1. bilinear regression

we have a dataset $z[\omega]$ that has noise $\eta[\omega]$, $z[\omega] = z0[\omega] + \eta[\omega]$,
where $\eta[\omega]$ is has power spectrum $\pi[\omega] = (1/\Omega)\langle\langle\eta[\omega]^*\eta[\omega]\rangle\rangle$,
where Ω is the total time which sets the coarse grainig of frequency $\Delta\omega = \frac{2\pi}{\Omega}$ (similar to crustal unit cell) . We fit this to a function $Z_{\lambda1,\lambda2,\ldots\lambda N}[\omega]$ where $\lambda1$, $\lambda2$, ... λN are fitting parameters. The best fit values $\Lambda1$, $\Lambda2$, ... ΛN of the parameters are determined by minimizing the goodness function

$$(1) \qquad G = \int d\omega\; W[\omega]\,(z[\omega] - Z_{\{\lambda_i\}}[\omega])^*\,(z[\omega] - Z_{\{\lambda_i\}}[\omega]) + c.c. \;\rightarrow \min$$

where $W[\omega]$ is weight function. The best fit values $\Lambda1$, $\Lambda2$, ... ΛN of the parameters depend on the noise $\eta[\omega]$ in the data and therefore the values of $\Lambda1$, $\Lambda2$, ... ΛN will be spread over a finite range when averaged over all configuration of the noise.

Question :

given the power spectrum $\pi[\omega]$ of the noise $\eta[\omega]$, what is the variance of each parameter , $\langle\langle\delta\Lambda1^2\rangle\rangle$, $\langle\langle\delta\Lambda2^2\rangle\rangle$, ... $\langle\langle\delta\Lambda N^2\rangle\rangle$ averaged over the noise $\eta[\omega]$. What is the distribution function of the $\Lambda1$, $\Lambda2$, ... ΛN given the distribution function of the noise ?

Answer :

Noise with power spectrum

(2 a) $\pi[\omega] = (1/\Omega)\langle\langle\eta[\omega]^*\eta[\omega]\rangle\rangle$ has a probability distribution function

$$(2) \qquad \mathbb{P}\{\eta[\omega]\} = \text{const} \times e^{-\int d\omega \frac{\eta[\omega]^* \times \eta[\omega]}{\pi[\omega]}}$$

the probability distribution function of the values of the best fit parametrs $\wp\{\Lambda_i + \delta\Lambda_i\}$ for small deviations $\delta\Lambda_i$ away from their avarage Λ_i is given by the same function where $\eta[\omega]$ is replaced by the difference of the fit $Z_{\{\lambda_i\}}[\omega]$ and the data $z[\omega]$, $\eta[\omega] \rightarrow z[\omega] - Z_{\{\Lambda_i+\delta\Lambda_i\}}[\omega]$

$$(3) \qquad \wp\{\delta\Lambda_i\} = \text{const} \times e^{-\int d\omega \frac{(z[\omega] - Z_{\{\Lambda_i+\delta\Lambda_i\}}[\omega])^*\,(z[\omega] - Z_{\{\Lambda_i+\delta\Lambda_i\}}[\omega])}{\pi[\omega]}}$$

For small $\delta\Lambda_i$ we can expand the exponent

$$(4) \qquad G_{\eta\,\{\lambda_i\}}\{z[\omega]\} = \int d\omega \frac{(z[\omega] - Z_{\{\lambda_i\}}[\omega])^*\,(z[\omega] - Z_{\{\lambda_i\}}[\omega])}{\pi[\omega]}$$

as

(5) $G_\eta\{\Lambda_i + \delta\Lambda_i\} = A_\eta + (B_\eta)_i\, \delta\Lambda_j + \delta\Lambda_i\, (C_\eta)_{ij}\, \delta\Lambda_j$,

where $(C_\eta)_{ij}$ is the matrix of second derivatives

(6) $$(C_\eta)_{ij} = \left(\frac{d^2 G_\eta}{d\lambda_i\, d\lambda_j}\right)_{\lambda_i = \Lambda_i},$$

the variances $\langle\langle\delta\Lambda_i\, \delta\Lambda_j\rangle\rangle$ =
$\langle\delta\Lambda_i\, \delta\Lambda_j\rangle - \langle\delta\Lambda_i\rangle\, \langle\delta\Lambda_j\rangle$ of the best fit parameters are given by

(7) $\langle\langle\delta\Lambda_i\, \delta\Lambda_j\rangle\rangle = (C_\eta{}^{-1})_{ij}$

In particular,

(8) $$\begin{aligned} \langle\langle\delta\Lambda 1^2\rangle\rangle &= (C_\eta{}^{-1})_{11} \\ \langle\langle\delta\Lambda 2^2\rangle\rangle &= (C_\eta{}^{-1})_{22} \\ &\ldots \\ \langle\langle\delta\Lambda N^2\rangle\rangle &= (C_\eta{}^{-1})_{NN} \end{aligned}$$

2. noise in the calorimeter response function R[ω]

the calorimeter response $R[\omega]$ is

(2.1) $$R[\omega] = \frac{\Delta T_{Therm}[\omega]}{\Delta P_{heater}[\omega]}$$
$$= \frac{\Delta T_{Therm}[\omega]}{\Delta V_{heater}[\omega/2]\, \Delta I_{heater}[\omega/2]}$$
$$= \left(\frac{dlogT_{therm}}{dlogR_{therm}}\right) \frac{\Delta V_{Therm}[\omega+\omega 1] / \Delta I_{Therm}[\omega 1]}{\Delta V_{heater}[\omega/2]\, \Delta I_{heater}[\omega/2]}$$

where $\alpha = \left(\frac{dlogT_{therm}}{dlogR_{therm}}\right)$ is a thermometer calibration factor

each of the four quantities , $\Delta V_{Therm}[\omega+\omega 1]$, $\Delta I_{Therm}[\omega 1]$, $\Delta V_{heater}[\omega/2]$, $\Delta I_{heater}[\omega/2]$ are independently measured by the lockin -- theroefore, each accumulating its noise -- and the calorimeter responce = Eq. (2.1) is calculated digitally using results of these four measurements.

We now consider noise in $R[\omega]$ as a result of the noise in $\Delta V_{Therm}[\omega+\omega 1]$, $\Delta I_{Therm}[\omega 1]$, $\Delta V_{heater}[\omega/2]$.

