PHYSICS Presentation of question

RITHUVARNA. K.M
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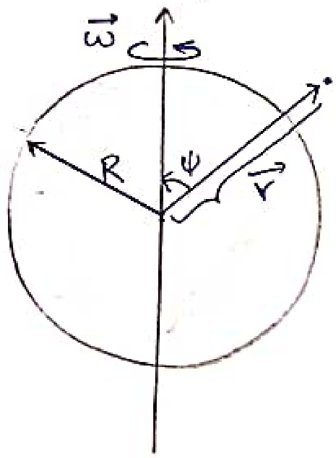
Q. A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\vec{\omega}$. Find the vector potential it produces at point $\vec{r}$.



ans.

⇒ The surface charge on spherical shell is σ.

⇒ When the spherical shell is set spinning at angular velocity $\vec{\omega}$, the charge present on the surface is also spining along with it, that is it is moving. Due to this surface current density will be produced.

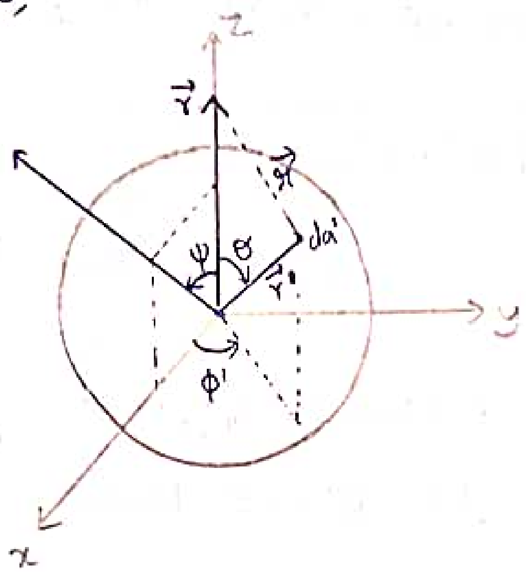
⇒ Due to this surface current density, a magnetic field is produced in the sphere. If magnetic field is produced, the magnetic vector potential will also be there.

$$\because \vec{B} = \vec{\nabla} \times \vec{A}.$$

here in this question we want to find the value of magnetic vector potential.

⇒ It might seem natural to set the polar axis along $\vec{\omega}$, but in fact the integration is easier if we let $\vec{r}$ lie on the z axis, so that $\vec{\omega}$ is tilted at an angle ψ.

We may as well orient the x-axis so that $\vec{\omega}$ lies in xz plane, as shown below,



⇒ Now, there can be a point on z axis, where the magnetic field due to surface current produced on a small area element da' can be felt.

⇒ So, the source point ($\vec{r}'$ from origin) and field point ($\vec{r}$ from origin) is what shown in the figure. The seperation vector is $\vec{\mathscr{r}}$.

⇒ The angle between the z-axis and the axis about which the sphere rotates is ψ.

⇒ The spherical coordinate of the source point is (r', θ', ϕ')

⇒ So now we want to find the magnetic potential at that point (on z axis) due to the surface current produced on the sphere.

⇒ This point can be inside or outside sphere. If $r < R$ then it is inside and if $r > R$ it is outside sphere.

we know, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{K(\vec{r}\,')\,da'}{\mathscr{r}}$ —— ①.

where $\vec{\mathscr{r}}$ is the separation vector.

da' → small area element of the source coordinate

$K(\vec{r}\,')$ → surface current density.

$\mathscr{r} = |\vec{\mathscr{r}}| = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$ (finding unknown side of a triangle).

(r' is R here) surface of shell.

Value of $d\vec{a}$ in terms of spherical coordinate is

$d\vec{a}' = R^2\sin\theta'\,d\theta'\,d\phi'$ $\left(d\vec{a}' = R\sin\theta'\,d\phi' \cdot R\,d\theta' = R^2\sin\theta'\,d\theta'\,d\phi'\right)$

we know the formulas,

$\vec{K} = \sigma\vec{v}$ & $\vec{v} = \vec{\omega}\times\vec{r}\,'$

(The velocity of a point $\vec{r}\,'$ in a rotating rigid body).

$\therefore \vec{K} = \sigma(\vec{\omega}\times\vec{r}\,')$

put in ① $\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\sigma(\vec{\omega}\times\vec{r}\,')\,d\vec{a}'}{\mathscr{r}}$

$\vec{w}$ is in xz plane, it can be resolved as, → $w\sin\psi\,\hat{x} + w\cos\psi\,\hat{z}$

$$\therefore \vec{v} = \vec{\omega}\times\vec{r}\,' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w\sin\psi & 0 & w\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\phi' \end{vmatrix}$$

$$\vec{\omega}\times\vec{r}\,' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w\sin\psi & 0 & w\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\phi' \end{vmatrix}$$

where $\vec{\omega}$ & $\vec{r}\,'$ are resolved into components

$$\Rightarrow \quad \vec{v} = R\omega\Big[-(\omega\cos\psi\,\sin\theta'\cos\phi')\hat{x} + (\cos\psi\,\sin\theta'\cos\phi' - \sin\psi\,\cos\theta')\hat{y} + (\sin\psi\,\sin\theta'\sin\phi')\hat{z}\Big] \quad \text{—— (2)}$$

Now,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')\,da'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')\,R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma\vec{v}\,da'}{\mathscr{r}} = \frac{\mu_0}{4\pi}\int \frac{\sigma(\vec{\omega}\times\vec{r}\,')R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma\left(R\omega\left[-(\omega\cos\psi\sin\theta'\cos\phi')\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{y} + (\sin\psi\sin\theta'\sin\phi')\hat{z}\right]\right)R^2\sin\theta'\,d\theta'\,d\phi'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}$$

only this term left.

