Problem Set 7

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26 January 2015

1. (a) (i) By definition, $\underline{I}_a^b(f)$ is the supremum of the set $\{L_f(P)\}$ over all partitions P of $[a, b]$. Thus $\underline{I}_a^b(f)$ is an upper bound for the set of lower sums, and so for any partition P, $L_f(P) \leq \underline{I}_a^b(f)$.

   (ii) Note that $\underline{I}_a^b(f)$ is the least upper bound of $\{L_f(P)\}$. Also, for any $\epsilon > 0$, $\underline{I}_a^b(f) - \epsilon < \underline{I}_a^b(f)$. That is, the value $\underline{I}_a^b(f) - \epsilon$ is less than the least upper bound of $\{L_f(P)\}$, thus it is not an upper bound for this set. Negating the statement for upper bound, there exists a partition P such that $\underline{I}_a^b(f) - \epsilon < L_f(P)$.

   (b) Since $L_f(P) \leq J$ for all partitions P of $[a, b]$ from property (i), J is an upper bound of the set $\{L_f(P)\}$. From property (ii), any $K < J$ is not an upper bound for $\{L_f(P)\}$. Thus J is the least upper bound for the set $\{L_f(P)\}$, and so by definition $J = \underline{I}_a^b(f)$.

2. (a) Suppose $P = \{a, x_1, x_2, \ldots, x_{n-1}, c, x_{n+1}, x_{n+2}, \ldots, x_{n+k}, b\}$ for some $n, k \in \mathbb{N}$. Then

$$Q = \{a, x_1, x_2, \ldots, x_{n-1}, c\} \text{ and } R = \{c, x_{n+1}, x_{n+2}, \ldots, x_{n+k}, b\}$$

   So the left side of the equation we need to prove is

$$\begin{aligned} L_f(P) = &\left[\inf_{x \in [a,\, x_1]} f(x)\right](x_1 - a) + \left[\inf_{x \in [x_1,\, x_2]} f(x)\right](x_2 - x_1) + \ldots \\ &+ \left[\inf_{x \in [x_{n-1},\, c]} f(x)\right](c - x_{n-1}) + \left[\inf_{x \in [c,\, x_{n+1}]} f(x)\right](x_{n+1} - c) \\ &+ \left[\inf_{x \in [x_{n+1},\, x_{n+2}]} f(x)\right](x_{n+2} - x_{n+1}) + \ldots + \left[\inf_{x \in [x_{n+k},\, b]} f(x)\right](b - x_{n+k}) \end{aligned}$$

But note that the right side of the equation is the sum of the following terms:

i. $L_f(Q) = \left[\inf_{x\in[a,\, x_1]} f(x)\right](x_1 - a) + \left[\inf_{x\in[x_1,\, x_2]} f(x)\right](x_2 - x_1)$

$$+ \ldots + \left[\inf_{x\in[x_{n-1},\, c]} f(x)\right](c - x_{n-1})$$

ii. $L_f(R) = \left[\inf_{x\in[c,\, x_{n+1}]} f(x)\right](x_{n+1} - c) + \left[\inf_{x\in[x_{n+1},\, x_{n+2}]} f(x)\right](x_{n+2} - x_{n+1})$

$$+ \ldots + \left[\inf_{x\in[x_{n+k},\, b]} f(x)\right](b - x_{n+k})$$

which are clearly the terms of $L_f(P)$. In words, this tells us that Q is the partition P restricted to the interval $[a, c]$ and R is the partition P restricted to the interval $[c, b]$. Thus $L_f(P) = L_f(Q) + L_f(R)$.

(b) The first statement is true. Note that from part (a), $L_f(P') = L_f(Q) + L_f(R)$. Also, $P \subseteq P'$, that is, P' is a refinement of P with the point $c \in [a, b]$ added. Suppose $c \in P$; then $P = P'$, so $L_f(P) = L_f(P')$. Alternatively, suppose $c \notin P$. Then some subinterval produced by P, say $[x_n, x_{n+1}]$, is split by P' into the subintervals $[x_n, c]$ and $[c, x_{n+1}]$. But note that

$$\inf_{x\in[x_n,x_{n+1}]} f(x) \leq \inf_{x\in[x_n,c]} f(x) \quad \text{and} \quad \inf_{x\in[x_n,x_{n+1}]} f(x) \leq \inf_{x\in[c,x_{n+1}]} f(x)$$

so in the case that $c \notin P$, $L_f(P) \leq L_f(P')$. Thus in general, $L_f(P) \leq L_f(P')$. Then $L_f(P) \leq L_f(Q) + L_f(R)$, as required.

