At $t = 0$ we have a spherical cloud of radius R and total charge Q, comprising N point-like particles. Each particle has charge $q = Q/N$ and mass m. The particle density is uniform, and all particles are at rest.

a) Evaluate the electrostatic potential energy of a charge located at a distance $r < R$ from the center at $t = 0$.

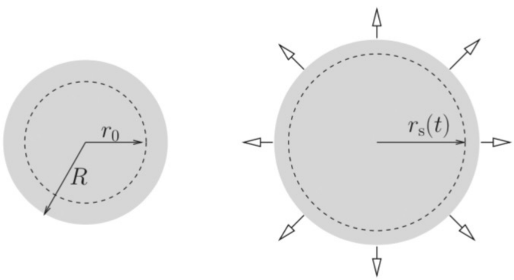


Fig. 1.4

b) Due to the Coulomb repulsion, the cloud begins to expand radially, keeping its spherical symmetry. Assume that the particles do not overtake one another, i.e., that if two particles were initially located at $r_1(0)$ and $r_2(0)$, with $r_2(0) > r_1(0)$, then $r_2(t) > r_1(t)$ at any subsequent time $t > 0$. Consider the particles located in the infinitesimal spherical shell $r_0 < r_s < r_0 + \mathrm{d}r$, with $r_0 + \mathrm{d}r < R$, at $t = 0$. Show that the equation of motion of the layer is

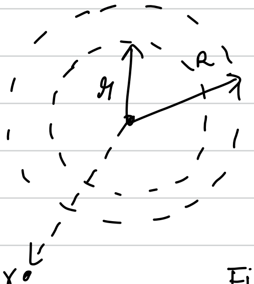
$$m\frac{\mathrm{d}^2 r_s}{\mathrm{d}t^2} = k_e \frac{qQ}{r_s^2}\left(\frac{r_0}{R}\right)^3 \tag{1.16}$$

c) Find the initial position of the particles that acquire the maximum kinetic energy during the cloud expansion, and determinate the value of such maximum energy.

their final kinetic energy. Compare the total kinetic energy with the potential energy initially stored in the electrostatic field.

e) Show that the particle density remains spatially uniform during the expansion.

a)



As N is large, it can be approximated to a continuous charge distribution.

$$V = -\int_{\infty}^{R} E\cdot dl + -\int_{R}^{r} E\cdot dl$$

Field outside the sphere $= \frac{1}{4\pi\varepsilon_0}\times\frac{Q}{r^2}$

$$-\frac{Q}{4\pi\varepsilon_0}\int^{R}\frac{1}{r^2}\,dr \qquad \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{R} = \frac{Q}{4\pi\varepsilon_0 R}$$

Field inside the sphere

$$E\times 4\pi r^2 = \frac{Q\times\frac{r^3}{R^3}}{\varepsilon_0}$$ ↳ Gauss's Law

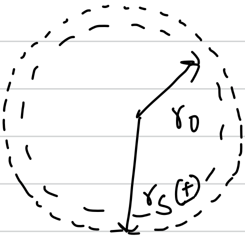
$$E = \frac{Qr}{4\pi\varepsilon_0 R^3} \qquad -\frac{Q}{4\pi\varepsilon_0 R^3}\int_{R}^{r} r\cdot dr$$

$$\frac{Q}{4\pi\varepsilon_0 R^3}\left[\frac{R^2}{2}-\frac{r^2}{2}\right]$$

Total potential at the point $= \frac{QR^2}{4\pi\varepsilon_0 R^3} + \frac{Q}{4\pi\varepsilon_0 R^3}\left[\frac{R^2}{2}-\frac{r^2}{2}\right]$

$$\frac{Q}{4\pi\varepsilon_0 R^3}\left[\frac{3}{2}R^2 - \frac{r^2}{2}\right]$$

b)



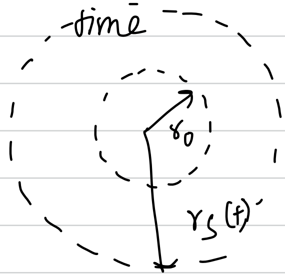
field at the infinitesmal layer at r_s

charge enclosed = constant throghout time for the layer $= Q\left(\frac{r_0}{R}\right)^3$

$$E \times 4\pi r_s^2 = \frac{Q}{\varepsilon_0}\left(\frac{r_0}{R}\right)^3$$

$$E = \frac{Q}{4\pi\varepsilon_0 r_s^2}\left(\frac{r_0}{R}\right)^3$$

at some later time



F on one particle $= m\cdot\frac{d^2 r_s}{dt^2} = \frac{qQ}{4\pi\varepsilon_0 r_s^2}\left(\frac{r_0}{R}\right)^3$

c) $\int F\cdot dr = \Delta KE$

$$= \frac{Qq}{4\pi\varepsilon_0}\left(\frac{r_0}{R}\right)^3 \int_{r_0}^{\infty} \frac{1}{r_s^2}\cdot dr$$

$$= \frac{Qq}{4\pi\varepsilon_0}\left(\frac{r_0}{R}\right)^3 \times \frac{1}{r_0} = \frac{Qq}{4\pi\varepsilon_0}\frac{r_0^2}{R^3}$$

KE max $= \frac{Qq}{4\pi\varepsilon_0}\frac{R^2}{R^3} = \frac{Qq}{4\pi\varepsilon_0 R}$

PE for outer layer $= \frac{Qq}{4\pi\varepsilon_0 R}$

Outer layer only → initial P.E = final K.E

not true for other layers, as final potential will not be 0

Innermost point does not move at all

d) Particle density remains spatially uniform through time

introduce a new variable $x = \frac{r_s}{r_0}$

$$\frac{dx}{dt} = \frac{dr_s}{dt}\frac{1}{r_0}$$

$$\frac{d^2x}{dt^2} = \frac{d^2r_s}{dt^2}\frac{1}{r_0}$$

$$m \cdot \frac{d^2x}{dt^2}\cancel{r_0} = \frac{qQ\,x^2\cancel{r_0}}{4\pi\varepsilon_0 R^3}$$

$$\frac{d^2r_s}{dt^2} = r_0 \cdot \frac{d^2x}{dt^2} \qquad m \cdot \frac{d^2x}{dt^2} = \frac{qQ\,x^2}{4\pi\varepsilon_0 R^3}$$

take 2 shells, with initial radii r_{10} & r_{20},

numbers of particles contained between them $= \frac{N \times (r_{10}^3 - r_{20}^3)}{R^3}$

this remains constant, because layers do not overtake each other

So at a time t, the particle density

$$= \frac{N/R^3\,(r_{10}^3 - r_{20}^3)}{\frac{4}{3}\pi(r_{1s}^3 - r_{2s}^3)}$$

$$r_{1s} = x(t)\,r_{10}$$
$$r_{2s} = x(t)\,r_{20}$$

$$= \frac{N/R^3\,\cancel{(r_{10}^3 - r_{20}^3)}}{\frac{4}{3}\pi\cancel{(r_{10}^3 - r_{20}^3)}\,x(t)^3}$$

→ so $x(t)$ is independent of r_0 → valid for all particles in the cloud

⇒ independent of choice of shells → spatially uniform ⇒ the layers with more particles are further apart