Feynman's Paradox

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1 Introduction

Feynman's disk paradox was a thought experiment that said if a free-floating disk of charges surrounded a solenoid with current flowing through it, the disk would rotate when the current was shut off due to a transfer of angular momentum from the crossed electric and magnetic fields..

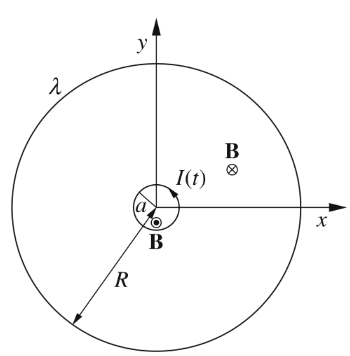
Problem

A non-conducting ring of radius R is at rest on the xy plane, with its center at the origin of the coordinate system. The ring has mass m, negligible thickness, and an electric charge Q distributed uniformly on it, so that the ring has a linear charge density λ =Q/$(2\pi a)$. The ring is free to rotate around its axis without friction. A superconducting circular ring of radius $a \ll R$, coaxial to the charged ring and carrying an electric current I_0, also lies on the xy plane, as shown. At time $t = 0$ the superconducting loop is heated above its critical temperature,and switches to normal conductivity. Consequently, its current decays to zero according to a law $I = I(t)$.

a) Neglecting self-induction effects, evaluate the angular velocity = (t) of the charged ring as a function of the current $I(t)$ in the smaller ring. Evaluate the final angular velocity ${}_f$, and the final angular momentum L_f, of the charged ring.

b) Evaluate the magnetic field at the ring center, B_c, generated by the rotation of the ring.

c) Discuss how the results of a) are modified by taking the "self-inductance" L of the charged ring into account.



S-6.6 Feynman's "Paradox"

a) The mutual inductance M between the charged ring and the superconducting ring is, assuming $a \ll R$ (see Problem 6.12),

$$M = 4\pi k_{\mathrm{m}} b_{\mathrm{m}} \frac{\pi a^2}{2R}\,. \tag{S-6.54}$$

Thus, when a current $I(t)$ is circulating in the smaller ring of radius a, the magnetic flux through the charged ring is

$$\Phi_I = MI(t) = 4\pi k_{\mathrm{m}} b_{\mathrm{m}} \frac{\pi a^2}{2R} I(t)\,. \tag{S-6.55}$$

If Φ_I is time-dependent, it gives origin to an induced electric field $\mathbf{E}_I$, whose line-integral around the charged ring is

$$\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -b_{\mathrm{m}} \frac{\mathrm{d}\Phi_I}{\mathrm{d}t} = -4\pi k_{\mathrm{m}} b_{\mathrm{m}}^2 \frac{\pi a^2}{2R}\, \partial_t I(t)\,. \tag{S-6.56}$$

Due to the symmetry of our problem, field $\mathbf{E}_I$ is azimuthal on the xy plane, and independent of ϕ. Its magnitude on the charged ring is thus

$$E_I = \frac{1}{2\pi R}\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R^2}\,\partial_t I(t)\,, \tag{S-6.57}$$

and the force exerted on an infinitesimal element $\mathrm{d}\ell$ of the charged ring is

$$\mathrm{d}\mathbf{f} = \mathbf{E}_I \lambda\,\mathrm{d}\ell = -\hat{\boldsymbol{\phi}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R^2}\,\lambda\,\mathrm{d}\ell\,\partial_t I(t)\,, \tag{S-6.58}$$

corresponding to a torque $\mathrm{d}\boldsymbol{\tau}$ about the center of the ring

$$\mathrm{d}\boldsymbol{\tau} = \mathbf{r}\times\mathrm{d}\mathbf{f} = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\,\lambda\,\mathrm{d}\ell\,\partial_t I(t)\,. \tag{S-6.59}$$

The total torque on the charged ring is thus

$$\boldsymbol{\tau} = \int \mathrm{d}\boldsymbol{\tau} = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\,\lambda\, 2\pi R\,\partial_t I(t) = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\, Q\,\partial_t I(t)\,, \tag{S-6.60}$$

where $Q = 2\pi R\lambda$ is the total charge of the ring. The equation of motion for the charged ring is thus

$$mR^2\frac{\mathrm{d}\omega}{\mathrm{d}t} = \boldsymbol{\tau} = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\, Q\,\partial_t I(t)\,, \tag{S-6.61}$$

where mR^2 is the moment of inertia of the ring. The solution for $\omega(t)$ is

$$\omega(t) = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q \int_0^t \partial_t I(t')\,\mathrm{d}t' = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q\,[I_0 - I(t)]\,, \tag{S-6.62}$$

and the final angular velocity is

$$\omega_\mathrm{f} = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q I_0 = \begin{cases} \dfrac{\mu_0 a^2 Q}{4mR^3}\, I_0\,, & \text{SI,} \\[2ex] \dfrac{\pi a^2 Q}{c^3 mR^3}\, I_0\,, & \text{Gaussian,} \end{cases} \tag{S-6.63}$$

corresponding to a final angular momentum

$$L_\mathrm{f} = mR^2\omega_\mathrm{f} = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2 Q}{R}\, I_0 = \begin{cases} \dfrac{\mu_0 a^2 Q}{R}\, I_0\,, & \text{SI,} \\[2ex] \dfrac{\pi a^2 Q}{c^3 R}\, I_0\,, & \text{Gaussian,} \end{cases} \tag{S-6.64}$$

