MAT230 Notes

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1 Introduction

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2 Separable ODEs

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3 Exact ODEs and Integrating Factors

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4 Linear ODEs and Bernoulli's Equation

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- Solution of $y' + ky = 0$ is $y = ce^{-kx}$.

5 Orthogonal Trajectories

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6 Existence and Uniqueness of Solutions

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7 Homogeneous 2nd Order Linear ODEs

- A second order ODE is called linear if it can be written in the form $y'' + p(x)y' + q(x)y = r(x)$, and nonlinear otherwise.
- If $r(x) = 0$, then the equation is homogeneous, and nonhomogeneous if $r(x) \neq 0$.
- **Fundamental Theorem for Homogeneous Linear ODE:** For a homogeneous linear ODE, if y_1 and y_2 are solutions on an open interval I, then $c_1y_1 + c_2y_2$ is also a solution on I, for all $c_1, c_2 \in \mathbb{R}$. In other words, any linear combination of solutions is also a solution.
- For a second order homogeneous linear ODE, an initial value problem has two initial conditions: $y(x_0) = K_0$ and $y'(x_0) = K_1$.
- Two functions y_1 and y_2 on an open interval I are called linearly independent when $k_1y_1(x) + k_2y_2(x) = 0$ for all $x \in I \implies k_1 = k_2 = 0$.
- A basis of solutions on I is a pair of linearly independent solutions on I.

- **Reduction of Order:** Suppose we know that y_1 is a solution to $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Let $U = \frac{1}{y_1^2} e^{-\int p(x)dx}$. Then a second solution, linearly independent to y_1, is given by $y_2 = y_1 \int U dx$.

8 Homogeneous Linear ODEs with constant coefficients

- Consider the ODE $y'' + ay' + by = 0$, where $a, b \in \mathbb{R}$ are constants. Then its characteristic equation is given by $\lambda^2 + a\lambda + b = 0$. The roots of this equation are $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$ and $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$.
- Suppose the discriminant of the characteristic equation is greater than 0 and thus λ_1 and λ_2 are two distinct real roots. Then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ form a basis of solutions on any interval, and the general solution is given by $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
- Suppose the discriminant is 0 and $\lambda = -\frac{a}{2}$ is one real repeated root. Then $y_1 = e^{-\frac{a}{2}x}$ is one solution, and another one can be found through reduction of order, giving $y_2 = xe^{-\frac{a}{2}x}$. So the general solution is $y = c_1 e^{-\frac{a}{2}x} + c_2 x e^{-\frac{a}{2}x}$.
- Suppose the discriminant is less than 0 and $\lambda_1 = -\frac{a}{2} + i\omega$ and $\lambda_2 = -\frac{a}{2} - i\omega$ are two complex roots. Then $y_1 = e^{-\frac{a}{2}x}\cos(\omega x)$ and $y_2 = e^{-\frac{a}{2}x}\sin(\omega x)$, where $\omega^2 = b - \frac{1}{4}a^2$. Hence the general solution is $y = e^{-\frac{a}{2}x}(c_1\cos(\omega x) + c_2\sin(\omega x))$.

9 Euler-Cauchy Equations

- Euler-Cauchy equations are ODEs of the form $x^2y'' + axy' + by = 0$, where $a, b \in \mathbb{R}$ are constants. The associated auxiliary equation is given by $m^2 + (a-1)m + b = 0$, whose roots are $m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}$ and $m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$.
- Suppose the discriminant is greater than 0 and we have two real roots m_1 and m_2. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ constitute a basis of solutions, making the general the solution $y = c_1 x^{m_1} + c_2 x^{m_2}$.
- Suppose the discriminant is 0 and $m = \frac{1}{2}(1-a)$ is a real repeated root. Then $y_1 = x^{\frac{1-a}{2}}$ is a solution, and another one can be found through reduction of order: $y_2 = \ln(x)x^{\frac{1-a}{2}}$. So the general solution is $y = x^{\frac{1-a}{2}}(c_1 + c_2\ln(x))$.
- Suppose the discriminant is less than 0 and we have two complex roots $m_1 = \frac{1}{2}(1-a) + i\omega$ and $m_2 = \frac{1}{2}(1-a) - i\omega$. Then $y_1 = x^{\frac{1}{2}(1-a)}\cos(\omega\ln(x))$ and $y_2 = x^{\frac{1}{2}(1-a)}\sin(\omega\ln(x))$. So the general solution is $y = c_1 x^{\frac{1}{2}(1-a)}\cos(\omega\ln(x)) + c_2 x^{\frac{1}{2}(1-a)}\sin(\omega\ln(x))$.

