Physics Presentation

Electromagnetism and Special Relativity

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1 The (easy) problem

Two infinite parallel line charges, with line charge density λ each, are separated by a distance d. They are constrained to move in opposite directions with velocity v relative to the ground frame. What is the ratio of the magnetic to the electric force (per unit length) exerted by one charge onto the other?

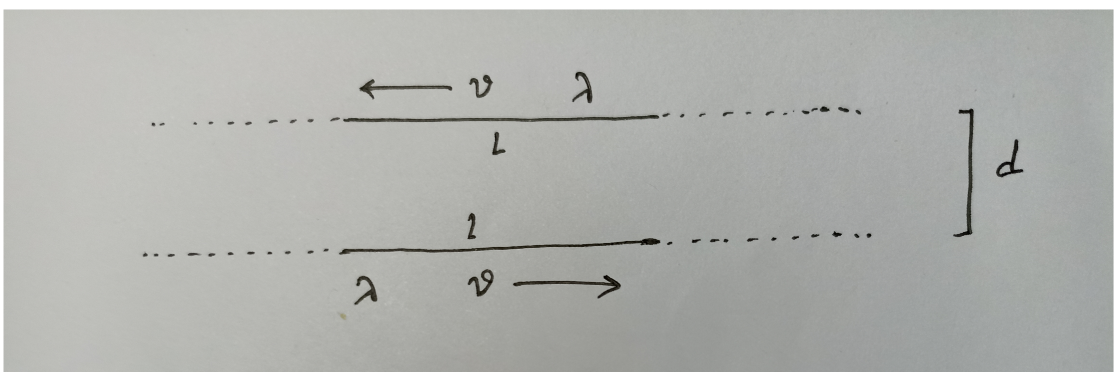

Figure 1: The problem setup (in $\mathcal{S}$)

Solution

Let us consider the upper line charge. The moving charges constitute a current $I = \lambda v$. This current, in accordance with Ampère's law, generates a magnetic field

$$B = \frac{\mu_0 I}{2\pi d} = \frac{\mu_0 \lambda v}{2\pi d}$$

at a distance d from the wire. By the right-hand rule, this field acts outwards from the plane of the charges, for points below the line charge. This field results in a force on the other line charge. Using

$$\mathbf{F}_B = q(\mathbf{v} \times \mathbf{B})$$

the force due to the magnetic field on a segment of length l (and charge λl) is given by

$$F_B = qv\frac{\mu_0 \lambda v}{2\pi d} = \lambda l v \frac{\mu_0 \lambda v}{2\pi d} = \frac{\mu_0 \lambda^2 v^2 l}{2\pi d}$$

which points downward, away from the top charge.

The electric field generated by the top line charge is given by Gauss's law.

$$E = \frac{\lambda}{2\pi\varepsilon_0 d}$$

Using

$$\mathbf{F}_E = q\mathbf{E}$$

the force due to the electric field on a segment of length l is given by

$$F_E = q\frac{\lambda}{2\pi\varepsilon_0 d} = \lambda l \frac{\lambda}{2\pi\varepsilon_0 d} = \frac{\lambda^2 l}{2\pi\varepsilon_0 d}$$

This force, naturally, also points downward, away from the top charge.

Therefore, the ratio of the magnetic force and electric force is given by

$$\frac{F_B}{F_E} = \frac{\frac{\mu_0 \lambda^2 v^2 l}{2\pi d}}{\frac{\lambda^2 l}{2\pi\varepsilon_0 d}} = \varepsilon_0 \mu_0 v^2 = \frac{v^2}{c^2}$$

where c is the speed of light. The equality

$$\frac{1}{c^2} = \varepsilon_0 \mu_0$$

arises by decoupling Maxwell's equations and obtaining two wave equations for $\mathbf{E}$ and $\mathbf{B}$, from which the speed of propagation of electromagnetic waves is obtained.

The total force per length l acting on either line charge is

$$F = F_E + F_B = \left(1 + \frac{v^2}{c^2}\right) F_E = \left(1 + \frac{v^2}{c^2}\right) \frac{\lambda^2 l}{2\pi\varepsilon_0 d}$$

as F_E and F_B act in the same direction.

2 The (not so easy) twist

The ratio

$$\frac{F_B}{F_E} = \frac{v^2}{c^2}$$

is reminiscent of the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

in special relativity. It can be shown that even though the magnetic force vanishes in the frame of either line charge, the total force per length l acting on either charge is invariant and is attributable to the electric field alone.

