2 CALCULUS OF FINITE DIFFERENCES

2.1 Introduction.

The calculus of finite difference deals with the changes in the values of the dependent variable with the change of the independent variable. Finite difference method is successfully applied in numerical analysis in interpolation, numerical differentiation and integration etc. In this chapter, we introduce and discuss different types of difference operators, their relations, fundamental theorem of difference calculus and their applications.

2.2. Finite differences.

Let $y = f(x)$ be a real-valued function of x defined in an interval $[a,b]$ and its values are known for $(n+1)$ equally spacing points x_i $(i = 0, 1, 2, \cdots, n)$ such that $x_i = x_\circ + ih$ $(i = 0, 1, 2, \cdots, n)$ where $x_\circ = a, x_n = b$ and h is the *space length*. Then x_i $(i = 0, 1, 2, \cdots, n)$ are called *nodes* and the corresponding values y_i are termed as *entries*.

We now introduce the concept of various type differences in order to find the values of $f(x)$ or its derivative for some intermediate values of x in $[a,b]$.

2.3. Forward Differences.

The differences $y_1 - y_\circ, y_2 - y_1, y_3 - y_2, \cdots, y_n - y_{n-1}$ for the entries $y_\circ, y_1, y_2, \cdots, y_{n-1}, y_n$ are called *first forward differences* and are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \cdots, \Delta y_{n-1}$ respectively. Thus we have

$$\Delta y_i = y_{i+1} - y_i \quad (i = 0, 1, 2, \cdots, n-1) \qquad \ldots \qquad (1)$$

where Δ is called *forward difference operator*. In general forward difference operator is defined by

$$\Delta f(x) = f(x+h) - f(x). \qquad \ldots \qquad (2)$$

Similarly, the higher order forward differences are define as

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i \qquad \cdots \qquad (3)$$

$$\cdots \qquad \cdots$$

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

where $i = 0, 1, 2, \cdots, n-1$ and r $(1 \le r \le n)$ is a positive integer.

Now $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

Hence, for n^{th} order forward difference, we have

$$\Delta^n y_0 = y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2} + \ldots + (-1)^n y_0 \qquad \cdots \qquad (4)$$

We can calculate the above forward differences very easily with the help of the following tables, called forward difference table:

Table 1 : Forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

As an illustration consider the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	0			
		1		
3	1		–4	
		–3		22
5	–2		18	
		15		–35
7	13		–17	
		–2		
9	11			

From the table, we have

$$x_0 = 1, y_0 = 0, \Delta y_0 = 1, \Delta^2 y_0 = -4, \ \Delta^2 y_0 = 22$$
$$x_1 = 3, y_1 = 1, \Delta y_1 = -3, \Delta^2 y_1 = 18, \ \Delta^3 y_1 = -35$$
$$x_2 = 5, y_2 = -2, \Delta y_2 = 15, \Delta^2 y_2 = -17$$

and so on.

2.4. Some properties of Δ.

If a and b be any two constants, then

(i) $\Delta a = 0$

(ii) $\Delta\{af(x)\} = a\Delta f(x)$

(iii) $\Delta\{af(x) \pm bg(x)\} = a\Delta f(x) \pm b\Delta g(x)$

(iv) $\Delta[f(x)g(x)] = f(x)\Delta g(x) + \Delta f(x).\, g(x+h)$

$$= f(x+h)\Delta g(x) + \Delta f(x) g(x)$$

(v) $\Delta\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Proof. **(i)** Let $f(x) = a$

$\therefore \quad \Delta a = \Delta f(x) = f(x+h) - f(x) = a - a = 0$

(ii) $\Delta\{af(x)\} = af(x+h) - af(x)$

$$= a\{f(x+h) - f(x)\}$$

$$= a\ \Delta f(x).$$

(iii) $\Delta\{af(x) \pm bg(x)\}$

$$= \{af(x+h) \pm bg(x+h)\} - \{af(x) \pm bg(x)\}$$

$$= a\{f(x+h) - f(x)\} \pm b\{g(x+h) - g(x)\}$$

$$= a\Delta f(x) \pm b\Delta g(x)$$

(iv) Left as an exercise.

(v) Left as an exercise.

Ex.2. Find $\Delta^2\left(ax^2+bx+c\right)$

[*W.B.U.T.,M(CS)-301,2007*]

Solution. $\Delta\left(ax^2+bx+c\right)$

$$=\left\{a(x+h)^2+b(x+h)+c\right\}-\left(ax^2+bx+c\right)$$

$$=2axh+ah^2+bh$$

$$\therefore\ \Delta^2\left(ax^2+bx+c\right)$$

$$=\left\{2ah(x+h)+ah^2+bh\right\}-\left(2axh+ah^2+bh\right)$$

$$=2ah^2$$

Ex.3. Evaluate $\Delta^2 \cos 2x$

[*M.A.K.A.U.T.,M(CS)-401,2014, W.B.U.T.-2008*]

Solution. $\Delta \cos 2x = \cos 2(x+h) - \cos 2x$

$$=-2\sin(2x+h)\sin h$$

$$\therefore\ \Delta^2 \cos 2x = -2\sin h\ \ \Delta\ \sin(2x+h)$$

$$=-2\sin h\{\sin(2x+2h+h)-\sin(2x+h)\}$$

$$=-2\sin h.\,2\cos(2x+2h)\sin h$$

$$=-4\sin h\ \cos 2(x+h)$$

2.5. Fundamental theorem of difference calculus.

Theorem : The n^{th} order difference of a polynomial $P(x)$ of degree n is constant and its $(n+1)^{th}$ order difference varishes.

Proof. Let $P(x)=a_0x^n+a_1x^{n-1}+.....+a_{n-1}x+a_n$ where $a_0, a_1, ..., a_n$ are constant and $a_0 \neq 0$ be a polynomial of degree n.

Then since

$$\Delta x^n = (x+h)^n - x^n = \sum_{i=1}^{n} \binom{n}{i} x^{n-i} h^i$$

which is polynomial of degree $n-1$, we have

$$\Delta P(x) = a_0 \Delta x^n + a_1 \Delta x^{n-1} + \ldots + a_{n-1} \Delta x + \Delta a_n$$

$$= a_0 nh\, x^{n-1} + b_1 x^{n-2} + \ldots + b_{n-2} x + b_{n-1}, \text{ say}$$

Then $\Delta P(x)$ is a polynomial of degree $n-1$.

Similary,

$$\Delta^2 P(x) = a_0 nh\, \Delta x^{n-1} + b_1 \Delta x^{n-2} + \ldots\ldots + b_{n-2} \Delta x + \Delta b_{n-1}$$

$$= a_0 n(n-1) h^2 . x^{n-2} + c_1 x^{n-3} + \ldots\ldots + c_{n-3} x + c_{n-2}, \text{ say}$$

which is a polynomial of degree $n-2$.

Proceeding in this way, we have

$$\Delta^n P(x) = a_0 n(n-1) \ldots 2.1 h^n$$

$$= a_0 n! h^n, \text{ which is constant}$$

Hence $\Delta^{n+1} P(x) = 0$

Ex.4. Evaluate $\Delta^3 P(x)$ where $P(x) = 5x^3 - 6x + 11$, taking $h = 2$.

Solution. $\Delta^3 P(x) = \Delta^3 \{5x^3 - 6x + 11\}$

$$= 5\Delta^3 x^3 - 6\Delta^3 x + \Delta^3 11$$

$$= 5.3!\ 2^3 - 6 \times 0 + 0 = 240$$

2.6. Backward differences.

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots\ldots, y_n - y_{n-1}$ are called the ***first backward differences*** and denoted by $\nabla y_1, \nabla y_2, \ldots\ldots, \nabla y_n$ respectively.

Thus we have

$$\nabla y_i = y_i - y_{i-1}, \quad i = 1, 2, \ldots\ldots, n \qquad \ldots \quad (5)$$

where ∇ is called the *backward difference operator*. In general backward differece operator is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

Similarly, the higher order backward differences are defined as

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}$$
$$\nabla^3 y_i = \nabla^2 y_i - \nabla^2 y_{i-1}$$

and so on.

Now $\nabla y_1 = y_1 - y_0$

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Hence for n^{th} order backward difference, we have

$$\nabla^n y_n = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^n y_0 \qquad \cdots \qquad (6)$$

We can calculate the different order backward differences very quickly with the help of the following table, called *backward difference table* :

Table 2 : Back difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

Ex.5. Construct the backward difference table of $y = x^2 + 4$ for $x = 1, 3, 5, 7, 9$ and find the values of $\nabla^2 f(5)$, $\nabla^2 f(7)$ and $\nabla^3 f(9)$.

