Q) A ball of radius R carries a positive charge whose volume density depends only on a separation r from the ball's centre as $\rho = \rho_0(1 - r/R)$, where ρ_0 is a constant. Assuming the permittivities of the ball and the environment to be equal to unity, find:

a) the magnitude of the electric field strength as a function of the distance r both inside and outside the ball;

b) the maximum intensity E_{max} & the corresponding distance r_m.

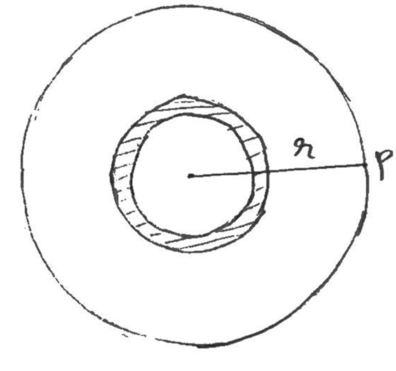
SOLUTION:

→ By using Gauss's Law

$$\int \vec{E} \cdot d\vec{s} = \frac{q_{in}}{\varepsilon_0} \quad \text{or} \quad \int \vec{E} \cdot d\vec{s} = \frac{\int dq}{\varepsilon_0}$$

a)

CASE 1: When $r < R$



$$\int \vec{E} \cdot d\vec{s} = \frac{q_{in}}{\varepsilon_0}$$

$$\int \vec{E} \cdot ds \cdot \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{q_{in}}{\varepsilon_0} \Rightarrow E = \frac{q_{in}}{4\pi r^2 \varepsilon_0} \quad \text{—(1)}$$

q_{in} = Total electric charge enclosed in a sphere of radius r.

$$q_{in} = \int_0^r \rho 4\pi r^2 dr = \int_0^r \rho_0\left(1 - \frac{r}{R}\right) 4\pi r^2 dr$$

$$= 4\pi\rho_0 \left[\int_0^r r^2 dr - \int_0^r \frac{r^3}{R} dr\right] = 4\pi\rho_0\left[\frac{r^3}{3} - \frac{r^4}{4R}\right]$$

From Eq. (1)

$$E = \frac{4\pi\rho_0}{4\pi r^2 \varepsilon_0}\left[\frac{r^3}{3} - \frac{r^4}{4R}\right] \Rightarrow E = \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^2}{4R}\right]$$

is radially outward.

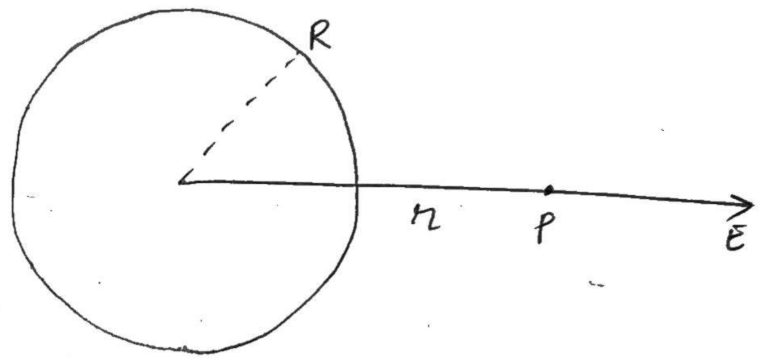
$$\therefore \boxed{E = \frac{\rho_0 r}{3\varepsilon_0}\left[1 - \frac{3r}{4R}\right]}$$

CASE-2 : When $r = R$

$$E = \frac{\rho_0}{\varepsilon_0}\left[\frac{R}{3} - \frac{R^2}{4R}\right] = \frac{\rho_0 R}{\varepsilon_0}\left[\frac{4-3}{12}\right]$$

$$\Rightarrow \boxed{E = \frac{\rho_0 R}{12\varepsilon_0}} \rightarrow \text{radially outward.}$$

CASE-3 : When $r > R$



$$q_{in} = \int_0^R \rho_0\left(1 - \frac{r}{R}\right) 4\pi r^2 dr$$

$$q_{in} = 4\pi\rho_0\left[\frac{R^3}{3} - \frac{R^4}{4R}\right]$$

$$= 4\pi\rho_0 R^3\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\cancel{4}\pi\rho_0 R^3}{\cancel{12}_3} = \frac{\pi\rho_0 R^3}{3}$$

From Eq. (1)

$$E = \frac{q_{in}}{4\pi r^2 \varepsilon_0} = \frac{\cancel{\pi}\rho_0 R^3}{3\times 4\cancel{\pi}\varepsilon_0 r^2} = \frac{\rho_0 R^3}{12\varepsilon_0 r^2} \therefore \boxed{E = \frac{\rho_0 R^3}{12\varepsilon_0 r^2}}$$

b) Electric Field outside the sphere decreases with increasing r.

$\therefore$ Electric Field is maximum inside the sphere.

$$E = \frac{\rho_0 r}{3\varepsilon_0}\left[1 - \frac{3r}{4R}\right]$$

For E_{max}, $\frac{dE}{dr} = 0$

Then $1 - \frac{3r}{2R} = 0 \Rightarrow r = \frac{2R}{3}$

$\therefore$ At $r = \frac{2R}{3}$, Electric Field will be maximum.

$$E_{max} = \frac{\rho_0\left(\frac{2R}{3}\right)}{3\varepsilon_0}\left[1 - \frac{\cancel{3}}{4\cancel{R}}\left(\frac{2\cancel{R}}{\cancel{3}}\right)\right] = \frac{\rho_0 R}{9\varepsilon_0} \quad \therefore \boxed{E_{max} = \frac{\rho_0 R}{9\varepsilon_0}}$$