Real Analysis: Sequences and Series

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Created: 11th January 2022
Last updated: 20th June 2022

1 Convergent Sequences

- A sequence $\{p_n\}$ in metrix space X is said to converge if there exists $p \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq n \implies d(p_n, p) < \epsilon$. In this case, we say $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.
- If $\{p_n\}$ does not converge, it diverges.
- **If $p, p' \in X$ and $\{p_n\}$ converges to p and p', then $p = p'$.** *Proof:* Let $\epsilon \geq 0$ be given. Then there exist integers N and N' such that $n \geq N \implies d(p_n, p) < \frac{\epsilon}{2}$ and $n \geq N' \implies d(p_n, p') < \frac{\epsilon}{2}$. Let $N^\circ = \max(N, N')$. So if $n \geq N^\circ$ then $d(p, p') \leq d(p, p_n) + d(p', p_n) < \epsilon$. Since ϵ was arbitrary, we get $d(p, p') = 0$. ■
- **If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.** *Proof:* Suppose $p_n \to p$. There exists $N \in \mathbb{N}$ such that $n \geq N \implies d(p_n, p) < 1$. Let $r = \max(1, d(p_1, p), d(p_2, p), ..., d(p_N, p))$. Then $d(p_n, p) < r$ for all $n \in \mathbb{N}$. ■
- **If $E \subset X$ and p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $\lim_{n\to\infty} p_n = p$.** *Proof:* Since p is a limit point, for each $n \in \mathbb{N}$ there exists $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. Given $\epsilon > 0$, choose N so that $N > \frac{1}{\epsilon}$. Then $n \geq N \implies n \geq \frac{1}{\epsilon} \implies \epsilon \geq \frac{1}{n} \implies d(p_n, p) < \epsilon$. So $p_n \to p$. ■
- **Suppose $\{s_n\}$ and $\{t_n\}$ are sequences in $\mathbb{C}$, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then:**
  1. $\lim_{n\to\infty} s_n + t_n = s + t$. *Proof:* Given $\epsilon > 0$, there exist integers N_1, N_2 such that $n \geq N_1 \implies |s_n - s| < \frac{1}{\epsilon}$ and $n \geq N_2 \implies |t_n - t| < \frac{1}{\epsilon}$. Let $N_3 = \max(N_1, N_2)$. Then $n \geq N_3 \implies |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \epsilon$. ■
  2. $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} c + s_n = c + s$, **for all** $c \in \mathbb{C}$. *Proof:* Given $\epsilon > 0$, there exists N such that $n \geq N \implies |s_n - s| < \epsilon$ which implies that $|cs_n - cs| < \epsilon$ and $|(c + s_n) - (c + s)| < \epsilon$. ■
  3. $\lim_{n\to\infty} s_n t_n = st$. *Proof:* Use the identity $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$. Given $\epsilon > 0$, there exist integers N_1 and N_2 such that $n \geq N_1 \implies |s_n - s| < \sqrt{\epsilon}$ and $n \geq N_2 \implies |t_n - t| < \sqrt{\epsilon}$. If we let $N = \max(N_1, N - 2)$, then $n \geq N \implies |(s_n - s)(t_n - t)| < \epsilon$ and thus $\lim_{n\to\infty}(s_n - s)(t_n - t) = 0$. By taking the limit of both sides of the identity, we get $\lim_{n\to\infty} s_n t_n - st = 0$. ■
  4. $\lim_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$**, where $s_n \neq 0$ for all $n \in \mathbb{N}$.** *Proof:* Choose m such that $|s_n - s| < \frac{1}{2}|s|$. Then $|s_n| > \frac{1}{2}|s|$. Given $\epsilon > 0$ there exists an integer $N > m$ such that $n \geq N \implies |s_n - s| < \frac{1}{2}|s|^2\epsilon$. So for $n \geq N$, $\left|\dfrac{1}{s_n} - \dfrac{1}{s}\right| = \left|\dfrac{s_n - s}{s_n s}\right| < \dfrac{2}{|s|^2}|s_n - s| < \epsilon$. ■
- A sequence is called a null sequence if its limit is 0.
- **If $a_n = k$ for all $n \geq K$ for some natural number K, then $\lim_{n\to\infty} a_n = k$.** *Proof:* Let $\epsilon > 0$. Then, $n \geq K \implies d(a_n, k) = 0 < \epsilon$. ■
- **Let $a_n \to l$ and $b_n \to m$ ($l, m \in \mathbb{R}$). Then:**

