Q. A conical surface carries a uniform surface charge density σ. The height of the cone is h as is the radius of the top. Find the potential difference between points A (the vertex) and B (the centre of the top).

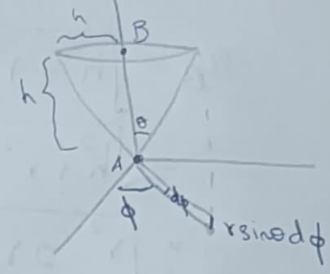
Ans In the given cone,

radius of the top $= h$

Height $= h$

$\therefore$ The angle $\theta = \tan^{-1}\left(\frac{h}{h}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}$



We need to find the potential difference between the points A and B

ie $\cancel{V_{AB}}$ $V_B - V_A$.

To find V_A

Let us consider a infinitesimal area dA ~~charge~~ on the cone at a distance r. The electric potential at A due to that infinitesimal charge is,

$$dV_a = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dA}{r_A} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dA}{r}$$

$dA = (dr)(r\sin\theta\, d\phi)$ using spherical coordinates.

$$V_a = \frac{1}{4\pi\varepsilon_0}\int_0^{h\sqrt{2}}\int_0^{2\pi}\frac{\sigma\, r\sin\theta\, d\phi\, dr}{r}$$

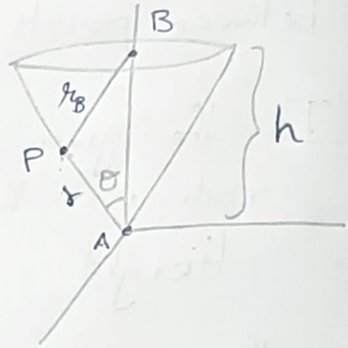
$$= \frac{1}{4\pi\varepsilon_0}2\pi\int_0^{h\sqrt{2}}\frac{\sigma}{\sqrt{2}}\,dr \qquad \left(\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\frac{2\pi\, h\sqrt{2}}{\sqrt{2}} = \frac{\sigma h}{2\varepsilon_0}$$

To find V_B

Similar to how we went to find potential at A

$$dV_B = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\, dA}{r_B}$$



$$V_B = \frac{1}{4\pi\varepsilon_0}\iint \frac{\sigma\, r\sin\theta\, d\phi\, dr}{r_B}$$

$$r_B = \sqrt{h^2 + r^2 - 2hr\cos\frac{\pi}{4}}$$ (cosine rule on ΔABP)

$$= \sqrt{h^2 + r^2 - \sqrt{2}hr}$$

$$= \sqrt{r^2 - 2r\frac{h}{\sqrt{2}} + \left(\frac{h}{\sqrt{2}}\right)^2 + \left(\frac{h}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\left(r - \frac{h}{\sqrt{2}}\right)^2 + \left(\frac{h}{\sqrt{2}}\right)^2}.$$

$$V_b = \frac{1}{4\pi\varepsilon_0}\int_0^{h\sqrt{2}}\int_0^{2\pi} \frac{\sigma \frac{r}{\sqrt{2}}\, d\phi\, dr}{\sqrt{\left(r - \frac{h}{\sqrt{2}}\right)^2 + \left(\frac{h}{\sqrt{2}}\right)^2}}$$

put $r - \frac{h}{\sqrt{2}} = \frac{h}{\sqrt{2}}\tan x$

$dr = \frac{h}{\sqrt{2}}\sec^2 x\, dx$

$r = \frac{h}{\sqrt{2}}(1 + \tan x)$

limits of integral

$r = h\sqrt{2}$

$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

$r = 0$

$\Rightarrow \tan x = -1 \Rightarrow x = -$

$$= \frac{2\pi}{4\pi\varepsilon_0}\int_{-\pi/4}^{\pi/4} \frac{\sigma \frac{h}{\sqrt{2}\cdot\sqrt{2}}(1+\tan x)\frac{h}{\sqrt{2}}\sec^2 x\, dx}{\sqrt{\left(\frac{h}{\sqrt{2}}\right)^2(\tan^2 x + 1)}}$$

$$= \frac{\sigma}{2\varepsilon_0}\int_{-\pi/4}^{\pi/4} \frac{\frac{h^2}{2\sqrt{2}}(1+\tan x)\sec^2 x\, dx}{\frac{h}{\sqrt{2}}\sec x}$$

$$= \frac{\sigma}{2\varepsilon_0}\frac{h}{2}\int_{-\pi/4}^{\pi/4}(\sec x + \sec x\tan x)\,dx$$

$$= \frac{\sigma h}{4\varepsilon_0}\Big[\ln(\sec x + \tan x) + \sec x\Big]_{-\pi/4}^{\pi/4}$$

$$= \frac{\sigma h}{4\varepsilon_0}\ln(\sqrt{2}+1) + \sqrt{2} - \ln(\sqrt{2}-1) - \sqrt{2}$$

$$= \frac{\sigma h}{4\varepsilon_0} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\therefore V_B - V_A = \frac{\sigma h}{4\varepsilon_0} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \frac{\sigma h}{2\varepsilon_0}$$

$$= \frac{\sigma h}{2\varepsilon_0}\left[\frac{1}{2} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 1\right]$$