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Batch - 7

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Child - Langmuir Law

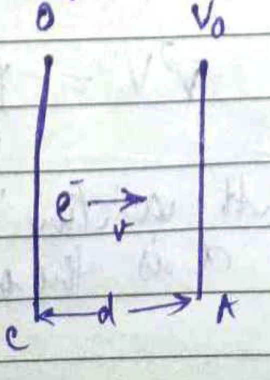
PH1213 Presentation

We all know that, according to Ohm's Law, current flowing through a conductor will be directly proportional to the voltage of the conductor, i.e, the potential diff across. ∴, $\boxed{I \propto V}$.

But what about space current; is it the same as that of conductor situation, does displacement current (or current through a vacuum) also follow the same Ohm's Law. Interestingly, the answer is NO. & we will figure out what the actual relation is in such cases by this simple question.

Q.) Given two plane parallel electrodes seperated by space with length 'd', at voltages 0 & V_0, find the current density if an unlimited supply of $e^{\ominus}$s at rest is supplied to the lower potential electrode. Neglect collisions.

Soln)



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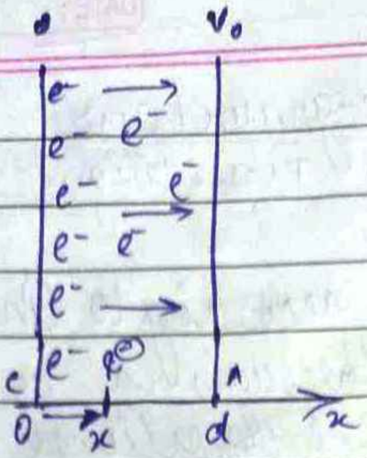
Solu^n.)



As an unlimited supply of $e^{\ominus}$s is present.

$\therefore \nabla^2 V \neq 0$ {laplacian eq^n not applicable}

By Poisson's Eq^n,

$$\therefore \boxed{\nabla^2 V = -\frac{\rho(x)}{\epsilon_0}} \quad -① $$

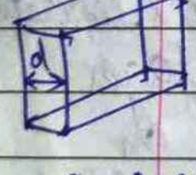
Now, As,

$$e\,V(x) = \frac{1}{2} m v^2$$

$$\Rightarrow \boxed{v = \sqrt{\frac{2eV(x)}{m}}} \quad -(II)$$

One important point

$$\boxed{J = \rho(x) \cdot v(x)} \quad -(III)$$



$\because$ As $J = \rho(x) \cdot v(x)$

$$\Rightarrow \rho(x) = \frac{J}{v(x)}$$

$$\Rightarrow \boxed{\rho(x) = \frac{J}{\sqrt{\frac{2eV(x)}{m}}}}$$

$\hookrightarrow$ (IV)

As J is constant, as $v \uparrow$ along x, $\rho(x)$ decreases.

From, Poisson's eq^n ①

$$\nabla^2 V = -\frac{\rho(x)}{\epsilon_0}$$

At a time 't' in space, x is the only variable here.

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$$\Rightarrow \frac{\partial^2 V}{\partial x^2} = -\rho/\epsilon_0$$

$$\Rightarrow \boxed{\frac{d^2V}{dx^2} = -\rho/\epsilon_0} \quad \text{— (IV)}$$

$\Downarrow$

$\Rightarrow$ we need to solve this diff eqⁿ for $V(x)$.

To solve, let $\frac{dV}{dx} = r$

$\therefore$ By substituting,

$$\frac{dr}{dx} = -\rho/\epsilon_0$$

Multiply both sides by r.

$$\Rightarrow r\frac{dr}{dx} = -\rho/\epsilon_0 \, r$$

$$\Rightarrow r\frac{dr}{\cancel{dx}} = -\rho/\epsilon_0 \frac{dV}{\cancel{dx}}$$

$$\Rightarrow \int_0^{r(x)} r\,dr = \int_0^{V(x)} -\rho(x)/\epsilon_0 \, dV$$

$$\Rightarrow \frac{r^2}{2} = \frac{-1}{\epsilon_0}\int_0^{V(x)} \frac{J}{\sqrt{\frac{2eV(x)}{m}}}\, dV$$

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$$\gamma^2 = \frac{-2J}{\epsilon_0 \sqrt{\frac{2e}{m}}} \int_0^{V(x)} \frac{dV}{\sqrt{V}}$$

$$\Rightarrow \gamma^2 = \frac{-2}{\epsilon_0} J \sqrt{\frac{m}{2e}} \times 2\sqrt{V(x)}$$

$$\Rightarrow \gamma^2 = -4\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}} \times \underbrace{\sqrt{V(x)}}_{(V^{1/2})}$$

$$\Rightarrow \boxed{\gamma = \underbrace{\left(\sqrt{-\frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}}\right)}_{K'} . V^{1/4}}$$

Putting $\gamma = \frac{dV}{dx}$

$$\frac{dV}{dx} = K' V^{1/4}$$

$$\Rightarrow \int_0^{V(x)} \frac{dV}{V^{1/4}} = K' \int_0^x dx$$

$$\Rightarrow \frac{V^{-1/4+1}}{-1/4+1} = K' x$$

$$\Rightarrow \boxed{\frac{4}{3} V_{(x)}^{3/4} = K' x} \quad \text{(vi)}$$

$$\Rightarrow 4V^{3/4} = 3 \times \sqrt{\frac{-2J}{\epsilon_0}\sqrt{\frac{m}{2e}}} \times x$$

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Substituting $x=d$ in eqn (vi)

At $x=d$; $V(x)=V_0$

$$\therefore \boxed{\frac{4}{3}V_0^{3/4} = K'd} \quad \text{— (vii)}$$

Dividing (vi) by (vii)

$$\left(\frac{V(x)}{V_0}\right)^{3/4} = \frac{x}{d}$$

$$\Rightarrow \frac{V(x)}{V_0} = \left(\frac{x}{d}\right)^{4/3}$$

$$\Rightarrow \boxed{V(x) = V_0\left(\frac{x}{d}\right)^{4/3}}$$

{ Clearly $V(x) \neq V_0(x/d)$ {As in the (If $\nabla^2 V = 0$) case of conductors} }

In eqn (vii)

$$\frac{4}{3}V_0^{3/4} = \sqrt{\frac{-4J}{\epsilon_0}\sqrt{\frac{m}{2e}}}\,.d$$

$$\Rightarrow \frac{16}{9d^2}V_0^{3/2} = \frac{-4J}{\epsilon_0} \times \sqrt{\frac{m}{2e}}$$

$$\Rightarrow \underbrace{-\frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}}_{\to K'' = \text{constant.}}\,.V_0^{3/2} = J$$

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$J = K^4 V_0^{3/2}$

As $I = J \cdot A$

$\Rightarrow \frac{I}{A} = (K^4) V_0^{3/2}$

$\Rightarrow I = K V_0^{3/2}$

$\Rightarrow \boxed{I \propto V_0^{3/2}}$

$\Downarrow$

$\Rightarrow$ Child Langmuir's Law.

(As opposed to $I \propto V$ in Ohm's Law)