1. **If $a_n \geq 0$ for all $n \geq K$, for some $K \in \mathbb{N}$, then $l \geq 0$.** *Proof:* Suppose $l < 0$. Let $\epsilon = \frac{|l|}{2}$. Since $a_n \geq 0$, $|a_n - l| \geq |l| > \epsilon$ for all $n \in \mathbb{N}$, which is a contradiction. So $l \geq 0$. ■
2. **If $a_n \leq b_n$ for all $n \geq K$, for some $K \in \mathbb{N}$, then $l \leq m$.** *Proof:* Let $c_n = b_n - a_n$. Then $c_n \geq 0$ for all $n \geq K$. Since $c_n \to m - l$, $m - l \geq 0$. ■
3. **If $a_n \leq \alpha$ for all $n \geq K$, for some $K \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then $l \leq \alpha$.** *Proof:* Let $c_n = \alpha - a_n$. Then $c_n \geq 0$ for all $n \geq K$. Since $c_n \to \alpha - l$, $\alpha - l \geq 0$. ■
4. **If $a_n \geq \alpha$ for all $n \geq K$, for some $K \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then $l \geq \alpha$.** *Proof:* Let $c_n = a_n - \alpha$. The proof follows similarly as above. ■

- **Sandwich/Squeeze Theorem (V1): Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. If $0 \leq b_n \leq ka_n$, for all $n \geq K$, for some $k \in \mathbb{R}, K \in \mathbb{N}$, and $a_n \to 0$, then $b_n \to 0$.** *Proof:* Let $\epsilon > 0$. As $a_n \to 0$, there exists M such that $n \geq M \implies |a_n| < \frac{\epsilon}{|k|+1}$. Then, $n \geq \max(K, M) \implies |b_n| \leq |ka_n| = |k||a_n| < |k|\frac{\epsilon}{|k|+1} < \epsilon$. Thus, $b_n \to 0$. ■

- **Sandwich/Squeeze Theorem (V2): Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $\mathbb{R}$. If $c_n \leq b_n \leq a_n$, for all $n \geq K$, for some $k \in \mathbb{R}, K \in \mathbb{N}$, and $a_n \to \alpha$, $c_n \to \alpha$, then $b_n \to \alpha$.** *Proof:* For all $n \geq K$, $0 \leq b_n - c_n \leq a_n - c_n$. Since $(a_n - c_n) \to 0$, $(b_n - c_n) \to 0 \implies b_n \to \alpha$. ■

- **Let $\{a_n\}$ be a sequence in $\mathbb{R}$. If $a_n \to \alpha$, then $|a_n| \to |\alpha|$.** *Proof:* Let $\epsilon > 0$. There exists K such that $n \geq K \implies |a_n - \alpha| < \epsilon$. By the triangle inequality, $||a_n| - |\alpha|| \leq |a_n - \alpha|$, and thus $|a_n| \to |\alpha|$. ■

2 Subsequences

- Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers where $n_1 < n_2 < n_3 < ...$ and so on. Then the sequence $\{p_{n_i}\}$ is a subsequence of $\{p_n\}$. $\{p_{n_i}\}$ converges, its limit is a subsequential limit of $\{p_n\}$.

- $n_k \geq k$ for all $k \in \mathbb{N}$.

