

# Physics Tutorial

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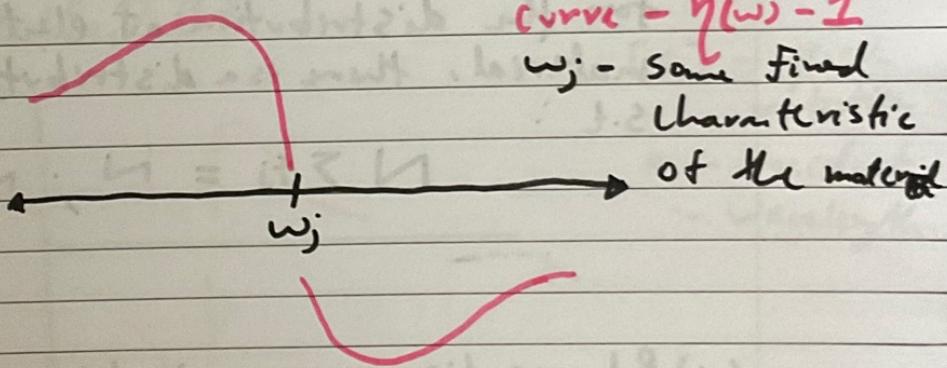
Q7 It is known from optics that the refractive index depends on the incident wavelength/frequency of light.

One approximate formula is Cauchy's formula:

$$\eta = 1 + A \left( 1 + \frac{B}{x^2} \right)$$

which implies that lower the wavelength, higher the  $\eta$ .

However, it has also been observed that :



curve -  $\eta(\omega) - 1$   
 $w_j$  - some fixed characteristic of the material

A widely invoked model for the origin of refractive index ( $\eta$ ) is the Lorentz oscillator model.

The model makes the following key assumptions:

- Electrons, in a non-conductor, are bound by a potential where the following approximation can be made:  $U(n + \Delta n) \underset{\text{qbr.}}{\approx} C + k(\Delta n)^2$

(iii) For a given wave velocity  $w/k$  :

$$\eta = \frac{Ck}{\omega}$$

(iii) Loss of energy due to radiation & other damping effects can be taken as a linear damping.

(iv) For Ubinding, when we write :

$$F_{\text{binding}} = m\omega_j^2 \Delta x$$

for the distribution of electrons in the material, there is a distribution of  $\omega_j$ 's  
s.t :

$$N \sum f_j = N ; N - \text{total no. of e}^- \text{s in the material}$$

(v) Polarisation is given by taking the real part of :

$$P = \frac{Nq^2}{m} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right) E$$

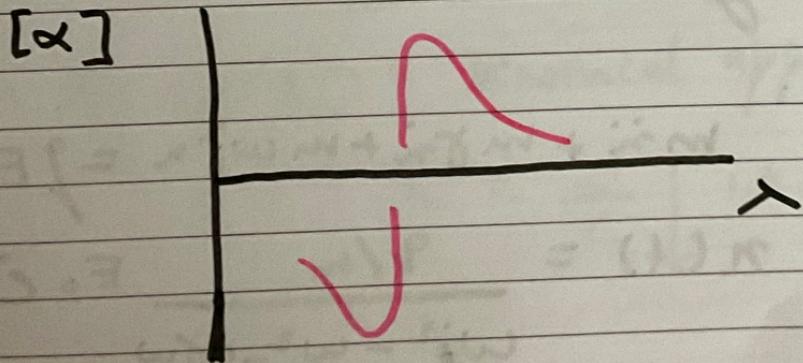
where  $E$  will be the complex representation of the electric field.

Also :  $P = \epsilon_0 \chi_e E$

Use the Lorentz oscillator model to explain the graph and show that Cauchy's formula is not contradictory to the graph.

Optional (if time permits) :

- (i) Using this model, qualitatively explain why chiral molecules can rotate linearly polarised light, despite being oriented randomly in a solution.
- (ii) For optically active compounds, the following graph is called the Cotton Effect.



$[\alpha]$  - optical rotation  
 $\lambda$  - incident wavelength.

Explain this graph.

Solu.

