SHUBAN RAMESH GUPTA 20221251
Physics presentation:

Green's Reciprocity Theorem

Motivating: Green's reciprocity theorem goes as follows: if there exists a charge distribution ρ_1 with voltage distribution V_1, and a completely independent (in absence of ρ_1) distribution ρ_2 with potential V_2,

$$\int_{\text{all space}} \rho_1 V_2 \, d\tau = \int_{\text{all space}} \rho_2 V_1 \, d\tau$$

Proving:
let E_1 be field due to ρ_1 and E_2 due to ρ_2.

$$\int E_1 \cdot E_2 \, d\tau = \int -E_1 \cdot \vec{\nabla} V_2 \, d\tau \quad (E = -\vec{\nabla} V)$$

$$\int \vec{\nabla} (E_1 \cdot V_2) d\tau = \int -E_1 \cdot \vec{\nabla} V_2 \, d\tau \int V_2 \vec{\nabla} E_1 \, d\tau$$

$$\therefore \int E_1 \cdot E_2 \, d\tau = \int V_2 \nabla \cdot E_1 \, d\tau - \int \vec{\nabla} (E_1 \cdot V_2) d\tau$$

$$= \int V_2 \nabla \cdot E_1 \, d\tau - \oint \cancel{E_1 V_2} \, da$$

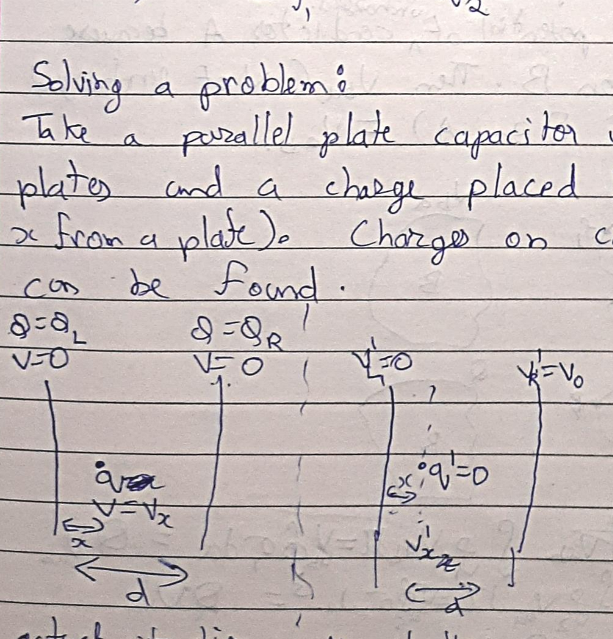
For all space, surface integral $\to 0$

$$\nabla \cdot E_1 = \rho_1 / \varepsilon_0$$

$$\therefore \int E_1 E_2 \, d\tau = \int V_2 \rho_1 / \varepsilon_0 \, d\tau = \int V_1 \rho_2 / \varepsilon_0 \, d\tau$$

Hence, $\int V_2 \rho_1 d\tau = \int V_1 \rho_2 \, d\tau$.

Solving a problem:
Take a parallel plate capacitor with grounded plates and a charge placed in between (distance x from a plate). Charges on conducting plates can be found.



actual situation ; New situation

From actual situation,

$$\int V_1 \rho_2 \, d\tau = V_L \int_L \rho'_L \, d\tau + V_R \int_R \rho'_R \, d\tau + V_x \int_x \rho'_x \, d\tau$$

$$V_L = V_R = \rho'_x = 0$$

$$\therefore \int V_1 \rho_2 \, d\tau = 0$$

$$\int V_2 \rho_1 \, d\tau = V'_L \int_L \rho \, d\tau + V'_R \int_R \rho_R \, d\tau + V'_x \int_x \rho \, d\tau$$

$V'_L = 0$ $\quad V'_R = V_0$ $\quad \int_R \rho_R \, d\tau = Q_R$ $\quad \int_x \rho \, d\tau = q$

$$V'_x = \frac{V_0 x}{d}$$

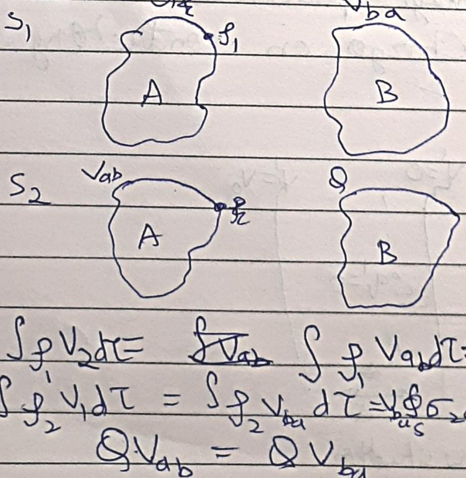
$$\therefore V_0 Q_R + \frac{V_0 x q}{d} = 0$$

$$Q_R = -q\, x/d$$ ~~by sy~~

For Q_L, take $x = d-x$. $Q_L = -q\left(1 - \frac{x}{d}\right)$

Another property: if we take 2 uncharged conductors and put charge on either, the other

IF V_{ab} is potential of uncharged conductor A because of charge Q on B. Then V_{ba} (V of uncharged B because of Q on A) $= V_{ab}$.



$$\int \rho_1 V_2 \, d\tau = \cancel{\int V_{ab}} \int \rho_1 V_{ab} \, d\tau = V_{ab} \oint \sigma_1 \, da = Q V_{ab}$$

$$\int \rho_2 V_1 \, d\tau = \int \rho_2 V_{ba} \, d\tau = V_{ba} \oint_S \sigma_2 \, da = Q V_{ba}$$

$$Q V_{ab} = Q V_{ba}$$

$$V_{ab} = V_{ba}.$$

Usage: Helps simplify problems involving conductors where exact charge distribution is not necessary.
Forms of reciprocity theorem used to analyse electrical networks. $\left(\int J_1 \cdot E_2 \, dV = \int E_1 \cdot J_2 \, dV \right)$