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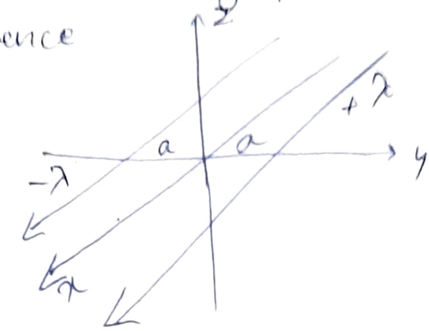
Q) Two infinetely long wires running parallel to the ~~x-ax~~ x-axis carry uniform charge densities $+\lambda$ and $-\lambda$

a) find the potential at any point (x, y, z) using origin as your reference

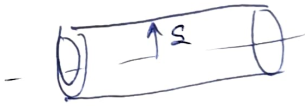


b) Show that equipotential surfaces are circular cylinders, and locate the radius of the cylinder corresponding to a given potential V_0

Solution

First, let's find the potential at a distance s from an infitely long straight wire with uniform line charge λ.

$$\oint \vec{E} \cdot da = 2\pi s l = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$



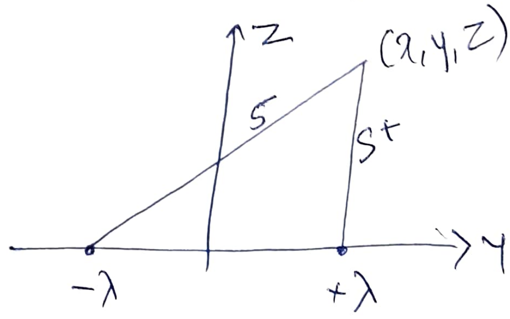
$$E = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

$$V(s) = -\int \vec{E} \cdot d\vec{l} = -\int_a^s \frac{1}{4\pi\epsilon_0} \frac{2\lambda\, ds}{s} = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s}{a}\right) \quad (1)$$

In this case,

$$V_+ = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_+}{a}\right)$$

$$V_- = +\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_-}{a}\right)$$



$$V_{total} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{S_-}{S_+}\right) = V_1 + V_2$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y+a)^2 + z^2}}{\sqrt{(y-a)^2 + z^2}}\right)$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right) = V$$

b) $V = V_0$ $\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2} = e^{(4\pi\epsilon_0 V_0)/\lambda} = \text{constant (say } K) - (2)$

$$y^2 + 2ay + a^2 + z^2 = K(y^2 + a^2 - 2ay + z^2)$$

$$y^2(K-1) + a^2(K-1) - 2ay(K+1) + z^2(K-1) = 0$$

$$y^2 + a^2 + z^2 - 2ay\left(\frac{K+1}{K-1}\right) = 0 \quad - (3)$$

Equation of a circle with $(y_0, 0)$ and radius R

$$(y - y_0)^2 + z^2 = R^2$$

$$y^2 + z^2 + (y_0^2 - R^2) - 2yy_0 = 0 \quad - (4)$$

Comparing (3) & (4)

$$y_0 = a\left(\frac{K+1}{K-1}\right)$$

$$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2$$

$$= a^2\left(\frac{K+1}{K-1}\right)^2 - a^2$$

$$= a^2\left[\frac{(K+1)^2}{(K-1)^2} - 1\right]$$

$$= \frac{a^2(4K)}{(K-1)^2}$$

$$R = \frac{2a\sqrt{K}}{|K-1|}$$

Substitute from ②

$$R = \frac{2a\, e^{2\pi \epsilon_0 V_0/\lambda}}{\cancel{2a}\; e^{4\pi\epsilon_0 V_0/\lambda} - 1}$$

$$R = a \frac{2}{e^{\frac{2\pi\epsilon_0 V_0}{\lambda}} - e^{-2\pi\epsilon_0 V_0/\lambda}} =$$

$$= a \frac{\cancel{2}\; 1}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} \qquad \sinh = \frac{e^{x} - e^{-x}}{2}$$

$$= a \,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)$$

* Potential does not depend on circle
* Equipotential surfaces are cylinder or circles in y-z plane since extending along x axis gives us our cylinder.
* Axis location $\Rightarrow y_0 = a\left(\frac{K+1}{K-1}\right)$