Hidden momentum

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According to special theory of relativity, the laws of physics are invariant in each inertial frame of reference. Incorrect applications of Maxwell's equations and Lorentz force law can give rise to paradoxes which are inconsistent with the theory of relativity. Here one such case is discussed, and its resolution is presented based on 'hidden' momentum of magnetic dipoles.

1. Introduction

The Maxwell's equations and Lorentz force law together make up the 5 pillars of electromagnetism. Maxwell's equations are Lorenz invariant, and are as given below:

$$\nabla . E = \rho / \epsilon_0$$

$$\nabla x E = \frac{-\partial B}{\partial t}$$

$$\nabla . B = 0$$

$$\nabla x B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

The electric field and magnetic fields transform, when seen by observers in different frames of reference according to the transformation formulae

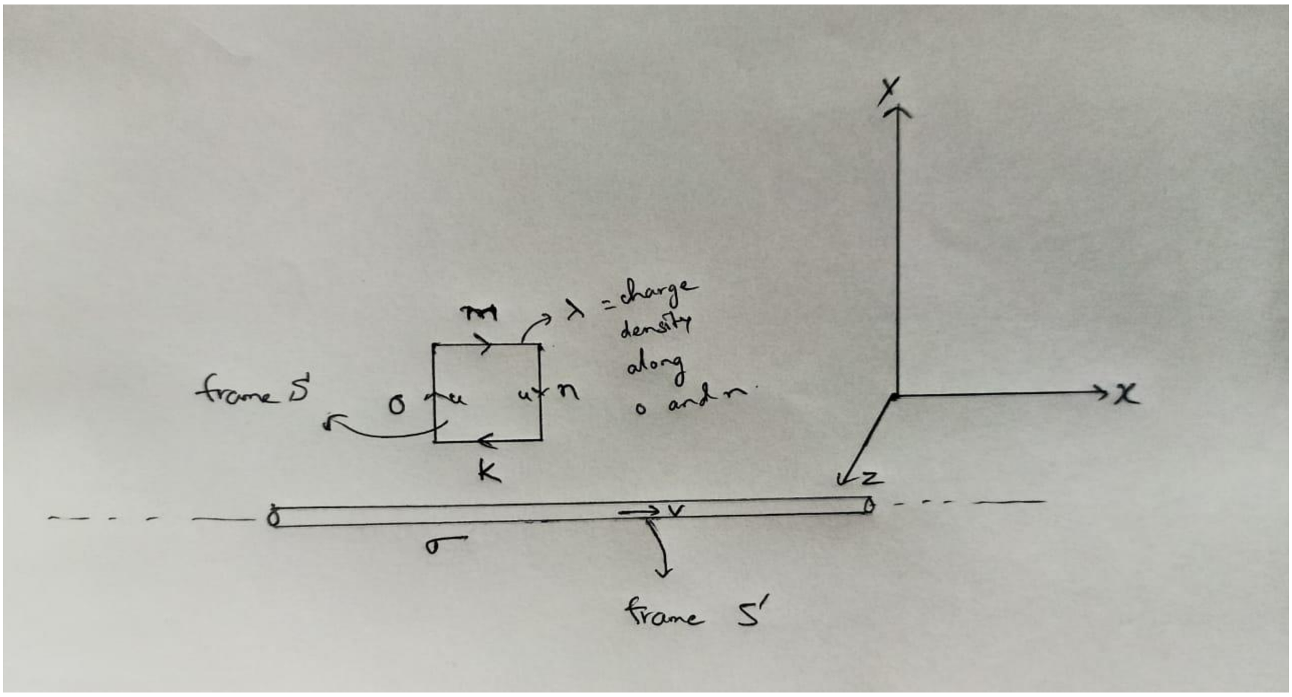
$$E'_x = E_x \quad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z - vB_y)$$

$$B'_x = B_x \quad B'_y = \gamma(B_y - \frac{v}{c^2} E_z) \quad B^{z\prime}{}_z = \gamma(B_z - \frac{v}{c^2} E_y)$$

Here v is the velocity of the frame with respect to another frame and γ is the Lorentz factor given by $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$.

