The Limit of $x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$ as x Tends to Zero

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The tower of exponents in the title poses some interesting challenges for devotees of L'Hôpital's rule. We show in this note that as x tends to 0 from the right, the expression in the title tends to 0 when the number of x's in that expression is odd and to 1 when the number of x's is even.

Let $^{0}x = 1$, and define recursively $^{n}x = x^{(n-1}x)}$, $n = 1, 2, 3, \cdots$, so that, for example,

$$^{4}x = x^{(x^{(x^{x})})} = x^{x^{x^{x}}}.$$

The expression ^{n}x is sometimes called the n-th hyperpower of x. Notice that the nesting of the parentheses from the left is quite important. It is clear that $\lim_{x \to 0^{+}} {}^{0}x = \lim_{x \to 0^{+}} 1 = 1$ and $\lim_{x \to 0^{+}} {}^{1}x = \lim_{x \to 0^{+}} x = 0$ and it is a standard calculus exercise to show that

$$\lim_{x \to 0^{+}} {}^{2}x = \lim_{x \to 0^{+}} x^{x} = e^{(\lim_{x \to 0^{+}} x \ln x)} = e^{0} = 1.$$

The reader should fill in the details using either L'Hôpital's rule or remark 1 below. This $1, 0, 1$ pattern persists.

THEOREM 1. *For every nonnegative integer k, we have*

$$\lim_{x \to 0^{+}} {}^{2k}x = 1 \qquad\qquad (A, k)$$

$$\lim_{x \to 0^{+}} {}^{2k+1}x = 0. \qquad\qquad (B, k)$$

Proof. Since $^{n}x = e^{(n-1}x \ln x)}$, for every positive integer k the above two statements are equivalent, respectively, to

$$\lim_{x \to 0^{+}} {}^{2k-1}x \ln x = 0 \qquad\qquad (A', k)$$

$$\lim_{x \to 0^{+}} {}^{2k}x \ln x = -\infty. \qquad\qquad (B', k)$$

We saw above that equations (A, 0), (B, 0), and (A, 1) hold. To see how the general induction proof should go, we will first establish the next two equations explicitly.

To verify equation (B, 1) we see that there holds the equivalent (B', 1):

$$\lim_{x \to 0^{+}} {}^{2}x \ln x = -\infty,$$

since $\lim_{x \to 0^{+}} {}^{2}x = 1$ and $\lim_{x \to 0^{+}} \ln x = -\infty$.

To confirm equation (A, 2), note that from $\lim_{x \to 0^{+}} {}^{2}x = 1$, it follows that for x sufficiently small, $^{2}x > 1/2$. Thus for $x \in (0, 1)$ sufficiently small we have

$$^{3}x = x^{(2}x)} < x^{1/2}.$$

From this and L'Hôpital's rule (alternatively, See Remark 1 below) we have

$$\lim_{x\to 0^+}|^3x\ln x|\le \lim_{x\to 0^+}|x^{1/2}\ln x|=\lim_{x\to 0^+}\left|\frac{(\ln x)'}{(x^{-1/2})'}\right|=\lim_{x\to 0^+}|-2x^{1/2}|=0,$$

which is the equivalent equation (A', 2).

Passing to the general proof, assume now that equations (A, k) and (B, k) hold. It suffices to establish equations (A', k + 1) and (B', k + 1).

By induction hypothesis (A, k), $\lim_{x\to 0^+}{}^{2k}x=1$. It follows that for x sufficiently small, $^{2k}x>1/2$. Thus for $x\in(0,1)$ sufficiently small we have

$$x^{(2^k x)}<x^{1/2}.$$

From this and L'Hôpital's rule (alternatively, see Remark 1 below) we have

$$\lim_{x\to 0^+}|^{2(k+1)-1}x\ln x|=\lim_{x\to 0^+}|x^{(2^k x)}\ln x|\le\lim_{x\to 0^+}|x^{1/2}\ln x|=\lim_{x\to 0^+}\left|\frac{\ln x}{x^{-1/2}}\right|$$

$$=\lim_{x\to 0^+}\left|\frac{(\ln x)'}{(x^{-1/2})'}\right|=\lim_{x\to 0^+}\left|\frac{x^{-1}}{-\frac{1}{2}x^{-3/2}}\right|=\lim_{x\to 0^+}2x^{1/2}=0,$$

which is equation (A', k + 1). Of course the equivalent equation (A, k + 1) also holds. It is now almost immediate that

$$\lim_{x\to 0^+}{}^{2(k+1)}x\ln x=-\infty,$$

since $\lim_{x\to 0^+}{}^{2(k+1)}x=1$ by equation (A, k + 1) and $\lim_{x\to 0^+}\ln x=-\infty$. This is equation (B', k + 1).   QED

Remarks.

