2 Orthogonality, Orthogonal and Orthonormal Bases

- Two vectors u and v are called orthogonal if $\langle u, v \rangle = 0$. This is denoted by $u \perp v$.

3 Orthogonal Projection and Gram-Schmidt Orthogonalization

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