Student: Arjun Vikram
Username: arjvik
ID#: 36783

Problem 1

		1	4	5		3	1	4
4	5	3	2	1	4	5	2	3
2	1	4	5	3	2	1	4	5
5	3	2		4	5	3		

Problem 3

To begin, we note that because $1 = d_1 < d_2 < d_3 < \cdots < d_k = n$, $d_i - d_{i-1} < d_i$. Thus, we observe that $d_i - d_{i-1} \in \{d_j | 1 \leq j < n\}$. We can rearrange this to get $d_i \in \{d_{i-1} + d_j | 1 \leq j < 1\}$ (which can also be written as $d_i = d_{i-1} + d_j$ for some $1 \leq j < i$).

We can now look at d_2. We note that all values of $n > 1$ have at least two factors, so there must be a d_2. By this previous observation, $d_2 \in \{1 + 1\} \implies d_2 = 2$. Therefore, all juicy numbers must be even. With this, we see that $n = 2$ is a square-free juicy number.

Similarly, we look at d_3 now. We note that there exists a d_3 only when n is not prime, however as we saw earlier, the only prime value of n that works is 2, as n must be even. Since we have already considered the case where n is 2, we can assume that there exists a d_3 from hereon. We see that $d_3 \in \{3, 4\}$ based on our original observation. However, since we are only considering square-free juicy numbers, we can omit the case where $d_3 = 4$, as any number with 4 as a factor is not square-free. Therefore, if it exists, $d_3 = 3$.

We can now look at d_4. We note that if n is square-free and has at least 3 factors, it must have a 4th factor (only square numbers have an odd number of factors), so d_4 must exist. From our original observation, $d_4 \in \{4, 5, 6\}$. Again, we do not need to consider the case where $d_4 = 4$, as we are only considering square-free juicy numbers. The other two cases are $d_4 = 5$ and $d_4 = 6$. Let us consider these cases separately.

Case 1: $d_4 = 5$

So far, $D = \{1, 2, 3, 5\}$.

We know that both 2 and 3 are factors of n, so d_5 must equal 6. Because n is square-free and has at least 5 factors, it must have a 6th factor. From our original observation, $d_6 \in \{7, 8, 9, 11, 12\}$. We note that 8, 9, and 12 do not work because n must be square-free. Therefore $d_4 = 5 \implies d_6 \in \{7, 11\}$.

Note: I know that this case leads nowhere, but I am not sure how to prove it. I understand that I will not receive full credit without a proof here.

Case 2: $d_4 = 6$

So far, $D = \{1, 2, 3, 6\}$.

Here, we note that if n has only 4 factors, $n = 6$. We see that $n = 6$ is a square-free juicy number. For all other values of n in this case, there must be a d_5. $d_5 \in \{7, 8, 9, 12\}$ from our original observation. If $d_5 \in \{8, 9, 12\}$, we see that n is not square-free. Thus, we see that $d_5 = 7$.

We now look at the possible values of d_6. From our original observation, we know that $d_6 \in \{8, 9, 10, 13, 14\}$. We note that 8 and 9 do not work because n must be square-free. We also note that 10 will not work because 5 is not one of our earlier divisors. We break the remaining two numbers into two cases.

Case 2.1: $d_6 = 13$

Note: I know that this case leads nowhere, but I am not sure how to prove it. I understand that I will not receive full credit without a proof here.

Case 2.2: $d_6 = 14$

So far, $D = \{1, 2, 3, 6, 7, 14\}$.

We note that d_7 must exist as $n \neq 14$ (given that $3|n$). From our original observation, we know that $d_7 \in 15, 16, 17, 20, 21, 28$. From the square-free condition, we see that d_7 can not be 16, 20, or 28. We also see that d_7 can not be 15 because 5 was not one of our earlier divisors. Again, we consider the remaining two numbers in two separate cases.

Case 2.2.1: $d_7 = 17$

Note: I know that this case leads nowhere, but I am not sure how to prove it. I understand that I will not receive full credit without a proof here.

Case 2.2.2: $d_7 = 21$

So far, $D = \{1, 2, 3, 6, 7, 14, 21\}$.

We see that $d_8 \in \{22, 23, 24, 27, 28, 35, 42\}$ from our original observation. We know that d_8 can not be 24, 27, or 28 because of the squarefree condition, and also that d_8 can not be 22 since 11 is not already a divisor and d_8 can not be 35 because 5 is not already a divisor. Again, we will split the remaining numbers into cases.

2.2.2.1: $d_8 = 23$

Note: I know that this case leads nowhere, but I am not sure how to prove it. I understand that I will not receive full credit without a proof here.

Case 2.2.2.2: $d_8 = 42$

So far, $D = \{1, 2, 3, 6, 7, 14, 21, 42\}$.

Here, we note that it is possible for d_8 to be the last divisor of n (as D right now contains all the divisors of 42), and we see that $n = 42$ is indeed a square-free juicy number.

