Method of images

Name: Nikhil T. H

Reg No: 20211141

Question

(a) A point charge q is situated a distance a from the centre of a grounded conducting sphere of radius R. Find the potential outside the sphere.

(b) How will you handle the case of a sphere at any potential V0 (relative, of course, to infinity)? What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a neutral conducting sphere.

Answer

Concept:

Explained in class:

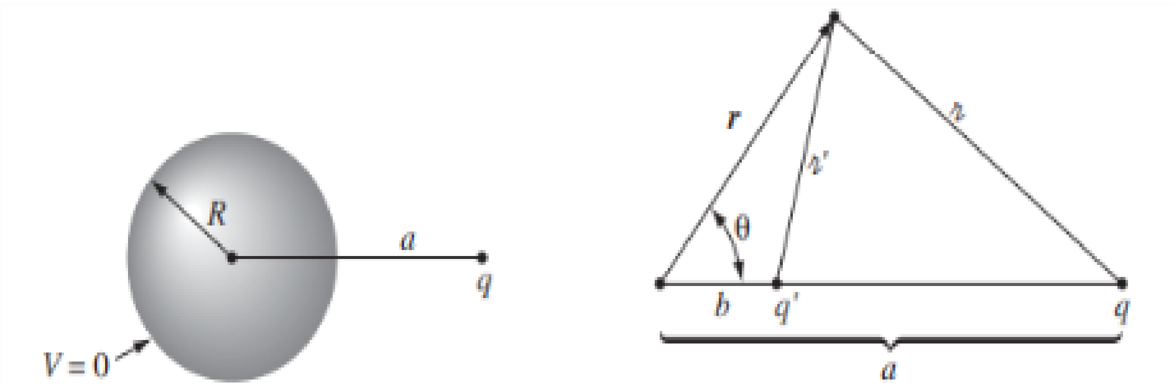
Uniqueness theorem (conditions when applicable)

Why force between conductor and charge equal to force between charge and q1.

How grounded conductor concept is modified and applied to neutral conductor.

Solution:

(a)



(a) From Fig. 3.13: $\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta}$; $\mathscr{r}' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$. Therefore:

$$\begin{aligned}\frac{q'}{\mathscr{r}'} &= -\frac{R}{a}\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} \quad \text{(Eq. 3.15), while } b = \frac{R^2}{a} \text{ (Eq. 3.16).}\\ &= -\frac{q}{\left(\frac{a}{R}\right)\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}} = -\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2+R^2-2ra\cos\theta}}.\end{aligned}$$

Therefore:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}}+\frac{q'}{\mathscr{r}'}\right) = \boxed{\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right\}}.$$

Clearly, when $r = R$, $V \to 0$.
(b) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$. Therefore,

$$\begin{aligned}\sigma(\theta) &= -\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left\{-\frac{1}{2}(r^2+a^2-2ra\cos\theta)^{-3/2}(2r-2a\cos\theta)\right.\\ &\quad\left.+\frac{1}{2}\left(R^2+(ra/R)^2-2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r-2a\cos\theta\right)\right\}\Bigg|_{r=R}\\ &= -\frac{q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta)+\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}\end{aligned}$$

$$\begin{aligned} &= \frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]\\ &= \boxed{\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}.}\\ q_{\text{induced}} &= \int\sigma\,da = \frac{q}{4\pi R}(R^2-a^2)\int(R^2+a^2-2Ra\cos\theta)^{-3/2}R^2\sin\theta\,d\theta\,d\phi\\ &= \frac{q}{4\pi R}(R^2-a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2}\right]\Bigg|_0^\pi\\ &= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right].\end{aligned}$$

But $a > R$ (else q would be *inside*), so $\sqrt{R^2+a^2-2Ra} = a - R$.

$$\begin{aligned} &= \frac{q}{2a}(a^2-R^2)\left[\frac{1}{(a+R)}-\frac{1}{(a-R)}\right] = \frac{q}{2a}\left[(a-R)-(a+R)\right] = \frac{q}{2a}(-2R)\\ &= \boxed{-\frac{qR}{a} = q'.}\end{aligned}$$

(c) The force on q, due to the sphere, is the same as the force of the image charge q', to wit:

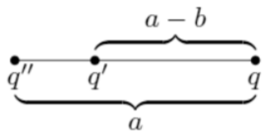
$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}.$$

To bring q in from infinity to a, then, we do work

$$W = \frac{q^2R}{4\pi\epsilon_0}\int_\infty^a\frac{\bar{a}}{(\bar{a}^2-R^2)^2}\,d\bar{a} = \frac{q^2R}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2-R^2)}\right]\Bigg|_\infty^a = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2R}{2(a^2-R^2)}}.$$

(b)

Place a second image charge, q'', at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$;



$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$

For a *neutral* sphere, $q' + q'' = 0$.

$$
\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0} q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2} \right) = \frac{qq'}{4\pi\epsilon_0} \left(-\frac{1}{a^2} + \frac{1}{(a-b)^2} \right) \\
&= \frac{qq'}{4\pi\epsilon_0} \frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)}{4\pi\epsilon_0} \frac{(R^2/a)(2a - R^2/a)}{a^2(a-R^2/a)^2} \\
&= -\boxed{\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a} \right)^3 \frac{(2a^2 - R^2)}{(a^2 - R^2)^2}}.
\end{aligned}
$$