Solution. The backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1	5			
3	13	8		
5	29	16	8	
7	53	24	8	0
9	85	32	8	0

From the table, we have

$\nabla^2 f(5) = 8,\ \nabla^2 f(7) = 8,\ \nabla^3 f(9) = 0$

Ex.6. Show that $\Delta.\nabla \equiv \Delta - \nabla$

[*W.B.U.T., CS-312, 2004, 2007, M(CS)-301,2014, M(CS)-401,2014*]

Solution. We have

$$\Delta f(x) = f(x+h) - f(x)$$

and $\nabla f(x) = f(x) - f(x-h)$

$$\begin{aligned}
\therefore\ \Delta\cdot\nabla f(x) &= \Delta\left[f(x) - f(x-h)\right] \\
&= \Delta f(x) - \Delta f(x-h) \\
&= \Delta f(x) - \left[f(x) - f(x-h)\right] \\
&= \Delta f(x) - \nabla f(x) \\
&= (\Delta - \nabla) f(x)
\end{aligned}$$

$\therefore \Delta\cdot\nabla \equiv \Delta - \nabla$

2.7. Shift Operator

The *Shift operator* is denoted by E and is defind as

$Ef(x) = f(x+h)$, h being the spacing

$\therefore E^2 f(x) = Ef(x+h) = f(x+2h)$

$E^3 f(x) = Ef(x+2h) = f(x+3h)$

In this way, in general, we have

$$E^n f(x) = f(x+nh) \qquad \ldots (7)$$

The *inverse shift operator* E^{-1} is defined by

$$E^{-1} f(x) = f(x-h)$$

and in general, we have

$$E^{-n} f(x) = f(x-nh) \qquad \ldots (8)$$

Since $\Delta f(x) = f(x+h) - f(x)$, it follows that

$$Ef(x) = f(x+h) = \Delta f(x) + f(x) = (\Delta + 1) f(x)$$

so that

$$E \equiv \Delta + 1$$

$$\text{i.e. } \Delta \equiv E - 1 \qquad \ldots (9)$$

Again $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$= E^2 f(x) - 2Ef(x) + f(x)$$

$$= (E-1)^2 f(x)$$

Hence $\Delta^2 \equiv (E-1)^2$ ... (10)

In this way, in general, we have

$$\Delta^n \equiv (E-1)^n \quad \text{... (11)}$$

Also $E^{-1}f(x) = f(x-h) = f(x) - \nabla f(x)$

$$[\because \nabla f(x) = f(x) - f(x-h)]$$

$$= (1-\nabla)f(x)$$

$$\therefore E^{-1} \equiv 1 - D \quad \text{... (12)}$$

Newton-Gregory formula.

We have $f(x+nh) = E^n f(x)$

$$= (1+\Delta)^n f(x)$$

$$= \left\{1 + \binom{n}{1}\Delta + \binom{n}{2}\Delta^2 + \ldots + \binom{n}{n}\Delta^n\right\} f(x)$$

$$= f(x) + \binom{n}{1}\Delta f(x) + \binom{n}{2}\Delta^2 f(x) + \ldots + \binom{n}{n}\Delta^n f(x) = \sum_{i=0}^{n}\binom{n}{i}\Delta^i f(x)$$

Thus $f(x+nh) = \sum_{i=0}^{n}\binom{n}{i}\Delta^i f(x)$ ... (13)

is known as *Newton-Gregory formula.*

Ex.7. If $y(0) = -1$, $y(1) = 3$, $y(2) = 8$, $y(3) = 13$ find, $y(6)$

Solution. We construct the following difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	−1			
		4		
1	3		1	
		5		−1
2	8		0	
		5		
3	13			

3

From Newton-Gregory formula (13), we have

$$f(x+nh)=\sum_{i=0}^{n}\binom{n}{i}\Delta^i f(x)$$

$$\therefore\ f(6)=f(0+6\times 1)=\sum_{i=0}^{6}\binom{6}{i}\Delta^i f(0)$$

$$=\left\{1+\binom{6}{1}\Delta+\binom{6}{2}\Delta^2+\binom{6}{3}\Delta^3+\ldots\right\}f(0)$$

$$=f(0)+6\Delta f(0)+15\Delta^2 f(0)+20\Delta^3 f(0)+\ldots..$$

$$=-1+6\times 4+15\times 1+20(-1)$$

$$=18$$

$$\therefore\ y(6)=18$$

Ex.8. Prove that $E\equiv e^{hD}$

Solution. We have

$$E\,f(x)=f(x+h)$$

$$=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots..,\text{ using Taylor's expansion}$$

$$=f(x)+hDf(x)+\frac{h^2}{2!}D^2 f(x)+\ldots..$$

$$=\left(1+hD+\frac{h^2D^2}{2!}+\ldots.\right)f(x)$$

$$=e^{hD}f(x)$$

$$\therefore E\equiv e^{hD}$$

Note. From the above result, we have

$$1+\Delta\equiv e^{hD}$$

$$\therefore\ hD=\log(1+\Delta)$$

$$\therefore\ D\equiv\frac{1}{h}\left(\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3-\frac{1}{4}\Delta^4+\ldots.\right)$$

Ex.9. Show that $D = \frac{1}{h}\log\left(\frac{1}{1-\nabla}\right)$.

Solution. We know

$$E \equiv e^{hD}$$

$$\therefore\ E^{-1} \equiv e^{-hD}$$

$$\text{or,}\ 1-\nabla \equiv e^{-hD} \qquad \left[\because\ E^{-1} = 1-\nabla\right]$$

Taking logarithm both sides, we get

$$\log(1-\nabla) \equiv -hD$$

$$\therefore\ D \equiv -\frac{1}{h}\log(1-\nabla)$$

$$\equiv \frac{1}{h}\log(1-\nabla)^{-1}$$

$$\therefore\ D \equiv \frac{1}{h}\log\left(\frac{1}{1-\nabla}\right)$$

2.8. Central difference and Average Operator

The central difference operator is denoted by δ and is defind as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) \qquad \ldots (14)$$

We can write (14) as

$$\delta f(x) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)$$

$$\therefore\ \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}} \qquad \ldots (15)$$

The average operator μ is defind as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right] \qquad \ldots (16)$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) f(x)$$

$$\therefore\ \mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) \qquad \ldots (17)$$

Ex.10. Show that $\Delta - \nabla = \delta^2$

Solution. From (14), we have

$$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$\therefore \delta^2 \equiv E - 2 + E^{-1}$$

$$\equiv (1+\Delta) - 2 + (1-\nabla) \qquad \left[\because E \equiv 1+\Delta,\ E^{-1} \equiv 1-\nabla\right]$$

$$\equiv \Delta - \nabla$$

$$\therefore \Delta - \nabla \equiv \delta^2$$

Ex.11. Prove that $hD \equiv \sinh^{-1}(\mu\delta)$, where μ is the average operator and D is the differentiation operator.

Solution. We have

$$\mu\delta \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(e^{hD} - e^{-hD}\right) \quad \left[\because E \equiv e^{hD}\right]$$

$$= \sinh(hD)$$

This implies

$$hD = \sin h^{-1}(\mu\delta)$$

Ex.12. $\mu^2 = \frac{1}{4}\left(\delta^2 + 4\right)$

Solution. We know,

central difference operator

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

and average operator

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

$$\delta^2 = E + E^{-1} - 2$$

$$\therefore\ E + E^{-1} = \delta^2 + 2 \qquad \dots \quad (1)$$

Again,

$$\mu^2 = \frac{1}{4}\left(E + E^{-1} + 2\right)$$

$$\therefore \ \mu^2 = \frac{1}{4}\left(\delta^2 + 4\right), \quad \text{by (1)}$$

2.9. Evaluation of missing terms in given datas.

Suppose we are given n values of $y = f(x)$ out of equispaced value of nodes $x_0, x_1, x_2, \ldots, x_n$. Let the unknown value of y be k. Since n values of y are known, we can assume $y = f(x)$ to be a polynomial of degree $n-1$ in x. Then the n^{th} order difference of $f(x)$, i.e., $\Delta^n f(x) = 0$, from where we can determine the value of k.

Ex.13. Estimate the missing term in the following tables:

x :	0	1	2	3	4
$f(x)$:	1	3	9	–	81

Solution. Since we are given four values of y, so we take $y = f(x)$ to be a polynomial of degree 3 in x so that

$$\Delta^4 f(x) = 0$$

i.e., $(E-1)^4 f(x) = 0$

i.e., $E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$

$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

$$\left[\because E^n f(x) = f(x+nh), h = 1\right]$$

Putting $x = 0$, we get

$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$$

or, $81 - 4f(3) + 6 \times 9 - 4 \times 3 + 1 = 0$

$$\therefore \ f(3) = 31$$

2.10. Divided differences.

Let $y = f(x)$ be a real valued function of x defind in a finite interval $[a,b]$ and let $y_i = f(x_i)$, $(i = 0, 1, 2, \ldots n)$ be the functional values corresponding to the distinct nodes $x_i (i = 0, 1, 2, \ldots n)$ not necessarily equispaced.

