Electrostatics $\xLeftrightarrow{\textit{Relativity}}$ Magnetism

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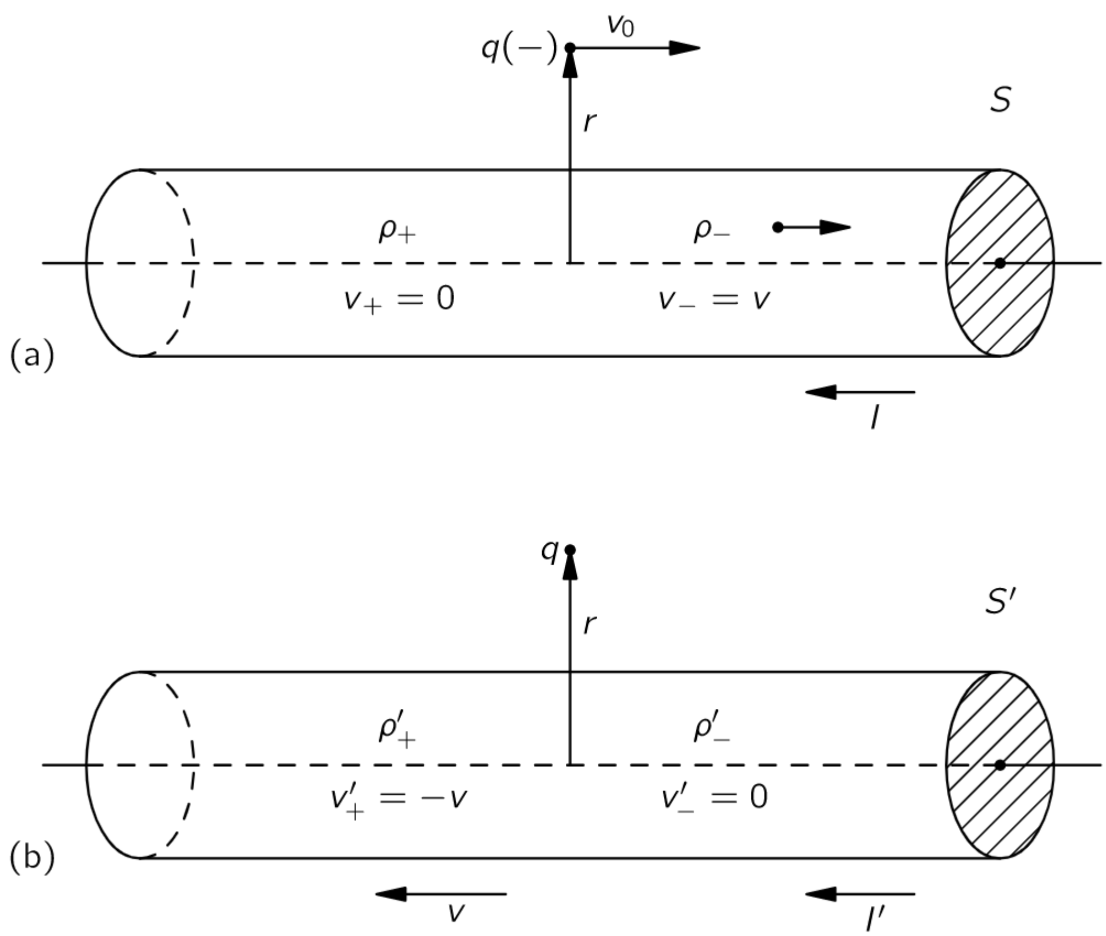
Motion of electron in two reference frames

Consider a negative charge q which moves with velocity v_0 (where v_0 is the drift velocity of electrons) parallel to a current-carrying wire. Analyzing motion of the electron in two reference frames: S fixed with respect to the wire, as in part (a) of the figure, and S' which moves with a velocity v_0, hence remains fixed with respect to the particle, as in part (b).



Failure of classical mechanics

In the S frame, there is clearly a magnetic force on the particle with magnitude $\mathbf{F} = qv_0 \times \mathbf{B}$, directed toward the wire, so if the charge were moving freely, we would observe it curve in toward the wire. The effect of electric field can be neglected, assuming steady flow of charges, with no accumulation of charge densities.

But in the S' frame, there can be no magnetic force on the particle, because its velocity is zero. Implies, the negative charge would stay motionless in the S' frame, violating the fundamental laws of physics.

