7 Numerical Solution of Ordinary Differential Equation

7.1 Introduction:

Differential equation are involved in many problems in Engineering and Science. In this chapter, we discuss various numerical methods for solving ordinary differential equations. Our aim is to study the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y_0 = y(x_0) \qquad \dots \qquad (1)$$

in which $f(x, y)$ is a continuous function of x and y in some domain D of the xy-plane and (x_0, y_0) is a given point in D. The condition $y_0 = f(x_0)$ is known as the initial condition. Sufficient conditions for the existence and uniqueness of the solution of the equation (1) are the well known *Lipschitz conditions* given by

(i) $f(x, y)$ is defined and continuous in D, the region containing (x_0, y_0)

(ii) there exists a constant L such that

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2| \qquad \dots \qquad (2)$$

for all $(x, y_1), (x, y_2) \in D$.

We now proceed to consider numerical techniques for solving (1) at a sequence of points $x_i = x_0 + ih$, called the *mesh points*, h being the step length. Let y_i be the approximation to the exact solution $y(x_i)$ of (1). A continuous approximation to y is then obtained by interpolating the data points (x_i, y_i).

7.2. Euler's method.

We shall now describe a method, known as Euler's method, which gives the solution in the form of a set of tabulated values. In single step method, we determine a function $\phi(x, y; h)$ of x, y and h (the step length) depending on $f(x, y)$ and its derivatives such that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0(h^{p+1}), \qquad \dots \qquad (3)$$

where p is a positive integer, called the order of the method.

A general single-step method of order p can be obtained by expanding $y(x+h)$ by Taylor's theorem as follows:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \dots + \frac{h^p}{p!}y^{(p)}(x)$$
$$+ \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x+\theta h),\ 0<\theta<1 \quad \dots \quad (4)$$

When $p = 1$, we have, from (4)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x+\theta h),\ 0<\theta<1 \quad \dots \quad (5)$$

so that $\phi(x, y; h) = y'(x) = f(x, y)$ ... (6)

Let $y_n = y(x_n)$ and $y_{n+1} = y(x_{n+1}) = y(x_n + h), (n = 1, 2, \dots)$

Then neglecting the last term in (5) and putting $x = x_n$, we have

$$y_{n+1} = y_n + hy'(x_n)$$
$$= y_n + h\,f(x_n, y_n), \text{ by (6)}$$

Thus we get

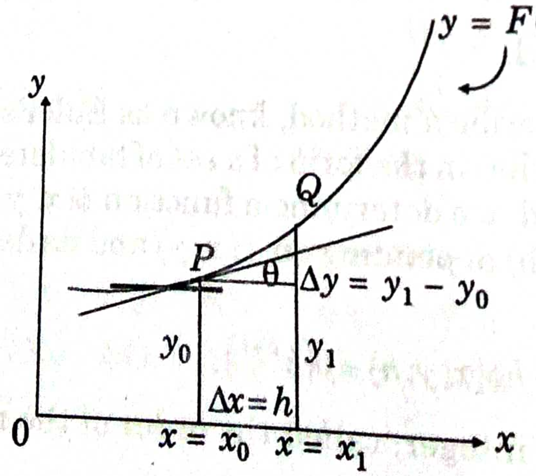
$$y_{n+1} = y_n + h\,f(x_n, y_n),\ n = 0, 1, 2, \dots \quad \dots \quad (7)$$

This is the general *recursion formula for Euler's method* which is a single step method of order 1.

