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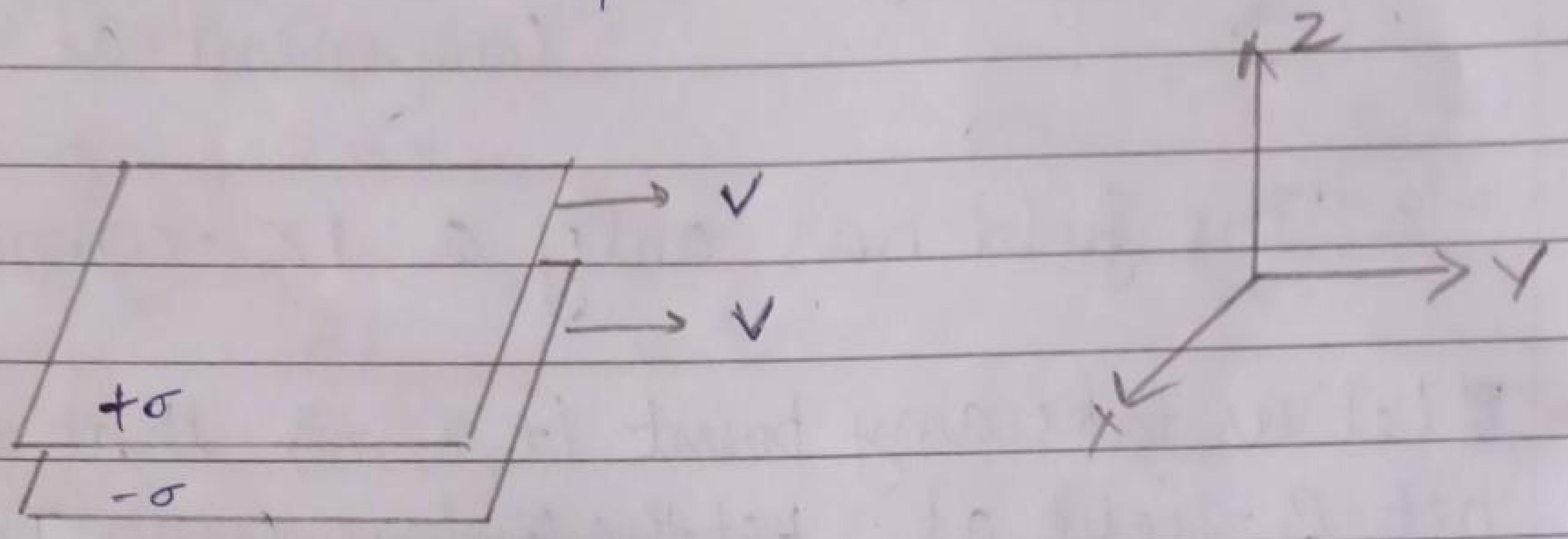
Physics Presentation