2. The Problem

1.



Consider a very small stationary magnetic dipole of a square current loop, and an infinite line charge of line charge density σ moving at velocity v, which is kept at a distance d. First, we consider the case when the direction of magnetic dipole is perpendicular to the line current. In the frame of reference of the magnetic dipole, say S, the line current gives rise to a magnetic field.

$B = \frac{\mu_0 \sigma v}{2\pi d}$ here $\sigma' = \frac{\sigma}{\gamma(v)}$

Due to this magnetic field the dipole experiences a torque given by

$$\tau = MxB = \frac{\mu_0 \sigma \lambda l^2 u v}{2\pi d} \hat{\imath}$$

Now coming to the frame of reference of the line current, say S'. In this frame, there is no magnetic field because of the current. The torque on the magnetic dipole is solely due to electrostatic forces. The net current at any point in the square loop must be same for a steady state. Therefore

$$\lambda_m u_m = \lambda_k u_k$$

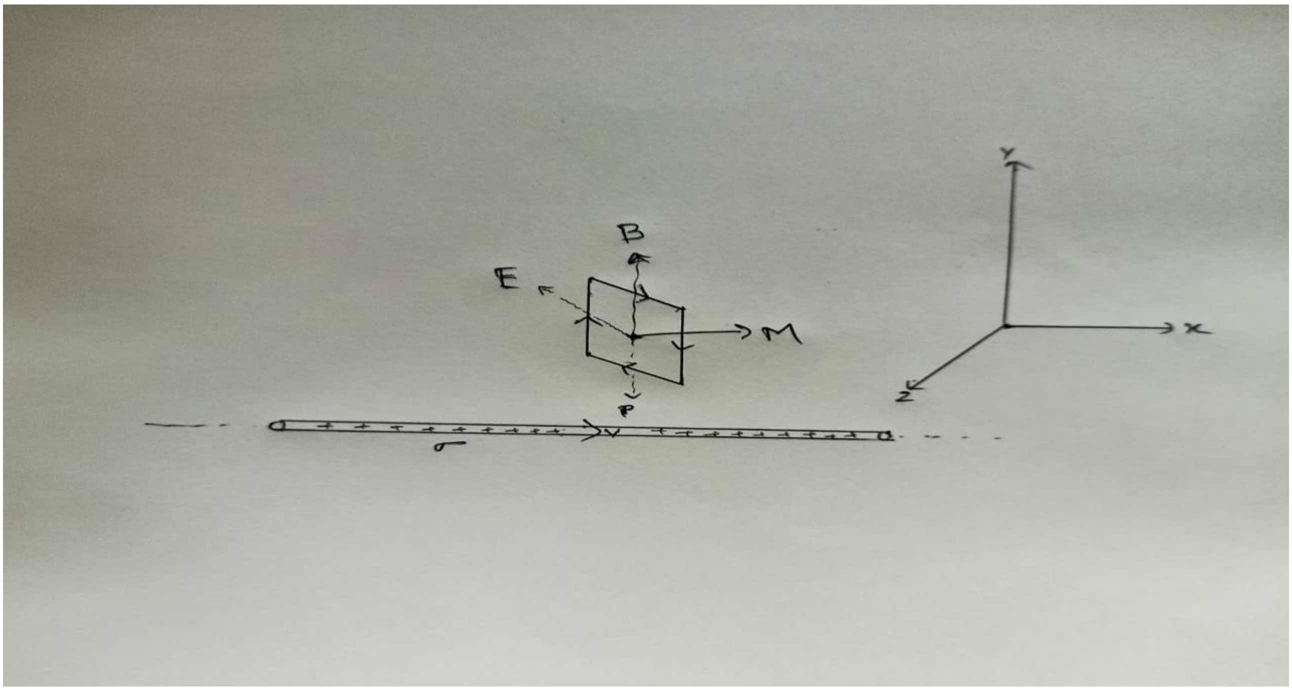
Here $u_m$and $u_k$can be found using some analysis and Einstein's velocity addition rule. On further calculations we get (see ref.2)

