Question :

Consider a toroidal coil of arbitrary and uniform cross section, with winding that is uniform and tight enough for each turn to be considered to be a closed loop.

Prove that the magnetic field is circumferential at all points.

Solution:

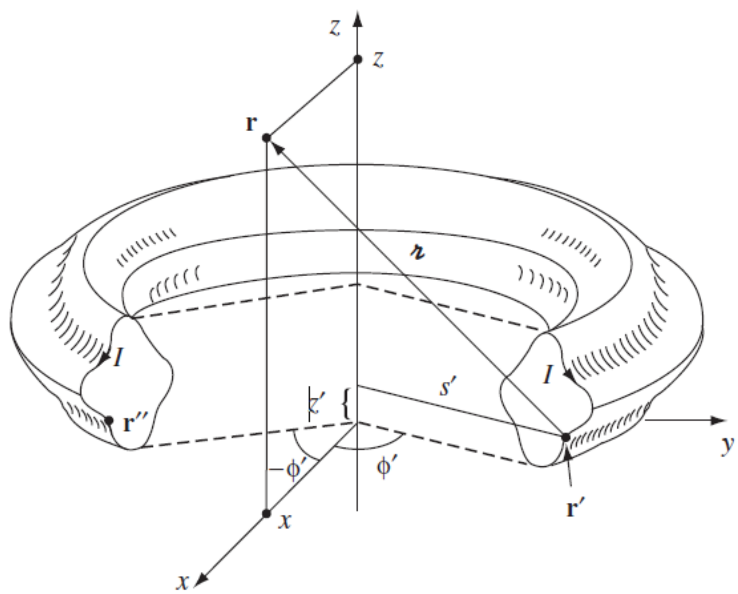
Proof. According to the Biot-Savart law, the field at r due to the current element

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{\mathbf{I} \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} dl'.$$

We may as well put $\mathbf{r}$ in the xz plane (Fig. 5.39), so its Cartesian components are

(x, 0, z), while the source coordinates are:

$$\mathbf{r}' = (s' \cos\phi', s' \sin\phi', z').$$



Then:

$$\boldsymbol{\mathscr{r}} = (x - s' \cos\phi', -s' \sin\phi', z - z').$$

Since the current has no ϕ component $\mathbf{I} = I_s \,\hat{\mathbf{s}} + I_z \,\hat{\mathbf{z}}$, or (in Cartesian coordinates)

$$\mathbf{I} = (I_s \cos\phi',\, I_s \sin\phi',\, I_z).$$

Accordingly,

$$\mathbf{I} \times \boldsymbol{\mathscr{r}} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ I_s \cos\phi' & I_s \sin\phi' & I_z \\ (x - s'\cos\phi') & (-s'\sin\phi') & (z - z') \end{bmatrix}$$

$$= \left[\sin\phi' \left(I_s(z-z') + s' I_z\right)\right] \hat{\mathbf{x}} + \left[I_z(x - s'\cos\phi') - I_s \cos\phi'(z-z')\right] \hat{\mathbf{y}}$$

$$+ \left[-I_s x \sin\phi'\right] \hat{\mathbf{z}}.$$

But there is a symmetrically situated current element at $\mathbf{r}''$, with the same s', the same $\mathscr{r}$, the same dl', the same I_s, and the same I_z, *but negative* ϕ' (Fig. 5.39). Because $\sin\phi'$ changes sign, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ contributions from $\mathbf{r}'$ and $\mathbf{r}''$ cancel, leaving only a $\hat{\mathbf{y}}$ term. Thus the field at $\mathbf{r}$ is in the $\hat{\mathbf{y}}$ direction, and in general the field points in the $\hat{\boldsymbol{\phi}}$ direction. □

Now that we know the field is circumferential, determining its magnitude is ridiculously easy. Just apply Ampère's law to a circle of radius s about the axis of the toroid:

$$B 2\pi s = \mu_0 I_{\text{enc}},$$

and hence

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\boldsymbol{\phi}}, & \text{for points inside the coil,} \\ \mathbf{0}, & \text{for points outside the coil,} \end{cases} \tag{5.60}$$

where N is the total number of turns.