We define divided differences for nodes $x_i (i = 0, 1, 2, \ldots n)$ of $f(x)$ as follows :

The first order divided difference of $f(x)$ for nodes x_0, x_1 is denoted by $f[x_0, x_1]$ and defined by

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{y_0 - y_1}{x_0 - x_1} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f[x_1, x_0]$$

Similarly, the first order divided difference for nodes x_1, x_2 is

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f[x_2, x_1]$$

In general, we have

$$f[x_i, x_{i+1}] = \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}}$$

$$= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$= f[x_{i+1}, x_i],\ i = 1, 2, \ldots n \qquad \ldots \quad (18)$$

The second order divided difference of $f(x)$ for x_0, x_1, x_2 is denoted by $f[x_0, x_1, x_2]$ and defined by

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$$

$$= f[x_2, x_1, x_0] \qquad \ldots \quad (19)$$

In general, the n^{th} order divided difference of $f(x)$ for $x_0, x_1, x_2, \cdots x_n$ is defined by

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_0, x_1, \ldots\ldots, x_{n-1}] - f[x_1, x_2, \ldots\ldots, x_n]}{x_0 - x_n}$$

$$= f[x_n, \ldots\ldots\ldots, x_1, x_0] \qquad \ldots \quad (20)$$

We can calculate the above different order divided differences very easily with the help of the following table, called *divided difference table.*

Table 3 : Divided difference table

x	$f(x)$	1st order	2nd order	3rd order	4th order
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$f(x_2)$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$f(x_3)$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$f(x_4)$				

Some properties of divided differences

(a) Divided differences are symmetric with respect to their arguments

Proof. This property is evident from (18), (19) and (20). Hence the arguments in a divided difference can be written in an arbitrary order. Thus we can write

$$f[x_0,x_1,x_2] = f[x_2,x_0,x_1] = f[x_0,x_2,x_1] \text{ etc.}$$

(b) Divided difference of a constant is zero.

Proof. Let $f(x) = \text{constant} = c$, say

Then $f[x_0, x_1] = \dfrac{f(x_0) - f(x_1)}{x_0 - x_1} = \dfrac{c - c}{x_0 - x_1} = 0$

(c) Divided difference of $k\,f(x)$ where k is constant, is k times the divided difference of $f(x)$.

Proof. Let $g(x) = kf(x)$

Then $g[x_0, x_1] = \dfrac{g(x_0) - g(x_1)}{x_0 - x_1} = \dfrac{kf(x_0) - kf(x_1)}{x_0 - x_1}$

$$= k\frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= kf[x_0, x_1].$$

(d) Divided difference of $f(x) \pm g(x)$ is the sum (or difference) of the corresponding divided difference of $f(x)$ and $g(x)$.

Proof. Let $F(x) = f(x) \pm g(x)$

Then $F[x_0, x_1] = \dfrac{F(x_0) - F(x_1)}{x_0 - x_1}$

$$= \frac{\{f(x_0) \pm g(x_0)\} - \{f(x_1) \pm g(x_1)\}}{x_0 - x_1}$$

$$= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \pm \frac{g(x_0) - g(x_1)}{x_0 - x_1}$$

$$= f[x_0, x_1] \pm g[x_0, x_1]$$

(e) k^{th} order divided difference of x^n is

(i) a polynomial of degree $n - k$ if $k < n$

(ii) a constant if $k = n$

(iii) zero if $k > n$.

Proof. Left as an exercise.

Ex.14. If $f(x) = \dfrac{1}{x}$ whose arguments are x_0, x_1, x_2, prove that

$$f[x_0, x_1, x_2] = \frac{1}{x_0\, x_1\, x_2}$$

Solution. We have

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{\frac{1}{x_0} - \frac{1}{x_1}}{x_0 - x_1} = -\frac{1}{x_0 x_1}$$

Similarly

$$f[x_1, x_2] = -\frac{1}{x_1 x_2}$$

$$\therefore\ f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}$$

$$= \frac{-\frac{1}{x_0 x_1} + \frac{1}{x_1 x_2}}{x_0 - x_2}$$

$$= \frac{1}{x_0 x_1 x_2}.$$

Ex.15. Construct the divided difference table for the following data :

x :	2	4	5	7	8
y :	3	43	138	778	1515

Solution. The divided difference table is

x	y	1st order	2nd order	3rd order	4th order
2	3				
		20			
4	43		25		
		95		10	
5	138		75		1
		320		16	
7	778		139		
		737			
8	1515				

2.11. Propagation of errors in a difference table

The given initial data are affected with round off error which lie between the limits $\pm\frac{1}{2}$ in the last significant figure. We now proceed to find how these initial round off errors affect the successive differeces in a difference table.

Let $y = f(x)$ be a real-valued function of x in $[a, b]$ and $y_i = f(x_i)$ be the exact value of $f(x)$ corresponding to the node x_i. If ε_i be the corresponding round off error, then the entered value of y_i is $y_i - \varepsilon_i$ and the difference table proceeds as follows :

Table 4 : Propagation of error in a difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	$y_0 - \varepsilon_0$			
		$\Delta y_0 - \varepsilon_1 + \varepsilon_0$		
x_1	$y_1 - \varepsilon_1$		$\Delta^2 y_0 - \varepsilon_2 + 2\varepsilon_1 - \varepsilon_0$	
		$\Delta y_1 - \varepsilon_2 + \varepsilon_1$		$\Delta^3 y_0 - \varepsilon_3 + 3\varepsilon_2 - 3\varepsilon_1 + \varepsilon_0$
x_2	$y_2 - \varepsilon_2$		$\Delta^2 y_1 - \varepsilon_3 + 2\varepsilon_2 - \varepsilon_1$	
		$\Delta y_2 - \varepsilon_3 + \varepsilon_2$		
x_3	$y_3 - \varepsilon_3$			

Now, $\quad \Delta \varepsilon_i = \varepsilon_{i+1} - \varepsilon_i$

$$\Delta^2 \varepsilon_i = \varepsilon_{i+2} - 2\varepsilon_{i+1} + \varepsilon_i$$

and in general,

$$\Delta^r \varepsilon_i = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k}, \quad i = 0, 1, 2, \ldots, n$$

so that $\Delta^r (y_i - \varepsilon_i) = \Delta^r y_i - \Delta^r \varepsilon_i = \Delta^r y_i - \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k}$

In the last significant figures, we have

$$\left| \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varepsilon_{i+r-k} \right| \le \sum_{k=0}^{r} \binom{r}{k} \left| \varepsilon_{i+r-k} \right|$$

$$\le \frac{1}{2} \sum_{k=0}^{r} \binom{r}{k} = 2^{r-1}$$

Thus the error in the computed value of $\Delta^r y_i$ varies between the limits $\pm 2^{r-1}$ in the last significant figures. In a difference table, this accumulation of errors in successive higher order differences is known as propagation of errors in a difference table :

Now if the step length h is small, then it is seen that the successive differences gradually decrease in significant figures so that the errors gradually increases in the successive differences. This is due to the propagation of errors. Proceeding through the table a stage is reached at which the computed differences are of the same order as their errors and so these differences are unrealistic. This stage is called *noise level.* For this reason, the computation of differences is to be stopped just before one gets differences with only one significant figure.

In the above, we have cosidered only round-off errors. But, sometimes we face another type of error, known as *accidental error.* If there is an accidental error ε, say, presumbly large, in one of the functional values, then this error is propagated through the difference table, as shown in the following table :

Table -5

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0 + \varepsilon$	
x_2	y_2		$\Delta^2 y_1 + \varepsilon$		$\Delta^4 y_0 - 4\varepsilon$
		$\Delta y_2 + \varepsilon$		$\Delta^3 y_1 - 3\varepsilon$	
x_3	$y_3 + \varepsilon$		$\Delta^2 y_2 - 2\varepsilon$		$\Delta^4 y_1 + 6\varepsilon$
		$\Delta y_3 - \varepsilon$		$\Delta^3 y_2 + 3\varepsilon$	
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^2 y_2 - 4\varepsilon$
		Δy_4		$\Delta^3 y_3 - \varepsilon$	
x_5	y_5		$\Delta^2 y_4$		
		Δy_5			
x_6	y_6				

From the above table it is clear that the errors in successive differences grow in size and hence the computed differences behave very irregularly after some stage. The presence of accidental error in the difference table may be diagonosed from this irregular behaviour. Here the propogation of errors are found to be confined within a cone, called diverging cone, from erroneous entry. Also it should be noted that the coefficient of ε in a column, say $\Delta^n y$ are binomial coefficients in the expansion of $(1-x)^n$, e.g. the coefficient of ε in $\Delta^3 y$ are ${}^3C_0, -{}^3C_1, {}^3C_2, -{}^3C_3$. i.e., $1, -3, 3, -1$.