Resolution through Analysis

S frame of reference

On evaluating the magnetic field due to a current carrying conductor, $\mathbf{B} = \dfrac{1}{4\pi\varepsilon_0 c^2}\dfrac{2I}{r}$

Hence, force on the negative charge $\mathbf{F} = qv_0\mathbf{B} = \dfrac{1}{4\pi\varepsilon_0 c^2}\dfrac{2qIv_0}{r}$

Since, current in the wire $I = \mathbf{J} \cdot \mathbf{A}$, where $\mathbf{A}$ is the cross-sectional area, where $\mathbf{J} = \rho_- v_0$, where ρ_- is the charge density of the conduction electrons, moving with the drift velocity v_0

Hence, the force in S frame $\mathbf{F} = \dfrac{q}{2\pi\varepsilon_0}\dfrac{\rho_- A}{r}\dfrac{v_0^2}{c^2}$

S′ frame of reference

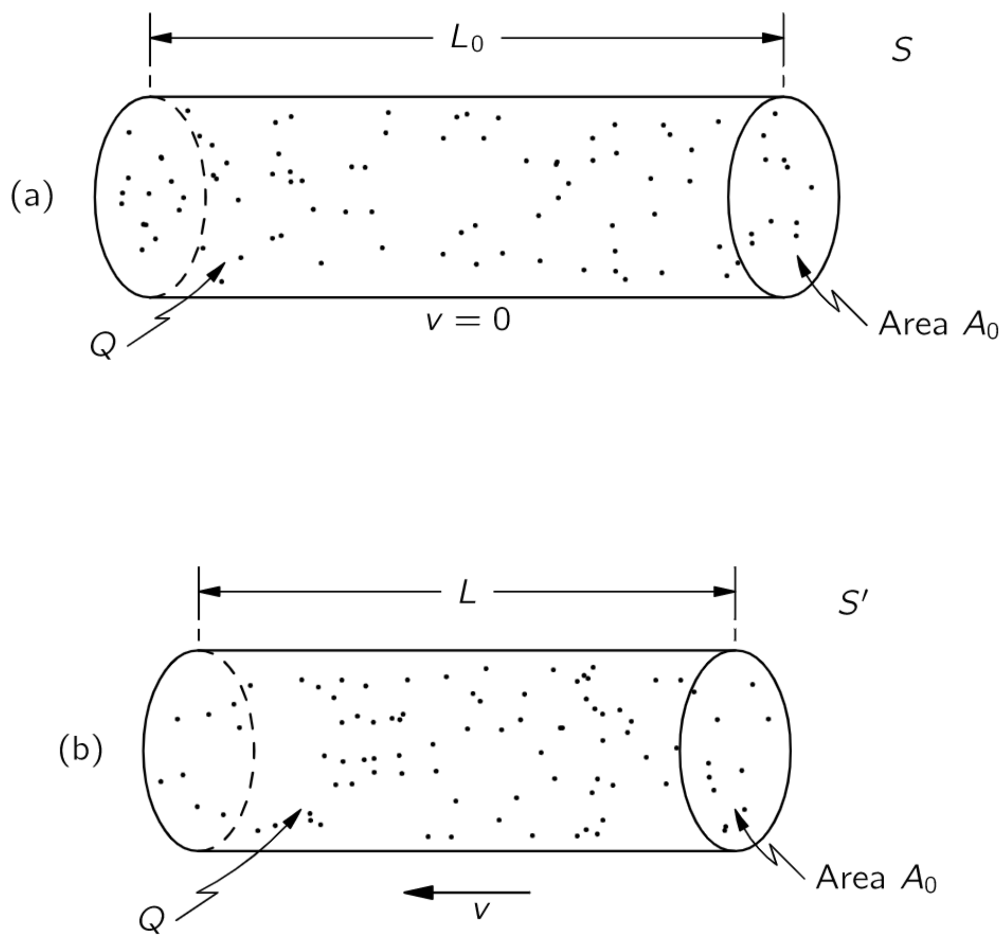
In S', the particle is at rest and the wire is running past (toward the left in the figure) with the speed v_0. The positive charges (charge density ρ_+) moving with the wire will make some magnetic field $\mathbf{B}'$ at the particle. Since the particle is at rest, there is no magnetic force.

If there is any force on the particle, it must come from an electric field, produced by the moving wire. This is only possible if the wire is charged locally, but since the wire is neutral, the current appears to be charged when set in motion.

Relativity to the rescue

Experimentally, the charge q is observed to be invariant under Lorentz transformations, and conservation of charge is validated. Since, charge is independent of frame of reference, in any frame the charge density of a distribution of electrons is just proportional to the number of electrons per unit volume, where volume can change because of the relativistic contraction of distances.

