Linear Algebra: Vector Spaces

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1 Vector Spaces

- A vector space V is a set, along with two operations, vector addition and scalar multiplication, satisfying the following properties:
  1. $v + w = w + v$ for all $v, w \in V$.
  2. $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
  3. There exists a zero vector $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
  4. For every $v \in V$ there exists $w \in V$ such that $v + w = 0$. Usually denoted $-v$.
  5. $1 \cdot v = v$ for all $v \in V$.
  6. $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$, for all $v \in V$, for all scalars α, β.
  7. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$, for all $u, v \in V$, for all scalars α.
  8. $(\alpha + \beta) \cdot v = \alpha \cdot v + \alpha \cdot v$, for all $v \in V$, for all scalars α, β.
- **The zero vector is unique.** *Proof:* Let $0, 0' \in V$ such that $0 + v = v$ and $0' + v = v$ for all $v \in V$. Then $0 = 0 + 0' = 0'$. ■
- **Additive inverses are unique.** *Proof:* Let $v \in V$. Let $u, w \in V$ such that $v + w = v + u = 0$. Then $v + w + w = v + u + w \implies (v + w) + w = (v + w) + u \implies w = u$. ■
- $0 = 0 \cdot v$ **for all** $v \in V$**.** *Proof:* $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 = 0 \cdot v$. ■
- $-v = (-1) \cdot v$ **for all** $v \in V$**.** *Proof:* $(-1) \cdot v + v = (-1) \cdot v + 1 \cdot v = (-1 + 1) \cdot v = 0$. Thus $(-1) \cdot v = -v$. ■
- The scalars are always from a field, usually $\mathbb{R}$ or $\mathbb{C}$.
- An $m \times n$ matrix is a rectangular array with m rows and n columns. Entries of a matrix are denoted a_{ij} or $(A)_{i,j}$, where i is the row and j is the column.
- Given a matrix A, its transpose A^T is the matrix formed by transforming the rows of A into columns. Formally, $(A)_{i,j} = (A^T)_{j,i}$.

2 Linear Combinations and Bases

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