Comments

Partitioning of Variance and Contribution or Importance of a Variable: A Visit to a Graduate Seminar[1]

Recently, a friend of mine related that he had had an occasion to visit a graduate seminar on "Research Methods" at an unspecified university. The topic of discussion for this session of the seminar was "Partitioning of Variance and Contribution or Importance of a Variable." One of the enthusiastic graduate students had selected this topic for special study and he was presenting his findings and conclusions to the members of the seminar. The following is the report of what happened. This is, of course, a second-hand report.

"Ladies and gentlemen of the seminar," said the speaker, "I have surveyed the literature related to the Contribution of a Variable and I wish to present a summary of my findings."

"I discovered that there have been several important contributions to this area of investigation. However, some additional ideas seem needed so I not only will review some old ideas but indicate some new directions that will add even more meaning to the old ideas."

The Simple Contribution of a Variable

"The first concept I will review is the Simple Contribution of a Variable.

[1]Comment on an article by Charles E. Werts, "The Partitioning of Variance in School Effects Studies." *American Educational Research Journal* 5: 311-318; May 1968.

The Simple Components of the predictable variance can be calculated from equation (1).

$$(1) \quad R^2_{y.123\ldots c} = \sum_{i=1}^{c} (b_{yi}\, r_{yi}) = b_{y1}\, r_{y1} + b_{y2}\, r_{y2} + \ldots + b_{yc}\, r_{yc}$$

Where

$R^2_{y.123\ldots c}$ = The squared multiple correlation coefficient resulting from predicting y from predictors 1 through c. This value is equal to the proportion of the total variance of y 'accounted for' by all predictors.

b_{yi} = the standardized partial regression coefficient of variable y.
r_{yi} = the correlation coefficient between variable y and variable i.

Since $R^2_{y.123\ldots c}$, the total variance accounted for by all predictors, is computed as the sum of several terms, it has been the practice to 'extract meaning' from each term. Consider the general term

$$b_{yi}\, r_{yi}$$

Since the subscripts y and i appear in this term, I have discovered that the astute analyst is compelled to identify this term with variables y and i. Otherwise, he would be extremely naive."

"Therefore, the obvious and natural description of the term

$$b_{yi}\, r_{yi}$$

is (1) the 'contribution of variable i to prediction of y' or (2) 'the variance of y due to variable i' or (3) 'the proportion of total variance of y accounted for by predictor i."

"Some writers have obtained more meaning from the following definition of

Relative Contribution:

$$w_{yi} \text{ (relative contribution)} = (b_{yi}\, r_{yi}) \,/\, R^2_{y.123\ldots c}$$

Then it would be said that w_{yi} is the *'relative contribution'* or *'relative variance'* or *'relative weight'* due to variable i."

"These interpretations," said the speaker, "certainly make sense." "However, past writers have payed little attention to those terms that have negative values or those terms that have *values greater than one.*

"To further clarify the situation, I have introduced some new ideas and definitions.

The following four conditions are defined:

(1) If $0 \leqslant b_{yi}\, r_{yi} \leqslant 1.00$
The value $b_{yi}\, r_{yi}$ will be called the *real variance of y due to predictor i or the real contribution of predictor i.* The abbreviation will be RC (real contribution)

(2) If $1.00 < b_{yi}\, r_{yi}$
The value $b_{yi}\, r_{yi}$ will be called the *extra-spacial real variance* of y due to predictor i or the *extra-spacial real contribution* of predictor i. The abbreviation will be ESRC.

(3) If $-1.00 \leqslant b_{yi}\, r_{yi} < 0$
The value $b_{yi} r_{yi}$ will be called the *imaginary variance* of y due to predictor i or the *imaginary contribution* of predictor i. The abbreviation will be IC.

(4) If $b_{yi} r_{yi} < -1.00$
The value $b_{yi} r_{yi}$ will be called the *extra-spacial imaginary* variance of y due to predictor i or the *extra-spacial imaginary contribution* of predictor i. The abbreviation will be ESIC.

Similar definitions are appropriate for *Relative Contributions*

$$w_{yi} = (b_{yi} r_{yi}) \; / \; R^2_{y.123\,...\,c}$$

The expressions to be used with w_{yi} are

(1) *real relative contribution* (R^2C)

(2) *extra-spacial real relative* contribution (ESR^2C)

(3) *imaginary relative* contribution (IRC)

(4) *extra-spacial imaginary relative* contribution (ESIRC)

"Several examples will illustrate the elegance and utility of these definitions."

