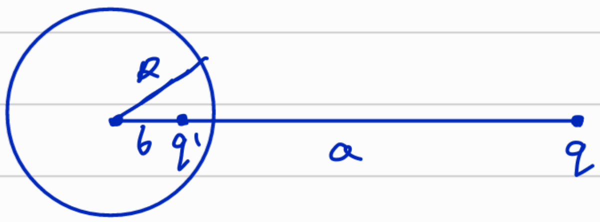
Q Two identical spherical conductors each of radius r, separated by a distance a, are both given charges Q. Find the force between the two spheres. (Find the 1^{st} order correction to coulomb's law which only holds for point charges)

Ans Part A

Consider a different system of a point charge q kept at a distance a from a grounded spherical conductor of radius R.



Placing an imaginary point charge of magnitude $q' = -\frac{R}{a} q$ at a distance

$b = \frac{R^2}{a}$ will cause the surface

of the conductor to have 0 potential as the potential due to q and q' cancel out in the spherical region. (appendix)

Part B

For a 1^{st} order approximation in the given system, divide Q into 2 parts q_1 and q_2 such that q_2 of one sphere acts as the image of q_1 of the other sphere.

That is to say, the charges on the conductors got divided to make their surface equipotential.

$\therefore$ We have $q_1 + q_2 = Q$

$$q_2 = -\frac{r}{a} q_1 = -\alpha q_1 \left[\alpha = \frac{r}{a}\right]$$

and $b = \dfrac{r^2}{a} = a\alpha^2$

from part A

$$\Rightarrow \quad q_1 = \frac{Q}{1-\alpha}, \quad q_2 = \frac{-Q\alpha}{1-\alpha}$$

$\therefore$ The force on one conductor = force on q_1 + force on q_2

$$= \frac{Kq_1^2}{a^2} + \frac{2Kq_1q_2}{(a-b)^2} + \frac{Kq_2^2}{(a-2b)^2}$$

$$= \frac{KQ^2}{a^2(1-\alpha)^2}\left[1 - \frac{2\alpha}{(1-\alpha^2)^2} + \frac{\alpha^2}{(1-2\alpha^2)^2}\right]$$

which after binomial approximation for small α gives: (appendix)

$$F = \frac{KQ^2}{a^2}\left(1 - \frac{4r^3}{a^3}\right)$$

Part C

This approximation is just of order 1 due to the fact that even though potential due to q_1 of conductor A on conductor B is cancelled by q_2 of conductor B, there is no charge to cancel the potential due to q_2 of A on B.

This would require further division of Q in 3 parts which can be found using

$$q_1 + q_2 + q_3 = Q$$

$$q_2 = -\alpha q_1 \quad \text{and} \quad q_3 = -q_2\left(\frac{r}{a-b}\right) = -q_2 \frac{\alpha}{1-\alpha^2}$$

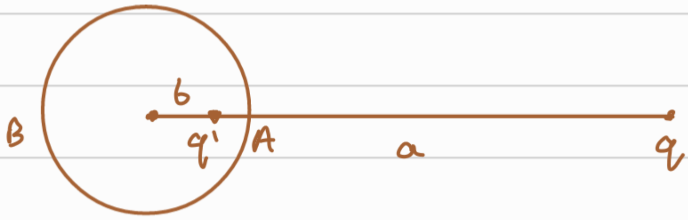
$$\Rightarrow q_1 = Q\,\frac{1-\alpha^2}{1-\alpha+\alpha^3}, \quad q_2 = -Q\,\frac{\alpha(1-\alpha^2)}{1-\alpha+\alpha^3}, \quad q_3 = \frac{Q\alpha^2}{1-\alpha+\alpha^3}$$

and the net force (2^{nd} order approximation) can be found using $F = F_{q_1} + F_{q_2} + F_{q_3}$

This method can be extended to n^{th} order and the force approximation will converge to the actual force as $n \to \infty$

Appendix :

Part A :



$$V_A = V_B = 0$$

$$\Rightarrow \quad \frac{kq}{a-r} + \frac{kq'}{r-b} = 0$$

$$q' = \left(\frac{r-b}{r-a}\right) q$$

and

$$\frac{kq}{r+a} + \frac{kq'}{r+b} = 0$$

$$q' = -\left(\frac{r+b}{r+a}\right) q$$

$$\Rightarrow \quad \left(\frac{r-b}{r+b}\right)\left(\frac{r+a}{r-a}\right) = -1$$

$$r^2 - ab - \cancel{rb + ra} = -r^2 + ab - \cancel{rb + ra}$$

$$b = \frac{r^2}{a}$$

$$q' = \frac{r\left(1 - \frac{r}{a}\right) q}{r - a}$$

$$q' = -\frac{r}{a} q$$

Part B

$$F = \frac{kq_1^2}{a^2} + \frac{2kq_1q_2}{(a-b)^2} + \frac{kq_2^2}{(a-2b)^2}$$

$$= kQ^2\left(\frac{1}{a^2(1-\alpha)^2} - \frac{2\alpha}{(a-a\alpha^2)^2(1-\alpha)^2} + \frac{\alpha^2}{(a-2a\alpha^2)^2(1-\alpha)^2}\right)$$

$$= \frac{kQ^2}{a^2(1-\alpha)^2}\left[1 - \frac{2\alpha}{(1-\alpha^2)^2} + \frac{\alpha^2}{(1-2\alpha^2)^2}\right]$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 \dots$$

$$F = \frac{kQ^2}{a^2}(1+2\alpha+3\alpha^2 \dots)\left[1 - 2\alpha(1+2\alpha^2 \dots) + \alpha^2(1+4\alpha^2 \dots)\right]$$

$$= \frac{kQ^2}{a^2}(1+2\alpha+3\alpha^2+4\alpha^3 \dots)(1-2\alpha+\alpha^2-4\alpha^3 \dots)$$

$$= \frac{kQ^2}{a^2}\left(1-4\alpha^3+O(\alpha^4)\right)$$