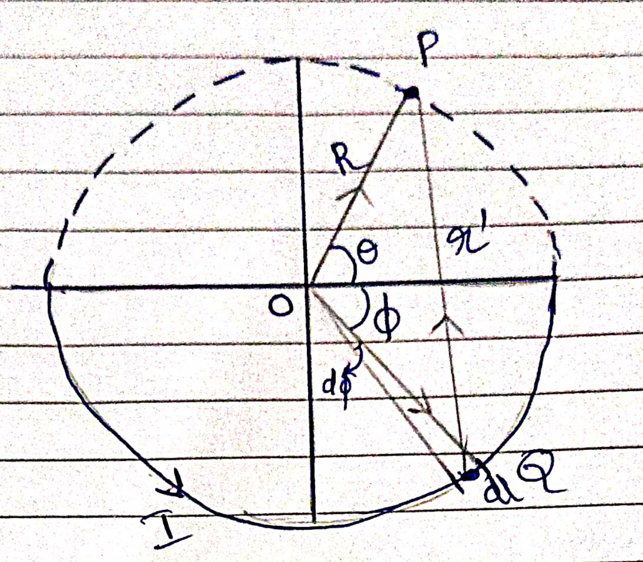
Q. A semicircular wire carries a steady current I. Find the magnetic field at a point P on the other semicircle.



From Biot-Savart's law:

Field at a point due to a current carrying conductor is given by

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl} \times \vec{r}}{r^3}$$

- Taking an infinitesimally small element dl on the conductor, making an angle ϕ with the horizontal.

$$d\phi = \frac{dl}{R} \implies dl = R\,d\phi$$

$$dl = (R\sin\phi\,\hat{i} + R\cos\phi\,\hat{j})\,d\phi$$

$$dl = R\,d\phi\,(\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

- Let $\vec{r'}$ be the ⊥ from dl to P.

From the diagram:

$$\vec{r'} = OP - OQ$$

$$OP = R = R(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$OQ = R\left(\cos\phi\,\hat{i} + \sin\phi(-\hat{j})\right)$$
$$R\left(\cos\phi\,\hat{i} - \sin\phi\,\hat{j}\right) = OQ.$$

$$\vec{r} = R\left[(\cos\theta - \cos\phi)\hat{i} + (\sin\theta + \sin\phi)\hat{j}\right]$$

- $$\vec{dl} \times \vec{r} = \begin{vmatrix} i & j & k \\ \sin\phi & \cos\phi & 0 \\ (\cos\theta - \cos\phi) & (\sin\theta + \sin\phi) & 0 \end{vmatrix} \quad R^2 d\phi$$

$$= \hat{k}\left[\sin\theta\sin\phi + \sin^2\phi - (\cos\theta\cos\phi - \cos^2\phi)\right] R^2 d\phi$$
$$R^2\left[(\sin\theta\sin\phi - \cos\theta\cos\phi) + 1\right]$$
$$R^2\left[1 - \cos(\theta + \phi)\right] d\phi\,\hat{k}.$$

- To find B, integrate over the semi-circle from $0 \to \pi$.

$$B = \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{\vec{dl} \times \vec{r}}{|r|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{R^2(1 - \cos(\theta + \phi))\, d\phi}{|r|^3}$$

$$|r|^3 = \left(\sqrt{R^2\left[(\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2\right]}\right)^3$$
$$= \left(2R^2(1 - \cos(\theta + \phi))\right)^{3/2}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{R^2(1 - \cos(\theta + \phi))}{\left(2R^2(1 - \cos\theta + \phi)\right)^{3/2}}\, d\phi$$

$$= \frac{\mu_0 I\, R^2}{4\pi\, 2\sqrt{2}\, R^3}\int_0^{\pi} \frac{(1 - \cos(\theta + \phi))}{(1 - \cos(\theta + \phi))^{3/2}}\, d\phi$$

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$$= \frac{\mu_0 I}{8\sqrt{2}\pi R} \int_0^{\pi} \frac{1}{(1-\cos(\theta+\phi))^{1/2}} \, d\phi$$

$$= \frac{\mu_0 I}{8\sqrt{2}\pi R} \int_0^{\pi} \frac{1}{\left[2\sin^2\left(\frac{\theta+\phi}{2}\right)\right]^{1/2}}$$

$$= \frac{\mu_0 I}{16\pi R} \int_0^{\pi} \operatorname{Cosec}\left(\frac{\theta+\phi}{2}\right) d\phi .$$

$$\because \int \operatorname{Cosec} x = \ln\left|\tan \frac{x}{2}\right| + C$$

$$= \frac{\mu_0 I}{16\pi R} \left[2 \ln \left(\tan \frac{(\theta+\phi)}{4}\right)\right]_0^{\pi}$$

$$= \frac{\mu_0 I}{8\pi R} \ln \left(\frac{\tan\left(\frac{\theta+\pi}{4}\right)}{\tan\left(\frac{\theta}{4}\right)}\right) \hat{K}$$