Example							Contribution		Relative Contribution	
No.	r_{y1}	r_{y2}	r_{12}	b_{y1}	b_{y2}	$R^2_{y.12}$	$(r_{y1} b_{y1})$	$(r_{y2} b_{y2})$	w_{y1}	w_{y2}
1	.400	.000	.707	.800	—.566	.320	.320 (RC)	.000 (RC)	1.000 (R^2C)	.000 (R^2C)
2	.200	.700	.800	—1.000	1.500	.850	—.200 (IC)	1.050 (ESRC)	—.235 (IRC)	1.235 (ESR^2C)
3	.100	.800	.500	—.400	1.000	.760	—.040 (IC)	.800 (RC)	—.053 (IRC)	1.053 (ESR^2C)

"In example one, *variable 2 makes no real contribution* (.000). Only the unsophisticated person would inquire about the meaning of *that* statement! The unsophisticated might say, surely it couldn't be that variable 2 makes "no difference" and can be eliminated from consideration. The $R^2_{y.1}$ is $(.400)^2 = .16$ using variable *1* alone. When variable *2* is added to variable *1*, $R^2_{y.12} = .320$ is *twice* as large as $R^2_{y.1} = .160$.

It seems that variable 2 *does make a difference* in some sense. But the unsophisticated should not proceed too far with such ideas."

"In example two, variable 1 makes *imaginary contribution* (IC) and variable two makes *extra-spacial real contribution* (ESRC). Research workers can see that these new interpretive definitions can be of great value in both pure and applied research. Meaningful and useful decisions can be made and actions taken as a result of these ideas.

In example three, variable 1 makes an *imaginary contribution* (IC) but variable 2 makes a *real contribution* (RC). That really *is* an exciting finding!"

The speaker then spoke with enthusiasm, "The comparison and interpretation of these three different situations,

RC and RC
IC and ESRC
IC and RC

can provide an exciting dissertation topic for some member of the class."

"So much for the Simple Contribution. I shall now consider the ideas of Joint Contribution."

The Joint Contribution Between Variables

"The discovery of the Simple Contribution was followed by the discovery of the Joint Contribution Between Variables." "Consider next the computing expression

$$R^2_{y.123\ldots c} = \sum_{i=1}^{c} \left(b_{yi} \left[\sum_{j=1}^{c} r_{ij}\, b_{yi} \right] \right)$$

where

r_{ij} = the correlation between variables i and j

Then

$$R^2_{y.123\ldots c} = \sum_{i=1}^{c} \sum_{j=1}^{c} (b_{yi}\, r_{ij}\, b_{yj})$$

$$(2)\quad R^2_{y.123\ldots c} = \sum_{i=1}^{c} (b_{yi})^2 + 2 \sum_{i=1}^{c-1} \left[\sum_{j=i+1}^{c} (b_{yi}\, r_{ij}\, b_{yj}) \right]$$

As in the Simple situation, since $R^2_{y.123\ldots c}$ is computed as the sum of several terms, it is only natural to extract meaning from each term.

First, consider the general term

$$(b_{yi})^2$$

Since only the subscripts y and i appear in the term, it is natural to refer to

$$(b_{yi})^2$$

as "the variance of y due to variable i independent of other variables" or "the contribution of variable i to the prediction of y independent of other variables."

Next observe the general term

$$2b_{yi}r_{ij}b_{yj}$$

Since this term involves subscripts y, i, and j, it is appropriate and meaningful to consider this term as the *joint* contribution. Therefore, the value of

$$2b_{yi}r_{ij}b_{yj}$$

is "the variance of y due to variables i and j *jointly,* or "the *joint* contribution of variables i and j to the prediction of y."

Since the causal relationship between variables i and j usually is unknown, the joint i-j variance is uninterpretable; i.e., we do not know whether it represents the influence of i, or j, or both. However, if both i and j are input variables, the joint i-j variance can be ascribed to input."

"The definitions given for increasing insight about the Simple Contribution can be used in the interpretation of Joint and Independent contributions.

An excellent example of the interpretation of *Joint Variance* can be found in *Partitioning of Variance in School Effects Studies* by Charles Werts:

In his example, Werts did not discuss the important meaning of the *negative* terms involved in the summation. For example, the *Joint Contribution* of Median Family Income (i) and Mean Reading Comprehension Score (j) to prediction of Percentage of Males who Enter College (y) is —.01325229936. (See Werts' article, p. 315.)

