Name - Keshav Kothari
Reg No - 20221140

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Batch - B2 - T3

Physics Tutorial Problem

→ Problem Title - Legendre Polynomials and Laplace's Equation in Spherical Coordinates

☆ In spherical coordinates, Laplace's equation reads -

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

→ Assuming azimuthal symmetry $\left(\frac{\partial V}{\partial\phi}=0\right)$ -

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

→ Applying seperation of variables technique, let $V(r,\theta)=R(r)\,\varnothing(\theta)$ -

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\varnothing\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\varnothing}{d\theta}\right)=0$$

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Since the first term depends only on r and the second only on θ, it follows that each must be a constant -

Let (1) $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) > 0$

(2) $\frac{1}{\phi \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) = -l(l+1) < 0$

where $l \in R - [-1, 0]$

$\Downarrow$

$l(l+1) \in R^+$

(1) Solving (1), we get -

$R(r) = Ar^l + \frac{B}{r^{l+1}}$; A, B are arbitrary integration constants

(2) $\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) = -l(l+1)\sin\theta\,\phi$

$\boxed{\text{General Differential Form of Legendre Polynomials}} \rightarrow \boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (n)(n+1)y = 0}$

$$\cos\theta\,\phi' + \sin\theta\,\phi'' + l(l+1)\sin\theta\,\phi = 0$$

$$\Rightarrow \frac{\cos\theta}{\sin\theta}\phi' + \phi'' + l(l+1)\phi = 0$$

☆ $$\frac{\partial\phi}{\partial\theta} = -\sin\theta\frac{\partial\phi}{\partial\cos\theta}$$

☆ $$\frac{\partial^2\phi}{\partial\theta^2} = \frac{\partial^2\phi}{\partial\cos\theta^2}\sin^2\theta - \frac{\partial\phi}{\partial\cos\theta}\cos\theta$$

→ Substituting, we get (let $\frac{\partial\phi}{\partial\cos\theta} = t'$)

$$\sin^2\theta\, t'' - 2\cos\theta\, t' + l(l+1)\phi = 0$$

→ Comparing with the general formula, we see that $\cos\theta = x$

$$\Rightarrow \boxed{\phi = P_l(\cos\theta)}$$

Legendre Polynomial of Degree 'l'

$\boxed{\text{Rodrigues Formula}} \rightarrow \boxed{P_l(x) = \frac{1}{2^l\, \lfloor l}\left(\frac{d}{dx}\right)^l (x^2-1)^l}$

Legendre Polynomials — Orthogonal + Complete

Definition – A set of orthogonal polynomials P_n of degree 'n' such that-

$$\int_{-1}^{+1} P_m P_n = 0 \quad \forall \; n \neq m$$

$$P_n(1) = 1 \quad \forall \; n$$

Significance – Legendre Polynomials crop up in several different branches of Science – from electrostatics to the Hydrogen Atom – and are particularly closely associated with Spherical Coordinates.
Their orthogonality and completeness plays an absolutely essential role in their widespread applicability in science.