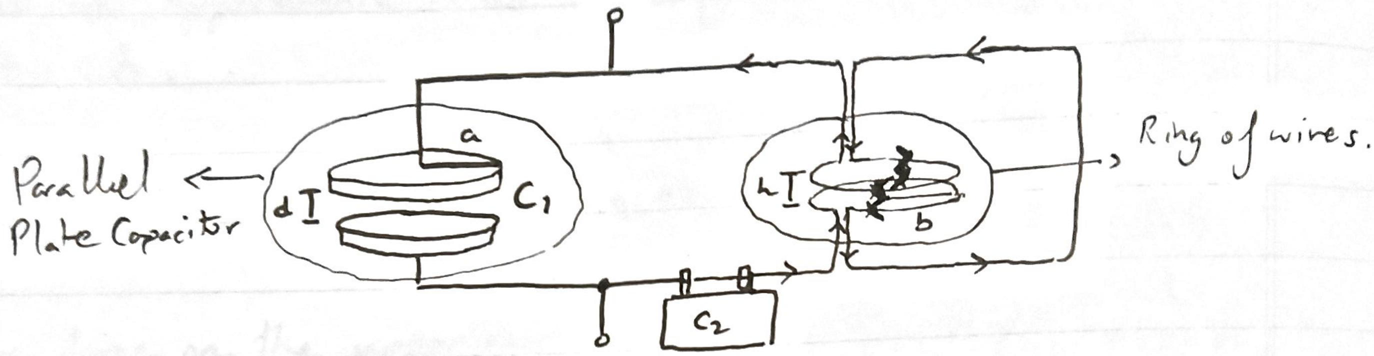
Determing the value of $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ (c)



$V = \mathcal{E}\cos(\omega t)$

An AC voltage of frequency $f \left(= \frac{\omega}{2\pi}\right)$ is applied to the parallel plate capacitor of capacitance C_1 and to the capacitor of capacitance C_2. The voltage across both the capacitors is $\mathcal{E}\cos(\omega t)$.

f, or $\left(\frac{C_2}{C_1}\right)$ can be adjusted so that the time average downward electric force on the upperplate, by the bottom plate exactly balances the time average downward magnetic force on the upper ring, by the bottom ring.

① ~~Electric force on the plate~~ Magnetic force on the ring

- Charge on C_2, $Q = V \cdot C_2 = \mathcal{E} C_2 \cos(\omega t)$

$\Rightarrow$ Current through C_2, $I = \frac{d}{dt}(Q) = -\omega \mathcal{E} C_2 \sin(\omega t)$

I flows through the rings as they are in series.

- Computing the magnetic force between two rings is complicated, so we approximate it as two long parallel straight wires, provided $h << b$.

Since force per unit length of a straight wire $\frac{F}{L}$ is $\frac{\mu_0 I^2}{2\pi h}$, the force on the upper ring due to the bottom ring, $F_m = \left(\frac{F}{L}\right) L$

$$\therefore F_m = \left(\frac{\mu_0 I^2}{2\pi h}\right)(2\pi b) = \mu_0 \frac{b}{h}\left(\omega \mathcal{E} C_2 \sin(\omega t)\right)^2$$

$$\Rightarrow \langle F_m \rangle_t = \frac{\mu_0 b}{2h}(\omega \mathcal{E} C_2)^2 \qquad \left[\because \langle \sin^2(\omega t) \rangle_T = \frac{1}{2}\right]$$

② Electric force on the plate

- Electric field between the plates, $E = \frac{V}{d} = \frac{\mathcal{E}}{d}\cos(\omega t)$

Since $C_1 = \frac{\epsilon_0 A}{d}$, $E = \frac{\mathcal{E} C_1}{\epsilon_0 A}\cos(\omega t)$.

- The force on the upper plate due to the bottom plate, F_e

$= (\text{Charge on the upper plate, } q) \times \underbrace{(\text{Electric field due to the bottom plate})}_{= E/2}$

- $q = \sigma A$. Since $\left(\frac{E}{2}\right) = \frac{\sigma}{2\epsilon_0}$, $q = E\epsilon_0 A$.

$$\therefore F_e = (q)(E/2)$$
$$= (E\epsilon_0 A)(E/2)$$
$$= \frac{E^2 \varepsilon_0 A}{2}$$
$$= \frac{\mathcal{E}^2 C_1^2}{2\epsilon_0 \pi a^2} \cos^2(\omega t) \qquad [\because A = \pi a^2]$$
$$\Rightarrow \langle F_e \rangle_t = \frac{\mathcal{E}^2 C_1^2}{4\epsilon_0 \pi a^2} \qquad \left[\because \langle \cos^2(\omega t)\rangle_T = \frac{1}{2}\right]$$

③ Balancing the forces

- When the forces are balaced, we get :

$$\langle F_e \rangle_T = \langle F_m \rangle_T$$

$$\Rightarrow \frac{\mathcal{E}^2 C_1^2}{4\epsilon_0 \pi a^2} = \frac{\mu_0 b}{2h} \omega^2 \mathcal{E}^2 C_2^2$$
$$\Rightarrow \frac{1}{\mu_0 \epsilon_0} = 2\left(\frac{b}{h}\right)\left(\frac{C_2}{C_1}\right)^2 \omega^2 (\pi a^2)$$
$$= 8\pi^3 a^2 \left(\frac{b}{h}\right)\left(\frac{C_2}{C_1}\right)^2 f^2$$
$$\Rightarrow c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = (8\pi^3)^{1/2} a \left(\frac{b}{h}\right)^{1/2} \left(\frac{C_2}{C_1}\right) f$$

Setting reasonable values of a, $\frac{b}{h}$, $\frac{C_2}{C_1}$,

$a = 0.1\,m$,

$\frac{b}{h} = 10$,

$\frac{C_2}{C_1} = 10^6$,

sweeping for f, the set-up balances around $f \approx 60.2\,Hz$, which gives c as $\sim 3 \cdot 10^8$.

Note: If we had N rings, the magnetic force on the upper ring increases by a factor of N^2, which decreases the necessary value for f, or $\left(\frac{C_2}{C_1}\right)$ by a factor of N.