Shinjini Mondal – 20221247 – BG

Q Two infinitely - long grounded metal plates, ~~again~~ at $y=0$ and $y=a$ are connected at $x=\pm b$ by metal strips maintained at a constant potential V_0 as shown in fig. (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

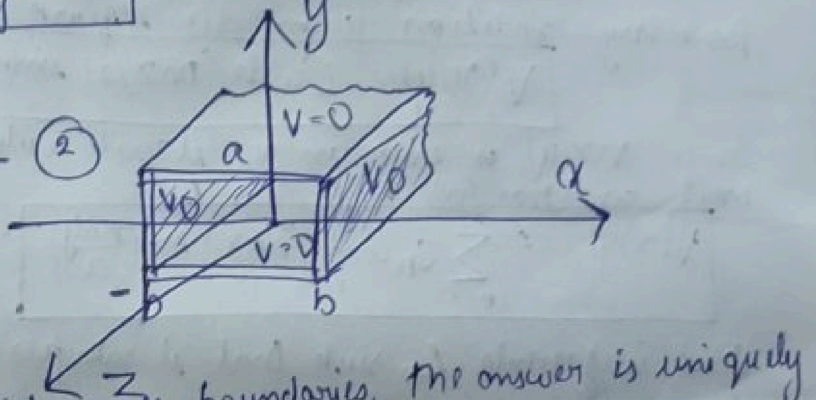
The configuration is independent of $z \Rightarrow$ it is a two-dimensional problem in mathematical terms

Laplace's equation $\longrightarrow$

$$\boxed{\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0} \quad - \quad (1)$$

Boundary Conditions $\longrightarrow$

$$\boxed{\begin{array}{ll} \text{i)} & V=0, \text{ when } y=0 \\ \text{ii)} & V=0, \text{ when } y=a \\ \text{iii)} & V=V_0 \text{ when } x=b \\ \text{iv)} & V=V_0, \text{ when } x=-b \end{array}} \quad - \quad (2)$$



Since the potential is specified on all boundaries, the answer is uniquely determined $\longrightarrow$ (First Uniqueness Theorem)

We look for solutions in the form of products —

$$\boxed{V(x,y)=X(x)Y(y)} \quad - \quad (3)$$

Majority of solutions to Laplace's Equation do not have such a form. But here, since the solution we get is very special, so by pasting them together we ~~get~~ can construct the general solution.

Separate the variables

$$\frac{\partial^2(XY)}{\partial x^2}+\frac{\partial^2(XY)}{\partial y^2}=0$$

$$\Rightarrow Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0$$

Dividing both sides by $X(x)Y(y)$,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0$$

Putting (3) in (1)

$$\boxed{\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0} \quad - \quad (4)$$

1st term depends only on x, second term depends only on y.

Eq^n of the form $\longrightarrow$

$$\boxed{f(x)+g(y)=0} \quad - \quad (5)$$

This is true only when f and g are both constant.

$$\rightarrow \boxed{\frac{1}{X}\frac{d^2X}{dx^2}=C_1} \text{ and } \boxed{\frac{1}{Y}\frac{d^2Y}{dy^2}=C_2} \text{ with } \boxed{C_1+C_2=0} \quad - \quad (6)$$

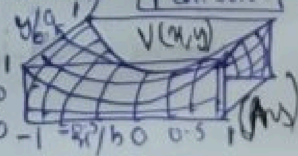
collect all the x - dependence into one term and all the y-dependence into another by dividing through V or $X(x)\,Y(y)$.

One of these constants is positive, the other negative (or both are 0)

In this case C_1 is positive & C_2 is negative (Reason → later)

$$\Rightarrow \boxed{\frac{d^2X}{dx^2}=k^2X} \qquad \boxed{\frac{d^2Y}{dy^2}=-k^2Y} \quad - \quad (7)$$

Function



(Ans)

Partial differential eq^n converted into two ordinary differential equations — lot easier to solve.

$$\boxed{X(x) = Ae^{kx} + Be^{-kx}} , \quad \boxed{Y(y) = C\sin ky + D\cos ky}$$

$$\boxed{V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right)} \quad - (8)$$

Appropriate separable solution to Laplace's Equation ⟶

The region doesnot extends to $x = \infty$, so e^{kx} is perfectly acceptable, so $A \neq 0$

The situation is symmetric w.r.t. x, so $V(-x,y) = V(x,y)$

$\Rightarrow \boxed{A = B}$ Using,

$$\boxed{e^{kx} + e^{-kx} = 2\cosh kx}$$

absorbing $2A$ into C and D,

$$\boxed{V(x,y) = \cosh kx\,(C\sin ky + D\cos ky)} \quad - (9)$$

Condition (i) demands $D = 0$, so, $V(x,y) = Ce^{-kx}\sin ky$

Condition (ii) yields $\sin ka = 0$

$\Rightarrow \boxed{k = \frac{n\pi}{a}}$, $n = 1, 2, 3, \ldots$

$\left(\begin{matrix}\sin ka = 0 \\ \Rightarrow ka = n\pi\end{matrix} \Rightarrow k = n\pi/a\right)$

Boundary conditions (i) and (ii) require $D = 0$, $k = n\pi/a$, so

$$\boxed{V(x,y) = C\cosh(n\pi x/a)\sin(n\pi y/a)} \quad (10)$$

Since $V(x,y)$ is even in x, it will automatically meet condition (iv) if it fits (iii).

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)} \quad - (11)$$

Pick coefficients C_n such that it satisfies condition (iii)

$$V(b,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = V_0$$

We can solve it by Fourier trick ⟶ Multiply (11) by $\sin\left(\frac{n'\pi y}{a}\right)$ (where n' is a positive integer) and integrate from 0 to a ⟶

$$\boxed{\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy} \quad - (12)$$

Integrating ⟶

$$\boxed{\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) = \begin{cases} 0 & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n \end{cases}} \quad - (13)$$

All terms in series dropout except $n = n'$, so left side of Eqn (12) reduces to ~~C_n~~ $(a/2)\,C_n\cosh\left(\frac{n\pi b}{a}\right)$

$$\therefore C_n\cosh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy$$

$$\therefore \text{Potential } V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5}\frac{1}{n}\frac{\cosh(n\pi x/a)\sin(n\pi y/a)}{\cosh(n\pi b/a)}$$

(strip at $x = \pm b$ has constant potential V_0 ⟶ it is insulated from the grounded plates at $y = 0$ and $y = a$)

$$\Rightarrow C_n\cosh\left(\frac{n\pi b}{a}\right) = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \frac{2V_0}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$