The truncation error of Euler's method is given by

$$\frac{h^2}{2}.y''(x_n + \theta h),\ 0<\theta<1 \text{ and } n = 1, 2, \dots$$

The geometrical illustration of Euler's method is given below



Let $y = F(x)$ be the solution of (1) and its graph be as shown in the adjoining figure. Since a very small portion of a smooth curve can be thought of as a line segment, so we can write

$$\frac{\Delta y}{\Delta x} = \tan\theta$$

$\therefore$ From the adjacent figure, we have

$$\Delta y = \left(\frac{dy}{dx}\right)_p . \Delta x \text{ and } y_1 = y_0 + \Delta y$$

$$\therefore y_1 = y_0 + \left(\frac{dy}{dx}\right)_p . \Delta x$$

or, $y_1 = y_0 + hf(x_0, y_0)$

which is the approximate value of y for $x = x_1$. On the same lines, the approximate value of y for $x = x_2$ is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

Thus, in general we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Note. A great disadvantage of the method lies in the fact that if h is not small enough then the method yields erroneous result; on the otherhand, if h is taken too small enough then the method becomes very slow.

Example.1. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$, find y for $x = 0.1$ by Euler's method, correct upto 4 decimal places, taking step length $h = 0.02$.

[*W.B.U.T., CS-312, 2007*]

Solution. Here $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.02$

$\therefore$ From (7), we get

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 1 + 0.02\left(\frac{1-0}{1+0}\right)$$

$\therefore y(0.02) = 1.02$

$$y(0.04) = y_2 = y_1 + h\,f(x_1, y_1)$$

$$= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.039231$$

Similarly $y(0.06) = 1.039231 + 0.02\,\dfrac{1.039231 - 0.04}{1.03923 + 0.04}$

$$= 1.057748$$

$$y(0.08) = 1.057748 + 0.02 \times 0.892641$$

$$= 1.075601$$

$$y(0.10) = 1.075601 + 0.02 \times 0.861544 = 1.092832$$

$\therefore y(0.1) = 1.0928$, correct upto 4 decimal places.

7.3. Modified Euler's Method.

To remove the drawback to some extent, we shall discuss modified Euler's method starting with the initial value y_0 an approximate value for y_1 is computed from the Euler's method as

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) \qquad \ldots \qquad (8)$$

Then to get the second approximation for y_1 we replace $f(x_0, y_0)$ in (8) by the average value of $f(x_0, y_0)$ and $f\left(x_1, y_1^{(0)}\right)$.

Thus the second approximation for y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

Similarly, third approximation for y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

Thus, in general

$$y_n^{(k)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f\left(x_n, y_n^{(k-1)}\right)\right], \quad k = 1, 2, 3, \ldots$$

is used to approximate y_n $\qquad \ldots \qquad (9)$

Example.2. Given $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^2}$, $y(1) = 1$. Evaluate $y(1.2)$ by modified Euler's method correct upto 4 decimal places.

[*W.B.U.T., CS-312, 2003, 2004, 2006, M(CS)-301, 2015, M(CS)-401, 2013, 2015*]

Solution. Here $f(x, y) = \dfrac{1}{x^2} - \dfrac{y}{x}$, $x_0 = 1, y_0 = 1$

Let $h = 0.1$ so that $x_1 = 1 + 0.1 = 1.1$

$\therefore y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1 \times (1 - 1) = 1$

$\therefore$ From (9), we get

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[(1-1) + \left\{\frac{1}{(1.1)^2} - \frac{1}{1.1}\right\}\right]$$

$$= 1 + 0.05(-0.08264)$$

$$= 0.99587$$

$$\therefore y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.1}{2}\left[(1-1) + \left(\frac{1}{(1.1)^2} - \frac{0.99587}{1.1}\right)\right]$$

$$= 1 + 0.05(-0.078888)$$

$$= 0.99606$$

Similarly $y_1^{(3)} = 1 + 0.05\left[(1-1) + \left(\dfrac{1}{(1.1)^2} - \dfrac{0.99606}{1.1}\right)\right]$

$$= 1 + 0.05(-0.079063)$$

$$= 0.99607$$

Hence $y_1 = y(1.1) \simeq 0.9961$

$\therefore x_1 = 1.1, y_1 = 0.9961$

$$\therefore f(x_1, y_1) = \frac{1}{(1.1)^2} - \frac{0.9961}{1.1} = -0.079$$

$$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$= 0.9961 + 0.1 \times (-0.079)$$

