PH1213 Tutorial Question

SHARAN K

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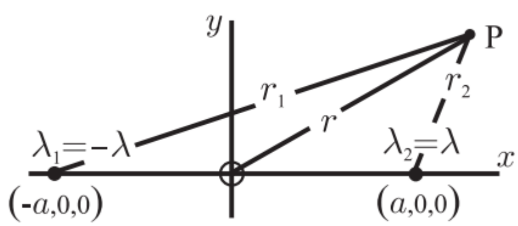
1 Question

Two infinitely long opposite line charges $\pm\lambda$ are separated by a distance 2a and lie parallel to the z-axis of Cartesian coordinates

1. Find the expression for the electric potential at an arbitrary point P(x, y, z).

2. Prove that all equipotential surfaces (with the exception of the $\phi = 0$ plane) and all field lines are circular cylinders.



Solution

$$\Phi(r) - \Phi(r_0) = \int_{r_0}^{r} E.dr = -\frac{\lambda}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0} ln\left[\frac{r_0}{r}\right]$$

We define $\Phi(r_0) = 0$,

$$\Phi(r) = \frac{\lambda}{2\pi\epsilon_0} ln\left[\frac{r_0}{r}\right]$$

Due to the principle of superposition, we have $\Phi(r) = \Phi_{\lambda_1} + \Phi_{\lambda_2}$. Thus,

$$\Phi(r) = \frac{\lambda}{2\pi\epsilon_0} ln\left[\frac{r_1}{r_2}\right]$$

Clearly,

$$r_1 = \sqrt{(x+a)^2 + y^2}$$
$$r_2 = \sqrt{(x-a)^2 + y^2}$$
$$\Longrightarrow \Phi(r) = \frac{\lambda}{4\pi\epsilon_0} ln\left[\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right]$$

On an equipotential surface, the term inside the log equals a positive constant. Then,

$$\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} = \alpha^2$$

By substituting and rearranging terms we get,

$$\left[x - a\left(\frac{\alpha^2+1}{\alpha^2-1}\right)\right]^2 + y^2 = \left(\frac{2a\alpha}{|\alpha^2-1|}\right)^2$$

Thus, the equipotential surfaces are circular cylinders centered on the x axis with radius $r_0 = \left|\frac{2a\alpha}{\alpha^2-1}\right|$

Taking,

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{\partial\Phi/\partial y}{\partial\Phi/\partial x}$$

Differentiating $\Phi(r)$,

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2 - a^2}$$

Substituting polar coordinates,

$$\frac{dy}{dx} = \frac{2r^2 \sin\theta\cos\theta}{r^2(\cos^2\theta - \sin^2\theta) - a^2}$$

Now, writing the LHS as,

$$\frac{dy}{dx} = \frac{\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial\theta}d\theta}{\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta}$$

Equating both equations,

$$\frac{2r^2 \sin\theta\cos\theta}{r^2(\cos^2\theta - \sin^2\theta) - a^2} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta}$$

Solving for $dr/d\theta$ we get,

$$\int \frac{r^2 + a^2}{r^2 - a^2}\frac{dr}{r} = \int \cot\theta d\theta$$

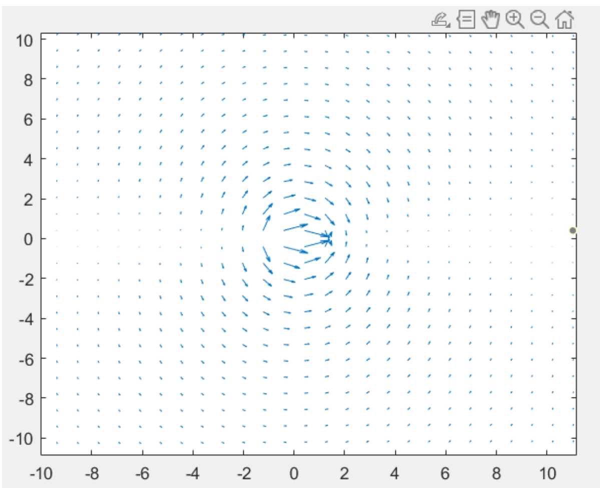
After integration,

$$ln\left[\frac{r^2 - a^2}{ra}\right] = ln(\sin\theta) + ln\beta$$

Translating to cartesian coordinates,

$$x^2 + \left(y - \frac{1}{2}\beta a\right)^2 = \left(a\sqrt{1 + \frac{1}{4}\beta^2}\right)^2$$

Thus, the field lines are circular cylinders centered at $(0, \frac{1}{2}\beta a)$ with radius $r_0 = a\sqrt{1 + \frac{1}{4}\beta^2}$



The above figure is a plot of field lines.