If d_8 is not the last divisor, we see that $d_9 \in \{43, 44, 45, 48, 49, 56, 63, 84\}$. Again, using similar reasoning to before, $d_9 \notin \{44, 45, 48, 49, 56, 63, 84\}$ (because those would either introduce non-square-free factors or require factors to be present earlier that aren't). Thus, $d_9 = 43$.

So now, $D = \{1, 2, 3, 6, 7, 14, 21, 42, 43\}$

We know that d_{10} must exist because 43 is not divisible by 42. Thus, from our original observation, we see that $d_{10} \in \{44, 45, 46, 49, 50, 57, 64, 85, 86\}$. Again, using similar reasoning to before, we see that d_{10} is either 57 or 86. We break these into two more cases.

Case 2.2.2.2.1: $d_10 = 57$

Note: I know that this case leads nowhere, but I am not sure how to prove it. I understand that I will not receive full credit without a proof here.

Case 2.2.2.2.2: $d_{10} = 86$

We can continue on like this, and eventually we will see that 1806 is a square-free juicy number. There will be more branched-off cases that I do not know how to verify. Eventually, you get to a case where none of the possibilities for divisors work.

To verify that the only 4 square-free juicy numbers are 2, 6, 42, and 1806, I wrote a program. The following python program was used to identify square-free juicy numbers:

```python
#!/usr/bin/env python3
from math import ceil

# Output a list of factors of n
def factors(n):
    l = []
    # Simply loop over all integers in the range [1, n]
    # and test divisibility (this could have been more optimized)
    for i in range(1,n+1):
        if n % i == 0:
            # i is a factor, add it to our list of factors
            l.append(i)
    return l

# Tell whether or not n is juicy given its factors f
def juicy(n, f):
    # Loop through factors d_2 to d_k
    for j in range(1, len(f)):
        # Ensure that d_j - d_{j-1} is a divisor
        if not ((f[j] - f[j-1]) in f):
            return False
    return True

```

```python
# Tell whether or not n is square-free given its factors f
def squarefree(n, f):
    # Loop through all squares i^2 from 0 to n (i <= sqrt(n))
    for i in range (2, ceil(n**0.5)+1):
        # If i^2 is a factor of n, n is not square-free
        if i**2 in f:
            return False
    return True

# The upper bound of our simulation (1,000,000)
MAX = 1_000_000

# Main loop, checking integers n in [2, MAX]
for n in range(2, MAX):
    # Factorize n
    f = factors(n)
    # Check if n is a square-free juicy number
    if juicy(n, f) and squarefree(n, f):
        print(n)
```

When this program is run, the output is

```
2
6
42
1806
```

While this simulation has a bound, we can see a pattern in it which can be used to explain the lack of any square-free juicy number after 1806. We notice that from the results of our simulation, if n is a square-free number, it appears that $n(n+1)$ will be a square-free juicy number, as long as $n+1$ is prime. This holds true up until $n = 1806$, where the sequence stops since 1807 is not prime. I am not sure how to prove this conjecture.

Problem 4

We approach this problem by placing $\triangle FIG$ on a coordinate plane. We place F on the origin, and rotate the triangle such that G is on the positive x-axis. We then assign variables to the coordinates of I and G. Let $I = (a, b)$ and $G = (c, 0)$. Since we have rotated the triangle such that $\overline{FG}$ is the x-axis, we know that D must also be on the x-axis. Let $D = (d, 0)$.

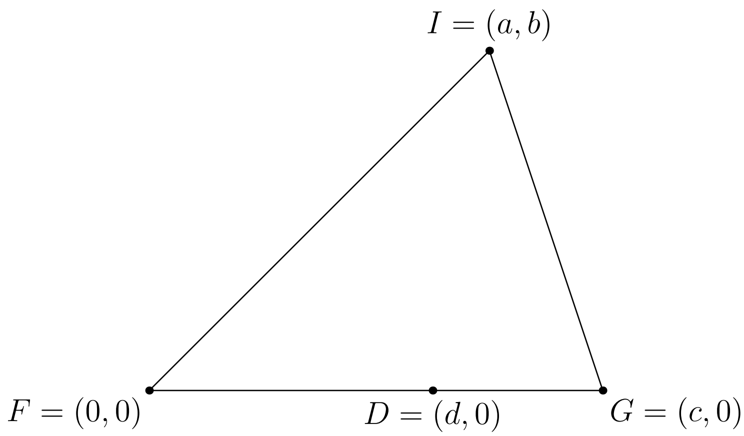

Figure 1: Placing $\triangle FIG$ on the coordinate plane

We see that the slope of line FI is $\frac{b}{a}$, and that the slope of line IG is $\frac{-b}{c-a}$. Thus, we know that the slope of a line perpendicular to FI must be $\frac{-a}{b}$ and the slope of a line perpendicular to IG must be $\frac{c-a}{b}$.

Let A be the midpoint of $\overline{FD}$ and B be the midpoint of $\overline{DG}$. We see that $A = \left(\frac{d}{2}, 0\right)$ and $B = \left(\frac{c+d}{2}, 0\right)$.