$\lambda_m = \lambda_o(1 - \frac{uv}{c^2})$ and $\lambda_k = \lambda_0(1 + \frac{uv}{c^2})$

The torque on the Magnetic dipole due to the electric field is

$\tau = (F_k - F_m)\, l/2 = \frac{\sigma' \lambda l^2 uv}{c^2}\hat{\imath}$ just as we expected from Lorentz transformation.

2.



Now consider similar parameters as before, but this time we consider the magnetic dipole to be parallel to the line of current. When we observe the magnetic dipole from *S*, we see that it experiences a torque due to the magnetic field of the line current.

$$\tau = \frac{\mu_0 \sigma l^2 uv}{2\pi d}\hat{k}$$

In the *S'* frame, there is no magnetic field. Also as velocity of charges is perpendicular to the relative velocity of frames *S* and *S'*, the charge density on opposite sides of the square loop does not change. We might conclude that there is no torque on the magnetic dipole. This conclusion can't be correct as if observer in one inertial frame notices a torque on the magnetic dipole, then observers in every other inertial frame must also conclude on the same thing. So, where did we go wrong?

The resolution lies in 'hidden' momentum of the magnetic dipole. A magnetic dipole carries linear momentum in the presence of an electric field, even when it is not moving. This hidden momentum is given by

$$p = \frac{-MxE}{c^2}$$

In our previous case, the hidden momentum comes out to be 0 as electric field and Magnetic dipole moment are parallel to each other. Here the hidden momentum comes out to be

$$p = \frac{-\lambda\sigma' l^2 u}{2\pi\epsilon_0 c^2}\hat{j}$$

Hidden angular momentum of the dipole is **r x p**, and the torque comes out to be **v x p**

$$\tau = v\ x\ p = -v\hat{\imath}\,x\ \frac{-\lambda\sigma' l^2 u}{2\pi\epsilon_0 c^2}\hat{j} = \frac{\lambda\sigma' l^2 uv}{2\pi\epsilon_0 c^2}\hat{k}$$

as we expect from the Lorentz transformations. Thus, the theory remains consistent.

3. Remarks

1.Here we have considered an example where we see the powers of hidden momentum using relativistic effects. A more general way to look at it comes from the Center of Energy theorem. Center of Energy theorem guarantees that the center of energy of an isolated system remains at rest when its total linear momentum vanishes. The expression of center of energy theorem is as given below.

$$v_E = \frac{dR_e}{dt} = \frac{c^2 P_{total}}{Utotal}$$

The derivation is mathematically cumbersome and hence omitted. (For highly enthusiastic people, see ref.5)

2. The Amperian model of dipole needs to use hidden momentum to get rid of paradoxes which arise when observing a system from different frames. In Gilbert model where magnetic monopoles are considered, the expression for torque arises naturally from the Lorentz force law

For magnetic monopoles, Lorentz force law would read

$$F = q_m(B - \frac{1}{c^2}(vxE))$$

The torque would hence be

$$\tau = mxb - mx(\frac{vxE}{c^2})$$

On expanding the 2nd term in the above expression, we get the missing torque.

References.

1. David J Griffiths, *Introduction to Electrodynamics 4th Edition*, section 11.2.4.
2.Bedford and Krumm, "*On the origin of magnetic dynamics*",American Journal Of Physics **54,** 1036 (1986)
3. V. Namias, "*Electrodynamics of moving Dipole: The case of the missing torque*", American Journal of Physics **57**, 171 (1989)
4.David J Griffiths, "*Mansuripur's Paradox*", American Journal of Physics **81**, 570 (2013)
5.Andrew Zangwill, *Modern Electrodynamics* , section 15.7