- If $n_k = k + 1$, then $\{a_{n_k}\}$ is called the 1-tail of $\{a_n\}$.

- **$\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.** *Proof:* Suppose every subsequence of $\{p_n\}$ converges to p. Then since $\{p_n\}$ is also a subsequence of itself, $\{p_n\}$ converges to p. Conversely, suppose $\{p_n\}$ converges to p and let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$. Given $\epsilon > 0$, there exists an integer M such that $n \geq M \implies |p_n - p| < \epsilon$. Now choose some integer $N \in \{n_k\}$ such that $N > M$. Then $n \geq N \implies |p_{n_k} - p| < \epsilon$, so $\{p_{n_k}\}$ converges to p. ■

- **Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.**

3 Infinite Limits and Properly Divergent Sequences

- Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that given $K \in \mathbb{R}$, there exists M such that $n \geq M \implies a_n > K$. In this case, we say that $a_n \to \infty$, or a_n properly diverges to ∞.

- Similarly, let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that given $K \in \mathbb{R}$, there exists M such that $n \geq M \implies a_n < K$. In this case, we say that $a_n \to -\infty$, or a_n properly diverges to $-\infty$.

- $a_n \to \infty$ if and only if $-a_n \to -\infty$.

- If $a_n \to \infty$, then $\{a_n\}$ is bounded below but not above. If $a_n \to -\infty$, then $\{a_n\}$ is bounded above but not below.

- $\{a_n\}$ has a subsequence which tends to ∞ if and only if $\{a_n\}$ is unbounded above. $\{a_n\}$ has a subsequence which tends to $-\infty$ if and only if $\{a_n\}$ is unbounded below.

- **$a_n \to \infty$ if and only if every subsequence of $\{a_n\}$ tends to ∞. $a_n \to -\infty$ if and only if every subsequence of $\{a_n\}$ tends to $-\infty$.** *Proof:* If every subsequence of a_n tends to ∞, then $a_n \to \infty$. Conversely, let $a_n \to \infty$. Let a_{n_k} be a subsequence of a_n. Given $K \in \mathbb{R}$, there exists M such that $n \geq M \implies a_n > K$. Therefore $k \geq M \implies n_k \geq M \implies a_{n_k} > K$. Thus $a_{n_k} \to \infty$. Proof for the $-\infty$ case is analogous. ■

4 Cauchy Sequences

- A sequence $\{p_n\}$ in a metric space X is a Cauchy sequence if for every $\epsilon > 0$, there is an integer N such that $d(p_m, p_n) < \epsilon$ if $m, n \geq N$.
- **In any metric space X, every convergent sequence is a Cauchy sequence.** *Proof:* Let $\{p_n\}$ be a sequence in X. If $p_n \to p$, then for all $\epsilon > 0$ there is an integer N such that $n \geq N \implies d(p_n, p) < \epsilon$. Then $d(p_n, p_m) \leq d(p, p_n) + d(p, p_m) < 2\epsilon$ whenever $n \geq N$ and $m \geq N$. So $\{p_n\}$ is a Cauchy sequence. ■
- **In $\mathbb{R}^k$, every Cauchy sequence converges.**
- A metric space in which every Cauchy sequence converges is said to be complete.
- A sequence $\{s_n\}$ in $\mathbb{R}$ is said to be monotonically increasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.
- A sequence $\{s_n\}$ in $\mathbb{R}$ is said to be monotonically decreasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.
- **Monotone Convergence Theorem: If $\{s_n\}$ is monotonic, then $\{s_n\}$ converges if and only if it is bounded.** *Proof:* Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$. Since $\{s_n\}$ is bounded, let $s = \sup E$. Then $s_n \leq s$ for all $n \in \mathbb{N}$. For every $\epsilon > 0$, there exists an integer N such that $s - \epsilon < s_N \leq s$ since if it were not so, then $s - \epsilon$ would be an upper bound for E. Since $\{s_n\}$ is increasing, $n \geq N \implies s - \epsilon < s_n \leq s < s + \epsilon$, and so $\{s_n\}$ converges to s. The converse has already been proved previously, and the proof where $\{s_n\}$ is decreasing is analogous. ■

