Problem 2.33

Problem 2.33 Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x axis, each a distance a from its nearest neighbors. Find the work per particle required to assemble this system. [*Partial Answer:* $-\alpha q^2/(4\pi\epsilon_0 a)$, for some dimensionless number α; your problem is to determine α. It is known as the **Madelung constant**. Calculating the Madelung constant for 2- and 3-dimensional arrays is much more subtle and difficult.]

$$W = \int_a^b F \cdot dl = -Q \int_a^b E \cdot dl = Q\left[V(b) - V(a)\right]$$

$$W = Q\left[V(r) - V(\infty)\right]$$
$$= Q\,V(r)$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$ [V(r) due to point charge]

Work done on placing 1st charge

$$W_1 = 0$$

$$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$$

$$W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

$$W_4 = \frac{1}{4\pi\varepsilon_0} q_4 \left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right]$$

$$\vdots$$

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j>1}^{n} \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j\neq 1}^{n} \frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j \neq 1}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{ij}} \right)$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)}$$

considering a single side we get

$$W = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{(-1)^n q^2}{na}$$

considering both sides [multiplication by 2] we get

$$W = 2 \cdot \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{(-1)^n q^2}{na}$$

finally we get

$$W = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{a} \sum_{1}^{\infty} \frac{(-1)^n}{n}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

$$\ln 2 = \sum_{1}^{\infty} \frac{(-1)^n}{n}$$ [Putting $x = 1$]

Solution $$\boxed{W = \frac{q^2 \ln 2}{4\pi\varepsilon_0 a}}$$