Physics Tutorial – Question on Electrostatics

Adapted from the book 'David J. Griffiths – Introduction to Electrodynamics (4th Edition)'

Question statement:

Imagine that new and extraordinarily precise measurements have revealed an error in Coulomb's law. The actual force of interaction between two point charges is found to be

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}.\left(1+\frac{R}{\lambda}\right).e^{-(R/\lambda)}\hat{R} \;-\; (i)$$

where λ is a new constant of nature (it has dimensions of length, obviously, and is a huge number—say half the radius of the known universe—so that the correction is small, which is why no one ever noticed the discrepancy before). You are charged with the task of reformulating electrostatics to accommodate the new discovery. Assume the principle of superposition still holds.

Sub-questions:

1) What is the electric field of a charge distribution 'ρ'?

Answer:

By the definition of the Electric field:-

$$\vec{E} = \frac{\vec{F}}{q_{test}}$$

In this question we can specify

$q_1 \Rightarrow \rho . dV = dq$ (Volume charge distribution)

$q_2 = q_{test}$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \left[\rho(\vec{r}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\left(1+\frac{\mathscr{r}}{\lambda}\right)e^{-(\mathscr{r}/\lambda)}\right]dV$$

2) Does this electric field admit a scalar potential? Explain briefly how you reached your conclusion. (No formal proof necessary – just a persuasive argument)

Answer:

Yes, the Electric field does have a scalar potential associated with it.

This field is entirely radial and depends only on the distance from the source r.

Hence, the curl of the Electric field is zero.

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

So, it admits a scalar potential 'V' such that,

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

This would hold true for a point charge, a system of discrete charges or a continuous charge distribution (by principle of superposition)

3) Find the potential of a point charge q. Use ∞ as your reference point.

Answer:

By definition:

$$V = -\int \vec{E} \cdot \vec{dl}$$

In this case, the limits would be defined as r and ∞

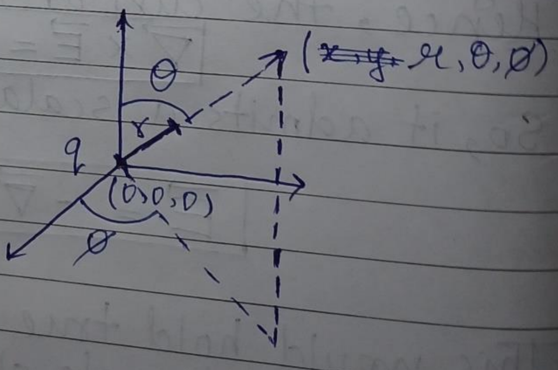
We can consider the point charge 'q' to be situated at the origin of the coordinate system.

$$\therefore\ V = -\int_{\infty}^{r} E \cdot dr = \frac{-q}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{e^{-r/\lambda}}{r^2}\left(1+\frac{r}{\lambda}\right) dr$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\int_{\infty}^{r} \frac{e^{-r/\lambda}}{r^2}\,dr + \int_{\infty}^{r} \frac{e^{-r/\lambda}}{\lambda r}\,dr\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[-\frac{e^{-r/\lambda}}{r}\Big|_{\infty}^{r} \cancel{- \frac{1}{\lambda}\int_{\infty}^{r} \frac{e^{-r/\lambda}}{r}} \cancel{+ \frac{1}{\lambda}\int_{\infty}^{r} \frac{e^{-r/\lambda}}{r}\,dr}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}$$



$$\boxed{V = \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}}$$

4) For a point charge q at the origin, show that:

$$\oint_S E.da + \frac{1}{\lambda^2}\int_V V.d\tau = \frac{q}{\mathcal{E}_0}$$

where S is the surface, V is the volume, of any sphere centred at q

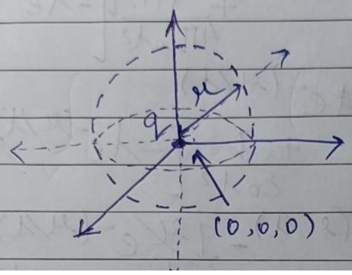
Answer:

Show that:

$$\oint_S E.da + \frac{1}{\lambda^2}\int_V V d\tau = \frac{q}{\varepsilon_0}$$

(for a point charge at the origin)

The Gaussian surface is a sphere centred at the origin



Radius of sphere = r

LHS:-

$$\oint_S \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-(r/\lambda)} r^2 \sin\theta \, d\theta \, d\phi$$

$$+ \int_V \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Area element in Spherical coordinates = $r^2 \sin\theta \, d\theta \, d\phi$

Volume element in spherical coordinates = $r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \cdot \frac{q}{4\pi\varepsilon_0}\left(1+\frac{r}{\lambda}\right)e^{-(r/\lambda)}$$

$$+\frac{q}{4\pi\varepsilon_0\lambda^2}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{r} e^{-r/\lambda} \cdot r \, dr$$

$$\frac{4\pi \cdot q}{4\pi\varepsilon_0}\left(1+\frac{r}{\lambda}\right)e^{-(r/\lambda)} + \frac{4\pi \cdot q}{4\pi\varepsilon_0\lambda^2}\left[-\lambda e^{-r/\lambda} r - \lambda^2 e^{-r/\lambda}\right]_0^r$$

$$\frac{q}{\varepsilon_0}e^{-(r/\lambda)} + \frac{qr}{\varepsilon_0\lambda}e^{-(r/\lambda)} + \frac{q}{\varepsilon_0\lambda^2}\left[-\lambda e^{-(r/\lambda)} r - \lambda^2 e^{-r/\lambda} + \lambda^2\right]$$

