Question: If the number of surfaces are fixed in a position and a given total charge is fixed on a surface, then the electrostatic energy in the region bounded by the surfaces in an absolute minimum where the charges are so placed that every surface is an equipotential as happens in the case of conductors.

Solution:

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \\ \vec{\nabla}\times\vec{E} &= 0 \\ \vec{E} &= -\vec{\nabla}\phi \end{aligned}\right\} \quad \text{---(i)}$$

Consider n conductors with total charge q_i

Total charge q_i $(i = 1, 2, 3, \dots n)$ $\vec{E}$ will be created by a distribution charge st.

$$\phi_i = \text{constant}$$

$$\int_S \vec{E}\cdot\hat{n}\,da = \frac{q_i}{\varepsilon_0}$$

Let $\vec{E}'$ be another possible electric field which satisfies

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E}' &= \frac{\rho}{\varepsilon_0} \\ \vec{\nabla}\times\vec{E}' &= 0 \\ \vec{E}' &= -\vec{\nabla}\phi \end{aligned}\right\} \quad \text{---(ii)}$$

So, $\int \vec{E}' \cdot \hat{n}\, da = \frac{q_i}{\varepsilon_0}$ ... (iii)

So, we have

$$\left.\begin{aligned} \vec{\nabla}\cdot(\vec{E}'-\vec{E}) &= 0 \\ \int_S (\vec{E}'-\vec{E})\cdot \hat{n}\, da &= 0 \end{aligned}\right\} \quad \text{.... (iv)}$$

Let W and W' be electrostatic energies corresponding to E and E'.

So, $W' - W = \frac{\varepsilon_0}{2} \int_V |\vec{E}'|^2 d\tau - \int_V |\vec{E}|^2 d\tau$

$$= \frac{\varepsilon_0}{2} \int_V \left(|\vec{E}'|^2 - |\vec{E}|^2\right) d\tau \quad \text{..... (v)}$$

Since, it involves two square fields integration, the use of equation (v) is inhibited.

So, rewriting it as

$$|\vec{E}'|^2 - |\vec{E}|^2 = |\vec{E}'-\vec{E}|^2 + 2\vec{E}\cdot(\vec{E}'-\vec{E}) \quad \text{.... (iv)}$$

First term in the integral is obviously non-negative
So, focusing on second term (leaving the factor 2)

We have

$$\vec{E}\cdot(\vec{E}'-\vec{E}) = -\vec{\nabla}\phi\cdot(\vec{E}'-\vec{E})$$

$$= -\left\{\vec{\nabla}\left[\phi(\vec{E}'-\vec{E})\right] - \phi\,\vec{\nabla}\cdot(\vec{E}'-\vec{E})\right\}$$

From (iv), 2^{nd} term is zero

$$\int_V \vec{E}\cdot(\vec{E}'-\vec{E})\,d\tau = -\int \vec{\nabla}(\phi\vec{E}' - \phi\vec{E})\,d\tau$$

$$= -\int_S \phi(\vec{E}'-\vec{E})\cdot\hat{n}\,da \quad \text{(Divergence theorem)}$$

Splitting last integral into n integrals,

$$\int_V \vec{E}\cdot(\vec{E}'-\vec{E})\,d\tau = -\sum_{i=1}^{n}\phi_i \int_{S_i}(\vec{E}'-\vec{E})\cdot\hat{n}\,da$$

So, the integral in the curved parantheses from above relation is zero.

So, $$W' - W = \frac{\varepsilon_0}{2}\int_V |\vec{E}'-\vec{E}|^2\,d\tau > 0$$

$$\Rightarrow W' > W$$

So, for any electrostatic field $\vec{E}'$ is different from the one due to equipotential surface.