On The Electrodynamics Of Moving Bodies

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1 The Question

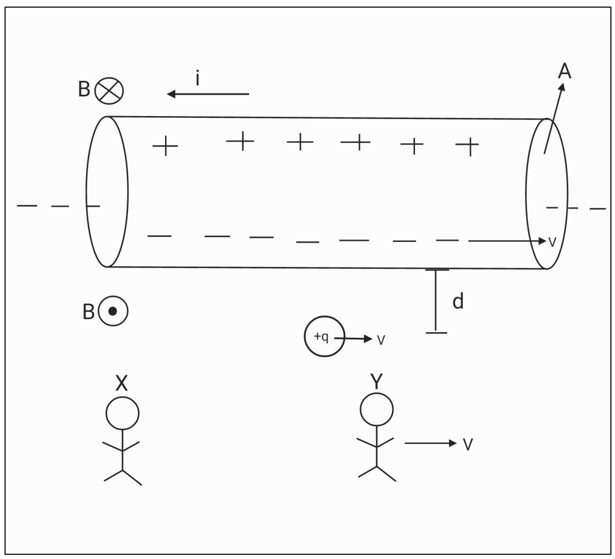

Figure 1: X's Frame

Consider a cylindrical current carrying wire of cross-sectional area A, and a charge q moving with the same velocity as the electrons in the wire, at a distance d from the wire. We also have two observers, X and Y. X is stationary in the ground frame, while Y is moving with the same velocity as the electrons and the charge q.

In X's frame, there's a magnetic field around the wire, and due to this magnetic field, a magnetic force acts on the moving charge q, which causes it to move away from the wire initially.

In Y's frame, the electrons are stationary but the positive charges are moving with a velocity v in the opposite direction which produces the same magnetic

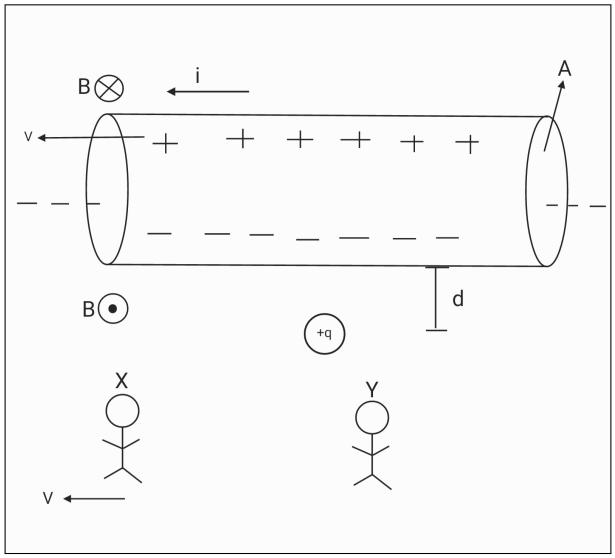

Figure 2: Y's Frame

field as seen in X's frame. But this time in Y's frame, the charge q is stationary, and hence no magnetic force can act on the charge q.

Therefore, observers X and Y have two different initial observations of the charge q. According to X the charge q and the wire are moving away from each other initially, but according to observer Y there is no force which can cause this initial separation.

How do we resolve this paradox?

2 The Solution

Here I propose my solution, where I consider the wire to be charged even in X's frame (due to the effect of length contraction on the moving electrons), i.e., the wire will be charged, but this time it will have a negative charge density. I will mathematically show how the forces in X's frame and Y's frame come out to follow the Force Transformation Law while considering electric forces even in X's frame.

In X's Frame:

Assume $\lambda_{-} = neA$, where λ_{-} represents the linear charge density of the electrons.

Then, using length contraction, the new linear charge density of electrons, λ'_- is given by:

$$\lambda'_- = \gamma\lambda_-$$

Now, the net linear charge density is given by:

$$\lambda = \lambda_+ + \lambda'_-$$

Note that if there was no current in the wire:

$$\lambda = \lambda_+ + \lambda_- = 0 \implies \lambda_+ = -(\lambda_-)$$

Therefore, we can write the net linear charge density as:

$$\lambda = \lambda_+ + \lambda'_- \implies \lambda = -(\lambda_-) + \gamma\lambda_- \implies \lambda = (\gamma - 1)\lambda_-$$

We know that, the magnetic field produced by an infinitely long wire is given by:

$$B = \frac{\mu_0 i}{2\pi d}$$

But, $i = neAv = \lambda'_- v$, where n represents the electron number density.

$$\overrightarrow{B} = \frac{\mu_0 \lambda'_- v}{2\pi d}\ \hat{k} \implies \overrightarrow{B} = \frac{\mu_0 \gamma\lambda_- v}{2\pi d}\ \hat{k}$$

We also know that,

$$\overrightarrow{F_B} = q(\overrightarrow{v} \times \overrightarrow{B})$$

But in this case, as we have v perpendicular to B, we can simply write

$$\overrightarrow{F_B} = -qvB\ \hat{j}$$

Therefore,

$$\boxed{\overrightarrow{F_B} = -\frac{\mu_0 \gamma\lambda_- v^2 q}{2\pi d}\ \hat{j}}$$

Note that this F_B is moving the charge away from the wire.

