* Question : [Taken from I.E. Irodov - Basic Laws of Electromagnetism]

● **4.10.** A capacitor consists of two fixed plates in the form of a semicircle of radius R and a movable plate of thickness h made of a dielectric with the dielectric constant ε, placed between them. The latter plate can freely rotate about the axis O (Fig. 4.7) and practically fills the entire gap between the fixed plates. A constant voltage U is maintained between the plates. Find the moment M about the axis O of forces acting on the movable plate when it is placed as shown in the figure.

* Figure :

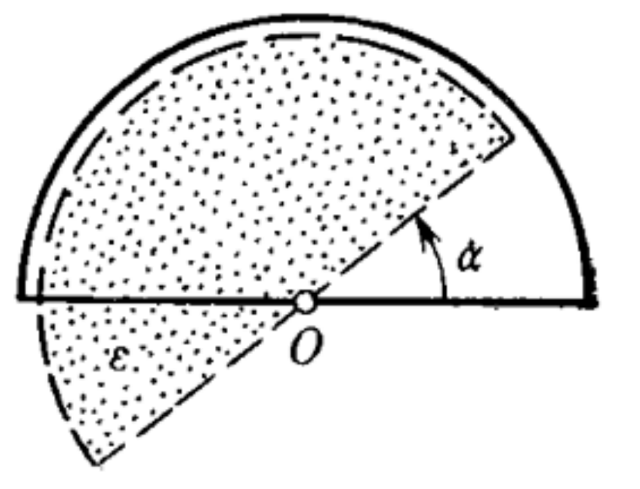

Fig. 4.7

* Solution :

→ The work performed by the moment of forces (M) upon the rotation of the plate through an angle element $d\alpha$ is equal to the decrease in energy of the system at q = constant.

$$\therefore M_z \, d\alpha = -dW\Big|_{q=\text{const.}} \quad ; \text{ where } W = \frac{q^2}{2C}$$

$$\Rightarrow M_z = -\frac{dW}{d\alpha} = -\frac{dW}{dC}\cdot\frac{dC}{d\alpha}$$

$$= -\frac{d}{dC}\left(\frac{q^2}{2C}\right)\cdot\frac{dC}{d\alpha} = \frac{q^2}{2C^2}\cdot\frac{dC}{d\alpha} \quad \text{——— (1)}$$

* For the given figure, capacitance $C = C_1 + C_2$, where C_1 = capacitance of the part with the dielectric

C_2 = capacitance of the part without the dielectric

Also, area of the sector with an angle α is determined as $A = \frac{\alpha R^2}{2}$ (R = radius of semi-circle)

$$\therefore C_1 = \frac{\epsilon\epsilon_0(\pi-\alpha)R^2}{2h} \quad \text{and} \quad C_2 = \frac{\epsilon_0\alpha R^2}{2h}$$

$$\Rightarrow C = C_1 + C_2 = \frac{\epsilon\epsilon_0(\pi-\alpha)R^2}{2h} + \frac{\epsilon_0\alpha R^2}{2h} = \frac{\epsilon_0 R^2}{2h}\left[\epsilon(\pi-\alpha)+\alpha\right]$$

Differentiating w.r.t. α,

$$\frac{dC}{d\alpha} = \frac{\epsilon_0 R^2}{2h}\left[-\epsilon+1\right] = \frac{\epsilon_0(1-\epsilon)R^2}{2h}$$

Substituting $\frac{dC}{d\alpha}$ in eq. (1) and taking $C = \frac{q}{U}$ (where U is given to be the const. potential between the two plates of the capacitor)

$$M_z = \frac{U^2}{2}\cdot\frac{\epsilon_0 R^2}{2h}(1-\epsilon) = -\frac{(\epsilon-1)\epsilon_0 R^2 U^2}{4h} < 0$$

The negative sign of M_z indicates that the moment of force is acting clockwise, opposite to the direction in which the angle α is measured.

∴ The moment has a tendency to pull the dielectric inside the capacitor.

To note → M_z is independent of the angle α. However at equilibrium, when $\alpha \to 0$, $M_z \to 0$. This discrepancy is due to the fact that edge effects exist at the edges of the capacitor plates, which has been ignored in the solution of this problem (for a substantial value of the angle α). However, for very small values of α, edge effects cannot be ignored.