Ques: Justify Earnshaw's theorem and Prove that a charge +q in centre of cube having charge +Q at each corner is in unstable equilibrium.

Ans:

Earnshaw's theorem = "It is impossible for a point charge (or a pole) to have a position of stable equilibrium when an inverse square law relates the force and distance".

Proof:

consider a particle in electric field $E(x,y,z)$:

force on particle at point x_0, y_0, z_0 $= F_0(x_0, y_0, z_0) = E(x_0, y_0, z_0) \times q$.

force point charge (unit) $=$ $F_0(x_0 y_0 z_0) = E_0(x_0, y_0, z_0)$

if this point is a point of stable equilibrium, then

$$F(x_0, y_0, z_0) = 0 \qquad -①$$

and for stability

$$\nabla \cdot F(x_0, y_0, z_0) < 0 \qquad -②$$

However, if $\vec{F}$ is an irrotational field then

$$F(x,y,z) = -\nabla \varphi(x,y,z) \qquad -③$$

$\varphi \rightarrow$ scalar potential

These condition are satisfied by electric field with its potential V.

$$\boxed{E = -\nabla V} \quad -④$$

necessary condition for stable equilibrium in terms of scalar potential are:

$$\nabla \Psi(x,y,z) = 0 \quad -⑤$$

$$\nabla^2 \Psi(x,y,z) > 0 \quad -6$$

In charge free region, field intensity vector E(x,y,z) is solenoidal as well as rotational, implies

$$\nabla \cdot E(x,y,z) = 0 \quad -⑦$$
$$\nabla \times E(x,y,z) = 0 \quad -8$$

~~from~~ on combining eq ⑧ and ④

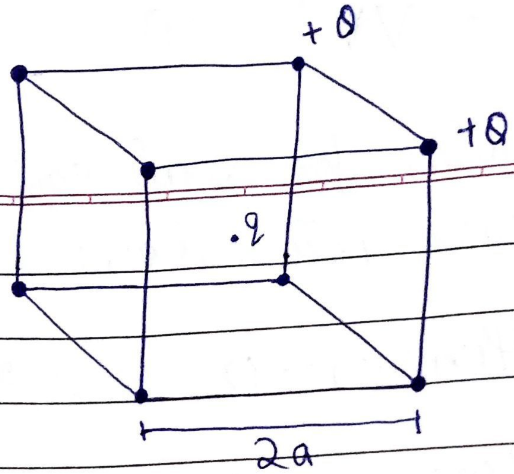
$$-\nabla \times \nabla \cancel{V} = 0$$
$$\boxed{\nabla^2 V(x,y,z) = 0} \quad ⑨$$

which doesn't satisfy eq ⑥ hence not condition of stability

For any charged particle of charge Q, it is true

$$F(x,y,z) = Q E(x,y,z)$$
from ⑦ $$\nabla \cdot F(x,y,z) = 0$$

hence, a charged particle can't be in stable equilibrium under electric field alone.



q = unit charge

∴ By Earnshaw's theorem, we can conclude that unit charge present at centre will be in unstable equilibrium

⇒ my solution and view to solve this problem.

we wish to prove that unit charge is in unstable equilibbrium so ~~not~~ net force be zero at position

Since 8 charges are placed symmetrically opposite to each other

so $\boxed{F_{net} = 0}$

so

it is in equilibrium

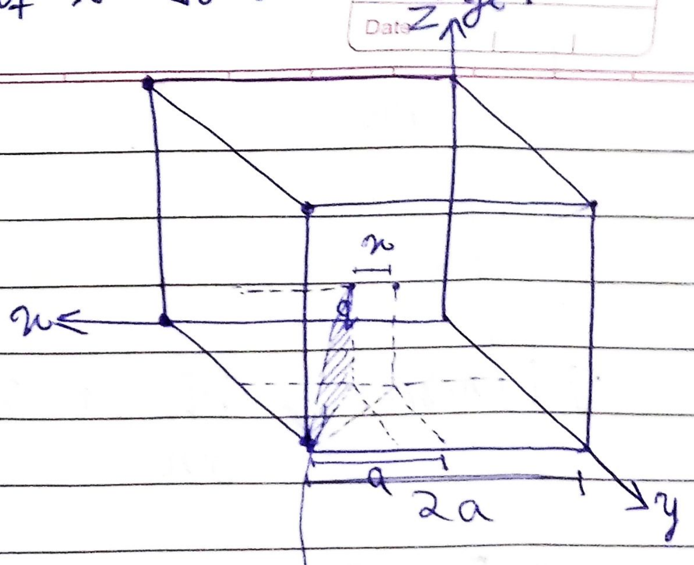
For unstability

⇒ if it is unstable, the direction of force should be along the direction of motion on little displacement. and should be +ve.

