THE PHYSICS OF A GRADIENT

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THEORY BEHIND MY QUESTION

- The core concept of the question that I have chosen to present, is the gradient, which is an essential part of vector calculus, and finds numerous applications in Electrostatics.

What is a gradient?

When the del operator ∇ is followed by a scalar field, the result of the operation is called the gradient of the field. What does the gradient tell you about a scalar field? Two important things: the magnitude of the gradient indicates how quickly the field is changing over space, and the direction of the gradient indicates the direction in which the field is increasing most quickly with distance. So although the gradient operates on a scalar field, the result of the gradient operation is a vector, with both magnitude and direction. Thus, if the scalar field represents terrain height, the magnitude of the gradient at any location tells you how steeply the ground is sloped at that location, and the direction of the gradient points uphill along the steepest slope.

GEOMETRICAL INTERPRETATION OF THE GRADIENT

Geometrical Interpretation of the Gradient:

- Like any vector, the gradient has magnitude and direction.
- The gradient ∇T points in the direction of the maximum increase of the function T.
- The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

MAIN IDEAS BEHIND THE SUBPARTS OF MY QUESTION

- Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the direction of the gradient. Now measure the slope in that direction (rise over run). That is the magnitude of the gradient. (Here the function we're talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only two variables, not three, but the geometrical meaning of the gradient is easier to grasp in two dimensions.)
- What would it mean for the gradient to vanish? If $\nabla T = 0$ at (x,y,z), then $dT = 0$ for small displacements about the point (x, y,z). This is, then, a stationary point of the function $T\,(x,y,z)$. It could be a maximum (a summit), a minimum (a valley), a saddle point (a pass), or a "shoulder." This is analogous to the situation for functions of one variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

QUESTION

Exploring The Hill

Q. The height of a certain hill, named Mount Gradient, is given by

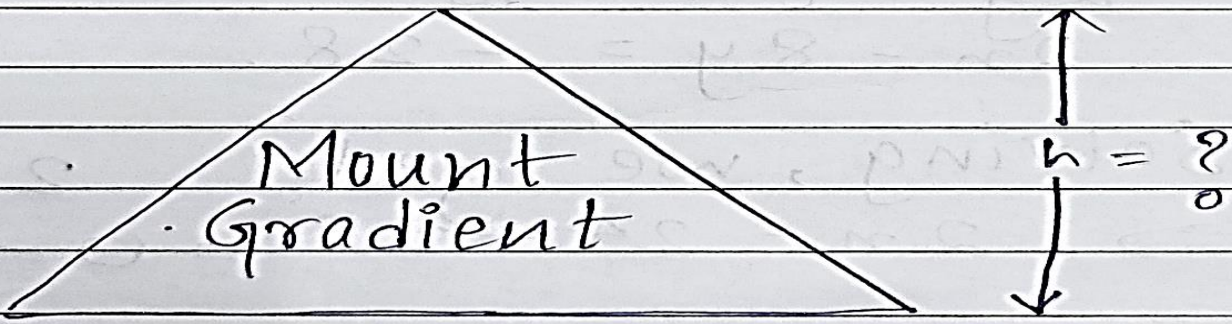
$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is the distance (in miles) north, x is the distance east of South Hadley.

a) Where is the top of the hill located?

b) How high is the hill?

c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?



SOLUTION

a) We first find the gradient of the height.

$$\nabla h = \frac{\partial h}{\partial x}\hat{x} + \frac{\partial h}{\partial y}\hat{y}$$

$$h = 20xy - 30x^2 - 40y^2 - 180x + 280y + 120$$

$$\nabla h = (20y - 60x - 180)\hat{x} + (2x - 8y + 28)\hat{y}$$

We want to find the top of the hill, which ~~will~~ essentially is the summit point.

As discussed earlier,

equate the gradient to zero.

which would give 2 eq^n.

$$2y - 6x = 18$$
$$2x - 8y = -28.$$

Solving, we get $22y = 66 \Rightarrow \boxed{y = 3}$

$\Longrightarrow 2x - 24 + 28 = 0 \Longrightarrow \boxed{x = -2}$

Hence, the top is 3 miles north, 2 miles west, of South Hadley.

b) Putting in $x = -2, y = 3$.