The noise in the 4 quantities is not uncorrelated because the of the coupling on the calorimeter and in the wiring between calorimetr andf the lockin. First, assume that only one of the four , $\Delta V_{Therm}[\omega + \omega 1]$, is noisy. Then the noise $\eta[\omega]$ in $R[\omega]$ is

$$(2.2)\quad \eta[\omega] = \delta R[\omega] = \delta V_{Therm}[\omega + \omega 1] \frac{\alpha / \Delta I_{Therm}[\omega 1]}{\Delta V_{heater}[\omega / 2]\, \Delta I_{heater}[\omega / 2]}$$

the noise $\delta V_{Therm}[\omega + \omega 1]$ includes
(1) the thermal noise on the thermometer
resistor (probably small at cryo temperatures)
(2) the vibration - in - field and antenna noise in the < calorimeter → lockin > wires
(2) amplifyer noise on the channels processing $\Delta V_{Therm}[\omega + \omega 1]$

we need to estimate the power spectrum $\pi_{Vtherm}[\omega] = \frac{1}{\Omega} \langle\langle \delta V_{Therm}[\omega + \omega 1]^* \delta V_{Therm}[\omega + \omega 1]\rangle\rangle$,
of $\delta V_{Therm}[\omega + \omega 1]$. The assumption of the white noise
is not satisfactory as in the calorimneter spectroscoipy the
frequency changes by 10^5 over the measured frequency range.

On top of that, $\eta[\omega]$ has an additional ω - dependent factor,
$\eta[\omega] = \phi_{Therm}[\omega]\, \delta V_{Therm}[\omega]$, where

$$(2.3)\quad \phi_{Therm}[\omega] = \frac{\alpha / \Delta I_{Therm}[\omega 1]}{\Delta V_{heater}[\omega / 2]\, \Delta I_{heater}[\omega / 2]}$$

which is not constant in frequency because the amplitudes of the AC
components of $\Delta I_{heater}[\omega / 2]$ and $\Delta I_{Therm}[\omega 1]$ are contunuously adjusrtedin the
experiment as frequency is swept to maintain comfortable S / N conditions

Ovearall, the power spectrum $\pi[\omega] =$
$(1 / \Omega) \langle\langle \eta[\omega]^* \eta[\omega]\rangle\rangle$ in the calorimeter response $R[\omega]$ is given by

$$(2.4)\quad \pi[\omega] = (1 / \Omega) \langle\langle \eta[\omega]^* \eta[\omega]\rangle\rangle = \left| \phi_{Therm}[\omega] \right|^2 \frac{1}{\Omega} \langle\langle \delta V_{Therm}[\omega + \omega 1]^* \delta V_{Therm}[\omega + \omega 1]\rangle\rangle$$

If all four voltages are noisy then (2.4) will have a
corresponding contribution form nise in each of them as well as
terms coming from the cross - neise between different channels.

3. question –– what is the power spectrum of the noise in the thermometer voltage $\frac{1}{\Omega}\langle\langle\, \delta V_{\mathrm{Therm}}[\omega+\omega 1]^{*}\, \delta V_{\mathrm{Therm}}[\omega+\omega 1]\rangle\rangle$ after lockin amplification ?

? $\frac{\mathrm{nV}}{\sqrt{\mathrm{Hz}}}$ or equivalent temperature ? What is its frequency dependence in the range between 0.01 Hz and 4 kHz ?

4. Derivation of Eq.(3)

For a given data $z[\omega]$, the best fit value $\Lambda_1, \Lambda_1, \ldots \Lambda_N$, of parameters $\lambda 1, \lambda 2, \ldots, \lambda N$ can be calculated as an average value over the probability distribution

$$(10) \qquad P[\{\lambda_1, \ldots, \lambda_{1N}\}] = e^{\frac{G0_T\{z[\omega]\}}{T}}\, e^{-\frac{G_{\{\lambda_i\}}\{z[\omega]\}}{T}}$$

$$e^{-\frac{G0_T\{z[\omega]\}}{T}} = \int d\lambda_1 \ldots d\lambda_{1N}\; e^{-\frac{G_{\lambda 1,\lambda 2,\ldots,\lambda N}\{z[\omega]\}}{T}}$$

where T is a parameter. The average value of each parameter $\langle\lambda_i\rangle_T$ at the temperature T is determined from

$$(11) \qquad \langle\lambda_i\rangle_T = e^{\frac{G0_T\{z[\omega]\}}{T}} \int d\lambda_1 \ldots d\lambda_{1N}\; \lambda_i\, e^{-\frac{1}{T} G_{\{\lambda_i\}}\{z[\omega]\}}, \text{ etc.}$$

the best fit values $\Lambda_1, \Lambda_1, \ldots \Lambda_N$, are equal to the average value of parameters $\lambda 1, \lambda 2, \ldots$ in the limit $T \to 0$.

$$(12) \qquad \Lambda_i = \lim_{T\to 0} \langle\lambda_i\rangle_T$$

The averages $\langle\lambda_i\rangle_T$ -- as well as the best fit Λ_i -- depend on the data and noise $z[\omega] + \eta[\omega]$ and therefore will be distributed over the finite range even in the limit of $T \to 0$. For a given noise configuration $\eta[\omega]$,

$$(13)\quad P[\{\lambda_1,\ ..,\ \lambda_{1\,N}\};\ T,\ \eta[\omega]] = e^{\frac{G0_T\{z[\omega]+\eta[\omega]\}}{T}}\ e^{-\frac{G_{\{\lambda_i\}}\{z[\omega]+\eta[\omega]\}}{T}}$$

Becasue noise is random,
the values of the Λ_i are themselves distributed over a vinite range when we consider all configurations of noise. The probability distribution of the noise is

$$(14)\quad \mathbb{P}\{\eta[\omega]\} = \text{const}\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}$$

where $\pi[\omega] = \langle\eta[\omega]^*\times\eta[\omega]\rangle$ is propoertional to the power spectrum. The probability distribution $\wp\{\Lambda_i\}$ for the best fit values Λ_i is

$$(15)\quad \wp\{\Lambda_i\} = \int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\ \mathbb{P}\{\eta[\omega]\}\times P[\{\lambda_i\};\ T,\ \eta[\omega]]$$

$$= \text{const}\times\int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}\ e^{\frac{G0_T\{z[\omega]+\eta[\omega]\}}{T}}\ e^{-\frac{G_{\{\lambda_i\}}\{z[\omega]+\eta[\omega]\}}{T}}$$

in the limit of $T\to 0$ the function $G0_T\{z[\omega]+\eta[\omega]\}$ depends very weakly on $\eta[\omega]$ (see SI 1) and therefore we can absorb it into the definition of the const. Therefore,
at small T the probability distribution $\wp\{\Lambda_i\}$ is

$$(16)\quad \wp\{\Lambda_i\} = e^{\frac{F_T}{T}}\times\int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}\ e^{-\frac{G_{\{\lambda_i\}}\{z[\omega]+\eta[\omega]\}}{T}}$$

$$e^{-\frac{F_T}{T}} = \int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\int d\lambda_1\ ..\ d\lambda_{1\,N}\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}\ e^{-\frac{G_{\{\lambda_i\}}\{\eta[\omega]\}}{T}}$$

and all averages are done with

$$(17)\quad \langle\ ...\ \rangle = \int d\lambda_1\ ..\ d\lambda_{1\,N}\ \wp\{\Lambda_i\}\ (\ ...)$$