So the algebraic sum of the errors in any difference column is zero and the maximum error in the differences is the same horizontal line which corresponds to the erroneous tabular value. All these observation help us to detect an error and make the necessary correction.

Ex.16 . Use the difference table to locate and correct the error in the following tabulated values :

x:	0	1	2	3	4
y:	5	8	10	20	29

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	5			
		3		
1	8		-1	
		2		9
2	10		8	
		10		-9
3	20			
		9	-1	
4	29			

From the above table, it is clear that the pattern of $\Delta^2 y$ are quite irregular and irregularity starts arround the horizontal line corresponding to the value of $y = 10$. Also it is observed that the third order differences follow the binomial coefficient pattern and the sum of the 3rd difference is zero.

Thus from the third difference pattern we get

$-3\varepsilon = 9$

$\therefore\ \varepsilon = -3$

So there exist an error -3 in the entry for $x = 2$ and the corresponding true value of y is

$10-(-3) = 13$.

ILLUSTRATIVE EXAMPLES

Ex.1. Show that $\Delta \log f(x) = \log\left[1+\dfrac{\Delta f(x)}{f(x)}\right]$

where Δ is the forward difference operator

[*W.B.U.T., CS-312, 2008,2009*]

Solution. We have $\Delta \log f(x) = \log f(x+h) - \log f(x)$, h being the space length

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log\left\{\frac{\Delta f(x) + f(x)}{f(x)}\right\} \qquad [\because \Delta f(x) = f(x+h) - f(x)]$$

$$= \log\left\{1+\frac{\Delta f(x)}{f(x)}\right\}$$

Ex.2. Evaluate

(i) $\Delta\left\{\dfrac{2^x}{(x+1)!}\right\}$, taking $h = 1$

(ii) $\Delta^2(2x+1)$, taking $h = 1$

(iii) $\Delta^2\left(\dfrac{1}{x^2+5x+6}\right)$, taking $h = 1$

Solution. **(i)** $\Delta\left\{\dfrac{2^x}{(x+1)!}\right\} = \dfrac{2^{x+1}}{(x+2)!} - \dfrac{2^x}{(x+1)!}$

$$= \frac{2^x}{(x+1)!}\left\{\frac{2}{x+2} - 1\right\} = \frac{x\,2^x}{(x+2)!}$$

(ii) $\Delta^2(2x+1)$

$$= \{2(x+1)+1\} - (2x+1), \text{ taking } h = 1$$

$$= 2$$

$$\therefore \Delta^2(2x+1) = \Delta(2) = 0$$

(iii) $\Delta\left(\dfrac{1}{x^2+5x+6}\right) = \Delta\left(\dfrac{1}{x+2} - \dfrac{1}{x+3}\right)$

$$= \left(\frac{1}{x+3} - \frac{1}{x+4}\right) - \left(\frac{1}{x+2} - \frac{1}{x+3}\right)$$

$$= \frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}$$

$$\therefore \Delta^2\left(\frac{1}{x^2+5x+6}\right) = \Delta\left(\frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}\right)$$

$$= \left(\frac{2}{x+4} - \frac{1}{x+5} - \frac{1}{x+3}\right) - \left(\frac{2}{x+3} - \frac{1}{x+4} - \frac{1}{x+2}\right)$$

$$= \frac{3}{x+4} - \frac{1}{x+5} - \frac{3}{x+3} + \frac{1}{x+2}$$

$$= -\frac{3}{(x+3)(x+4)} + \frac{3}{(x+2)(x+5)}$$

$$= \frac{6}{(x+2)(x+3)(x+4)(x+5)}$$

Ex.3. Show that $\Delta^n e^x = (e-1)^n e^x$

Solution. We have

$$\Delta e^x = e^{x+1} - e^x, \text{ taking } h = 1$$

$$= (e-1)e^x$$

$$\therefore \Delta^2 e^x = (e-1)\Delta e^x = (e-1)(e-1).e^x = (e-1)^2 e^x$$

So the result holds for $n = 1, 2$

Let the given result is true for $n = m$, i.e.,

$$\Delta^m e^n = (e-1)^m e^x$$

$$\therefore \Delta\left(\Delta^m e^n\right) = (e-1)^m \Delta e^x$$

or, $\Delta^{m+1} e^x = (e-1)^m.(e-1)e^x = (e-1)^{m+1} e^x.$

Hence the result is also true for $n = m+1$, provided it holds for $n = m$. But the result is valid for $n = 1, 2$. Therefore, by the method of induction, the result $\Delta^n e^x = (e-1)^n e^x$ is valid for all values of n.

Ex.4. Prove that $\left(\dfrac{\Delta^2}{E} e^x\right) \dfrac{Ee^x}{\Delta^2 e^x} = e^x$

Solution. We have

$$\left(\frac{\Delta^2}{E} e^x\right) \frac{Ee^x}{\Delta^2 e^x} = \left(\Delta^2 E^{-1} e^x\right) \frac{e^{x+h}}{\Delta^2 e^x}$$

$$= \left(\Delta^2 e^{x-h}\right) \frac{e^{x+h}}{\Delta^2 e^x} = e^{-h} \Delta^2 e^x \frac{e^x . e^h}{\Delta^2 e^x}$$

$$= e^{-h} . e^x . e^h$$

$= e^x$, h being the space length.

Ex.5. Show that $\Delta\left(\dfrac{f_m}{g_m}\right) = \dfrac{\Delta f_m . g_m - f_m \Delta g_m}{g_m g_{m+1}}$

Solution. $\Delta\left(\dfrac{f_m}{g_m}\right) = \dfrac{f_{m+1}}{g_{m+1}} - \dfrac{f_m}{g_m} = \dfrac{f_{m+1} g_m - f_m g_{m+1}}{g_m g_{m+1}}$

$$= \frac{(f_{m+1} - f_m) g_m - f_m (g_{m+1} - g_m)}{g_m g_{m+1}} = \frac{g_m \Delta f_m - f_m \Delta g_m}{g_m g_{m+1}}$$

Ex.6. If $f(x) = u(x)\, v(x)$, show that

$$f(x_0, x_1) = u(x_0) v(x_0, x_1) + u(x_0, x_1) v(x_1).$$

Solution. Here $f(x) = u(x) v(x)$

$$\therefore\ f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{u(x_0) v(x_0) - u(x_1) v(x_1)}{x_0 - x_1}$$

$$= \frac{u(x_0)\{v(x_0) - v(x_1)\} + v(x_1)\{u(x_0) - u(x_1)\}}{x_0 - x_1}$$

$$= u(x_0) \frac{v(x_0) - v(x_1)}{x_0 - x_1} + v(x_1) \frac{u(x_0) - u(x_1)}{x_0 - x_1}$$

$$= u(x_0) v(x_0, x_1) + v(x_1) u(x_0, x_1).$$

Ex.7. Prove that $\delta \equiv \Delta E^{-\frac{1}{2}}$ and hence prove that

$$E \equiv \left(\frac{\Delta}{\delta}\right)^2$$

Solution. We have

$$\Delta E^{-\frac{1}{2}} f(x)$$

$$= \Delta f\left(x - \frac{h}{2}\right), \text{ h being the spacing}$$

$$= f\left(x - \frac{h}{2} + h\right) - f\left(x - \frac{h}{2}\right)$$

$$= f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) = \delta f(x)$$

$$\therefore\ \Delta E^{-\frac{1}{2}} = \delta$$

$$\therefore\ E^{-\frac{1}{2}} \equiv \frac{\delta}{\Delta}$$

$$\therefore\ E \equiv \left(\frac{\Delta}{\delta}\right)^2$$

Ex.8. Evaluate $\left(\frac{\Delta^2}{E}\right)x^4$, spacing being one

Solution. $\left(\frac{\Delta^2}{E}\right)x^4 = \left\{\frac{(E-1)^2}{E}\right\}x^4$

$$= \left(\frac{E^2 - 2E + 1}{E}\right)x^4 = \left(E - 2 + E^{-1}\right)x^4$$

$$= Ex^4 - 2x^4 + E^{-1}x^4$$

$$= (x+1)^4 - 2x^4 + (x-1)^4$$

$$\left[\because\ h = 1,\ Ef(x) = f(x+h),\ E^{-1}f(x) = f(x-h)\right]$$

$$= 12x^2 + 2$$

Ex.9. Taking $h = 2$ what will be the value of

$$\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$$

Solution. Since $(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)$ is a polynomial of degree 10 with leading term abcd x^{10}, so