10 Existence and Uniqueness of Solutions, Wronskian

- **Existence and Uniqueness Theorem:** Consider the IVP $y'' + p(x)y' + q(x)y = 0$, where $y(x_0) = K_0$ and $y'(x_0) = K_1$. If p and q are continuous on some open interval I and $x_0 \in I$, then this IVP has a unique solution $y(x)$ in I.
- Let the above ODE have continuous coefficients p and q on some open interval I. Then solutions y_1 and y_2 on I are linearly independent if and only if their Wronskian, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ is equal to 0 at some x_0 in I. Further, if $W(y_1, y_2) = 0$ at some $x_0 \in I$, it is equal to 0 on all of I. Thus for linear independence on I we simply need a point where $W \neq 0$.

11 Nonhomogeneous 2nd Order Linear ODEs

- Consider the 2nd order nonhomogeneous linear ODE $y'' + p(x)y' + q(x)y = r(x)$. Its general solution on an open interval I is of the form $y(x) = y_h + y_p(x)$, where $y_h = c_1y_1(x) + c_2y_2(x)$ is the general solution of the associated homogeneous equation on I, and $y_p(x)$ is any solution of the nonhomogeneous equation on I.

- **Method of Undetermined Coefficients:** Let $y'' + ay' + by = r(x)$ be a nonhomogeneous ODE with constant coefficients. Then, depending on the form of $r(x)$ we can determine $y_p(x)$ with the following table.
$$\begin{cases} r(x) = ke^{\gamma x} & y_p(x) = Ce^{\gamma x} \\ r(x) = kx^n & y_p(x) = K_n x^n + K_{n-1}x^{n-1} + ... + K_1 x + K_0 \\ r(x) = k\cos(\omega x) & y_p(x) = K\cos(\omega x) + M\sin(\omega x) \\ r(x) = k\sin(\omega x) & y_p(x) = K\cos(\omega x) + M\sin(\omega x) \\ r(x) = ke^{\alpha x}\cos(\omega x) & y_p(x) = e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x)) \\ r(x) = ke^{\alpha x}\sin(\omega x) & y_p(x) = e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x)) \end{cases}$$

- When $y_p(x)$ has been determined, its constants can be found by substituting it into the ODE. If $r(x)$ is the sum of two functions from the left column of the table, then $y_p(x)$ will be the sum of the corresponding functions on the right column.

- **Method of Variation of Parameters:** Let $y''+p(x)y'+q(x)y = r(x)$ be a 2nd order nonhomogeneous linear ODE, where p, q are continuous on some open interval I. Let y_1 and y_2 be basis of solutions for the associated homogeneous equation. Then we can find $y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$, where W is the Wronskian of y_1 and y_2.

12 Higher Order Homogeneous Linear ODEs

- An nth order homogeneous linear ODE is of the form $y^{(n)}+p_{n-1}(x)y^{(n-1)}+...+p_1(x)y'+p_0(x)y = 0$.

- A general solution is of the form $y(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x)$, where $y_1, y_2, ..., y_n$ form a basis of solutions.

- An IVP consists of n initial conditions: $y(x_0) = K_0$, $y'(x_0) = K_1$,...,$y^{(n-1)}(x_0) = K_{n-1}$.

- **Existence and Uniqueness Theorem:** If the coefficients $p_0, p_1, ..., p_{n-1}$ of the above IVP are continuous on some open interval I and $x_0 \in I$, then it has a unique solution $y(x)$ in I.

- The Wronskian of solutions $y_1, y_2, ..., y_n$ is given by the nth order determinant $W(y_1, y_2, ..., y_n) = \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{vmatrix}$. The solutions $y_1, y_2, ..., y_n$ are linearly dependent on I if and only if $W = 0$ at some point in I.

13 Higher Order Homogeneous Linear ODEs with Constant Coefficients

- Consider an nth order homogeneous linear ODE $y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1y' + a_0y = 0$, where $a_{n-1}, ..., a_1, a_0 \in \mathbb{R}$ are constants. The characteristic equation is given by $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$.

- Suppose $\lambda_1, ..., \lambda_n$ are real and distinct roots of the characteristic equation. Then the general solution is $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + ... + c_ne^{\lambda_n x}$.

- If λ is a real root of multiplicity m, then the corresponding linearly independent solutions are $e^{\lambda x}$, $xe^{\lambda x}$, $x^2e^{\lambda x}$, ... , $x^{m-1}e^{\lambda x}$.

- If complex roots occur, they do so in conjugate pairs. So let $\lambda = \gamma + i\omega$ and $\overline{\lambda} = \gamma - i\omega$ be such a pair of roots. Then $y_1 = e^{\gamma x}\cos(\omega x)$ and $y_2 = e^{\gamma x}\sin(\omega x)$.

- If $\lambda = \gamma + i\omega$ is a complex of root of multiplicity 2, then so is $\overline{\lambda}$, so the solutions would be $e^{\gamma x}\cos(\omega x)$, $e^{\gamma x}\sin(\omega x)$, $xe^{\gamma x}\cos(\omega x)$ and $xe^{\gamma x}\sin(\omega x)$.

14 Nonhomogeneous Linear ODEs

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