$$\cancel{\frac{q}{\varepsilon_0}e^{-(r/\lambda)}} + \cancel{\frac{qr}{\varepsilon_0\lambda}e^{-(r/\lambda)}} - \cancel{\frac{qr}{\varepsilon_0\lambda}e^{-(r/\lambda)}} - \frac{q}{\varepsilon_0}e^{(-r/\lambda)} + \frac{q}{\varepsilon_0}$$

$$RHS = \frac{q}{\varepsilon_0}$$

Henceproved

5) Show that this result generalizes:

$$\oint_S E.da + \frac{1}{\lambda^2}\int_V V.d\tau = \frac{Q_{enc}}{\mathcal{E}_0}$$

for any charge distribution. (This is the next best thing to Gauss' Law, in the new "Electrostatics")

Answer:

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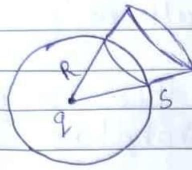
We will now consider a non-spherical surface?



This is a dent in the sphere pushing a patch of area $R^2\sin\theta d\theta d\phi$ to $S^2\sin\theta d\theta d\phi$ of radius S.

To prove the expression, let us consider the change of $\oint_S \vec{E}.d\vec{a} + \frac{1}{\lambda^2}\int V d\tau$

$$\Delta\oint \vec{E}.d\vec{a} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{S^2}\left(1+\frac{S}{\lambda}\right)e^{-(S/\lambda)}S^2\sin\theta d\theta d\phi - \frac{1}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-(R/\lambda)}R^2\sin\theta d\theta d\phi\right\}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta d\theta d\phi$$

$$\Delta\frac{1}{\lambda^2}\int V.d\tau = \frac{1}{\lambda^2}\frac{q}{4\pi\varepsilon_0}\int\frac{e^{-(r/\lambda)}}{r}r^2\sin\theta d\theta d\phi dr$$

$$= \frac{-1}{\lambda^2}\frac{q}{4\pi\varepsilon_0}\sin\theta d\theta d\phi\int_R^S e^{-r/\lambda}.r\,dr$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta d\theta d\phi$$

$$\Updownarrow$$

$$\frac{-q}{4\pi\varepsilon_0}\sin\theta d\theta d\phi\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\Big|_R^S$$

Change in $\frac{1}{\lambda^2}\int V d\tau$ exactly compensates change in $\oint E.da$ and we get $\frac{1}{\varepsilon_0}q$ for the total using the dented sphere, just as we did with the perfect sphere.

Any closed surface can be built up by successive distortions of the sphere, so the result holds true for all shapes.

By superposition, if there are many charges inside, the total is $\frac{Q_{enc}}{\varepsilon_0}$

Charges outside the Gaussian sphere don't contribute since

$$\Delta \nabla \oint_S \vec{E} \cdot da + \Delta \nabla \int_V v \cdot d\tau = 0$$

The "new" Gauss Law holds true for any charge configuration.

6) Draw the triangle diagram for this world, putting in all the appropriate formulae.

Answer:

We need to refer to the Gauss Law in the differential form to complete the triangle diagram.

We have already proved that:

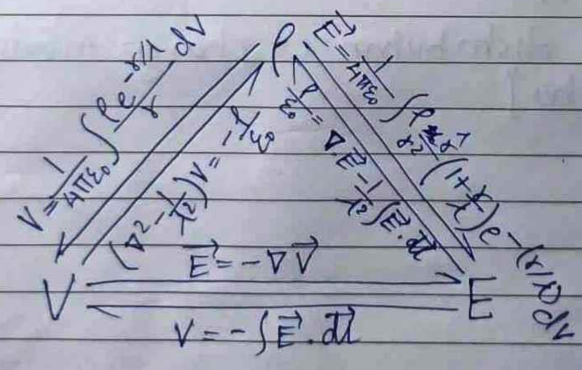
$$\oint_S \vec{E}\cdot d\vec{s} + \frac{1}{\lambda^2}\int_V V\,d\tau = \frac{Q_{enc}}{\varepsilon_0}$$

Using Stokes' Law of divergence of a gradient and putting $\vec{E} = -\vec{\nabla} V$:

$$-\int_T \nabla^2 V\,d\tau + \frac{1}{\lambda^2}\int_T V\,d\tau = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow \int_T \left(\nabla^2 - \frac{1}{\lambda^2}\right) V\,d\tau = -\int \frac{\rho}{\varepsilon_0}\,d\tau$$

$$\Rightarrow \left(\nabla^2 - \frac{1}{\lambda^2}\right) V = -\frac{\rho}{\varepsilon_0}$$



7) Show that *some* of the charge on a conductor distributes itself uniformly over the volume, with the remainder on the surface.

Answer:

Again, we can refer to Gauss' Law in its differential form.

Since E_{net} is zero inside (otherwise charge would move and in such a direction as to cancel the field), V is constant (inside) and hence ρ is uniform throughout the volume.

Any "extra" charge must reside on the surface.

The fraction at the surface depends on λ and on the shape of the conductor.

We know $V = -\int \vec{E} \cdot \vec{dl}$

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot \vec{E} - \frac{1}{\lambda^2} \int \vec{E} \cdot \vec{dl}$$

$$\therefore \rho = \frac{V \varepsilon_0}{\lambda^2} = \text{const.}$$

[Uniform distribution of charge inside the conductor]

NOTE: In all the answers 𝓇 has been used instead of *R* which is a part of equation *i*.