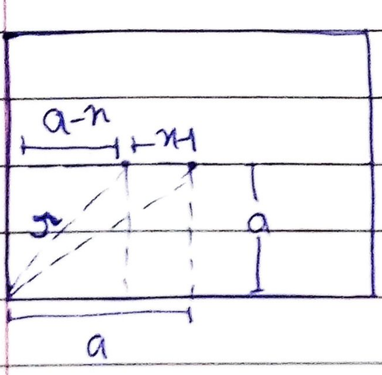
$$\boxed{(\nabla \cdot \vec{F}(x,y,z)) > 0}$$

for unstability

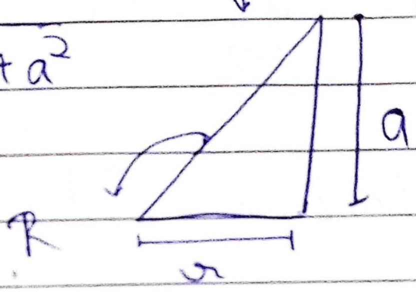
Lets give a displacement of x to unit charge, in direction $\hat{x}$



the x y plane



$r = \sqrt{(a-x)^2 + a^2}$



$R = \sqrt{r^2 + a^2}$

$\boxed{R = \sqrt{3a^2 + x^2 - 2ax}}$ — (I)

for four charges towards unit charge approached, distance $R_1 = \sqrt{3a^2 + x^2 - 2ax}$ — (I)

similarly, for other four charges from which it moved away

$R_2 = \sqrt{3a^2 + x^2 + 2ax}$ — (II)

net potential of displaced unit charge

$$V_{net} = \frac{4Q}{4\pi\varepsilon_0\sqrt{3a^2+x^2-2ax}} + \frac{4Q}{4\pi\varepsilon_0\sqrt{3a^2+x^2+2ax}}$$

$$V_{net} = \frac{Q}{\pi\varepsilon_0}\left(\frac{1}{\sqrt{3a^2+x^2-2ax}} + \frac{1}{\sqrt{3a^2+x^2+2ax}}\right)$$ — (III)

Since displacement is only in x-direction

$$F = -\frac{dV}{dx}$$

$$F = -\frac{dV}{dx} = \left(\frac{d}{dx}\left(\frac{Q}{\pi\varepsilon_0}\left(\frac{1}{\sqrt{3a^2+x^2-2ax}} + \frac{1}{\sqrt{3a^2+x^2+2ax}}\right)\right.\right.$$

$$-\frac{dV}{dx} = -\frac{d}{dx}\left[\frac{Q}{\pi\varepsilon_0}\left(\frac{1}{\sqrt{3a^2+x^2-2ax}} + \frac{1}{\sqrt{3a^2+x^2+2ax}}\right)\right]$$

$$= \frac{-Q}{\pi\varepsilon_0}\left(\frac{d}{dx}\frac{1}{\sqrt{3a^2+x^2-2ax}} + \frac{d}{dx}\frac{1}{\sqrt{3a^2+x^2+2ax}}\right)$$

$$= \frac{-Q}{\pi\varepsilon_0}\left[\frac{-1}{2}\frac{(2x-2a)}{(3a^2+x^2-2ax)^{3/2}} + \left(\frac{-1}{2}\right)\frac{(2x+2a)}{\sqrt{3a^2+x^2+2ax}}\right)$$

$$F_{net} = \frac{Q}{\pi\varepsilon_0}\left[\frac{(x-a)}{(3a^2+x^2-2ax)^{3/2}} + \frac{(x+a)}{(3a^2+x^2+2ax)^{3/2}}\right]$$

$$= \frac{Q}{\pi\varepsilon_0}\left[\frac{(a+x)}{(3a^2+x^2+2ax)^{3/2}} - \frac{(a-x)}{(3a^2+x^2-2ax)^{3/2}}\right]$$

For a very small displacement, (x^2-2ax) can be neglected.

$$= \frac{Q}{\pi\varepsilon_0}\left[\frac{(a+x)}{(3a^2)^{3/2}} - \frac{(a-x)}{(3a^2)^{3/2}}\right]$$

$$\vec{F}_{net} = \frac{Q}{4\pi\varepsilon_0}\frac{2x}{(3a^2)^{3/2}}\,\hat{i}$$

$$\vec{F}_{net} = \frac{2xQ}{\pi\varepsilon_0 (3a^2)^{3/2}}\,\hat{i}$$

$$\nabla\cdot F(x,y,z) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{2Qx}{\pi\varepsilon_0(3a^2)^{3/2}}\right) + 0 + 0$$

$$= \frac{2Q}{\pi\varepsilon_0(3a^2)^{3/2}} > 0$$

hence

$$\boxed{\nabla\cdot F(x,y,z) > 0}$$

hence it is in unstable equilibrium.