To evaluate $\wp\{\Lambda_i\}$ we need to integrate over all noise configurations

$$(18)\quad \int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}\ e^{-\frac{G_{\{\lambda_i\}}\{z[\omega]+\eta[\omega]\}}{T}} =$$

$$\int\mathcal{D}\eta[\omega]\times\mathcal{D}\eta[\omega]^*\ e^{-\int d\omega\ \frac{\eta[\omega]^*\times\eta[\omega]}{\pi[\omega]}}\ e^{-\int d\omega\ W[\omega]\ \frac{(\eta[\omega]-(Z[\omega]-z[\omega]))^*(\eta[\omega]-(Z[\omega]-z[\omega]))}{T}} =$$

The integral is a product of simple gaussian integrals at each ω,
which are evaluated using an identity

$$(19)\quad \int d\eta d\eta^*\ e^{-\frac{\eta^*\eta}{T2}-\frac{(\eta-a)^*(\eta-a)}{T1}} = \int d\eta d\eta^*\ e^{-\frac{\frac{T1}{T2}\eta^*\eta-(\eta-a)^*(\eta-a)}{T1}}$$

$$= \int dx dy\ e^{-\frac{\frac{T1}{T2}(\eta1^2+\eta2^2)+(\eta1-i\,\eta2-a1+i\,a2)(\eta1+i\,\eta2-a1-i\,a2)}{T1}}$$

$$= \int dx dy\ e^{-\frac{\frac{T1}{T2}(\eta1^2+\eta2^2)+2\left((a1-\eta1)^2+(a2-\eta2)^2\right)}{T1}}$$

$$= \int dxdy\, e^{-\frac{\frac{2\ T1}{T1+2\ T2}\ a1^2 + \left(2+\frac{T1}{T2}\right)\left(\eta 1 - \frac{2\ a1}{\left(2+\frac{T1}{T2}\right)}\right)^2 + (\eta 1 \to \eta 2,\ a1 \to a2)}{T1}}$$

$$= \frac{\pi\ T1}{\left(2 + \frac{T1}{T2}\right)}\ e^{-\frac{2\ (a1^2 + a2^2)}{T1+2\ T2}}$$

Note that the parameters T1 and T2 enter as a sum only in the exponent (the prefactor is not important). This reflects the fact that when T1 goes to 0, the value of a is very close to η, however, distribution of a is still broad because η is distributed broadly. In other words, the temperature T rather then being zero, is set by the power spectrum $\pi[\omega]$.

We have

(20) $$\lim_{T\to 0} \int \mathcal{D}\eta[\omega] \times \mathcal{D}\eta[\omega]^*\ e^{-\int d\omega \frac{\eta[\omega]^* \times \eta[\omega]}{\pi[\omega]}}\ e^{-\frac{G_{\{\lambda_i\}}\{\eta[\omega]\}}{T}} =$$

$$\lim_{T\to 0} \int \mathcal{D}\eta[\omega] \times \mathcal{D}\eta[\omega]^*\ e^{-\int d\omega \frac{\frac{T/W[\omega]}{\pi[\omega]}\eta[\omega]^* \times \eta[\omega] + \left(\eta[\omega] - \left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)\right)^* \left(\eta[\omega] - \left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)\right)}{T/W[\omega]}} =$$

$$\text{const}\ e^{-\int d\omega \frac{\left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)^* \left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)}{\pi[\omega]}}$$

Finally, the probablity distribution $\wp\ \{\Lambda_i\}$ for the parameters λ_i when $z[\omega]$ is characterized by the noise power spectrum $\pi[\omega]$ is

(21) $$\wp\ \{\Lambda_i\} = e^F\ e^{-G_{\eta\,\{\lambda_i\}}\{z[\omega]\}},$$

$$e^{-F} = \int d\lambda_1 \,..\, d\lambda_{1\,N}\ e^{-G_{\eta\,\{\lambda_i\}}\{z[\omega]\}},$$

where the "noise-goodness" $G_{\eta\,\{\lambda_i\}}\{z[\omega]\}$ is determined by by the properties of the noise alone,

(22) $$G_{\eta\,\{\lambda_i\}}\{z[\omega]\} = \int d\omega \frac{1}{\pi[\omega]} \left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)^* \left(Z_{\{\lambda_i\}}[\omega] - z[\omega]\right)$$

Note that $G_{\eta\,\{\lambda_i\}}\{z[\omega]\}$ is equivalent to a specific choice of Weight function in the goodness function discussed above. The noise spectrum $\langle \eta[\omega]^* \times \eta[\omega] \rangle = \pi[\omega]$ simply sets the (frequency-- dependent) value of T in the representation of the best fit value of parameters.

Note that one could obtain $\wp\ \{\Lambda_i\}$ by simply replacing $\eta[\omega]$ with $Z_{\{\lambda_i\}}[\omega] - z[\omega]$ in the noise distribution functioon $\mathbb{P}\{\eta[\omega]\}$,

(21 a) $$\wp\ \{\Lambda_i\} = \mathbb{P}\{\eta[\omega] \to Z_{\{\lambda_i\}}[\omega] - z[\omega]\}$$

The averages $\langle\langle \Lambda_i \rangle\rangle$ and the variances $\left\langle\left\langle \delta\Lambda_i^2 \right\rangle\right\rangle$ of the fit parameters are determined in a standard way by their probility distribution $\wp\ \{\Lambda_i\}$

(23) $$\left\langle\left\langle\delta\Lambda_i^{\,2}\right\rangle\right\rangle = \left\langle\Lambda_i^{\,2}\right\rangle - \left\langle\Lambda_i\right\rangle^2 ,$$
$$\left\langle\Lambda_i^{\,2}\right\rangle = \int d\lambda_1 \,..\, d\lambda_{1\,N}\ (\lambda_i)^2\, \wp\ \{\Lambda_i\} ,$$
$$\left\langle\Lambda_i\right\rangle = \int d\lambda_1 \,..\, d\lambda_{1\,N}\ (\lambda_i)\, \wp\ \{\Lambda_i\}$$

We will now calculate the variance $\left\langle\left\langle\delta\Lambda_i^{\,2}\right\rangle\right\rangle$. Near the minimum we expand $G_\eta\ \{\Lambda_i + \delta\Lambda_i\}$ as

(24) $$G_\eta\ \{\Lambda_i + \delta\Lambda_i\} = A_\eta + \left(B_\eta\right)_i \delta\Lambda_j + \delta\Lambda_i \left(C_\eta\right)_{ij} \delta\Lambda_j ,$$

where $\left(C_\eta\right)_{ij}$ is the matrix of second derivatives

(25) $$\left(C_\eta\right)_{ij} = \frac{d^2\, G_\eta}{d\delta\Lambda_i\, d\delta\Lambda_j}$$

of G_η near its minimum. We note that if we used a different goodness function G_W to find the best fit values Λ_i for parameters λ_i,

(26) $$\Lambda_i \leftarrow \left\{\frac{dG_W}{d\Lambda_i} = 0\right\}$$

then these values are not at the minimum of $G_\eta\ \{\Lambda_i\}$ and therefore $\left(B_\eta\right)_i$ are nonzero.

