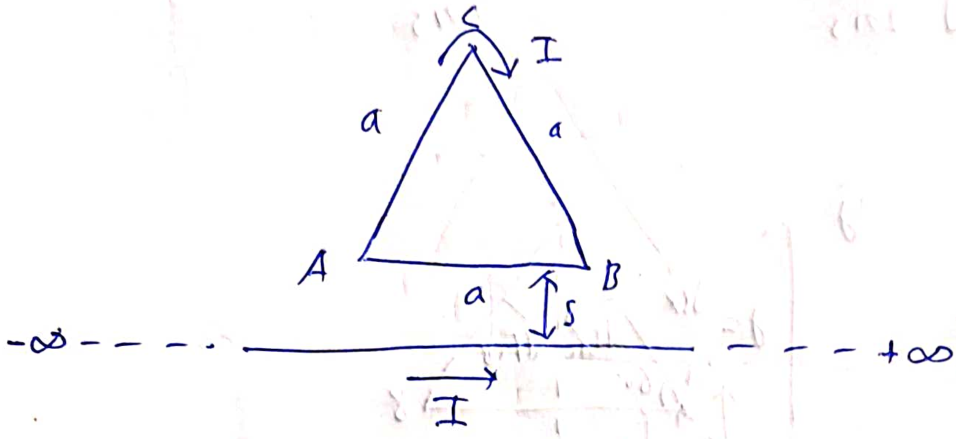
Q- Find the force on the triangular loop in the given figure below?

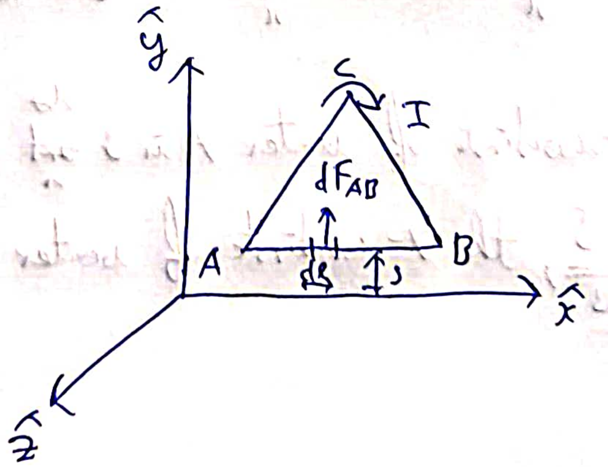


Soln. The triangular loop has sides of length a and is s distance away from an infinitely long current carrying wire

From Biot - savart's law, we know that magnetic field around an infinite straight current carrying conductor is $B = \frac{\mu_0 I}{2\pi s} (\hat{z})$ → pointing into the page

where B is the ~~current~~ magnetic field, I is the current and s is the distance from the conductor.

We will find force on each side of the triangular loop and then we will add them.



Force on side AB:

$\vec{dl} = -dx\,\hat{x}$ (we take the $\vec{dl}$ element in that direction ~~o~~ along which current is flowing)

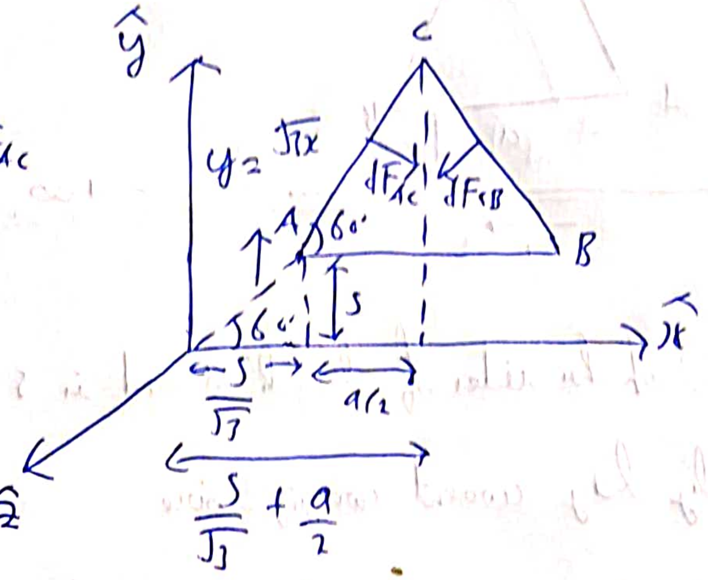
$$\vec{dF}_{BA} = I(\vec{dl} \times \vec{B})$$

$$\vec{dF}_{BA} = I\left(-dx\,\hat{x} \times \frac{\mu_0 I}{2\pi s}\hat{z}\right) = \frac{\mu_0 I^2}{2\pi s} dx\,\hat{y}$$

$$F_{BA} = \int dF_{BA} = \int \frac{\mu_0 I^2}{2\pi s} dx\, \hat{y} = \frac{\mu_0 I^2 a}{2\pi s} \hat{y}$$

Force on BC and AC:

The x components of dF_{AC} and dF_{AC} and dF_{CB} cancels out.



→ Since the triangular loop is equilateral therefore each angle is 60°. So if we place the origin in such a way so that if we imagine extending the left side of the triangle it passes through the origin, this makes a linear function with zero y-intercept, the ratio of y-coordinate to x-coordinate will be the tangent of 60 degree.

$$\tan 60° = \frac{y}{x} \Rightarrow \sqrt{3} = \frac{y}{x} \qquad y = \sqrt{3}\, x$$

→ The y-coordinate of vertex A is s ~~and~~ so the x-coordinate of vertex A will be $\frac{s}{\sqrt{3}}$, the x-coordinate of vertex C will be $\left(\frac{s}{\sqrt{3}} + \frac{a}{2}\right)$.

On the left side, $B = \frac{\mu_0 I}{2\pi y} \hat{z}$
(AC)

$$dF_{AC} = I(\vec{dl} \times \vec{B}) = I(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \times \left(\frac{\mu_0 I}{2\pi y}\hat{z}\right)$$

$$= \frac{\mu_0 I^2}{2\pi y}(-dx\,\hat{y} + dy\,\hat{x})$$

→ But the x component cancels the corresponding term from the right side and,

$$F_y = \frac{-\mu_0 I^2}{2\pi}\int_{S/\sqrt{3}}^{(S/\sqrt{3}+a/2)} \frac{1}{y}\,dx$$

$$(\because y = \sqrt{3}x)$$

$$= \frac{-\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(\frac{S/\sqrt{3} + a/2}{S/\sqrt{3}}\right)$$

$$= \frac{-\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1 + \frac{\sqrt{3}a}{2S}\right)$$

→ The force on the right side (BC) is same as,

$$F_{AC} + F_{BC} = \frac{-\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1+\frac{\sqrt{3}a}{2S}\right) + \left[\frac{-\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1+\frac{\sqrt{3}a}{2S}\right)\right]$$

$$= \frac{-2\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1+\frac{\sqrt{3}a}{2S}\right)$$

Now, on adding F_{BA}, F_{AC} and F_{CB} we get,

$$\text{Total force} = F_{BA} + F_{AC} + F_{BC}$$

$$= \frac{\mu_0 I^2}{2\pi S} - \frac{\mu_0 I^2}{2\sqrt{3}\pi} \ln\left(1+\frac{\sqrt{3}a}{2S}\right)$$

$$= \boxed{\frac{\mu_0 I^2}{2\pi}\left[\frac{a}{S} - \frac{2}{\sqrt{3}} \ln\left(1+\frac{\sqrt{3}a}{2S}\right)\right]}$$