

# Real Analysis I: Sequences and Series

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## 1 Convergent Sequences

- A sequence  $\{p_n\}$  in metric space  $X$  is said to converge if there exists  $p \in X$  such that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies d(p_n, p) < \epsilon$ . In this case, we say  $\lim_{n \rightarrow \infty} p_n = p$  or  $p_n \rightarrow p$ .
- If  $\{p_n\}$  does not converge, it diverges.
- **If  $p, p' \in X$  and  $\{p_n\}$  converges to  $p$  and  $p'$ , then  $p = p'$ .** *Proof:* Let  $\epsilon \geq 0$  be given. Then there exist integers  $N$  and  $N'$  such that  $n \geq N \implies d(p_n, p) < \frac{\epsilon}{2}$  and  $n \geq N' \implies d(p_n, p') < \frac{\epsilon}{2}$ . Let  $N^\circ = \max(N, N')$ . So if  $n \geq N^\circ$  then  $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$ . Since  $\epsilon$  was arbitrary, we get  $d(p, p') = 0$ . ■
- **If  $\{p_n\}$  converges, then  $\{p_n\}$  is bounded.** *Proof:* Suppose  $p_n \rightarrow p$ . There exists  $N \in \mathbb{N}$  such that  $n \geq N \implies d(p_n, p) < 1$ . Let  $r = \max(1, d(p_1, p), d(p_2, p), \dots, d(p_N, p))$ . Then  $d(p_n, p) < r$  for all  $n \in \mathbb{N}$ . ■
- **If  $E \subset X$  and  $p$  is a limit point of  $E$ , then there is a sequence  $\{p_n\}$  in  $E$  such that  $\lim_{n \rightarrow \infty} p_n = p$ .** *Proof:* Since  $p$  is a limit point, for each  $n \in \mathbb{N}$  there exists  $p_n \in E$  such that  $d(p_n, p) < \frac{1}{n}$ . Given  $\epsilon > 0$ , choose  $N$  so that  $N > \frac{1}{\epsilon}$ . Then  $n \geq N \implies n \geq \frac{1}{\epsilon} \implies \epsilon \geq \frac{1}{n} \implies d(p_n, p) < \epsilon$ . So  $p_n \rightarrow p$ . ■
- **Suppose  $\{s_n\}$  and  $\{t_n\}$  are sequences in  $\mathbb{C}$ , and  $\lim_{n \rightarrow \infty} s_n = s$ ,  $\lim_{n \rightarrow \infty} t_n = t$ . Then:**
  1.  $\lim_{n \rightarrow \infty} s_n + t_n = s + t$ . *Proof:* Given  $\epsilon > 0$ , there exist integers  $N_1, N_2$  such that  $n \geq N_1 \implies |s_n - s| < \frac{\epsilon}{2}$  and  $n \geq N_2 \implies |t_n - t| < \frac{\epsilon}{2}$ . Let  $N_3 = \max(N_1, N_2)$ . Then  $n \geq N_3 \implies |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$ . ■
  2.  $\lim_{n \rightarrow \infty} cs_n = cs$ ,  $\lim_{n \rightarrow \infty} c + s_n = c + s$ , for all  $c \in \mathbb{C}$ . *Proof:* Given  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N \implies |s_n - s| < \epsilon$  which implies that  $|cs_n - cs| < \epsilon$  and  $|(c + s_n) - (c + s)| < \epsilon$ . ■
  3.  $\lim_{n \rightarrow \infty} s_n t_n = st$ . *Proof:* Use the identity  $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$ . Given  $\epsilon > 0$ , there exist integers  $N_1$  and  $N_2$  such that  $n \geq N_1 \implies |s_n - s| < \sqrt{\epsilon}$  and  $n \geq N_2 \implies |t_n - t| < \sqrt{\epsilon}$ . If we let  $N = \max(N_1, N_2)$ , then  $n \geq N \implies |(s_n - s)(t_n - t)| < \epsilon$  and thus  $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$ . By taking the limit of both sides of the identity, we get  $\lim_{n \rightarrow \infty} s_n t_n - st = 0$ . ■
  4.  $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$ , where  $s_n \neq 0$  for all  $n \in \mathbb{N}$ . *Proof:* Choose  $m$  such that  $|s_n - s| < \frac{1}{2}|s|$ . Then  $|s_n| > \frac{1}{2}|s|$ . Given  $\epsilon > 0$  there exists an integer  $N > m$  such that  $n \geq N \implies |s_n - s| < \frac{1}{2}|s|^2\epsilon$ . So for  $n \geq N$ ,  $\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s_n s} \right| < \frac{2}{|s|^2} |s_n - s| < \epsilon$ . ■

## 2 Subsequences

- Given a sequence  $\{p_n\}$ , consider a sequence  $\{n_k\}$  of positive integers where  $n_1 < n_2 < n_3 < \dots$  and so on. Then the sequence  $\{p_{n_i}\}$  is a subsequence of  $\{p_n\}$ .  $\{p_{n_i}\}$  converges, its limit is a subsequential limit of  $\{p_n\}$ .

