Cycle detection problem

1 The problem

Let function f maps finite set S to itself:

$$f : S \to S \tag{1}$$

Choose initial value $a \in S$ and build a sequence by applying f iteratively:

$$x_0 = a \tag{2}$$

$$x_{i+1} = f(x_i), \quad i = 0, 1, \ldots \tag{3}$$

Since S is finite the sequence gets back to the older value:

$$x_\nu = x_\mu \quad (\nu > \mu) \tag{4}$$

and then cyclically repeats values $x_\mu, \ldots, x_{\nu-1}$.

Mathematical details are presented below. The problem in question is to find loop parameters: cycle start index μ and its length λ.

2 Mathematical analysis

Let ν be *the largest* index such that values

$$x_0, x_1, \ldots, x_{\nu-1} \tag{5}$$

are all different. This means the value x_ν already appeared in the sequence before, at some index $\mu < \nu$:

$$x_\nu = x_\mu \tag{6}$$

Let

$$\lambda = \nu - \mu \tag{7}$$

By applying function f to both sides of equation

$$x_{\mu+\lambda} = x_\mu \tag{8}$$

we have

$$x_{\mu+\lambda+1} = x_{\mu+1} \tag{9}$$

Apply function f again:

$$x_{\mu+\lambda+2} = x_{\mu+2} \tag{10}$$

By induction we conclude

$$x_{i+\lambda} = x_i \tag{11}$$

for any index $i \geqslant \mu$.

For future reference let us prove the following statement:

Theorem 1. *Given two indices* $i < j$ *we have:* $x_i = x_j$ *if and only if*

$$i \geqslant \mu \tag{12}$$

and

$$\lambda \mid j - i \tag{13}$$

3 Floyd's hare and tortoise algorithm

4 Brent's algorithm