Cycle detection problem

1 The problem

Let function f maps finite set S to itself:

$$f : S \to S \tag{1}$$

Choose initial value $a \in S$ and build a sequence by applying f iteratively:

$$x_0 = a \tag{2}$$

$$x_{i+1} = f(x_i), \quad i = 0, 1, \ldots \tag{3}$$

Since S is finite, the sequence contain duplicates.

2 The analysis

Let ν be *the largest* index such that values

$$x_0, x_1, \ldots, x_{\nu-1} \tag{4}$$

are all different. This means the value x_ν already appeared in the sequence before, at some index $\mu < \nu$:

$$x_\nu = x_\mu \tag{5}$$

Let

$$\lambda = \nu - \mu \tag{6}$$

By applying function f to both sides of equation

$$x_{\mu+\lambda} = x_\mu \tag{7}$$

we have

$$x_{\mu+\lambda+1} = x_{\mu+1} \tag{8}$$

Apply function f again:

$$x_{\mu+\lambda+2} = x_{\mu+2} \tag{9}$$

By induction we conclude

$$x_{i+\lambda} = x_i \tag{10}$$

for any index $i \geqslant \mu$.

So the sequence

$$x_\mu, x_{\mu+1}, \dots \tag{11}$$

is periodic. The period is λ. The values

$$x_\mu, x_{\mu+1}, \dots, x_{\mu+\lambda-1} \tag{12}$$

repeats from position $\mu + \lambda$, then from $\mu + 2\lambda$, etc.

The cycle detection problem is to find cycle starting position μ and its length λ.