To calculate the averages $\left\langle\left\langle\delta\Lambda_i^{\,2}\right\rangle\right\rangle$ we use a characteristic function $X\ \{\xi_i\}$ defined via

(27) $$e^{-X\{\xi_i\}} = \int d\delta\Lambda_1 \,..\, d\delta\Lambda_N\ e^{-G_\eta\{\Lambda_i+\delta\Lambda_i\} + i\,\delta\Lambda_i\,\xi_i}$$

(28) $$X\ \{\xi_i\} - A_\eta$$
$$= -\log \int d\delta\Lambda_1 \,..\, d\delta\Lambda_N\ e^{-(B_\eta)_i\,\delta\Lambda_j - \delta\Lambda_i\,(C_\eta)_{ij}\,\delta\Lambda_j + i\,\delta\Lambda_i\,\xi_i}$$
$$= -\log \int d\delta\Lambda_1 \,..\, d\delta\Lambda_N\ e^{-\delta\Lambda_i\,(C_\eta)_{ij}\,\delta\Lambda_j + (i\,\xi_i - (B_\eta)_i)\,\delta\Lambda_j}$$
$$= -\log \int d\delta\Lambda_1 \,..\, d\delta\Lambda_N\ e^{-\delta\Lambda_i\,(C_\eta)_{ij}\,\delta\Lambda_j + (i\,\xi_i - (B_\eta)_i)\,\delta\Lambda_j}$$
$$= -\log\left(\sqrt{\frac{\pi}{\det\left(C_\eta\right)}}\ e^{-\frac{1}{4}\,(i\,\xi_i - (B_\eta)_i)\,\left((C_\eta)^{-1}\right)_{ij}\,\left(i\,\xi_j - (B_\eta)_j\right)}\right)$$
$$= \frac{1}{4}\left(i\,\xi_i - \left(B_\eta\right)_i\right)\left(\left(C_\eta\right)^{-1}\right)_{ij}\left(i\,\xi_j - \left(B_\eta\right)_j\right) + \frac{1}{2}\log\det\left(C_\eta\right)$$

The averages are calculated as derivatives of the characteristic function $X\ \{\xi_i\}$,

(29) $$\left\langle\delta\Lambda_i\right\rangle = \frac{1}{i}\left(e^{X\{\xi_i\}}\ \frac{d}{d\xi_j}\ e^{-X\{\xi_i\}}\right)_{\xi\to 0} = -\frac{1}{i}\left(\frac{dX}{d\xi_j}\right)_{\xi\to 0}$$

(30) $\langle \delta\Lambda_i\, \delta\Lambda_j \rangle$

$$= \left(\frac{1}{i}\right)^2 \left(e^{X\{\xi_i\}} \frac{d}{d\xi_i} \frac{d}{d\xi_j} e^{-X\{\xi_i\}}\right)_{\xi\to 0}$$

$$= \left(\frac{1}{i}\right)^2 \left(e^{X\{\xi_i\}} \frac{d}{d\xi_i} \left(-\frac{dX\{\xi_i\}}{d\xi_j} e^{-X\{\xi_i\}}\right)\right)_{\xi\to 0}$$

$$= \left(\frac{1}{i}\right)^2 \left(e^{X\{\xi_i\}} \left(-\frac{d^2X\{\xi_i\}}{d\xi_i\, d\xi_j} + \frac{dX\{\xi_i\}}{d\xi_i} \frac{dX\{\xi_i\}}{d\xi_j}\right) e^{-X\{\xi_i\}}\right)_{\xi\to 0}$$

$$= \left(\frac{1}{i}\right)^2 \left(-\frac{d^2X}{d\xi_i\, d\xi_j} + \frac{dX}{d\xi_i} \frac{dX}{d\xi_j}\right)_{\xi\to 0}$$

$$= \left(\frac{1}{i}\right)^2 \left(-\frac{d^2X\{\xi_i\}}{d\xi_i\, d\xi_j}\right)_{\xi\to 0} + \left(\frac{1}{i}\right)^2 \langle \delta\Lambda_i \rangle \langle \delta\Lambda_j \rangle$$

(31) $\langle\langle \delta\Lambda_i\, \delta\Lambda_j \rangle\rangle$

$$= \langle (\delta\Lambda_i - \langle \delta\Lambda_i \rangle)(\delta\Lambda_j - \langle \delta\Lambda_j \rangle) \rangle$$

$$= \langle \delta\Lambda_i\, \delta\Lambda_j \rangle - \langle \delta\Lambda_i \rangle \langle \delta\Lambda_j \rangle$$

$$= \left(-\frac{d^2X}{d\xi_i\, d\xi_j} + \frac{dX}{d\xi_i} \frac{dX}{d\xi_j}\right)_{\xi\to 0} - \left(-\left(\frac{dX}{d\xi_j}\right)_{\xi\to 0}\right)\left(-\left(\frac{dX}{d\xi_j}\right)_{\xi\to 0}\right)$$

$$= \left(\frac{1}{i}\right)^2 \left(-\frac{d^2X}{d\xi_i\, d\xi_j}\right)_{\xi\to 0}$$

$$= \frac{1}{4} \left((C_\eta)^{-1}\right)_{ij}$$

SI 0

$$\delta G = \delta\Lambda\, A\, \delta\Lambda + B\, \delta\Lambda$$

$$e^{-\frac{X\{\xi_i\}}{T}} = \int d\delta\Lambda_1\, d\delta\Lambda_2 \ldots d\delta\Lambda_N\ e^{-\frac{\delta\Lambda A \delta\Lambda + B \delta\Lambda}{T} + i\, \delta\Lambda_i\, \xi_i}$$

$$\delta\Lambda\, A\, \delta\Lambda + B\, \delta\Lambda + i\, \delta\Lambda\, \xi =$$

$$\left(\delta\Lambda + \frac{1}{2} A^{-1} B + \frac{i}{2} A^{-1} \xi\right) A \left(\delta\Lambda + \frac{1}{2} A^{-1} B + \frac{i}{2} A^{-1} \xi\right) - \frac{1}{4} (B + i\,\xi)\, A^{-1} (B + i\,\xi) =$$

$$X = \frac{1}{4} (B + i\,\xi)\, A^{-1} (B + i\,\xi)$$

5. dependence of $e^{-\frac{G0_T\{z[\omega] + \eta[\omega]\}}{T}}$ on $\eta[\omega]$?