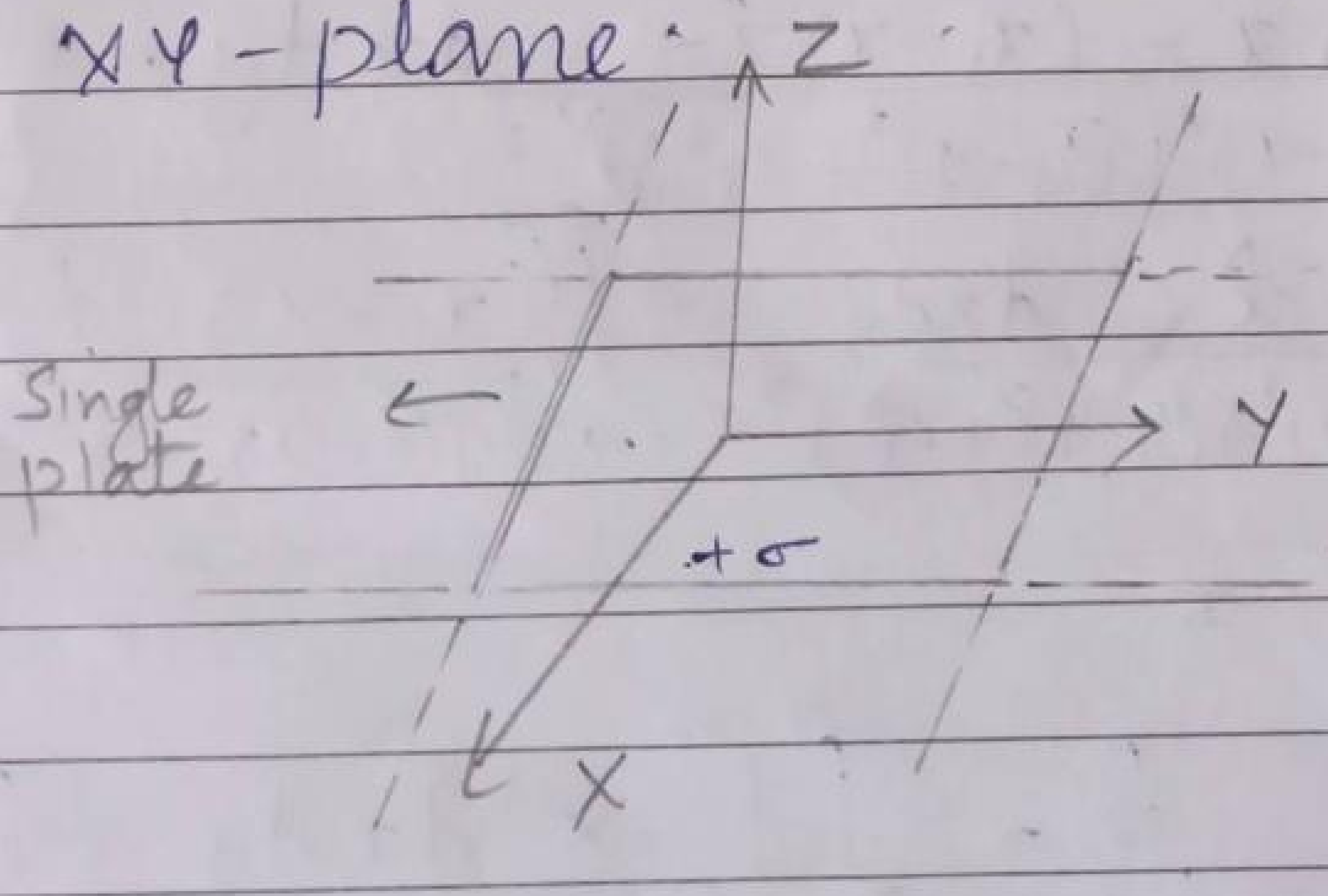
A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in the figure.



a) Find the magnetic field between the plates and also above and below them.
b) Find the magnetic force per unit area on the upper plate, including the direction.

Ans: Consider that the velocity of the plates is in the y direction.

a) Since the dimension of the plates is not mentioned we assume that the capacitor is infinite in the XY-plane.



