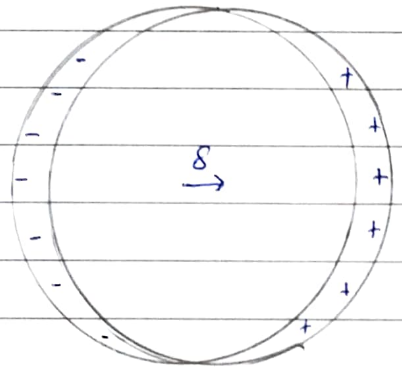
Overlapping Charged Spheres

→ We assume that a neutral sphere of radius R can be regarded as the superposition of two "rigid" spheres: One of uniform charge density $+\rho_0$, comprising of nuclei of the atoms, and a second sphere of the same radius but of negative uniform charge density $-\rho_0$, comprising the electrons. We further assume that its possible to shift the two spheres relative to each other by a quantity δ, as shown in the figure without perturbing the internal structure of either sphere.

Find the electrostatic field generated by the global charge distribution



a) in the "inner" region, where the two spheres overlap

b) in the "outer" region i.e. outside both spheres discussing the limit of small displacements.

Solution

a) Electrostatic field at any point inside the sphere is the vector addition of field generated by the individual spheres by the principle of superposition.

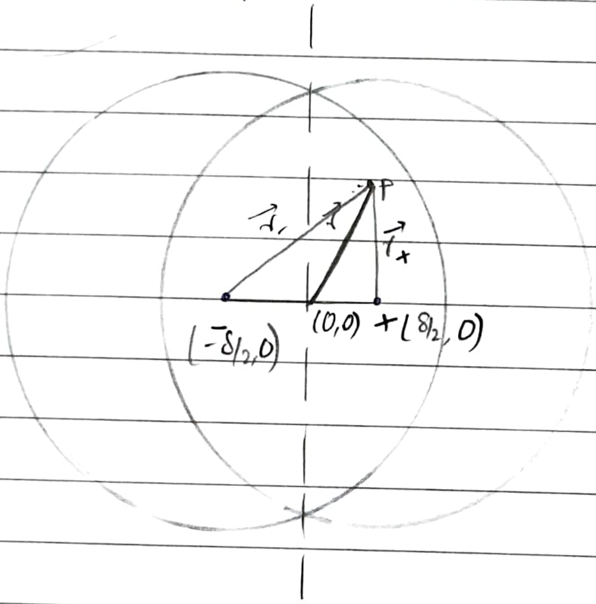
We know that the field generated by a single uniformly charged sphere at any interior point

$$\vec{E}(\vec{r}) = \frac{1}{\cancel{4\pi}\epsilon_0} \frac{\rho \vec{r}}{3}$$

where

ρ = charge density

$\vec{r}$ = position vector relative to center



From the given figure

$$\vec{r} = \vec{r}_+ + \frac{\vec{\delta}}{2}$$

$$\vec{r} - \frac{\vec{\delta}}{2} = \vec{r}_+$$

$$\vec{r}_+ = \vec{r} - \frac{\vec{\delta}}{2}$$

also,

$$\vec{r}_- = \vec{r} + \frac{\vec{\delta}}{2}$$

$$\vec{E}_+ = \frac{1}{\cancel{4\pi}\epsilon_0} \frac{\rho_0 \vec{r}_+}{3}$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0} \frac{\rho_0}{3} \left(\vec{r} - \frac{\vec{\delta}}{2} \right)$$

$$\vec{E}_- = \frac{-1}{\cancel{4\pi}\epsilon_0} \frac{\rho_0 \vec{r}_-}{3}$$

$$= \frac{-1}{\cancel{4\pi}\epsilon_0} \frac{\rho_0}{3} \left(\vec{r} + \frac{\vec{\delta}}{2} \right)$$

$$\vec{E}_{ins} = \vec{E}_{+} + \vec{E}_{-}$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0}\frac{\rho_0}{3}\left(\vec{r} - \frac{\vec{\delta}}{2}\right) - \frac{1}{4\pi\epsilon_0}\frac{\rho_0}{3}\left(\vec{r} + \frac{\vec{\delta}}{2}\right)$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0}\frac{\rho_0}{3}\left[\cancel{\vec{r}} - \frac{\vec{\delta}}{2} - \cancel{\vec{r}} - \frac{\vec{\delta}}{2}\right)$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0}\frac{\rho_0}{3}\left(-\vec{\delta}\right)$$

$$= -\frac{1}{\cancel{4\pi}\epsilon_0}\frac{\rho_0}{3}\left(\vec{\delta}\right)$$

b) The electrostatic field generated by a uniform charged sphere, with volume charge density ρ_0 outside the volume of the sphere is equivalent to considering a point charge at the center of the sphere with a charge

$$Q = \frac{4\pi R^3}{3}\rho_0$$

⇒ The field at any point outside both spheres is the sum of its electric fields due to the positive sphere and negative sphere, both treated as point charges at their center

