

# Algebra II: Introduction to Module Theory

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## 1 Basic Definitions

- Let  $R$  be a ring. Let  $M$  be an abelian group with respect to an operation  $+$ .  $M$  is a left-module on  $R$  if there exists an action of  $R$  on  $M$ , i.e, a map  $R \times M \rightarrow M$ , denoted  $rm$ , such that:

1.  $(r + s)m = rm + sm$ , for all  $r, s \in R$  and all  $m \in M$ .
2.  $(rs)m = r(sm)$ , for all  $r, s \in R$  and all  $m \in M$ .
3.  $r(m + n) = rm + rn$ , for all  $r \in R$  and all  $m, n \in M$ .
4. If  $R$  has unity, then  $1m = m$  for all  $m \in M$ .

A right-module on  $R$  can be defined analogously.

- Modules over a field  $\mathbb{F}$  and vector spaces over  $\mathbb{F}$  are the same thing.
- Let  $R$  be a ring and  $M$  be an  $R$ -module. An  $R$ -submodule of  $M$  is a subgroup  $N$  of  $M$  which is closed under the action of ring elements, i.e,  $rn \in N$  for all  $r \in R$  and all  $n \in N$ . A submodule of  $M$  thus just a subset of  $M$  which is itself a module with the same operations.
- If  $R$  is a field, then submodules are the same thing as subspaces.
- Every  $R$ -module  $M$  has at least two submodules:  $M$  itself, and  $\{0\}$ , the trivial submodule.
- Let  $R$  be any commutative ring. Then  $R$  is a module on itself, where the action is simply regular multiplication in  $R$ . In this case the submodules of  $R$  would simply be the ideals of  $R$ .
- If  $M$  is an  $R$ -module and  $S$  is a subring of  $R$  with  $1_S = 1_R$ , then  $M$  is also an  $S$ -module.
- Let  $\mathbb{F}$  be a field and let  $n \in \mathbb{Z}^+$ . The affine  $n$ -space over  $\mathbb{F}$  is  $\mathbb{F}^n = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F}\}$ . It is a module/vector space over  $\mathbb{F}$ , with addition and scalar multiplication defined componentwise.
- Let  $R$  be a ring with unity and let  $n \in \mathbb{Z}^+$ . Define  $R^n = \{(a_1, a_2, \dots, a_n) : a_i \in R\}$ . This is an  $R$ -module with componentwise operations. It is called the free module of rank  $n$  over  $R$ .
- If  $M$  is an  $R$ -module and if  $I$  is an ideal of  $R$ , and if  $am = 0$  for all  $a \in I$ , and all  $m \in M$ , then we say that  $M$  is annihilated by  $I$ . Here,  $M$  can be made into an  $(R/I)$ -module with the operation  $(r + I)m = rm$ . Since  $am = 0$  for all  $a \in I$ , this is well defined. When  $I$  is a maximal ideal, then  $M$  is a vector space over the field  $R/I$ .
- Let  $A$  be an abelian group. For any  $n \in \mathbb{Z}$  and  $a \in A$ , define  $na = a + a + \dots + a$  ( $n$  times) if  $n > 0$ ,  $na = 0$  if  $n = 0$ , and  $na = -a - a - \dots - a$  ( $n$  times) if  $n < 0$ . This makes  $A$  into a  $\mathbb{Z}$ -module, and shows that every abelian group is a  $\mathbb{Z}$ -module. Additionally,  $\mathbb{Z}$ -submodules are just subgroups of  $A$ .
- **Submodule Criterion:** Let  $R$  be a ring and  $M$  be an  $R$ -module. Then,  $N \subseteq M$  is a submodule of  $M$  if and only if  $N \neq \emptyset$  and  $x + ry \in N$  for all  $r \in R$  and all  $x, y \in N$ . *Proof:* Suppose  $N$  is a submodule. Then  $0 \in N$  so  $N \neq \emptyset$ . Additionally,  $N$  is closed under addition and  $rn \in N$  for all  $r \in R, n \in N$ . Conversely, suppose  $N \neq \emptyset$  and  $x + ry \in N$  for all  $x, y \in N$  and all  $r \in R$ . Let  $r = -1$ . Then by the subgroup criterion,  $N$  is an (additive) subgroup of  $M$ . So  $0 \in N$ . Let  $x = 0$ . Then  $N$  is closed under the action of ring elements and is therefore a submodule. ■
- Let  $M$  be an  $R$ -module.  $m \in M$  is called a torsion element if  $rm = 0$  for some nonzero element  $r \in R$ .  $\text{Tor}(M)$  denotes the set of all torsion elements in  $M$ .

## 2 Quotient Modules and Module Homomorphisms

- Let  $R$  be a ring and  $M, N$  be  $R$ -modules. A map  $\Phi : M \rightarrow N$  is an  $R$ -module homomorphism if  $\Phi(x + y) = \Phi(x) + \Phi(y)$  for all  $x, y \in M$  and  $\Phi(rx) = r\Phi(x)$  for all  $r \in R, x \in M$ .
- $\text{Hom}(M, N)$  denotes the set of all module homomorphisms from  $M$  to  $N$ .
- $\mathbb{Z}$ -module homomorphisms are the same as abelian group homomorphisms.
- Let  $\Phi, \Psi \in \text{Hom}(M, N)$  and define  $\Phi + \Psi$  as  $(\Phi + \Psi)(m) = \Phi(m) + \Psi(m)$  for all  $m \in M$ . Then  $\text{Hom}(M, N)$  is an abelian group. If  $R$  is a commutative ring, then define  $r\Phi$  as  $(r\Phi)(m) = r\Phi(m)$  for all  $m \in M$ . With this action,  $\text{Hom}(M, N)$  is itself an  $R$ -module.
- Since  $\Phi \circ \Psi \in \text{Hom}(M, N)$  whenever  $\Phi, \Psi \in \text{Hom}(M, N)$ , using function composition as the multiplication operation,  $\text{Hom}(M, N)$  is a ring with unity.
- The ring  $\text{Hom}(M, M)$  is called the endomorphism ring of  $M$  and its elements are called endomorphisms. It is also denoted  $\text{End}(M)$ .
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## 3 Generation of Modules, Direct Sums, Free Modules

## 4 Tensor Products of Modules

## 5 Exact Sequences; Projective, Injective and Flat Modules