5 Upper and Lower Limits

- We define the extended real numbers, $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$, where $-\infty < r < \infty$ for all $r \in \mathbb{R}$. The arithmetic operations of $\mathbb{R}$ are partially extended to $\tilde{\mathbb{R}}$:
  1. $a + \infty = \infty + a = \infty$ for $a \neq -\infty$
  2. $a - \infty = -\infty + a = -\infty$ for $a \neq \infty$
  3. $\infty - \infty$ is not defined.
- Let $\{s_n\}$ be a sequence in $\mathbb{R}$ with the property that for every $M \in \mathbb{R}$ there is an integer N such that $n \geq N \implies s_n \geq M$. Then we say that $s_n \to \infty$. Similarly, if for every $M \in \mathbb{R}$ there an integer N such that $n \geq N \implies s_n \leq M$, we say that $s_n \to -\infty$.
- Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We define the limit superior and limit inferior of $\{a_n\}$ as such:
  1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
  2. If $\{a_n\}$ is bounded above, then let $M_k = \sup \{a_k, a_{k+1}, a_{k+2}, ...\}$. Then $\limsup a_n = \lim\limits_{k\to\infty} M_k$.
  3. If $a_n \to -\infty$, then $M_k \to -\infty$ and $\limsup a_n = -\infty$.
  4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
  5. If $\{a_n\}$ is bounded below, then let $m_k = \inf \{a_k, a_{k+1}, a_{k+2}, ...\}$. Then $\liminf a_n = \lim\limits_{k\to\infty} m_k$.
  6. If $a_n \to \infty$, then $m_k \to \infty$ and $\liminf a_n = \infty$.
- An alternate definition follows:
  1. $\limsup a_n = \infty$ if and only if $\{a_n\}$ is unbounded above.
  2. If $\{a_n\}$ is bounded above, and there exists $u \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n < u + \epsilon$ and there exist infinitely many n where $a_n > u - \epsilon$, then $\limsup a_n = u$.

3. Otherwise, $\limsup a_n = -\infty$.
4. $\liminf a_n = -\infty$ if and only if $\{a_n\}$ is unbounded below.
5. If $\{a_n\}$ is bounded below, and there exists $l \in \mathbb{R}$ such that, for all $\epsilon > 0$, there exists an integer M where $n \geq M \implies a_n > l - \epsilon$ and there exist infinitely many n where $a_n < l + \epsilon$, then $\liminf a_n = l$.
6. Otherwise, $\liminf a_n = \infty$.

- Another equivalent definition: Let $\mathbb{S}$ be the set containing all subsequential limits of a_n, including ∞ and $-\infty$. Then $\limsup a_n = \sup \mathbb{S}$ and $\liminf a_n = \inf \mathbb{S}$. These numbers exist since $\mathbb{S}$ is non-empty. If $\{a_n\}$ is bounded, then there exists at least one real subsequential limit. If a_n is unbounded in either direction, then there exist subsequences that diverge in either direction.

- $\liminf a_n \leq \limsup a_n$. *Proof:* If $\limsup a_n = \infty$ or $\liminf a_n = -\infty$, we are done. So suppose $\limsup a_n = -\infty$. Then $\sup \mathbb{S} = -\infty$ and thus $\mathbb{S} = \{-\infty\}$. So $\liminf a_n = -\infty$. If $\liminf a_n = \infty$, then by similar reasoning we can show that $\limsup a_n = \infty$. So let $\limsup a_n = \alpha \in \mathbb{R}$ and let $\liminf a_n = \beta \in \mathbb{R}$. $\alpha = \sup \mathbb{S}$ and $\beta = \inf \mathbb{S}$, so $\alpha \leq \beta$. ■

- $a_n \to \infty$ **if and only if** $\liminf a_n = \limsup a_n = \infty$**.** *Proof:* Suppose $a_n \to \infty$. Then for all $\alpha \in \mathbb{R}$, there exists K such that $n \geq K \implies a_n > \alpha$. So a_n is bounded below.

