Fields in Dielectrics

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Q. Consider two media of differing electric permittivity ϵ_α and ϵ_β whose interface is a plane extending to infinity. A point charge q is placed at d distance from the interface in medium α. What is the field produced by the charge?

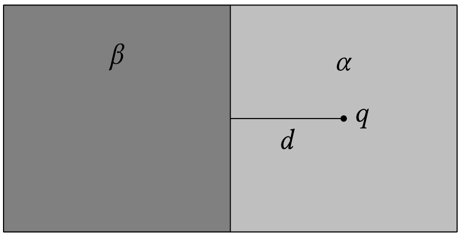


Figure 1: The System

Let us consider that the plane of the interface is the xy-plane, and that the charge lies on the z-axis. Medium α is found in the region where $z > 0$ and medium β is found in the region where $z < 0$.

First we will list the constraints on $\mathbf{E}$. Then, using the method of images, we will obtain a field that satisfies these conditions. By the First Uniqueness Theorem, this will be the only possible field that fulfils the constraints.

The conditions that $\mathbf{E}$ must satisfy are as follows;
in medium α,

$$\epsilon_\alpha \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{Eq 1a}$$

in medium β,

$$\epsilon_\beta \nabla \cdot \mathbf{E} = 0 \tag{Eq 1b}$$

and throughout the entire region,

$$\nabla \times \mathbf{E} = 0 \tag{Eq 1c}$$

Eq 1a and Eq 1b are statements of Gauss's law in the two media. The charge density is 0 throughout medium β as it has no free charges, and therefore no bound charges. Eq 1c comes from the fact that the electric field is conservative. We also have a set of boundary conditions at $z = 0$ that are as follows;

$$\mathbf{E}_x\left(z = 0_+\right) = \mathbf{E}_x\left(z = 0_-\right) \tag{Eq 2a}$$

$$\mathbf{E}_y\left(z = 0_+\right) = \mathbf{E}_y\left(z = 0_-\right) \tag{Eq 2b}$$

$$\epsilon_\alpha \mathbf{E}_z\left(z = 0_+\right) = \epsilon_\beta \mathbf{E}_z\left(z = 0_-\right) \tag{Eq 2c}$$

Let us take a quick detour to understand where these conditions come from. First we must understand how dielectrics react to electric fields. Since a dielecrtic by definition has no free electrons, it cannot entirely cancel an external field. Instead, the atoms of a dielectric get polarized by the field; the electrons shift their orbits such that the net force on the nucleus and on the electrons becomes zero. In doing so they form dipoles and we say that the medium is polarized. Polarization of a medium is the "density" of the dipole moment, it is the dipole moment per unit volume.

$$\mathbf{P} = \frac{\mathbf{p}}{V} \tag{Eq 3}$$

In linear dielectrics, this field is linearly proportional to the net electric field on the medium.

$$\mathbf{P} = \epsilon\chi\mathbf{E} \tag{Eq 4}$$

The polarization of a body can also be expressed in terms of bound charges; on the surface of the material as well as inside the bulk. To see how, let us start with a toy model; a line n of identical dipoles $\mathbf{p} = q\mathbf{d}$ arranged head to tail. The intermediate charges cancel out, and the system acts as one large dipole $\mathbf{p}_{net} = nq\mathbf{d}$, with a $-q$ charge on the tail and a $+q$ charge on the head.

Now, imagine a cylinder of dielectric, polarized linearly along its axis, in effect a bundle of these lines of dipoles. The charge accumulated on either end of this cylinder is similarly

$$\begin{aligned} q &= \frac{p}{l} \\ &= \frac{P\ Vol.}{l} \\ &= PA \end{aligned} \tag{Eq 5}$$

And the surface charge density is

$$\sigma_b = P \tag{Eq 6}$$

However, if we cut the cylinder obliquely, at say an angle θ, then the area over which the charge is spread is now $A_{end} = \frac{A}{\cos\theta}$, but the charge is still PA. Hence, the surface charge density in general is

$$\begin{aligned} \sigma_b &= P\cos\theta \\ &= \mathbf{P}\cdot\hat{\mathbf{n}} \end{aligned} \tag{Eq 7}$$

But what if the polarization varies along the length? Then we will have accumulation of charges along the length. This accumulated charge will be the negative of the charge pushed through the boundary of the volume.

$$\begin{aligned} \int_{Vol} \rho_b\, d\tau = -\oint_S \mathbf{P}\cdot d\mathbf{a} &= -\int_{Vol} \nabla\cdot\mathbf{P}\, d\tau \\ \implies \rho_b &= -\nabla\cdot\mathbf{P} \end{aligned} \tag{Eq 8}$$

We can now write Gauss's Law in an interesting manner.

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon} \\ \epsilon\nabla\cdot\mathbf{E} &= \rho \\ &= \rho_b + \rho_f \\ \nabla\cdot\epsilon\mathbf{E} - \rho_b &= \rho_f \\ \nabla\cdot\epsilon\mathbf{E} + \nabla\cdot\mathbf{P} &= \rho_f \\ \nabla\cdot(\epsilon\mathbf{E} + \mathbf{P}) &= \rho_f \end{aligned} \tag{Eq 9}$$

Here, ρ_f is the free charge density, that does not arise from polarization. The quantity $\epsilon\mathbf{E} + \mathbf{P}$ is referred to as the electric displacement field, denoted by $\mathbf{D}$.

