Ch 4 Notes: 14-28

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EXAMPLE : IDEAL DIFFERENTIATOR

$$y_c(t) = \frac{d}{dt}\left(x_c(t)\right)$$

RECALL THE CTFT PROPERTY RELATED TO TIME DIFFERENTIATION

$$y_c(t) = \frac{d}{dt}\left(x_c(t)\right) \xleftrightarrow{CTFT} Y_c(j\Omega) = \underbrace{j\Omega}_{H_c(j\Omega)} \cdot X_c(j\Omega)$$

THIS IS THE FREQUENCY RESPONSE OF AN IDEAL CT DIFFERENTIATOR

THE DT REALIZATION CAN MATCH THIS FREQUENCY RESPONSE OVER A RANGE OF FREQUENCIES DETERMINED BY THE SAMPLING RATE (OR, EQUIVALENTLY, SAMPLING PERIOD):

$$H_{eff}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/T \\ 0, & \text{ELSE} \end{cases}$$

USING $\omega = \Omega T$, OR $\Omega = \omega/T$, THE DT SYSTEM SHOULD HAVE THE FOLLOWING FREQ. RESPONSE

$$H(e^{j\omega}) = \frac{j\omega}{T}, \quad |\omega| < \pi$$

TO DESIGN THE FILTER, WE CAN FIND ITS

... DTFT.

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TO DESIGN THE FILTER, WE CAN FIND ITS IMPULSE RESPONSE BY APPLYING INVERSE DTFT:

$$h[n] = DTFT^{-1}\{H(e^{j\omega})\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{j\omega}{T}\, e^{j\omega n}\, d\omega$$

$$= \frac{j}{2\pi T}\int_{-\pi}^{\pi} \omega \cdot e^{j\omega n}\, d\omega$$

INTEGRATE BY PARTS:

$$\int u\, dv = uv - \int v\, du$$

$$u = \omega \Rightarrow du = d\omega$$

$$dv = e^{j\omega n} d\omega \Rightarrow v = \frac{1}{jn} e^{j\omega n}$$

$$= \frac{j}{2\pi T}\left[\left(\omega \cdot \frac{1}{jn} e^{j\omega n}\right)_{\omega=-\pi}^{\omega=\pi} - \int_{-\pi}^{\pi} \frac{1}{jn} e^{j\omega n} \cdot d\omega\right]$$

$$= \frac{j}{2\pi T}\left[\frac{\pi}{jn} e^{j\pi n} + \frac{\pi}{jn} e^{-j\pi n} - \frac{1}{jn} \cdot \frac{1}{jn} e^{j\omega n}\Big|_{\omega=-\pi}^{\omega=\pi}\right]$$

$$= \frac{j}{2\pi T}\left[\frac{\pi}{jn}\left(e^{j\pi n} + e^{-j\pi n}\right) - \frac{1}{(jn)^2}\left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{1}{Tn} \cdot \frac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right) + \frac{\overset{= -\frac{1}{j}}{j}}{2\pi T n^2}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

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$$= \frac{1}{Tn}\cdot\frac{1}{2}\underbrace{\left(e^{\cdot\cdot}+e\right)}_{\cos(\pi n)} + \frac{}{2\pi T n^2}\left(e \quad\right)$$

$$= \frac{1}{nT}\cos(\pi n) - \frac{1}{\pi n^2 T}\cdot\underbrace{\frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right)}_{\sin(\pi n)}$$

$$= \frac{1}{nT}\cos(\pi n) - \frac{1}{\pi n^2 T}\sin(\pi n)$$

$$= \frac{\pi n\cos(\pi n) - \sin(\pi n)}{\pi n^2 T}$$

FOR $n=0$, WE HAVE A $\frac{0}{0}$ SITUATION, SO WE NEED TO CHECK WHETHER THE RATIO IS WELL-DEFINED.

LET $x = \pi n$, THEN EXAMINE THE LIMIT OF THE RATIO AS $x \to 0$

$$\frac{\pi n\cos(\pi n) - \sin(\pi n)}{\pi n^2 T} = \frac{\pi}{T}\cdot\frac{\pi n\cos(\pi n) - \sin(\pi n)}{(\pi n)^2}$$

$$= \frac{\pi}{T}\cdot\frac{x\cos(x) - \sin(x)}{x^2}$$

RECALL L'HÔPITAL'S RULE:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

So

$$\lim_{x\to 0} \frac{x\cdot\cos(x) - \sin(x)}{x^2} = \lim_{x\to 0} \frac{(x\cdot\cos(x) - \sin(x))'}{(x^2)'}$$

$$= \lim_{x\to 0} \frac{(x\cdot\cos(x))' - (\sin(x))'}{2x}$$

$$= \lim_{x\to 0} \frac{x\cdot(\cos(x))' + (x)'\cdot\cos(x) - \cos(x)}{2x}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}\cdot\sin(x) + \cancel{\cos(x)} - \cancel{\cos(x)}}{2\cancel{x}}$$

$$= \lim_{x\to 0} -\frac{1}{2}\sin(x) = -\frac{1}{2}\lim_{x\to 0}\sin(x)$$

$$= 0$$

So

$$h[n] = \begin{cases} 0, & n = 0 \\ \dfrac{\pi n\cos(\pi n) - \sin(\pi n)}{\pi n^2 T}, & n \neq 0 \end{cases}$$

$\sin(\pi n) \to 0$ FOR n INTEGER

$$= \begin{cases} 0, & n = 0 \\ \dfrac{\cancel{\pi}\cancel{n}\cos(\pi n)}{\cancel{\pi}n^{\cancel{2}}T}, & \text{ELSE} \end{cases}$$

$$= \begin{cases} 0, & n = 0 \\ \dfrac{\cos(\pi n)}{nT}, & \text{ELSE} \end{cases}$$

$$\cos(\pi n) = \ldots 1, -1, -1, 1, -1 \ldots \quad (-1)^n$$

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$$\cos(\pi n) = \ldots 1, -1, -1, 1, -1 \ldots \quad (-1)^n$$

$$n = \quad -2, -1, 1, 2, 3, \ldots$$

$$\cos(\pi n) = (-1)^n$$

$$h[n] = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^n}{nT}, & \text{ELSE} \end{cases}$$

EXAMPLE DT LOWPASS FILTER DESIGN BY IMPULSE INVARIANCE

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \text{ELSE} \end{cases}$$

CT IMPULSE RESPONSE

$$h_c(t) = CTFT^{-1}\{H_c(j\Omega)\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H_c(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \cdot \frac{1}{jt} \cdot e^{j\Omega t} \Big|_{\Omega = -\Omega_c}^{\Omega = \Omega_c}$$

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$$= \frac{1}{2\pi} \cdot \frac{1}{jt} \left(e^{j\Omega_c t} - e^{-j\Omega_c t} \right)$$

$$= \frac{1}{\pi t} \cdot \underbrace{\frac{1}{2j} \left(e^{j\Omega_c t} - e^{-j\Omega_c t} \right)}_{\sin(\Omega_c t)}$$

$$= \frac{\sin(\Omega_c t)}{\pi t}$$

WE SAMPLE THIS CT IMPULSE RESPONSE TO GET DT IMPULSE RESPONSE:

$$h[n] = T \cdot h_c(nT) = T \cdot h_c(t)\Big|_{t=nT}$$

$$= \cancel{T} \cdot \frac{\sin(\Omega_c \cdot n \cdot T)}{\pi \cdot n \cdot \cancel{T}} = \frac{\sin(\overbrace{\Omega_c T}^{\omega_c} \cdot n)}{\pi n}$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{ELSE} \end{cases}$$