$$= 0.98819$$

$\therefore$ From (9), we have

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98819}{1.2}\right]$$

$$= 0.98569$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98569}{1.2}\right]$$

$$= 0.98580$$

Similarly,

$$y_2^{(3)} = 0.9961 + 0.05\left[-0.079 + \frac{1}{(1.2)^2} - \frac{0.98580}{1.2}\right]$$

$$= 0.985797$$

Thus $y_2 \simeq 0.9858$, correct upto four decimal places

$\therefore y(1.2) \simeq 0.9858$

7.4. Runge-Kutta method.

This method is one of the most widely used methods to obtain greater accuracy and most suitable in case when computation of higher order derivatives is complicated. In single step method, it follows from (3) that

$$y(x+h) = y(x) + h\phi(x, y; h) + 0\left(h^{p+1}\right) \quad \ldots \quad (10)$$

When $p = 2$, we get from (8)

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x+0h), 0 < 0 < 1 \ldots \quad (11)$$

so that $\phi(x, y; h) = y'(x) + \frac{h}{2}y''(x)$

$$= f + \frac{h}{2}(f_x + f\ fy) \quad \ldots \quad (12)$$

In an 2-stage Runge-Kutta method, we set

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + \alpha h, y + \beta k_1)$$

$$k = w_1 k_1 + w_2 k_2 \quad \ldots \quad (13)$$

The constants α, β, w_1 and w_2 are determined so that (12) agree with Taylor's series of order as high as possible.

$\therefore$ From (10), we get

$$y(x+h) = y(x) + k + 0\left(h^3\right) \quad \ldots \quad (14)$$

From (12) and (13), it follows that

$$f + \frac{1}{2}h\left(f_x + ff_y\right) = w_1 f(x, y) + w_2 f(x + \alpha h, y + \beta k_1)$$

$$= \omega_1 f + w_2\left(f + \alpha h f_x + \beta k_1 f_y\right) + 0\left(h^2\right)$$

$$= (\omega_1 + \omega_2)f + h\left(\omega_2 \alpha f_x + \omega_2 \beta f f_y\right) + 0\left(h^2\right) \quad [\because k_1 = hf]$$

which is true for all values of the constant α, β, ω_1 and ω_2 and therefore, for arbitrary f. Thus we have

$$w_1 + w_2 = 1 \quad \ldots \quad (15)$$

$$w_1 \alpha = w_2 \beta = 1/2$$

Any set of values of the constants α, β, w_1, w_2 satisfying (15) gives a one-parameter family of solutions and each of these corresponds to a 2-stage Runge-Kutta method of order 2.

A possible solution of (15) is

$$w_1 = w_2 = \frac{1}{2}, \alpha = \beta = 1$$

Thus $k_1 = hf(x, y)$

$$k_2 = hf(x+h, y+k_1)$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{h}{2}[f(x, y) + f(x+h, y+hf(x, y))]$$

$$y(x+h) = y(x) + k + 0(h^3)$$

Hence the iterative formula is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right],$$

$$n = 0, 1, 2, \ldots \qquad \ldots \quad (16)$$

This is know as *Runge-Kutta method of order 2* with truncation error of order h^3

In the similar manner the *Runge-Kutta method of order 4* can be written as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \qquad \ldots \quad (17)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3), \; n = 0, 1, 2, \ldots$$

Here the truncation error is of order h^5

Note. **(i) The advantage of this method is that the method is stable and self starting. It is easy to change the step size h for higher order accuracy.**

(ii) For this method several evaluations of the first derivative are require and so the method is time consuming. Most disadvantage of this method is that neither the truncation errors nor the estimates of them are obtained in the computation procedure.

