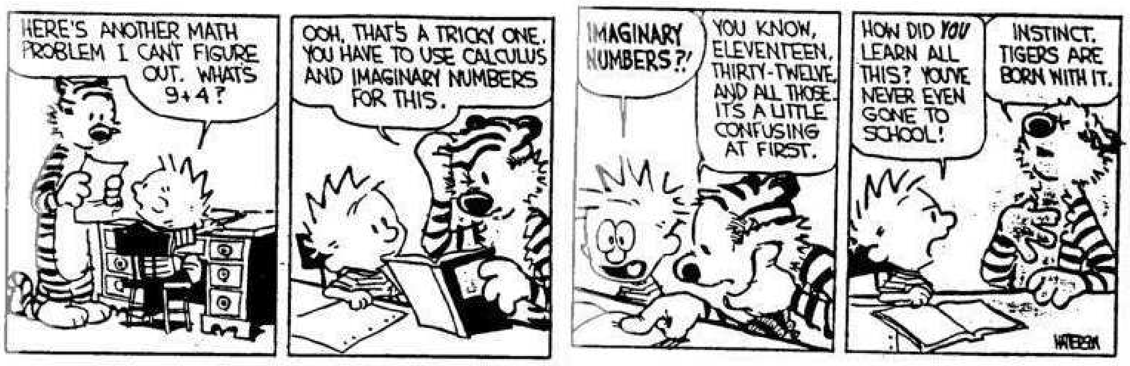
AMC Club Holiday Review 2019-2020

Compiled by Arjun Vikram

“Students, warm-up! ” - Mr. Newton



How to use this review

This review contains several problems from various areas of math tested on the AMC exam.

Before beginning the problems in a section, look back at any notes you may have on that subject. Check our website (sem-amc-club.tk) to view old notes packets. For geometry, you may want to skim your notes from your actual geometry class last year.

Try out the warm-up in each section before starting the exercises, and make sure to check your answers. The warm-up contains some important formulas that you should either memorize or be able to derive in a few seconds. These are not the only formulas you will need to know however.

Attempt to solve each problem before looking at the solution. If you are stumped on a problem, take a break and look at it again. Don't jump to reading the solution.

Solve each problem on scratch paper, and be organized with your work, so you can stay organized on the real test.

Quick tips about the AMC 10

The Test: The AMC is made of 25 multiple-choice (5-answer) questions covering a range of topics from algebra, combinatorics, geometry, and number theory. You do not get a calculator on the exam.

Timing: You get 75 minutes for 25 questions, however it is unlikely that you will answer all 25 questions. Budget your time based on your ability (look at last year's scores to see how many questions you answered).

Scoring: You earn

- 6 points for each correct answer
- 1.5 points for each skipped question
- 0 points for each incorrect answer

This means you should only guess if you have eliminated at least two answers.

Algebra Warm-up

1. What are the formulas for
    - i. Sum of an arithmetic series with n terms, with first term a_1 and last term a_n?
    - ii. Sum of a geometric series with n terms, with first term a_1 and ratio r?
    - iii. Sum of a geometric series with ∞ terms, with first term a_1 and ratio $-1 < r < 1$?
2. What are Vieta's formulas for:
    - a quadratic $ax^2 + bx + c$ with roots r, s:
        - i. Sum of roots $r + s =$
        - ii. Product of roots $rs =$
    - a cubic $ax^3 + bx^2 + cx + d$ with roots r, s, t:
        - iii. Sum of roots $r + s + t =$
        - iv. Sum of pairwise products of roots (aka taken two at a time) $rs + st + rt =$
        - v. Product of roots $rst =$
    - a general polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$:
        - vi. Sum of roots =
        - vii. Product of roots =
        - vii. Sum of roots taken k at a time =

Algebra Exercises

1. If m men can do a job in d days, then $m + r$ men can do the job in how many days?

$$\text{(A) } d+r \quad \text{(B) } d-r \quad \text{(C) } \frac{md}{m+r} \quad \text{(D) } \frac{d}{m+r} \quad \text{(E) } \frac{(m+r)d}{m}$$

Solution

2. From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this 45 boys leave. There are then 5 girls for each boy. The number of girls at the start was

$$\text{(A) } 40 \quad \text{(B) } 43 \quad \text{(C) } 29 \quad \text{(D) } 50 \quad \text{(E) } 55$$

