Physics Problem Presentation

Prove that a ~~[illegible]~~ vector field can be decomposed into a conservative and solenoidal field, and also show that it can uniquely be determined by its divergence, curl and its boundary conditions.

Solution:

Consider the Field $F(r)$

then; $F(r) = \int_V F(r')\,\delta^3(r-r')\,d\tau'$

$$\delta^3(r-r') = \frac{-1}{4\pi}\nabla^2\left(\frac{1}{|r-r'|}\right)$$

$$F(r) = \int_V F(r')\left(\frac{-1}{4\pi}\nabla^2\left(\frac{1}{|r-r'|}\right)\right)d\tau'$$

$$= \frac{-1}{4\pi}\nabla^2\int_V \frac{F(r')}{|r-r'|}\,d\tau'$$

$$= \frac{-1}{4\pi}\nabla\left(\nabla\cdot\int_V \frac{F(r')}{|r-r'|}\,d\tau'\right) + \frac{1}{4\pi}\nabla\times\left(\nabla\times\int_V \frac{F(r')}{|r-r'|}\,d\tau'\right)$$

Clearly, the first term is conservative and the second is solenoidal.

Each of these can be split into two terms, out of which the ones having $\nabla\cdot F(r')$ and $\nabla\times F(r')$ will disappear, as they are w.r.t the unprimed coords.

Thus; $F(r) = \frac{-1}{4\pi}\nabla\left(\int_V F(r')\cdot\nabla\left(\frac{1}{|r-r'|}\right)d\tau'\right) - \frac{1}{4\pi}\nabla\times\left(F(r')\times\nabla\left(\frac{1}{|r-r'|}\right)d\tau'\right)$

$$= \frac{1}{4\pi}\nabla\left(\int_V F(r')\cdot\nabla'\left(\frac{1}{|r-r'|}\right)d\tau'\right) + \frac{1}{4\pi}\nabla\times\left(F(r')\times\nabla'\left(\frac{1}{|r-r'|}\right)d\tau'\right)$$

$$= \frac{1}{4\pi}\nabla\left(\int_V \nabla'\cdot\left(\frac{F(r')}{|r-r'|}\right)d\tau' - \int_V \frac{\nabla' F(r')}{|r-r'|}d\tau'\right) + \frac{1}{4\pi}\nabla\times\left(\int_V \nabla'\times\left(\frac{F(r')}{|r-r'|}\right)d\tau' - \int_V \frac{\nabla'\times F(r')}{|r-r'|}d\tau'\right)$$

Using the divergence and stokes theorems, the first term of each can be converted to surface integrals, which go to zero over all space, if F is well-behaved.

Thus;

$$F(r) = \frac{-1}{4\pi} \int_V \left(\nabla' \cdot F(r')\right) \nabla\left(\frac{1}{|r-r'|}\right) d\tau' - \frac{1}{4\pi} \int_V \left(\left(\nabla' \times F(r')\right) \times \nabla\left(\frac{1}{|r-r'|}\right)\right) d\tau'$$

Thus; $F(r)$ is described completely by $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$, given proper boundary conditions.