Physics Presentation (Batch-1)

Electrostatics [Method of Images]

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20211021)

THE METHOD OF IMAGES :-

The method of image charges also known as method of mirror charges is a basic problem solving tool in Electrostatics.

The validity of the method of image charges rests upon a corollary of the uniqueness theorem, which states that the electric potential in a volume V is uniquely determined if both the charge density throughout the region and the value of the electric potential on all boundaries are specified.

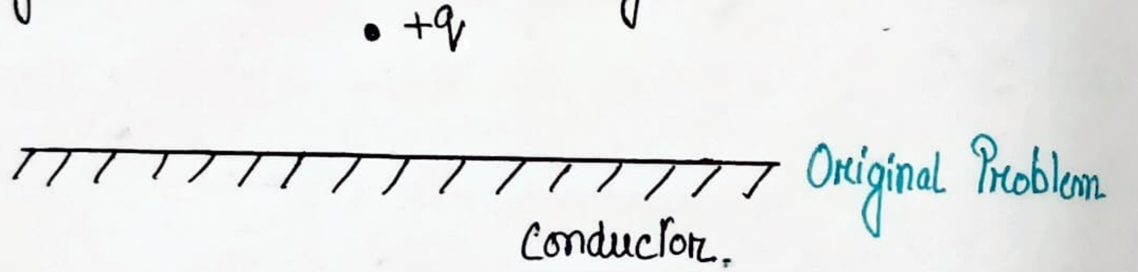
Method of images replaces the original boundary by appropriate image charges in lieu of a formal solution of Poisson's or Laplace's equation so that the original problem is greatly simplified.

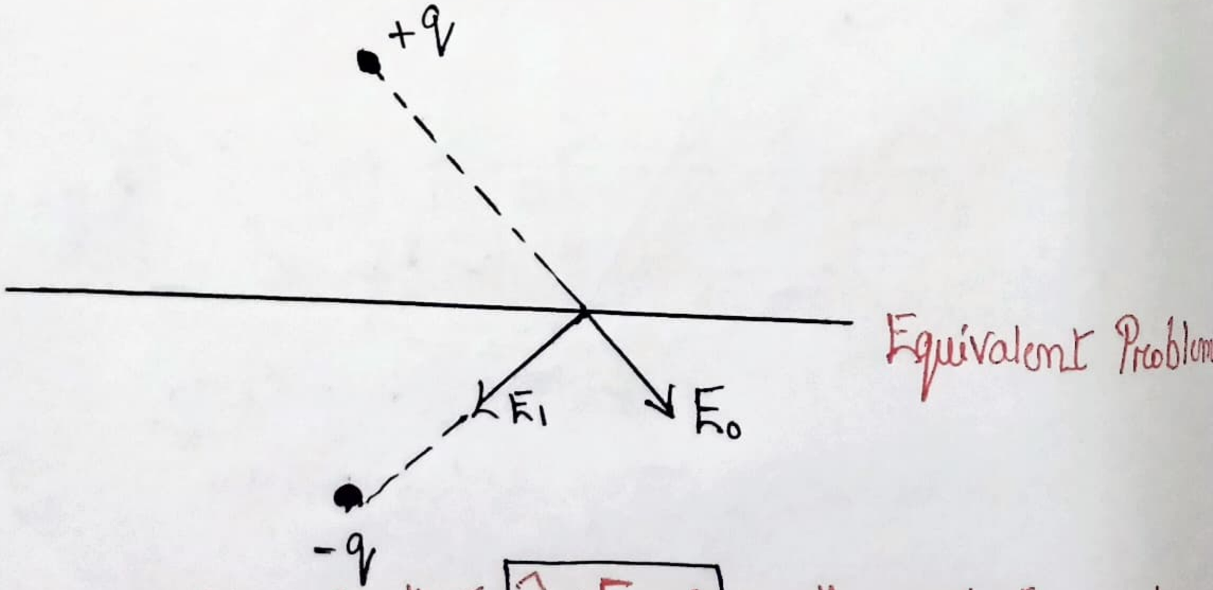
The basic principle of the method of images is the uniqueness theorem.

As long as the solution satisfies Poisson's or Laplace's Equation and the solution satisfies the given boundary condition, the simplest solution should be taken.

Method of Images | Image Theory for a flat conductor surface or a half space is quite easy to derive. To see that, we can start with electrostatics theory of putting a positive charge above a flat plane. As mentioned before, for electrostatics, the plane or half space does not have to be a perfect conductor, but only a conductor (or a metal). The tangential static electric field on the surface of the conductor has to be zero.

The tangential static electric field can be canceled by putting an image charge of opposite sign at the mirror location of the original charge. This is shown in the figure below.





The boundary condition is that $\boxed{\hat{n} \times E = 0}$ on the conductor surface.