- $\{p_n\}$  converges to  $p$  if and only if every subsequence of  $\{p_n\}$  converges to  $p$ . *Proof:* Suppose every subsequence of  $\{p_n\}$  converges to  $p$ . Then since  $\{p_n\}$  is also a subsequence of itself,  $\{p_n\}$  converges to  $p$ . Conversely, suppose  $\{p_n\}$  converges to  $p$  and let  $\{p_{n_k}\}$  be a subsequence of  $\{p_n\}$ . Given  $\epsilon > 0$ , there exists an integer  $M$  such that  $n \geq M \implies |p_n - p| < \epsilon$ . Now choose some integer  $N \in \{n_k\}$  such that  $N > M$ . Then  $n \geq N \implies |p_{n_k} - p| < \epsilon$ , so  $\{p_{n_k}\}$  converges to  $p$ . ■
- **Bolzano-Weierstrass Theorem:** Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

### 3 Cauchy Sequences

- A sequence  $\{p_n\}$  in a metric space  $X$  is a Cauchy sequence if for every  $\epsilon > 0$ , there is an integer  $N$  such that  $d(p_m, p_n) < \epsilon$  if  $m, n \geq N$ .
- **In any metric space  $X$ , every convergent sequence is a Cauchy sequence.** *Proof:* Let  $\{p_n\}$  be a sequence in  $X$ . If  $p_n \rightarrow p$ , then for all  $\epsilon > 0$  there is an integer  $N$  such that  $n \geq N \implies d(p_n, p) < \epsilon$ . Then  $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$  whenever  $n \geq N$  and  $m \geq N$ . So  $\{p_n\}$  is a Cauchy sequence. ■
- **In  $\mathbb{R}^k$ , every Cauchy sequence converges.**
- A metric space in which every Cauchy sequence converges is said to be complete.
- A sequence  $\{s_n\}$  in  $\mathbb{R}$  is said to be monotonically increasing if  $s_n \leq s_{n+1}$  for all  $n \in \mathbb{N}$ .
- A sequence  $\{s_n\}$  in  $\mathbb{R}$  is said to be monotonically decreasing if  $s_n \geq s_{n+1}$  for all  $n \in \mathbb{N}$ .
- **If  $\{s_n\}$  is monotonic, then  $\{s_n\}$  converges if and only if it is bounded.** *Proof:* Suppose  $s_n \leq s_{n+1}$ . Let  $E$  be the range of  $\{s_n\}$ . Since  $\{s_n\}$  is bounded, let  $s = \sup E$ . Then  $s_n \leq s$  for all  $n \in \mathbb{N}$ . For every  $\epsilon > 0$ , there exists an integer  $N$  such that  $s - \epsilon < s_N \leq s$  since if it were not so, then  $s - \epsilon$  would be an upper bound for  $E$ . Since  $\{s_n\}$  is increasing,  $n \geq N \implies s - \epsilon < s_n \leq s < s + \epsilon$ , and so  $\{s_n\}$  converges to  $s$ . The converse has already been proved previously, and the proof where  $\{s_n\}$  is decreasing is analogous. ■