- $a_n \to -\infty$ **if and only if** $\liminf a_n = \limsup a_n = -\infty$**.** *Proof:*

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exists** M **such that** $n \geq M \implies a_n < v+\epsilon$**, then** $v \geq \limsup a_n$**.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n < v + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n > \alpha - \frac{1}{2}\delta = v + \frac{1}{2}\delta$, so we have a contradiction. Thus, $v \geq \limsup a_n$. ■

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exist infinitely many** n **such that** $a_n > v - \epsilon$**, then** $v \leq \limsup a_n$**.** *Proof:* Let $\limsup a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n < \alpha + \frac{1}{2}\delta$, and so we have a contradiction. Thus $v \leq \limsup a_n$. ■

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exists** M **such that** $n \geq M \implies a_n > v-\epsilon$**, then** $v \leq \liminf a_n$**.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v > \alpha$. Then $v = \alpha + \delta$, where $\delta > 0$. There exists N such that $n \geq N \implies a_n > v - \frac{1}{2}\delta = \alpha + \frac{1}{2}\delta$. But there also exist infinitely many n such that $a_n < \alpha + \frac{1}{2}\delta$, so we have a contradiction. Thus $v \leq \liminf a_n$. ■

- **If there exists** $v \in \mathbb{R}$ **such that given** $\epsilon > 0$**, there exist infinitely many** n **such that** $a_n < v + \epsilon$**, then** $v \geq \liminf a_n$**.** *Proof:* Let $\liminf a_n = \alpha$, and suppose $v < \alpha$. Then $\alpha = v + \delta$, where $\delta > 0$. There exist infinitely many n such that $a_n < v + \frac{1}{2}\delta = \alpha - \frac{1}{2}\delta$. But there also exists N such that $n \geq N \implies a_n > \alpha - \frac{1}{2}\delta$, so we have a contradiction. Thus $v \geq \liminf a_n$. ■

- **For a sequence** $\{a_n\}$ **in** $\mathbb{R}$**,** $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$ **if and only if** $\limsup a_n = \liminf a_n = a$**.** *Proof:* Let $\limsup a_n = \liminf a_n = a$. Then for all $\epsilon > 0$, there exist integers M, N such that $n \geq M \implies a_n < a + \epsilon$ and $n \geq N \implies a_n > a - \epsilon$. Let $P = \max(M, N)$. Then $n \geq P \implies |a_n - a| < \epsilon$. Conversely, suppose $a_n \to a$. For all $\epsilon > 0$, there exists K such that $n \geq K \implies a - \epsilon < a_n < a + \epsilon$. Thus $a \leq \liminf a_n$ and $a \geq \limsup a_n$. Since $\liminf a_n \leq \limsup a_n$, we have $\limsup a_n = \liminf a_n = a$. ■

- $\liminf(-a_n) = -\limsup a_n$. *Proof:* Let $\limsup a_n = \alpha$. Then, for every $\epsilon > 0$ there exists M such that $n \geq M \implies a_n < \alpha + \epsilon$ and there exist infinitely many n such that $a_n > \alpha - \epsilon$. So for every $\epsilon > 0$, there exists M such that $n \geq M \implies -a_n > -\alpha - \epsilon$ and there exist infinitely many n such that $-a_n < -\alpha + \epsilon$. So $\liminf(-a_n) = -\alpha$. ■

- **Let** $a_n \leq b_n$ **for all** $n \geq K$**. Then** $\limsup a_n \leq \limsup b_n$**.** *Proof:* If $\limsup b_n = \infty$ then we are done.