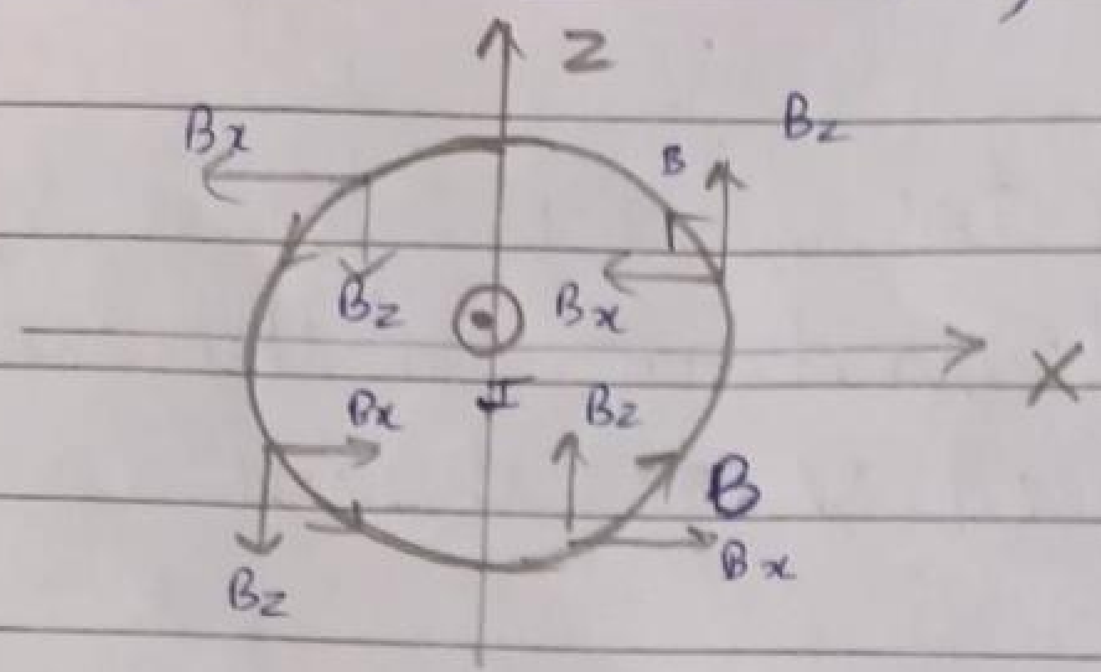
According to Biot-Savart's Law,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times \vec{n}\, da'}{|n|^3}$$

where $\vec{n} = \vec{r} - \vec{r}\,'$, $\vec{r}$ is the vector distance from the origin to the field point and $\vec{r}\,'$ is the vector distance from origin to the source charge.

We know, $\vec{K} = \sigma v \hat{y}$

Since $\vec{B}$ is $\perp^r$ to $\vec{K}$, $\vec{B}$ cannot have a Y-component.



From this figure, ~~the~~ any vertical component of Magnetic field at +X is cancelled by the corresponding component at −X.

∴ The field has only a X-component.

Let us take any point (x_1, y_1, z_1) that we want to find the net B-field at. ~~Let the surface curren~~ The field is produced due to the surface current at the general point $(x, y, 0)$.

$$\vec{n} = (x_1 - x)\hat{x} + (y_1 - y)\hat{y} + z_1\hat{z}$$

$$\vec{K}(\vec{r}') \times \vec{n} = \sigma v\, \hat{y} \times ((x_1 - x)\hat{x} + (y_1 - y)\hat{y} + z_1\hat{z}) = (z_1\hat{x} - (x_1 - x)\hat{z})\sigma v.$$

Integrating all over XY plane,

$$\vec{B}(\vec{r}) = \frac{\mu_0 \sigma v}{4\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\vec{K}(\vec{r}) \times \vec{n}}{((x_1-x)^2 + (y_1-y)^2 + z_1^2)^{3/2}}\, dx\, dy.$$

$$= \frac{\mu_0 \sigma v}{4\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z_1\hat{x} - (x_1-x)\hat{z}}{((x_1-x)^2 + (y_1-y)^2 + z_1^2)^{3/2}}\, dx\, dy.$$

$$= \frac{\mu_0 \sigma v}{4\pi}\left[\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z_1\hat{x}\, dx\, dy}{((x_1-x)^2 + (y_1-y)^2 + z_1^2)^{3/2}}}_{\mathrm{I}} - \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x_1-x)\,\hat{z}\, dx\, dy}{((x_1-x)^2 + (y_1-y)^2 + z_1^2)^{3/2}}}_{\mathrm{II}}\right]$$

$$\mathrm{II} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x_1-x)\, dx\, dy}{((x_1-x)^2 + (y_1-y)^2 + z_1^2)^{3/2}}\,\hat{z} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{X\, dx\, dy}{(X^2 + Y^2 + z_1^2)^{3/2}}\,\hat{z}, \quad X = x_1 - x, \; Y = y_1 - y$$

$$= \int_0^{\infty}\int_0^{2\pi} \frac{(r \sin\theta \cos\phi)\, r\, dr\, d\phi}{(r^2 + z_1^2)^{3/2}}.$$ (converting to polar coordinates).

