Real Analysis II: Riemann-Stieltjes Integral

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1 Definition and Existence of the Integral

- Let $[a,b]$ be an interval. A partition of $[a,b]$ is a finite set of points $x_0, x_1, ..., x_n$, where $a = x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n = b$. Here $\Delta x_i = x_i - x_{i-1}$, $(i = 1, 2, .., n)$.
- $\mathbb{P}(I)$ denotes the set of all partitions of an interval I.
- Let P, Q be partitions. Q is called a refinement of P if $P \subseteq Q$.
- Let f be a bounded real function defined on $[a,b]$. For each partition P of $[a,b]$, let:
  1. $M_i = \sup f(x)$, $(x_{i-1} \leq x \leq x_i)$
  2. $m_i = \inf f(x)$, $(x_{i-1} \leq x \leq x_i)$
  3. $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$
  4. $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$
  5. $\overline{\int_a^b} f = \inf \{U(P,f) : P \in \mathbb{P}([a,b])\}$
  6. $\underline{\int_a^b} f = \sup \{L(P,f) : P \in \mathbb{P}([a,b])\}$

  Here, $U(P,f)$ and $L(P,f)$ are called upper and lower sums respectively. $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ are called the upper and lower integrals.
- Since f is bounded on $[a,b]$, there exist $M, m \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Thus $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$, for any partition P of $[a,b]$.
- **Let $P, Q \in \mathbb{P}(I)$. If Q is a refinement of P, then $L(P,f) \leq L(Q,f)$ and $U(P,f) \geq U(Q,f)$.** *Proof:* Suppose $Q = P \cup \{x^*\}$. Then $x^* \in [x_{i-1}, x_i]$ for some $x_i \in P$. Let $w_1 = \inf \{f(x) : x \in [x_{i-1}, x^*]\}$ and $w_2 = \inf \{f(x) : x \in [x^*, x_i]\}$. Since $w_1 \geq m_i$ and $w_2 \geq m_i$, $L(Q,f) \geq L(P,f)$. If Q contains k more points than P, then repeat this reasoning k times. The proof for $U(Q,f)$ is analogous. ■
- **For all $P, Q \in \mathbb{P}(I)$, $L(P,f) \leq U(Q,f)$.** *Proof:* $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. Thus $L(P,f) \leq L(P \cup Q, f) \leq U(P \cup Q, f) \leq U(Q,f)$. ■
- If $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ are equal, then we say that f is Riemann integrable on $[a,b]$. Their common value is denoted by $\int_a^b f$.
- $R(I)$ denotes the set of all Riemann integrable functions on an interval I.
- **Riemann Criterion for Integrability: Let f be bounded on interval I. $f \in R(I)$ if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P,f) - L(P,f) < \epsilon$.** *Proof:* Suppose that given $\epsilon > 0$, there exists partition P such that $U(P,f) - L(P,f) < \epsilon$. Since $U(P,f) \geq \overline{\int} f$ and $-L(P,f) \geq \underline{\int} f$, $\overline{\int} f - \underline{\int} f \leq U(P,f) - L(P,f) < \epsilon$. Thus $\overline{\int} f = \underline{\int} f$ and $f \in R(I)$. Conversely, suppose $f \in R(I)$. Let $\epsilon > 0$. Since $\underline{\int} f = \overline{\int} f$, there exists a partition P_1 such that $U(P_1, f) < \int f + \frac{\epsilon}{2}$ and a partition P_2 such that $L(P_2, f) > \int f - \frac{\epsilon}{2}$. Thus, $U(P_1, f) - L(P_2, f) < \epsilon$. Let $P = P_1 \cup P_2$. Then $U(P,f) \leq U(P_1, f)$ and $-L(P,f) \leq -L(P_2, f)$. So $U(P,f) - L(P,f) \leq U(P_1, f) - L(P_2, f) < \epsilon$. ■
- **Corollary: Let $\epsilon > 0$. If there exists a partition $P \in \mathbb{P}(I)$ such that $U(P,f) - L(P,f) < \epsilon$, then $U(Q,f) - L(Q,f) < \epsilon$ for every refinement Q of P.**

2 Properties of the Integral

- **All constant functions are Riemann-integrable.** *Proof:* Let $\alpha \in \mathbb{R}$ and let $f(x) = \alpha$ for all $x \in [a, b]$. Let $P \in \mathbb{P}[a, b]$, $P = \{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$. Then $M_i = m_i = \alpha$ for all $1 \leq i \leq n$. Thus, $U(P, f) = L(P, f) = \alpha(b - a)$. Therefore $\int_a^b f = \alpha(b - a)$. ■

- **If f, g are Riemann integrable on $[a, b]$, then so are $pf + qg$ ($p, q \in \mathbb{R}$), $|f|$, f^2, fg, and $\dfrac{f}{g}$ (provided g is bounded away from 0, i.e, there exists an $m > 0$ such that $|g(x)| > m$ on $[a, b]$).**

  1. $\int_a^b (pf + qg) = p \int_a^b g + q \int_a^b g$. *Proof:*
  2. $|f|$. *Proof:*
  3. f^2. *Proof:*
  4. fg. *Proof:*
  5. $\dfrac{f}{g}$. *Proof:*

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3 Integration and Differentiation

4 Integration of Vector-valued Functions

5 Rectifiable Curves