Solution

3. What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$

$$\text{(A) } -\frac{2004}{2003} \quad \text{(B) } -1 \quad \text{(C) } \frac{2003}{2004} \quad \text{(D) } 1 \quad \text{(E) } \frac{2004}{2003}$$

Solution

4. Let $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$. What is $a + b + c + d$?

$$\text{(A) } -5 \quad \text{(B) } -\frac{10}{3} \quad \text{(C) } -\frac{7}{3} \quad \text{(D) } \frac{5}{3} \quad \text{(E) } 5$$

Solution

5. Find the value of x that satisfies

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}$$

$$\text{(A) } 2 \quad \text{(B) } 3 \quad \text{(C) } 5 \quad \text{(D) } 6 \quad \text{(E) } 9$$

Solution

6. There are 100 players in a single-elimination tennis tournament. Single-elimination means that a player who loses a match is eliminated. In the first round, the strongest 28 players are given a bye, and the remaining 72 players are paired off to play. After each round, the remaining players play in the next round. The tournament continues until only one player remains unbeaten. The total number of matches played is:

$$\text{(A) prime} \quad \text{(B) divisible by 2} \quad \text{(C) divisible by 5} \quad \text{(D) divisible by 7} \quad \text{(E) divisible by 11}$$

Solution

7. Two non-zero real numbers $a,\ b$ satisfy $ab = a - b$. What is the value of $\frac{a}{b} + \frac{b}{a} - ab$?

$$\text{(A) } -2 \quad \text{(B) } -\frac{1}{2} \quad \text{(C) } \frac{1}{3} \quad \text{(D) } \frac{1}{2} \quad \text{(E) } 2$$

Solution

8. Al, Betty, and Claire split $\$1000$ among them to be invested in different ways. Each begins with a different amount. After one year, they have a total of $\$1500$. Betty and Claire have both doubled their money, whereas Al has managed to lose $\$100$. What was Al's original portion?

$$\text{(A)}\ \$250 \qquad \text{(B)}\ \$350 \qquad \text{(C)}\ \$400 \qquad \text{(D)}\ \$450 \qquad \text{(E)}\ \$500$$

Solution

9. Brenda and Sally run in opposite directions on a circular track, starting at diametrically opposite points. They first meet after Brenda has run 100 meters. They next meet after Sally has run 150 meters past their first meeting point. Each girl runs at a constant speed. What is the length of the track in meters?

$$\text{(A)}\ 250 \qquad \text{(B)}\ 300 \qquad \text{(C)}\ 350 \qquad \text{(D)}\ 400 \qquad \text{(E)}\ 500$$

Solution

10. The first four terms in an arithmetic sequence are $x+y$, $x-y$, xy, and x/y in that order. What is the fifth term?

$$\text{(A)}\ -\frac{15}{8} \qquad \text{(B)}\ -\frac{6}{5} \qquad \text{(C)}\ 0 \qquad \text{(D)}\ \frac{27}{20} \qquad \text{(E)}\ \frac{123}{40}$$

Solution

Algebra Answer Key

Warm-up:

1. Series Formulas:
    - a. $n \cdot (\frac{a_1 + a_n}{2})$
    - b. $a_1 \cdot \frac{1 - r^n}{1 - r}$
    - c. $a_1 \cdot \frac{1}{1 - r}$
2. Vieta's Formulas:
    - i. $\frac{-b}{a}$
    - ii. $\frac{c}{a}$
    - iii. $\frac{-b}{a}$
    - iv. $\frac{c}{a}$
    - v. $\frac{-d}{a}$
    - vi. $\frac{-a_{n-1}}{a_n}$
    - vii. $(-1)^n \frac{a_0}{a_n}$
    - viii. $(-1)^k \frac{a_{n-k}}{a_n}$

Exercises:

1. C
2. A
3. B
4. B
5. B
6. E
7. E
8. C
9. C
10. E

Combinatorics Warm-up

1. What are the formulas for (n items, k slots)
    - i. Permutations with reputation
    - ii. Permutations without reputation
    - iii. Combinations without repetition
    - iv. Combinations with repetition (you may have to derive this one, remember the ice cream shop analogy with robots)
2. Define a one-is-to-one correspondence (aka bijection) and describe how it is used in AMC problems