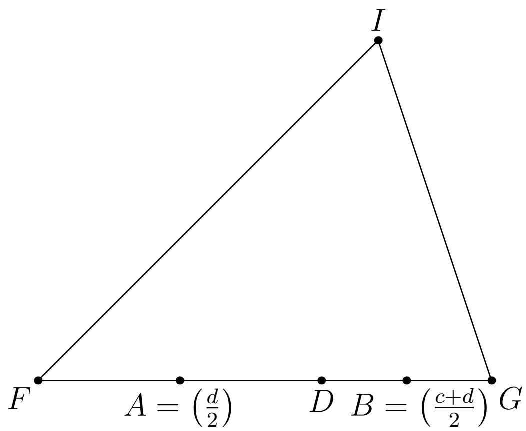

Figure 2: Adding points A and B

Let the line perpendicular to FI passing through A be ℓ_1, and the line perpendicular to FG passing through B be ℓ_2.

We see that the equation of ℓ_1 is

$$y = \frac{-a}{b}\left(x - \frac{d}{2}\right),$$

and that the equation of ℓ_2 is

$$y = \frac{c-a}{b}\left(x - \frac{c+d}{2}\right).$$

If we solve for the intersection of ℓ_1 and ℓ_2, we see that

$$T = \left(\frac{-a+c+d}{2}, \frac{a(a-c)}{2b}\right).$$

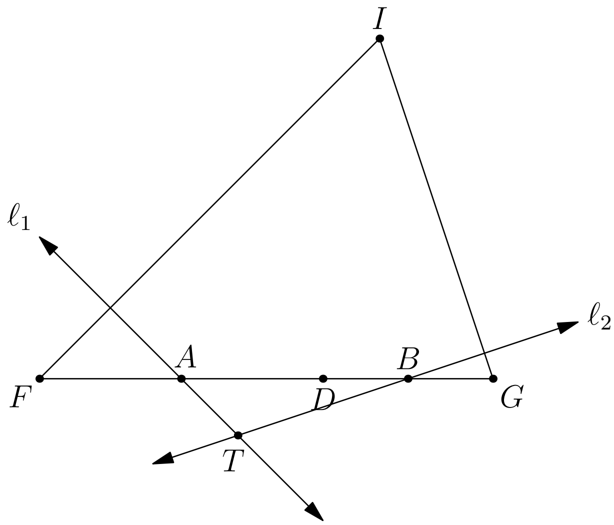

Figure 3: Drawing ℓ_1 and ℓ_2

Now we look at the two parts of our "if and only if".

For T to be equidistant from F and G ($FT = GT$), T must lie on the perpendicular bisector of $\overline{FG}$, which is the line $x = \dfrac{c}{2}$. This means that

$$\begin{aligned} \frac{-a+c+d}{2} &= \frac{c}{2} \\ \frac{-a+d}{2} &= 0 \\ a &= d \end{aligned}$$

This shows that $FT = GT \iff a = d$ (this is a biconditional statement because all of our manipulations are reversible).

Next, for $\overline{ID} \perp \overline{FG}$, we see that since $\overline{FG}$ is horizontal, $\overline{ID}$ must be vertical. This means that I and D must have the same x-coordinates, so $a = d$. This shows that $\overline{ID} \perp \overline{FG} \iff a = d$ (again, this is biconditional).

Therefore, since we have shown that $FT = GT \iff a = d \iff \overline{ID} \perp \overline{FG}$, we can use the transitive property to show that $\boxed{FT = GT \iff \overline{ID} \perp \overline{FG}}$. $\square$

Problem 5

Note: this is an incomplete solution, it contains only a few observations. I understand that I will not receive full credit for this.

Firstly, we introduce a bit of shorthand notation: let $f^2(x) = (f(x))^2$. (This is similar to the notation used in trigonometry to notate raising functions to powers).

We begin by noting that $f(x)$ is an odd function by the following process:

$$\begin{aligned} f(a+b) \cdot f(a-b) &= f^2(a) - f^2(b) \\ f(a-b) &= \frac{f^2(a) - f^2(b)}{f(a+b)} \end{aligned}$$

$$\begin{aligned} f(b+a) \cdot f(b-a) &= f^2(b) - f^2(a) \\ f(b-a) &= \frac{f^2(b) - f^2(a)}{f(b+a)} \\ &= -\frac{f^2(a) - f^2(b)}{f(a+b)} \\ &= -f(a-b) \end{aligned}$$

If we substitute in $x = a - b$ (meaning $-x = b - a$), we see that $f(-x) = -f(x)$. This implies that $f(-0) = f(0) = -f(0)$, so $f(0) = 0$. If we use the fact that $f(x) = f(x + 2\pi)$, we can see that $-f(x) = f(-x) = f(2\pi - x)$, so $f(x) = -f(2\pi - x)$. This allows us to see that $f(\pi) = -f(2\pi - \pi) = -f(\pi)$, so $f(\pi) = 0$ as well.