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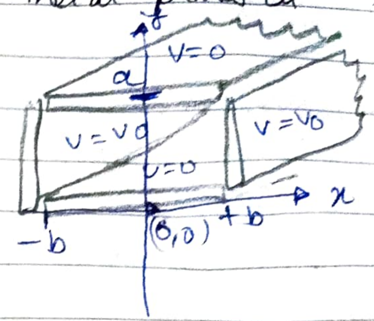
B4

Problem

~~Introduction~~ . Two infinitely long grounded metal plates at $y=0$, $y=a$,

Connected by metal strips at $x = \pm b$ {insulated at corners} at Constant $V = V_0$



find $V_{(x,y)} = ?$

Solution

Boundary condition,

$V = V_0$ at $x = +b$

$V = V_0$ at $x = -b$

$V = 0$ at $y = 0$

$V = 0$ at $y = a$

we have $\nabla^2 V = 0$ {charge density is 0}

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ {It can be reduced to two dimension as system is independent of Z}

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

assuming my soln to be $V_{(x,y)} = X_{(x)} \times Y_{(y)}$ {Seperation of variables • Special soln}

satisfying in eqn, we get →

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$ {dividing by V}

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$f(x) + g(y) = 0$$ {Two function of x and y both are independent variable} {only possible solution is both are constant}

$f(x) = C_1$

$g(y) = C_2$

So $C_1 + C_2 = 0$

$C_1 = -C_2$

let $C_1 = k^2$, $C_2 = -k^2$

$$X_{(x)} = Ae^{kx} + Be^{-kx}, \quad Y_{(y)} = (C\,S(ky) + D\,C(ky))$$

{where $S(\theta) = \sin\theta$, $C(\theta) = \cos\theta$}

Now, satisfying our boundary condition,
we get, at $x = \pm b$ $V = V_0$

we get $A = B$

let them be absorbed in C and D

for $y = 0, a$ $V = 0$

$D = 0$. and $k = \frac{n\pi}{a}$ {to make $\sin(ka) = 0$}

rewriting soln

$$V_{(x,y)} = C \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$ {$e^{kx} + e^{-kx} = 2\cosh$ absorb 2 in C and D}

General solution to the problem be like,

$$V_{(x,y)} = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

Using fourier series method {as Arka did in Class}

$$V_0 = V_{(b,y)} = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(n\pi y/a\right)$$

$$\int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right)$$ {multiplying $\sin(n'\pi y/a)$ both side}

which reduces to

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0 & \text{If } n \text{ is even} \\ \frac{4V_0}{n\pi} & \text{If } n \text{ is odd} \end{cases}$$

final solution looks like

$$\boxed{V_{(x,y)} = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a)}$$

→ choose image if plotted looks like a saddle