Algebra II: Introduction to Module Theory

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1 Basic Definitions

- Let R be a ring. Let M be an abelian group with respect to an operation $+$. M is a left-module on R if there exists an action of R on M, i.e, a map $R \times M \to M$, denoted rm, such that:

  1. $(r+s)m = rm + sm$, for all $r, s \in R$ and all $m \in M$.
  2. $(rs)m = r(sm)$, for all $r, s \in R$ and all $m \in M$.
  3. $r(m+n) = rm + rn$, for all $r \in R$ and all $m, n \in M$.
  4. If R has unity, then $1m = m$ for all $m \in M$.

  A right-module on R can be defined analogously.

- Modules over a field $\mathbb{F}$ and vector spaces over $\mathbb{F}$ are the same thing.
- Let R be a ring and M be an R-module. An R-submodule of M is a subgroup N of M which is closed under the action of ring elements, i.e, $rn \in N$ for all $r \in R$ and all $n \in N$. A submodule of M thus just a subset of M which is itself a module with the same operations.
- If R is a field, then submodules are the same thing as subspaces.
- Every R-module M has at least two submodules: M itself, and $\{0\}$, the trivial submodule.
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