

Cycle detection problem

1 The problem

Let function f maps finite set S to itself:

$$f : S \rightarrow S \quad (1)$$

Choose initial value $a \in S$ and build a sequence by applying f iteratively:

$$x_0 = a \quad (2)$$

$$x_{i+1} = f(x_i), \quad i = 0, 1, \dots \quad (3)$$

Since S is finite the sequence gets back to the older value:

$$x_\nu = x_\mu \quad (\nu > \mu) \quad (4)$$

and then cyclically repeats values $x_\mu, \dots, x_{\nu-1}$.

Mathematical details are presented below. The problem in question is to find loop parameters: cycle start index μ and its length λ .

2 Mathematical analysis

Let ν be *the largest* index such that values

$$x_0, x_1, \dots, x_{\nu-1} \quad (5)$$

are all different. This means the value x_ν already appeared in the sequence before, at some index $\mu < \nu$:

$$x_\nu = x_\mu \quad (6)$$

Let

$$\lambda = \nu - \mu \quad (7)$$

By applying function f to both sides of equation

$$x_{\mu+\lambda} = x_{\mu} \quad (8)$$

we have

$$x_{\mu+\lambda+1} = x_{\mu+1} \quad (9)$$

Apply function f again:

$$x_{\mu+\lambda+2} = x_{\mu+2} \quad (10)$$

By induction we conclude

$$x_{i+\lambda} = x_i \quad (11)$$

for any index $i \geq \mu$.

For future reference let us prove the following statement:

Theorem 1. *Given two indices $i < j$ we have: $x_i = x_j$ if and only if*

$$i \geq \mu \quad (12)$$

and

$$\lambda \mid j - i \quad (13)$$