Real Analysis I: Continuity

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1 Limits of Functions

- Let X and Y be metric spaces. Suppose $E \subset X$, f maps E into Y, and p is a limit point of E. Then, $\lim_{x \to p} f(x) = q$ if there exists a point $q \in Y$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, p) < \delta \implies d_Y(f(x), q) < \epsilon$, where $x \in E$. Restated in terms of $\mathbb{R}$: Let $f : D \to \mathbb{R}$, where D is a non-empty subset of $\mathbb{R}$. $\lim_{x \to p} f(x) = q$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in D$, $|x - p| < \delta \implies |f(x) - q| < \epsilon$.
- **$\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and $p_n \to p$.** *Proof:* Suppose $\lim_{x \to p} f(x) = q$. Choose an appropriate $\{p_n\}$ in E. Then, there exists $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also there exists N such that $n \geq N$ implies $0 < d_X(p_n, p) < \delta$. Therefore, for $n \geq N$, $d_Y(f(p_n), q) < \epsilon$, and thus $f(p_n) \to q$. Conversely, suppose $\lim_{x \to p} f(x) \neq q$. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \in E$ such that $d_Y(f(x), q) \geq \epsilon$ but $0 < d_X(x, p) < \delta$. Let $\delta_n = \frac{1}{n}$, $n \in \mathbb{N}$. Then the corresponding x_n is a sequence in E which converges to p and $x_n \neq p$, but $f(x_n)$ does not converge to q. ■
- If f has a limit at p, then this limit is unique.
- **Algebra of Limits: Let $E \subset X$ be a metric space and f, g be complex functions on E. Let p be a limit point of E, $\lim_{x \to p} f(x) = A$, and $\lim_{x \to p} g(x) = B$. Then,**

  1. $\lim_{x \to p} (f + g)(x) = A + B$.
  2. $\lim_{x \to p} (fg)(x) = AB$.
  3. $\lim_{x \to p} (\frac{f}{g})(x) = \frac{A}{B}$, if $B \neq 0$.

  These properties follow from the analogous properties of sequences.

2 Continuous Functions

- Let X and Y be metric spaces. Suppose $E \subset X$, f maps E into Y, and p is a limit point of E. Then, f is said to be continuous at p if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \epsilon$, i.e, $\lim_{x \to p} f(x) = f(p)$.
- If f is continuous at every point of E, then it is said to be continuous on E.
- **Suppose X, Y, Z are metric spaces, $E \subset X$, $f : E \to Y$, $g : f(E) \to Z$, and $h : E \to Z$, $h(x) = g(f(x))$. If f is continuous at $p \in E$ and g is continuous at $f(p)$, then h is continuous at p.** *Proof:* Let $\epsilon > 0$ be given. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that $d_Z(g(y), g(f(p))) < \epsilon$ if $d_Y(y, f(p)) < \eta$ and $y \in f(E)$. Since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$ and $x \in E$. It follows that $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$ if $d_X(x, P) < \delta$. Thus h is continuous at p. ■
- If f, g are continuous at a limit point p, then so are $f + g$, fg, and $\frac{f}{g}$ if $g(p) \neq 0$. This follows from the algebra of limits.

3 Continuity and Compactness

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4 Continuity and Connectedness

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5 Discontinuities

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6 Monotonic Functions

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7 Infinite Limits and Limits at Infinity

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