PH1213 - Introduction to Electrodynamics
Tutorial Problem

May 5, 2022

Problem

Theorems in Vector Calculus

$$\int_V (\nabla \cdot A) d\tau = \int_S A.nda \tag{1}$$

Divergence Theorem (2)

$$\int_S (\nabla \times A).nda = \oint_C A.dl \tag{3}$$

Stoke's Theorem (4)

Problem Statement

Principle of Superposition

Charge Distribution

Gauss Law

Electrostatics-Problem

◇ Suppose we have a spherical charged surface ,whose surface charge distribution varies as cosine of the polar angle(θ), as $\sigma = \sigma_0 \cos(\theta)$.Find the electric field inside the surface.Obviously finding the solution using integration is bit mathematical.Hint : By creating this charge distribution from known distribution.

Using Principle of Superposition

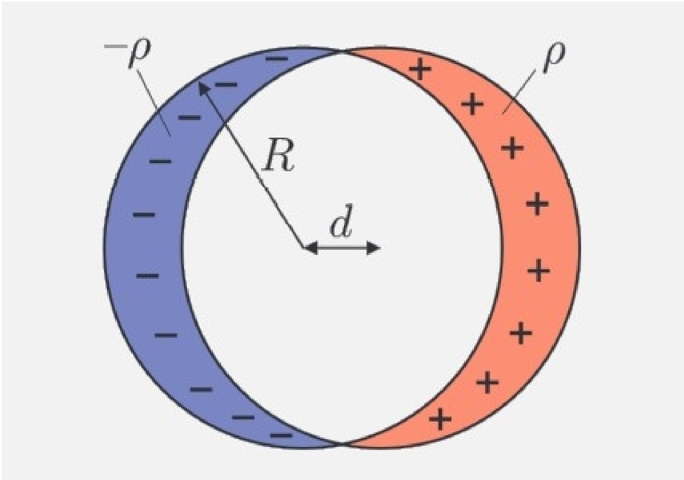

Figure 1: Generalised

The validity of superposition principle is a consequence of linearity of Maxwell's Equations that describes the electromagnetic phenomena.
For example

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Let $\vec{E_1}$&$\vec{E_2}$ are solutions to the equations which means;
$\nabla \cdot \vec{E_1} = \rho_1$ and $\nabla \cdot \vec{E_2} = \rho_2$, then by linearity of the differential operator ,

$$\nabla \cdot \left(\vec{E_1} + \vec{E_2}\right) = \frac{1}{\epsilon_0}(\rho_1 + \rho_2)$$

Therefore,

$$\vec{E_1} + \vec{E_2}$$

is a solution

Generalized question : We have two identical non-conducting spheres of volume charge density $+\rho\& -\rho$which are partially overlapping (the region of overlap will have zero charge density),Determine the electric field inside the region of overlapping.

1 Solution

1.1 Field inside the overlapping region

The electric field produced by the system shown in the figure 1 , can be thought of two independent spheres of opposite charge densities combine together(given).So the electric field produced inside can be decomposed as if the fields where produced by independently by the spheres and by superposition of these Electric Fields we get the net electric field.
We are very much familiar to find the electric field produced by a solid sphere of charge density ρ.
It is derived from Gauss's Law as follows.

$$\oint \vec{E}.d\vec{A} = \frac{Q_i}{\epsilon_0}$$

Integral Form.
Considering a Gaussian Surface , concentric to the sphere , but inside (Because we need to calculate it for some point inside the sphere itself).

$$E * 4 * \pi * r^2 = \rho * \frac{4}{3} * \pi * r^3 / \epsilon_0$$

Simplifying and writing it in vector form

$$\vec{E} = \frac{\rho}{3\epsilon_0}\vec{r}$$

Now we move on to original problem.
Let P be any point inside the region of overlapping of spheres,$\vec{d}$ be the vector joining negative to positive center.Let's denote $\vec{r^+}$,the vector joining the point with the positive center and $\vec{r^-}$ be the vector that joins the point P with negative center.