Now, we will be introducing the electric field into X's frame. As explained earlier, this electric field arises due to the length contraction of the electrons, which in turn results in a net negative charge density on the wire.

We know that, the electric field produced by an infinitely long charged wire is given by:

$$E = \frac{\lambda}{2\pi\epsilon_0 d}$$

Substituting the value of λ, we get:

$$\overrightarrow{E} = \frac{(\gamma - 1)\lambda_-}{2\pi\epsilon_0 d} \ \hat{j}$$

Hence, the Electric Force on the point charge will be:

$$\overrightarrow{F_E} = q\overrightarrow{E}$$

Which gives us:

$$\boxed{\overrightarrow{F_E} = \frac{q(\gamma-1)(\lambda_-)}{2\pi\epsilon_0 d} \ \hat{j}}$$

Note that this force is pulling the charge towards the wire, as the net linear charge density is negative in polarity whereas the moving charge is positive.

Therefore, we can write the total force on the charge in X's frame as :

$$\overrightarrow{F_X} = \overrightarrow{F_B} + \overrightarrow{F_E} \implies \boxed{\overrightarrow{F_X} = \left(-\frac{\mu_0\gamma\lambda_- v^2 q}{2\pi d} + \frac{q(\gamma-1)(\lambda_-)}{2\pi\epsilon_0 d}\right) \ \hat{j}}$$

In Y's Frame:

This frame will only contain electric interactions, as the positive charge is not in motion. The positive charges in the wire are in motion and therefore they will be length contracted.

The new linear charge density of the positive charges, λ'_+ is given by:

$$\lambda'_+ = \gamma\lambda_+$$

Note that if there was no current in the wire:

$$\lambda = \lambda_+ + \lambda_- = 0 \implies \lambda_- = -(\lambda_+) \tag{1}$$

Therefore, we can write the net linear charge density as:

$$\lambda = \lambda'_+ + \lambda_- \implies \lambda = \gamma\lambda_+ - \lambda_+ \implies \lambda = (\gamma - 1)\lambda_+$$

As mentioned previously, the electric field produced by an infinitely long charged wire is given by:

$$E = \frac{\lambda}{2\pi\epsilon_0 d}$$

Substituting the value of λ, we get:

$$\overrightarrow{E'} = \frac{(\gamma - 1)\lambda_+}{2\pi\epsilon_0 d} \ \hat{j}$$

Hence, the Electric Force on the point charge will be:

$$\overrightarrow{F'_E} = q\overrightarrow{E'}$$

Which gives us:

$$\boxed{\overrightarrow{F'_E} = \frac{q(\gamma-1)(\lambda_+)}{2\pi\epsilon_0 d} \ \hat{j}}$$

Note that this force is pushing the charge away from the wire, as the net linear charge density as well as the point charge is positive.

As there are no other forces in this frame, the total force in Y's frame is:

$$\overrightarrow{F_Y} = \overrightarrow{F'_E} \implies \boxed{\overrightarrow{F_Y} = \frac{q(\gamma-1)(\lambda_+)}{2\pi\epsilon_0 d} \ \hat{j}}$$

If our analysis is correct, $\overrightarrow{F_X}$ and $\overrightarrow{F_Y}$, must satisfy the Force Transformation Law, i.e.:

$$\overrightarrow{F_Y} = \frac{\overrightarrow{F_X}}{\sqrt{1 - v^2/c^2}} \implies \overrightarrow{F_Y} = \gamma \ \overrightarrow{F_X}$$

Therefore, we will show that: $F_X/F_Y = \sqrt{1 - v^2/c^2} = 1/\gamma$

$$\frac{F_X}{F_Y} = \frac{-\dfrac{\mu_0 \gamma \lambda_- v^2 q}{2\pi d} + \dfrac{q(\gamma - 1)(\lambda_-)}{2\pi\epsilon_0 d}}{\dfrac{q(\gamma - 1)(\lambda_+)}{2\pi\epsilon_0 d}}$$

$$\implies \frac{\gamma v^2}{c^2(\gamma - 1)} - 1 \implies \frac{\gamma v^2 - \gamma c^2 + c^2}{c^2(\gamma - 1)}$$

On dividing numerator and denominator by c^2:

$$\implies \frac{-\gamma(1 - v^2/c^2) + 1}{\gamma - 1} \implies \frac{1 - 1/\gamma}{\gamma - 1} \implies \frac{(1 - 1/\gamma)}{\gamma(1 - 1/\gamma)} \implies \boxed{1/\gamma}$$

Therefore, we have shown that the forces F_X and F_Y satisfy the Force Transformation Law.

3 Closing Remarks

Thus, this question portrays how Special Relativity can be used to show that the Electric and Magnetic Fields are just two sides of the same coin.

I was quite enthralled by this question, because it is not inherently obvious that Relativity should be used in resolving the paradox, but when it is used, it fits in perfectly. It really makes you think of how our world would work if the Speed of Light was more comparable to our everyday speeds.

I thoroughly enjoyed the process of finding this question and coming up with a solution for it. Electrodynamics and Relativity both are beautiful subjects in their own right, but their union is truly phenomenal.

Thank you!