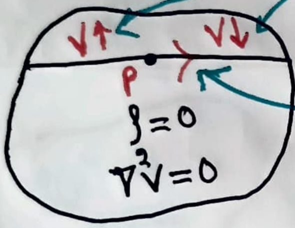
Basic Concepts

- $\nabla \cdot \vec{E} = \rho/\varepsilon_0$
- $\bar{E} = -\nabla V$
- $\nabla \cdot (\nabla \vec{V}) = -\rho/\varepsilon_0$
- $\nabla^2 V = -\rho/\varepsilon_0$

If $\rho = 0 \Rightarrow \nabla^2 V = 0$ (Laplace's Equation)

* If $\nabla^2 V = 0$ in a region

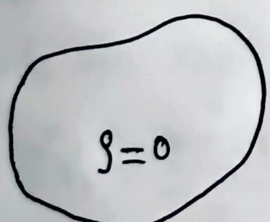
Compared to potential at point P [where P is any point inside the Region]



Direction of Electric field

(V cannot be maximum or minimum inside the region)

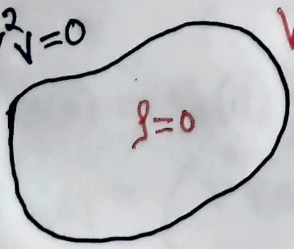
Uniqueness Theorem

- 

$V=0$ on the surface

Then $V=0$ everywhere in the volume.

- $\nabla^2 V = 0$



$V_{surface} = V_0(\vec{r})$ known at all points on the surface

Let us consider different functional forms of potential inside the region.

$V_1(\bar{r})$, $V_2(\bar{r})$

$V_3(\bar{r}) = V_1(\bar{r}) - V_2(\bar{r})$ [Consider]

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 \quad \left(\begin{matrix} \nabla^2 V_1 = 0 \\ \nabla^2 V_2 = 0 \end{matrix} \text{ By Laplace's Equation} \right)$$

$$= 0 - 0$$

$$\Rightarrow \nabla^2 V_3 = 0$$

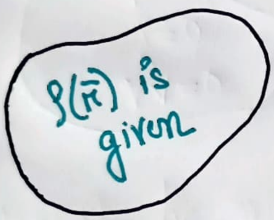
$V_3(\vec{r})$ at Surface $= 0$ [As $V_1 = V_2 = V_0$ at surface]

So $V_3(\vec{r}) = 0$ [at all points in the Volume]

$$\Rightarrow V_1(\vec{r}) = V_2(\vec{r})$$

Therefore different functional forms of potential are not possible.

Hence V is uniquely specified.

- 

$V = V_0(\vec{r})$ at the Surface

Let us consider different functional forms of potential $[V_1(\vec{r}), V_2(\vec{r})]$ inside the region.

$$\nabla^2 V_1 = \frac{-\rho(\vec{r})}{\varepsilon_0}$$

$$\nabla^2 V_2 = \frac{-\rho(\vec{r})}{\varepsilon_0}$$

$V_3(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r})$ [Consider]

$$\nabla^2 V_3(\vec{r}) = \left(\nabla^2 V_1(\vec{r}) - \nabla^2 V_2(\vec{r}) \right)$$

$$\nabla^2 V_3(\vec{r}) = \left(-\frac{\rho(\vec{r})}{\varepsilon_0} \right) - \left(-\frac{\rho(\vec{r})}{\varepsilon_0} \right) = 0$$

$\nabla^2 V_3 = 0 \quad V_3(\vec{r})_{[\text{At Surface}]} = 0 \quad [\because V_1 = V_2 = V_0, \text{ at surface}]$

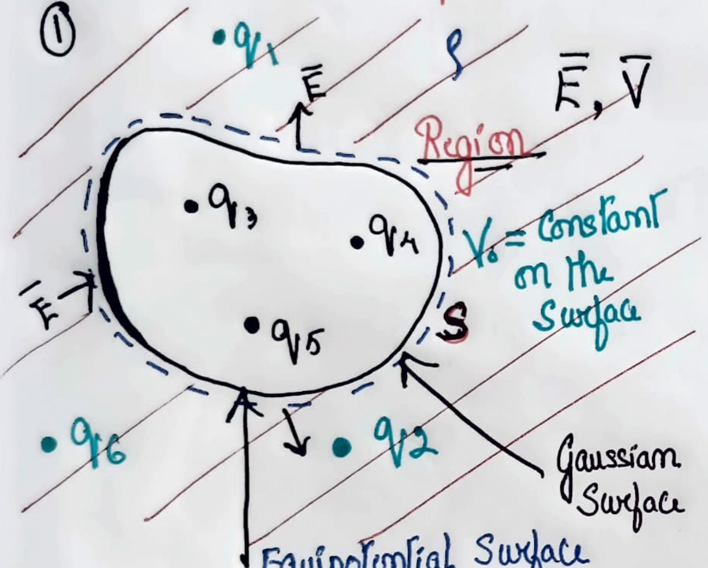
So $V_3(\bar{r}) = 0$ [at all points in the volume]

