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Batch 3

Q. Suppose the actual force of interaction between two point charges is

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}\left(1+\frac{r}{\lambda}\right) e^{-r/\lambda}\,\hat{r}$$

where λ is a constant and is of the order of the half of the radius of the universe so that the force ~~~~ reduces to coulomb's law for 'r' small enough. Formulate the new laws of electrostatics.

Soln:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\left(1+\frac{r}{\lambda}\right) e^{-r/\lambda}\,\hat{r}$$

For a system of charges, by superposition principle, we have

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2}\left(1+\frac{r_i}{\lambda}\right) e^{-\frac{r_i}{\lambda}}\,\hat{r}_i$$

For a continuous charge distribution,

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')\,\hat{r}}{r^2}\left(1+\frac{r}{\lambda}\right) e^{-\frac{r}{\lambda}}\,d\tau}$$

As the electric field is radial and symmetric,

$$\vec{\nabla} \times \vec{E} = 0$$

$\Rightarrow \int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$ is independent of path and therefore it has a scalar potential such that,

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

$$V(\vec{r}) - V(\infty) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l}$$

as $\int \vec{E}\cdot d\vec{l}$ is independent of path, we choose the path along radial line $d\vec{l} = dr\,\hat{r}$

$$\Rightarrow V(\vec{r}) - 0 = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}\,dr$$

let $-\frac{r}{\lambda} = t \Rightarrow dt = -\frac{dr}{\lambda}$

$$\Rightarrow V(\vec{r}) = \frac{-q}{4\pi\epsilon_0}\int_{-\infty}^{-\frac{r}{\lambda}} \frac{1}{(-\lambda t)^2}(1-t)e^{t}(-\lambda\,dt)$$

$$= \frac{1}{\lambda}\frac{q}{4\pi\epsilon_0}\int_{-\infty}^{-\frac{r}{\lambda}}\left(\frac{1}{t^2}-\frac{1}{t}\right)e^{t}\,dt$$

$$= \frac{1}{\lambda}\frac{q}{4\pi\epsilon_0}\left(\frac{e^{t}}{t}\right]_{-\infty}^{-\frac{r}{\lambda}}$$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}}$$

Now, Let us calculate the equivalent Gauss's law in this universe, for a single charge at origin,

$$\oint_S \vec{E}\cdot d\vec{a} = \iint \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}} \times r^2\sin\theta\,d\theta\,d\phi$$

$$= \frac{q}{4\pi\epsilon_0}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= \frac{q}{\epsilon_0}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}$$

which is not constant and thus the "usual" Gauss' law does not hold. But if we calculate volume integral of potential

$$\int_V V\,d\tau = \iiint \frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r} \times r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{q}{4\pi\epsilon_0}\int r e^{-r/\lambda}\,dr \int \sin\theta\,d\theta \int d\phi$$

$$= \frac{q}{\epsilon_0}\int r e^{-r/\lambda}\,dr$$

By simple substitution $t = -\frac{r}{\lambda}$ and then applying integration-by-parts we see that,

$$\int_V V d\tau = (-\lambda^2)\frac{q}{\epsilon_0}\left[\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} - 1\right]$$

$$\Rightarrow \quad -\frac{1}{\lambda^2}\int_V V d\tau = \underbrace{\frac{q}{\epsilon_0}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda}}_{\oint_S \vec{E}\cdot d\vec{a}} - \frac{q}{\epsilon_0}$$

$$\Rightarrow \quad \boxed{\oint_S \vec{E}\cdot d\vec{a} + \frac{1}{\lambda^2}\int_V V d\tau = \frac{q}{\epsilon_0}}$$

We obtained this for a single charge at origin but, in general, it applies to any distribution of charge and any surface and not only a sphere.

Now by divergence theorem,

$$\int_V (\vec{\nabla}\cdot\vec{E})d\tau + \frac{1}{\lambda^2}\int_V V d\tau = \int \frac{\rho}{\epsilon_0} d\tau$$

$$\Rightarrow \int_V \left(\vec{\nabla}\cdot\vec{E} + \frac{V}{\lambda^2}\right) d\tau = \int \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

Since this is valid for any volume, the integrands must be same,

$$\boxed{\vec{\nabla}\cdot\vec{E} + \frac{V}{\lambda^2} = \frac{\rho}{\epsilon_0}} \quad \text{—— (1)}$$

Suppose there is some conductor with free charges as well as some extra charge residing on it.

As $\vec{E}$ is zero inside a conductor,

$$\vec{\nabla}\cdot\vec{E} = 0$$

Also, $\vec{E} = -\vec{\nabla} V \Rightarrow V = \text{constant}$

As $\vec{\nabla}\cdot\vec{E} = 0$, $\frac{V}{\lambda^2} = \frac{\rho}{\epsilon_0}$ and therefore

ρ = charge density inside the conductor should also be constant.