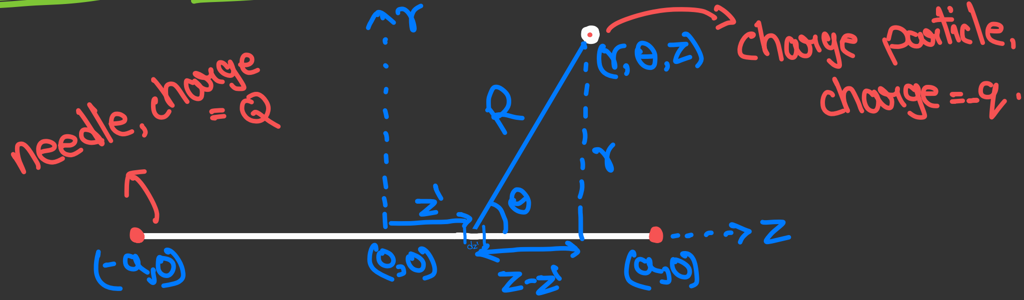
KNITTING NEEDLE AND CHAGED DROPLET



Questions we wish to Answer :-

① How is the motion in $z=0$ plane?

② What is the trajectory of the particle, when a sufficiently low velocity is imparted to it in z-direction?

③ What is the minimum velocity for the particle to not be trapped inside the Electric Bottle?

Solution :-

charge density $= \dfrac{Q}{2a} = \lambda$

$\Rightarrow$ Electric Potential $V(r,\theta,z) = \displaystyle\int_{-a}^{a} \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dz'}{R}$

$$= \int_{-a}^{a} \frac{\lambda}{4\pi\varepsilon_0} \frac{dz'}{\left[r^2+(z-z')^2\right]^{1/2}}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0} \ln\left[\sqrt{r^2+(z-z')^2} + z - z'\right]\Bigg|_{-a}^{a}$$

$$\Rightarrow \boxed{V_{(r,z)} = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z+a+R_1}{z-a+R_2}\right)} \; ; \; R_1 = \sqrt{r^2+(z+a)^2}$$

$$R_2 = \sqrt{r^2+(z-a)^2}$$

Now to analyze motion in $z=0$ plane, Let us find $\vec{E}$ in $z=0$:-

By symmetry, we can see that $\vec{E}$ has only radial component.

$$\Rightarrow E_{(r,z=0)} = \frac{-\partial V_{(r,z=0)}}{\partial r}\hat{r} = \frac{-\partial}{\partial r}\left[\frac{\lambda}{4\pi\varepsilon_0} \ln \frac{a+\sqrt{r^2+a^2}}{\sqrt{r^2+a^2}-a}\right]\hat{r}$$

$$\Rightarrow \boxed{\vec{E}_{(r,z=0)} = \frac{\lambda}{4\pi\varepsilon_0} \frac{2a}{r\sqrt{r^2+a^2}}\hat{r}}$$

$\Rightarrow$ A charged particle with $-q$ charge and mass m, will have a circular orbit around the needle in $z=0$ with, say, radius r_0 and velocity $v_0 = r_0\omega_0$

$$\Rightarrow \quad m\omega_0^2 r_0 = |\vec{E}|q$$

$$m\omega_0^2 r_0 = \frac{\lambda}{4\pi\varepsilon_0} \frac{2aq}{r_0\sqrt{r_0^2+a^2}}$$

$$\boxed{\omega_0^2 = \frac{\lambda}{4\pi\varepsilon_0} \frac{2aq}{m r_0^2\sqrt{r_0^2+a^2}}}$$

Before we answer the next questions, Let us analyze Angular momentum of the particle moving in a random path around the needle:- $\longrightarrow$

$\vec{r} = r\hat{r} + z\hat{z}$

We Know that, $\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$; $\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$; $\frac{d\hat{z}}{dt} = 0$

$$\Rightarrow \vec{v} = \frac{d}{dt}(\vec{r}) = \frac{d}{dt}(r\hat{r} + z\hat{z})$$

$$\Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

$$\vec{L} = \vec{r} \times m\vec{v} = (r\hat{r} + z\hat{z}) \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z})$$

$$\Rightarrow \boxed{\vec{L} = -mrz\dot{\theta}\hat{r} + m(\dot{r}z - r\dot{z})\hat{\theta} + mr^2\dot{\theta}\hat{z}}$$

With Some tedious math,

$$\Rightarrow \boxed{\frac{d\vec{L}}{dt} = -mz(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{r} + m(\ddot{r}z + r\ddot{z} - rz\dot{\theta})\hat{\theta} + mr(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{z}}$$