If we take a length L_0 of the wire, in which there is a charge density ρ of stationary charges, it will contain the total charge $Q = \rho L_0 A_0$. If the same charges are observed in a different frame to be moving with velocity v_0, they will all be found in a piece of the material with the shorter length $L = L_0 \left(\sqrt{1 - \frac{v_0^2}{c^2}}\right)$ (Length contraction)



Since dimensions transverse to the motion are unchanged, $A = A_0$. Since the charge is invariant, $Q = Q_0$, implies $\rho = \dfrac{\rho_0}{\sqrt{1-\frac{v_0^2}{c^2}}}$.

For the positive charge density ρ_+ of the wire, in S', $\rho'_+ = \dfrac{\rho_+}{\sqrt{1-\frac{v_0^2}{c^2}}}$

Since the negative charges are at rest in S', they have their rest density ρ'_- in this frame. Hence, for the conduction electrons, in S frame $\rho_- = \dfrac{\rho'_-}{\sqrt{1-\frac{v_0^2}{c^2}}}$

Hence, in S' frame, $\rho' = \rho'_+ + \rho'_- = \dfrac{\rho_+}{\sqrt{1-\frac{v_0^2}{c^2}}} + \rho_-\left(\sqrt{1-\frac{v_0^2}{c^2}}\right)$

Since the stationary wire is neutral, $\rho_- = -\rho_+$, implies $\rho' = \rho_+\dfrac{\frac{v_0^2}{c^2}}{\sqrt{1-\frac{v_0^2}{c^2}}}$

Evaluating the electrostatic force in $\mathbf{S'}$

On evaluating the electric field due to a current carrying conductor, $\mathbf{E} = \dfrac{\rho' A}{2\pi\varepsilon_0 r}$

Since $\rho' = \rho_+\dfrac{\frac{v_0^2}{c^2}}{\sqrt{1-\frac{v_0^2}{c^2}}}$, hence, $\mathbf{E} = \dfrac{\rho_+ A}{2\pi\varepsilon_0 r}\dfrac{\frac{v_0^2}{c^2}}{\sqrt{1-\frac{v_0^2}{c^2}}}$

Hence, force on the negatively charged particle $\mathbf{F}' = \dfrac{q\rho_+ A}{2\pi\varepsilon_0 r}\dfrac{\frac{v_0^2}{c^2}}{\sqrt{1-\frac{v_0^2}{c^2}}}$ is towards the wire.

Comparing the force in S frame, $\mathbf{F} = \dfrac{q}{2\pi\varepsilon_0}\dfrac{\rho_- A}{r}\dfrac{v_0^2}{c^2}$

Hence, force in S' frame $\mathbf{F}' = \dfrac{q}{2\pi\varepsilon_0}\dfrac{\rho_+ A}{r}\dfrac{v^2}{c^2}\dfrac{1}{\sqrt{1-\frac{v_0^2}{c^2}}} = \dfrac{\mathbf{F}}{\sqrt{1-\frac{v_0^2}{c^2}}}$

Decoding the result

The electrostatic force of attraction in S' is $\mathbf{F}' = \dfrac{\mathbf{F}}{\sqrt{1-\dfrac{v_0^2}{c^2}}}$,

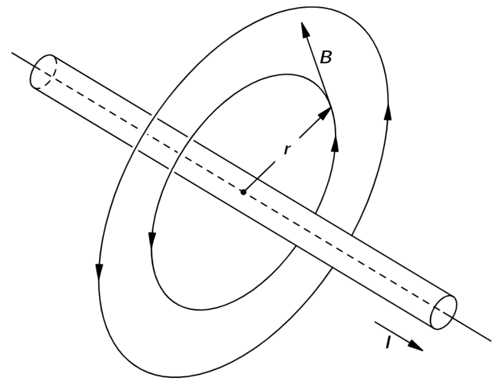
where $\mathbf{F} = \dfrac{q}{2\pi\varepsilon_0}\dfrac{\rho_- A}{r}\dfrac{v_0^2}{c^2}$ is the magnetic force of attraction in S frame of reference.

For the small drift velocities $v_0 << c$ we have been considering, the two forces are equal. According to Feynman, we understand that magnetism and electricity are just "two ways of looking at the same thing."

If we had chosen another coordinate system, we would have found a different mixture of **E** and **B** fields. Electric and magnetic forces are part of one physical phenomenon: the electromagnetic interactions of particles. The separation of this interaction into electric and magnetic parts depends very much on the reference frame chosen for the description, bridged by Einstein's special theory of relativity.