$$\Delta^{10}\left\{(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right\}$$

$$= \Delta^{10}\{abcdx^{10} + \text{other terms of degree} \le 9\}$$

$$= abcd\ \Delta^{10}(x^{10}) + 0 \ = abcd\ h^{10} \times 10!$$

$$= abcd \times 2^{10} \times 10!$$

Ex.10. If f_i is the value of $f(x)$ at $x = x_i$ where $x_i = x_0 + ih$, $(i = 1, 2, 3, \ldots)$ and $h > 0$, prove that

$$f_i = E^i f_0 = \sum_{j=0}^{i} \binom{i}{j} \Delta^j f_0 \quad (j < i)$$

Solution. We have

$$f_i = f(x_i) = f(x_0 + ih) = E^i f(x_0) = E^i f_0 = (1+\Delta)^i f_0$$

$$= \left\{1 + \binom{i}{1}\Delta + \binom{i}{2}\Delta^2 + \ldots + \binom{i}{i}\Delta^i\right\} f_0$$

$$= 1 + \binom{i}{1}\Delta f_0 + \binom{i}{2}\Delta^2 f_0 + \ldots + \binom{i}{i}\Delta^i f_0$$

$$= \sum_{j=0}^{i} \binom{i}{j} \Delta^j f_0$$

Ex.11. Find y_7, given $y_0 = 0$, $y_1 = 7$, $y_2 = 26$, $y_3 = 63$, $y_4 = 124$

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		
		61			
4	124				

Here $x_0 = 0, h = 1, n = 7$

$$\therefore y_7 = f(0+7.1) = y_0 + \binom{7}{1}\Delta y_0 + \binom{7}{2}\Delta^2 y_0 + \binom{7}{3}\Delta^3 y_0 + \ldots$$

$$= 0 + 7\times 7 + 21\times 12 + 35\times 6 + 35\times 0 = 511$$

Ex.12. By constructing a difference table, find the sixth term of the series 8, 12, 19, 29, 42, ...

[*W.B.U.T., CS-312, 2004*]

Solution. Let the sixth term of the series be p. Then we construct the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	8				
		4			
2	12		3		
		7		0	
3	19		3		0
		10		0	
4	29		3		$p-58$
		13		$p-58$	
5	42		$p-55$		
		$p-42$			
6	p				

Since 5 entries are given, $\Delta^4 y$ must be constant

$\therefore p - 58 = 0$

$\therefore p = 58$

$\therefore$ So the sixth term of the series is 58

Ex.13. Show that $\Delta^{n+1} f(x_0) \simeq h^{n+1} f^{n+1}(x_0)$ if h is very small

Solution. We have

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$$

$$\text{or, } \lim_{h\to 0} \frac{\Delta f(x_0)}{h} = f'(x_0)$$

$$\Rightarrow \Delta f(x_0) \simeq h f'(x_0)$$

Again, $\lim\limits_{h\to 0}\dfrac{f'(x_0+h)-f'(x_0)}{h}=f''(x_0)$

or, $\lim\limits_{h\to 0}\dfrac{\Delta f'(x_0)}{h}=f''(x_0)$

$\Rightarrow \Delta f'(x_0)\simeq hf''(x_0)\Rightarrow \Delta\left(hf'(x_0)\right)\simeq h^2 f''(x_0)$

$\Rightarrow \Delta\left(\Delta f(x_0)\right)\simeq h^2 f''(x_0)\Rightarrow \Delta^2 f(x_0)\simeq h^2 f''(x_0)$

Repeating the process, we have by induction

$$\Delta^{n+1}f(x_0)\simeq h^{n+1}f^{n+1}(x_0).$$

Ex.14. Given $u_0+u_8=1.9243,\ u_1+u_7=1.9590,$

$u_2+u_6=1.9823,\ u_3+u_5=1.9956$. Find u_4

Solution. As we are given 8 values of $u(x)$, so $u(x)$ is a polynomial of degree 7, so that

$\Delta^8 u(x)=0$

and hence in particular,

$\Delta^8 u_0=0$

i.e., $(E-1)^8 u_0=0$

i.e., $\left(E^8-8E^7+28E^6-56E^5+70E^4-56E^3+28E^2-8E+1\right)u_0=0$

i.e., $u_8-8u_7+28u_6-56u_5+70u_4-56u_3+28u_2-8u_1+u_0=0$

i.e., $(u_0+u_8)-8(u_1+u_7)+28(u_2+u_6)-56(u_3+u_5)+70u_4=0$

i.e., $1.9243-8\times 1.9590+28\times 1.9823-56\times 1.9956+70u_4=0$

i.e. $u_4=\dfrac{69\cdot 9969}{70}$

$\therefore u_4=0.999955$

Ex.15. Prove that $f(4)=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$

[*W.B.U.T., C.S.-312, 2008, M.A.K.A.U.T., M(CS)-401,2014*]

Solution. We have $\Delta f(3)=f(4)-f(3)$

$\therefore f(4)=f(3)+\Delta f(3)$

$=f(3)+\Delta\{f(2)+\Delta f(2)\}\quad [\because \Delta f(2)=f(3)-f(2)]$

$$= f(3) + \Delta f(2) + \Delta^2 f(2)$$

$$= f(3) + \Delta f(2) + \Delta^2\{f(1) + \Delta f(1)\} \quad [\because \Delta f(1) = f(2) - f(1)]$$

$$= f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$$

Thus $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$

Ex.16. Show that $\nabla y_{n+1} = h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\right] D y_n$ where D is the differential operator.

Solution. We have

$$\nabla y_{n+1} = y_{n+1} - y_n = (E-1)y_n = \left(e^{hD} - 1\right)y_n \qquad [\because E \equiv e^{hD}]$$

$$= \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\right)y_n$$

$$= h\left(1 + \frac{1}{2}hD + \frac{1}{6}h^2D^2 + \ldots\right)Dy_n$$

Also, $E \equiv e^{hD}$ gives

$e^{-hD} \equiv E^{-1} = 1 - \nabla$ so that

$-hD = \log_e(1-\nabla)$

$$= -\nabla - \frac{1}{2}\nabla^2 - \frac{1}{3}\nabla^3 \ldots$$

$$\therefore hD = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots$$

Hence
$$\nabla y_{n+1} = h\left[1 + \frac{1}{2}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right) + \frac{1}{6}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots\right)^2 + \ldots\right]Dy_n$$

$$= h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \ldots\right]Dy_n$$

Ex.17. Show that $\Delta^r y_k = \nabla^r y_{k+r}$

[W.B.U.T., C.S-312, 2006, M.A.K.A.U.T., M(CS)-401, 2013]

Solution. $\nabla^r y_{k+r}$

$$= \left(1-E^{-1}\right)^r y_{k+r} \qquad \left[\because \nabla \equiv 1-E^{-1}\right]$$

$$= \left(\frac{E-1}{E}\right)^r y_{k+r} = (E-1)^r \left(E^{-r} y_{k+r}\right)$$

$$= (E-1)^r y_{k+r-r} = (E-1)^r y_k$$

$$= \Delta^r y_k \qquad [\because \Delta \equiv E-1]$$

Ex.18. Show that if Δ operates on n, then

$$\Delta\binom{n}{x+1} = \binom{n}{x} \text{ and hence } \sum_{n=1}^{N}\binom{n}{x} = \binom{n+1}{x+1} - \binom{1}{x+1}$$

[*M.A.K.A.U.T., M(CS)-401,2013*]

Solution. $\Delta\binom{n}{x+1} = \binom{n+1}{x+1} - \binom{n}{x+1}$

$$= \frac{(n+1)!}{(n-x)!(x+1)!} - \frac{n!}{(n-x-1)!(x+1)!}$$

$$= \frac{n!}{(n-x-1)!(x+1)!}\left(\frac{n+1}{n-x} - 1\right)$$

$$= \frac{n!}{(n-x-1)!(x+1)!}\,\frac{x+1}{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} = \binom{n}{x}$$

$$\therefore \sum_{n=1}^{N}\binom{n}{x} = \sum_{n=1}^{N}\Delta\binom{n}{x+1}$$

$$= \left\{\binom{2}{x+1} - \binom{1}{x+1}\right\} + \left\{\binom{3}{x+1} - \binom{2}{x+1}\right\}$$

$$+ \ldots + \left\{\binom{N+1}{x+1} - \binom{N}{x+1}\right\}$$

$$= \binom{N+1}{x+1} - \binom{1}{x+1}$$

Ex.19. Show that $\Delta^m\left(\frac{1}{x}\right)=\frac{(-1)^m\, m!\,h^m}{x(x+h)(x+2h)\dots(x+mh)}$

