PH1213

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Q Consider the following potential of a point charge q at the origin.

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-r/\lambda}}{r}$$

Where q is at the origin, and λ is ~~at~~ a constant.

Such a potential arises because a charged ~~particle~~ particle q will attract oppositely charged particles, or repel like charges in the volume around it, ~~cause~~ causing a cloud of charge density to screen/mask the charge of the original ~~for~~ particle.

Determine the ~~C~~ charge distribution $\rho(\vec{r})$ associated with the electric field $E(\vec{r})$, associated with the potential $V(\vec{r})$.

Find the ~~t~~ net charge in all of space involved.

Ans. We can find the Electric Field ~~$\vec{E}(\vec{r})$~~ $\vec{E}(\vec{r})$ by taking the gradient of $-V(\vec{r})$, since $\vec{\nabla}\times\vec{E} = 0$.

$$\vec{E} = -\vec{\nabla} V(\vec{r})$$

$$= -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}\right)$$

(We use ~~sp~~ spherical coordinates, because the potential is spherically symmetric)

(The value of V depends on the distance from the origin, not the direction)

$$= -\left(\frac{q}{4\pi\varepsilon_0}\frac{\left(-\frac{1}{\lambda}e^{-r/\lambda}\cdot r - e^{-r/\lambda}\right)}{r^2}\hat{r} + 0 + 0\right)$$

$$= +\frac{q\, e^{-r/\lambda}}{4\pi\varepsilon_0 r^2}\left(\frac{r}{\lambda}+1\right)\hat{r}$$

$$= \frac{q\, e^{-r/\lambda}}{4\pi\varepsilon_0 r^2}\left(\frac{r}{\lambda}+1\right)\hat{r}$$

We can then use Gauss' Law in differential form to find charge ~~[illegible]~~ density.

$$\frac{\rho}{\varepsilon_0} = \vec{\nabla} \cdot \vec{E}$$

$$\Rightarrow \rho(\vec{r}) = \varepsilon_0 \left(\vec{\nabla} \cdot \left(\frac{q e^{-r/\lambda}}{4\pi\varepsilon_0 r^2} \left(\frac{r}{\lambda}+1\right)\hat{r} + 0\hat{\theta} + 0\hat{\phi} \right)\right)$$

$$= \varepsilon_0 \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2 \cdot q e^{-r/\lambda} (1+\frac{r}{\lambda})}{4\pi\varepsilon_0 r^2} \right)\right)$$

$$= \frac{\varepsilon_0}{r^2} \left(\frac{q}{4\pi\varepsilon_0} \left(\frac{\partial}{\partial r} e^{-r/\lambda} \left(\frac{r}{\lambda}+1\right)\right)\right)$$

$$= \frac{q}{4\pi r^2} \left(-\frac{1}{\lambda} e^{-\frac{r}{\lambda}} \left(\frac{r}{\lambda}+1\right) + \frac{1}{\lambda} e^{-r/\lambda} \right)$$

$$= \frac{-q}{4\pi r^2} \frac{e^{-\frac{r}{\lambda}}}{\lambda} \left(\frac{r}{\lambda} + 1 - 1 \right)$$

$$\rho(\vec{r}) = \frac{-q e^{-\frac{r}{\lambda}}}{4\pi r \lambda^2} \rightarrow (1)$$

But we know that there is a point charge at the ~~centre~~ origin which is not present in this expression, because $E(\vec{r})$ & $V(\vec{r})$ is not defined at the origin

Therefore, we represent the density of the charge at the origin with a Dirac-delta function

$$\rho(\vec{r}) = q\,\delta^3(\vec{r}) - \frac{q\,e^{-\frac{\lambda}{r}}}{4\pi r\lambda^2}$$

We could also get expression ① from the potential directly, using the Laplacian:

$$\rho(\vec{r}) = -\varepsilon_0 \nabla^2 V(\vec{r})$$

But taking Laplacian involves a lot of arithmetic in spherical coordinates, and we get more information about the system by finding $\vec{E}(\vec{r})$ first.

To find the net charge in the system ~~&~~ overall space, we integrate as follows:

$$Q = \int_{\text{All space}} \rho(\vec{r})\,d\tau$$

$$= \int \left(q\,\delta^3(\vec{r}) - \frac{q\,e^{-r/\lambda}}{4\pi r\lambda^2}\right) d\tau$$

$$= \int \left(q\,\delta^3(\vec{r})\right) d\tau - \int \frac{q\,e^{-r/\lambda}}{4\pi r\lambda^2}\,d\tau$$

$$= q - \int_{r=0}^{\infty}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{q\,e^{-r/\lambda}}{4\pi r\lambda^2}\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= q - q\iint \left[\frac{r e^{-r/\lambda}(-\cos\theta)}{4\pi\lambda^2}\right]_{\theta=0}^{\theta=\pi} d\phi\, dr$$

$$= q - q \int \int_{\phi=0}^{2\pi} \frac{2re^{-r/\lambda}}{4\pi\lambda^2} \, d\phi \, dr$$

$$= q - q \int \left[\frac{2re^{-r/\lambda}\phi}{4\pi\lambda^2} \right]_{\phi=0}^{\phi=2\pi} dr$$

$$= q - q \int_{r=0}^{\infty} \frac{r}{\lambda} e^{-r/\lambda} \frac{dr}{\lambda}$$

Let $\frac{r}{\lambda} = x$

$dx = \frac{dr}{\lambda}$

$$= q - q \int_{x=0}^{\infty} x e^{-x} \, dx$$

$$= q - q \left[-xe^{-x} - e^{-x} \right]_0^{\infty}$$

$$= q - q(1)$$

$$= 0$$

We see that the net charge of the system throughout space is zero.

This also means that the net charge of the cloud ~~s~~ that is charged is equal to the point charge, but of opposite sign.

Such potentials are seen where there are mobile charge carriers around the charge, damping the electric field around the charge