$$= \sin\theta \int_0^{\infty} \frac{r^2\, dr}{(r^2 + z_1^2)^{3/2}} \int_0^{2\pi} \cos\phi\, d\phi. = \sin\theta \int_0^{\infty} \frac{r^2\, dr}{(r^2 + z_1^2)^{3/2}} \left[\sin\phi\right]_0^{2\pi}$$

$$= \underline{\underline{0}}.$$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z_1 \hat{x}\, dx\, dy}{((x_1-x)^2+(y_1-y)^2+z_1^2)^{3/2}} = z_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\, dy \quad \hat{x}}{(x^2+y^2+z_1^2)^{3/2}}$$

$$= z_1 \int_0^{\infty}\int_0^{2\pi} \frac{r\, dr\, d\phi}{(r^2+z_1^2)^{3/2}} \quad \hat{x} \quad (\because x^2+y^2=r^2).$$

$$= z_1 \cdot 2\pi \int_0^{\infty} \frac{r\, dr}{(r^2+z_1^2)^{3/2}}. \qquad \text{Let } r^2+z_1^2 = k. \quad 2r\,dr = dk.$$

$$= z_1 \pi \int_{z_1^2}^{\infty} \frac{dk}{k^{3/2}} = z_1 \pi \left[\frac{-2}{\sqrt{k}}\right]_{z_1^2}^{\infty}.$$

$$= -2\pi z_1 \left(0 - \frac{1}{\sqrt{z_1^2}}\right)$$

$$= \frac{2\pi z_1}{|z_1|}\, \hat{x}.$$

$$\therefore \vec{B} = \frac{\mu_0 \sigma v}{2\cancel{4}\pi} \times \frac{2\pi z_1}{|z_1|}\, \hat{x}.$$

$\therefore$ (for the top plate)

$$\vec{B} = \begin{cases} \dfrac{\mu_0 \sigma v}{2}\, \hat{x} & (z > 0, \text{ aiming out of the page} \\ -\dfrac{\mu_0 \sigma v}{2}\, \hat{x} & (z < 0, \text{ aiming into the page}. \end{cases}$$

Using similar calculation, the field produced by the bottom plate $(-\sigma)$

$$\bar{B} = \begin{cases} -\dfrac{\mu_0 \sigma v}{2}\, \hat{x} & (z > 0, \text{ aiming into the page} \\ \dfrac{\mu_0 \sigma v}{2}\, \hat{x} & (z < 0 \text{ aiming out of the page} \end{cases}$$

Above and below both plates the two fields cancel. Between the plates, they add up to $-\mu_0 \sigma V$, pointing into the page.

ie, $\vec{B} = \begin{cases} -\mu_0 \sigma v \hat{x} & \text{(between the plates} \\ 0 & \text{(above and below the plates)} \end{cases}$

b) The Lorentz force law says that

$$\vec{F}_{mag} = \int (K \times B)\, da$$

So force per unit area $f = K \times B$

Here $K = \sigma v \hat{y}$,

$\vec{B}$ (on the upper plate) $= -\frac{\mu_0 \sigma v}{2} \hat{x}$ (The field above the upper plate is cancelled).

$\therefore f_m = \frac{\mu_0 \sigma^2 v^2}{2} \hat{z}$ (Pointing upwards).

~~This can also be~~ The direction can also be calculated from right hand rule.