Real Analysis: Basic Topology

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1 Countability

- If there exists a bijection between sets A and B, then A and B are said to be in one-one correspondence.
- For any $n \in \mathbb{N}$, let $J_n = \{1, 2, 3, ..., n\}$. If A is in one-one correspondence with J_n for some n, then A is finite. The empty set is also considered finite.
- A is countable if it is in one-one correspondence with $\mathbb{N}$.
- A is uncountable if it is neither finite nor countable.
- A is at most countable if it is either finite or countable.
- **Every infinite subset of a countable set A is countable itself.** *Proof:* Suppose $E \subset A$ and E is infinite. Arrange the elements of A into a sequence $\{x_n\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Let n_k be the kth smallest integer such that $x_{n_k} \in E$. Putting $f(k) = x_{n_k}$, we get a one-one correspondence between $\mathbb{N}$ and E. So E is countable. ■
- **Let $\{E_n\}$ be a collection of countable sets, and let $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.** *Proof seems to rely on some diagram so I couldn't include it here.*
- **Corollary: Suppose A is at most countable, and $\forall a \in A$, B_a is at most countable. Then $T = \bigcup_{a \in A} B_a$ is at most countable.**
- **Let A be countable, and let B_n be the set of all n-tuples $(a_1, a_2, ..., a_n)$ where $a_k \in A$ and the members of the tuples are not necessarily distinct. Then B_n is countable.** *Proof:* Since $B_1 = A$, it is countable. Suppose that B_{n-1} is countable. The elements of B_n have the form (b, a), where $b \in B_{n-1}$ and $a \in A$. For every fixed b, the set of pairs (b, a) is in one-one correspondence with A and thus countable. So B_n is a countable union of countable sets, and thus countable itself. We have our proof by induction. ■
- **Corollary: $\mathbb{Q}$ is countable.** *Proof:* Every rational number is of the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. This is in one-one correspondence with the set of n-tuples (a, b), which is countable. So $\mathbb{Q}$ is countable. ■

2 Metric Spaces

- A set X is said to be a metric space if for all $p, q \in X$, there is an associated real number $d(p, q)$ such that

  1. $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$
  2. $d(p, q) = d(q, p)$
  3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$

  Any function with the above properties is a distance function, or a metric.
- If $a_i < b_i$ for $i = 1, 2, 3, ...k$, the set of all points $(x_1, x_2, ..., x_k)$ in $\mathbb{R}^k$ such that $a_i \leq x_i \leq b_i$ is called a k-cell. A 1-cell is thus an interval and a 2-cell is a rectangle.

