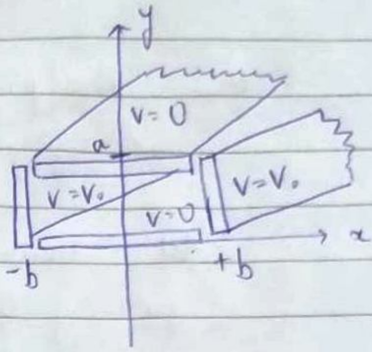
NAME - RACHIT PRATHAM
BATCH - B-3 (T6)
ROLL NO. - 20221212

Problem- Two infinately long grounded metal plates at $y=0$, $y=a$ connected by metal strips at $x=\pm b$ at constant $V=V_0$ (insulated at corners). Find $V(x,y)$.



Solution- Boundary condition,

$V=V_0$ at $x=+b$
$V=V_0$ at $x=-b$
$V=0$ at $y=0$
$V=0$ at $y=a$

we have, $\nabla^2 V = 0$ ($\because$ charge density is 0)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ (It can be reduced to two dimension as system is independent of z)

assuming my solution to be $V(x,y) = X(x) \times Y(y)$ (separation of variable, special solution)

Satisfying in eqn, we get -

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$ (dividing by V)

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow f(x) + g(y) = 0$$

(Two function of x and y both are independent variable only possible solution is both are constant)

$\Rightarrow$ $f(x) = C_1$, $g(y) = C_2$
such that, $C_1 + C_2 = 0$
$C_1 = -C_2$

let $c_1 = k^2$, $c_2 = -k^2$

$X(x) = Ae^{kx} + Be^{-kx}$, $Y(y) = C\sin(ky) + D\cos(ky)$

Now, satisfying our boundary condition,

we get, from $x = \pm b$, $V = V_0$

we get $A = B$

let them be absorbed in C and D.

for $y = 0, a$, $V = 0$

$D = 0$ and $k = \frac{n\pi}{a}$ (to make $\sin(ka) = 0$)

rewriting solution,

$$V(x,y) = C\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

($e^{kx} + e^{-kx} = 2\cosh x$, absorb 2 in C and D)

General solution to the problem be like,

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

using fourier series method (as Arka did in class)

$$V_0 = V(b,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right)dy = \int_0^a \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)$$

(multiplying $\sin\left(\frac{n'\pi y}{a}\right)$ both side)

which reduces to, $C_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0 & \text{, if } n \text{ is even} \\ \frac{4V_0}{n\pi} & \text{, if } n \text{ is odd} \end{cases}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin\left(\frac{n\pi y}{a}\right)}$$

(it's image looks like a saddle)