Combinatorics Exercises

1. A fair die is rolled six times. The probability of rolling at least a five at least five times is

$$(\mathrm{A})\ \frac{2}{729} \quad (\mathrm{B})\ \frac{3}{729} \quad (\mathrm{C})\ \frac{12}{729} \quad (\mathrm{D})\ \frac{13}{729} \quad (\mathrm{E})\ \text{None of these}$$

   Solution

2. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

$$(\mathrm{A})\ 22 \quad (\mathrm{B})\ 25 \quad (\mathrm{C})\ 27 \quad (\mathrm{D})\ 28 \quad (\mathrm{E})\ 729$$

   Official Solution My Solution (please read)

3. A point (x, y) is randomly picked from inside the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$, and $(0, 1)$. What is the probability that $x < y$?

$$(\mathrm{A})\ \tfrac{1}{8} \quad (\mathrm{B})\ \tfrac{1}{4} \quad (\mathrm{C})\ \tfrac{3}{8} \quad (\mathrm{D})\ \tfrac{1}{2} \quad (\mathrm{E})\ \tfrac{3}{4}$$

   Solution

4. Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?

$$(\mathrm{A})\ \tfrac{1}{9} \quad (\mathrm{B})\ \tfrac{1}{8} \quad (\mathrm{C})\ \tfrac{1}{6} \quad (\mathrm{D})\ \tfrac{2}{11} \quad (\mathrm{E})\ \tfrac{1}{5}$$

   Solution

5. Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

$$(\mathrm{A})\ \tfrac{1}{4} \quad (\mathrm{B})\ \tfrac{3}{8} \quad (\mathrm{C})\ \tfrac{4}{7} \quad (\mathrm{D})\ \tfrac{5}{7} \quad (\mathrm{E})\ \tfrac{3}{4}$$

   Solution

6. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

$$(\mathrm{A})\ 202 \quad (\mathrm{B})\ 223 \quad (\mathrm{C})\ 224 \quad (\mathrm{D})\ 225 \quad (\mathrm{E})\ 234$$

   Solution

7. A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

$$(\mathrm{A})\ 10^4 \cdot 26^2 \quad (\mathrm{B})\ 10^3 \cdot 26^3 \quad (\mathrm{C})\ 5 \cdot 10^4 \cdot 26^2$$
$$(\mathrm{D})\ 10^2 \cdot 26^4 \quad (\mathrm{E})\ 5 \cdot 10^3 \cdot 26^3$$

   Solution

8. Henry's Hamburger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any collection of condiments. How many different kinds of hamburgers can be ordered?

$$\text{(A) } 24 \quad \text{(B) } 256 \quad \text{(C) } 768 \quad \text{(D) } 40,320 \quad \text{(E) } 120,960$$

Solution

9. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

$$\text{(A) } \frac{11}{20} \quad \text{(B) } \frac{4}{7} \quad \text{(C) } \frac{81}{140} \quad \text{(D) } \frac{3}{5} \quad \text{(E) } \frac{17}{28}$$

Solution

10. Compute the number of 7-digit positive integers that start or end (or both) with a digit that is a (nonzero) composite number.

$$\text{(A) } 3 \cdot 10^6 \quad \text{(B) } 6 \cdot 10^6 \quad \text{(C) } 9 \cdot 10^6 \quad \text{(D) } 10 \cdot 10^6 \quad \text{(E) } 12 \cdot 10^6$$

Solution

11. Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

$$\text{(A) } 45 \quad \text{(B) } 63 \quad \text{(C) } 64 \quad \text{(D) } 201 \quad \text{(E) } 1005$$

Solution

Combinatorics Answer Key

Warm-up:

1. Permutations/Combinations formulas:
    - i. n^k
    - ii. $nPk = \dfrac{n!}{(n-k)!}$
    - iii. $\dbinom{c}{k} = nCk = \dfrac{n!}{k!(n-k)!}$
    - iv. $\dbinom{n+k-1}{n-1} = \dbinom{n+k-1}{k}$
2. A one-is-to-one correspondence is a mapping between two sets such that every element from the first set is mapped to exactly one element from the second set, and vice versa. For instance, you can form a one-is-to-one correspondence between the sets $\{Su, M, T, W, Th, F, S\}$ and $\{1, 2, 3, 4, 5, 6, 7\}$ using the mapping function which takes a day and outputs where it is in the week. Another example of a one-is-to-one correspondence is used in the derivation of the combinations with repetitions formula, where we form a correspondence between combinations of ice cream in a bowl allowing repetitions and permutations of arrows and stars in a line.

