PH1223 - Electricity and Magnetism
Tutorial Problem

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Consider a uniformly charged spherical shell of radius R and surface charge density σ. Consider a point just outside the sphere (at an infinitesimal distance outside the sphere).

i) Calculate the electric field at this point by using the result that outside the sphere, the field at a distance x is $Q/4\pi\epsilon_0 x^2$, where Q is the total charge on the sphere. You should obtain a value of σ/ϵ_0.

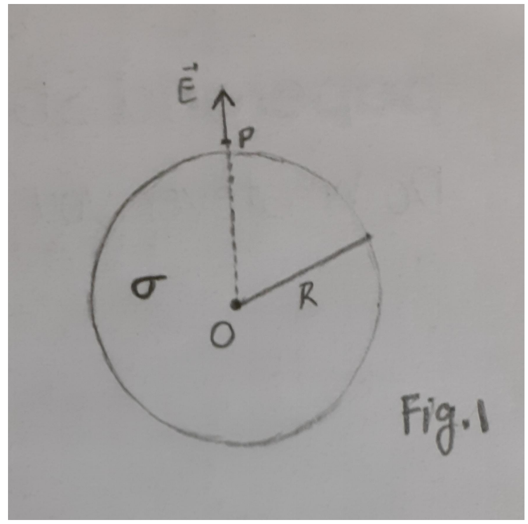
ii) Calculate the electric field by breaking up the shell into rings sharing a common axis that passes through the centre of the shell and the point, and integrating the field contributions from all rings. Use the result that the electric field due to a ring of radius r and line charge density lambda on its axis at a distance x from the centre is given by

$$E = \frac{\lambda r z}{2\epsilon_0 \left(r^2+z^2\right)^{3/2}}$$, pointing along the axis.

iii) Using the method in (ii), we obtain a value of $\sigma/2\epsilon_0$ for the electric field, which is incorrect. Moreover, we can use a similar argument to show that the field at an infinitesimal distance inside the shell is also $\sigma/2\epsilon_0$ (which is false since the field inside the sphere is 0).
Explain why the method is incorrect, and suggest how to modify the integral to obtain a correct result.

Solution

Part (i):



The electric field outside the sphere at a distance x from the centre O is

$$E = Q/4\pi\epsilon_0 x^2$$

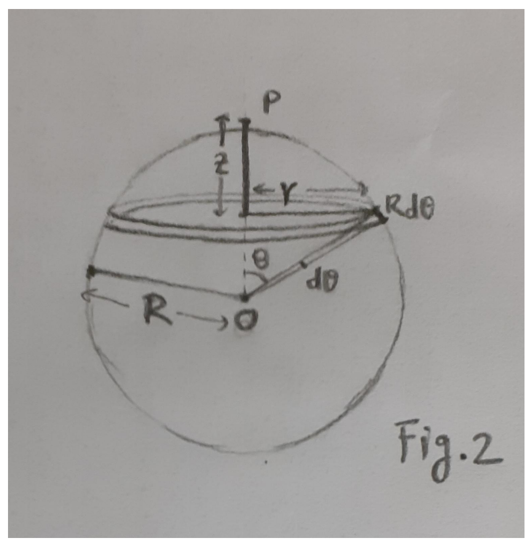
where Q is the total charge on the sphere.
Now, since the radius is R and surface charge density is σ,

$$Q = \sigma A = \sigma 4\pi R^2$$

Also, since the point P is infinitesimally close to the surface of our sphere, we can take $x \approx R$

Thus, $E = \frac{\sigma}{\epsilon_0}$.

Part (ii):



Consider a ring lying on the sphere, making an angle of θ with the axis passing through O and P. The centre of the ring is at a distance $R\cos\theta$ from the centre, and the radius of the ring is given by $r = R\sin\theta$ (Fig. 2).

We can approximate the ring to be of thickness $Rd\theta$. Thus, the effective charge per unit length of the ring is given by $\lambda = \sigma R d\theta$.

Since the point is infinitesimally far from the surface of the sphere, we can treat the distance from the centre to the point as R. Thus, the distance of the point from the centre of the ring is $z = R(1 - \cos\theta)$.

