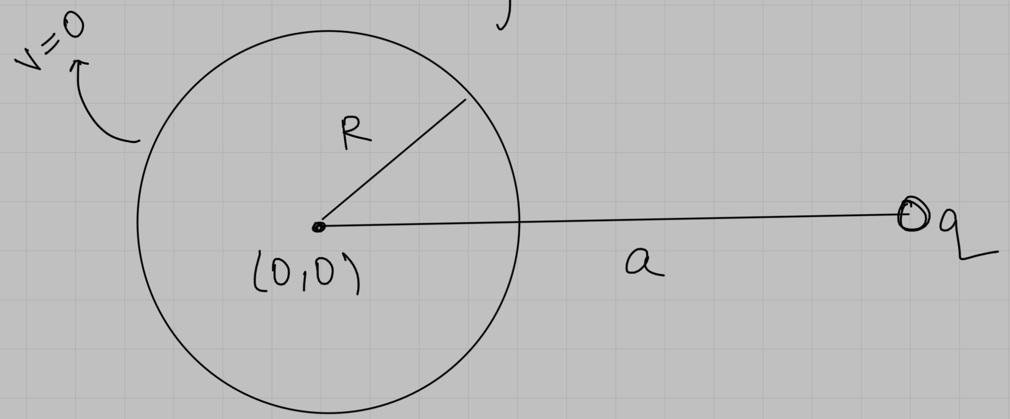
1Q) There exists a grounded conducting sphere of Radius R. A charge q is placed at a distance a from it as shown below:



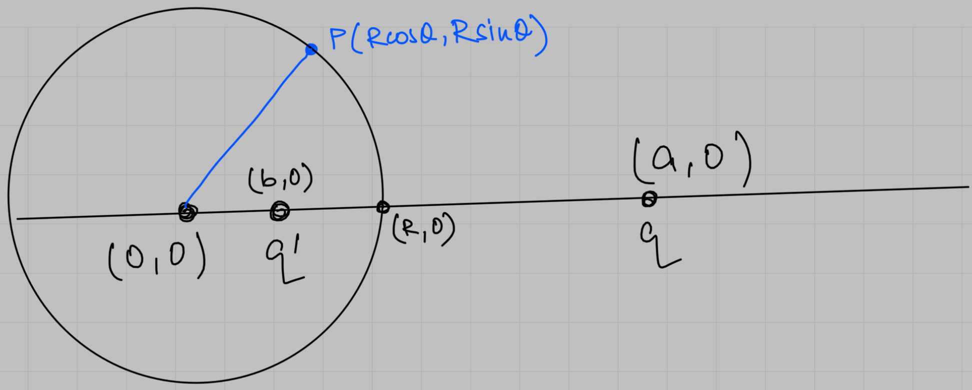
Find the potential at a coordinate $\vec{r}$, which is outside the sphere.

2Q) Now, if the sphere was not grounded, but held at a potential V_0, find the force of attraction on q. Analyse what this tells about the system.

Solution

1) We need to find the potential outside the sphere. Hence, image charge q' must be placed inside the sphere somewhere.

Understandably, to have $V=0$ on surface of sphere, q' has to be $-ve$, and has to be closer to q which is $+ve$.



∴ For all θ, there would exist a solution and value of q' and b to satisfy $V=0$ on sphere surface.

$$\therefore V = 0 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(R\cos\theta - a)^2 + R^2\sin^2\theta}} + \frac{q'}{\sqrt{(R\cos\theta - b)^2 + R^2\sin^2\theta}}\right]$$

solving this is difficult.

But it can be simplified. If there exists only one solution when I solve for $\theta = 0$ and $\theta = \pi$, then that one solution only is the solution that must be applicable for all θ.