$\Rightarrow V_1(\bar{r}) = V_2(\bar{r})$

Therefore different functional forms of potential are not possible

Hence Potential is uniquely specified

Concept Behind Method of Images



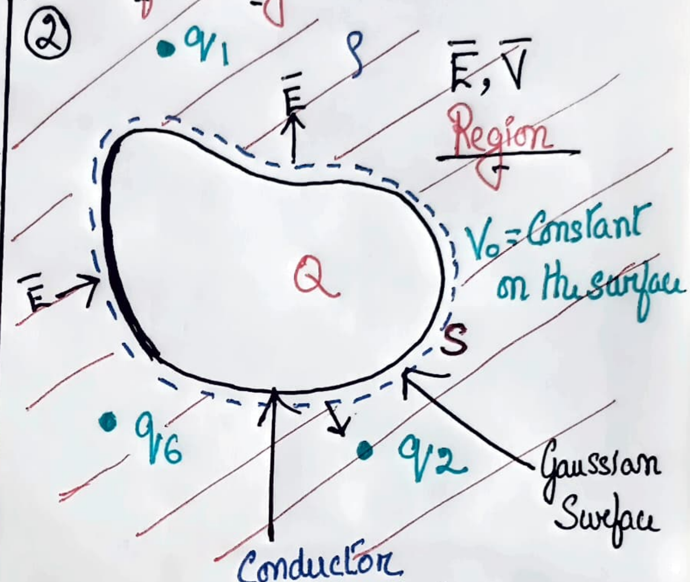
Boundary Surface $\Rightarrow$ 'S' and Infinity

- Let us consider a region of interests which has two boundary surface which is 'S' and infinity.
- Boundary Surface 'S' is an equipotential surface

Region of Interests contains charge q_1, q_6 and q_2.

- By drawing a gaussian surface over surface 'S' we get flux

Since $\oint \bar{E} \cdot d\bar{a} = \dfrac{\Sigma q_{in}}{\epsilon_0}$



Boundary Surface $\Rightarrow$ 'S' and Infinity

- Let us consider a metallic conductor of same dimensions of the equipotential surface as shown in figure 1 and generate the same physical conditions as shown in figure 1.

(For this we add charge Q and make the potential of the conductor V_0)

Region of Interests contain charge q_1, q_6, and q_2.

- By drawing a gaussian surface over surface 'S' we get Flux $\oint \bar{E} \cdot d\bar{a} = \dfrac{Q}{\epsilon_0}$

Since Region of Interests for both the figures (1 and 2) have same charge distribution ($\rho(\vec{r})$ is same), same potential at the surface (V_0) Therefore fields get specified uniquely [Uniqueness Theorem].
Hence Electric fields ($\vec{E}$) and Potential ($\vec{V}$) at the regions of both the figures are same.

$$\therefore \oint_{(\text{figure } 1)} E \cdot da = \oint_{(\text{figure } 2)} E \cdot da$$

$$\frac{\sum q_{in}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

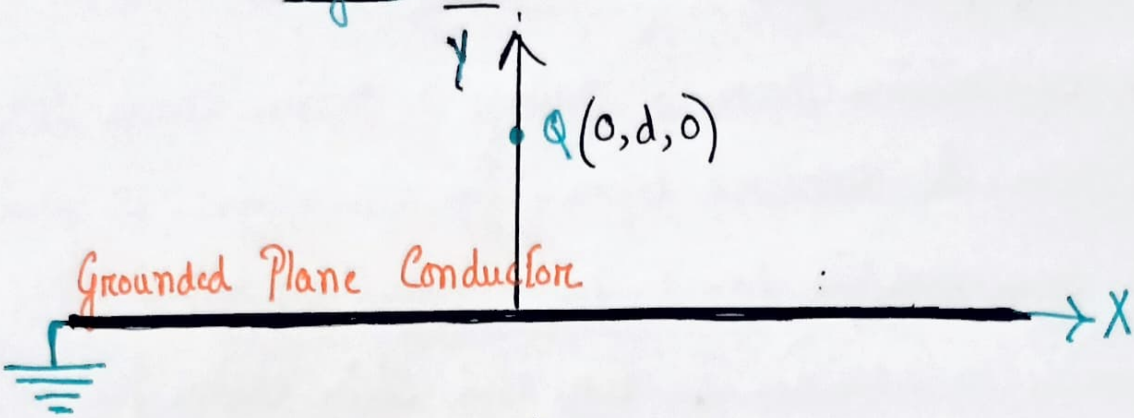
$$\therefore \boxed{Q = \sum q_{in}}$$

* From the above discussion we conclude that -

For a given charge distribution for which we can choose an equipotential surface and exactly if we place a conductor on that location of equipotential surface (the same amount of charge present in the original distribution is also provided to the conductor) then Electric field and potential outside will not change (for both the scenerios, Electric field and potential will be same).

