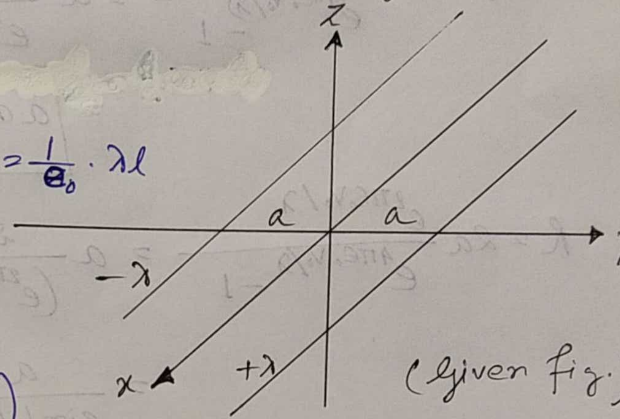
1. Two infinitely long wires running parallel to the $x$-axis, carry uniform charge densities $+\lambda$ and $-\lambda$.

(a) Find the potential at any point $(x, y, z)$, using the origin as your reference.

(b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential $V_0$.

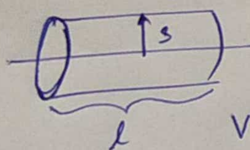



(Given fig.)

Solution: $\oint E \cdot da$

$= E \cdot 2\pi s \cdot l$

$= \frac{1}{\epsilon_0} \cdot Q_{enc.} = \frac{1}{\epsilon_0} \cdot \lambda l$




$\Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$

$V(s) = -\int_a^s \left(\frac{1}{4\pi\epsilon_0} \frac{2\lambda}{s}\right) d\vec{s}$

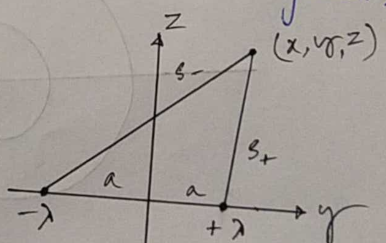
$= -\frac{1}{4\pi\epsilon_0} 2\lambda \ln\left(\frac{s}{a}\right)$

$= -\frac{1}{2\pi\epsilon_0} \cdot \lambda \ln\left(\frac{s}{a}\right)$

↳ (a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_+}{a}\right)$, $s_+$ = distance from $\lambda_+$

" " $-\lambda$ " $V_- = +\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_-}{a}\right)$, $s_-$ = distance from $\lambda_-$

Total $V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_-}{s_+}\right)$




Now, $s_+ = \sqrt{(y-a)^2 + z^2}$ and $s_- = \sqrt{(y+a)^2 + z^2}$

$\Rightarrow V(x, y, z) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right)$

$= \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]$

(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k$ = constant

i.e., $y^2 + 2ay + a^2 + z^2 = k(y^2 - 2ay + a^2 + z^2)$

$\Rightarrow y^2(k-1) + a^2(k-1) - 2ay(k+1) - z^2(k-1) = 0$

$\Rightarrow y^2 + z^2 + a^2 - 2ay\left(\frac{k+1}{k-1}\right) = 0$

↳ equation for a circle with centre at $(y_0, 0)$ and radius $R$ is —

$(y-y_0)^2 + z^2 = R^2 \Rightarrow y^2 + z^2 + (y_0^2 - R^2) - 2yy_0 = 0$

Evidently, the equipotential ~~s~~ are circles, with $y_0 = a\left(\frac{k+1}{k-1}\right)$ &

$$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - a^2$$

$$= a^2\,\frac{(k^2+2k+1-k^2+2k-1)}{(k-1)^2} = a^2\,\frac{4k}{(k-1)^2}$$

$$\Rightarrow R = \frac{2a\sqrt{k}}{|k-1|}$$

In terms of $V_0$ —

$$y_0 = a\,\frac{e^{(4\pi\epsilon_0 V_0/\lambda)}+1}{e^{(4\pi\epsilon_0 V_0/\lambda)}-1} = a\,\frac{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}}$$

$$= \boxed{a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$

$$R = 2a\,\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\,\frac{2}{\left(e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}\right)}$$

$$= \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} = \boxed{a\,\mathrm{cosech}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$