Linear Algebra: Inner Product Spaces

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1 Introduction

- An inner product is a function that assigns a scalar to a pair of vectors.
- Let $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. The norm of a vector in $\mathbb{R}^n$ is defined as $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. The inner product of two vectors is defined as $\langle x, y \rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$.
- Let V be a vector space over $\mathbb{C}$. An inner product in V must satisfy the following properties:
  1. **Conjugate symmetry:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
  2. **Linearity:** $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in V$ and all $\alpha, \beta \in \mathbb{C}$.
  3. **Non-negativity:** $\langle x, x \rangle \geq 0$ for all $x \in V$.
  4. **Non-degeneracy:** $\langle x, x \rangle = 0$ if and only if $x = 0$.

  If such a function exists, then V together with its inner product is an inner product space.
- Given an inner product space, the norm of a vector x is defined as $\|x\| = \sqrt{\langle x, x \rangle}$.
- **$\langle x, 0 \rangle = 0$ for all vectors x.** *Proof:* $\langle x, 0 \rangle = \langle x, x - x \rangle = \langle x, x \rangle + \langle x, -x \rangle = \langle x, x \rangle - \langle x, x \rangle = 0$. ■
- **Let $x \in V$. Then $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in V$.** *Proof:*
- **Corollary: Let x, y be vectors in an inner product space V. $x = y$ if and only if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$.** *Proof:*
- **Suppose two operators $A, B : X \to Y$ satisfy**
- **Cauchy-Schwarz Inequality:** $|\langle x, y \rangle| \leq \|x\| \|y\|$**.** *Proof:*
- **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$**.** *Proof:*
- **Polarization Identities: Let $x, y \in V$. If V is a real inner product space, then $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$. If V is a complex inner product space, then $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$.** *Proof:*
- **Parallelogram Identity:** $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$**.** *Proof:*
- It is easy to show that the norm satisifies the following properties:
  1. **Homogeneity:** $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all vectors v, all scalars α.
  2. **Non-negativity:** $\|v\| \geq 0$ for all vectors v.
  3. **Non-degeneracy:** $\|v\| = 0$ if and only if $v = 0$.

  A vector space equipped with a function satisfying the above properties, along with the triangle inequality, is called a normed space.

2 Orthogonality, Orthogonal and Orthonormal Bases

- Two vectors u and v are called orthogonal if $\langle u, v \rangle = 0$. This is denoted by $u \perp v$.
- **Pythagorean Identity: If $u \perp v$, then $\|u\| + \|v\| = \|u\|^2 + \|v\|^2$.** *Proof:*
- A vector v is said to be orthogonal to a subspace E if $v \perp w$ for all $w \in E$.
- Subspaces E and F are said to be orthogonal if all vectors in E are orthogonal to all vectors in F.
- **Let E be spanned by $v_1, v_2, ..., v_n$. Then $v \perp E$ if and only if $v \perp v_k$ for all k.** *Proof:*
- A system of vectors $v_1, v_2, ..., v_n$ is called orthogonal if $v_i \perp v_j$ whenever $i \neq j$. If $\|v_k\| = 1$ for all k, the system is called orthonormal.
- **Generalized Pythagorean Identity: Let $v_1, v_2, ..., v_n$ be orthogonal. Then, $\left\| \sum_{k=1}^{n} \alpha_k v_k \right\|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \|v_k\|^2$.** *Proof:*
- **Any orthogonal system $v_1, v_2, ..., v_n$ of nonzero vectors is linearly independent.** *Proof:*
- If an orthogonal or orthonormal system $v_1, v_2, ..., v_n$ is also a basis, it is called an orthogonal or orthonormal basis.
- Let $v_1, v_2, ..., v_n$ be an orthogonal basis for V, and let $x \in V$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$. Taking the inner product of both sides with v_1, we get $\langle x, v_1 \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n, v_1 \rangle = \alpha_1 \langle v_1, v_1 \rangle = \alpha_1 \|v_1\|^2$ (as $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$). Thus $\alpha_1 = \dfrac{\langle x, v_1 \rangle}{\|v_1\|^2}$. The same process can be used to find all other coordinates for any vector in this basis, i.e, $\alpha_k = \dfrac{\langle x, v_k \rangle}{\|v_k\|^2}$.

3 Orthogonal Projection and Gram-Schmidt Orthogonalization

- Let E be a subspace of inner product space V. For a vector $v \in V$, its orthogonal projection onto E, denoted $P_E v$, is a vector $w \in E$ such that $v - w \perp E$.
- **Let $w = P_E v$. Then for all $x \in E$, $\|v - w\| \leq \|v - x\|$. If $\|v - x\| = \|v - w\|$ for some $x \in E$, then $x = w$.** *Proof:*
- **Let $v_1, v_2, ..., v_n$ be an orthogonal basis in E. Then the orthogonal projection of v in E is given by $P_E v = \sum_{k=1}^{n} \alpha_k v_k$, where $\alpha_k = \dfrac{\langle v, v_k \rangle}{\|v_k\|^2}$.** *Proof:*
- **Gram-Schmidt Orthogonalization Algorithm:** Let $x_1, x_2, ..., x_n$ be a linearly independent system in inner product space V. Then we can find an orthogonal system $v_1, v_2, ..., v_n$ in V such that $\operatorname{span}(x_1, x_2, ..., x_n) = \operatorname{span}(v_1, v_2, ..., v_n)$. Additionally, for all $r \leq n$, we get $\operatorname{span}(v_1, v_2, ..., v_r) = \operatorname{span}(x_1, x_2, ..., x_r)$. First, let $v_1 = x_1$. Let $E_1 = \operatorname{span}(v_1) = \operatorname{span}(x_1)$. Define $v_2 = x_2 - P_{E_1} x_2 = x_2 - \dfrac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1$. Let $E_2 = \operatorname{span}(v_1, v_2)$. Clearly $\operatorname{span}(x_1, x_2) = E_2$. Define $v_3 = x_3 - P_{E_2} x_3 = x_3 - \dfrac{\langle x_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle x_3, v_2 \rangle}{\|v_2\|^2} v_2$. Let $E_3 = \operatorname{span}(v_1, v_2, v_3)$. Again, it is clear that $\operatorname{span}(x_1, x_2, x_3) = E_3$. As $x_3 \notin E_2$, $v_3 \neq 0$. (If x_3 had been in E_2, so would v_3, which would imply that $\langle v_3, v_3 \rangle = 0$ and thus $v_3 = 0$).