The beauty of inverse-square law electrostatic force

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Various implications of the inverse-square law have been presented. An unconventional geometrical method of calculating the E-field due to a uniform straight linear charge distribution has been presented which arises due to $1/r^2$ dependence of the E-field. A full understanding of this result can lead to a deeper appreciation of symmetry in a seemingly asymmetric system. It has been shown that how the ratio of gravitational to electric forces being a constant does not cause something like nuclear fission due to the r-dependence of the ratio of nuclear to electrostatic forces. Finally, an insight has been given on how the inverse-square law is much easier to test than those which may depend on some other powers of r.

I. E-field DUE TO A STRAIGHT LINE CHARGE *(thin rod)*

The problem is to find the electric field everywhere due to a thin rod of length L with a uniform charge density λ. Without loss of generality, place the rod on the x-axis from $x=a$ to $x=b$ with $b=a+L$. (Fig. 1)

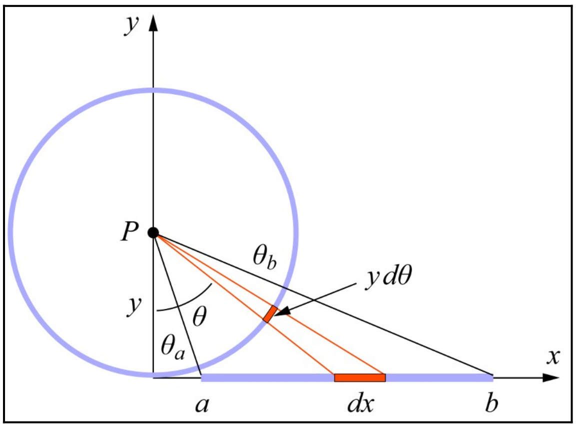


Fig. 1

$x = y\tan\theta$

$dx = y\,d\theta/\cos^2\theta$

$r = y/\cos\theta$ where r is the distance between the point P and the element dx.

The infinitesimal contribution dE at a point P on the y-axis, due to the charge $dq = \lambda dx$ on the x-axis, is given by

$$dE = \frac{k\,dq}{r^2} = \frac{k\lambda\,dx}{r^2} = \frac{k\lambda\,y\,d\theta/\cos^2\theta}{y^2/\cos^2\theta} = \frac{k[\lambda(y\,d\theta)]}{y^2}$$

The above expression corresponds to the infinitesimal contribution from an arc segment of radius y and arc length $y d\theta$ with the same linear charge density λ. In other words, the electric field contribution from the charge on the x-axis can be mapped to the contribution by hypothetical charges on a circular segment of radius y, as shown in Fig. 1. For a circular arc, the symmetry axis is well defined. Thus, the total electric field due to a circular segment is along the direction bisecting the arc. If the lines connecting the ends are defined by the angles θ_a

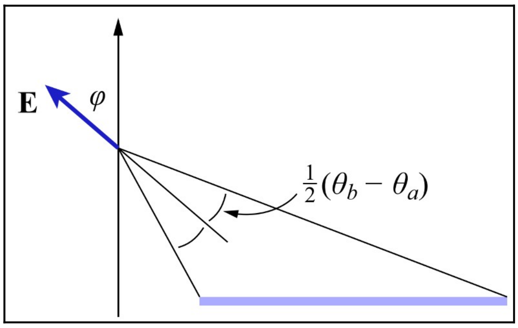


Fig. 2

and θ_b, then the arc is defined by the angular spread of $\theta_b - \theta_a$, and the bisecting line will be pointing in the direction $\theta_a + \frac{\theta_b - \theta_a}{2} = \frac{\theta_a + \theta_b}{2}$. (Fig. 2)

$$\cot\phi = \frac{E_y}{-E_x} = \frac{-(\sin\theta_b - \sin\theta_a)}{\cos\theta_b - \cos\theta_a} = \cot\frac{1}{2}(\theta_b + \theta_a)$$

The magnitude of the total electric field can be calculated for the simple arc to be

$$E = \frac{2k\lambda}{y}\sin\frac{1}{2}(\theta_b - \theta_a)$$

Where is the beauty of inverse-square law here? Let us see:

Let us see the general case where the dependence on r is as $1/r^n$. For this case,

$$dE = \frac{kdq}{r^n} = \frac{k\lambda\, dx}{r^n} = \frac{k\lambda\, y d\theta/\cos^2\theta}{y^n/\cos^n\theta} = \frac{k[\lambda(y d\theta)]}{y^n}\cos^{n-2}\theta$$

We can clearly see that the above expression does not correspond to the infinitesimal contribution from an arc segment of radius y and arc length $yd\theta$ with the same linear charge density λ. Only when $n=2$, $\cos^{n-2}\theta$ factor vanishes and we get the mapping from a uniformly charged rod to that of a circular arc.

Thus, the inverse-square law grants us the luxury to map the electric field from an infinitesimal charged line segment dx to that of an infinitesimal arc-segment of a fixed radius. The symmetry axis of the arc is easily defined. We have hence found a hidden symmetry in a seemingly asymmetric system.