Appendix

Magnetic field due to a current carrying conductor



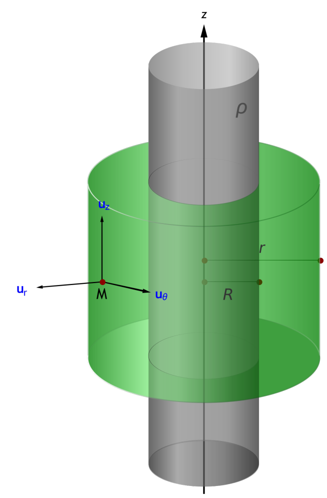
For an Amperian loop of radius r, $\oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 I_{enc}$

From the symmetry, $\mathbf{B}$ has the same magnitude at all points on a circle concentric with the wire, implies $\oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mathbf{B} \cdot 2\pi r$

Hence, $\mathbf{B} \cdot 2\pi r = \mu_0 I$, hence $\mathbf{B} = \dfrac{\mu_0 I}{2\pi r}$

Since $\mu_0 \varepsilon_0 = \dfrac{1}{c^2}$, implies $\mathbf{B} = \dfrac{1}{4\pi\varepsilon_0 c^2} \dfrac{2I}{r}$

Electric field due to a current carrying conductor



For an Gaussian cylinder of radius r and length l,
$\oint \mathbf{E} \cdot d\mathbf{S} = \dfrac{1}{\varepsilon_0} \int \rho \, dV$

From the symmetry, $\mathbf{E}$ has the same magnitude at all points on a circle concentric with the wire, implies $\oint \mathbf{E} \cdot d\mathbf{S} = \mathbf{E} \cdot 2\pi r L$

Hence, $\mathbf{E} \cdot 2\pi r L = \dfrac{\rho A L}{\varepsilon_0}$, hence $\mathbf{E} = \dfrac{\rho A}{2\pi\varepsilon_0 r}$

Special relativity and Lorentz transformations

Special relativity is based on postulates, that the laws of physics are invariant in all inertial frames of reference, and the speed of light in vacuum is the same for all observers, regardless of the motion of the light source or observer. Through the discovery of Lorentz's transformations, all physical laws (including Maxwell's laws) look the same in every inertial frame of reference. Under Lorentz's transformations, invariance of speed of light, charge, and relativity of space, time are observed.

Lorentz transformations

Define an event to have coordinates (x, y, z) at time t in system S and (x', y', z') at time t' in a reference frame moving at a velocity v with respect to that frame, S'. Then,

$$t' = \gamma \left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma\,(x - vt)$$
$$y' = y \qquad z' = z$$

where $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$, and c is the speed of light

The consequences of special relativity can be derived from the Lorentz transformation equations. These transformations, and hence special relativity, lead to different physical predictions than those of Newtonian mechanics at all relative velocities, and most pronounced when relative velocities become comparable to the speed of light.

Time Dilation

The time lapse between two events is not invariant from one observer to another, but is dependent on the relative speeds of the observers' reference frames. Suppose a clock is at rest in the S. The location of the clock on two different ticks is then characterized by $\Delta x = 0$. For a clock in S', $\Delta t' = \gamma\,\Delta t = \dfrac{\Delta t}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

This shows that the time $(\Delta t')$ between the two ticks as seen in the frame in which the clock is moving (S'), is longer than the time (Δt) between these ticks as measured in the rest frame of the clock (S).

Length contraction

The dimensions of an object as measured by one observer may be smaller than the results of measurements of the same object made by another observer. Suppose a measuring rod is at rest and aligned along the x axis in S. In this system, the length of this rod is written as Δx. To measure the length of this rod in the system S', in which the rod is moving, the distances x' to the end points of the rod must be measured simultaneously in that system S'. In other words, the measurement is characterized by $\Delta t' = 0$, hence $\Delta x' = \dfrac{\Delta x}{\gamma} = \Delta x \left(\sqrt{1 - \dfrac{v^2}{c^2}}\right)$.

This shows that the length ($\Delta x'$) of the rod as measured in the frame in which it is moving (S'), is shorter than its length (Δx) in its own rest frame (S).

Maxwell Equations

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int \rho \, \mathrm{d}\mathbf{V} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\oint \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{B} \cdot \mathrm{d}\mathbf{S} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \left(\int \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{E} \cdot \mathrm{d}\mathbf{S} \right) \qquad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

References

- *Classical electromagnetism and Special Relativity: Wikipedia*
- *Special Theory of Relativity: Wikipedia*
- *How Special Relativity Makes Magnets Work: Veritasium, YouTube*