$$(S1)\ e^{-\frac{G0_T\{z[\omega]+\eta[\omega]\}}{T}}$$

$$= \int d\lambda 1\, d\lambda 2 \ldots d\lambda N\ e^{-\frac{G_{\lambda 1,\lambda 2,\ldots,\lambda N}\{z[\omega]+\eta[\omega]\}}{T}}$$

$$= \int d\lambda 1\, d\lambda 2 \ldots d\lambda N$$

$$e^{-\frac{1}{T}\int d\omega\, W[\omega]\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]-\eta[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j}d\Lambda_i\, d\Lambda_j\right)^*\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]-\eta[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j}d\Lambda_i\, d\Lambda_j\right)}$$

$$= \int d\lambda 1\, d\lambda 2 \ldots d\lambda N\ e^{-\frac{1}{T}\int d\omega\, W[\omega]\times\left(A[\omega]-\eta[\omega]+B_i[\omega]\, d\Lambda_i+C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)^*\times\left(A[\omega]-\eta[\omega]+B_i[\omega]\, d\Lambda_i+C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)}$$

we difine Λ_i as the best fit values of noiseless data ,
$\eta[\omega] = 0$, defined by N equations for the extremum

$$\frac{d\, G_{\lambda 1,\lambda 2,\ldots,\lambda N}\{z[\omega]+\eta[\omega]\}}{d\ \lambda_i}\Big|_{\Lambda_i} = \int d\omega\, W[\omega]\left(\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}\right)^*\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]\right) + c.c.\ \Big|_{\Lambda_i}$$

$$= \int d\omega\ W[\omega]\left(B_i[\omega]^*\, A[\omega] + B_i[\omega]\, A[\omega]^*\right)\Big|_{\Lambda_i}$$

$$= 0$$

where $A[\omega] = Z_{\{\Lambda_i\}}[\omega]$ and $B_i[\omega] = \frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}$.

We write

$$\int d\omega\ W[\omega]\times\left(A[\omega]-\eta[\omega]+B_i[\omega]\, d\Lambda_i+C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)^*\times$$
$$\left(A[\omega]-\eta[\omega]+B_i[\omega]\, d\Lambda_i+C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)$$

$$= \int d\omega\ W[\omega]\ \Big($$
$$(A[\omega]-\eta[\omega])^*\times(A[\omega]-\eta[\omega])$$
$$+\ (A[\omega]-\eta[\omega])^*(B_i[\omega]\, d\Lambda_i) + c.c.$$
$$+\left(B_i[\omega]^*\, B_j[\omega]+(A[\omega]-\eta[\omega])^*\, C_{ij}[\omega]\right) d\Lambda_i\, d\Lambda_j + c.c.$$
$$\Big)$$

$$= \int d\omega\ W[\omega]\ \Big($$
$$(A[\omega]-\eta[\omega])^*\times(A[\omega]-\eta[\omega])$$
$$-\ (\eta[\omega]^*\, B_i[\omega] + c.c.)\ d\Lambda_i$$
$$+\left(B_i[\omega]^*\, B_j[\omega]+(A[\omega]-\eta[\omega])^*\, C_{ij}[\omega] + c.c.\right) d\Lambda_i\, d\Lambda_j$$
$$\Big)$$

$$= \int d\omega\ W[\omega]\ \Big($$
$$(A[\omega]-\eta[\omega])^*\times(A[\omega]-\eta[\omega])$$

$$+\left(B_i[\omega]^*\, B_j[\omega]+(A[\omega]-\eta[\omega])^*\, C_{ij}[\omega] + c.c.\right)$$
$$\left(d\Lambda_i - \frac{1}{2}\left(B_i[\omega]^*\, B_j[\omega]+(A[\omega]-\eta[\omega])^*\, C_{ij}[\omega] + c.c.\right)^{-1}(\eta[\omega]^*\, B_i[\omega] + c.c.)\right)$$

$$\left(d\Lambda_j - \frac{1}{2}\left(B_i[\omega]^* B_j[\omega] + (A[\omega] - \eta[\omega])^* C_{ij}[\omega] + c.c.\right)^{-1}(\eta[\omega]^* B_i[\omega] + c.c.)\right)$$

$$- \frac{1}{4}\left(\eta[\omega]^* B_j[\omega] + c.c.\right)\left(B_i[\omega]^* B_j[\omega] + (A[\omega] - \eta[\omega])^* C_{ij}[\omega] + c.c.\right)^{-1}$$

$$\left(\eta[\omega]^* B_j[\omega] + c.c.\right)$$

$$\Big)$$

Therefore,

(S1) =

$$\int d\lambda 1\, d\lambda 2 \ldots d\lambda N\; e^{-\frac{1}{T}\int d\omega\; W[\omega]\times\left(A[\omega] - \eta[\omega] + B_i[\omega]\, d\Lambda_i + C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)^*\times\left(A[\omega] - \eta[\omega] + B_i[\omega]\, d\Lambda_i + C_{ij}[\omega]\, d\Lambda_i\, d\Lambda_j\right)} =$$

$$= \sqrt{\frac{1}{\mathrm{Det}_{ij}\left[\frac{1}{T}\int d\omega\; W[\omega]\left(B_i[\omega]^* B_j[\omega] + (A[\omega] - \eta[\omega])^* C_{ij}[\omega] + c.c.\right)\right]}} \times$$

$$\times\; e^{-\frac{1}{T}\int d\omega\; W[\omega]\times(A[\omega] - \eta[\omega])^*\times(A[\omega] - \eta[\omega])}$$

$$\times\; e^{-\frac{1}{T}\frac{1}{4}\int d\omega\; W[\omega]\times\left(\eta[\omega]^* B_j[\omega] + c.c.\right)\left(B_i[\omega]^* B_j[\omega] + (A[\omega] - \eta[\omega])^* C_{ij}[\omega] + c.c.\right)^{-1}\left(\eta[\omega]^* B_j[\omega] + c.c.\right)}$$

The last two factors are equivalent to gaussian distribution of $\eta[\omega]$

const $\times\, e^{-\int d\omega\, \frac{\eta[\omega]^*\eta[\omega]}{p[\omega]}}$, where $p[\omega] \propto T$ and, therefore, it vanishes in the $T \to 0$ limit.

Therefore, we can assume that the normalization factor $e^{-\frac{G\Theta_T\{z[\omega] + \eta[\omega]\}}{T}}$ depends weakly on $\eta[\omega]$ in thje $T \to 0$ limit.