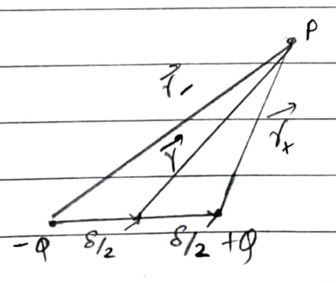
→ for $r \gg \delta$ we can treat this as a dipole with the dipole moment

$$|p| = Q\delta = \frac{4\pi R^3}{3}\rho_0\,\delta$$

From the figure

$$\vec{r}_+ = \vec{r} - \frac{\vec{\delta}}{2}$$

$$\vec{r}_- = \vec{r} + \frac{\vec{\delta}}{2}$$



Potential at point P outside the sphere

$$V(\vec{r}) = \frac{kQ}{\left|\vec{r} - \frac{\vec{\delta}}{2}\right|} - \frac{kQ}{\left|\vec{r} + \frac{\vec{\delta}}{2}\right|}$$

$\left[\begin{array}{l} k = 1/4\pi\epsilon_0 \\ V(\vec{r}) = V_+ + V_- \end{array} \right]$

(By superposition)

$$V(\vec{r}) = \frac{kQ}{\left(r^2 + \frac{\delta^2}{4} - r\delta\cos\theta\right)^{1/2}} - \frac{kQ}{\left(r^2 + \frac{\delta^2}{4} + r\delta\cos\theta\right)^{1/2}}$$

Since $r \gg \delta$ we can ignore the $\frac{\delta^2}{4}$ term

$$\therefore V(\vec{r}) = \frac{kQ}{\left(r^2 - r\delta\cos\theta\right)^{1/2}} - \frac{kQ}{\left(r^2 + r\delta\cos\theta\right)^{1/2}}$$

By binomial approximation

$$V(\vec{r}) = \frac{kQ}{r\left(1 - \frac{1}{2}\frac{\delta\cos\theta}{r}\right)} - \frac{kQ}{r\left(1 + \frac{1}{2}\frac{\delta\cos\theta}{r}\right)}$$

$$= \frac{kQ\left(\cancel{1} + \frac{1}{2}\frac{\delta\cos\theta}{r} - \cancel{1} + \frac{1}{2}\frac{\delta\cos\theta}{r}\right)}{r}$$

$$= \frac{kQ\,\delta\cos\theta}{r^2}$$

$\because Q\delta = p$

$$\therefore V(\vec{r}) = \frac{kp\cos\theta}{r^2}$$

$$= \frac{k}{r^2}\,p\cos\theta$$

We know that

$$\vec{E} = -\nabla V(\vec{r})$$

$$\therefore \vec{E} = -\frac{\partial V(\vec{r})}{\partial r} - \frac{1}{r}\frac{\partial V(\vec{r})}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial V(\vec{r})}{\partial \phi}$$

$$\vec{E} = -\left(\frac{-2kp\cos\theta}{r^3}\right)\hat{r} - \frac{1}{r}\left(-\frac{kp\sin\theta}{r^2}\right)\hat{\theta} - 0$$

$$= \frac{2kp\cos\theta\,\hat{r}}{r^3} + \frac{kp\sin\theta\,\hat{\theta}}{r^3}$$

$$\vec{E}(r,\theta) = \frac{kp}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \quad \text{— (1)}$$

This result is in terms of spherical polar coordinates

To convert it to coordinate free form

$$\vec{p}\cdot\hat{r} = p\cos\theta$$

$$\vec{p}\cdot\hat{\theta} = p\cos\left(\theta + \frac{\pi}{2}\right)$$

$$= -p\sin\theta$$

$$\vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$$

$$\vec{p} = p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta}$$

$$p\sin\theta\,\hat{\theta} = p\cos\theta\,\hat{r} - \vec{p}$$

substituting $p\cos\theta\,\hat{r}$ and $p\sin\theta\,\hat{\theta}$ in equation (1)

$$\vec{E} = \frac{k\cancel{p}}{r^3}\left(2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}\right)$$

$$= \frac{k}{r^3}\left(2(\vec{p}\cdot\hat{r})\hat{r} + p\cos\theta\,\hat{r} - \vec{p}\right)$$

$$= \frac{k}{r^3}\left(2(\vec{p}\cdot\hat{r})\hat{r} + (p\cdot\hat{r})\hat{r} - \vec{p}\right)$$

$$\boxed{\vec{E}_{out} = \frac{k}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$