    If there is a one is to one correspondence $A \leftrightarrow B$ between A and B, then the size of A and the size of B are equal ($|A| = |B|$). This means that we can make a one-is-to-one correspondence between a set that is more difficult to count and a set which is easier to count, and then simplify a counting problem.

Exercises:

1. D
2. D
3. A
4. D
5. C
6. D
7. C
8. C
9. E
10. B
11. A

Geometry Warm-up

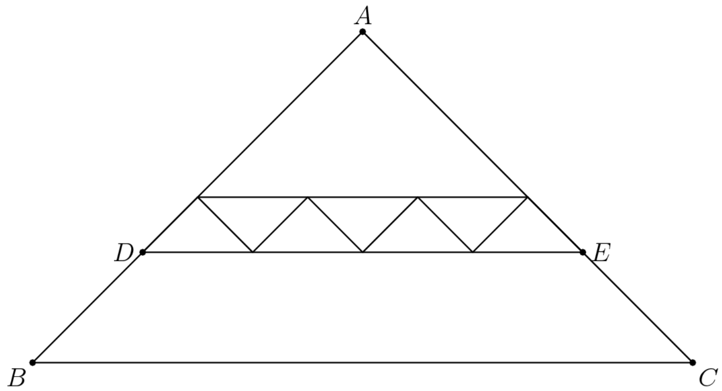
1. List all the area formulas you know for a triangle.
    - For brevity and lack of a better variable, use K to represent the area
    - Use the variables a, b, c for side lengths
    - Use the variables A, B, C for angles
    - Use the variable $s = \frac{a+b+c}{2}$ for the semiperimeter ($\frac{a+b+c}{2}$)
    - Use the variable r for the inradius
    - Use the variable R for the circumradius

2. What are the law of sines and cosines?

3. List all similarity and congruence conditions (sometimes called theorems)

Geometry Exercises

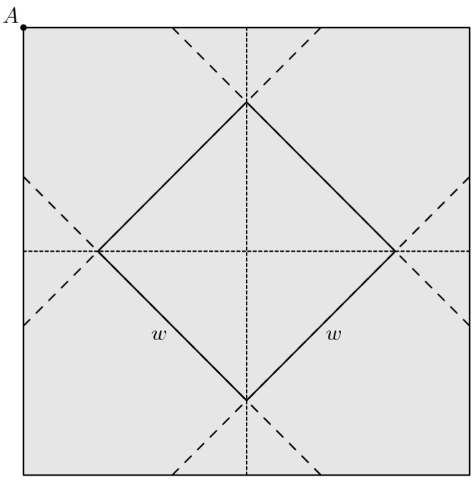
1. All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?

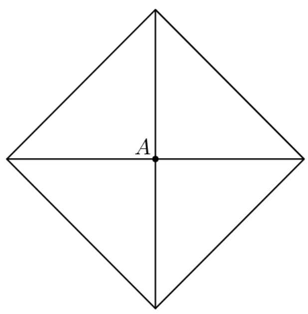


$$\textbf{(A) } 16 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 22 \qquad \textbf{(E) } 24$$

[Solution]

2. A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?





$$\textbf{(A) } 2(w+h)^2 \qquad \textbf{(B) } \frac{(w+h)^2}{2} \qquad \textbf{(C) } 2w^2+4wh \qquad \textbf{(D) } 2w^2 \qquad \textbf{(E) } w^2h$$

[Solution]

3. A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

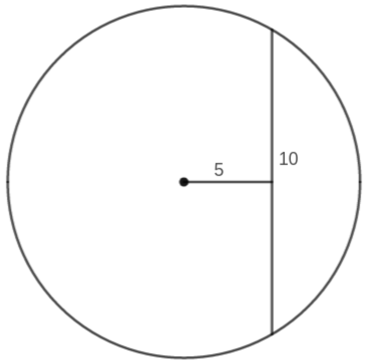


Diagram by Arjun

$$\text{(A) } 25\pi \qquad \text{(B) } 50\pi \qquad \text{(C) } 75\pi \qquad \text{(D) } 100\pi \qquad \text{(E) } 125\pi$$