We can calcutate $\frac{d\vec{L}}{dt}$ in a different way :-

$$\frac{d\vec{L}}{dt} = \tau = \vec{r} \times \vec{F} = \vec{r} \times (-q\vec{E}) = \vec{r} \times [q\nabla V(r,z)]$$

$$\Rightarrow \boxed{\frac{d\vec{L}}{dt} = (zF_r - rF_z)\hat{\theta}}$$

L_r & L_z = Const

$$\Rightarrow \boxed{L_z = mr^2\dot{\theta} = \text{Constant}}$$

$$\Rightarrow \dot{\theta} = \frac{L_z}{mr^2}$$

The trajectory when the particle is given a velocity v_{0z} perpendicular to $z=0$ plane, while the particle is orbiting in $z=0$ plane:-

When the velocity is given to the particle while it's orbiting, we can intuitively see that the particle will now follow an Helical path.

Now, the final Question!!

$\underbrace{\text{At } z=0,\ r=r_0,\ \dot{r}=0,\ r_0\omega_0=v_0,\ L_z=mr_0v_0,\ \dot{z}=v_{0z}}_{\text{Initial Conditions, at } z=0 \text{ plane, when velocity is given.}}$

In the subsequent motion, The total energy U, is conserved.

$$\Rightarrow U=\frac{1}{2}m(\vec{v})^2+V(r,z)=\frac{1}{2}m(\vec{v_0})^2+V(r_0,z=0)$$

$$U=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)-\frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{z+a+R_1}{z-a+R_2}\right)$$

Let $u=\frac{R_1+R_2}{2}$; R_1, R_2 were defined earlier.

$$\boxed{U=\frac{m}{2}\left(\dot{r}^2+\frac{L_z^2}{m^2r^2}+\dot{z}^2\right)-\frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{u+a}{u-a}\right)}$$

Energy is conserved.

$$\Rightarrow U_{(r_0, z=0)} = U_{(r,z)}$$

$$\frac{m}{2}\left(\frac{L_z^2}{m^2 r_0^2} + v_{0z}^2\right) - \frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{U_0+a}{U_0-a}\right) = \frac{m}{2}\left(\dot{r}^2 + \frac{L_z^2}{m^2 r^2} + \dot{z}^2\right) - \frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{U+a}{U-a}\right)$$

$$;\ U_0 = \sqrt{r_0^2 + a^2}$$

If there exist $\pm z_1$ for the helical motion, so that the particle is traped in between $\pm z_1 \rightarrow$ Electric bottle

$$\Rightarrow \text{At } z = z_1,\ \dot{r} = 0,\ \dot{z} = 0,\ r = r_1$$

$$\Rightarrow \frac{m}{2}\left(\frac{L_z^2}{m^2 r_0^2} + v_{0z}^2\right) - \frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{U_0+a}{U_0-a}\right) = \frac{m}{2}\left(0 + \frac{L_z^2}{m^2 r_1^2} + 0\right) - \frac{\lambda q}{4\pi\varepsilon_0}\ln\left(\frac{U_1+a}{U_1-a}\right)$$

$$;\ U_1 = \frac{\sqrt{r_1^2 + (z_1+a)^2} + \sqrt{r_1^2 + (z_1-a)^2}}{2}$$

for, $z_1 \rightarrow \infty$, Potential $\rightarrow 0$.

$$\Rightarrow \frac{L_z^2}{m^2 r_0^2} + v_{0z}^2 - \frac{\lambda q}{2\pi\varepsilon_0 m}\ln\left(\frac{U_0+a}{U_0-a}\right) = \frac{L_z^2}{m^2 r_1^2}$$

As, $r_1 < r_0$

$$\Rightarrow \boxed{v_{0z}^2 > \frac{\lambda q}{2\pi\varepsilon_0 m}\ln\left(\frac{\sqrt{r_0^2+a^2}+a}{\sqrt{r_0^2+a^2}-a}\right)}$$

$\rightarrow$ Velocity required to eject the droplet.

Thus turning points exist for,

$$v_{0z}^2 \leq \frac{\lambda q}{2\pi m\varepsilon_0} \ln\left(\frac{\sqrt{r_0^2+a^2}+a}{\sqrt{r_0^2+a^2}-a}\right).$$

References:

* Youtube → Science off the Sphere, Knitting needle
* J.D. Jackson, 3rd Edition