

Real Analysis I: Basic Topology

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1 Countability

- If there exists a bijection between sets A and B , then A and B are said to be in one-one correspondence.
- For any $n \in \mathbb{N}$, let $J_n = \{1, 2, 3, \dots, n\}$. If A is in one-one correspondence with J_n for some n , then A is finite. The empty set is also considered finite.
- A is countable if it is in one-one correspondence with \mathbb{N} .
- A is uncountable if it is neither finite nor countable.
- A is at most countable if it is either finite or countable.
- **Every infinite subset of a countable set A is countable itself.** *Proof:* Suppose $E \subset A$ and E is infinite. Arrange the elements of A into a sequence $\{x_n\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Let n_k be the k th smallest integer such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$, we get a one-one correspondence between \mathbb{N} and E . So E is countable. ■
- **Let $\{E_n\}$ be a collection of countable sets, and let $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.** *Proof seems to rely on some diagram so I couldn't include it here.*
- **Corollary: Suppose A is at most countable, and $\forall a \in A, B_a$ is at most countable. Then $T = \bigcup_{a \in A} B_a$ is at most countable.**
- **Let A be countable, and let B_n be the set of all n -tuples (a_1, a_2, \dots, a_n) where $a_k \in A$ and the members of the tuples are not necessarily distinct. Then B_n is countable.** *Proof:* Since $B_1 = A$, it is countable. Suppose that B_{n-1} is countable. The elements of B_n have the form (b, a) , where $b \in B_{n-1}$ and $a \in A$. For every fixed b , the set of pairs (b, a) is in one-one correspondence with A and thus countable. So B_n is a countable union of countable sets, and thus countable itself. We have our proof by induction. ■
- **Corollary: \mathbb{Q} is countable.** *Proof:* Every rational number is of the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. This is in one-one correspondence with the set of n -tuples (a, b) , which is countable. So \mathbb{Q} is countable. ■

2 Metric Spaces

- A set X is said to be a metric space if for all $p, q \in X$, there is an associated real number $d(p, q)$ such that
 1. $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$
 2. $d(p, q) = d(q, p)$
 3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$

Any function with the above properties is a distance function, or a metric.

- If $a_i < b_i$ for $i = 1, 2, 3, \dots, k$, the set of all points (x_1, x_2, \dots, x_k) in \mathbb{R}^k such that $a_i \leq x_i \leq b_i$ is called a k -cell. A 1-cell is thus an interval and a 2-cell is a rectangle.