$$\int_0^{2\pi}\sin\phi'\,d\phi' = 0 = \int_0^{2\pi}\cos\phi'\,d\phi'$$

$$\vec{A} = -\frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\int_0^{\pi}\frac{\cos\theta'\sin\theta'\,d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\,\hat{y}$$

Let $u=\cos\theta' \Rightarrow du = -\sin\theta'\,d\theta'$

$$\vec{A} = \frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\int_0^{\pi}\frac{\cos\theta'(-\sin\theta'\,d\theta')}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\,\hat{y}$$

For limits
when $\theta'=0 \Rightarrow u=\cos 0=1$
when $\theta'=\pi \Rightarrow u=\cos\pi=-1$

$$\vec{A} = \frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\int_1^{-1}\frac{u\,du}{\sqrt{R^2+r^2-2Rru}}\,\hat{y}$$

$$\vec{A} = -\frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\int_{-1}^{1}\frac{u\,du}{\sqrt{R^2+r^2-2Rru}}\,\hat{y} \quad \text{—— (3)}$$

take $v = R^2 + r^2 - 2Rru$

$\therefore dv = -2Rr\,du$

when $u = -1 \Rightarrow v = (R+r)^2$

$u = +1 \Rightarrow v = (R-r)^2$

$\therefore$ the integral becomes.

$$\int_{(R+r)^2}^{(R-r)^2} \frac{v^2 - R^2 - r^2}{\sqrt{v}\,(-2Rr)} \frac{dv}{-2Rr}$$

$$= \frac{1}{4R^2r^2} \int_{(R+r)^2}^{(R-r)^2} \sqrt{v}\,dv - \frac{(R^2+r^2)\,dv}{\sqrt{v}}$$

on evaluation this becomes

$$\frac{1}{4R^2r^2}\left[\frac{2v\sqrt{v}}{3} \Big|_{(R+r)^2}^{(R-r)^2} - (R^2+r^2)\,2\sqrt{v} \Big|_{(R+r)^2}^{(R-r)^2} \right.$$

$$= \frac{-1}{3R^2r^2} \left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r) \right]$$

Inside the sphere For $R > r$

$$= \frac{1}{-3R^2r^2} \left[(R^2+r^2+Rr)(R-r) - (R^2+r^2-Rr)(R+r) \right]$$

$$= -\frac{1}{3R^2r^2} \left[(R^3 - \cancel{rR^2} + \cancel{Rr^2} - r^3 + \cancel{rR^2} - \cancel{Rr^2}) \right.$$
$$\left. - (R^3 + \cancel{rR^2} + \cancel{Rr^2} + r^3 - \cancel{rR^2} - \cancel{Rr^2}) \right]$$

$$= \frac{-1}{3R^2r^2} \left[(R^3 - r^3) - (R^3 + r^3) \right]$$

$$\therefore \int_{-1}^{1} \frac{u\,du}{\sqrt{R^2+r^2-2Rru}} = \frac{2r^3}{3R^2r^2} = \frac{2r}{3R^2} \quad //$$

Inside the sphere for $R > r$.

Similary for $r > R$ can be done.

which gives,

$$\int_{-1}^{1} \frac{u\,du}{\sqrt{R^2+r^2-2Rru}} = \frac{2R}{3r^2} \quad \text{outside the sphere for } r > R.$$

Now, (3) becomes.

outside the sphere For ~~R>r~~ $R<r$

$$\vec{A} = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2}\left(\frac{2R}{3r^2}\right)\hat{y}$$

$$\vec{A} = \frac{\mu_0 R^3 \sigma(-\omega r \sin\psi\,\hat{y})}{2}\left(\frac{2R}{3r^3}\right) \quad \text{and} \quad \vec{\omega}\times\vec{r} = -\omega r \sin\psi\,\hat{y}$$

$$\vec{A} = \frac{\mu_0 R^4 \sigma(\vec{\omega}\times\vec{r})}{3r^3}$$

Inside the sphere for $R > r$

$$\vec{A} = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$$

i.e, $$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}), & \text{for points inside the sphere,} \\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}), & \text{for points outside the sphere} \end{cases}$$

extra

In natural spherical coordinates, r has angle θ making with z-axis.

Let ω is along z axis, angle between ω and r is θ. Then.

$$\vec{\omega}\times\vec{r} = -\omega r \sin\theta\,\hat{\phi}$$

$$\boxed{\vec{A} = \frac{\mu_0 R^4\sigma}{3r^3}\,\vec{\omega}\times\vec{r} = \frac{\mu_0 R^4\sigma}{3r^3}(-\omega r \sin\theta\,\hat{\phi}) = -\frac{\mu_0 \omega R^4 \sigma}{3r^2}\sin\theta\,\hat{\phi}}$$

$$\vec{A}_{(r,\theta,\phi)} = -\frac{\mu_0 \omega R^4 \sigma}{3r^2}\sin\theta\,\hat{\phi} \quad \text{for outside sphere.}$$

and inside the sphere.

$$\vec{A}(r,\theta,\phi) = -\frac{\mu_0 \omega R \sigma}{3} r \sin\theta \,\hat{\phi}$$

curiously, the field inside this spherical shell is uniform:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{2\mu_0 R \omega \sigma}{3} \left(\cos\theta \,\hat{r} - \sin\theta \,\hat{\theta}\right)$$

$$= \frac{2}{3} \mu_0 \sigma R \omega \,\hat{z} = \frac{2}{3} \mu_0 \sigma R \vec{\omega}$$