6. $\int d\eta d\eta^* e^{-\frac{\frac{T1}{T2}\eta^*\eta - (\eta - a)^*(\eta - a)}{T1}}$

the integral is done using identity

$$\int d\eta d\eta^*\, e^{-\frac{\frac{T1}{T2}\eta^*\eta - (\eta - a)^*(\eta - a)}{T1}}$$

$$= \int dxdy\, e^{-\frac{\frac{T1}{T2}(x^2+y^2) + (x - i y - a1 + i a2)(x + i y - a1 - i a2) + c.c.}{T1}}$$

$$= \int dxdy\, e^{-\frac{\frac{T1}{T2}(x^2+y^2) + 2\left((a1-x)^2 + (a2-y)^2\right)}{T1}}$$

$$= \int dxdy\, e^{-\frac{\frac{2\, a1^2\, T1}{T1+2\, T2} + \left(2+\frac{T1}{T2}\right)\left(x-\frac{2\, a1}{\left(2+\frac{T1}{T2}\right)}\right)^2 + (x\to y,\ a1\to a2)}{T1}}$$

$$= \frac{\pi\, T1}{\left(2 + \frac{T1}{T2}\right)}\, e^{-\frac{2\left(a1^2+a2^2\right)}{T1+2\, T2}}$$

$$(x - i\, y - a1 + i\, a2)(x + i\, y - a1 - i\, a2) +$$
$$(x + i\, y - a1 - i\, a2)(x - i\, y - a1 + i\, a2)$$
$$2\left((a1-x)^2 + (a2-y)^2\right)$$
$$\frac{T1}{T2}\left(x^2+y^2\right) + 2\left((a1-x)^2 + (a2-y)^2\right)$$

$$\left(2 + \frac{T}{T2}\right)\left(\frac{2\, a1^2\, T1\, T2}{(T+2\, T2)^2} + \left(x - \frac{2\, a1}{\left(2+\frac{T1}{T2}\right)}\right)^2\right)$$

$$\left(2 + \frac{T1}{T2}\right)\left(\frac{2\, a1^2\, T1\, T2}{(T1+2\, T2)^2} + \left(x - \frac{2\, a1}{\left(2+\frac{T1}{T2}\right)}\right)^2\right) = \left(\frac{T1}{T2}\, x^2 + 2\, (a1-x)^2\right)$$

0

$$\left(2 + \frac{T1}{T2}\right)\left(\frac{2\, a1^2\, T1\, T2}{(T1+2\, T2)^2}\right) = \frac{2\, a1^2\, T1}{T1+2\, T2}$$

0

$$\frac{2\, a1^2\, T1}{T1+2\, T2} + \left(2+\frac{T1}{T2}\right)\left(x - \frac{2\, a1}{\left(2+\frac{T1}{T2}\right)}\right)^2$$

$$(16)\quad \int dxdy\, e^{-\frac{\frac{T1}{T2}\left(x^2+y^2\right) + 2\left((a1-x)^2+(a2-y)^2\right)}{T1}} =$$

$$\int dxdy\, e^{-\frac{\frac{2\, a1^2\, T1}{T1+2\, T2} + \left(2+\frac{T1}{T2}\right)\left(x-\frac{2\, a1}{\left(2+\frac{T1}{T2}\right)}\right)^2 + (x\to y,\ a1\to a2)}{T1}} = \frac{\pi\, T1}{\left(2+\frac{T1}{T2}\right)}\, e^{-\frac{2\left(a1^2+a2^2\right)}{T1+2\, T2}}$$

7. 2 x2 matrix diagonalization and ellipse

$$\begin{pmatrix} \text{Cos}[\phi] \\ \text{Sin}[\phi] \end{pmatrix}^{\mathsf{T}} . \begin{pmatrix} a + b\,\text{Cos}[2\,\theta] & b\,\text{Sin}[2\,\theta] \\ b\,\text{Sin}[2\,\theta] & a - b\,\text{Cos}[2\,\theta] \end{pmatrix} . \begin{pmatrix} \text{Cos}[\phi] \\ \text{Sin}[\phi] \end{pmatrix} \text{ // Tr // FullSimplify}$$

$a + b\,\text{Cos}[2\,(\theta - \phi)]$

$$\begin{pmatrix} a + b\,\text{Cos}[2\,\theta] & b\,\text{Sin}[2\,\theta] \\ b\,\text{Sin}[2\,\theta] & a - b\,\text{Cos}[2\,\theta] \end{pmatrix} \text{ // Eigensystem // FullSimplify}$$

{{a - b, a + b}, {{-Tan[θ], 1}, {Cot[θ], 1}}}

{{a - b, a + b}, {{-Sin[θ], Cos[θ]}, {Cos[θ], Sin[θ]}}}

$$\left(\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} \sqrt{a-b} & 0 \\ 0 & \sqrt{a+b} \end{pmatrix} . \begin{pmatrix} \text{Cos}[\phi] \\ \text{Sin}[\phi] \end{pmatrix}\right)^{\mathsf{T}} . \begin{pmatrix} a + b\,\text{Cos}[2\,\theta] & b\,\text{Sin}[2\,\theta] \\ b\,\text{Sin}[2\,\theta] & a - b\,\text{Cos}[2\,\theta] \end{pmatrix} .$$
$$\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} \sqrt{a-b} & 0 \\ 0 & \sqrt{a+b} \end{pmatrix} . \begin{pmatrix} \text{Cos}[\phi] \\ \text{Sin}[\phi] \end{pmatrix} \text{ // Tr // FullSimplify}$$

$(a - b)\,(a + b)$

$$\left(\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} \frac{\sqrt{a-b}}{\sqrt{(a-b)\,(a+b)}} & 0 \\ 0 & \frac{\sqrt{a+b}}{\sqrt{(a-b)\,(a+b)}} \end{pmatrix}\right)^{\mathsf{T}} . \begin{pmatrix} a + b\,\text{Cos}[2\,\theta] & b\,\text{Sin}[2\,\theta] \\ b\,\text{Sin}[2\,\theta] & a - b\,\text{Cos}[2\,\theta] \end{pmatrix} .$$
$$\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} \frac{\sqrt{a-b}}{\sqrt{(a-b)\,(a+b)}} & 0 \\ 0 & \frac{\sqrt{a+b}}{\sqrt{(a-b)\,(a+b)}} \end{pmatrix} \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\left(\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} \frac{1}{\sqrt{a+b}} & 0 \\ 0 & \frac{1}{\sqrt{a-b}} \end{pmatrix}\right)^{\mathsf{T}} \text{ // Inverse}\right) -$$
$$\left(\begin{pmatrix} \sqrt{a+b} & 0 \\ 0 & \sqrt{a-b} \end{pmatrix} . \begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}\right)^{\mathsf{T}} \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left(\left(\begin{pmatrix}\sqrt{a+b} & 0\\ 0 & \sqrt{a-b}\end{pmatrix}.\begin{pmatrix}\mathrm{Cos}[\theta] & \mathrm{Sin}[\theta]\\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}\right)^{T}.\right.$$

$$\left.\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\left(\begin{pmatrix}\sqrt{a+b} & 0\\ 0 & \sqrt{a-b}\end{pmatrix}.\begin{pmatrix}\mathrm{Cos}[\theta] & \mathrm{Sin}[\theta]\\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}\right) - \right.$$

$$\left.\begin{pmatrix}a + b\,\mathrm{Cos}[2\,\theta] & b\,\mathrm{Sin}[2\,\theta]\\ b\,\mathrm{Sin}[2\,\theta] & a - b\,\mathrm{Cos}[2\,\theta]\end{pmatrix}\right) \mathbin{//} \mathrm{FullSimplify} \mathbin{//} \mathrm{MatrixForm}$$

