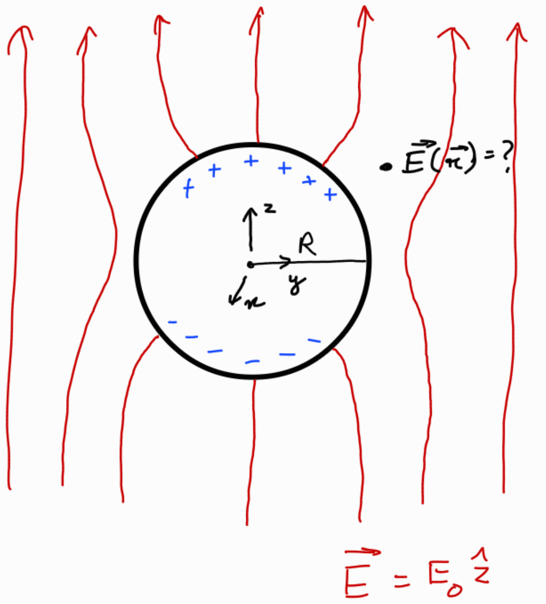
Potential around a conductor in an external electric field



Question :

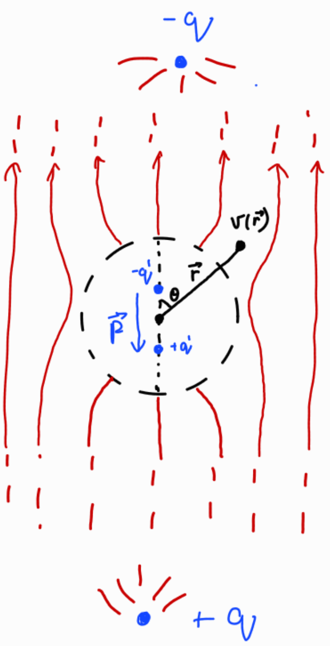
An uncharged metal sphere of radius R is kept in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$. Find the potential in the region outside the sphere.

Boundary conditions of system :

① $V = \text{const.}$ on sphere & along x-y plane (equipotential surface)
Let's take that constant to be 0.

② $V_{ext} \to -E_0 z + C$ at ∞ $\quad$ ($\because E \to E_0$ at ∞)
But $C = 0$ & $z = r\cos\theta$ $\quad \therefore V_{ext} = -E_0 r\cos\theta$ at ∞ $(r >> R)$

According to the uniqueness theorem, we can replace this system with another system of same boundary conditions.



Consider 2 charges +q and -q situated at z = -a & z = +a respectively, with $a \to \infty$

(The system retains the same boundary conditions)

Now we can calculate $\vec{E}_0$ in terms of q, R & a

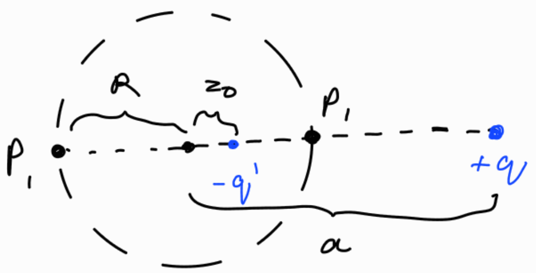
$$\therefore \vec{E}_0 \approx \left(\frac{Kq}{a^2} + \frac{Kq}{a^2}\right)\hat{z} = \frac{2Kq}{a^2}\hat{z} \quad \text{(around sphere)}$$

$$\boxed{E_0 = \frac{2Kq}{a^2}}$$

Now, using the method of images, imagine 2 charges $-q'$ & $+q'$ at $z = z_0$ & $z = -z_0$ respectively, such that the boundary conditions remain satisfied.
($-q'$ is the image of $+q$, & $+q'$ image of $-q$)

Doing so will give us a dipole at the centre of the sphere, pointing towards $-z$ axis. Now the potential outside can be written in terms of this dipole :

$$V_D(\vec{r}) = \frac{K p \cos(\pi - \theta)}{r^2} = \boxed{-\frac{K p \cos\theta}{r^2}}$$



p can be calculated by taking equations of potential on the equipotential surface.

At P_1 :

$$\frac{kq}{a+R} - \frac{kq'}{R+z_0} = 0$$

$$\Rightarrow qR + qz_0 = q'a + q'R \quad \text{— (1)}$$

At P_2 : $\frac{kq}{a-R} - \frac{Kq'}{R-z_0} = 0 \Rightarrow qR - qz_0 = q'a - q'R$ — (2)

$$(1) + (2) \Rightarrow 2qR = 2q'a \Rightarrow q' = \frac{qR}{a}$$

Substitute in (1) $\Rightarrow qR + qz_0 = \frac{qR}{a} \times a + \frac{qR}{a} \times R \Rightarrow z_0 = \frac{R^2}{a}$

$$\therefore \vec{p} = -(q')(2z_0)\hat{z} = -\frac{qR}{a} \times \frac{2R^2}{a}\hat{z} = \frac{-2qR^3}{a^2}\hat{z}, \quad p = \frac{-2qR^3}{a^2}$$

Hence, $V_D(\vec{r}) = -K\left(\frac{-2qR^3}{a^2}\right)\cos\theta \times \frac{1}{r^2} = \frac{2Kq}{a^2} \times \frac{R^3}{r^2}\cos\theta$

$$\boxed{V_D(\vec{r}) = E_0 \frac{R^3}{r^2}\cos\theta}$$

$$\therefore \quad V_{net}(\vec{r}) = V_{ext}(\vec{r}) + V_0(\vec{r})$$

$$\boxed{V_{net}(\vec{r}) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

Physical interpretation of result

The potential due to induced charges is quite similar to a dipole :

i) It falls off as $\frac{1}{r^2}$

ii) Angular dependence is given by $\cos\theta$

However, its dependence on geometry (radius of sphere) increases as R^3, as opposed to 'd' (linear) in dipoles.

The $\frac{1}{r^2}$ term only becomes significant in the proximity of the conductor, and the net result does not show radial symmetry.

Physical significance

1. Can be used to understand behaviour of capacitors in an external electric field
2. Can be used to optimally design lightning rods
3. Designing of electrodes in electrolysis reactions

And many more ...