Point Charge Over Grounded Plane Conductor

* By Direct Solution

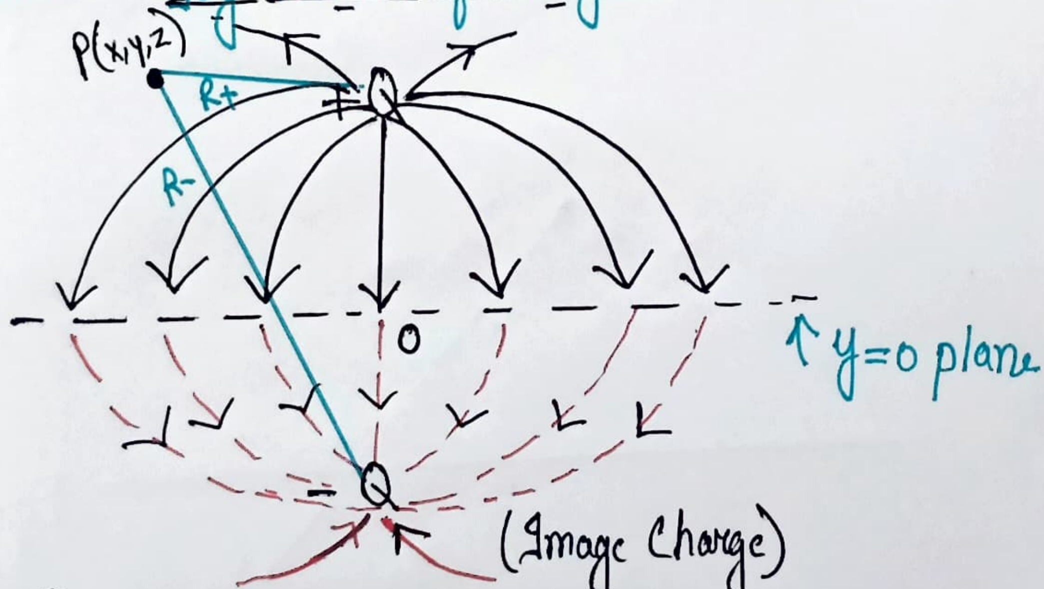


(a). Physical Arrangement

$$\boxed{V(x,y,z) = \frac{Q}{4\pi\varepsilon_0\sqrt{x^2+(y-d)^2+z^2}} + \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_s}{R_1}\,ds}$$

Where R_1 is the distance from ds to the point under consideration and S is the surface of the entire conducting plane.

* By Method of Images



(b). Image Charge and field Lines

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

* Only valid in the region of $y > 0$.

* Some Special Cases :-

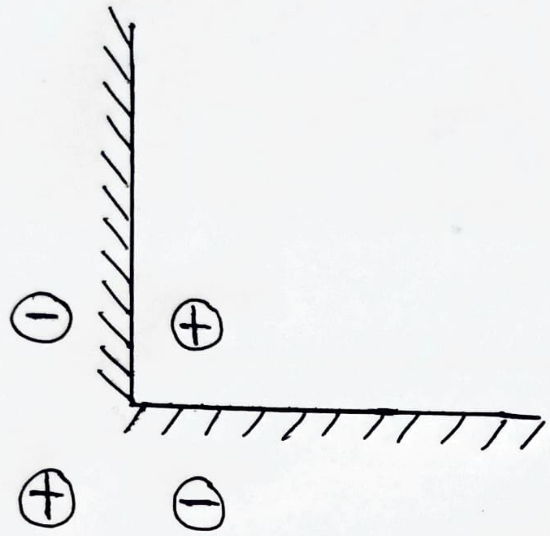
One curious case is for a static charge placed near a conductive sphere (or cylinder) as shown in figure below. A charge of $+Q$ reflects to a charge $-Q_I$ inside the sphere. For electrostatics, the sphere (or cylinder) need only be a conductor. However, this cannot be generalised to electrodynamics or a time-varying problem, because of the retardation effect: A time-varying dipole or charge will be felt at different points asymmetrically on the surface of the sphere from the original and image charges. Exact cancelation of the tangential electric field on the surface of the sphere or cylinder cannot occur for time-varying field.

When a static charge is placed over a dielectric interface, image theory can be used to find the closed form solution. This solution can be derived using Fourier transform technique. It can also be extended to multiple interfaces. But image theory cannot be used for the electrodynamic case due to the different speed of light in different media, giving rise to different retardation effects.

