Real Analysis I: Continuity

Arjun Vardhan

†

Created: 25th February 2022
Last updated: 3rd April 2022

1 Limits of Functions

- Let X and Y be metric spaces. Suppose $E \subset X$, f maps E into Y, and p is a limit point of E. Then, $\lim\limits_{x \to p} f(x) = q$ if there exists a point $q \in Y$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, p) < \delta \implies d_Y(f(x), q) < \epsilon$, where $x \in E$. Restated in terms of $\mathbb{R}$: Let $f : D \to \mathbb{R}$, where D is a non-empty subset of $\mathbb{R}$. $\lim\limits_{x \to p} f(x) = q$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in D$, $|x - p| < \delta \implies |f(x) - q| < \epsilon$.

- **$\lim\limits_{x \to p} f(x) = q$ if and only if $\lim\limits_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and $p_n \to p$.** *Proof:* Suppose $\lim\limits_{x \to p} f(x) = q$. Choose an appropriate $\{p_n\}$ in E. Then, there exists $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also there exists N such that $n \geq N$ implies $0 < d_X(p_n, p) < \delta$. Therefore, for $n \geq N$, $d_Y(f(p_n), q) < \epsilon$, and thus $f(p_n) \to q$. Conversely, suppose $\lim\limits_{x \to p} f(x) \neq q$. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \in E$ such that $d_Y(f(x), q) \geq \epsilon$ but $0 < d_X(x, p) < \delta$. Let $\delta_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then the corresponding x_n is a sequence in E which converges to p and $x_n \neq p$, but $f(x_n)$ does not converge to q. ■

- **If f has a limit at p, then this limit is unique.** *Proof:*

2 Continuous Functions

3 Continuity and Compactness

4 Continuity and Connectedness

5 Discontinuities

6 Monotonic Functions

7 Infinite Limits and Limits at Infinity