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}a + b\,\mathrm{Cos}[2\,\theta] & b\,\mathrm{Sin}[2\,\theta]\\ b\,\mathrm{Sin}[2\,\theta] & a - b\,\mathrm{Cos}[2\,\theta]\end{pmatrix} = \left(\begin{pmatrix}\sqrt{a+b} & 0\\ 0 & \sqrt{a-b}\end{pmatrix}.\begin{pmatrix}\mathrm{Cos}[\theta] & \mathrm{Sin}[\theta]\\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}\right)^{T}.$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\left(\begin{pmatrix}\sqrt{a+b} & 0\\ 0 & \sqrt{a-b}\end{pmatrix}.\begin{pmatrix}\mathrm{Cos}[\theta] & \mathrm{Sin}[\theta]\\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}\right)$$

errors

$$\frac{\delta G}{G} = \left(\frac{1}{a}\,\frac{\delta\lambda}{\lambda}\right)^2$$

$$\begin{pmatrix}\frac{\delta\lambda 1}{\lambda 1}\\ \frac{\delta\lambda 2}{\lambda 2}\end{pmatrix} =$$

$$\frac{1}{30}\begin{pmatrix}\mathrm{Cos}[\theta] & -\mathrm{Sin}[\theta]\\ \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}.\begin{pmatrix}\frac{1}{\sqrt{a+b}} & 0\\ 0 & \frac{1}{\sqrt{a-b}}\end{pmatrix}.\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{30}\begin{pmatrix}\mathrm{Cos}[\theta] & -\mathrm{Sin}[\theta]\\ \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta]\end{pmatrix}.\begin{pmatrix}\frac{1}{\sqrt{a+b}}\\ \frac{1}{\sqrt{a-b}}\end{pmatrix}$$

check

$$\int d\eta\, d\eta^{*}\, e^{-\frac{\eta^{*}\eta}{T2} - i\,\xi\,\eta}$$

$$= \int d\eta\, d\eta^{*}\, e^{-\frac{\left(\eta - \frac{T2}{2}\, i\,\xi\right)^{*}\left(\eta - \frac{T2}{2}\, i\,\xi\right)}{T2} + \frac{T2}{4}\,\xi^{*}\,\xi}$$

$$= \pi\, T2\, e^{\frac{T2}{4}\,\xi^{*}\,\xi}$$

8. digression on $\pi[\omega]$

Noise with power spectrum $\pi[\omega] = \langle\langle \eta[\omega]^{*}\,\eta[\omega]\rangle\rangle$ has a probability distribution function

$$(2)\quad \mathbb{P}\{\eta[\omega]\} = \mathrm{const} \times e^{-\int d\omega\, \frac{\eta[\omega]^{*}\times\eta[\omega]}{\pi[\omega]}}$$

whats missing is the total time . We assume that
the total time is Ω -- this is analogous to the crystal size

$$\eta[\omega_n] = \int dt\, \eta[t]\, e^{-i\,\omega_n\, t}$$

$$\eta[t] = \int \frac{d\omega}{2\pi}\, \eta[\omega]\, e^{i\,\omega\, t} = \frac{\Delta\omega}{2\pi} \sum \eta[\omega_n]\, e^{i\,\omega_n\, t}$$

the size of the "unit cell" in time is $\Delta\omega = \frac{1}{2\pi\Omega}$ -- this provides regularization

$\langle\langle \eta[t2] \times \eta[t1] \rangle\rangle$ is intesive (not proportional to "volume" Ω)
$\langle\langle \eta[\omega_n]^* \eta[\omega_n] \rangle\rangle$ is extensive (yes proportional to "volume" Ω)

$$<\text{power spectrum}> = G[\omega_n] = \frac{1}{\Omega} \langle\langle \eta[\omega_n]^* \eta[\omega_n] \rangle\rangle = (2\pi\Delta\omega) \langle\langle \eta[\omega_n]^* \eta[\omega_n] \rangle\rangle$$

< power spectrum >
is is intesive (not proportional to "volume" Ω) -- it is "per unit volume"

$$\langle\langle \eta[t2] \times \eta[t1] \rangle\rangle = \left(\frac{\Delta\omega}{2\pi}\right)^2 \sum \langle\langle \eta[\omega_n]^* \eta[\omega_m] \rangle\rangle\, e^{i\,(\omega_n+\omega_m)\frac{(t1-t2)}{2} + (\omega_n-\omega_m)\frac{(t1+t2)}{2}}$$

$$= \left(\frac{\Delta\omega}{2\pi}\right)^2 \sum \langle\langle \eta[\omega_n]^* \eta[\omega_n] \rangle\rangle\, \delta_{nm}\, e^{i\,(\omega_n+\omega_m)\frac{(t1-t2)}{2} + (\omega_n-\omega_m)\frac{(t1+t2)}{2}}$$

$$= \left(\frac{\Delta\omega}{2\pi}\right)^2 \sum \langle\langle \eta[\omega_n]^* \eta[\omega_n] \rangle\rangle\, e^{i\,\omega_n\,(t1-t2)}$$

$$= \left(\frac{\Delta\omega}{2\pi}\right) \int \frac{d\omega}{2\pi} \langle\langle \eta[\omega]^* \eta[\omega] \rangle\rangle\, e^{i\,\omega_n\,(t1-t2)}$$

9. relation between power spectrum and $\pi[\omega]$

the probability distribution $\mathbb{P}\{\eta[\omega_n]\} = \text{const} \times e^{-\int d\omega \frac{\eta[\omega]^* \times \eta[\omega]}{\pi[\omega]}}$ in a discreet form

$$\int d\omega\, \frac{\eta[\omega]^* \times \eta[\omega]}{\pi[\omega]} = \sum \Delta\omega\, \frac{\eta[\omega_n]^* \times \eta[\omega_n]}{\pi[\omega_n]}$$

$$\mathbb{P}\{\eta[\omega_n]\} = \text{const} \times e^{-\sum \Delta\omega \frac{\eta[\omega_n]^* \times \eta[\omega_n]}{\pi[\omega_n]}}$$

$$e^{-X\{\xi[\omega_n]\}} = \int \mathcal{D}\eta[\omega_n] \times \mathcal{D}\eta[\omega_n]^*\, e^{-\sum \Delta\omega \frac{\eta[\omega_n]^* \times \eta[\omega_n]}{\pi[\omega_n]} - \sum \Delta\omega\, (\xi[\omega_n]^* \times \eta[\omega_n] + \xi[\omega_n]\, \eta[\omega_n]^*)}$$

$$= \prod \int d\eta[\omega_n] \times d\eta[\omega_n]^*\, e^{-\Delta\omega \frac{\eta[\omega_n]^* \times \eta[\omega_n]}{\pi[\omega_n]} - \Delta\omega\, (\xi[\omega_n]^* \times \eta[\omega_n] + \xi[\omega_n]\, \eta[\omega_n]^*)}$$

$$\int d\eta[\omega_n] \times d\eta[\omega_n]^* \, e^{-\Delta\omega \frac{\eta[\omega_n]^* \times \eta[\omega_n]}{\pi[\omega_n]} - \Delta\omega \, (\xi[\omega_n]^* \times \eta[\omega_n] + \xi[\omega_n] \, \eta[\omega_n]^*)}$$