$V(\theta = 0) = 0$ and $V(\theta = \pi) = 0$

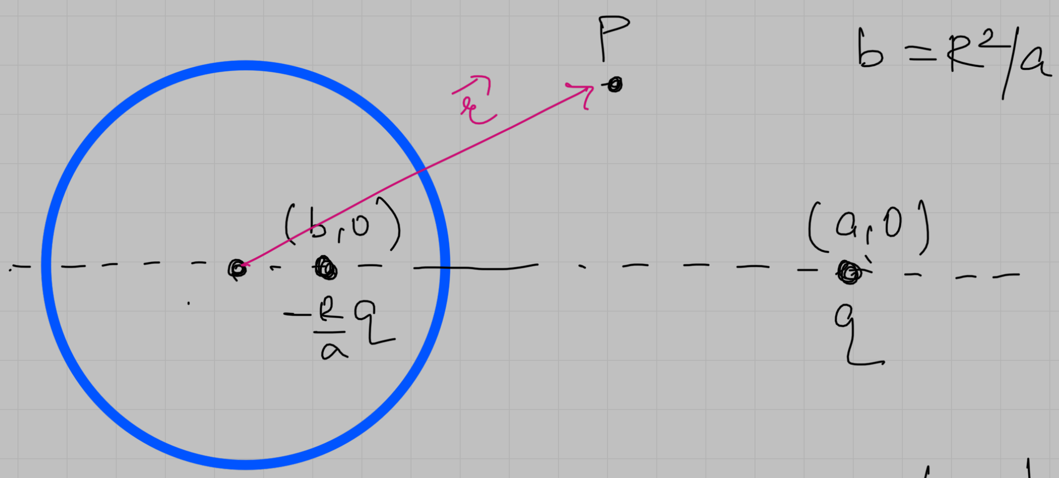
would give us the following:

$$\frac{q}{a-R} + \frac{q'}{R-b} = 0 \quad \text{———} \quad (1)$$

$$\frac{q}{a+R} + \frac{q'}{b+R} = 0 \quad \text{———} \quad (2)$$

From this, we get $b = \frac{R^2}{a}$ and $q' = -\frac{R}{a}q$.

$\therefore$ The system for calculations outside the sphere gets reduced to the following:



$$V(\vec{r}) = \frac{-qR}{4\pi\varepsilon_0 a}\left(\frac{1}{|\vec{r}-\vec{r}_{q'}|}\right) + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}-\vec{r}_q|}\right)$$

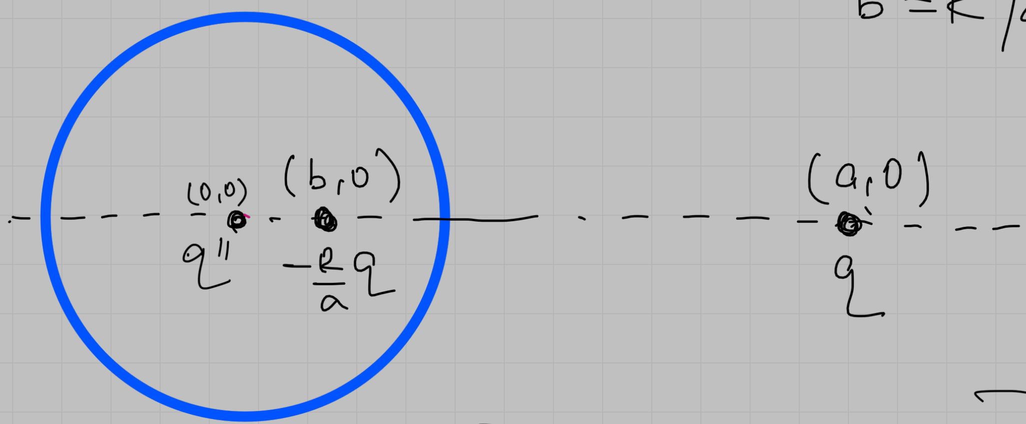
where $\vec{r}_q$ and $\vec{r}_{q'}$ are the charges' respective positions.

2) Now, instead of grounded to $V=0$, the sphere has to be maintained at $V=V_0$. This would only be possible by introduction of another image charge q'' such that it symmetrically affects the spherical boundary surface in question.

$$\longrightarrow \quad V_0 = \frac{1}{4\pi\varepsilon_0}\,\frac{q''}{R} \Rightarrow \underline{q'' = 4\pi\varepsilon_0 V_0 R}$$