We start by writing our damped oscillator equation, with incident light polarised along  $\hat{n}$  driving the oscillator:

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = gE$$

We can consider only:

$$E = E_0 e^{-i\omega t}$$

as we may build up all other solutions using Fourier's theorem.

This is a Non-Homogeneous 2nd Order O.D.E but we shall study only the Steady State Solution.

Now:

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = gE \cdot e^{-i\omega t}$$

$$\Rightarrow x(t) = \frac{g/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

We now have the polarisation:

$$P = \frac{Nq^2}{m} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) E$$

$$\rightarrow P = \epsilon_0 \chi_e E$$

This  $\chi_c$  now gives us :

$$\epsilon_r = 1 + \frac{N\eta^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\delta_j\omega}$$

Recall that in :

$$\nabla^2 E = \epsilon \gamma_0 \frac{\partial^2 E}{\partial t^2}$$

we have plane waves :

$$E(z,t) = E_0 e^{i(kz - \omega t)}$$

and :  $R = \sqrt{\epsilon_r} \frac{\omega}{c}$

By applying a binomial approximation on  $\sqrt{\epsilon_r}$   
we get :

$$k \approx \frac{\omega}{c} \left[ 1 + \frac{N\eta^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\delta_j\omega} \right]$$

Let  $k = k_0 + iK$ . We now have :

$$E(z,t) = E_0 e^{-Kz} e^{i(k_0 z - \omega t)}$$

where  $e^{-Kz}$  is attenuation of the  
incident wave.

We may now invoke that :

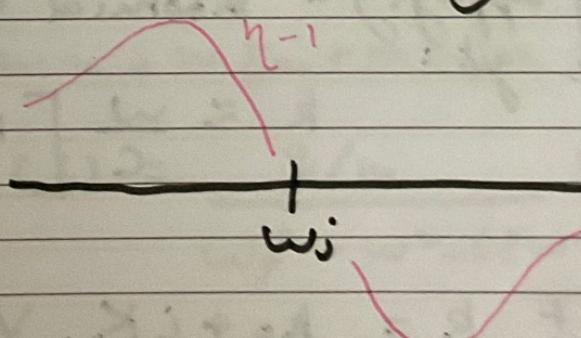
$$\eta = \frac{Ck_0}{\omega} \approx 1 + \frac{Ng^2}{2m\epsilon_0} \sum_j \frac{f_j(w_j^2 - \omega^2)}{(w_j^2 - \omega^2)^2 + \delta_j^2}$$

The term of interest is :

$$\eta^{-1} = \frac{Ng^2}{2m\epsilon_0} \sum_j \frac{f_j(w_j^2 - \omega^2)}{(w_j^2 - \omega^2)^2 + \delta_j^2}$$

as it introduces the frequency dependence in  $\eta$ .

Close to  $w_j = \omega$ , we have a resonance, which gives us our :



Curve. This behavioural regime is called anomalous dispersion.

Therefore, resonances can be induced in a material's electrons, due to incident light!

Anomalous dispersion and other predictions of this model have been tested and confirmed.

A more useful regime of this formula is when:

$$\frac{\omega}{\omega_j} \ll 1$$

In this regime:

$$\eta \approx 1 + \frac{Ng^2}{2m\omega_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

$$\rightarrow \eta \approx 1 + \frac{Ng^2}{2m\omega_0} \sum_j \frac{f_j}{\omega_j^2} \left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1}$$

A binomial approximation and setting the constant on the 2nd term equal to A yields:

$$\eta \approx 1 + A \left(1 + \frac{\beta}{\omega^2}\right)$$

which is Cauchy's formula.

Optional :

Despite random orientation in a solution, chiral molecules have an invariant handedness.

This means that for these molecules:

$$\epsilon^{\perp} \neq \epsilon^{\parallel}$$

where the permittivity for circularly polarised lights arent the same.

This is due to the handedness of oscillator arrangements.

Linearly polarised light may now be written as a superposition of the two circularly polarised beams. Now:

$$\epsilon^{\perp} \neq \epsilon^{\parallel}$$

Causes the light to rotate in one direction, upon recombination of the superposition.

The Cotton effect is due to the absorption peak of light polarised in one direction, at a characteristic wavelength.