$$= \int d\eta[\omega_n] \times d\eta[\omega_n]^* \, e^{-\frac{\Delta\omega}{\pi[\omega_n]} \, (\eta[\omega_n]^* \times \eta[\omega_n] - \pi[\omega_n] \times (\xi[\omega_n]^* \times \eta[\omega_n] + \xi[\omega_n] \, \eta[\omega_n]^*))}$$

$$= \int d\eta[\omega_n] \times d\eta[\omega_n]^* \, e^{-\frac{\Delta\omega}{\pi[\omega_n]} \left((\eta[\omega_n] - \pi[\omega_n] \times \xi[\omega_n])^* \, (\eta[\omega_n] - \pi[\omega_n] \times \xi[\omega_n]) - \pi[\omega_n]^2 \, (\xi[\omega_n])^* \, (\xi[\omega_n]) \right)}$$

$$= e^{\Delta\omega \, \pi[\omega_n] \, (\xi[\omega_n])^* \, (\xi[\omega_n])} \int d\eta[\omega_n] \times d\eta[\omega_n]^* \, e^{-\frac{\Delta\omega}{\pi[\omega_n]} \, ((\eta[\omega_n] - \pi[\omega_n] \times \xi[\omega_n])^* \, (\eta[\omega_n] - \pi[\omega_n] \times \xi[\omega_n]))}$$

$$= e^{\Delta\omega \, \pi[\omega_n] \, (\xi[\omega_n])^* \, (\xi[\omega_n])} \, \frac{\pi[\omega_n]}{\Delta\omega}$$

$$e^{-X\{\xi[\omega_n]\}} = \left(\prod \frac{\pi[\omega_n]}{\Delta\omega} \right) e^{\sum \Delta\omega \, \pi[\omega_n] \times \xi[\omega_n]^* \times \xi[\omega_n]}$$

$$= \left(\prod \frac{\pi[\omega_n]}{\Delta\omega} \right) e^{\int d\omega \, \pi[\omega] \times \xi[\omega]^* \times \xi[\omega]}$$

$$\langle\langle \eta[\omega_n]^* \, \eta[\omega_n] \rangle\rangle = \langle \eta[\omega_n]^* \, \eta[\omega_n] \rangle - \langle \eta[\omega_n]^* \rangle \, \langle \eta[\omega_n] \rangle$$

$$= \left(e^{X\{\xi[\omega_n]\}} \, \frac{1}{\Delta\omega^2} \, \frac{d}{d\xi[\omega_n]} \, \frac{d}{d\xi^*[\omega_n]} \, e^{-X\{\xi[\omega_n]\}} - \left(e^{X\{\xi[\omega_n]\}} \, \frac{1}{\Delta\omega} \, \frac{d}{d\xi[\omega_n]} \, e^{-X\{\xi[\omega_n]\}} \right)^2 \right) \Big|_{\{\xi^*[\omega_n]=0\}}$$

$$= \frac{1}{\Delta\omega^2} \, \frac{d}{d\xi[\omega_n]} \, \frac{d}{d\xi^*[\omega_n]} \, X\{\xi[\omega_n]\} = \frac{1}{\Delta\omega} \, \pi[\omega_n]$$

$$\langle\langle \eta[\omega_n]^* \, \eta[\omega_n] \rangle\rangle = \frac{1}{\Delta\omega} \, \pi[\omega_n]$$

$$S[\omega_n] = \frac{1}{\Omega} \, \langle\langle \eta[\omega_n]^* \, \eta[\omega_n] \rangle\rangle$$

$$\frac{1}{\Delta\omega} \, \pi[\omega_n] = \langle\langle \eta[\omega_n]^* \, \eta[\omega_n] \rangle\rangle = S[\omega_n] \, \Omega$$

$$\frac{1}{\Omega\Delta\omega} \, \pi[\omega_n] = S[\omega_n]$$

$$(2\pi) \, \pi[\omega_n] = S[\omega_n]$$

$$\pi[\omega_n] = \frac{1}{2\pi} \, S[\omega_n]$$

$$\left(e^{-a x^2 - b x - c} \, D\left[D\left[e^{a x^2 + b x + c}, x\right], x\right] - \left(e^{-a x^2 - b x - c} \, D\left[e^{a x^2 + b x + c}, x\right] \right)^2 \right) \text{ /. } x \to 0 \text{ // FullSimplify}$$

2 a

IGNORE

We now assume that noise is weak and we can taylor expand the fitting function $Z_{\{\lambda i\}}[\omega]$ in deviadions $\delta\lambda_i$

$Z_{\{\Lambda_i+\lambda_i\}}[\omega]$

$$\text{const}\int d\lambda 1 d\lambda 2 \,..\, d\lambda N\, e^{-\int d\omega\, \frac{(Z[\omega]-z0[\omega])^{*}\times(Z[\omega]-z0[\omega])}{T2}} =$$

$$\text{const}\int d\lambda 1 d\lambda 2 \,..\, d\lambda N\, e^{-\int d\omega\, \frac{\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)^{*}\left(Z[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)}{T}} =$$

$$\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)^{*}$$

$$\left(Z[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)$$

$$(Z_{\{\Lambda_i\}}[\omega]-z[\omega])^{*}\,(Z_{\{\Lambda_i\}}[\omega]-z[\omega])$$

$$+\left(\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}\right)^{*}(Z_{\{\Lambda_i\}}[\omega]-z[\omega])\, d\Lambda_i + i\,(\Lambda_i+d\Lambda_i)\,\xi_i$$

$$+\left(\left(\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i}\right)^{*}\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_j}+\left(\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j}\right)^{*}(Z_{\{\Lambda_i\}}[\omega]-z[\omega])\right) d\Lambda_i\, d\Lambda_j$$

$$(B_i + i\,\xi_i)\, d\Lambda_i + A_{ij}\, d\Lambda_i\, d\Lambda_j = d\Lambda_i\, A\, d\Lambda_j + (B_i + i\,\xi_i)\, d\Lambda_i$$

$$= \left(d\Lambda_j + (B_i + i\,\xi_i)\, A^{-1}\right) A \left(d\Lambda_j + A^{-1}\,(B_i + i\,\xi_i)\right) - (B_i + i\,\xi_i)\, A^{-1}\,(B_i + i\,\xi_i)$$

$$\int d\lambda 1 d\lambda 2 \,..\, d\lambda N\, e^{-\int d\omega\, \frac{\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)^{*}\left(Z_{\{\Lambda_i\}}[\omega]-z[\omega]+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i} d\Lambda_i+\frac{dZ_{\{\Lambda_i\}}[\omega]}{d\Lambda_i\, d\Lambda_j} d\Lambda_i\, d\Lambda_j\right)}{T}} =$$

$$e^{\frac{1}{T}\int d\omega\, (B_i + i\, T\,\xi_i)\, A^{-1}\,(B_i + i\, T\,\xi_i)} =$$