PH1213 Problem Presentation

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1 The Problem

Consider a particle of mass m and charge q suspended on a string of length L. At a distance d ($d > L$) under the point of suspension there is an infinite grounded conducting plane. Assuming the pendulum is displaced to amplitudes sufficiently small that Hooke's Law is valid and ignoring gravitational and magnetic effects, the objective is to compute the angular frequency of the pendulum's oscillation.

2 Initial Thoughts

The angular accelaration α of the oscillation can be found using the equation $\tau = mL^2\alpha$ where τ is the torque on the charge at a certain point in its motion. This torque is a result of the electrostatic force of attraction between the charge and the charges it induces in the conducting plane. This induced charge distribution and the force on the charge due to it are quite hard to determine from first principles. We shall see, however, that the method of images reduces this difficulty drastically.

3 The Solution

We shall assume that the infinte grounded conducted plane lies in the x-y plane and that the point of suspension of the charge q lies on the z-axis and is the given by the point $(0, 0, d)$. Figure 1 below, shows this pendulum displaced by an angle θ from it's mean position. We need to calculate the torque it experiences about the point of suspension for any θ

What we intend to do first is to solve Poisson's equation in the region $z \geq 0$ with a single point charge q at $(0, 0, d - L\cos\theta)$ subject to the following 2 boundary conditions:

1. $z = 0 \implies V = 0$ as the plane is grounded
2. $V \to 0$ far from the charge

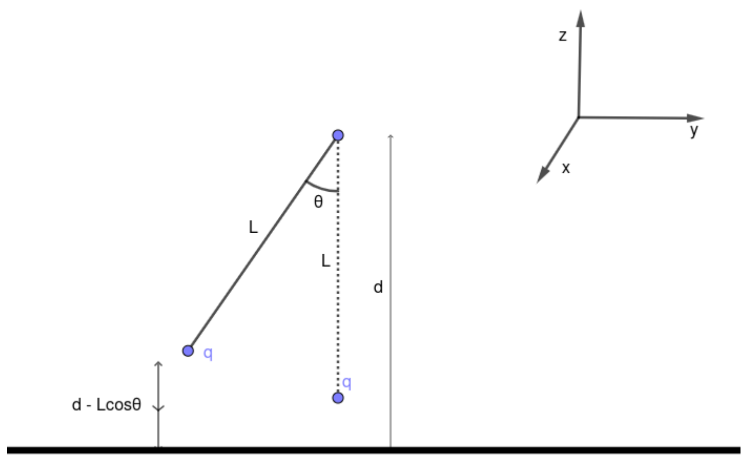

Figure 1: Original Case

Now, it is known that any potential field that satisfies these boundary conditions is the unique solution to Poisson's equation. Thus, we construct a completely different situation, that satisfies these boundary conditions and is easy to derive quantitative results from to find the potential field in the present case.

This new case that we consider is that without the conducting plane, but with another pendulum with the same geometry in the region below the plate, a reflection of the first pendulum through the x-y plane with charge $-q$ instead of $+q$. This new system is pictured in Figure 2 below.

The potential V at any point can now be written as

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-[d-L\cos\theta])^2}} - \frac{1}{\sqrt{x^2+y^2+(z+[d-L\cos\theta])^2}}\right]$$

It can easily be seen that this potential function satisfies the required boundary conditions and is hence the required solution. Since, the electric field and hence the force on q is determined completely by the potential, it is the same in the modified case as it is in the original. Now, this force can be written using Coulomb's Law as:

$$\mathbf{F}(\theta) = \frac{-1}{16\pi\epsilon_0}\frac{q^2}{(d-L\cos\theta)^2}\hat{\mathbf{z}}$$

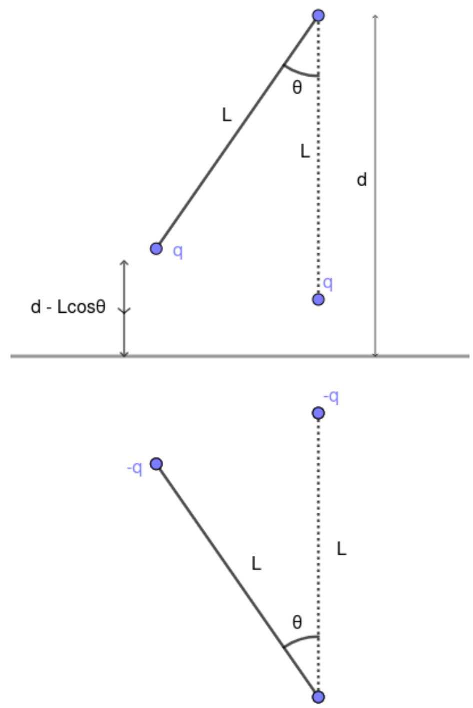

Figure 2: Modified Case

Now that we're equipped with the equation for the force on the charged particle when the pendulum is at some angular displacement θ, we can proceed to solve the main problem. Consider the pendulum displaced by some angle θ.

The magnitude of total force on it (which is vertical) in this situation is given by the result already derived as:

$$F(\theta) = \frac{-1}{16\pi\epsilon_0}\frac{q^2}{(d - L\cos\theta)^2}$$

From this the torque can be calculated as

$$\tau(\theta) = FL\sin\theta = \frac{-1}{16\pi\epsilon_0}\frac{q^2}{z(\theta)^2} = \frac{-q^2}{16\pi\epsilon_0}\frac{L\sin\theta}{(d - L\cos\theta)^2}$$

However, $\tau(\theta) = mL^2\ddot{\theta}$ from which we see that

$$\ddot{\theta} = \frac{-q^2}{16mL\pi\epsilon_0}\frac{\sin\theta}{(d - L\cos\theta)^2}$$

Now, for very small displacements, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. This allows us to write

$$\ddot{\theta} \approx \frac{-q^2}{16mL\pi\epsilon_0}\frac{\theta}{(d-L)^2}$$

This is of the form, $\ddot{\theta} = -\omega^2\theta$. From this, we get

$$\omega^2 = \frac{q^2}{16mL\pi\epsilon_0(d-L)^2}$$

This completes the solution to the problem.

4 Remarks

As the pendulum, moves over the conducting plane, the surface charge distribution on the plane changes. Determining the surface charge density on the plane and tracking these changes through time is far from straightforward. However, the method of images employed here helps circumvent this problem by enabling us to calculate the force on the charge at any displacement of the pendulum without having to worry about the charge distribution on the plate. This illustrates the power of the method images when used in a suitable situation.

A word must be said about eddy currents. The moving charge over the conducting plate would, in a real life situation induce eddy currents which would serve to damp the free oscillations of the pendulum. The solution presented here ignores this effect, which is justifiable for charges of small magnitude moving at slow velocities, as the magnitude of eddy currents they would induce would be reasonably negligible.