[W.B.U.T., C.S-312, 2007, M(CS)-301,2014]

Solution. We have $\Delta\left(\frac{1}{x}\right)=\frac{1}{x+h}-\frac{1}{x}=\frac{-h}{x(x+h)}=\frac{(-1)1!\,h}{x(x+h)}$

$$\Delta^2\left(\frac{1}{x}\right)=-h\left[\frac{1}{(x+h)(x+2h)}-\frac{1}{x(x+h)}\right]$$

$$=\frac{2h^2}{x(x+h)(x+2h)}$$

$$=\frac{(-1)^2\, 2!\,h^2}{x(x+h)(x+2h)}.$$

Thus the result is true for $m=1,2$. We suppose that the result is valid for $m=r$

$$\therefore\ \Delta^r\left(\frac{1}{x}\right)=\frac{(-1)^r\, r!\,h^r}{x(x+h)(x+2h)\dots(x+rh)}$$

Then $\Delta^{r+1}\left(\frac{1}{x}\right)$

$$=(-1)^r r!h^r\left[\frac{1}{(x+h)(x+2h)\dots\left(x+\overline{r+1}\,h\right)}-\frac{1}{x(x+h)\dots(x+rh)}\right]$$

$$=\frac{(-1)^r r!h^r}{(x+h)(x+2h)\dots(x+rh)}\left[\frac{1}{x+\overline{r+1}\,h}-\frac{1}{x}\right]$$

$$=\frac{(-1)^{r+1}(r+1)!\,h^{r+1}}{x(x+h)(x+2h)\dots\left(x+\overline{r+1}\,h\right)}$$

Hence the given result holds for $m=r+1$, provided it holds for $m=r$. But the result is ture for $m=1,2$. Hence by induction the result holds for any value of m.

Ex.20. Show that

(a) $u_{n+x} = u_n + \binom{x}{1}\Delta u_{n-1} + \binom{x+1}{2}\Delta^2 u_{n-2} + \binom{x+2}{3}\Delta^3 u_{n-3} + \ldots$

(b) $\Delta^n u_{x-n} = u_x - \binom{n}{1}u_{x-1} + \binom{n}{2}u_{x-2} - \binom{n}{3}u_{x-3} + \ldots$

Solution. (a) $u_{n+x} = E^x u_n = \left(\frac{1}{E}\right)^{-x} u_n$

$$= \left(\frac{E-\Delta}{E}\right)^{-x} u_n \quad [\because 1+\Delta = E]$$

$$= \left(1 - \Delta E^{-1}\right)^{-x} u_n$$

$$= \left[1 + x\Delta E^{-1} + \frac{x(x+1)}{2!}\Delta^2 E^{-2} + \frac{x(x+1)(x+2)}{3!}\Delta^3 E^{-3} + \ldots\right] u_n$$

$$= u_n + \binom{x}{1}\Delta E^{-1} u_n + \binom{x+1}{2}\Delta^2 E^{-2} u_n + \binom{x+2}{3}\Delta^3 E^{-3} u_n + \ldots$$

$$= u_n + \binom{x}{1}\Delta u_{n-1} + \binom{x+1}{2}\Delta^2 u_{n-2} + \binom{x+2}{3}\Delta^3 u_{n-3} + \ldots$$

(b) $\Delta^n u_{x-n} = \Delta^n E^{-n} u_x = \left(\Delta E^{-1}\right)^n u_x$

$$= \left\{(E-1)E^{-1}\right\}^n u_x \qquad [\because E = 1+\Delta]$$

$$= \left(1 - E^{-1}\right)^n u_x$$

$$= \left[1 - \binom{n}{1}E^{-1} + \binom{n}{2}E^{-2} - \binom{n}{3}E^{-3} + \ldots\right] u_x$$

$$= u_x - \binom{n}{1}E^{-1}u_x + \binom{n}{2}E^{-2}u_x - \binom{n}{3}E^{-3}u_x + \ldots$$

$$= u_x - \binom{n}{1}u_{x-1} + \binom{n}{2}u_{x-2} - \binom{n}{3}u_{x-3} + \ldots$$

Ex.21. Prove that

$$U_x = U_{x-1} + \Delta U_{x-2} + \Delta^2 U_{x-3} + \ldots \Delta^{n-1} U_{x-n} + \Delta^n U_{x-n}$$

[*W.B.U.T., C.S-312, 2006*]

Solution. $U_x - \Delta^n U_{x-n} = U_x - \Delta^n E^{-n} U_x$

$$= \left(1 - \Delta^n E^{-n}\right) U_x$$

$$= \frac{E^n - \Delta^n}{E^n} U_x$$

$$= \frac{1}{E^n}\left(\frac{E^n - \Delta^n}{E - \Delta}\right) U_n \qquad [\because E \equiv 1 + \Delta \Rightarrow E - \Delta \equiv 1]$$

$$= E^{-n}\left[E^{n-1} + \Delta E^{n-2} + \Delta^2 E^{n-3} + \ldots + \Delta^{n-1}\right] U_x$$

$$= \left[E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \ldots + \Delta^{n-1} E^{-n}\right] U_x$$

$$= U_{x-1} + \Delta U_{x-2} + \Delta^2 U_{x-3} + \ldots + \Delta^{n-1} U_{x-n}$$

$$\therefore\ U_x = U_{x-1} + \Delta U_{x-2} + \Delta^2 U_{x-3} + \ldots + \Delta^{n-1} U_{x-n} + \Delta^n U_{x-n}$$

Ex.22. Prove that $f[x_0, x_1, x_2] = 1$ if $f(x) = x^2$ where x_0, x_1, x_2 are distinct.

Solution. We have $f(x) = x^2$

$$\therefore f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{x_0^2 - x_1^2}{x_0 - x_1}$$

$$= x_0 + x_1$$

Similarly $f[x_1, x_2] = x_1 + x_2$

$$\therefore f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2} = \frac{(x_0 + x_1) - (x_1 + x_2)}{x_0 - x_2}$$

$$= \frac{x_0 - x_2}{x_0 - x_2} = 1$$

Ex.23. If $f(x)=\dfrac{1}{x^2}$ whose arguments are a, b, c, d in this order, prove that

$$f[a, b, c, d]=-\frac{abc+bcd+acd+abd}{a^2b^2c^2d^2}$$

Solution. Here $f(x)=\dfrac{1}{x^2}$

$$\therefore f[a, b]=\frac{f(a)-f(b)}{a-b}=\frac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{a-b}=-\frac{a+b}{a^2b^2}$$

Similarly

$$f[b, c]=-\frac{b+c}{b^2c^2}$$

$$\therefore f[a, b, c]=\frac{f[a, b]-f[b, c]}{a-c}$$

$$=\frac{-\dfrac{a+b}{a^2b^2}+\dfrac{b+c}{b^2c^2}}{a-c}$$

$$=\frac{ab+bc+ca}{a^2b^2c^2}$$

Similarly $f[b, c, d]=\dfrac{bc+cd+bd}{b^2c^2d^2}$

Thus $f[a, b, c, d]$

$$=\frac{f[a, b, c]-f[b, c, d]}{a-d}$$

$$=\frac{\dfrac{ab+bc+ca}{a^2b^2c^2}-\dfrac{bc+cd+bd}{b^2c^2d^2}}{a-d}$$

$$=-\frac{abc+bcd+acd+abd}{a^2b^2c^2d^2}$$

Ex.24 Find the missing value in the following table :

x	:	2	4	6	8	10
y	:	5.6	8.6	13.9	–	35.6

Solution. Since we are given four values of y, so we take $y = f(x)$ to be a polynomial of degree 3 in x so that

$\Delta^4 f(x) = 0$

i.e., $(E-1)^4 f(x) = 0$

i.e., $E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$

i.e., $f(x+8) - 4f(x+6) + 6f(x+4) - 4f(x+2) + f(x) = 0$

$$\left[\because\ E^n f(x) = f(x+nh),\ h = 2\right]$$

Putting $x = 2$, we get

$f(10) - 4f(8) + 6f(6) - f(4) + f(2) = 0$

or, $35.6 - 4f(8) + 6 \times 13.9 - 4 \times 8.6 + 5.6 = 0$

$\therefore\ f(8) = 22.55$

Ex.25. Find the missing term in the following table:

x	:0	1	2	3	4	5
y	:0	–	8	15	–	35

[W.B.U.T., C.S-312, 2007, M(CS)-301,2015]

Solution. Since we are given four values, therefore, we take $f(x)$ to be a polynomial of degree 3 in x so that

$\Delta^4 f(x) = 0$ for all values of x

$\therefore (E-1)^4 f(x) = 0$

i.e., $E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$

i.e., $f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$ ... (1)

Putting $x = 0$, we get

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

or, $f(4) - 4 \times 15 + 6 \times 8 - 4f(1) + 0 = 0$

or, $f(4) - 4f(1) = 12$ ... (2)

Again putting $x = 1$, we get

$$f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0$$

or, $35 - 4f(4) + 6 \times 15 - 4 \times 8 + f(1) = 0$

or, $4f(4) - f(1) = 93$ ... (3)

Solving (2) and (3) we get $f(1) = 3$, $f(4) = 24$.