Example.3. Use Runge-Kutta method of order 2 to calculate $y(0.2)$ for the equation

$$\frac{dy}{dx} = x + y^2,\ y(0) = 1$$

Solution. Here $f(x, y) = x + y^2,\ x_0 = 0,\ y_0 = 1$

We take $h = 0.1$. Then

$k_1 = hf(x_0, y_0) = 0.1 \times (0+1) = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1) = 0.1 f(0.1, 1.1)$

$= 0.1 \times \{0.1 + (1.1)^2\} = 1.31$

$\therefore$ From iterative formula (16) of Runge-Kutta method of order 2 we get

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.1 + 1.31)$$

$$= 1.1155$$

$\therefore y(0.1) \simeq 1.1155$

Thus $x_1 = 0.1, y_1 = 1.1155$

$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \{0.1 + (1.1155)^2\}$

$\simeq 0.1344$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= 0.1 \times f(0.2, 1.2499)$

$\simeq 0.1762$

$\therefore$ From (16), we get

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.1155 + \frac{1}{2}(0.1344 + 0.1762)$$

$$= 1.2708$$

Hence $y(0.2) \simeq 1.2708$

Example.4. Find $y(1.1)$ using Runge-Kutta method of fourth order, given that

$$\frac{dy}{dx} = y^2 + xy,\ y(1) = 1$$

[W.B.U.T., CS-312, 2005
M.A.K.A.U.T., MCS-401, 2014]

Solution. Here $f(x, y) = y^2 + xy,\ x_0 = 1, y_0 = 1$

Taking $h = 0.1$, we have

$$k_1 = hf(x_0, y_0) = 0.1(1^2 + 1 \times 1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= 0.1f(1.05, 1.1)$$

$$= 0.1\{(1.1)^2 + 1.05 \times 1.1\}$$

$$= 0.2365$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= 0.1f(1.05, 1.11825)$$

$$= 0.1\{(1.11825)^2 + 1.11825 \times 1.05\}$$

$$= 0.2425$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(1.1, 1.2425)$$

$$= 0.1\{(1.2425)^2 + 1.1 \times 1.2425\}$$

$$= 0.2910556$$

$\therefore$ From iterative formula (17) of Runge-Kutta method of order 4, we get

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.2 + 2 \times 0.2365 + 2 \times 0.2425 + 0.2910556)$$

$\simeq 1.2415$, correct upto four decimal places.

$\therefore y(1.1) \simeq 1.2415$

7.5. Predictor-Corrector methods.

In order to solve the differential equation (1), by this method, we first obtain the approximate value of $y_{n+1} = y(x_{n+1})$ by predictor formula and then improve this value by means of a corrector formula. It may be noted that the corrector formula is more accurate than the predictor one although it requires a companion of predictor formula and knowledge of the initial set of values $y_0, y_1, \dots y_n$

The simplest formula of this type is Euler's formula and the modified Euler's one is given by

$$y_{n+1}^{(p)} = y_n + hf(x_n, y_n) \qquad \dots \quad (18)$$

$$y_{n+1}^{(c)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(p)}\right)\right] \qquad \dots \quad (19)$$

The first is an open formula which can be used for predicting y_{n+1} and this value can be used to compute $f(x_{n+1}, y_{n+1})$ to get a corrector formula which can be used in an iterative manner.

Ex.5. Solve the equation $\frac{dy}{dx} = x + y$ with initial condition $y(0) = 1$, taking step length 0.1 to find $y(0.2)$ by predictor-corrector method. *[M.A.K.A.U.T., MCS-401, 2014]*

Solution. Here $f(x, y) = x + y,\quad x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$\therefore y_1^p = y_0 + h\ f(x_0, y_0)$$

$$= 1 + 0.1\ f(0, 1) = 1 + 0.1(0 + 1) = 1.1$$

$$y_1^{c(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(p)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[f(0, 1) + f(0.1, 1.1)]$$

$$= 1 + 0.05\,[(0+1) + (0.1+1.1)] = 1.11$$

$$y_1^{c(2)} = 1 + \frac{0.1}{2}\,[f(0.1) + f(0.1, 1.11)]$$

$$= 1 + 0.05\,[1 + (0.1 + 1.11)] = 1.1105$$

$$y_1^{c(3)} = 1 + \frac{0.1}{2}\,[1 + (0.1 + 1.1105)]$$

$$= 1.110525$$

$$\therefore\ y_1 = y(0.1) \simeq 1.1105$$

$$\therefore\ y_2^p = y_1 + h\,f(x_1, y_1)$$

$$= 1.1105 + 0.1\,f(0.1, 1.1105)$$

$$= 1.1105 + 0.1(0.1 + 1.1105) = 1.23155$$

$$y_2^{c(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_1^p\right)\right]$$