1. The proof may be made more elementary by proving that $\lim_{x\to 0^+}|x^{1/2}\ln x|=0$ without using L'Hôpital's rule. To do this we will need the fact that $0<\ln y<y$ for $y\in(1,\infty)$. This is geometrically evident if $\ln y$ is defined as $\int_1^y(dt/t)$, the area under the function $1/t$ and between $t=1$ and $t=y$. Now set $y=x^{-1/4}$. Then

$$\lim_{x\to 0^+}x^{1/2}\ln x=\lim_{y\to\infty}\frac{\ln y^{-4}}{y^2}=-4\lim_{y\to\infty}\left(\left(\frac{1}{y}\right)\left(\frac{\ln y}{y}\right)\right).$$

This is zero by the squeeze theorem, since $\lim_{y\to\infty}1/y=0$ and $0<\ln y<y$ implies $0<\ln y/y<1$. One can similarly show that $\lim_{x\to 0^+}x^\alpha\ln x=0$ for any $\alpha>0$.

2. Perhaps I should have credited *Macsyma*® as coauthor. While fooling around with it, I asked it to find successively the limits of ^2x through ^6x as x tended to 0 from the right. Since I was expecting all these limits to be 1, my first reaction to the results, 1, 0, 1, 0, and 1 was to erroneously think that I had discovered a bug in *Macsyma*®.

3. Let $^\infty x=\lim_{n\to\infty}{}^nx$. One might like to study $\lim_{x\to 0^+}{}^\infty x$. Unfortunately, $^\infty x$ exists only when $x\in[e^{-e},e^{1/e}]$. See the references for this. Reference [2] has a superb bibliography.

4. The phenomenon described by THEOREM 1 persists if we replace each x in $x^{\cdot^{\cdot^x}}$ by a function that is asymptotic to a power of x. More explicitly, the following generalization of THEOREM 1 has much the same proof. Let a be any real number and let $\{g_n\}$ be a sequence of functions such that there are constants $c_n>0$ and b_n

such that

$$\lim_{x \to a^+} \frac{g_n(x)}{b_n(x-a)^{c_n}} = 1.$$

Define $\{f_n(x)\}$ recursively by $f_0(x) = g_0(x) = 1$, and, for $n \geq 1$, $f_n(x) = g_n(x)^{f_{n-1}(x)}$. Then, for $k = 0, 1, 2, \ldots$,

$$\begin{cases} \lim_{x \to a^+} f_{2k}(x) = 1 \\[2ex] \lim_{x \to a^+} f_{2k+1}(x) = 0. \end{cases}$$

REFERENCES

1. J. M. de Villiers and P. N. Robinson, The interval of convergence and limiting functions of a hyperpower sequence, *Amer. Math. Monthly*, 93(1986) 13–23.
2. R. Arthur Knoebel, Exponentials reiterated, *Amer. Math. Monthly*, 88(1981) 235–252.

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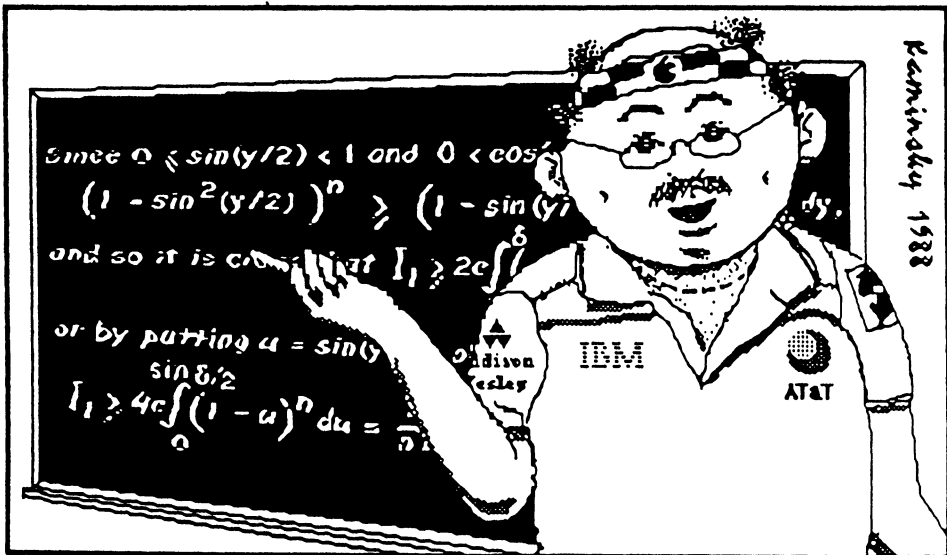
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