As a counterexample to the second statement, consider the function $f(x) = x$ on the interval $[0, 4]$. Let $P = \{0, 1, 3, 4\}$, $c = 2$, and thus $Q = \{0, 1, 2\}$ and $R = \{2, 3, 4\}$. Note that $L_f(P) = 0(1) + 1(2) + 3(1) = 5$ But $L_f(Q) = 0(1) + 1(1)$ and $L_f(R) = 2(1) + 3(1)$, and so $L_f(Q) + L_f(R) = 6 > L_f(P)$.

(c) Note that $T = \underline{I}_a^c(f) + \underline{I}_c^b(f)$, and by the definition of supremum, $\underline{I}_a^c(f) \geq L_f(Q)$ for any partition Q of $[a, c]$, and likewise $\underline{I}_c^b(f) \geq L_f(R)$ for any partition R of $[c, b]$. But we have shown in part (b) that given any partition P on $[a, b]$, there exist partitions Q and R on $[a, c]$ and $[c, d]$ respectively such that $L_f(P) \leq L_f(Q) + L_f(R)$. So we have

$$L_f(P) \leq L_f(Q) + L_f(R) \leq \underline{I}_a^c(f) + \underline{I}_c^b(f)$$

And taking the extremes of this inequality, $L_f(P) \leq \underline{I}_a^c(f) + \underline{I}_c^b(f) = T$. Thus $L_f(P) \leq T$ for any partition P of $[a, b]$.

(d) Note that $\underline{I}_a^c(f)$ and $\underline{I}_c^b(f)$ are the suprema of $L_f(Q)$ and $L_f(R)$ over all partitions Q and R of $[a, c]$ and $[c, b]$, respectively. From part 1 (a), for all $\epsilon/2 > 0$, there exist particular partitions Q' and R' of $[a, c]$ and $[c, b]$ such that

$$\underline{I}_a^c(f) - \epsilon/2 < L_f(Q') \tag{1}$$

$$\underline{I}_c^b(f) - \epsilon/2 < L_f(R') \tag{2}$$

Then, adding equations (1) and (2) together, we have that for any $\epsilon > 0$ given, there exist partitions Q' and R' of $[a, c]$ and $[c, b]$ respectively such that

$$\underline{I}_a^c(f) + \underline{I}_c^b(f) - \epsilon < L_f(Q') + L_f(R')$$

Finally, note that from part 2 (a), there exists a partition $P = Q' \cup R'$ over $[a, b]$ where $L_f(P) = L_f(Q') + L_f(R')$. Thus for any $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $T - \epsilon < L_f(P)$.

(e) In part 1 (a), we showed that given a number J and a bounded function f on $[a, b]$, if $L_f(P) \leq J$ for all partitions P of $[a, b]$ and if for every $\epsilon > 0$ there exists a partition P such that $J - \epsilon < L_f(P) \leq J$, then J is the lower integral of f on $[a, b]$. Note that in part 2 (c) we showed that if $T = \underline{I}_a^c(f) + \underline{I}_c^b(f)$ then $L_f(P) \leq T$ for any partition P of $[a, b]$. We also showed in part 2 (d) that given any $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $T - \epsilon < L_f(P)$. These are precisely the two conditions required in part 1 (a), so we have that $T = \underline{I}_a^b(f)$. Thus we have shown that

$$\underline{I}_a^b(f) = \underline{I}_a^c(f) + \underline{I}_c^b(f)$$

3. Assume f is integrable on $[a, c]$ and on $[c, b]$. This implies that $\underline{I}_a^c(f) = \overline{I}_a^c(f)$, and likewise, $\underline{I}_c^b(f) = \overline{I}_c^b(f)$. Adding these equations together, we have

$$\underline{I}_a^c(f) + \underline{I}_c^b(f) = \overline{I}_a^c(f) + \overline{I}_c^b(f) \tag{3}$$

