14.5.2 Eigen-decomposition of the Autocorrelation Matrix for Sinusoids in White Noise

In the previous discussion we assumed that the sinusoidal signal consists of p real sinusoids. For mathematical convenience we shall now assume that the signal consists of p complex sinusoids of the form

$$x(n) = \sum_{i=1}^{p} A_i e^{j(2\pi f_i n + \phi_i)} \tag{14.5.13}$$

where the amplitudes $\{A_i\}$ and the frequencies $\{f_i\}$ are unknown and the phases $\{\phi_i\}$ are statistically independent random variables uniformly distributed on $(0, 2\pi)$. Then the random process $x(n)$ is wide-sense stationary with autocorrelation function

$$\gamma_{xx}(m) = \sum_{i=1}^{p} P_i e^{j2\pi f_i m} \tag{14.5.14}$$

where, for complex sinusoids, $P_i = A_i^2$ is the power of the ith sinusoid.

Since the sequence observed is $y(n) = x(m) + w(n)$, where $w(n)$ is a white noise sequence with spectral density σ_w^2, the autocorrelation function for $y(n)$ is

$$\gamma_{yy}(m) = \gamma_{xx}(m) + \sigma_w^2 \delta(m), \qquad m = 0, \pm 1, \ldots, \pm(M-1) \tag{14.5.15}$$

Hence the $M \times M$ autocorrelation matrix for $y(n)$ can be expressed as

$$\mathbf{\Gamma}_{yy} = \mathbf{\Gamma}_{xx} + \sigma_w^2 \mathbf{I} \tag{14.5.16}$$

where $\mathbf{\Gamma}_{xx}$ is the autocorrelation matrix for the signal $x(n)$ and $\sigma_w^2 \mathbf{I}$ is the autocorrelation matrix for the noise. Note that if we select $M > p$, $\mathbf{\Gamma}_{xx}$ which is of dimension $M \times M$ is not of full rank, because its rank is p. However, $\mathbf{\Gamma}_{yy}$ is full rank because $\sigma_w^2 \mathbf{I}$ is of rank M.

In fact, the signal matrix $\mathbf{\Gamma}_{xx}$ can be represented as

$$\mathbf{\Gamma}_{xx} = \sum_{i=1}^{p} P_i \mathbf{s}_i \mathbf{s}_i^H \tag{14.5.17}$$

where H denotes the conjugate transpose and $\mathbf{s}_i$ is a signal vector of dimension M defined as

$$\mathbf{s}_i = [1, e^{j2\pi f_i}, e^{j4\pi f_i}, \ldots, e^{j2\pi(M-1)f_i}] \tag{14.5.18}$$

Since each vector (outer product) $\mathbf{s}_i \mathbf{s}_i^H$ is a matrix of rank 1 and since there are p vector products, the matrix $\mathbf{\Gamma}_{xx}$ is of rank p. Note that if the sinusoids were real, the correlation matrix $\mathbf{\Gamma}_{xx}$ would have rank $2p$.

Now, let us perform an eigen-decomposition of the matrix $\mathbf{\Gamma}_{yy}$. Let the eigenvalues $\{\lambda_i\}$ be ordered in decreasing value with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_M$ and let the corresponding eigenvectors be denoted as $\{\mathbf{v}_i, i = 1, \ldots, M\}$. We assume that the eigenvectors are normalized so that $\mathbf{v}_i^H \cdot \mathbf{v}_j = \delta_{ij}$. In the absence of noise the eigenvalues λ_i, $i = 1, 2, \ldots, p$, are nonzero while $\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_M = 0$. Furthermore, it follows that the signal correlation matrix can be expressed as

$$\mathbf{\Gamma}_{xx} = \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^H \tag{14.5.19}$$

Thus, the eigenvectors $\mathbf{v}_i$, $i = 1, 2, \ldots, p$ span the signal subspace as do the signal vectors $\mathbf{s}_i$, $i = 1, 2, \ldots, p$. These p eigenvectors for the signal subspace are called the *principal eigenvectors* and the corresponding eigenvalues are called the *principal eigenvalues*.

