Physics Tutorial Problem

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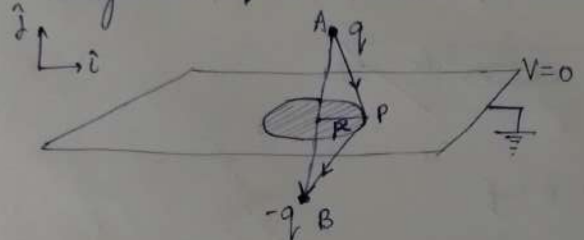
Problem: A charge q is placed at a height h from the grounded large conducting plate. Find the radius of the circular area containing charge $-q/2$ and has its centre at the foot of $\perp$ from the charge to the plate.

Solution: By method of images, we can imagine a charge B of magnitude $-q$ at a distance h under the plate.



Imagine that this area of disc fills in the charge $-q/2$. Let the radius of this disc be R.

Now, to write total Electric field on P,

Electric field at P = Electric field from A + Electric field from induced charges

by method of images, this $-q$ charge at B can act like that.

Electric field due to A at P

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{AP}}{(AP)^3}$$

Electric field due to B at P

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{PB}}{(PB)^3}$$

Net field at P $= \dfrac{q}{4\pi\epsilon_0}\dfrac{\vec{AP}+\vec{PB}}{(AP)^3}$ [here we could write so because due to symmetry, AP = PB in magnitude]

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{AB}}{(AP)^3}$$

$$= \frac{q(2h)}{4\pi\epsilon_0}\frac{1}{(h^2+R^2)^{3/2}}(-\hat{j})$$

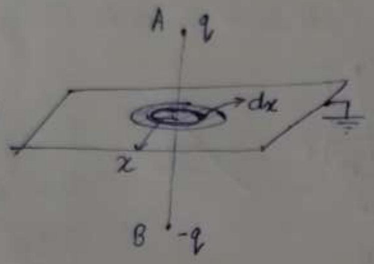
Now, since there net electric field is $\perp$ to the area (parallel to area vector),

$\oint E \cdot da = \frac{q}{\epsilon_0}$ (Gauss's Law) $\Rightarrow \epsilon_0 E = \sigma$ (where σ is surface charge density)

Hence, $\sigma = \frac{2qh\epsilon_0}{4\pi\epsilon_0(h^2+R^2)^{3/2}} = \frac{qh}{2\pi(h^2+R^2)^{3/2}}$

Now, we have to find x such that the net charge induced in that area is $-q/2$

To do that,
Let take a ring of radius x, thickness dx as follows in the diagram



Now, charge density on the ring $= (2\pi x\, dx) \times \frac{qh}{2\pi(h^2+x^2)^{3/2}}$

$dq = \frac{qh\, \cancel{2\pi} x\, dx}{\cancel{2\pi}(h^2+x^2)^{3/2}}$; $\int_0^{q/2} dq = \int_0^R \frac{qhx\, dx}{(h^2+x^2)^{3/2}}$

Let $h^2+x^2 = t^2$

$2x\,dx = 2t\,dt$, $x\,dx = t\,dt$, $x=0$ $t=h$ & $x=R$, $t=(h^2+R^2)^{1/2}$

Hence, our original equation becomes $\int_0^{q/2} dq = \int_h^{(R^2+h^2)^{1/2}} \frac{qh\,t\,dt}{t^3}$

Hence, $q/2 - 0 = \int_h^{(h^2+R^2)^{1/2}} \frac{qh\,dt}{t^3} = qh\left[-\frac{1}{t}\right]_h^{(h^2+R^2)^{1/2}}$

Hence, $+\cancel{q}/2 = \cancel{q}h\left[\frac{-1}{\sqrt{h^2+R^2}} - \left(\frac{-1}{h}\right)\right]$

Hence, $\frac{+1}{2h} = \frac{1}{h} - \frac{1}{\sqrt{h^2+R^2}}$, $\frac{1}{h} - \frac{1}{2h} = \frac{1}{\sqrt{h^2+R^2}}$

$$\Rightarrow \frac{1}{2h} = \frac{1}{\sqrt{h^2+R^2}} \; ; \; \frac{2h}{1} = \sqrt{h^2+R^2}$$

Squaring both sides,

$$4h^2 = h^2 + R^2 \, , \quad 3h^2 = R^2$$

Hence we have $\boxed{R = \sqrt{3}\,h}$