This important finding was not mentioned by Werts; but the new definitions that I have introduced above allow us to interpret this value as the *imaginary variance due jointly to Family Income and Reading Comprehension*. The *concept of imaginary variance* immediately suggests the introduction of complex numbers. The prospect of using complex numbers should excite those of you who are looking for a dissertation topic.

In example number one presented above

$$b_{y1}^2 = (.800)^2 = .64$$

$$b_{y2}^2 = (—.566)^2 = .32$$

$$2b_{y1}r_{12}b_{y2} = 2\ (.800)\ (.707)\ (—.566) = —.640$$

Then .64 is the *real independent variance* (or contribution, RIC) due to predictor *1* independent of predictor *2*, .32 is the *real independent variance* (or contribution, RIC) due to predictor *2* independent of predictor *1*.

But, —.640 is the *imaginary joint variance* (or contribution IJC) due to predictors *1* and *2*.

In example two above

$$b_{y1}^2 = (—1.00)^2 = 1.00$$

$$b_{y2}^2 = (1.5)^2 = 2.25$$

$$2b_{y1}r_{12}b_{y2} = 2\ (—1.00)\ (.800)\ (1.5) = —2.4$$

The value $b_{y1}^2 = 1.00$ is the *real contribution* of predictor *1 independent* of predictor *2* (RIC); 2.25 is the *extra-spacial real contribution* of predictor *2 independent* of predictor *1* (ESRIC). But now the full power of the new definition can be brought into play. The value —2.4 is the *extra-spacial imaginary joint contribution* of predictors *1* and *2* (ESIJC). The proper assignment of these meaningful names is essential for the sophisticated researcher.

The graduate student then says, "But further, more meaningful interpretations are needed. A more elegant meaningful concept of *Joint-Joint-Varibus-Personna* has been created and I will now unveil this important breakthrough."

The Joint-Joint-Varibus-Personna Contribution (J^2VPC)

With excitement never before seen in a graduate seminar, the speaker says, "Extensions of the above concepts were not easy; however, sudden insight lead me to the analysis I will describe.

Let

$$r_{ij} = \frac{1}{n} \sum_{k=1}^{n} z_{ki}\, z_{kj}$$

Where r_{ij} = correlation between predictors i and j
n = number of observations
z_{ki} = standardized score for person k on variable i
z_{kj} = standardized score for person k on variable j

Then

$$R^2_{y.123\,\ldots\,c} = \frac{1}{n} \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{m} (b_{yi}\, z_{ki}\, z_{kj}\, b_{yj})$$

And I examined the general term

$$\frac{1}{n} (b_{yi}\, z_{ki}\, z_{kj}\, b_{yj})$$

which contains the subscripts y, i, j, and k. A sudden insight struck me! I saw the subscripts and the idea came through. This term *must* be interpreted as the "variance of y due to variable i, variable j, and person k *jointly*" or "the *joint-joint contribution* of variable i, variable j, and person k to the prediction of y."

This elegant meaningful interpretation for

$$\frac{1}{n} (b_{yi}\, z_{ki}\, z_{kj}\, b_{yj})$$

is described as the *Joint-Joint Varibus-Personna Contribution* (J^2VPC)

The speaker then paused and said, "The potential of the concept of J^2VPC has not been realized and it is the development and application of J^2VPC that I have chosen for my dissertation topic. I hope that this new concept will provide new tools for more meaningful analyses in research."

"Before closing," said the speaker, "I would like to relate to you two laws that I have uncovered that may guide you in your future work.

Ward's First Law

If a *meaningful number* can be computed as the sum of several numbers, then each term of the sum must be *as meaningful as* or *more meaningful than the sum.*

Ward's Second Law

> If results from a meaningful analysis do not agree with expectations, then a more meaningful analysis must be performed.

The student then summarized by saying, "Meaningful interpretations can be supplied for the contributions of a variable to prediction. Occasionally, however, the assigned meaning may have little relationship to the problem that stimulated the analysis.

After a slight pause, the speaker said, "On second thought, it probably is more accurate to say that *most of the time* the assigned meaning has little relationship to the problem."

A moment of silence followed. The graduate student then looked at his professor's expression and then replied, "Thank you for your attention—and in closing *I think I'll switch my dissertation topic—I'd rather switch than fight!*

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