Note that from part 2, we showed that given any interval $[a, b]$ on which a function f is bounded, and for any point $c \in (a, b)$, we have $\underline{I}_a^b(f) = \underline{I}_a^c(f) + \underline{I}_c^b(f)$. Similar reasoning shows that $\overline{I}_a^b(f) = \overline{I}_a^c(f) + \overline{I}_c^b(f)$ for any $c \in (a, b)$. Thus, extending the equality of equation (3), we have that

$$\underline{I}_a^b(f) = \underline{I}_a^c(f) + \underline{I}_c^b(f) = \overline{I}_a^c(f) + \overline{I}_c^b(f) = \overline{I}_a^b(f)$$

And thus $\underline{I}_a^b(f) = \overline{I}_a^b(f)$, so f is integrable on $[a, b]$.

4. (i) Let us consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

on the interval $[0,2]$. For the lower sum, note that on any subinterval $[a,b] \subseteq [0,2]$, where $a \neq b$, $\inf_{x\in[a,b]} f(x) = 1$. Thus for any partition P of $[0,2]$,

$$L_f(P) = 1(2-0) = 2$$

So $\underline{I}_0^2(f) = \sup_P L_f(P) = 2$.

For the upper sum, note that for any partitions P and Q of $[0,2]$, $L_f(P) \leq U_f(Q)$ (proven in the notes by Dr. Gracia-Saz). But we have shown that for any partition P of $[0,2]$, $L_f(P) = 2$. Thus for any partition Q of $[0,2]$, $U_f(Q) \geq 2$. In words, $J = 2$ is a lower bound for the upper sum of f over all partitions of $[0,2]$. Now we seek to show that J is the greatest lower bound.

Consider the partition $R = \{0, 1-\delta, 1+\delta, 2\}$ for some $0 < \delta < 1$. Notice that the upper sum on this partition of $[0,2]$ is

$$\begin{aligned} U_f(R) &= 1(1-\delta) + 2(1+\delta-(1-\delta)) + 1(2-(1+\delta)) \\ &= 1-\delta+4\delta+1-\delta = 2+2\delta \end{aligned}$$

Let $\epsilon > 0$ be given. Using the notation above, set $\delta = \epsilon/4$. Then

$$U_f(R) = 2 + 2(\frac{\epsilon}{4}) = 2 + \frac{\epsilon}{2} < 2 + \epsilon$$

Thus for any $\epsilon > 0$, there exists a partition R of $[0,2]$ such that $2 \leq U_f(R) < 2+\epsilon$. This implies that $J = 2$ is the greatest lower bound for $U_f(R)$, so $\overline{I}_0^2(f) = 2$.

Thus we have that $\underline{I}_0^2(f) = \overline{I}_0^2(f) = 2$, so f is integrable on $[0,2]$.

(ii) Now we consider the function

$$g(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 2 & \text{if } x \in \mathbb{Q} \end{cases}$$

on the interval $[0,2]$. Note that on any subinterval $[a,b] \subseteq [0,2]$, where $a \neq b$, there exist both rational and irrational numbers in $[a,b]$. This implies that

$$\sup_{x\in[a,b]} g(x) = 2 \ \text{ and } \inf_{x\in[a,b]} g(x) = 1$$

Since the above is true for any subinterval $[a,b] \subseteq [0,2]$, we have that for any partition P of $[0,2]$,

$$U_g(P) = 2(2) = 4 \ \text{ and } L_g(P) = 1(2) = 2$$

And so

$$\overline{I}_0^2(g) = \inf_P U_g(P) = 4 \ \text{ and } \underline{I}_0^2(g) = \sup_P L_g(P) = 2$$

Thus $\overline{I}_0^2(g) \neq \underline{I}_0^2(g)$, so g is not integrable on $[0,2]$.

5. (a) A formula for x_i in terms of i and n is

$$x_i = x_0 + i\left(\frac{x_n - x_0}{n}\right)$$

given any $0 \le i \le n$.