The electric field due to this ring is given by

$$dE = \frac{\lambda r z}{2\epsilon_0 \left(r^2+z^2\right)^{3/2}} = \frac{(\sigma R d\theta)(R sin\theta)R(1-cos\theta)}{2\epsilon_0 \left(R^2 sin^2\theta + R^2(1-2cos\theta + cos^2\theta)\right)^{3/2}}$$
$$= \frac{\sigma\, sin\theta(1-cos\theta)d\theta}{4\sqrt{2}\epsilon_0\, (1-cos\theta)^{3/2}} = \frac{\sigma\, sin\theta\, d\theta}{4\sqrt{2}\epsilon_0 \sqrt{1-cos\theta}} = \frac{2\sigma\, sin(\theta/2)cos(\theta/2)\, d\theta}{8\epsilon_0\, sin(\theta/2)} = \frac{\sigma\, cos(\theta/2)\, d\theta}{4\epsilon_0}$$

dE points radially outward along the axis.

The total field obtained by integrating the expression with respect to θ from 0 to π is

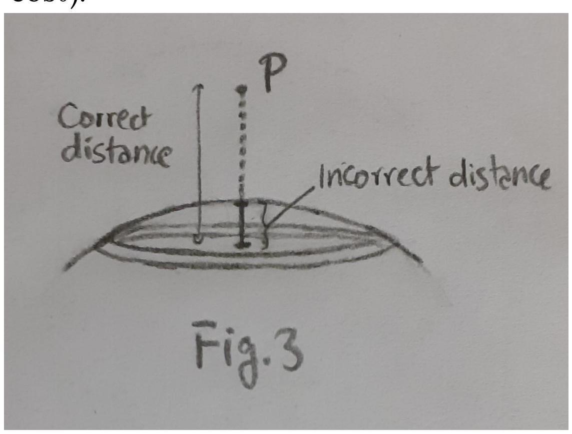
$$E = \int dE = \frac{\sigma}{4\epsilon_0}\int_0^{\pi} cos(\theta/2)\, d\theta = \frac{\sigma}{4\epsilon_0}\left[2 sin(\theta/2)\right]_0^{\pi} = \frac{\sigma}{2\epsilon_0}$$

E points radially outward.

Part (iii):

The above result is incorrect, because the same calculation would supposedly yield the field just inside the sphere too, where it is zero instead of $\frac{\sigma}{2\epsilon_0}$. The calculation does, however, give the next best thing, namely the average of these two values. We'll see why shortly.

The reason why the calculation is invalid is that it doesn't correctly describe the field due to rings on the sphere very close to the given point, that is, for rings characterised by $\theta \approx 0$. The closeup view in Fig. 3 shows that the distance from the centre of a ring to the given point is not equal to R(1 - cosθ).



What is true is that if we remove a thin strip from the top of the sphere (so we now have a gap in the circle representing the cross sectional view), then the above integral is valid for the remaining part of the cylinder. The thin strip contributes negligibly to the integral, since

$$\int_{\Delta}^{\pi} sin(\theta/2)d\theta = [sin(\theta/2)]_{\Delta}^{\pi} = 1 - sin(\Delta/2) \approx 1 \text{ for } \Delta \text{ small.}$$

So, we can say that the field due to the remaining part of the cylinder is equal to the above result of $\frac{\sigma}{2\epsilon_0}$. By superposition, the total field due to the entire cylinder is this field of $\frac{\sigma}{2\epsilon_0}$ plus the field due to the thin strip. But if the point in question is infinitesimally close to the cylinder, then the thin strip looks like an infinite plane, the field of which we know is $\frac{\sigma}{2\epsilon_0}$ in either direction.

The desired total field is then

$$E_{outside} = E_{sphere\ minus\ strip} + E_{strip} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

By superposition we also obtain the correct field just inside the shell:

$$E_{inside} = E_{sphere\ minus\ strip} - E_{strip} = \frac{\sigma}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0} = 0$$

The relative minus sign arises because the field from the sphere-minus-strip is continuous across the gap, but the field from the strip is not; it points in different directions on either side of the strip.

References: *Electricity and Magnetism*, Edward M. Purcell and David J. Morin