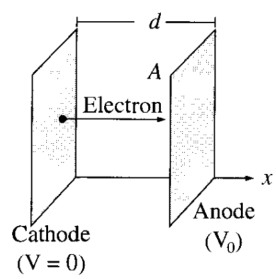
PH1213 Problem Presentation

GAURAV VERMA

20221102

1 Problem



In a vacuum diode, electrons are 'boiled' off a hot cathode at potential zero and accelerated across a gap to the anode, held at positive potential V_0. The cloud of moving electrons within the gap quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then, a steady current I flows in between the plates. Suppose $A >> d^2$ so that edge effects can be neglected. Find the potential $V(x)$ as a function of distance x.

2 The Strategy

We start by writing the equation for current in terms of charge density and speed of the electrons. Then we write the speed as a function of the potential. Then we use Poisson's Equation and eliminate ρ and v to get $V(x)$.

3 Solution

Let the electron moves at speed $v(x)$ at a distance x from the cathode, and $V(x)$ is the potential at that point. We write

$$eV(x) = \frac{1}{2}mv^2$$

$$v(x) = \sqrt{\frac{2e}{m}}V^{1/2} \tag{1}$$

Now we find an equation for the current I. A small charge dq in the vacuum can be written as

$$dq = \rho A dx$$

Therefore

$$I = \frac{dq}{dt} = \frac{\rho A dx}{dt} = \rho A v \tag{2}$$

By Poisson's Equation, we have

$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$

$$\rho = -\epsilon_0 \frac{d^2 V}{dx^2} \tag{3}$$

Putting equations (1) and (3) in equation (2) we get

$$I = -\epsilon_0 A V^{1/2} \sqrt{\frac{2e}{m}} \frac{d^2 V}{dx^2}$$

$$\Rightarrow \frac{d^2 V}{dx^2} = k V^{-1/2}, \qquad k = \frac{-I}{A\epsilon_0}\sqrt{\frac{m}{2e}}$$

$$\Rightarrow \frac{d^2 V}{dx^2} = \frac{k}{\sqrt{V}}$$

$$\Rightarrow V' \frac{dV'}{dx} = \frac{k}{\sqrt{V}} \frac{dV}{dx}$$

$$\Rightarrow V' dV' = k V^{-1/2} dV$$

$$\Rightarrow \int V' dV' = \int k V^{-1/2} dV$$

$$\Rightarrow \frac{V'^2}{2} = 2k\sqrt{V}$$

$$\Rightarrow V' = 2\sqrt{k} V^{1/4}$$

$$\Rightarrow \frac{dV}{V^{1/4}} = 2\sqrt{k} dx$$

$$\Rightarrow \int \frac{dV}{V^{1/4}} = \int 2\sqrt{k} dx$$

$$\Rightarrow V^{3/4} = \frac{3}{2}\sqrt{k} x$$

$$\Rightarrow \boxed{\mathrm{V} = \left(\frac{81 I^2 m}{32 \epsilon_0^2 e A^2}\right)^{1/3} x^{4/3}}$$

Therefore we have found potential as a function of x.