(b) The length of each subinterval in P_n is

$$\begin{aligned} d = x_{i+1} - x_i &= x_0 + (i+1)\left(\frac{x_n - x_0}{n}\right) - \left[x_0 + i\left(\frac{x_n - x_0}{n}\right)\right] \\ &= (i+1)\left(\frac{x_n - x_0}{n}\right) - i\left(\frac{x_n - x_0}{n}\right) \\ d &= \frac{x_n - x_0}{n} \end{aligned}$$

(c) Assuming that $x_0 = 1$ and $x_n = 3$, we know that f is increasing on $[x_0, x_n]$, and so the maximum of f on any subinterval $[x_{i-1}, x_i]$ is simply the value of f at x_i, or at the right endpoint of the subinterval. So

$$\max_{x \in [x_{i-1}, x_i]} f(x) = f(x_i) = x_i^2 = \left[x_0 + i\left(\frac{x_n - x_0}{n}\right)\right]^2$$

(d) We have

$$U_f(P_n) = \sum_{i=0}^{n-1}\left[x_0 + i\left(\frac{x_n - x_0}{n}\right)\right]^2\left[\frac{x_n - x_0}{n}\right]$$

for any $n \in \mathbb{N}$, where $x_0 = 1$ and $x_n = 3$.

(e) Using the formulas given and some basic algebra, we have

$$\begin{aligned} U_f(P_n) &= \sum_{i=0}^{n-1}\left[x_0 + i\left(\frac{x_n - x_0}{n}\right)\right]^2\left[\frac{x_n - x_0}{n}\right] \\ &= \left(\frac{x_n - x_0}{n}\right)\sum_{i=0}^{n-1}\left[x_0 + i\left(\frac{x_n - x_0}{n}\right)\right]^2 \\ &= \left(\frac{x_n - x_0}{n}\right)\sum_{i=0}^{n-1}\left(x_0^2 + \frac{2i}{n}(x_n - x_0) + \frac{i^2(x_n - x_0)^2}{n^2}\right) \\ &= \frac{x_n - x_0}{n}\left[x_0^2\sum_{i=0}^{n-1}1 + \frac{2(x_n - x_0)}{n}\sum_{i=0}^{n-1}i + \frac{(x_n - x_0)^2}{n^2}\sum_{i=0}^{n-1}i^2\right] \\ &= \frac{x_n - x_0}{n}\left[nx_0^2 + \frac{2(x_n - x_0)}{n}\frac{(n-1)(n)}{2} + \frac{(x_n - x_0)^2}{n^2}\frac{(n-1)(n)(2n-1)}{6}\right] \end{aligned}$$

Note that we have used that fact here that $\sum_{i=0}^{n-1} i = 0 + \sum_{i=1}^{n-1} i$, and likewise for i^2. Continuing our manipulations, we have that

$$\begin{aligned}
U_f(P_n) &= \frac{x_n - x_0}{n}\left[nx_0^2 + (x_n - x_0)(n-1) + \frac{(x_n - x_0)^2}{n}\frac{(n-1)(2n-1)}{6}\right] \\
&= \frac{x_n - x_0}{n}\left[nx_0^2 + (x_n - x_0)(n-1) + \frac{(x_n - x_0)^2}{n}\frac{(2n^2 - 3n + 1)}{6}\right] \\
&= (x_n - x_0)x_0^2 + (x_n - x_0)^2\left(\frac{n-1}{n}\right) + (x_n - x_0)^3\frac{(2n^2 - 3n + 1)}{6n^2}
\end{aligned}$$

Thus a compact formula for $U_f(P_n)$ is

$$U_f(P_n) = (x_n - x_0)x_0^2 + (x_n - x_0)^2\left(\frac{n-1}{n}\right) + (x_n - x_0)^3\frac{(2n^2 - 3n + 1)}{6n^2}$$

(f) Taking the limit of the equation above, we have

$$\lim_{n\to\infty} U_f(P_n) = (x_n - x_0)x_0^2 + (x_n - x_0)^2 + \frac{1}{3}(x_n - x_0)^3$$

since

$$\lim_{n\to\infty} \frac{n-1}{n} = 1 \text{ and } \lim_{n\to\infty} \frac{2n^2 - 3n + 1}{6n^2} = \frac{1}{3}$$

Finally, substituting $x_0 = 1$ and $x_n = 3$, we have

$$\lim_{n\to\infty} U_f(P_n) = 2(1)^2 + 2^2 + \frac{1}{3}(2)^3 = 2 + 4 + \frac{8}{3} = \frac{26}{3}$$

Hence

$$\lim_{n\to\infty} U_f(P_n) = \int_1^3 x^2 dx = \frac{26}{3}$$