independent of the mass m of the ring.
b) The rotating charged ring is equivalent to a circular loop carrying a current $I_{\mathrm{rot}} = Q\omega/2\pi$. Thus, after the current in the small ring is switched off, there is still a magnetic field due to the rotation of the charged ring. The final magnetic field at the center of the rings is

$$\mathbf{B}_{\mathrm{c}} = \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}}{2}\,\frac{I_{\mathrm{rot}}}{R} = \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}}{4\pi}\,\frac{Q\omega_{\mathrm{f}}}{R}$$
$$= \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}^2 b_{\mathrm{m}}^2 a^2 Q^2}{4mR^4}\,I_0 = \begin{cases} \hat{\mathbf{z}}\,\dfrac{\mu_0^2 a^2 Q^2}{64\pi^2 mR^4}\,I_0\,, & \text{SI,} \\ \hat{\mathbf{z}}\,\dfrac{a^2 Q^2}{4c^4 mR^4}\,I_0\,, & \text{Gaussian,} \end{cases} \tag{S-6.65}$$

parallel to the initial field $\mathbf{B}_0 = \hat{\mathbf{z}}\,k_{\mathrm{m}} I_0/(2a)$, in agreement with Lenz's law. We further have

$$\pi a^2 B_c = M I_{\mathrm{rot}}\,, \tag{S-6.66}$$

where M is the mutual inductance of the rings (S-6.54).
c) As seen above at point **b)**, the rotating charged ring generates a magnetic field all over the space. This field modifies the magnetic flux through the rotating ring itself, giving origin to self-induction. Let $\mathcal{L}$ be the "self-inductance" of the rotating ring. The magnetic flux generated by the rotating ring through itself is

$$\Phi_{\mathrm{rot}} = \frac{1}{b_{\mathrm{m}}}\,\mathcal{L}\,I_{\mathrm{rot}} = \frac{1}{b_{\mathrm{m}}}\,\mathcal{L}\,\frac{Q\omega}{2\pi}\,. \tag{S-6.67}$$

Correspondingly, (S-6.56) for the line integral of the electric field around the charged ring is modified as follows:

$$\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -b_{\mathrm{m}}\left(\frac{\mathrm{d}\Phi_I}{\mathrm{d}t} + \frac{\mathrm{d}\Phi_{\mathrm{rot}}}{\mathrm{d}t}\right) = -\frac{4\pi^2 k_{\mathrm{m}} b_{\mathrm{m}}^2 a^2}{2R}\,\partial_t I - \mathcal{L}\,\frac{Q}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\,. \tag{S-6.68}$$

The torque on the ring becomes

$$\boldsymbol{\tau} = -\hat{\mathbf{z}}\left(\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I + \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)\,, \tag{S-6.69}$$

and the equation of motion (S-6.61) becomes

$$mR^2\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I - \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\,,$$

or

$$\left(mR^2 + \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\right)\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I\,, \tag{S-6.70}$$

which is equivalent to (S-6.61) if we replace the mass of the charged ring by an effective value

$$m_{\mathrm{eff}} = m + \mathcal{L}\,\frac{Q^2 a^2}{2\pi R^2}\,. \tag{S-6.71}$$

Thus we obtain for the dependence of ω on $I(t)$

$$\omega(t) = k_{\mathrm{m}} b_{\mathrm{m}}^2\,\frac{\pi a^2 Q}{m_{\mathrm{eff}} R^3}\,[I_0 - I(t)]\,, \tag{S-6.72}$$

and for its final value

$$\omega_{\mathrm{f}} = k_{\mathrm{m}} b_{\mathrm{m}}^2\,\frac{\pi a^2 Q}{m_{\mathrm{eff}} R^3}\,I_0\,, \tag{S-6.73}$$

corresponding to a final angular momentum

$$\begin{aligned} L_{\mathrm{f}} = mR^2\omega_{\mathrm{f}} &= k_{\mathrm{m}} b_{\mathrm{m}}^2\,\frac{\pi a^2 Q}{R + \mathcal{L}a^2 Q^2/(2\pi m R)}\,I_0 \\ &= \begin{cases} \dfrac{\mu_0}{4\pi}\,\dfrac{\pi a^2 Q}{R + \mathcal{L}a^2 Q^2/(2\pi m R)}\,I_0\,, & \text{SI,} \\ \dfrac{1}{c^3}\,\dfrac{\pi a^2 Q}{R + \mathcal{L}a^2 Q^2/(2\pi m R)}\,I_0\,, & \text{Gaussian.} \end{cases} \end{aligned} \tag{S-6.74}$$

The final magnetic flux through the charged ring is

$$\Phi_{\mathrm{f}} = \frac{1}{b_{\mathrm{m}}}\,\mathcal{L}\,\frac{Q\omega_{\mathrm{f}}}{2\pi} = k_{\mathrm{m}} b_{\mathrm{m}}\,\frac{\mathcal{L}a^2 Q^2}{2mR^3 + \mathcal{L}Q^2 a^2 R/\pi}\,I_0\,, \tag{S-6.75}$$

and the approximations of point a) are valid only if

$$\Phi_{\mathrm{f}} \ll \Phi_0 = 4\pi k_{\mathrm{m}} b_{\mathrm{m}}\,\frac{\pi a^2}{2R}\,I_0\,, \quad \text{or} \quad \frac{\mathcal{L}Q^2}{4\pi^2 m R^2 + 2\pi\mathcal{L}Q^2 a^2} \ll 1. \tag{S-6.76}$$