In the frame of the top charge ($\bar{\mathcal{S}}$), the resultant velocity of the bottom charge is given by the Einstein velocity addition rule.

$$\bar{v} = \frac{v_1 - v'}{1 - \frac{v_1 v'}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}}$$

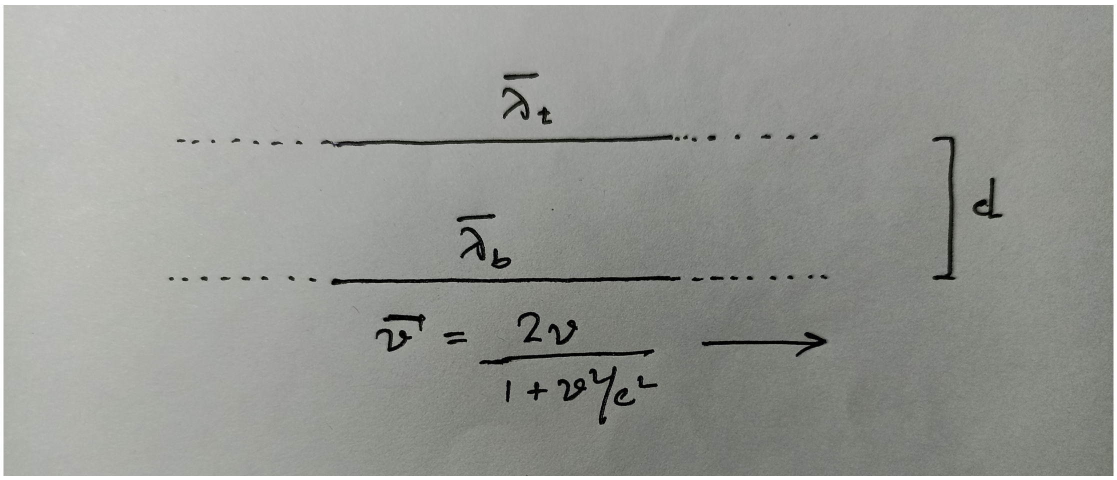

Figure 2: The problem setup in $\bar{\mathcal{S}}$

Here, v_1 is the velocity of the bottom charge in the ground frame ($\mathcal{S}$) and v' is the velocity of $\bar{\mathcal{S}}$ with respect to $\mathcal{S}$ (here, $v_1 = v$ and $v' = -v$).

In $\bar{\mathcal{S}}$, the bottom charge will appear length-contracted by a factor of

$$\bar{\gamma} = \frac{1}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}}$$

Due to the fact that charge is invariant across inertial frames, the charge density on the bottom charge will undergo a corresponding increase, with

$$\begin{aligned}
\bar{\lambda}_b &= \bar{\gamma}\lambda \\
&= \frac{1}{\sqrt{1 - \frac{\bar{v}^2}{c^2}}}\lambda \\
&= \frac{c}{\sqrt{c^2 - \bar{v}^2}}\lambda \\
&= \frac{c}{\sqrt{c^2 - \frac{4v^2}{\left(1+\frac{v^2}{c^2}\right)^2}}}\lambda \\
&= \frac{c\left(1 + \frac{v^2}{c^2}\right)}{\sqrt{c^2\left(1 + \frac{v^2}{c^2}\right)^2 - 4v^2}}\lambda
\end{aligned}$$

Also, the top charge is not moving in $\bar{\mathcal{S}}$, but is moving in $\mathcal{S}$ with velocity v. Hence, it will appear length-dilated in $\bar{\mathcal{S}}$, and the charge density will decrease to

$$\bar{\lambda}_t = \frac{1}{\gamma}\lambda = \sqrt{1 - \frac{v^2}{c^2}}\lambda$$

Therefore, the force per length l between the charges in $\bar{\mathcal{S}}$ is given by

$$\bar{F} = \bar{\lambda}_b l \frac{\bar{\lambda}_t}{2\pi\varepsilon_0 d} = \frac{\lambda^2 l}{2\pi\varepsilon_0 d} \frac{c\left(1 + \frac{v^2}{c^2}\right)\sqrt{1 - \frac{v^2}{c^2}}}{\sqrt{c^2\left(1 + \frac{v^2}{c^2}\right)^2 - 4v^2}}$$

The separation d remains constant as the contraction takes place only in the direction of relative motion. As in $\mathcal{S}$, the force acts away from the top charge.