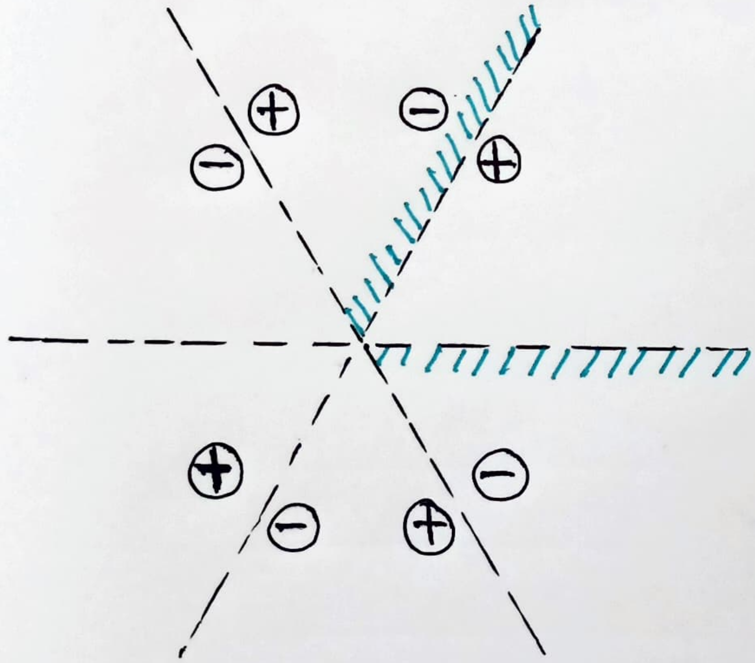
Image Theory for Multiple Images

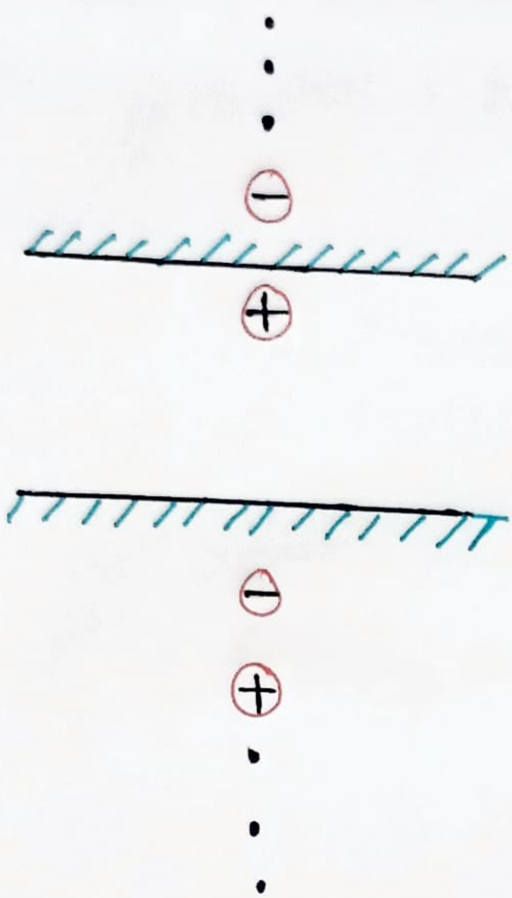
(Method of Images)

(a).

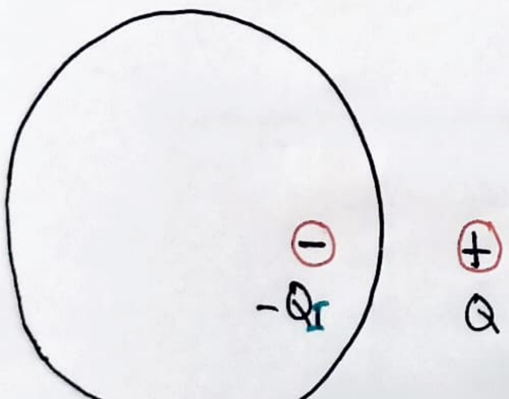


(b).

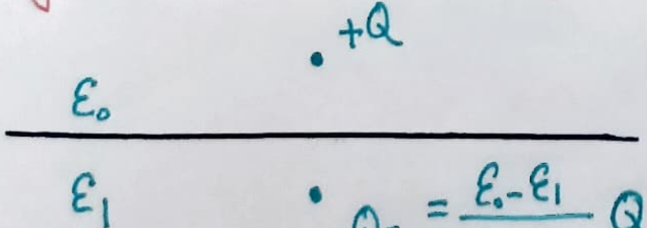




* Image Theory for a point charge near a cylinder or a sphere can be found in closed form



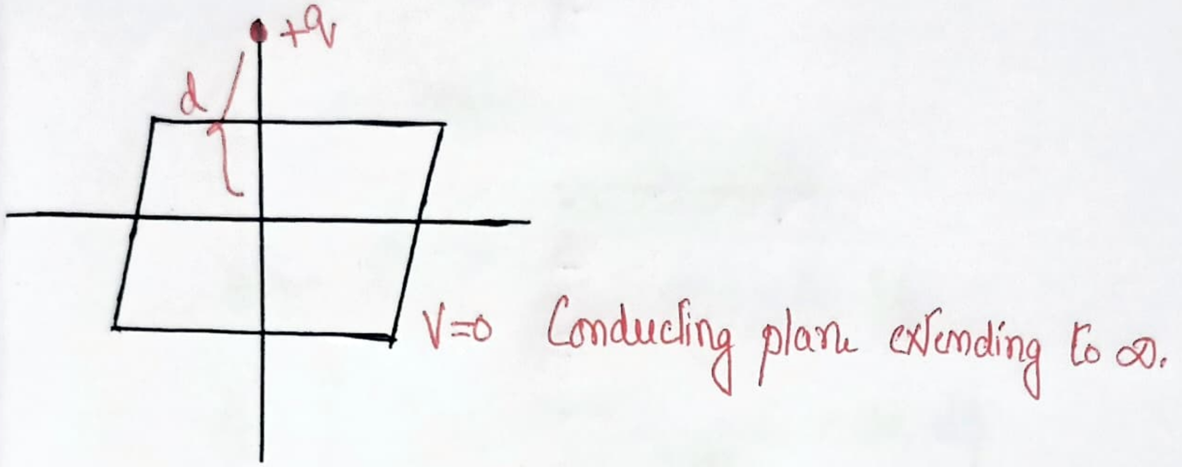
* A static charge over a dielectric interface can be found in closed form.

