PH1213

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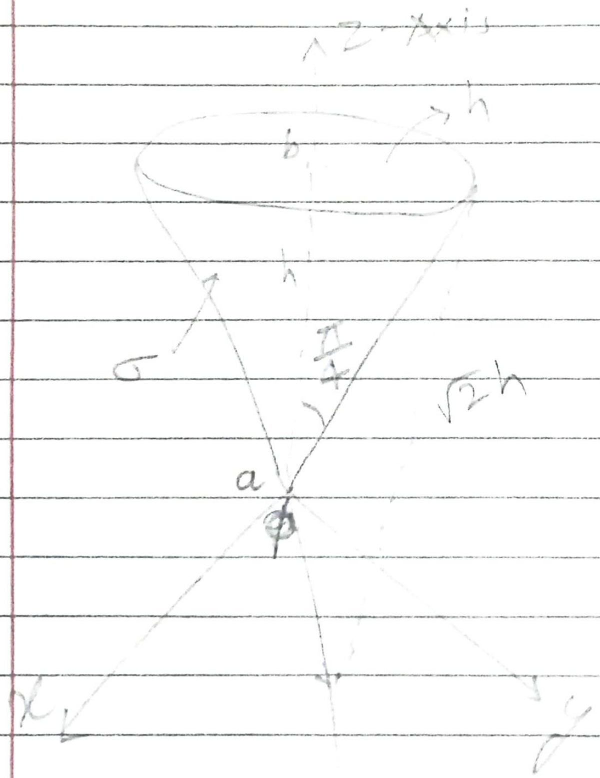
Ques A Conical surface (an empty ice-cream cone) Carries a uniform surface charge σ. The height of cone is 'h' as in the radius of the top. Find the potential difference between points 'a' (the Vertex) and 'b' (the center of the top).

Ans.



In this Question we are using Spherical co-ordinate $\theta = \frac{\pi}{4}$ (as Radius & height is same)

Calculating potential at point 'a'

We take a small area element 'da'

$$da = r \sin\theta \, dr \, d\phi$$

Here $\theta = \frac{\pi}{4}$ (constant)

So, $da = \frac{r \, dr \, d\phi}{\sqrt{2}}$

$$V(a) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{\sigma \, da}{r}$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\sqrt{2}h} \frac{\sigma \cancel{r} \, dr \, d\phi}{\sqrt{2} \cancel{r}}$$

$$\Rightarrow \frac{\sigma (\cancel{2}\cancel{\pi}) (\cancel{\sqrt{2}} h)}{\cancel{4}_2 \cancel{\pi} \epsilon_0 (\cancel{\sqrt{2}})}$$

$$V(a) = \frac{\sigma h}{2\epsilon_0}$$

~~Where σ - surface charge density~~

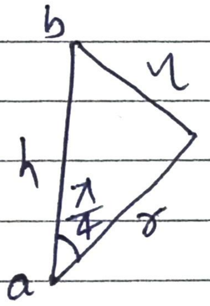
~~r -~~

Calculating potential at point 'b'

again considering small area element 'da'

$$da = \frac{r}{\sqrt{2}}\, dr\, d\phi$$

In this case we consider a new variable η (distance b/w point 'b' to surface of small area element on cone)



Using cosine Law

$$\eta^2 = r^2 + h^2 - 2rh\cos\frac{\pi}{4}$$

$$\eta = \sqrt{r^2 + h^2 - \frac{2}{\sqrt{2}}\, rh}$$

$$\eta = \sqrt{\left(r - \frac{h}{\sqrt{2}}\right)^2 + \frac{h^2}{2}}$$

$$V(b) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\sqrt{2}h} \frac{\sigma\left(\frac{r\, dr\, d\phi}{\sqrt{2}}\right)}{\sqrt{\left(r - \frac{h}{\sqrt{2}}\right)^2 + \frac{h^2}{2}}}$$

let ~~$r - \frac{h}{\sqrt{2}} = \frac{h}{\sqrt{2}} \tan\theta$~~

let $r - \frac{h}{\sqrt{2}} = \frac{h}{\sqrt{2}} \tan\alpha$

$$\frac{dr}{d\alpha} = \frac{h}{\sqrt{2}} \sec^2\alpha$$

$$dr = \frac{h}{\sqrt{2}} \sec^2\alpha \, d\alpha$$

So, $$V(b) = \frac{2\pi\sigma}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{2}}\right)\int_{-\pi/4}^{\pi/4} \frac{\frac{h}{\sqrt{2}}(1+\tan\alpha)\frac{h}{\sqrt{2}}\sec^2\alpha \, d\alpha}{\sqrt{\frac{h^2}{2}(\tan^2\alpha+1)}}$$

$$\Rightarrow \frac{2\pi\sigma}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{2}}\right)\int_{-\pi/4}^{\pi/4} \frac{\cancel{\frac{h}{\sqrt{2}}}(1+\tan\alpha)\frac{h}{\sqrt{2}}\sec^2\alpha \, d\alpha}{\cancel{\frac{h}{\sqrt{2}}}\sqrt{\sec^2\alpha}}$$

$$\Rightarrow \frac{\sigma}{2\epsilon_0}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{h}{\sqrt{2}}\right)\int_{-\pi/4}^{\pi/4} \sec\alpha(1+\tan\alpha)\, d\alpha$$

$$\Rightarrow \frac{\sigma h}{4\epsilon_0} 2\int_0^{\pi/4} \sec\alpha \, d\alpha$$

$\left[\because \int_{-a}^{a} \sec\alpha \tan\alpha \, d\alpha = 0 \text{ as } \cancel{\tan\alpha} \text{ it is odd function}\right]$

$$\Rightarrow \frac{\sigma h}{2\epsilon_0}\Big[\ln(\sec\alpha + \tan\alpha)\Big]_0^{\pi/4}$$

$$\Rightarrow \frac{\sigma h}{2\epsilon_0}\left(\ln\left(1+\sqrt{2}\right)\right)$$

So, $V(b) = \dfrac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right)$

So, $V(a) - V(b) = \dfrac{\sigma h}{2\epsilon_0} - \dfrac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right)$

$$\boxed{V(a) - V(b) = \frac{\sigma h}{2\epsilon_0}\left[1 - \ln\left(1+\sqrt{2}\right)\right]}$$