In $\mathcal{S}$, therefore, the force is transformed as

$$\begin{aligned}
F &= \frac{1}{\gamma}\bar{F} \\
&= \frac{\lambda^2 l}{2\pi\varepsilon_0 d} \frac{c\left(1 + \frac{v^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\sqrt{c^2\left(1 + \frac{v^2}{c^2}\right)^2 - 4v^2}} \\
&= \frac{\lambda^2 l}{2\pi\varepsilon_0 d} \frac{c\left(1 + \frac{v^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\sqrt{c^2\left(1 + \frac{v^4}{c^4} + \frac{2v^2}{c^2}\right) - 4v^2}} \\
&= \frac{\lambda^2 l}{2\pi\varepsilon_0 d} \frac{c\left(1 + \frac{v^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{c\sqrt{1 + \frac{v^4}{c^4} - \frac{2v^2}{c^2}}} \\
&= \frac{\lambda^2 l}{2\pi\varepsilon_0 d} \frac{\left(1 + \frac{v^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)^2}} \\
&= \frac{\lambda^2 l}{2\pi\varepsilon_0 d}\left(1 + \frac{v^2}{c^2}\right)
\end{aligned}$$

This force is exactly the same as the total force F obtained by considering magnetic and electric forces separately. Thus, what is perceived as a magnetic force in one frame is actually the consequence of the electric force in another frame, and both electric and magnetic forces are componenents of a single **electromagnetic force**.

3 Appendices

Derivation of the Einstein velocity addition rule

The Lorentz transformations between frames $\mathcal{S}$ and $\bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ moves at a velocity v' with respect to $\mathcal{S}$, parallel to the x-axis, are given by

$$\bar{x} = \gamma(x - v't)$$
$$\bar{y} = y$$
$$\bar{z} = z$$
$$\bar{t} = \gamma\left(t - \frac{v'}{c^2}x\right)$$

Let a particle move a distance dx in $\mathcal{S}$ in time dt. Its velocity v_1 in $\mathcal{S}$ is given by

$$v_1 = \frac{dx}{dt}$$

In $\bar{\mathcal{S}}$, it has moved a distance

$$d\bar{x} = \gamma(dx - v'dt)$$

in a time

$$d\bar{t} = \gamma\left(dt - \frac{v'}{c^2}dx\right)$$

according to the Lorentz transformations. Therefore its velocity in $\bar{\mathcal{S}}$ is given by

$$\bar{v} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - v'dt)}{\gamma\left(dt - \frac{v'}{c^2}dx\right)} = \frac{\frac{dx}{dt} - v'}{1 - \frac{v'}{c^2}\frac{dx}{dt}} = \frac{v_1 - v'}{1 - \frac{v_1 v'}{c^2}}$$

This is the Einstein velocity addition rule.

Derivation of the force transformation rule

The transformation equation for forces perpendicular to v' can be obtained as follows.

$$\bar{F}_\perp = \frac{d\bar{p}_\perp}{d\bar{t}} = \frac{dp_\perp}{\gamma\left(dt - \frac{v'}{c^2}dx\right)} = \frac{\frac{dp_\perp}{dt}}{\gamma\left(1 - \frac{v'}{c^2}\frac{dx}{dt}\right)} = \frac{F_\perp}{\gamma\left(1 - \frac{v'v_\parallel}{c^2}\right)}$$

Here, $d\bar{p}_\perp$ and $dp_\perp$ are the momenta in the respective frames. The momenta do not transform as they are perpendicular to the direction of relative motion. Now, if $v_\parallel$ is zero, then

$$\bar{F}_\perp = \frac{1}{\gamma}F_\perp$$

In the problem, $v_\parallel$ is zero in $\bar{\mathcal{S}}$ (as the top charge is stationary) instead of $\mathcal{S}$, and hence, the perpendicular forces transform as

$$F = \frac{1}{\gamma}\bar{F}$$

4 References

- Griffiths, D. J. (1999). *Introduction to Electrodynamics*, Chapter 12, pp. 495-564
- Feynman, Richard P. (1964). *The Feynman Lectures on Physics, Volume II*, Chapter 13-6