$h = 10(-12 - 12 - 36 + 36 + 84 + 12) = \boxed{720\,ft}$.

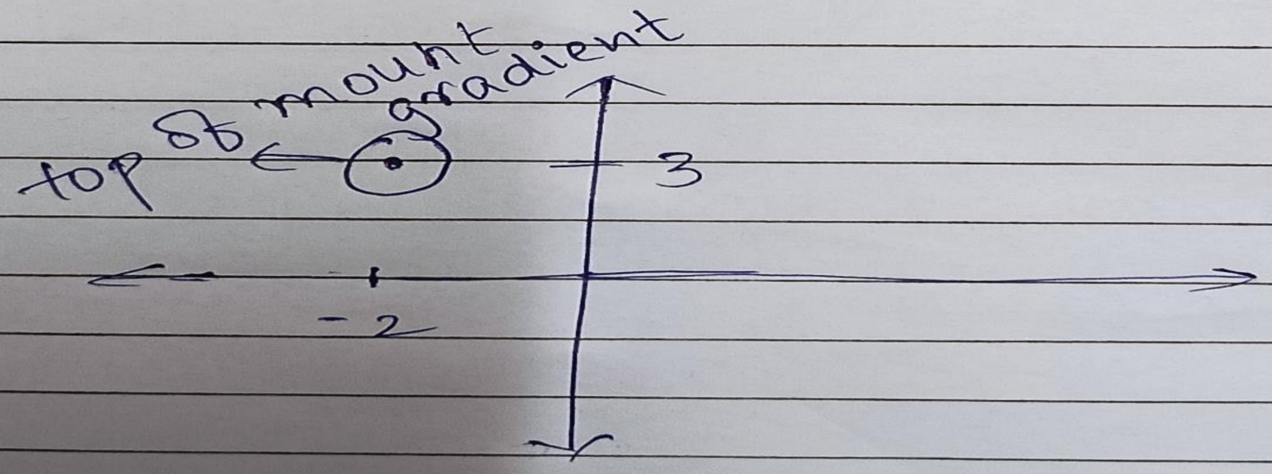
c) Putting in $x = 1, y = 1$

$\nabla h = 10(2 - 6 - 18)\hat{x} + (2 - 8 + 28)\hat{y}$

$= 220(-\hat{x} + \hat{y})$

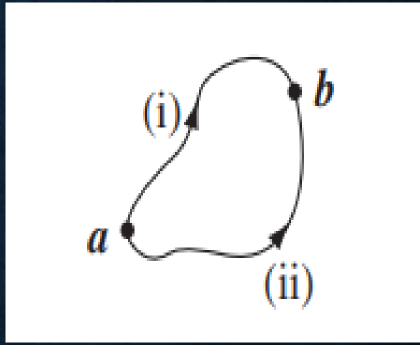
$|\nabla h| = 220\sqrt{2} \approx 311$ ft/mile.

direction : northwest.



APPLICATION OF GRADIENT IN ELECTROSTATICS

- By Stoke's Theorem, any vector whose curl is zero is equal to the gradient of some scalar.
- Because $\nabla \times E = 0$, the line integral of E around any closed loop is zero (that follows from Stokes' theorem). Because $\oint E.dl = 0$, the line integral of E from point a to point b is the same for all paths (otherwise you could go out along path (i) and return along path (ii)—Figure on the right—and obtain $\oint E.dl = 0$). Because the line integral is independent of path, we can define a function



$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.$$

APPLICATION OF GRADIENT IN ELECTROSTATICS

- Here O is some standard reference point on which we have agreed beforehand; V then depends only on the point r. It is called the electric potential. The potential difference between two points a and b is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

- Now, the fundamental theorem for gradients states that

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l},$$

so

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

APPLICATION OF GRADIENT IN ELECTROSTATICS

- Since, finally, this is true for any points a and b, the integrands must be equal:

$$\mathbf{E} = -\nabla V.$$

Small application of gradient.

$$V(x, y, z) = xy - 2y^2.$$

$$E = -\nabla \cdot V.$$

$$E = \frac{-\partial V}{\partial x} - \frac{\partial V}{\partial y}$$

$$= \left[-y\,\hat{i} + (-x + 4y)\,\hat{j} \right]$$