Combinatorics

Warm-up: What are the formulas for (n items, k slots)

- a. Permutations with reputation
- b. Permutations without reputation
- c. Combinations without repetition
- d. Combinations with repetition

Exercises

1. A fair die is rolled six times. The probability of rolling at least a five at least five times is

   $$\text{(A) } \frac{2}{729} \quad \text{(B) } \frac{3}{729} \quad \text{(C) } \frac{12}{729} \quad \text{(D) } \frac{13}{729} \quad \text{(E) None of these}$$

   Solution

2. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

   $$\text{(A) } \frac{1}{8} \quad \text{(B) } \frac{1}{4} \quad \text{(C) } \frac{3}{8} \quad \text{(D) } \frac{1}{2} \quad \text{(E) } \frac{3}{4}$$

   Solution

3. Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?

   $$\text{(A) } \frac{1}{9} \quad \text{(B) } \frac{1}{8} \quad \text{(C) } \frac{1}{6} \quad \text{(D) } \frac{2}{11} \quad \text{(E) } \frac{1}{5}$$

   Solution

4. Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

   $$\text{(A) } \frac{1}{4} \quad \text{(B) } \frac{3}{8} \quad \text{(C) } \frac{4}{7} \quad \text{(D) } \frac{5}{7} \quad \text{(E) } \frac{3}{4}$$

   Solution

5. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

   $$\text{(A) } 202 \quad \text{(B) } 223 \quad \text{(C) } 224 \quad \text{(D) } 225 \quad \text{(E) } 234$$

   Solution

6. A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

   $$\text{(A) } 10^4 \cdot 26^2 \quad \text{(B) } 10^3 \cdot 26^3 \quad \text{(C) } 5 \cdot 10^4 \cdot 26^2$$
   $$\text{(D) } 10^2 \cdot 26^4 \quad \text{(E) } 5 \cdot 10^3 \cdot 26^3$$

   Solution

7. Henry's Hamburger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any collection of condiments. How many different kinds of hamburgers can be ordered?

$$(\text{A})\ 24 \quad (\text{B})\ 256 \quad (\text{C})\ 768 \quad (\text{D})\ 40,320 \quad (\text{E})\ 120,960$$

Solution

8. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

$$(\text{A})\ \frac{11}{20} \quad (\text{B})\ \frac{4}{7} \quad (\text{C})\ \frac{81}{140} \quad (\text{D})\ \frac{3}{5} \quad (\text{E})\ \frac{17}{28}$$

Solution

9. Compute the number of 7-digit positive integers that start or end (or both) with a digit that is a (nonzero) composite number.

$$(\text{A})\ 3 \cdot 10^6 \quad (\text{B})\ 6 \cdot 10^6 \quad (\text{C})\ 9 \cdot 10^6 \quad (\text{D})\ 10 \cdot 10^6 \quad (\text{E})\ 12 \cdot 10^6$$

Solution

10. Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

$$(\text{A})\ 45 \quad (\text{B})\ 63 \quad (\text{C})\ 64 \quad (\text{D})\ 201 \quad (\text{E})\ 1005$$

Solution