Solution

4. Let $O = (0, 0)$ be the origin. Points A and B are selected on the graph of $y = -\frac{1}{2}x^2$ so that triangle ABO is equilateral. Find AB.

$$\text{(A) } 2 \qquad \text{(B) } 2\sqrt{3} \qquad \text{(C) } 4 \qquad \text{(D) } 4\sqrt{3} \qquad \text{(E) } 4\sqrt{5}$$

Solution

5. The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n.

Hint: you will need to use Triangle Inequality and some Log Properties

$$\text{(A) } 642 \qquad \text{(B) } 813 \qquad \text{(C) } 893 \qquad \text{(D) } 900 \qquad \text{(E) } 917$$

Solution

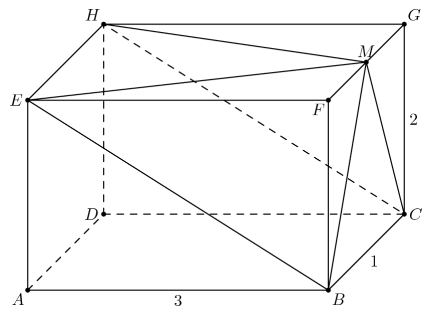
6. Side $\overline{AB}$ of $\triangle ABC$ has length 10. The angle bisector of A meets $\overline{BC}$ at D, and $CD = 3$. The set of all possible values of AC is an open interval (m, n). What is $m + n$?

Hint: you will need to use Triangle Inequality and Angle Bisector Theorem

$$\text{(A) } 16 \qquad \text{(B) } 17 \qquad \text{(C) } 18 \qquad \text{(D) } 19 \qquad \text{(E) } 20$$

Solution

7. In the rectangular parallelepiped shown, $AB = 3,\ BC = 1,\ CG = 2$. Point M is the midpoint of FG. What is the volume of the rectangular pyramid with base $BCHE$ and apex M?



$$\textbf{(A) } 1 \qquad \textbf{(B) } \frac{4}{3} \qquad \textbf{(C) } \frac{3}{2} \qquad \textbf{(D) } \frac{5}{3} \qquad \textbf{(E) } 2$$

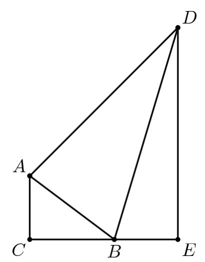
Solution

8. Right triangle $\triangle ABC$ has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

$$\textbf{(A) } 5 \qquad \textbf{(B) } 9 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 13 \qquad \textbf{(E) } 15$$

Solution

9. Triangle ABC has a right angle at C, $AC = 3$ and $BC = 4$. Triangle ABD has a right angle at A and $AD = 12$. Points C and D are on opposite sides of $\overline{AB}$. The line through D parallel to $\overline{AC}$ meets $\overline{CB}$ extended at E. If $\frac{DE}{DB} = \frac{m}{n}$, where m and n are relatively prime positive integers, then $m + n =$



$$\textbf{(A) } 25 \qquad \textbf{(B) } 128 \qquad \textbf{(C) } 153 \qquad \textbf{(D) } 243 \qquad \textbf{(E) } 256$$

Solution (look at Solution 2, especially note the author!)

10. Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

$$\textbf{(A) } \sqrt{2} \qquad \textbf{(B) } \frac{3}{2} \qquad \textbf{(C) } \frac{5}{3} \qquad \textbf{(D) } \sqrt{3} \qquad \textbf{(E) } 2$$

Solution

Geometry Answer Key

Warm-up:

1. Triangle formulas:
   - Standard formula (half times base times height): $K = \frac{1}{2}bh$
   - SAS area: $K = \frac{1}{2}ab\sin C$ (and cyclic permutations)
   - Heron's formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$
   - Inradius formula: $K = r \cdot s$
   - Circumradius formula: $K = \dfrac{abc}{4R}$
   - Equilateral Triangle formula: $K = \frac{\sqrt{3}}{4}a^2$
   - ...and several more advanced formulas
2. Law of Sines/Cosines:
   - (Extended) Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{2R}$
   - Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$ (and cyclic variants)
3. Similarity and Congruence:
   - Similarity: AA, ASA, SSS
   - Congruence: SSS, SAA, ASA, **Not ASS**

Exercises:

1. E
2. A
3. B
4. D
5. C
6. C
7. E
8. D
9. B
10. B

Number Theory Warm-up

1. What is SFFT (Simon's Favorite Factoring Trick)? Use it to find all non-negative integer solutions (a, b) to $ab + a + b = 3$.