In the presence of noise, the noise autocorrelation matrix in (14.5.16) can be represented as

$$\sigma_w^2 \mathbf{I} = \sigma_w^2 \sum_{i=1}^{M} \mathbf{v}_i \mathbf{v}_i^H \tag{14.5.20}$$

By substituting (14.5.19) and (14.5.20) into (14.5.16), we obtain

$$\begin{aligned} \mathbf{\Gamma}_{yy} &= \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=1}^{M} \sigma_w^2 \mathbf{v}_i \mathbf{v}_i^H \\ &= \sum_{i=1}^{p} (\lambda_i + \sigma_w^2) \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^{M} \sigma_w^2 \mathbf{v}_i \mathbf{v}_i^H \end{aligned} \tag{14.5.21}$$

This eigen-decomposition separates the eigenvectors into two sets. The set $\{\mathbf{v}_i, i = 1, 2, \ldots, p\}$, which are the principal eigenvectors, span the signal subspace, while the set $\{\mathbf{v}_i, i = p+1, \ldots, M\}$, which are orthogonal to the principal eigenvectors, are said to belong to the noise subspace. Since the signal vectors $\{\mathbf{s}_i, i = 1, 2, \ldots, p\}$ are in the signal subspace, it follows that the $\{\mathbf{s}_i\}$ are simply linear combinations of the principal eigenvectors and are also orthogonal to the vectors in the noise subspace.

In this context we see that the Pisarenko method is based on an estimation of the frequencies by using the orthogonality property between the signal vectors and the vectors in the noise subspace. For complex sinusoids, if we select $M = p+1$ (for real sinusoids we select $M = 2p+1$), there is only a single eigenvector in the noise subspace (corresponding to the minimum eigenvalue) which must be orthogonal to the signal vectors. Thus we have

$$\mathbf{s}_i^H \mathbf{v}_{p+1} = \sum_{k=0}^{p} v_{p+1}(k+1) e^{-j2\pi f_i k} = 0, \qquad i = 1, 2, \ldots, p \tag{14.5.22}$$

But (14.5.22) implies that the frequencies $\{f_i\}$ can be determined by solving for the zeros of the polynomial

$$V(z) = \sum_{k=0}^{p} v_{p+1}(k+1) z^{-k} \tag{14.5.23}$$

all of which lie on the unit circle. The angles of these roots are $2\pi f_i$, $i = 1, 2, \ldots, p$.

When the number of sinusoids is unknown, the determination of p may prove to be difficult, especially if the signal level is not much higher than the noise level. In theory, if $M > p + 1$, there is a multiplicity $(M - p)$ of the minimum eigenvalue. However, in practice the $(M - p)$ small eigenvalues of $\mathbf{R}_{yy}$ will probably be different. By computing all the eigenvalues it may be possible to determine p by grouping the $M - p$ small (noise) eigenvalues into a set and averaging them to obtain an estimate of σ_w^2. Then, the average value can be used in (14.5.9) along with $\mathbf{R}_{yy}$ to determine the corresponding eigenvector.

14.5.3 MUSIC Algorithm

The multiple signal classification (MUSIC) method is also a noise subspace frequency estimator. To develop the method, let us first consider the "weighted" spectral estimate

$$P(f) = \sum_{k=p+1}^{M} w_k |\mathbf{s}^H(f)\mathbf{v}_k|^2 \tag{14.5.24}$$

where $\{\mathbf{v}_k, k = p + 1, \ldots, M\}$ are the eigenvectors in the noise subspace, $\{w_k\}$ are a set of positive weights, and $\mathbf{s}(f)$ is the complex sinusoidal vector

$$\mathbf{s}(f) = [1, e^{j2\pi f}, e^{j4\pi f}, \ldots, e^{j2\pi(M-1)f}] \tag{14.5.25}$$

Note that at $f = f_i$, $\mathbf{s}(f_i) \equiv \mathbf{s}_i$, so that at any one of the p sinusoidal frequency components of the signal, we have

$$P(f_i) = 0, \qquad i = 1, 2, \ldots, p \tag{14.5.26}$$

Hence, the reciprocal of $P(f)$ is a sharply peaked function of frequency and provides a method for estimating the frequencies of the sinusoidal components. Thus

$$\frac{1}{P(f)} = \frac{1}{\sum_{k=p+1}^{M} w_k \left|\mathbf{s}^H(f)\mathbf{v}_k\right|^2} \tag{14.5.27}$$

Although theoretically $1/P(f)$ is infinite at $f = f_i$, in practice the estimation errors result in finite values for $1/P(f)$ at all frequencies.

The MUSIC sinusoidal frequency estimator proposed by Schmidt (1981, 1986) is a special case of (14.5.27) in which the weights $w_k = 1$ for all k. Hence

$$P_{\text{MUSIC}}(f) = \frac{1}{\sum_{k=p+1}^{M} \left|\mathbf{s}^H(f)\mathbf{v}_k\right|^2} \tag{14.5.28}$$

The estimates of the sinusoidal frequencies are the peaks of $P_{\text{MUSIC}}(f)$. Once the sinusoidal frequencies are estimated, the power of each of the sinusoids can be obtained by solving (14.5.11).