$\therefore$ The effective system for calculations outside the sphere would be:

$$q'' = 4\pi\varepsilon_0 V_0 R$$
$$b = R^2/a$$



$$\therefore \vec{F}_{on\ q} = \frac{q}{4\pi\varepsilon_0}\left[\frac{q''}{a^2} - \frac{Rq/a}{(a-b)^2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{4\pi\varepsilon_0 V_0 R}{a^2} - \frac{Rq}{a(a-b)^2}\right]$$

$$\vec{F}_{on\ q} = \frac{qV_0R}{a^2} - \frac{Rq^2}{4\pi\varepsilon_0 a(a-b)^2}$$

$$\therefore \vec{F}_q = \frac{qV_0R}{a^2} - \frac{Rq^2a}{4\pi\varepsilon_0(a^2-R^2)^2}$$

Analyse this equation (for $V_0 > 0$)

One part varies with $\frac{1}{a^2}$, while other term varies with $\frac{1}{a^3}$.

For very large 'a':

$$\vec{F_q} \approx \frac{qV_0R}{a^2} \quad \text{and} \quad V_0 = \frac{1}{4\pi\varepsilon_0} \frac{q''}{R}$$

$$\therefore \vec{F_q} \approx \frac{qq''}{4\pi\varepsilon_0 a^2}$$

This is as if there was an actual point charge at distance 'a' away.

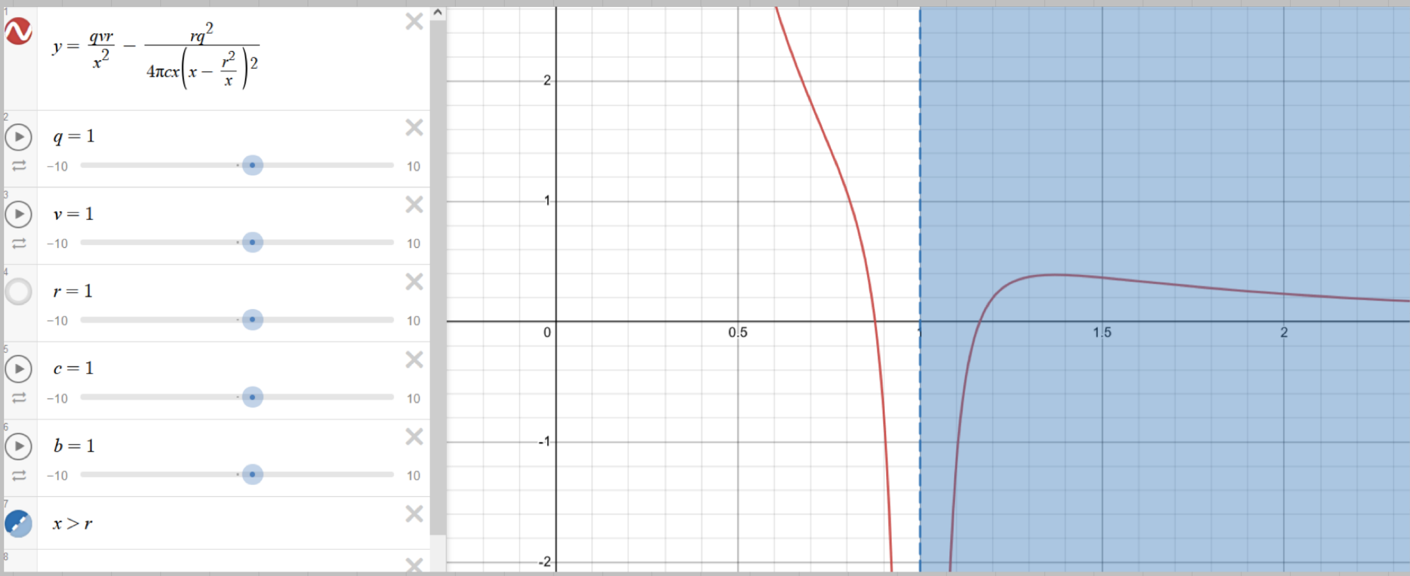
The factor where induced charges play a role fade away, as one might have thought intuitionally.

For 'a' close to 'R':

The $1/(a^2-R^2)^2$ term dominates {due to induced charges}, with the force now being attractive, instead of repulsive earlier.

This means that there should be a value of 'a' such that $\vec{F_q} = 0$

i.e. q is in equilibrium at that position.



Note that the above graph is only valid for $x > r$ because it does not stand valid for regions where an image charge has been introduced.

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