$$= 1.1105 + \frac{0.1}{2}\,[f(0.1, 1.1105) + f(0.2, 1.23155)]$$

$$= 1.1105 + 0.05\,[(0.1 + 1.1105) + (0.2 + 1.23155)]$$

$$= 1.1105 + 0.05\,(1.2105 + 1.43155)$$

$$= 1.242602$$

$$y_2^{c(2)} = 1.1105 + \frac{0.1}{2}\,[f(0.1, 1.1105) + f(0.2, 1.242602)]$$

$$= 1.1105 + 0.05(1.2105 + 1.442602)$$

$$= 1.243155$$

$$y_2^{c(3)} = 1.1105 + 0.05\,[1.2105 + (0.2 + 1.243155)]$$

$$= 1.243183$$

$\therefore\ y_2 = y(0.2) \simeq 1.2432$, correct upto four decimal places

Let us now proceed to discuss some multi-step methods which are of much practical use as predictor-corrector methods.

I. Adams-Bashforth method.

We consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

which when integrated over the range $[x_n, x_{n+1}]$ leads to

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dx \qquad \dots (20)$$

To evaluate the integral on the right hand side of (20), we can replace $f(x, y)$ by a polynomial which interpolates $f(x, y(x))$ at the $(k+1)$ equidistant points $x_{n-k}, x_{n-k+1} \cdots, x_n$; by the Newton's backward difference interpolation formula.

$$P_k(x) = \sum_{j=0}^{k} \binom{s+j-1}{j} \nabla^j f_n, \qquad \dots (21)$$

where $s = \dfrac{x - x_n}{h}$

Then, in virtue of (20) and (21), we get

$$y_{n+1} = y_n + h\sum_{j=0}^{k} \alpha_j \nabla^j f_n \quad (n \geq k) \qquad \dots (22)$$

where $\alpha_j = \int_0^1 \binom{s+j-1}{j} ds, \ (j = 0, 1, 2, \dots k+1) \ \dots \quad (23)$

The formula (22) is known as the $(k+1)$ step Adams-Bashforth formula and is an predictor formula of order $k+1$

A few values of α_j as obtained from (23) are

$$\alpha_0 = 1, \alpha_1 = \tfrac{1}{2}, \alpha_2 = \tfrac{5}{12}, \alpha_3 = \tfrac{3}{8}, \alpha_4 = \tfrac{251}{720}, \dots$$

Thus, for $k = 3$ we can rewrite (22) in the form

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n\right] \quad \ldots \quad (24)$$

Substituting the expressions of differences given by

$$\nabla f_n = f_n - f_{n-1}, \quad \nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$$

$$\nabla^3 f_n = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3},$$

we get

$$y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \quad \ldots \quad (25)$$

which is known as the *4-th step Adams-Bashforth formula in ordinate form.*

II. Adams-Moulton method.

This method can be dealt with along the same lines as in Adams-Bashforth method with the exception that instead of the interpolating points $x_{n-k}, x_{n-k+1}, \ldots, x_n$, we consider the points $x_{n-k+1}, x_{n-k+2}, \ldots, x_{n+1}$. Then, as usual integrating the equation

$$\frac{dy}{dx} = f(x, y)$$

over $[x_n, x_{n+1}]$ we get after simplification,

$$y_{n+1} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \text{ for } k = 3 \quad \ldots \quad (26)$$

This formula is known as the 4-th step Adams-Moulton corrector formula of order 4.