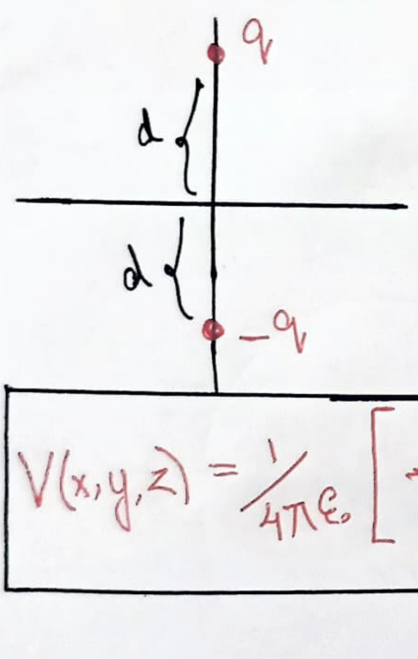


$$Q_I = \frac{\varepsilon_0 - \varepsilon_1}{\varepsilon_0 + \varepsilon_1} Q$$

* **The Classic Image Problem :-**

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. What is the potential in the region above the plane ?





$$\boxed{V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]}$$

$$\left(\begin{array}{l} V=0, \text{ when } z=0 \\ V\to 0, \ x^2+y^2+z^2 \gg d^2 \end{array}\right)$$

$$\boxed{\sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}}$$

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$\boxed{\sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

∴ Total Induced Charge

$$Q = \int \sigma \, da$$

$$\sigma(r) = \frac{-qd}{2\pi (r^2 + d^2)^{3/2}}$$

where $r^2 = x^2 + y^2$ $\quad da = r \, dr \, d\phi$

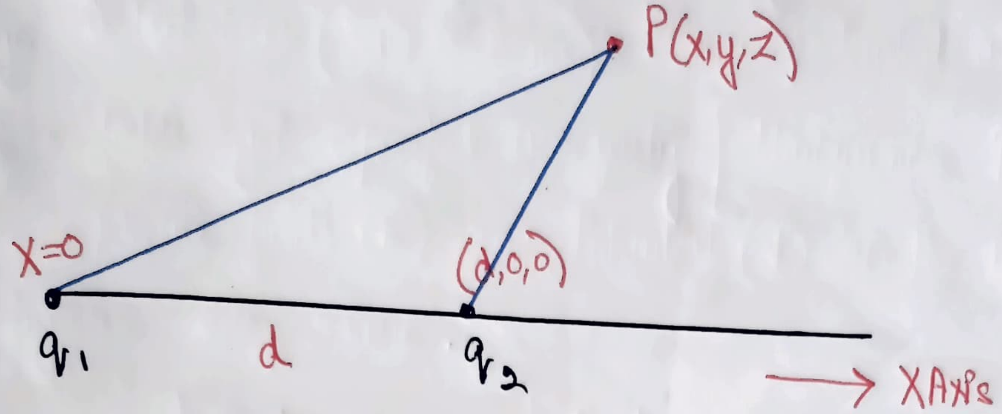
$$Q = \int_0^{2\pi} \int_0^{\infty} \frac{-qd}{2\pi (r^2 + d^2)^{3/2}} \, r \, dr \, d\phi$$

$$Q = \frac{qd}{\sqrt{r^2 + d^2}} \Bigg|_0^{\infty}$$

$$Q = -q$$

∴ The total charge induced on the plane is '$-q$'. ~~as~~

Concept Required for Solving Presentation Problem



Let us consider a distribution with two charges q_1 and q_2 placed a separation 'd' on X Axis. P is a point with position coordinates (x, y, z)

Total Potential at P

$$\frac{q_1}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}} + \frac{q_2}{4\pi\varepsilon_0\sqrt{(x-d)^2+y^2+z^2}} = V_0 = 0$$

(Consider Equipotential Surface at P with $V_0 = 0$)

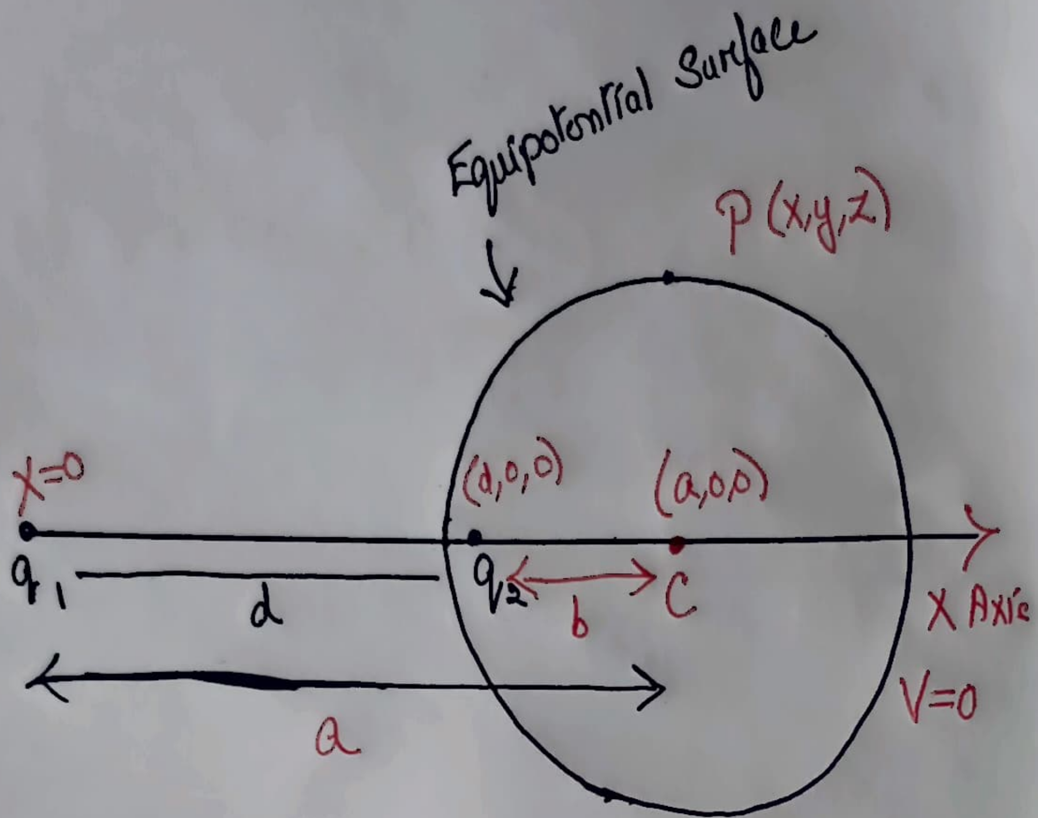
$$\frac{q_1}{\sqrt{x^2+y^2+z^2}} = \frac{-q_2}{\sqrt{(x-d)^2+y^2+z^2}}$$

