2019 AMC 10B Problems

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Problem 1

Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

$\textbf{(A) } \frac{5}{8} \qquad \textbf{(B) } \frac{4}{5} \qquad \textbf{(C) } \frac{7}{8} \qquad \textbf{(D) } \frac{9}{10} \qquad \textbf{(E) } \frac{11}{12}$

Problem 2

Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

$\textbf{(A) } 11 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 19 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 27$

Problem 3

In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

$\textbf{(A) } 66 \qquad \textbf{(B) } 154 \qquad \textbf{(C) } 186 \qquad \textbf{(D) } 220 \qquad \textbf{(E) } 266$

Problem 4

All lines with equation $ax+by=c$ such that a,b,c form an arithmetic progression pass through a common point. What are the coordinates of that point?

$\textbf{(A) } (-1,2) \qquad \textbf{(B) } (0,1) \qquad \textbf{(C) } (1,-2) \qquad \textbf{(D) } (1,0) \qquad \textbf{(E) } (1,2)$

Problem 5

Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y = x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y = x$. Which of the following statements is not always true?

$\textbf{(A)}$ Triangle $A'B'C'$ lies in the first quadrant.

$\textbf{(B)}$ Triangles ABC and $A'B'C'$ have the same area.

$\textbf{(C)}$ The slope of line AA' is -1.

$\textbf{(D)}$ The slopes of lines AA' and CC' are the same.

$\textbf{(E)}$ Lines AB and $A'B'$ are perpendicular to each other.

Problem 6

There is a real n such that $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 10 \qquad \textbf{(D)}\ 11 \qquad \textbf{(E)}\ 12$

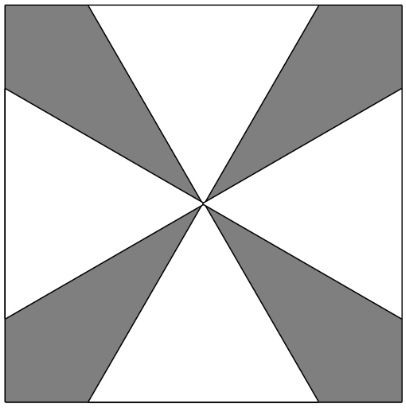
Problem 7

Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

$\textbf{(A)}\ 18 \qquad \textbf{(B)}\ 21 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 25 \qquad \textbf{(E)}\ 28$

Problem 8

The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



(A) 4 **(B)** $12-4\sqrt{3}$ **(C)** $3\sqrt{3}$ **(D)** $4\sqrt{3}$ **(E)** $16-\sqrt{3}$

Problem 9

The function f is defined by $f(x)=\lfloor|x|\rfloor-|\lfloor x\rfloor|$for all real numbers x, where $\lfloor r\rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

(A) $\{-1,0\}$ **(B)** The set of nonpositive integers **(C)** $\{-1,0,1\}$

(D) $\{0\}$ **(E)** The set of nonnegative integers

Problem 10

In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** infinitely many

Problem 11

Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9:1$, and the ratio of blue to green marbles in Jar 2 is $8:1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

(A) 5 **(B)** 10 **(C)** 25 **(D)** 45 **(E)** 50

Problem 12

What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

$\textbf{(A)}\ 11 \qquad \textbf{(B)}\ 14 \qquad \textbf{(C)}\ 22 \qquad \textbf{(D)}\ 23 \qquad \textbf{(E)}\ 27$

Problem 13

What is the sum of all real numbers x for which the median of the numbers $4,6,8,17,$ and x is equal to the mean of those five numbers?

$\textbf{(A)}\ -5 \qquad \textbf{(B)}\ 0 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ \frac{15}{4} \qquad \textbf{(E)}\ \frac{35}{4}$

Problem 14

The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 12 \qquad \textbf{(D)}\ 14 \qquad \textbf{(E)}\ 17$

Problem 17

A red ball and a green ball are randomly and independently tossed into bins numbered with positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k=1,2,3,\ldots.$ What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

$\textbf{(A)}\ \frac{1}{4} \qquad \textbf{(B)}\ \frac{2}{7} \qquad \textbf{(C)}\ \frac{1}{3} \qquad \textbf{(D)}\ \frac{3}{8} \qquad \textbf{(E)}\ \frac{3}{7}$

Hints to problems 12 onwards are on the back. Try solving the problems first before looking at the hints.

Solutions to this test will be posted on our website at sem-amc-club.tk.

Hints

Problem 12

First find the base-seven representation of 2019. Then look at how you could maximize the sum of the digits for a number less than 2019.

Problem 13

What are the possible positions of x in this sequence? What is the median and mean for each position? Equate them and solve for x. Don't forget to test if these solutions are extraneous.

Problem 14

Remember that $19!$ is divisible by all numbers $n \le 19$. What divisibility tests could you use to make this easier?

Problem 17

This problem requires you using the formula for the sum of an infinite geometric sequence:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \qquad \text{(given that } |r| < 1\text{)}$$

where a is the first term of the geometric sequence, and r is the common ratio between terms.

If you didn't know this formula before, try solving the problem without reading the rest of the hint.

What is the probability that the green ball lands in bin 1, and the red ball lands in a bin higher than that? What is the probability of the same thing for bin 2? Bin 3? Bin 4? What is the pattern? How can you add all of these numbers up (remember, there are an infinite number of bins)?