Ex.26. Find the missing term :

x	:1	2	3	4	5	6	7
$f(x)$:	2	4	8	–	32	64	128

[W.B.U.T., M(CS)-301, 2006]

Solution. Since we are given six values of y, we take $y = f(x)$ to be a polynomial of degree 5 in x so that

$$\Delta^6 f(x) = 0$$

i.e., $(E-1)^6 f(x) = 0$

i.e., $\left(E^6 - {}^6C_1E^5 + {}^6C_2E^4 - {}^6C_3E^3 + {}^6C_4E^2 - {}^6C_5E + 1\right)f(x) = 0$

i.e., $E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6Ef(x) + f(x) = 0$

$$f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0$$

$[\because h = 1 \text{ and } E^n f(x) = f(x+nh)]$

Putting $x = 1$, we get

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

or, $128 - 6 \times 64 + 15 \times 32 - 20f(4) + 15 \times 8 - 6 \times 4 + 2 = 0$

or, $20f(4) = 322$

$f(4) = 16.1$

Ex.27. If the 3rd order differences of $f(x)$ be constant and $f(-1)=-1$, $f(0)=0$, $f(1)=1$, $f(2)=8$ and $f(3)=27$ find $f(4)$ using difference table.

Solution. Let $f(4)=a$

We now construct the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
–1	–1			
		1		
0	0		0	
		1		6
1	1		6	
		7		6
2	8		12	
		19		a–58
3	27		a–46	
		a–27		
4	a			

Since 3rd order difference of $f(x)$ is constants, so we must have

$a-58=6$

$\therefore\ a=64$

$\therefore\ f(4)=64$

Ex.28. If $f(x)$ is a polynomial of degree 3 and $f(0)=-1$; $f(1)=5$ $f(2)=13$, $f(3)=36$, $f(4)=69$ where $f(2)$ is not correct, find the error in $f(2)$.

Solution. Let the correct value of $f(2)$ be $13+\varepsilon$

Then we construct the following difference table:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	–1			
		6		
1	5		$2+\varepsilon$	
		$8+\varepsilon$		$13-3\varepsilon$
2	$13+\varepsilon$		$15-2\varepsilon$	
		$23-\varepsilon$		$-5+3\varepsilon$
3	36		$10+\varepsilon$	
		33		
4	69			

Since $f(x)$ is a polynomial of degree 3, so $\Delta^3 f(x)$ must be constant

$\therefore\ 13-3\varepsilon=-5+3\varepsilon$

i.e. $6\varepsilon=18$

$\therefore\ \varepsilon=3$

So the error in $f(2)$ is 3 and correct value of $f(2)$ is 13+3 = 16

Exercise

I. Short Answer Questions

1. Prove that

(i) $(1+\Delta)(1-\nabla)\equiv 1$

(ii) $\Delta+\nabla\equiv\dfrac{\Delta}{\nabla}-\dfrac{\nabla}{\Delta}$

(iii) $E\cdot\Delta=\Delta\cdot E$ [*W.B.U.T., CS-312, 2013*]

(iv) $\delta\equiv E^{1/2}\nabla$ [*M.A.K.A.U.T., M(CS)-301, 2014*]

(v) $2\mu\delta\equiv\Delta+\nabla$

(vi) $1-e^{-hD}\equiv\nabla$

(vii) $\Delta.\nabla=\Delta-\nabla\equiv\delta^2$

2. Prove that

$$D\equiv\frac{1}{h}\log(1+\Delta)\equiv-\frac{1}{h}\log(1-\nabla)$$

3. Show that

$$D\equiv\frac{1}{h}\left[\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\ldots\right]$$

4. Taking $h=1$, find $(\Delta+\nabla)^2$ of the function x^2+x

5. Find $\Delta\left(\dfrac{x}{x^2-1}\right)$, taking $h=1$

6. Prove the following :

(i) $\left(\dfrac{\Delta^2}{E}\right)x^3=6xh^2$

(ii) $\Delta^2 \cos 2x = -4\sin^2 h \cos 2(x+h)$ [W.B.U.T., CS-312, 2008]

(iii) $\Delta^2(x^2+2x+5)=2$

(iv) $\Delta(x+\cos x)=\pi-2\cos x$, taking $h=\pi$

(v) $\Delta \tan^{-1}\left(\frac{n-1}{h}\right)=\tan^{-1}\frac{1}{2n^2}$

7. Evaluate :

(i) $\Delta^2(ax^2+bx+c)$ [W.B.U.T., CS-312, 2007]

(ii) $\Delta^2(e^{3x+5})$ (iii) $(\Delta-\nabla)x^2$

(iv) $\Delta\left(\frac{2^x}{x!}\right)$ (v) $\Delta\left(\frac{x}{x^2+7x+12}\right)$

8. Evaluate

(i) $\Delta^3\{(1-x)(1-2x)(1-3x)\}$

(ii) $\Delta^{10}\{(1-x)(1-x^2)(1-3x^3)(1-4x^4)\}$

9. Show that $\frac{\Delta^2 x^3}{Ex^3}=\frac{6}{(1+x)^2}$, taking $h=1$

10. Given $f(0)=580$, $f(1)=556$, $f(2)=520$ and $f(4)=385$, find $f(3)$.

11. If $y_1=1$, $y_2=3$, $y_3=7$, $y_4=13$ and $y_5=21$, find y_0

12. If $f(x)$ is a polynomial of degree 2 and $f(1)=3$, $f(2)=7$, $f(3)=13$, $f(4)=21$, find $f(5)$ using difference table

Answers

4. 8

7. (i) $2ah^2$ (ii) $(e^3-1)^2 e^{3x+5}$ (iii) $2h^2$

(iv) $\frac{2^x(1-x)}{(x+1)!}$ (v) $\frac{4}{x+5}-\frac{7}{x+4}+\frac{3}{x+3}$

8. (i) 36 (ii) $24\times 10!$

11. 1 10. 465

II. Long Answer Questions

1. Evaluate

(i) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$ [W.B.U.T.,CS-312, 2009]

(ii) $\Delta\left(\dfrac{x^2}{\sin^2 2x}\right)$, taking $h=1$

2. Show that

(i) $\Delta\left\{\dfrac{1}{f(x)}\right\} = -\dfrac{\Delta f(x)}{f(x)f(x+1)}$

(ii) $\Delta^n \alpha^{an+b} = \left(\alpha^{ah}-1\right)\alpha^{ax+b}$

3. If D stands for the differential operator $\dfrac{d}{dx}$, prove that

(i) $D \equiv \dfrac{1}{h}\left(\Delta - \dfrac{1}{2}\Delta^2 + \dfrac{1}{3}\Delta^3 - \ldots\right)$

(ii) $D \equiv \dfrac{1}{h}\left(\nabla + \dfrac{1}{2}\nabla^2 + \dfrac{1}{3}\nabla^3 + \ldots\right)$

4. If $y(0)=1$, $y(1)=4$, $y(2)=10$, $y(3)=22$; find $y(5)$.

5. Show that $f(E)e^x = e^x f(e)$ where $f(e)$ is a polynomial in E taking units as the interval of differencing

6. Find the missing term/terms in following tables :

(i)

x :	1	2	3	4	5
y :	7	–	27	40	55

(ii)

x :	1	2	3	4	5	6	7
y :	2	4	8	–	32	64	128

(iii)

x :	0	5	10	15	20	25
y :	6	10	–	17	–	31

[W.B.U.T., M(CS)-401, 2013]

7. Given that $y_0 + y_8 = 80$, $y_1 + y_7 = 10$, $y_2 + y_6 = 5$, $y_3 + y_5 = 10$, find y_4.

8. Prove for equally speced interpolating point

$$x_i = x_0 + ih \quad (h > 0, i = 0, 1, 2, \ldots n)$$

$$\Delta^k y_0 = \sum_{i=0}^{k} (-1)^i \binom{k}{i} y_{k-i}$$

9. Show that

$$\binom{n+1}{1} u_0 + \binom{n+1}{2} \Delta u_0 + \binom{n+1}{3} \Delta^2 u_0 + \ldots + \Delta^n u_0 = \sum_{i=0}^{n} u_i$$

10. Find the first term of the series whose second and subsequent terms are

15, 10, 7, 6, 7, 10, ...