### 4 Upper and Lower Limits

- We define the extended real numbers,  $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ , where  $-\infty < r < \infty$  for all  $r \in \mathbb{R}$ . The arithmetic operations of  $\mathbb{R}$  are partially extended to  $\tilde{\mathbb{R}}$ :
  1.  $a + \infty = \infty + a = \infty$  for  $a \neq -\infty$
  2.  $a - \infty = -\infty + a = -\infty$  for  $a \neq \infty$
  3.  $\infty - \infty$  is not defined.
- Let  $\{s_n\}$  be a sequence in  $\mathbb{R}$  with the property that for every  $M \in \mathbb{R}$  there is an integer  $N$  such that  $n \geq N \implies s_n \geq M$ . Then we say that  $s_n \rightarrow \infty$ . Similarly, if for every  $M \in \mathbb{R}$  there an integer  $N$  such that  $n \geq N \implies s_n \leq M$ , we say that  $s_n \rightarrow -\infty$ .
- Let  $\{a_n\}$  be a sequence in  $\mathbb{R}$ . We define the limit superior and limit inferior of  $\{a_n\}$  as such:
  1.  $\limsup a_n = \infty$  if and only if  $\{a_n\}$  is unbounded above.
  2. If  $\{a_n\}$  is bounded above, then let  $M_k = \sup \{a_k, a_{k+1}, a_{k+2}, \dots\}$ . Then  $\limsup a_n = \lim_{k \rightarrow \infty} M_k$ .
  3. If  $a_n \rightarrow -\infty$ , then  $M_k \rightarrow -\infty$  and  $\limsup a_n = -\infty$ .
  4.  $\liminf a_n = -\infty$  if and only if  $\{a_n\}$  is unbounded below.
  5. If  $\{a_n\}$  is bounded below, then let  $m_k = \inf \{a_k, a_{k+1}, a_{k+2}, \dots\}$ . Then  $\liminf a_n = \lim_{k \rightarrow \infty} m_k$ .
  6. If  $a_n \rightarrow \infty$ , then  $m_k \rightarrow \infty$  and  $\liminf a_n = \infty$ .
- An alternate definition follows:

1.  $\limsup a_n = \infty$  if and only if  $\{a_n\}$  is unbounded above.
  2. If  $\{a_n\}$  is bounded above, and there exists  $u \in \mathbb{R}$  such that, for all  $\epsilon > 0$ , there exists an integer  $M$  where  $n \geq M \implies a_n < u + \epsilon$  and there exist infinitely many  $n$  where  $a_n > u - \epsilon$ , then  $\limsup a_n = u$ .
  3. Otherwise,  $\limsup a_n = -\infty$ .
  4.  $\liminf a_n = -\infty$  if and only if  $\{a_n\}$  is unbounded below.
  5. If  $\{a_n\}$  is bounded below, and there exists  $l \in \mathbb{R}$  such that, for all  $\epsilon > 0$ , there exists an integer  $M$  where  $n \geq M \implies a_n > l - \epsilon$  and there exist infinitely many  $n$  where  $a_n < l + \epsilon$ , then  $\liminf a_n = l$ .
  6. Otherwise,  $\liminf a_n = \infty$ .
- Another equivalent definition: Let  $\mathbb{S}$  be the set containing all subsequential limits of  $a_n$ , including  $\infty$  and  $-\infty$ . Then  $\limsup a_n = \sup \mathbb{S}$  and  $\liminf a_n = \inf \mathbb{S}$ . These numbers exist since  $\mathbb{S}$  is non-empty. If  $\{a_n\}$  is bounded, then there exists at least one real subsequential limit. If  $a_n$  is unbounded in either direction, then there exist subsequences that diverge in either direction.
  - $\liminf a_n \leq \limsup a_n$ . *Proof:* If  $\limsup a_n = \infty$  or  $\liminf a_n = -\infty$ , we are done. So suppose  $\limsup a_n = -\infty$ . Then  $\sup \mathbb{S} = -\infty$  and thus  $\mathbb{S} = \{-\infty\}$ . So  $\liminf a_n = -\infty$ . If  $\liminf a_n = \infty$ , then by similar reasoning we can show that  $\limsup a_n = \infty$ . So let  $\limsup a_n = \alpha \in \mathbb{R}$  and let  $\liminf a_n = \beta \in \mathbb{R}$ .  $\alpha = \sup \mathbb{S}$  and  $\beta = \inf \mathbb{S}$ , so  $\alpha \leq \beta$ . ■
  - $a_n \rightarrow \infty$  **if and only if**  $\liminf a_n = \limsup a_n = \infty$ . *Proof:*
  - $a_n \rightarrow -\infty$  **if and only if**  $\liminf a_n = \limsup a_n = -\infty$ . *Proof:*
  - **If there exists  $v \in \mathbb{R}$  such that given  $\epsilon > 0$ , there exists  $M$  such that  $n \geq M \implies a_n < v + \epsilon$ , then  $v \geq \limsup a_n$ .** *Proof:* Let  $\limsup a_n = \alpha$ , and suppose  $v < \alpha$ . Then  $\alpha = v + \delta$ , where  $\delta > 0$ . There exists  $N$  such that  $n \geq N \implies a_n < v + \frac{1}{2}\delta$ . But there also exist infinitely many  $n$  such that  $a_n > \alpha - \frac{1}{2}\delta = v + \frac{1}{2}\delta$ , so we have a contradiction. Thus,  $v \geq \limsup a_n$ . ■
  - **If there exists  $v \in \mathbb{R}$  such that given  $\epsilon > 0$ , there exist infinitely many  $n$  such that  $a_n > v - \epsilon$ , then  $v \leq \limsup a_n$ .** *Proof:* Let  $\limsup a_n = \alpha$ , and suppose  $v > \alpha$ . Then  $v = \alpha + \delta$ , where  $\delta > 0$ . There exist infinitely many  $n$  such that  $a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$ . But there also exists  $N$  such that  $n \geq N \implies a_n < \alpha + \frac{1}{2}\delta$ , and so we have a contradiction. Thus  $v \leq \limsup a_n$ . ■
  - **If there exists  $v \in \mathbb{R}$  such that given  $\epsilon > 0$ , there exists  $M$  such that  $n \geq M \implies a_n > v - \epsilon$ , then  $v \leq \liminf a_n$ .** *Proof:* Let  $\liminf a_n = \alpha$ , and suppose  $v > \alpha$ . Then  $v = \alpha + \delta$ , where  $\delta > 0$ . There exists  $N$  such that  $n \geq N \implies a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$ . But there also exist infinitely many  $n$  such that  $a_n < \alpha + \frac{1}{2}\delta$ , so we have a contradiction. Thus  $v \leq \liminf a_n$ . ■
  - **If there exists  $v \in \mathbb{R}$  such that given  $\epsilon > 0$ , there exist infinitely many  $n$  such that  $a_n < v + \epsilon$ , then  $v \geq \liminf a_n$ .** *Proof:* Let  $\liminf a_n = \alpha$ , and suppose  $v < \alpha$ . Then  $\alpha = v + \delta$ , where  $\delta > 0$ . There exist infinitely many  $n$  such that  $a_n < v + \frac{1}{2}\delta = \alpha - \frac{1}{2}\delta$ . But there also exists  $N$  such that  $n \geq N \implies a_n > \alpha - \frac{1}{2}\delta$ , so we have a contradiction. Thus  $v \geq \liminf a_n$ . ■
  - **For a sequence  $\{a_n\}$  in  $\mathbb{R}$ ,  $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$  if and only if  $\limsup a_n = \liminf a_n = a$ .** *Proof:* Let  $\limsup a_n = \liminf a_n = a$ . Then for all  $\epsilon > 0$ , there exist integers  $M, N$  such that  $n \geq M \implies a_n < a + \epsilon$  and  $n \geq N \implies a_n > a - \epsilon$ . Let  $P = \max(M, N)$ . Then  $n \geq P \implies |a_n - a| < \epsilon$ . Conversely, suppose  $a_n \rightarrow a$ . For all  $\epsilon > 0$ , there exists  $K$  such that  $n \geq K \implies a - \epsilon < a_n < a + \epsilon$ . Thus  $a \leq \liminf a_n$  and  $a \geq \limsup a_n$ . Since  $\liminf a_n \leq \limsup a_n$ , we have  $\limsup a_n = \liminf a_n = a$ . ■
  - $\liminf(-a_n) = -\limsup a_n$ . *Proof:* Let  $\limsup a_n = \alpha$ . Then, for every  $\epsilon > 0$  there exists  $M$  such that  $n \geq M \implies a_n < \alpha + \epsilon$  and there exist infinitely many  $n$  such that  $a_n > \alpha - \epsilon$ . So for every  $\epsilon > 0$ , there exists  $M$  such that  $n \geq M \implies -a_n > -\alpha - \epsilon$  and there exist infinitely many  $n$  such that  $-a_n < -\alpha + \epsilon$ . So  $\liminf(-a_n) = -\alpha$ . ■
  - **Let  $a_n \leq b_n$  for all  $n \geq K$ . Then  $\limsup a_n \leq \limsup b_n$ .** *Proof:* If  $\limsup b_n = \infty$  then we are done.

- 5 Special Sequences
- 6 Series
- 7 Series of Non-negative Terms
- 8 Euler's Number
- 9 Root and Ratio Tests
- 10 Power Series
- 11 Summation by Parts
- 12 Absolute Convergence
- 13 Addition and Multiplication of Series
- 14 Rearrangements