2. What does it mean for two numbers to be coprime? Tell if the following numbers are coprime:
    - i. $2, 3$
    - ii. $6, 8$
    - iii. $1, 9$
    - iiii. $0, 5$

3. What is the euclidean algorithm for finding the GCD of two numbers? How can you use this to find the LCM?

Number Theory Exercises

1. A six place number is formed by repeating a three place number; for example, 256256 or 678678. Any number of this form is always divisible by

$$\text{(A) } 7 \text{ only} \quad \text{(B) } 11 \text{ only} \quad \text{(C) } 13 \text{ only} \quad \text{(D) } 101 \quad \text{(E) } 1001$$

Solution

2. The largest number by which the expression $n^3 - n$is divisible for all possible integral values of n, is:

$$\text{(A) } 2 \quad \text{(B) } 3 \quad \text{(C) } 4 \quad \text{(D) } 5 \quad \text{(E) } 6$$

Solution

3. In the base ten number system the number 526 means $5 \times 10^2 + 2 \times 10 + 6$. In the Land of Mathesis, however, numbers are written in the base r. Jones purchases an automobile there for 440 monetary units (abbreviated m.u). He gives the salesman a 1000 m.u bill, and receives, in change, 340 m.u. The base r is:

$$\text{(A) } 2 \quad \text{(B) } 5 \quad \text{(C) } 7 \quad \text{(D) } 8 \quad \text{(E) } 12$$

Solution

4. The number of solutions in positive integers of $2x + 3y = 763$ is:

$$\text{(A) } 255 \quad \text{(B) } 254 \quad \text{(C) } 128 \quad \text{(D) } 127 \quad \text{(E) } 0$$

Solution

5. The number of positive integers less than 1000 divisible by neither 5 nor 7 is:

$$\text{(A) } 688 \quad \text{(B) } 686 \quad \text{(C) } 684 \quad \text{(D) } 658 \quad \text{(E) } 630$$

Solution

6. A rectangular floor measures a by b feet, where a and b are positive integers and $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the whole floor. How many possibilities are there for the ordered pair (a, b)?

$$\text{(A) } 1 \quad \text{(B) } 2 \quad \text{(C) } 3 \quad \text{(D) } 4 \quad \text{(E) } 5$$

Solution

7. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

Note: this is an IMO problem, but is is widely regarded as the easiest IMO problem ever. Try it, you will be pleasantly surprised to see that you can sovle an IMO problem!

Solution

Number Theory Answer Key

Warm-up:

1. See our notes on SFFT (Lesson 4) at sem-amc-club.tk/Week4.pdf

   Solutions are $(0,3),(1,1),(3,0)$
2. Two numbers are coprime if they share no factors other than 1. A good rule of thumb is that a and b are coprime if and only if there exist integers p and q such that $ap+bq=1$ (Bezout's Lemma)
   - Coprime ($\gcd(2,3)=1$ and $\underline{-1}\cdot 2+\underline{1}\cdot 3=1$)
   - Not coprime ($\gcd(6,8)=2$)
   - Coprime ($\gcd(1,9)=1$ and $\underline{1}\cdot 1+\underline{0}\cdot 9=1$)
   - Not coprime ($gcd(0,5)=5$)
3. See [Euclidean Algorithm]() on AoPS.

   Note that $a\cdot b=\gcd(a,b)\cdot \operatorname{lcm}(a,b)$.

Exercises:

1. E
2. E
3. D
4. D
5. B
6. B
7. See solution at https://bit.ly/2sAJ3V5

Questions

Please contact Arjun Vikram if you have any questions about this problemset.

If you do not already have my number, text me at 912-999-0219.

Acknowledgements

Problems compiled by Arjun Vikram from SEM AMC Club

Thanks to Kenneth and Mahesh for finding some typos

Dedication

This problemset is dedicated to Mr. Newton.

It was inspired by his quote "Students, warm-up!".

Revision 2 - minor edits and typo fixes