11. Calculate the n^{th} divided difference of $\frac{1}{x}$ based on the points $x_0, x_1, \ldots, x_n$

12. Prove that [M.A.K.A.U.T., M(CS)-401, 2015]

(i) $u_0 + \frac{x}{1!} u_1 + \frac{x^2}{2!} u_2 + \ldots$

$$= e^x \left[u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \ldots \right]$$

(ii) $u_0 - u_1 + u_2 - \ldots = \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 - \frac{1}{16} \Delta^3 u_0 + \ldots$

13. If $f(x) = \frac{1}{x}$ whose arguments are a, b, c, d in this order, prove that $f[a, b, c, d] = -\frac{1}{abcd}$

14. If $f(x) = \sin x$, find the value of the divided difference

$$f\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$$

15. $y = f(x)$ is a polynomial of degree 5 with $y_0 = 0$, $y_1 = 3$, $y_2 = 14$, $y_3 = 45$, $y_4 = 84$, $y_5 = 155$, $y_6 = 258$. It is found that. there is an error in the value of y_3. Find the correct value of y_3

[W.B.U.T., CS-312, 2004, M(CS)-401, 2012]

16. If y is a polynomial of degree 3 and the values are as follows. Locate and correct the wrong value of y.

x :	0	1	2	3	4	5	6	7
y :	25	21	18	18	27	45	76	123

Answers

1. (i) $\dfrac{4}{(x+2)(x+3)(x+4)} + \dfrac{6}{(x+3)(x+4)(x+5)}$

(ii) $\dfrac{(2x+1)\sin 2x - 2x^2 \sin 1 \cos(2x+1)}{\sin 2(x+1) \sin 2x}$

4. 76 6. (i) 16 (iii) 13.25, 22.50 7. 6

10. 22 11. $\dfrac{(-1)^n}{x_0 x_1 \dots x_n}$ 16. error $= -1$, correct value of $y(3) = 19$

III. Multiple Choice Questions

1. $\Delta^3(y_0)$ may be expressed as which of the following terms?

(a) $y_3 - 3y_2 + 3y_1 - y_0$ (b) $y_2 - 2y_1 + y_0$

(c) $y_3 + 3y_2 + 3y_1 + y_0$ (d) none of these

[W.B.U.T., CS-312,2008, M(CS)-401,2016]

2. $\Delta^2(e^x)$ (taking $h = 1$) is equal to

(a) $(e-1)e^x$ (b) $(e-1)^2 e^{2x}$ (c) $(e-1)^2 e^x$ (d) e^x

3. The value of $\Delta^2(ax^2 + bx + c)$ is

(a) ah^2 (b) $2a$ (c) a (d) $2ah^2$

4. Taking $h = \pi$, $\Delta(x + \cos x)$ is equal to

(a) $\pi + 2\cos x$ (b) $x - \sin x$ (c) $\pi - 2\cos x$ (d) $1 - \sin x$

5. If the interval of differencing is unity and $f(x) = ax^2, a$ is a constant which of the following choices is wrong ?

(a) $\Delta f(x) = a(2x+1)$ (b) $\Delta^2 f(x) = 2a$

(c) $\Delta^3 f(x) = 2$ (d) $\Delta^4 f(x) = 0$

[W.B.U.T., CS-312, 2009, 2010, M(CS)-301, 2014, M(CS)-401,2014]

6. $\Delta(ab^{cx})$ is equal to (taking $h = 1$)

(a) $(b^{cx} - 1)ab^{cx}$ (b) ab^{cx}

(c) ab^{cx-h} (d) $(b^c - 1)^2 ab^{cx}$

7. The value of $\frac{\Delta^2}{E}(x^3)$ is

(a) x (b) $6x$ (c) $3x$ (d) x^2

8. Which of the following is true ?

(a) $\Delta^n x^n = (n+1)!$ (b) $\Delta^n x^n = n!$

(c) $\Delta^n x^n = 0$ (d) $\Delta^n x^n = n$

[W.B.U.T., CS-312, 2009]

9. If $f(x) = \frac{1}{x^2}$, then the divided differnce $f(a,b)$ is

(a) $\frac{a+b}{(ab)^2}$ (b) $-\frac{a+b}{(ab)^2}$ (c) $\frac{1}{a^2 - b^2}$ (d) $\frac{1}{a^2} - \frac{1}{b^2}$

[W.B.U.T., CS-312, 2009, 2010, CS-401, 2013]

10. $\Delta^2(ax^2 + bx) =$

(a) $2a$ (b) a (c) $a+b$ (d) $a-b$

11. $\frac{\Delta^2 x^3}{Ex^3}$ is equal to

(a) $\frac{3}{(1+x)^2}$ (b) $\frac{6}{(1+x)^2}$ (c) $\frac{1}{(1+x)^2}$ (d) $\frac{5}{(1+x)^2}$

12. The value of $\Delta^2\left(\frac{1}{x-1}\right)$ is

(a) $\frac{1}{x(x^2-1)}$ (b) $\frac{2}{x(x^2-1)}$ (c) $\frac{1}{x}-\frac{1}{x-1}$ (d) none

13. $(\Delta-\nabla)x^2$ is equal to (the notations have their usual meanings)

(a) h^2 (b) $-2h^2$

(c) $2h^2$ (d) none of these

[W.B.U.T., CS-312, 2009, M(CS)-301, 2015]

14. Which of the following is true ?

(a) $E=1-\Delta$ (b) $E=1+\Delta$ (c) $\Delta=1+E$ (d) $E=1/\Delta$

[W.B.U.T., CS-312, 2009, M(CS)-401, 2016, 2014]

15. $E\nabla \equiv$

(a) E (b) E^2 (c) Δ (d) Δ^{-1}

16. The $(n+1)$th order forward difference of the nth degree polynomial is

(a) $n!$ (b) $(n+1)!$

(c) 0 (d) none of these

[W.B.U.T., CS-312, 2008]

17. If $f(x)$ be a polynomial of degree n, then —— order difference is a constant.

(a) nth (b) $(n-1)$th (c) $(n+1)$th (d) none

[W.B.U.T., CS-312, 2003]

18. Taking $h=1$, the value of $\Delta^3[(1-x)(1-2x)(1-3x)]$ is

(a) 36 (b) 37 (c) −35 (d) −36

19. Which of the following is not true (the notations have their usual meanings) ?

(a) $\Delta=E-1$ (b) $\Delta.\nabla=\Delta-\nabla$ (c) $\frac{\Delta}{\nabla}=\Delta+\nabla$ (d) $\nabla=1-E^{-1}$

[W.B.U.T., CS-312, 2008, M(CS)-301, 2014]

20. If $f(3) = a + \Delta f(1) + \Delta^2 f(1)$ then $a =$

(a) $f(0)$ (b) $f(1)$ (c) $f(2)$ (d) $f(3)$

[M.A.K.A.U.T., M(CS)-401, 2014]

21. $\Delta\left(\frac{1}{x}\right)$ is equal to

(a) $\frac{1}{x(x+h)}$ (b) $\frac{1}{x+h}$ (c) $-\frac{h}{x+h}$ (d) $-\frac{h}{x(x+h)}$

22. Which relations are true?

(a) $E = 1 + \Delta,\ \Delta\nabla = \Delta - \nabla$

(b) $E = 1 - \Delta,\ \Delta\nabla = \Delta + \nabla$

(c) $E = 1 - \Delta,\ \Delta\nabla = \Delta - \nabla$

(d) $E = 1 + \Delta,\ \Delta\nabla = \Delta + \nabla$ [M.A.K.A.U.T., M(CS)-301, 2015]

23. The value of $(1+\Delta)(1-\nabla)$ is

(a) 0 (b) 1 (c) 2 (d) 3

[M.A.K.A.U.T., M(CS)-301, 2015]

24. If $f(x) = \frac{1}{x}$, the divided difference $[a,b,c]$ is

(a) $\frac{1}{a+b+c}$ (b) $\frac{1}{abc}$ (c) $\frac{1}{a^2+b^2}$ (d) $a+b+c$

[M.A.K.A.U.T., M(CS)-401, 2015]

25. If the interval of differencing is unity and $f(x) = ax^2$ (a is contant), which one of the following choices is wrong?

(a) $\Delta f(x) = a(2x+1)$ (b) $\Delta^2 f(x) = 2a$

(c) $\Delta^3 f(x) = 2$ (d) $\Delta^4 f(x) = 0$

[M.A.K.A.U.T., M(CS)-401, 2013]

Answers

1.a	2.c	3.d	4.c	5.c	6.d	7.b	8.b	9.b	10.a
11.b	12.b	13.c	14.b	15.c	16.c	17.a	18.d	19.c	20.c
21.d	22.a	23.b	24.b	25.c					