Q A dielectric sphere of radius R has variable permittivity throughout space described by

$$\varepsilon(r) = \begin{cases} \varepsilon_0 (R/r)^2 & r < R \\ \varepsilon_0 & r > R \end{cases}$$

The sphere is embedded in a constant external electric field $\vec{E} = E_0 \hat{z}$.

a) Show that $V(\vec{r})$ obeys

$$\left[\nabla^2 V + \frac{d \ln \varepsilon}{dr} \frac{\partial V}{\partial r} = 0 \right] \text{ for all } \vec{r}.$$

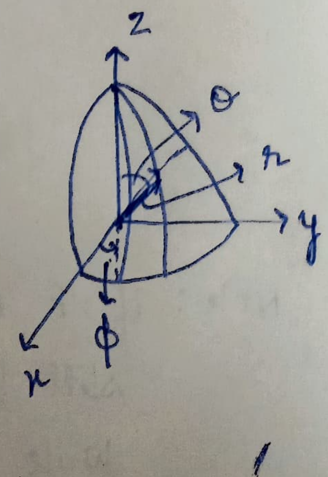
$\Rightarrow$ $\because$ there are no free charges

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon \vec{E})$$

$$= \vec{E} \cdot (\vec{\nabla} \varepsilon) + \varepsilon \vec{\nabla} \cdot \vec{E} = 0 \quad \rightarrow 1.1$$

The permittivity only depends of r (given in q.),

so we can write $\vec{\nabla}_\varepsilon = \frac{\partial \varepsilon}{\partial r} \hat{e}_r \rightarrow 1.2$



Putting 1.2 in 1.1,

$$0 = (\vec{\nabla} V) \cdot \hat{e}_r \frac{d\varepsilon}{dr} + \varepsilon \nabla^2 V$$

$$= \frac{\partial V}{\partial r} \frac{d\varepsilon}{dr} \frac{1}{\varepsilon} + \nabla^2 V$$

$$\Rightarrow \left[\frac{\partial V}{\partial r} \frac{d \ln \varepsilon}{dr} + \nabla^2 V = 0 \right]$$

(b) Explain why the solution can be written as

$$V(r,\theta) = \sum_{l=0}^{\infty} V_l(r) \{\hat{z}_{i_1} \dots \hat{z}_{i_l}\} \hat{r}_{i_1} \dots \hat{r}_{i_l}$$

Or $$V(r,\theta) = \sum_{l=0}^{\infty} V_l(r) P_l(\cos\theta)$$ [$P_l(\cos\theta)$ is the legendre polynomial]

$\Rightarrow$ External field is along z axis (given in q)

$\therefore$ there's an azimuthal symmetry, $\Rightarrow \dfrac{\partial V}{\partial \phi} = 0$

$\therefore$ $V = V(r,\theta)$

Legendre polynomials $P_l(\cos\theta)$ are a complete set of polar angles $0 < \theta < \pi$

$\Rightarrow$ for each value of r, $V(r,\theta)$ can be expanded in a legendre series.

(the coefficients will be functions of r).

$$V(r,\theta) = \sum_{l=0}^{\infty} V_l(r) P_l(\cos\theta)$$

Note: if E depended both on θ as well as r, the argument would still be valid (complete set of polar angles) & we could write $V(r,\theta)$ <u>but</u> the eq^n's would have become coupled to each other, $\therefore$ difficult to solve.