$$\vec{E_+} = \frac{\rho}{3\epsilon_0}\vec{r_+}$$

$$\vec{E_-} = -\frac{\rho}{3\epsilon_0}\vec{r_-}$$

The net Electric field at point P be the superposition of the fields produced by both spheres(stated above) , therefore

$$\vec{E_{net}} = E_+ \vec{+} E_-$$

$$\vec{E_{net}} = \frac{\rho}{3\epsilon_0}\vec{r_+} - \frac{\rho}{3\epsilon_0}\vec{r_-} = \frac{\rho}{3\epsilon_0}(\vec{r_+} - \vec{r_-})$$

From the figure it is clear from the figure that $\vec{r^+} - \vec{r^-} = -\vec{d}$
Therefore

$$\vec{E_{net}} = -\frac{\rho}{3\epsilon_0}\vec{d}$$

It is interesting to note that the overlapping field have constant electric field.This is also valid irrespective of the radii of the sphere.

We will now solve the problem of the given charge distribution , $\sigma = \sigma_0 \cos(\theta)$,where θ is the polar angle.

The solid line is the line joining the midpoint of the line joining centers to the surface of spheres.The line is intersecting the spheres, let the distance be d_+ and d_- and the distance between intersecting points be d.

$$d_+^2 + (\frac{a}{2})^2 - 2\frac{a}{2}d_+\cos(\theta) = R^2 \tag{5}$$

$$d_-^2 + (\frac{a}{2})^2 + 2\frac{a}{2}d_-\cos(\theta) = R^2 \tag{6}$$

$$d_+^2 - d_-^2 = a(d_+ + d_-)cos(\theta) \tag{7}$$

$$d = d_+ - d_- = acos(\theta) \tag{8}$$

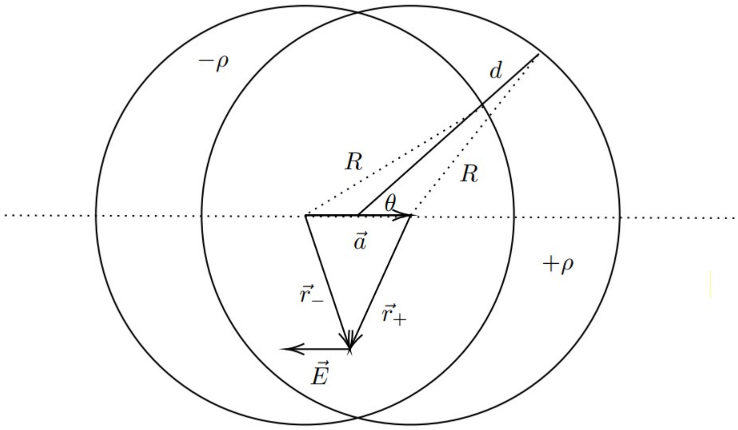

Figure 2:

When a is very small ,this configuration is equivalent to a sphere with radius R with no charge inside, but have charges on the surface with surface charge density:

$$\sigma(\theta) = \rho d = \rho a cos(\theta) = \sigma_0 cos(\theta)$$

So a charge distribution on the surface of a sphere in this form is exactly similar to the general problem we solved which generates an uniform electric field inside the sphere

$$\vec{E} = -\frac{\rho}{3\epsilon_0}\vec{a} = -\frac{\sigma_0}{3\epsilon_0}\hat{a}$$

Also we can find the electric field outside by approximating it as dipole for points which are far way.

We will get the same expression for electric field as dipole with dipole moment , $\vec{p} = Q\vec{a} = \rho\frac{4\pi R^3}{3}\vec{a} = \frac{4\pi R^3}{3}\sigma_0\hat{a}$:

Let the dipole be in y-z plane (for sake of derivation),
The expression for the components of Electric field is given by :

$$E_z = \frac{p}{4\pi\epsilon_0}\frac{3\cos^2(\theta) - 1}{r^3} \tag{9}$$

$$\tag{10}$$

The components of x and y can be combined to give transverse component (perpendicular to z axis ,it will be in x-y plane) :

$$E_\perp = \frac{p}{4\pi\epsilon_0}\frac{3cos(\theta)sin(\theta)}{r^3} \tag{11}$$

$$\tag{12}$$

This result we have derived can be useful in study of dielectrics For a dielectric sphere which retains polarization after an external electric field is applied, assuming this polarisation is due to small displacement of all positive charges with respect to negative charges then the electric field intensity inside is a constant and certain results can be shown.