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Batch 3

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Q. Two infinitely long wires running parallel to x-axis carry uniform charge densities $+\lambda$ and $-\lambda$.

a) Find potential at any point (x, y, z) using origin as reference

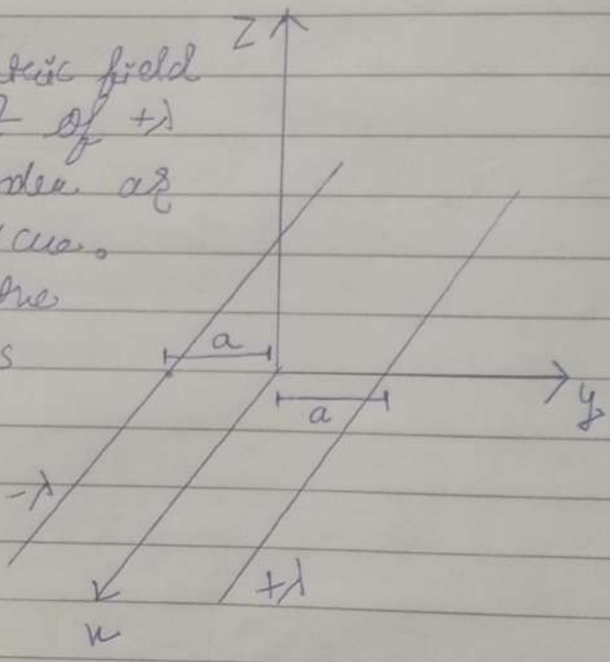
b) Equipotential surfaces for the system are circular cylinders. Locate the axis, radius and centre.

Solⁿ. Strategy :- find Electric field

a) 1. Trying to ~~potential~~ of $+\lambda$ wire taking cylinder as the gaussian surface.

2. Using this $\vec{E}$ in the formula $V(s) = -\int_a^s E\,ds$

to calculate potential. Using the exact similar method to find the potential for $-\lambda$ wire.



3. Adding the potentials of the two wires to get the potential at any point (x, y, z) from the origin

b) 1. Identifying the constant terms from the formula of a). ~~Use~~ Perform some arithmetic manipulations to get the geometric equation of the shape.

2. Identifying the shape to be circular cylinders further calculating axis, radius and centre of the same.

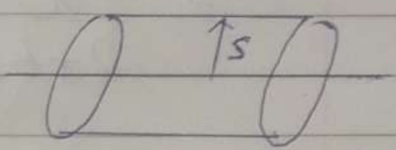
Formular :- Gauss' law :- $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enclosed}}{\epsilon_0}$

$$V(s) = -\int_{\infty}^{s} E\, ds$$

Solution :-

a) 1.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$



$$\vec{E} \cdot 2\pi s = \frac{1}{\epsilon_0} Q_{enc}$$

(The area vector and the emerging electric field are $\perp$ (dot product zero) thus not contributing)

$$Q_{enc} = \lambda l$$

$$\therefore \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s}$$

2. Now, let $s = a$

$$V(s) = -\int_{a}^{s} E\, ds = -\int_{a}^{s} \left(\frac{1}{4\pi\epsilon_0} \frac{2\lambda}{s}\right) ds.$$

$$= -\frac{1}{4\pi\epsilon_0} 2\lambda \left(\ln s - \ln a\right)$$

$$= -\frac{1}{4\pi\epsilon_0} 2\lambda \left(\ln \frac{s}{a}\right)$$

$\therefore$ Potential for $+\lambda = -\frac{1}{2\pi\epsilon_0}\left(\ln \frac{s_+}{a}\right)$ $\quad s_+ \rightarrow$ distance for $+\lambda$ wire

Similarly,
Potential for $-\lambda = +\frac{1}{2\pi\epsilon_0}\left(\ln \frac{s_-}{a}\right)$ $\quad s_- \rightarrow$ distance from $-\lambda$ wire

3.) $\therefore$ Total potential $= \frac{-\lambda}{2\pi\epsilon_0}(\ln s_+ - \ln a) + \frac{\lambda}{2\pi\epsilon_0}(\ln s_- - \ln a)$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\ln s_- - \cancel{\ln a} - \ln s_+ + \cancel{\ln a}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right)$$

