Problem 5.59

(a) Prove that the average magnetic field, over a sphere of radius R, due to steady currents inside the sphere, is

$$\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}, \tag{5.93}$$

where $\mathbf{m}$ is the total dipole moment of the sphere. Contrast the electrostatic result, Eq. 3.105. [This is tough, so I'll give you a start:

$$\mathbf{B}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{B}\, d\tau.$$

Write $\mathbf{B}$ as $(\nabla \times \mathbf{A})$, and apply Prob. 1.61(b). Now put in Eq. 5.65, and do the surface integral first, showing that

$$\int \frac{1}{\mathscr{r}}\, d\mathbf{a} = \frac{4}{3}\pi \mathbf{r}'$$

(see Fig. 5.65). Use Eq. 5.90, if you like.]

(b) Show that the average magnetic field due to steady currents *outside* the sphere is the same as the field they produce at the center.

Solution :-

(a) $$B_{avg} = \frac{\int B\, d\tau}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi R^3}\int (\nabla \times A)\, d\tau = -\frac{3}{4\pi R^3}\oint A \times da$$

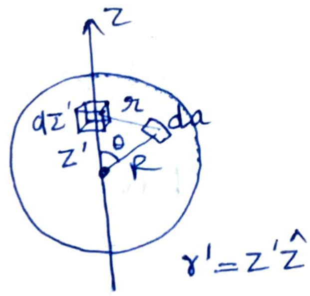
$$= -\frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\oint \left\{\int \frac{J}{r} d\tau'\right\} \times da$$

$$= -\frac{3\mu_0}{(4\pi)^2 R^3}\int J \times \left\{\oint \frac{1}{r} da\right\} d\tau'$$ {∵ There is steady current inside sphere}

$r' \longrightarrow$ Source pt.

$r \longrightarrow$ field pt.

To do surface integral, choose (x, y, z) co-ord. so that r' lies on z-axis –



Then $r = \sqrt{R^2 + (z')^2 - 2Rz'\cos\theta}$

$da = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

By symmetry, x and y componant must integrate to 0.

z componant of $\hat{r}$ is $\cos\theta$.

$$\therefore \oint \frac{1}{r} da = \hat{z}\int \frac{\cos\theta}{\sqrt{R^2 + (z')^2 - 2Rz'\cos\theta}} R^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi R^2 \hat{z}\int_0^\pi \frac{\cos\theta \sin\theta}{\sqrt{R^2 + (z')^2 - 2Rz'\cos\theta}} d\theta$$

Let $u = \cos\theta \implies du = -\sin\theta\, d\theta$

$$= 2\pi R^2 \hat{z}\int_{-1}^{1} \frac{u\, du}{\sqrt{R^2 + (z')^2 - 2Rz'u}}$$

on integrating –

$$= 2\pi R^2 \hat{z}\left\{\frac{-2\left[2(R^2 + (z')^2) + 2Rz'u\right]}{3\,(2Rz')^2}\sqrt{R^2 + (z')^2 - 2Rz'u}\right\}\Bigg|_{-1}^{1}$$

$$= -\frac{2\pi R^2 \hat{z}}{3(Rz')^2}\left\{\left[R^2 + (z')^2 + Rz'\right]|R - z'| - \left[R^2 + (z')^2 - Rz'\right]|R + z'|\right\}$$

$$= \begin{cases} \frac{4\pi}{3} z'\hat{z} = \frac{4\pi}{3} r' & (r' < R) \\ \frac{4\pi R^3}{3(z')^2}\hat{z} = \frac{4\pi}{3}\frac{R^3}{(r')^3} r', & (r' > R) \end{cases}$$

so for $\underline{\underline{(r' < R)}}$:-

$$B_{avg} = -\frac{3\mu_0}{(4\pi)^2 R^3}\frac{4\pi}{3}\int (J \times r')\, d\tau' = -\frac{\mu_0}{4\pi R^3}\int (J \times r')\, d\tau'$$

By putting $m = \frac{1}{2}\int (r' \times J)\, d\tau'$

$$B_{avg} = \frac{\mu_0}{4\pi}\frac{2m}{R^3}$$

(b) $(r' > R)$

$$B_{avg} = -\frac{3\mu_0}{(4\pi)^2 R^3}\frac{4\pi}{3} R^3 \int \left(J \times \frac{r'}{(r')^3}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \frac{J \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

where $\mathscr{r}$ now goes from source pt. to centre
$(\therefore \mathscr{r} = -r')$

$$\therefore \boxed{B_{avg} = B_{centre}}$$