For curious read:

This method, although very uncommon, has been known for quite sometime now. Edward Routh in his book *A Treatise on Analytical Statics with Numerous Examples* talks about a similar method for gravitational force.[1] In 1879, Lord Kelvin (William Thomson) and Peter Guthrie Tait published a solution using pure geometrical arguments without calculus in their *Treatise on Natural Philosophy*.[2] This method can be traced back to 1828 when George Green derived the ellipsoidal equipotential surface in *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* which was self-published and was ignored by mathematicians for two decades.[3][4]

II. THE RATIO OF FORCES

$$F_{nuclear} \propto \frac{Ae^{\frac{-r}{r_0}}}{r^2}$$

1 Edward Routh, *A Treatise on Analytical Statics with Numerous Examples* (Cambridge U.P., Cambridge, 1891 & 1892), Vol. II, pp. 4–6

2 W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, Part II (Cambridge U.P., Cambridge, 1879 & 1883), p. 27.

3 G. Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (Printed for the author by T. Wheelhouse, Nottingham, 1828), pp. 68–69.

4 N. M. Ferrers, *Mathematical Papers of the Late George Green* (MacMillan, London, 1871), pp. 329–330.

$$F_{electrostatic} \propto \frac{k}{r^2}$$

$$\frac{F_{nuclear}}{F_{electrostatic}} \propto A \frac{e^{\frac{-r}{r_0}}}{k}$$

Deep inside the nucleus (for $r << r_o$), $e^{\frac{-r}{r_0}} \simeq 1$ and thus, $F_{nuclear}/F_{electrostatic} = A/k$ in such a case. $A>>k$ and thus the nuclear force dominates. As r becomes larger and satisfies $r >> r_o$, the exponential $e^{\frac{-r}{r_0}}$ completely suppresses the A/k factor and the electro-repulsion wins.

For small nuclei system, the system is stable. But, as the atomic number increases, the size of the nucleus increases. The nuclear forces at the farthest points are too weak in such a case. Take the example of uranium, with 92 protons. The nuclear forces act mainly between each proton (or neutron) and its nearest neighbour, while the electrical forces act over larger distances, giving a repulsion between each proton and all of the others in the nucleus. The more protons in a nucleus, the stronger is the electrical repulsion, until, as in the case of uranium, the balance is so delicate that the nucleus is almost ready to fly apart from the repulsive electrical force. If such a nucleus is just "tapped" lightly (as can be done by sending in a slow neutron), it breaks into two pieces, each with positive charge, and these pieces fly apart by electrical repulsion. The energy which is liberated is the energy of the atomic bomb. This energy is usually called "nuclear" energy, but it is really "electrical" energy released when electrical forces have overcome the attractive nuclear forces.[5] What about other forces? The case in gravitational forces is very different.[6]

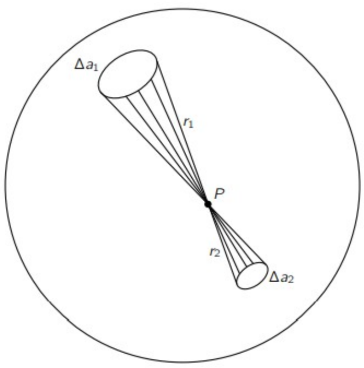
$$\frac{F_{gravitational}}{F_{electrostatic}} = \frac{4\pi\epsilon_0 G m_1 m_2}{q_1 q_2}$$

The ratio of the gravitational to electrostatic force is a constant for given masses and charges. If the electrostatic force would not have been an inverse-square law following force, our world would have been extremely different!

III. TESTING 1/r² DEPENDENCE

Gauss' law is a consequence of the $1/r^2$ dependence of the Coulomb force. Consider any point P inside a uniformly charged spherical shell. Imagine two symmetric cones having their apex at the point P and extending to the surface of the spherical shell where each cuts out a small surface area Δa_1 and Δa_2, respectively. Let the respective distances of these small area elements from the point P be r_1 and r_2. Using geometry,

$$\frac{\Delta a_2}{\Delta a_1} = \frac{r_2^2}{r_1^2}$$



Since the surface of the shell is uniformly charged, the charge on each of the area elements is proportional to the area of the element. That is to say,

5 *The Feynman Lectures on Physics*, Chapter 1 (Electromagnetism)

6 R. Shankar, *Fundamentals of Physics II*, 1.6 *The ratio of gravitational to electric forces*

$$\frac{\Delta q_2}{\Delta q_1} = \frac{\Delta a_2}{\Delta a_1}$$

If E_1 and E_2 are the E-fields at the point P due to area elements,

$$\frac{E_2}{E_1} = \frac{\Delta q_2 / r_2^{\,2}}{\Delta q_1 / r_1^{\,2}} = 1$$

The fields cancel exactly. Since all parts of the surface can be paired off in the same way, the total field at P is zero. But you can see that it would not be so if the exponent of r in Coulomb's law were not exactly two.

[7]Benjamin Franklin was the first to notice that the field inside a conducting shell is zero! When he reported his observation to Priestley, the latter suggested that it might be connected with an inverse-square law. If we write that the electrostatic force depends on $r^{-2+\epsilon}$, we can place an upper bound on ϵ. By this method Maxwell determined that ϵ was less than 1/10,000. The experiment was repeated and improved upon in 1936 by Plimpton and Lawton. They found that Coulomb's exponent differs from two by less than one part in a billion. Experimental test of Coulomb's law has been discussed in various papers.[8]

7 *The Feynman Lectures on Physics*, Chapter 5 (Application of Gauss' Law)
8 D. F. Bartlett, P. E. Goldhagen, and E. A. Phillips, *Phys. Rev.* D **2**, 483 *(published August 1, 1970)*