$$q_1^2\left[(x-d)^2+y^2+z^2\right] = q_2^2\left[x^2+y^2+z^2\right]$$

$$\left(x^2+y^2+z^2\right)\left(q_1^2-q_2^2\right) - 2q_1^2xd + q_1^2d^2 = 0$$

(This is the equation of sphere)

$$\left[\begin{array}{c}(x-a)^2+(y-b)^2+(z-c)^2 = R^2 \\ \text{Center } (a,b,c) \\ \text{Radius } R\end{array}\right]$$



$$a = \frac{q_1^2 d}{q_1^2 - q_2^2}$$

$$b = a - d$$

$$b = \frac{q_2^2 d}{q_1^2 - q_2^2}$$

$$R = -\frac{(q_1 q_2 d)}{(q_1^2 - q_2^2)}$$

$$\frac{a}{R} = -\frac{q_1}{q_2}$$

$$b = \frac{q_2^2 d}{q_1^2 - q_2^2}$$

* $\boxed{q_2 = -\frac{R}{a} q_1}$ (q_2 plays the role of Image Charge)

$$\frac{a}{b} = \frac{q_1^2}{q_2^2} = \frac{q_1^2}{\left(\frac{R}{a}\right)^2 q_1^2} = \frac{a^2}{R^2}$$

* $\boxed{b = \frac{R^2}{a}}$ (b is the distance of image charge from centre of sphere)

Based on the above discussion -

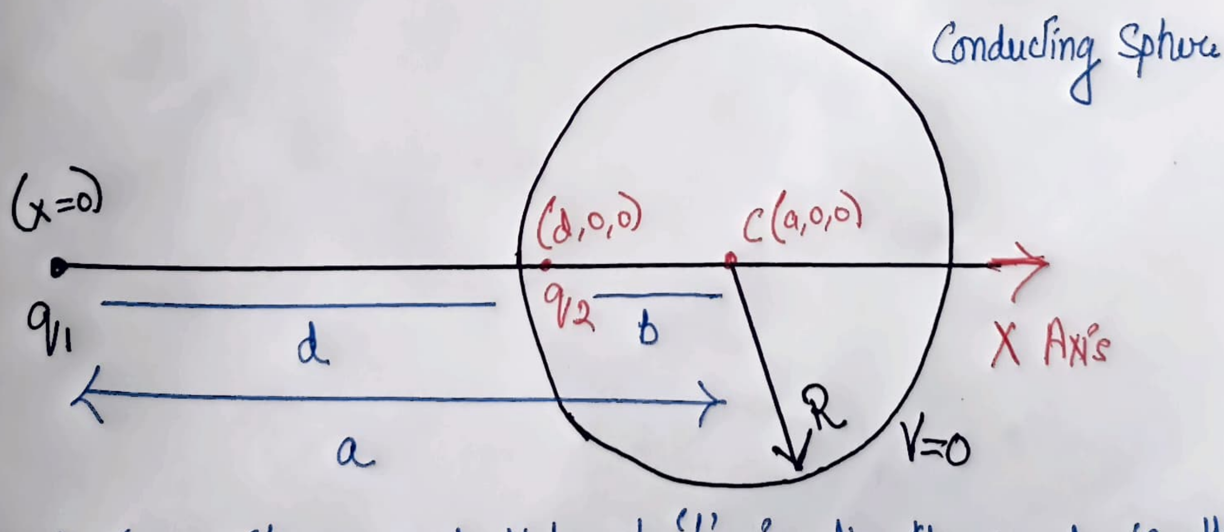
We can present a $\boxed{\text{new geometry}}$ where we consider a spherical surface of radius R centered at $C(a,0,0)$. (conducting sphere)