We can recast this in integral form as

$$\oint_S \mathbf{D} \cdot d\mathbf{a} = \int_{Vol} \rho_f \, d\tau \tag{Eq 10}$$

Let us apply this to a pill box at the interface $z = 0$. As we shrink the box, the free charge becomes zero and we consider the displacement vectors normal to the surface. So,

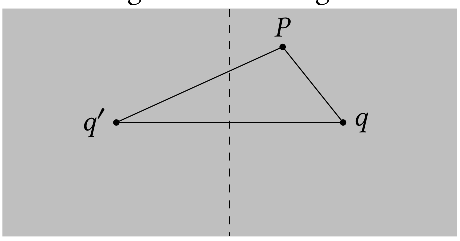
$$\begin{gathered} \mathbf{D}_{z=0_+} \cdot \hat{z} = \mathbf{D}_{z=0_-} \cdot \hat{z} \\ \epsilon_\alpha E_z(z = 0^+) + P_z(z = 0^+) \cdot \hat{z} = \epsilon_\beta E_z(z = 0^-) + P_z(z = 0^-) \cdot \hat{z} \end{gathered} \tag{Eq 11}$$

Now, $P_z(z = 0^+) \cdot \hat{z} = P_z(z = 0^-) \cdot \hat{z} = \sigma_b$ as we had shown before. Hence we get the third boundary condition

$$\epsilon_\alpha \mathbf{E}_z \left(z = 0_+\right) = \epsilon_\beta \mathbf{E}_z \left(z = 0_-\right) \tag{Eq 12}$$

We now return to the problem at hand. For simplicity, we will deal with the potential field of the system. Let us assume that the field in α is the same as that produced by two charges, the actual charge q and an image charge q' at $z = -d$, when the space is filled with medium α. This gives us the expression

Figure 2: The Images



$z = 0$

$$\phi(z > 0) = \frac{1}{2\tau\epsilon_0\epsilon_\alpha} \left(\frac{q}{R_1} + \frac{q'}{R_2} \right) \tag{Eq 13}$$

Here, $R_1 = \sqrt{x^2 + y^2 + (z - d)^2)}$ and $R_2 = \sqrt{x^2 + y^2 + (z + d)^2)}$. This field obeys Gauss's law in the entire region where $z > 0$, as $\nabla^2\phi = 0$ at all points apart from $(0, 0, d)$, where the charge q is located.

In the region $z < 0$, we assume that the potential is due to a charge q^* located at $(0, 0, d)$, when the space is filled with medium β. In this case, the field is

$$\phi(z > 0) = \frac{1}{2\tau\epsilon_0\epsilon_\beta} \frac{q^*}{R_1} \tag{Eq 14}$$

This field also satisfies Gauss's law throughout the region $z < 0$, and without any singularities. Hence, we fulfil conditions Eq 1a and Eq 1b.

We will now use the boundary conditions to pick appropriate values of q' and q^*. Eq 2a and Eq 2b are satisfied if

$$\frac{\partial}{\partial x}\phi(0_+) = \frac{\partial}{\partial x}\phi(0_-)$$
$$\frac{\partial}{\partial y}\phi(0_+) = \frac{\partial}{\partial y}\phi(0_-)$$

In this case it is useful to write the partial derivatives of $\frac{1}{R_1}$ and $\frac{1}{R_2}$.

$$\frac{\partial}{\partial x}\frac{1}{R_1} = \frac{-x}{R_1^3}$$
$$\frac{\partial}{\partial y}\frac{1}{R_1} = \frac{-y}{R_1^3}$$
$$\frac{\partial}{\partial z}\frac{1}{R_1} = \frac{-(z-d)}{R_1^3}$$

$$\frac{\partial}{\partial x}\frac{1}{R_2} = \frac{-x}{R_2^3}$$
$$\frac{\partial}{\partial y}\frac{1}{R_2} = \frac{-y}{R_2^3}$$
$$\frac{\partial}{\partial z}\frac{1}{R_2} = \frac{-(z+d)}{R_2^3}$$

At $z = 0$,

$$\left.\frac{\partial}{\partial x}\frac{1}{R_1}\right|_{z=0} = \left.\frac{\partial}{\partial x}\frac{1}{R_2}\right|_{z=0}$$
$$\left.\frac{\partial}{\partial y}\frac{1}{R_1}\right|_{z=0} = \left.\frac{\partial}{\partial y}\frac{1}{R_2}\right|_{z=0}$$
$$\left.\frac{\partial}{\partial z}\frac{1}{R_1}\right|_{z=0} = -\left.\frac{\partial}{\partial z}\frac{1}{R_2}\right|_{z=0}$$

From the first two equalities, we can see that Eq 2a and Eq 2b hold if

$$\frac{q+q'}{\epsilon_\alpha} = \frac{q^*}{\epsilon_\beta} \tag{Eq 15a}$$

And the third equality can be coupled with Eq 2c to give the condition

$$q - q' = q^* \tag{Eq 15b}$$

From here it is merely a bit of linear algebra to obtain the values of q' and q^*.

$$q' = \frac{\epsilon_\alpha - \epsilon_\beta}{\epsilon_\alpha + \epsilon_\beta} q \tag{Eq 16a}$$

$$q^* = \frac{2\epsilon_\beta}{\epsilon_\alpha + \epsilon_\beta} q \tag{Eq 16b}$$

We can put these back into the expression for the scalar field, and take it's gradient to obtain the electric field.