From the figure, the expression for s_+ and s_- is calculated as

$$s_+ = \sqrt{(y-a)^2+(z-0)^2+(x-x)^2} = \sqrt{(y-a)^2+z^2}$$

$$s_- = \sqrt{(y-a)^2+(z-0)^2+(x-x)^2}\ \sqrt{(y+a)^2+z^2}$$

$$\therefore V_{(x,y,z)} = \frac{\lambda}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right]$$

$$= \boxed{V_{(x,y,z)} = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]}$$

b) For equipotential surfaces

V is constant

$$\therefore \frac{V\times 4\pi\epsilon_0}{\lambda} = \ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]$$

$$\therefore e^{\frac{4V\pi\epsilon_0}{\lambda}} = \left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right] = K \text{ (constant)}$$

Further simplifying

$$y^2+2ay+a^2+z^2 = k(y^2-2ay+a^2+z^2)$$

$$= y^2(k-1)+z^2(k-1)+a^2(k-1)-2ay(k+1)=0$$

$$\Rightarrow y^2+z^2+\cancel{a^2}-2ay\left(\frac{k+1}{k-1}\right) = -a^2$$

Add $\left[a\frac{(k+1)}{(k-1)}\right]^2$ on both sides

and further solving we get,

$$\left[y-a\frac{(k+1)}{(k-1)}\right]^2+z^2 = a^2\left[\frac{4k}{(k-1)}\right]$$

Comparing with Equation of circle in y-z plane

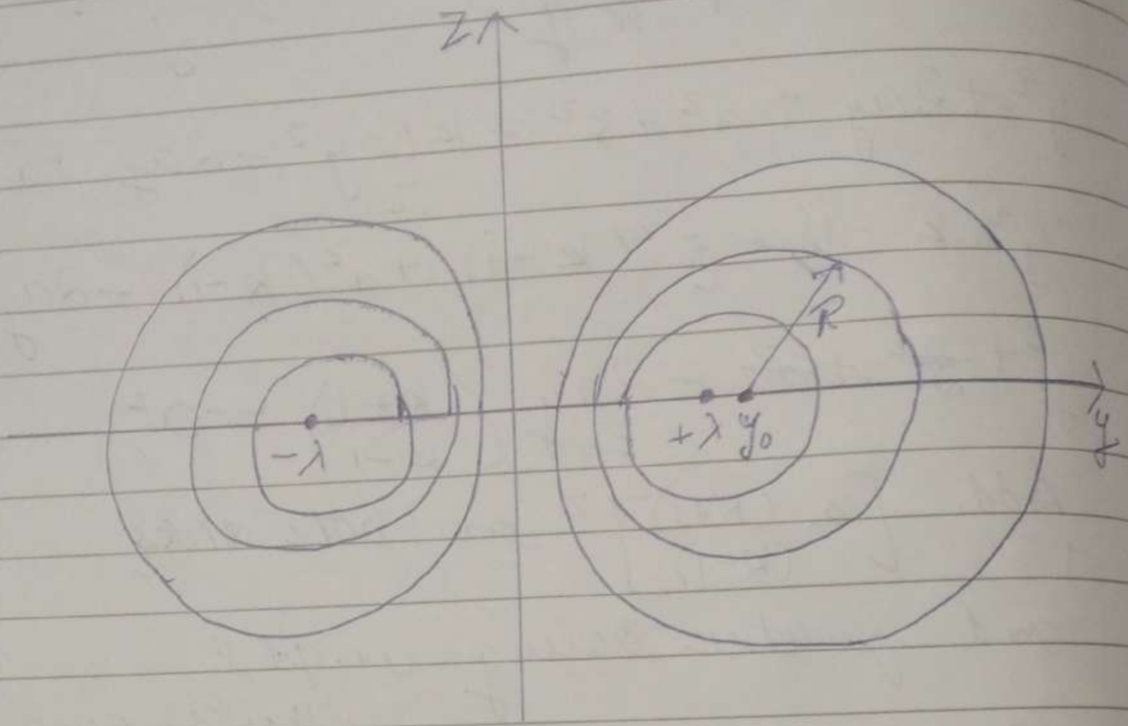
$$(y-y_0)^2+(z-z_0)^2 = R^2$$

$\therefore$ The Equipotential of the system form circles (not concentric). with

$$\boxed{y_0\,(\text{centre}) = \left(\frac{a(k+1)}{(k-1)}, 0\right)}$$ and

$$\text{Radius} = \boxed{R = \frac{2a\sqrt{k}}{k-1}}$$

Thus representing a circular cylinder with an axis $\parallel^{el}$ to the x-axis.



we can further convert the centre (y_0, z) and radius (R) into hyperbolic functions.

$$y_0 = a\,\frac{e^{4\pi\epsilon_0 V_0/\lambda}+1}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\,\frac{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}$$

$$= a \coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)$$

$$R = 2a\,\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\,\frac{2}{\left(e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}\right)}$$

$$= \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}$$

$$= a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)$$