Number Theory Exercises

1. A six place number is formed by repeating a three place number; for example, 256256 or 678678. Any number of this form is always divisible by

   $$\text{(A) } 7 \text{ only} \quad \text{(B) } 11 \text{ only} \quad \text{(C) } 13 \text{ only} \quad \text{(D) } 101 \quad \text{(E) } 1001$$

   Solution

2. The largest number by which the expression $n^3 - n$is divisible for all possible integral values of n, is:

   $$\text{(A) } 2 \quad \text{(B) } 3 \quad \text{(C) } 4 \quad \text{(D) } 5 \quad \text{(E) } 6$$

   Solution

3. In the base ten number system the number 526 means $5 \times 10^2 + 2 \times 10 + 6$. In the Land of Mathesis, however, numbers are written in the base r. Jones purchases an automobile there for 440 monetary units (abbreviated m.u). He gives the salesman a 1000 m.u bill, and receives, in change, 340 m.u. The base r is:

   $$\text{(A) } 2 \quad \text{(B) } 5 \quad \text{(C) } 7 \quad \text{(D) } 8 \quad \text{(E) } 12$$

   Solution

4. The number of solutions in positive integers of $2x + 3y = 763$ is:

   $$\text{(A) } 255 \quad \text{(B) } 254 \quad \text{(C) } 128 \quad \text{(D) } 127 \quad \text{(E) } 0$$

   Solution

5. The number of positive integers less than 1000 divisible by neither 5 nor 7 is:

   $$\text{(A) } 688 \quad \text{(B) } 686 \quad \text{(C) } 684 \quad \text{(D) } 658 \quad \text{(E) } 630$$

   Solution

6. A rectangular floor measures a by b feet, where a and b are positive integers and $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the whole floor. How many possibilities are there for the ordered pair (a, b)?

   $$\text{(A) } 1 \quad \text{(B) } 2 \quad \text{(C) } 3 \quad \text{(D) } 4 \quad \text{(E) } 5$$

   Solution

7. Prove that the fraction $\dfrac{21n + 4}{14n + 3}$ is irreducible for every natural number n.

   Note: this is an IMO problem, but is is widely regarded as the easiest IMO problem ever. Try it, you will be pleasantly surprised to see that you can sovle an IMO problem!

   Solution