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Physics Presentation Problem

Q. Suppose you have two separated conductors. If you charge up a by an amount Q (leaving b uncharged), the resulting potential of b is, say, V_{ab}. On the other hand, if you put the same charge Q on conductor b, the potential would be V_{ba}. Show that
$$V_{ab} = V_{ba}.$$

Soln. For this problem, we first prove an important theorem known as Green's Reciprocity theorem. It says that if a charge distribution $\rho_1(r)$ produces a potential $V_1(r)$ and another charge distribution $\rho_2(r)$ produces a potential $V_2(r)$, then

$$\int_{\text{all space}} \rho_1 V_2 \, d\tau = \int_{\text{all space}} \rho_2 V_1 \, d\tau$$

Proof: Let $I = \int E_1 . E_2 \, d\tau$
$$= \int (\nabla V_1).(\nabla V_2) \, d\tau$$

But $\nabla.(V_1 \nabla V_2) = (\nabla V_1).(\nabla V_2) + V_1(\nabla^2 V_2)$

$$\Rightarrow I = \int \nabla.(V_1 \nabla V_2) d\tau - \int V_1(\nabla^2 V_2) \, d\tau$$
$$= \oint_S V_1(\nabla V_2) da + \frac{1}{\epsilon_0}\int V_1 \rho_2 \, d\tau$$

Since surface integral is over a huge sphere at

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infinity, $V_1, V_2 \to 0$

So, $I = \frac{1}{\epsilon_0} \int V_1 \rho_2 \, d\tau$

By the same argument, with 1 and 2 reversed,

$$I = \frac{1}{\epsilon_0} \int V_2 \rho_1 \, d\tau$$

So, $\boxed{\int V_1 \rho_2 \, d\tau = \int V_2 \rho_1 \, d\tau}$ qed.

Now, for the conductors, in case I; $Q_a = \int_a \rho_1 \, d\tau = Q$; $Q_b = \int_b \rho_2 \, d\tau = 0$;

$V_{1b} \equiv V_{ab}$.

in case II; $Q_a = \int_a \rho_2 \, d\tau = 0$, $Q_b = \int_b \rho_2 \, d\tau = Q$;

$V_{2a} \equiv V_{ba}$

Now, $\int V_1 \rho_2 \, d\tau = V_{1a} \int_a \rho_2 \, d\tau + V_{1b} \int_b \rho_2 \, d\tau = V_{ab} Q$

$\int V_2 \rho_1 \, d\tau = V_{2a} \int_a \rho_1 \, d\tau + V_{2b} \int_b \rho_1 \, d\tau = V_{ba} Q$

By Green's Reciprocity Theorem,

$$Q V_{ab} = Q V_{ba}$$

$$\Rightarrow \boxed{V_{ab} = V_{ba}}$$ qed