PH1213 Presentation

Rahul S. Hejjaji

20221214

1 The Problem

A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has 3 grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential, $V_0(y)$ (a thin layer of insulation at the two corners prevents shorting).

a) Develop a general formula for the potential inside the pipe.

b) Find the potential explicitly, for the case $V_0(y) = V_0$(a constant)

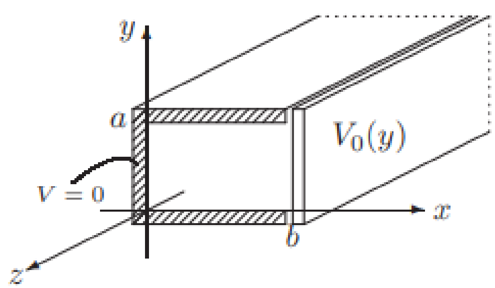

Figure 1: The Setup

2 The Solution

It is apparent from the configuration that the potential inside the pipe is independent of z. This problem requires that we solve Lapalace's equation in 2 dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

subject to the following boundary conditions

$$V(x,0) = 0 \tag{1}$$

$$V(x,a) = 0 \tag{2}$$

$$V(0,y) = 0 \tag{3}$$

$$V(b,y) = V_0(y) \tag{4}$$

As the boundary conditions are independent of each other, Laplace's equation can be solved by separating the variables. Thus, we assume a solution of the form

$$V(x,y) = V(x)V(y)$$

We know that this yields a solution of the form

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky)$$

From condition (1) we find that $D = 0$. From condition (3), we find that $B = -A$. Finally from (2) we see that $ka = n\pi$ where n is a natural number. Using these observations, we get

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\left(\sin\frac{n\pi y}{a}\right) = (2AC)\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (1), (2), (3) is

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{5}$$

Now, we determine the coefficients C_n using condition (4). Since we have

$$V(x,y) = \sum C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

we find, using Fourier's trick, that

$$C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$\implies C_n = \frac{2}{a\sinh\left(n\pi b/a\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

Thus, the solution to part a) is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \left[\frac{2}{a \sinh(n\pi b/a)} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy \right] \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

Now we proceed to solve part b). Since $V_0(y) = V_0$, a constant, it can be brought outside the integral arrived at in the previous part. So, we have

$$C_n = \frac{2V_0}{a \sinh(n\pi b/a)} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

After integrating, we find that for even n, $C_n = 0$ and for odd n,

$$C_n = \frac{4V_0}{n\pi \sinh(n\pi b/a)}$$

Plugging this into (5) we get

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{1,3,5\ldots}^{\infty} \frac{\sinh(n\pi x/a) \sin(n\pi y/a)}{n \sinh(n\pi b/a)}}$$