PH-1213

INTRODUCTION TO ELECTRODYNAMICS

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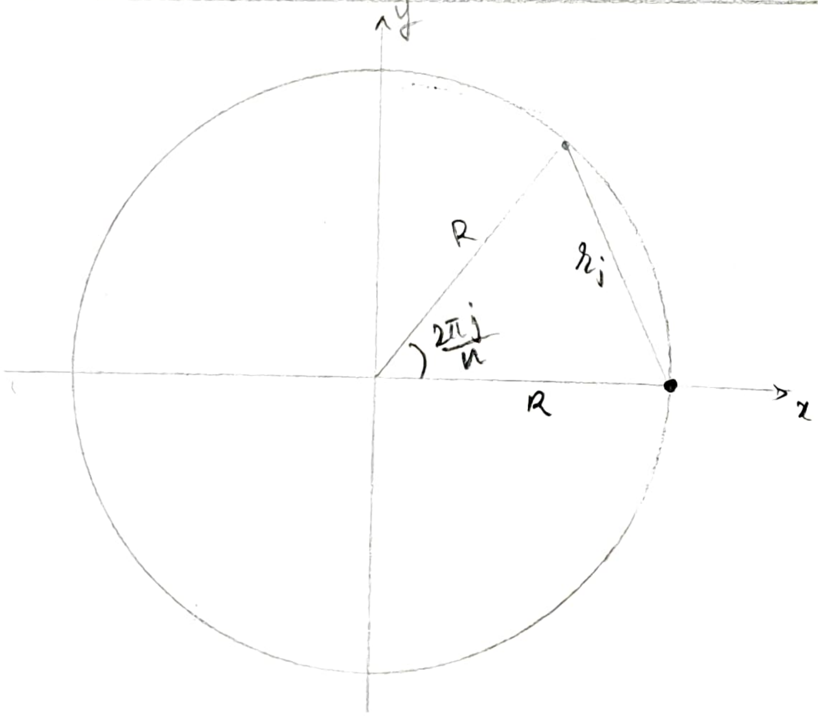
QUESTION:

What is the minimum-energy configuration for a system of N equal point charges on or inside a circle of radius R? Because the charge on a conductor goes to the surface, you might think the N charges would arrange themselves (uniformly) around the circumference. Show to the contrary that for N = 12 it is better 11 on the circumference & one at the centre. How about for N = 11 (is the energy lower if you put all 11 around the circumference, or if you put 10 on the circumference & one at the centre)?

(Intro. to electrodynamics. D.J. Griffith. Chapter 2 Prob. 2.61)

Solution:



Suppose that n-charges are evenly spaced around the circle, with the j^{th} particle at an angle $2\pi j/n$.

We know that the work done / energy required to construct a discrete charge system is

$$W_n = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

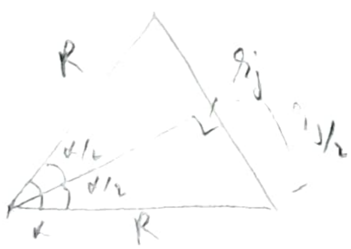
here as all charges are equal

$$W_n = n\frac{1}{2}qV$$

$V \Rightarrow$ potential due to the $(n-1)$ charges on the n^{th} charge

$$V = \frac{1}{4\pi\varepsilon_0} q \sum_{j=1}^{n-1} \frac{1}{r_j} \qquad r_j = 2R\sin\left(\frac{j\pi}{n}\right)$$

~~by sine rule~~



$$\longrightarrow \frac{r_j}{2} = R\sin\left(\frac{2j\pi}{2n}\right)$$

$$r_j = 2R\sin(j\pi/n)$$

Now,

$$W_n = \frac{n}{2}q\left(\frac{1}{4\pi\varepsilon_0}q\sum_{j=1}^{n-1}\frac{1}{r_j}\right)$$

$$\Rightarrow W_n = \frac{n}{2}\frac{q^2}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{1}{2R\sin(j\pi/n)}$$

$$\Rightarrow W_n = \frac{q^2}{4\pi\varepsilon_0 R}\frac{n}{4}\sum_{j=1}^{n-1}\frac{1}{\sin(j\pi/n)}$$

$$\Rightarrow \cancel{W_n = \frac{q^2}{4\pi\varepsilon_0 R}\left(\frac{n}{4}\right)\Omega_n} \qquad W_n = \frac{q^2}{4\pi\varepsilon_0 R}\Omega_n$$

$$\text{where } \Omega_n = \left(\sum_{j=1}^{n-1}\frac{1}{\sin(j\pi/n)}\right)\frac{n}{4}$$

Now lets calculate $\Omega_{10}, \Omega_{11}, \Omega_{12}$, this helps us in understanding how it works. (using calculators)

$$\Omega_{10} = \frac{10}{4} \sum_{j=1}^{9} \frac{1}{\sin(j\pi/10)} = 38.6245$$

$$\Omega_{11} = \frac{11}{4} \sum_{j=1}^{10} \frac{1}{\sin(j\pi/11)} = 48.5757$$

$$\Omega_{12} = \frac{12}{4} \sum_{j=1}^{11} \frac{1}{\sin(j\pi/12)} = 59.8074.$$

If $(n-1)$ charges are on the circle & the n^{th} charge is at the centre, the total energy is.

$$W_n = \left[\Omega_{n-1} + (n-1)\right] \frac{q^2}{4\pi\varepsilon_0 R}$$

~~this is, because of the centre charge apply a equal~~

& this is, because of the presence of the centre charge assembling the other charges takes $\frac{(n-1)q^2}{4\pi\varepsilon_0 R}$

So, for

$n=11$ $\quad \Omega_{10} + 10 = 38.6245 + 10 = \boxed{48.6245} > \Omega_{11}$

& $n=12$ $\quad \Omega_{11} + 11 = 48.5757 + 11 = \boxed{59.5757} < \Omega_{12}$

Thus a lower energy is achieved for 11 charges if they are all on the rim, but in case of 12 charges its better to put one at the centre.

— Thank You!!! —