- If $x \in \mathbb{R}^k$ and the open (or closed) ball with center at x and radius r is defined as $\{y \in \mathbb{R}^k : \|y - x\| < r\}$ (or $\|y - x\| \leq r$).
- $E \subset \mathbb{R}^k$ is convex if for all $x, y \in E$, $\lambda x + (1 - \lambda)y \in E$, where $0 < \lambda < 1$.
- Suppose X is a metric space. Then,
  1. A neighborhood of $p \in X$ is the set $N_r(p) = \{q \in X : d(p, q) < r\}$, where $r > 0$.
  2. p is a limit point of $E \subset X$ if for all $r > 0$, there exists $q \in N_r(p)$ such that $p \neq q$ and $q \in E$.
  3. If $p \in E$ and p is not a limit point of E, then p is an isolated point of E.
  4. E is closed if every limit point of E is a point of E.
  5. p is an interior point of E if there exists $r > 0$ such that $N_r(p) \subset E$.
  6. E is open if every point of E is an interior point of E.
  7. E is perfect if E is closed and every point of E is a limit point of E.
  8. E is bounded if there exists $M \in \mathbb{R}$ and $q \in E$ such that for all $p \in E$, $d(p, q) < M$.
  9. E is dense in X if every point of X is a limit point of E, or a point of E or both.
- **Every neighborhood is an open set.** *Proof:* Let $E = N_r(p)$, and let $q \in E$. Then, there exists $h \in \mathbb{R}$ such that $d(p, q) = r - h$. For all points s such that $d(q, s) < h$, we then have $d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r$, so $s \in E$. Thus for all $q \in E$, $N_h(q) \subset E$. Therefore q is an interior point of E and E is an open set. ■
- **If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.** *Proof:* Suppose there exists a neighborhood N of p containing finitely many points of E, and let $q_1, q_2, ..., q_m$ be those points. Let $r = \min(d(p, q_n))$, where $1 \leq n \leq m$. The neighborhood $N_r(p)$ contains no point of E other than p, contradicting p being a limit point of E. ■
- **Corollary: A finite set has no limit points.**
- **A set is open if and only if its complement is closed.** *Proof:* Suppose E^c is closed, and let $x \in E$. Then $x \notin E^c$ and x is not a limit point of E^c. So there exists a neighborhood N such that $N \cap E^c$ is empty, and so x is an interior point of E. Thus E is open. Conversely, suppose E is open, and let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so x is not an interior point of E. Since E is open, $x \in E^c$. Thus E^c is closed. ■
- **Corollary: A set is closed if and only if its complement is open.**
- **For any collection $\{G_a\}$ of open sets, $\bigcup_a G_a$ is open.** *Proof:* Let $G = \bigcup_a G_a$. If $x \in G$, then $x \in G_a$ for some a. Since x is an interior point of G_a, it is also an interior point of G, and thus G is open. ■
- **For any collection $\{F_a\}$ of closed sets, $\bigcap_a F_a$ is closed.** *Proof:* Since $(\bigcap_a F_a)^c = \bigcup_a ({F_a}^c)$, and since ${F_a}^c$ is open, $\bigcap_a F_a$ is closed. ■
- **For any finite collection $G_1, G_2, ..., G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.** *Proof:* Let $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x, with radii r_i such that $N_i \subset G_i$, $1 \leq i \leq n$. Let $r = \min(r_1, r_2, ..., r_n)$. Then $N_r(x) \subset G_i$ for $1 \leq i \leq n$, so $N_r(x) \subset H$. So H is open. ■
- **For any finite collection $F_1, F_2, ..., F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.** *Follows from the previous statement.*
- If X is a metric space, $E \subset X$, then E^l denotes the set of all limit points of E in X and $\bar{E} = E \cup E^l$ is the closure of E.
- **$\bar{E}$ is closed; $\bar{E} = E$ if and only if E is closed; $\bar{E} \subset F$ for every closed set F such that $E \subset F$.** *Proof:* If $p \notin \bar{E}$ then $p \notin E$ and p is not a limit point of E. So there exists a neighborhood N of p that does not intersect E. Thus $\bar{E}^c$ is open and $\bar{E}$ is closed. If $\bar{E} = E$, then E is closed. Conversely, if E is closed, then $E^l \subset E$ and thus $E = \bar{E}$. If F is closed and $E \subset F$, then $F^l \subset F$ and so $E^l \subset F$. So $\bar{E} \subset F$. ■

- **Let E be a nonempty subset of $\mathbb{R}$ that is bounded above. Let $y = \sup E$. Then, $y \in \bar{E}$ and $y \in E$ if E is closed.** *Proof:* If $y \in E$ then $y \in \bar{E}$ so assume $y \notin E$. For every $h > 0$ there exists $x \in E$ such that $y - h < x < y$. If this were not so, then $y - h$ would be an upper bound for E. Thus y is a limit point of E. Hence $y \in \bar{E}$. ■

- Suppose $Y \subset X$. A subset E of Y is open relative to Y if for each $p \in E$ there exists an $r > 0$ such that $q \in E$ whenever $d(p,q) < r$ and $q \in Y$.