We place a charge q_1 at a distance 'a' from center of the sphere (Along X Axis). To maintain potential ($V=0$) an image charge q_2 is formed valued -

$$\boxed{q_2 = \frac{-R}{a} q_1}$$ (Magnitude of Image Charge)

q_2 is formed at a distance 'b' from center of the sphere -

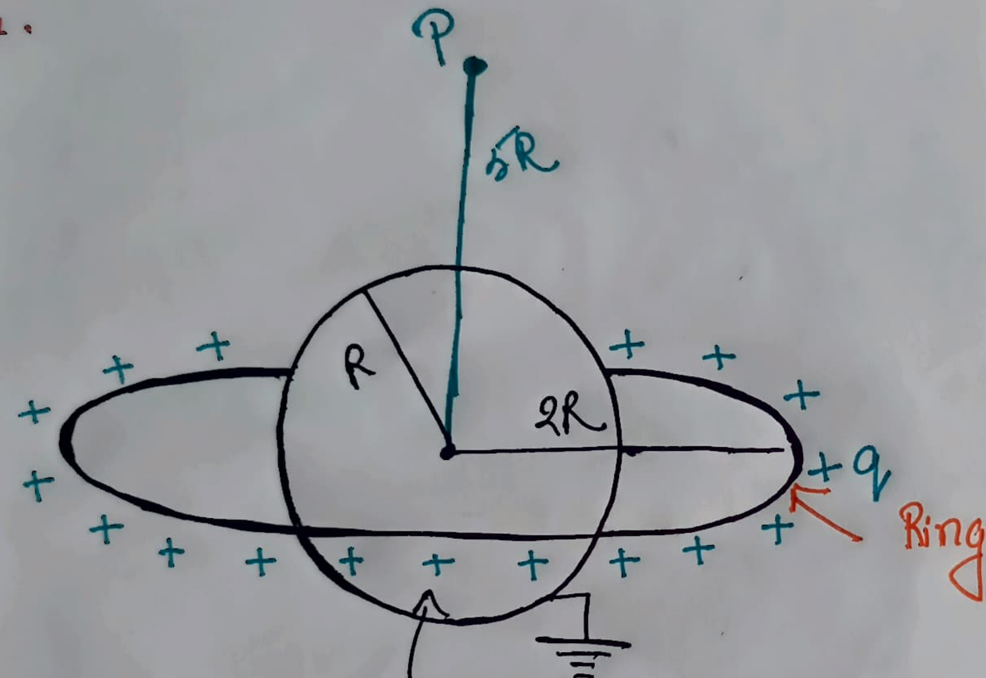
$$\boxed{b = \frac{R^2}{a}}$$ (Where a is distance of original charge from center of sphere)



* Concept of Image Charge and Value of 'b' (Formula) is directly used in the presentation problem.

(MAIN PROBLEM OF THE PRESENTATION)

Q. A grounded metallic sphere of radius R is surrounded by a concentric ring of radius 2R. The ring carries a charge 'q' uniformly distributed on its length. Find the electric potential at a point on the axis of the ring at a distance 5R from the centre.



Grounded Metallic Sphere

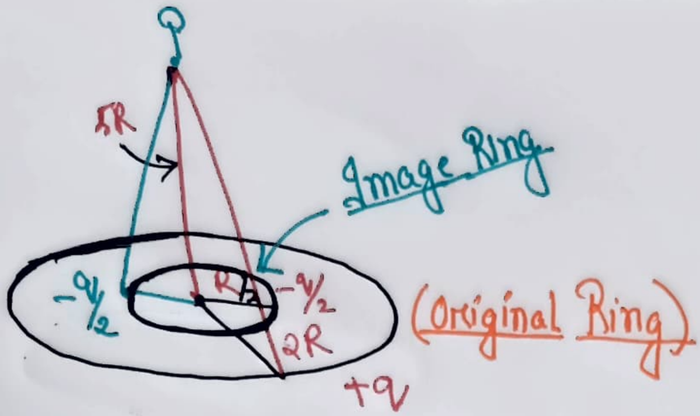
Image Charge $= -\dfrac{Rq}{a}$ [Where R = Radius of the Sphere
a = Distance of Charge from centre of Sphere

$= -\dfrac{R}{2R}dq$

$= -\dfrac{dq}{2}$

$b = \dfrac{R^2}{a} = \dfrac{R^2}{2R} = \dfrac{R}{2}$ [Where b = Distance of Image Charge from Centre of Sphere]

Derivation for Image Charge and Distance (b) is shown in previous concept.



Potential due to Original Ring at point P

$$= \frac{q}{4\pi\varepsilon_0 \sqrt{(5R)^2 + (2R)^2}}$$

$$= \frac{q}{4\pi\varepsilon_0 R\sqrt{29}}$$

Potential due to the Image Ring at point P

$$= \frac{-q}{2} \frac{1}{4\pi\varepsilon_0 \sqrt{(5R)^2 + \left(\frac{R}{2}\right)^2}}$$

$$= \frac{-q}{4\pi\varepsilon} \frac{1}{\sqrt{100R^2 + R^2}}$$

$$= -\frac{q}{4\pi\varepsilon_0} \frac{1}{R\sqrt{101}}$$

$\therefore$ Potential ~~due~~ at point P [Overall]

$$= \frac{q}{4\pi\varepsilon_0 R\sqrt{29}} + \left(\frac{-q}{4\pi\varepsilon_0 R\sqrt{101}}\right)$$

$$V_P = \frac{q}{4\pi\varepsilon_0 R}\left(\frac{1}{\sqrt{29}} - \frac{1}{\sqrt{101}}\right)$$

References :-

- Griffiths, D. J. (1999). Introduction to Electrodynamics, (Potentials) page - (124-126)