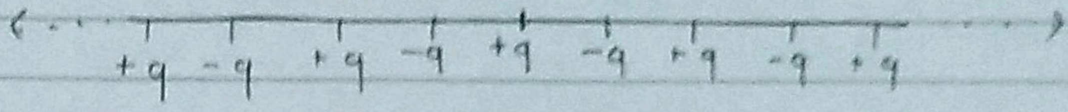
Q. Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x-axis, each a distance a from its nearest neighbours. Find the work per particle required to assemble this system.
[Partial Answer: $-\alpha q^2/4\pi\epsilon_0 a$), for some dimensionless number α; your problem is to determine α. It is known as the Madelung constant.]



For total energy required to construct the above configuration we have,

$$W = \frac{1}{2} \sum q_i \, V(r_i)$$

This expression is going to be infinite since there are infinitly many charges present in this system. Therefore, we will focus on finding the energy required per particle to construct this system.

$\therefore$ We know for some arbitary particle q_i

$$W_i = \frac{1}{2} q_i \, V(r_i) \quad \text{—— (I)}$$

Now,

$$V(r_i) = \frac{1}{4\pi\epsilon_0} \frac{-q}{a} \times 2 + \frac{1}{4\pi\epsilon_0} \frac{q}{2a} \times 2 + \frac{1}{4\pi\epsilon_0} \frac{(-q)}{3a} \times 2 + \cdots$$

$$= \frac{2q}{4\pi\epsilon_0} \left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots \right) \quad \text{—— (1)}$$

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After taking out all the common quantities in eq^n ① we can clearly see a pattern forming.

Appendix A

Taylor series expansion of $\ln(1+x)$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Substituting $x=1$ we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\Rightarrow -\ln 2 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{———} \quad (a)$$

Substituting eq^n ① with eq^n ⓐ from Appendix A

We get,

$$V(r_i) = \frac{2q}{4\pi\varepsilon_0}\left(-\ln 2\right)$$

Now substituting $V(r_i) = 2q/4\pi\varepsilon_0 \left(-\ln 2\right)$ in eq^n Ⓘ we get:

$$W_i = \tfrac{1}{2}\, q_i\, V(r_i)$$

$$= \frac{1}{2}\, q \left(\frac{2q}{4\pi\varepsilon_0 a}\right)\left(-\ln 2\right)$$

$$\boxed{W_i = \frac{-q^2}{4\pi\varepsilon_0 a}\ln 2} \quad \text{Ⓑ}$$

$\Rightarrow$ This expression gives energy req. per particle to construct the system.

Comparing Eq^n Ⓑ with $\left(-\alpha q^2/4\pi\varepsilon_0 a\right)$

We can clearly infer that $\boxed{\alpha = \ln 2}$

Ans.