Linear Algebra: Introduction to Matrices

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1 Introduction

- An $m \times n$ matrix has m rows and n columns. a_{ij} represents the entry at the ith row and jth column. If $A = [a_{ij}]$ is an $m \times n$ matrix with entries from field $\mathbb{F}$, we say $A \in \mathbb{M}_{m,n}(\mathbb{F})$.
- A is called a square matrix if $m = n$.
- A is called a zero matrix if $a_{ij} = 0$ for all i and j.
- Let $A \in \mathbb{M}_n(\mathbb{F})$. Then the entries $a_{11}, a_{22}, ..., a_{nn}$ are called the diagonal entries of A. These constitute the principal diagonal of A.
- Let $A \in \mathbb{M}_n(\mathbb{F})$. If all non-diagonal entries are 0, then A is said to be a diagonal matrix. It is denoted $\text{diag}(a_{11}, a_{22}, ..., a_{nn})$.
- If $A = \text{diag}(1, 1, ..., 1)$, then A is the identity matrix, denoted I_n or I.

2 Matrix Operations

- Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$. Then the transpose of A, denoted A^T, is an $n \times m$ matrix with $(A^T)_{ij} = a_{ji}$ for all i, j.
- Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$. Then the conjugate transpose of A, denoted A^*, is an $n \times m$ matrix with $(A^*)_{ij} = \overline{a_{ji}}$ for all i, j. Here, $\overline{a_{ji}}$ is the complex conjugate of a_{ji}.
- For any matrix, $(A^*)^* = A$ and $(A^T)^T = A$.
- Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ and Let $B = [b_{ij}] \in \mathbb{M}_{n,r}(\mathbb{C})$. The product of A and B, denoted AB, is the matrix $C = [c_{ij}] \in \mathbb{M}_{m,r}(\mathbb{C})$ such that for $1 \leq i \leq m$, $1 \leq j \leq r$, $c_{ij} = \displaystyle\sum_{k=1}^{n} a_{ik}b_{kj}$. That is, c_{ij} is the dot product of the ith row of A with the jth column of B.
- AB is only defined if the number of columns in A is equal to the number of rows in B.
- Two square matrices A and B are said to commute if $AB = BA$.
- Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $B \in \mathbb{M}_n(\mathbb{C})$ is said to be the left inverse of A if $BA = I_n$. $C \in \mathbb{M}_n(\mathbb{C})$ is said to be the right inverse of A if $AC = I_n$.
- A is said to be invertible if there exists $B \in \mathbb{M}_n(\mathbb{C})$ such that $AB = BA = I_n$.
- **Let $A \in \mathbb{M}_n(\mathbb{C})$. If A has a right and left inverse, then they are equal.** *Proof:* Let B be the left and C be the right inverse of A. Then $B = BI_n = B(AC) = (BA)C = I_nC = C$. ■
- The inverse matrix of A is unique and denoted A^{-1}.
- **Let A and B be invertible. Then,**
  1. **$(A^{-1})^{-1} = A$.** *Proof:*
  2. **$(AB)^{-1} = B^{-1}A^{-1}$.** *Proof:*
  3. **$(A^*)^{-1} = (A^{-1})^*$.** *Proof:*

3 Special Matrices

- If $A = \alpha I$ for some $\alpha \in \mathbb{F}$, then A is called a scalar matrix.
- Let $A \in \mathbb{M}_n(\mathbb{F})$. If $a_{ij} = 0$ for $i > j$, then A is said to be an upper triangular matrix. If $a_{ij} = 0$ for $i < j$, then A is said to be an lower triangular matrix. A is said to be a triangular matrix if it is either of the two.
- Let $A \in \mathbb{M}_n(\mathbb{R})$. Then,
  1. A is called symmetric if $A^T = A$.
  2. A is called skew-symmetric if $A^T = -A$.
  3. A is called orthogonal if $A^{-1} = A^T$.
  4. A is said to be a permutation matrix if it has exactly one non-zero entry, namely 1, in each row and column.
- Let $A \in \mathbb{M}_n(\mathbb{C})$. Then,
  1. A is called normal if $A^*A = AA^*$.
  2. A is called Hermitian if $A^* = A$.
  3. A is called skew-Hermitian if $A^* = -A$.
  4. A is called unitary if $A^{-1} = A^*$.
- A matrix A is called idempotent if $A^2 = A$.
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4 Submatrices