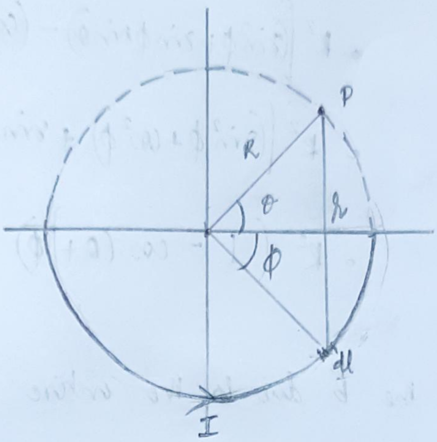
A semicircular wire carries a steady current I (it must be hooked up to some other wires to complete the circuit, but we are not concerned with them here) Find the magnetic field at a point P on the other semicircle?



Magnetic field at a point due to a current carrying conductor is given by the Biot-Savart's law

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl} \times \vec{r}}{r^3}$$

Take an ~~...~~ infinitesimally small current element dl on the semicircular conductor. Let this make an angle ϕ ~~from~~ with the horizontal

$\therefore \quad dl = R\, d\phi$, where R is the radius of the semicircular wire

$$dl = R \sin\phi\, d\phi\, \hat{i} + R\cos\phi\, d\phi\, \hat{j}$$
$$= R\, d\phi\, (\sin\phi\, \hat{i} + \cos\phi\, \hat{j})$$

Let $\vec{r}$ be the perpendicular from the dl to the point P.

$$\vec{r} = OP - OQ$$
$$= R(\cos\theta\, \hat{i} + \sin\theta\, \hat{j}) - R(\cos\phi\, \hat{i} - \sin\phi\, \hat{j})$$
$$= R\left((\cos\theta - \cos\phi)\hat{i} + (\sin\theta + \sin\phi)\hat{j}\right)$$

$$\vec{dl} \times \vec{r} = \begin{vmatrix} i & j & k \\ \sin\phi & \cos\phi & 0 \\ (\cos\theta - \cos\phi) & (\sin\theta + \sin\phi) & 0 \end{vmatrix} R^2 \, d\phi$$

$$= R^2 \left[(\sin^2\phi + \sin\phi \sin\theta) - (\cos\theta\cos\phi - \cos^2\phi) \right] d\phi \, \hat{k}$$

$$= R^2 \left[(\sin^2\phi + \cos^2\phi) + \sin\phi \sin\theta - \cos\phi\cos\theta \right] d\phi \, \hat{k}$$

$$= R^2 \left(1 - \cos(\theta + \phi) \right) d\phi \, \hat{k}$$

Now to find the $\vec{B}$ due to the entire semicircle we use ~~integrate~~ Biot-Savartz law

$$B = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl} \times \vec{r}}{r^3}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{R^2 (1 - \cos(\theta + \phi)) \, d\phi}{|r|^3}$$

$$|r|^3 = \left[\sqrt{R^2 \left[(\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2 \right]} \right]^3$$

$$= \left[R^2 \left[\cancel{2}\cos^2\theta + \cos^2\phi + \sin^2\theta + \sin^2\phi - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi \right] \right]^{3/2}$$

$$= \left[R^2 \left(2 - 2\left[\cos\theta\cos\phi - \sin\theta\sin\phi \right] \right) \right]^{3/2}$$

$$= \left[2R^2 \left(1 - \cos(\theta + \phi) \right) \right]^{3/2}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{\pi} \frac{R^2(1-\cos(\theta+\phi))\, d\phi}{[2R^2(1-\cos(\theta+\phi))]^{3/2}}$$

$$= \frac{\mu_0 I \times R^2}{4\pi \times R^3 \times 2\sqrt{2}} \int_0^{\pi} \frac{1-\cos(\theta+\phi)}{[1-\cos(\theta+\phi)]^{3/2}}\, d\phi$$

$$= \frac{\mu_0 I}{8\sqrt{2}\pi R} \int_0^{\pi} \frac{1}{(1-\cos(\theta+\phi))^{1/2}}\, d\phi$$

$$= \frac{\mu_0 I}{8\sqrt{2}\pi R} \int_0^{\pi} \frac{1\, d\phi}{\left(2\sin^2\left(\frac{\theta+\phi}{2}\right)\right)^{1/2}}$$

$$= \frac{\mu_0 I}{16\pi R} \int_0^{\pi} \operatorname{cosec}\left(\frac{\theta+\phi}{2}\right) d\phi$$

$$\because \int \operatorname{cosec} x = \ln\left|\tan\frac{x}{2}\right| + c$$

$$= \frac{\mu_0 I}{16\pi R} \left[2\ln\left[\tan\left(\frac{\theta+\phi}{4}\right)\right]\right]_0^{\pi}$$

$$= \frac{\mu_0 I}{8\pi R} \ln\left[\frac{\tan\left(\frac{\theta+\pi}{4}\right)}{\tan\left(\frac{\theta}{4}\right)}\right] \hat{k}$$