- **Suppose $Y \subset X$. $E \subset Y$ is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.** *Proof:* Suppose E is open relative to Y. Then for all $p \in E$, there exists $r_p > 0$ such that $d(p,q) < r_p$, $q \in Y \implies q \in E$. Let $V_p = \{q \in X : d(p,q) < r_p\}$, and let $G = \bigcup_{p \in E} V_p$. G is thus an open subset of X. Since $p \in V_p$ for all $p \in E$, $E \subset G \cap Y$. Also, $V_p \cap Y \subset E$ for all $p \in E$, so $G \cap Y \subset E$. Thus $E = Y \cap G$. Conversely, if G is open in X and $E = G \cap Y$, for all $p \in E$ there exists a neighborhood V_p such that $V_p \subset G$. Then $V_p \cap Y \subset E$, and thus E is open relative to Y. ■

3 Compact Sets

- A collection $\{G_a\}$ of open subsets of metric space X is an open cover of $E \subset X$ if $E \subset \bigcup_a G_a$.

- $K \subset X$ is compact if every open cover of K contains a finite subcover. That is, for every collection of open sets $\{G_a\}$ such that $K \subset \bigcup_a G_a$, there exist finitely many indices $a_1, a_2, ..a_n$ such that $K \subset \bigcup_{i=1}^n G_{a_i}$.

- **Let $K \subset Y \subset X$. Then K is compact relative to X if and only if it is compact relative to Y.** *Proof:* Suppose K is compact relative to X. Let $\{V_\alpha\}$ be a collection of sets open relative to Y such that $K \subset \bigcup_\alpha V_\alpha$. There exist open sets G_α in X such that $V_\alpha = Y \cap G_\alpha$ for all α. Since K is compact relative to X, $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup .... \cup G_{\alpha_n}$. Since $K \subset Y$, $K \subset V\alpha_1 \cup V_{\alpha_2} \cup .... \cup V_{\alpha_n}$. So K is compact relative to Y. Conversely, suppose K is compact relative to Y. Let $\{G_\alpha\}$ be an open cover of K in X and let $V_\alpha = Y \cap G_\alpha$. Then $K \subset V\alpha_1 \cup V_{\alpha_2} \cup .... \cup V_{\alpha_n}$ and since $V\alpha \subset G_\alpha$, K is compact relative to X. ■

- **Compact subsets of metric spaces are closed.** *Proof:* Let K be a compact subset of metric space X. Suppose $p \in X$, $p \notin K$. If $q \in K$, let V_q and W_q be neighborhoods of p and q respectively, of radius less than $\frac{1}{2}d(p,q)$. Since K is compact, there exist finitely many points $q_1, q_2, ..., q_n \in K$ such that $K \subset W_{q_1} \cup W_{q_2} \cup ... \cup W_{q_n} = W$. Let $V = V_{q_1} \cap V_{q_2} \cap ... \cap V_{q_n}$. Then V is a neighborhood of p that does not intersect W, so $V \subset K^c$. Thus p is an interior point of K^c and K^c is open. ■

- **Closed subsets of compact sets are compact.** *Proof:* Suppose $F \subset K \subset X$, where F is closed relative to X and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is added to $\{V_\alpha\}$, we get an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K and thus also covers F. If $F^c \in \Phi$, we can remove it while still retaining an open cover of F. Thus F is compact. ■

- **Corollary: If F is closed and K is compact, then $F \cap K$ is compact.** *Proof:* $F \cap K$ is closed, and since $F \cap K \subset K$, $F \cap K$ is compact. ■

- **If $\{K_\alpha\}$ is a collection of compact subsets of metric space X such that the intersection of every finite subcollection of it is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty.** *Proof:*

- **Corollary: If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n$ $(n = 1, 2, 3...)$, then $\bigcap_1^\infty K_n$ is not empty.**

- **If E is an infinite subset of a compact set K, then E has a limit point in K.** *Proof:*

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- **Heine-Borel Theorem: Let $E \subset \mathbb{R}^n$. Then each of the following statements implies the other two:**

  1. **E is closed and bounded.**
  2. **E is compact.**

3. **Every infinite subset of E has a limit point in E.**

Proof:

4 Connected Sets

- Two subsets A and B of a metric space X are separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
- $E \subset X$ is connected if it is not the union of two nonempty separated sets.