6 Special Sequences

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7 Series

- Given a sequence $\{a_n\}$, let $S_n = \sum_{k=0}^{n} a_k$. Then, $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$. We say that $\sum_{n=0}^{\infty} a_n$ converges if and only if S_n converges. If S_n properly diverges to $\pm\infty$, then $\sum_{n=0}^{\infty} a_n$ properly diverges.
- S_n is called the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$.
- The Cauchy criterion can be restated in terms of series. S_n converges if and only if for all $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies |S_n - S_m| < \epsilon \implies \left|\sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k\right| < \epsilon \implies \left|\sum_{k=n}^{m} a_k\right| < \epsilon$.
- If we let $m = n$, then we get $|a_n| < \epsilon$. Thus, if $\sum a_n$ converges, then $a_n \to 0$. The converse is not necessarily true.
- $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.
- $\sum a_n$ is said to be conditionally convergent if $\sum |a_n|$ diverges but $\sum a_n$ converges.
- Let $x \in R$. Then, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. So $x = x^+ - x^-$ and $|x| = x^+ + x^-$.
- **Let a_n be a sequence that is ultimately non-negative, and let A_n be the sequence of its partial sums. Then $\sum a_n$ converges if and only if A_n is bounded above.** *Proof:* Suppose $\sum a_n$ converges. Then A_n converges and is thus bounded. Conversely, suppose A_n is bounded above. Since a_n is ultimately non-negative, A_n is ultimately monotonically increasing. Thus A_n and $\sum a_n$ converge. ■
- **Basic Comparison Test: If $|a_n| \leq b_n$ for $n \geq N_1$, and if $\sum b_n$ converges, then $\sum a_n$ converges. If $c_n \geq d_n \geq 0$ for $n \geq N_2$, and if d_n diverges, then c_n diverges. Here, N_1, N_2 are fixed integers.** *Proof:* Suppose $\sum b_n$ converges. Given $\epsilon > 0$, there exists K such that $m \geq n \geq K \implies \left|\sum_{k=n}^{m} b_k\right| < \epsilon$. Thus, $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} b_k \leq \left|\sum_{k=n}^{m} b_k\right| < \epsilon$. So $\sum a_n$ converges. Now, suppose $\sum d_n$ diverges. If $\sum c_n$ converges, then $\sum d_n$ must also converge. So $\sum c_n$ diverges. ■
- **Comparison Test V1: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$, $\alpha, \beta > 0$ such that $n > M \implies \alpha a_n \leq b_n \leq \beta a_n$, then $\sum b_n$ converges if and only if $\sum a_n$ converges.** *Proof:* Suppose $\sum b_n$ converges. Since $|\alpha a_n| = \alpha a_n \leq b_n$ for $n > M$, $\sum \alpha a_n$ converges and thus $\sum a_n$ converges. Conversely, suppose $\sum a_n$ converges. Since $|b_n| = b_n \leq \beta a_n$ for $n > M$, $\sum b_n$ converges. ■
- **Comparison Test V2: If a_n and b_n are ultimately non-negative, and if there exist $M \in \mathbb{N}$ such that $n > M \implies 0 \leq \frac{b_n}{b_{n+1}} \leq \frac{a_n}{a_{n+1}}$, then $\sum a_n$ converges if $\sum b_n$ converges.** *Proof:* Suppose $\sum b_n$ converges.
- **Comparison Test V3: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.** *Proof:*
- **Comparison Test V4: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 = \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} < \infty$, then $\sum a_n$ converges if $\sum b_n$ converges.** *Proof:*
- **Comparison Test V5: If a_n is ultimately non-negative and b_n is ultimately positive, and if $0 < \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} = \infty$, then $\sum b_n$ converges if $\sum a_n$ converges.** *Proof:*
- **Limit Comparison Test: Let x_n and y_n be strictly positive sequences and $r = \lim \frac{x_n}{y_n}$. If $r \neq 0$ then $\sum x_n$ converges if and only if $\sum y_n$ converges. If $r = 0$ then $\sum x_n$ converges if $\sum y_n$ converges.** *Proof:* Let $r \neq 0$. Since $r = \lim \frac{x_n}{y_n}$, there exists K such that $n \geq K \implies \frac{1}{2}r \leq \frac{x_n}{y_n} \leq 2r \implies (\frac{1}{2}r)y_n \leq x_n \leq (2r)y_n$. Comparison Test V1 gives us the desired result. Now let $r = 0$. Then there exists K such that $0 < x_n \leq y_n$ for $n \geq K$. The Basic Comparison Test gives the desired result. ■