- If $x \in \mathbb{R}^k$ and the open (or closed) ball with center at x and radius r is defined as $\{y \in \mathbb{R}^k : \|y - x\| < r\}$ (or $\|y - x\| \leq r$).
- $E \subset \mathbb{R}^k$ is convex if for all $x, y \in E$, $\lambda x + (1 - \lambda)y \in E$, where $0 < \lambda < 1$.
- Suppose X is a metric space. Then,
 1. A neighborhood of $p \in X$ is the set $N_r(p) = \{q \in X : d(p, q) < r\}$, where $r > 0$.
 2. p is a limit point of $E \subset X$ if for all $r > 0$, there exists $q \in N_r(p)$ such that $p \neq q$ and $q \in E$.
 3. If $p \in E$ and p is not a limit point of E , then p is an isolated point of E .
 4. E is closed if every limit point of E is a point of E .
 5. p is an interior point of E if there exists $r > 0$ such that $N_r(p) \subset E$.
 6. E is open if every point of E is an interior point of E .
 7. E is perfect if E is closed and every point of E is a limit point of E .
 8. E is bounded if there exists $M \in \mathbb{R}$ and $q \in E$ such that for all $p \in E$, $d(p, q) < M$.
 9. E is dense in X if every point of X is a limit point of E , or a point of E or both.
- **Every neighborhood is an open set.** *Proof:* Let $E = N_r(p)$, and let $q \in E$. Then, there exists $h \in \mathbb{R}$ such that $d(p, q) = r - h$. For all points s such that $d(q, s) < h$, we then have $d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r$, so $s \in E$. Thus for all $q \in E$, $N_h(q) \subset E$. Therefore q is an interior point of E and E is an open set. ■
- **If p is a limit point of E , then every neighborhood of p contains infinitely many points of E .** *Proof:* Suppose there exists a neighborhood N of p containing finitely many points of E , and let q_1, q_2, \dots, q_m be those points. Let $r = \min(d(p, q_n))$, where $1 \leq n \leq m$. The neighborhood $N_r(p)$ contains no point of E other than p , contradicting p being a limit point of E . ■
- **Corollary: A finite set has no limit points.**
- **A set is open if and only if its complement is closed.** *Proof:* Suppose E^c is closed, and let $x \in E$. Then $x \notin E^c$ and x is not a limit point of E^c . So there exists a neighborhood N such that $N \cap E^c$ is empty, and so x is an interior point of E^c . Thus E^c is open. Conversely, suppose E is open, and let x be a limit point of E^c . Then every neighborhood of x contains a point of E^c , so x is not an interior point of E . Since E is open, $x \in E^c$. Thus E^c is closed. ■
- **Corollary: A set is closed if and only if its complement is open.**
- **For any collection $\{G_a\}$ of open sets, $\bigcup_a G_a$ is open.** *Proof:* Let $G = \bigcup_a G_a$. If $x \in G$, then $x \in G_a$ for some a . Since x is an interior point of G_a , it is also an interior point of G , and thus G is open. ■
- **For any collection $\{F_a\}$ of closed sets, $\bigcap_a F_a$ is closed.** *Proof:* Since $(\bigcap_a F_a)^c = \bigcup_a (F_a^c)$, and since F_a^c is open, $\bigcap_a F_a$ is closed. ■
- **For any finite collection G_1, G_2, \dots, G_n of open sets, $\bigcap_{i=1}^n G_i$ is open.** *Proof:* Let $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x , with radii r_i such that $N_i \subset G_i$, $1 \leq i \leq n$. Let $r = \min(r_1, r_2, \dots, r_n)$. Then $N_r(x) \subset G_i$ for $1 \leq i \leq n$, so $N_r(x) \subset H$. So H is open. ■
- **For any finite collection F_1, F_2, \dots, F_n of closed sets, $\bigcup_{i=1}^n F_i$ is closed.** *Follows from the previous statement.*
- If X is a metric space, $E \subset X$, then E^l denotes the set of all limit points of E in X and $\bar{E} = E \cup E^l$ is the closure of E .
- **\bar{E} is closed; $\bar{E} = E$ if and only if E is closed; $\bar{E} \subset F$ for every closed set F such that $E \subset F$.** *Proof:* If $p \notin \bar{E}$ then $p \notin E$ and p is not a limit point of E . So there exists a neighborhood N of p that does not intersect E . Thus \bar{E}^c is open and \bar{E} is closed. If $\bar{E} = E$, then E is closed. Conversely, if E is closed, then $E^l \subset E$ and thus $E = \bar{E}$. If F is closed and $E \subset F$, then $F^l \subset F$ and so $E^l \subset F$. So $\bar{E} \subset F$. ■

- Let E be a nonempty subset of \mathbb{R} that is bounded above. Let $y = \sup(E)$. Then, $y \in \bar{E}$ and $y \in E$ if E is closed. *Proof:* If $y \in E$ then $y \in \bar{E}$ so assume $y \notin E$. For every $h > 0$ there exists $x \in E$ such that $y - h < x < y$. If this were not so, then $y - h$ would be an upper bound for E . Thus y is a limit point of E . Hence $y \in \bar{E}$. ■
- Suppose $Y \subset X$. A subset E of Y is open relative to Y if for each $p \in E$ there exists an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

3 Compact Sets

- A collection $\{G_a\}$ of open subsets of metric space X is an open cover of $E \subset X$ if $E \subset \bigcup_a G_a$.
- $K \subset X$ is compact if every open cover of K contains a finite subcover. That is, for every collection of open sets $\{G_a\}$ such that $K \subset \bigcup_a G_a$, there exist finitely many indices a_1, a_2, \dots, a_n such that $K \subset \bigcup_{i=1}^n G_{a_i}$.

4 Connected Sets

- Two subsets A and B of a metric space X are separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
- $E \subset X$ is connected if it is not the union of two nonempty separated sets.