Thus another set of predictor-corrector formula which is commonly used is given below:

$$y_{n+1}^{(p)} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37y_{n-2} - 9f_{n-3}] \quad (n > 3) \quad \ldots \quad (27)$$

$$y_{n+1}^{(c)} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \quad (n > 3) \ldots \quad (28)$$

Example.6. Given that $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$, $y(0.1) = 1.0954$, $y(0.2) = 1.1852$, $y(0.3) = 1.2649$. Find $y(0.4)$ by Adams-Moulton formula.

Solution. Here $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

and $y_0 = 1, y_1 = 1.0954, y_2 = 1.1852, y_3 = 1.2649$

Also $f(x, y) = y - \frac{2x}{y}$

Hence we obtain

$$f_0 = f(x_0, y_0) = 1, f_1 = f(x_1, y_1) = 0.9128$$

$$f_2 = f(x_2, y_2) = 0.8451, f_3 = f(x_3, y_3) = 0.7906$$

For $n = 3$, (27) and (28) give respectively

$$y_4^p = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \quad \cdots \quad (1)$$

and

$$y_4^c = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \quad \cdots \quad (2)$$

Then from (1), we get

$$y_4^p = 1.2649 + \frac{0.1}{24}(55 \times 0.7906 - 59 \times 0.8451 + 37 \times 0.9128 - 9 \times 1)$$

$$\simeq 1.3415$$

and hence $\tilde{f}_4 = f(x_4, y_4^p) = 1.3415 - \frac{2 \times 0.4}{1.3415} = 0.7452$

$\therefore$ From (2), we get

$$y_4^{c(1)} = 1.2649 + \frac{0.1}{24}(9 \times 0.7452 + 19 \times 0.7906 - 5 \times 0.8451 + 0.9128)$$

$$\simeq 1.3416$$

Hence $\tilde{f}_4 = f\left(x_4, y_4^{c(1)}\right) = 1.3416 - \dfrac{2 \times 0.4}{1.3416} = 0.7453$

$$\therefore y_4^{c(2)} = 1.2649 + \frac{0.1}{24}(9 \times 0.7453 + 19 \times 0.7906 - 5 \times 0.8451 + 0.9128)$$

$$= 1.3416$$

$\therefore y_4^{c(1)} = y_4^{c(2)} = 1.3416$, correct to four decimal places.

Hence $y(0.4) = 1.3416$

III. Milne's method.

The multistep method due to Milne is obtained by integration over more than one step. Integration of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

over the range $[x_{n-3}, x_{n+1}]$ gives

$$y_{n+1} = y_{n-3} + \int_{x_{n-3}}^{x_{n+1}} f(x, y)dx$$

Evaluating this integral by 3 point Newton-Cotes quadrature rule and neglecting error term, we get

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \dots \quad (29)$$

which is the 4-step explicit recursion formula of order 4, known as *Milne's prodictor formula of order 4.*

On the other hand, if we integrate the differential equation $\dfrac{dy}{dx} = f(x, y)$ over the range $[x_{n-1}, x_{n+1}]$, we get

$$y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} f(x, y)dx$$

Evaluating this integral by Simpson's one-third rule with interpolating points x_{n-1}, x_n, x_{n+1} and neglecting error term, we obtain

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \quad \dots \quad (30)$$

which is the 2-step implicit recursion formula of order 4, called *Milne's corrector formula of order 4.*

Example.7. Compute $y(0.4)$ by Milne's predictor-corrector method from the equation

$$\frac{dy}{dx} = xy + y^2,$$

given that $y(0) = 1, y(0.1) = 1.1169, y(0.2) = 1.2773, y(0.3) = 1.5040$

[*M.A.K.A.U.T., M(CS)-401, 2015*]

Solution. We have $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$ and $y_0 = 1, y_1 = 1.1169, y_2 = 1.2773, y_3 = 1.5040$

Also $f(x, y) = xy + y^2$

$\therefore f_0 = f(x_0, y_0) = 1, f_1 = f(x_1, y_1) = 1.3591, f_2 = f(x_2, y_2) = 1.8869$