8 Series of Non-negative Terms

- **If $0 \leq x < 1$, then $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, this series diverges.** *Proof:* If $x \neq 1$, then $X_n = \displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n-1}}{1-x}$. If $0 \leq x < 1$, then $\displaystyle\lim_{n\to\infty} \frac{1-x^{n-1}}{1-x} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x = 1$, then the sum is $1 + 1 + 1 + ...$ which diverges. If $x > 1$ then $\dfrac{1-x^{n-1}}{1-x}$ diverges. ■

- **$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.** *Proof:* If $p \leq 0$, then $\dfrac{1}{n^p}$ does not tend to 0, and thus the series diverges.

9 Euler's Number

- We define Euler's number as: $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.

- $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. *Proof:*

- **e is irrational.** *Proof:*

10 Root and Ratio Tests

- **Root Test: Let (a_n) be a sequence in $\mathbb{R}$, and let $\limsup |a_n|^{\frac{1}{n}} = l$. Then, $\sum a_n$ diverges if $l > 1$, $\sum a_n$ converges if $l < 1$, and the test is inconclusive if $l = 1$.** *Proof:* Suppose $l < 1$. Choose x such that $l < x < 1$. Then there exists M such that $n \geq M \implies |a_n|^{\frac{1}{n}} > x \implies |a_n| > x^n$. Since $x < 1$, $\sum x^n$ converges. So $\sum a_n$ converges absolutely by the comparison test. If $l > 1$, then there exist infinitely many n such that $|a_n| > 1$. Therefore a_n does not tend to 0 and thus $\sum a_n$ diverges. ■

- **Ratio Test: Let (a_n) be a sequence in $\mathbb{R}$ that is ultimately non-zero. Let $\liminf \left|\dfrac{a_{n+1}}{a_n}\right| = r$, and $\limsup \left|\dfrac{a_{n+1}}{a_n}\right| = R$. Then, $\sum a_n$ diverges if $r > 1$, $\sum a_n$ converges absolutely if $R < 1$, and the test is inconclusive if $r \leq 1 \leq R$.** *Proof:* Suppose $R < 1$. Then there exists $x \in \mathbb{R}$ such that $R < x < 1$. Thus there exists K such that $n \geq K \implies \left|\dfrac{a_{n+1}}{a_n}\right| < x$. Since $x = \dfrac{x^{n+1}}{x^n}$, we have $\left|\dfrac{a_{n+1}}{a_n}\right| < \dfrac{x^{n+1}}{x^n} \implies \left|\dfrac{a_n}{a_{n+1}}\right| > \dfrac{x^n}{x^{n+1}}$. Since $\sum x^n$ converges, by V2 of the comparison test, $\sum |a_n|$ converges. Now, suppose $r > 1$. Let $\epsilon = \frac{1}{2}(r-1)$. Then, there exists M such that $n \geq M \implies \left|\dfrac{a_{n+1}}{a_n}\right| > r - \epsilon > 1 \implies |a_{n+1}| > |a_n|$. Therefore $|a_n|$ does not tend to 0. So a_n also does not tend to 0 and $\sum a_n$ diverges. ■

11 Power Series

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12 Summation by Parts

13 Absolute Convergence

14 Addition and Multiplication of Series

15 Rearrangements