$f_3 = f(x_3, y_3) = 2.7132$

Now putting $n = 3$ in (29) and (30), we get

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \qquad \ldots (1)$$

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \qquad \ldots (2)$$

$\therefore$ From (1) and (2),

$$y_4^{p(1)} = 1 + \frac{4 \times 0.1}{3}(2 \times 1.3591 - 1.8869 + 2 \times 2.7132)$$
$$= 1.8344$$

and hence $f_4^{p(1)} = f\left(x_4, y_4^{p(1)}\right) = 4.0988$

$$\therefore y_4^{c(1)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.0988)$$
$$= 1.8386$$

and so $f_4^{c(1)} = f\left(x_4, y_4^{c(1)}\right) = 4.1159$

Hence from (2),

$$y_4^{c(2)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.1159)$$
$$= 1.8391$$

Hence $f_4^{c(2)} = f\left(x_4, y_4^{c(2)}\right) = 4.1182$

$$\therefore y_4^{c(3)} = 1.2773 + \frac{0.1}{3}(1.8869 + 4 \times 2.7132 + 4.1182)$$

$$= 1.8392$$

$$\therefore y_4^{c(2)} = y_4^{c(3)} = 1.839, \text{ correct upto 3 decimal places}$$

Hence $y(0.4) \simeq 1.839$

7.6. Finite difference method.

The finite difference method which in also known as net method is a popular method for solving boundary value problems. In this method, the derivatives are replaced by finite difference relations and then solving the resulting system of equations by a standard procedure. In general, for better accuracy,central differences are preferred to replace the derivatives.

Thus

$$y'(x_i) = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \ldots \qquad (31)$$

$$y''(x_i) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \text{ etc.}$$

To solve the boundary value problem, defined by

$$y''(x) + p(x)y' + q(x)y = r(x) \qquad \ldots \qquad (32)$$

with the boundary conditions

$$y(x_0) = a$$

$$y(x_n) = b,$$

we divide the interval $[a, b]$ into n equal subintervals of width h so that the end points are $x_0 = a, x_n = b$ and the interior mesh points are

$$x_i = x_0 + ih, i = 1, 2, \ldots n$$

The corresponding values of y at these points are denoted by

$$y_i = y(x_i) = y(x_0 + ih), \; i = 1, 2, \ldots n$$

At the point $x = x_i$ we get from (32),

$$y_i'' + p_i y_i' + q_i y_i = r_i$$

with $y_0 = a, y_n = b,\ i = 1, 2, \dots n-1$

where $p_i = p(x_i), q_i = q(x_i)$ and $r_i = r(x_i)$

Substituting the expression for y_i' and y_i'' the equation (29) gives

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p_i \frac{y_{i+1} - y_{i-1}}{2h} + q_i y_i = r_i$$

$$\text{or, } \left(1 - \frac{h}{2} p_i\right) y_{i-1} + \left(-2 + q_i h^2\right) y_i + \left(1 + \frac{h}{2} p_i\right) y_{i+1} = r_i h^2,$$

$$i = 1, 2, \dots n-1$$

This linear system of equations can be solved by using any of the elimination methods. The solution of this system constitutes an approximate solution of the boundary value problem defined by (32)

Example.8. Solve the equation

$$\frac{d^2y}{dx^2} + y = 0$$

with $y(0) = 0, y(1) = 1$, using finite difference method taking $h = 0.25$. [*M.A.K.A.U.T., M(CS)-301, 2015*]

Solution. The given equation is approximated as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i = 0,\ i = 1, 2, \dots n-1$$

i.e. $y_{i-1} + \left(h^2 - 2\right) y_i + y_{i+1} = 0,\ i = 1, 2, 3$ ... (1)

which, together with the boundary conditions $y_0 = 0, y_n = 1$, comprise a system of $(n+1)$ equations for the $(n+1)$ unknown $y_0, y_1, y_2, \dots y_n$

Choosing $h = 0.25$ i.e. $n = 4$, the above system of equations are

$$y_0 - 1.9375 y_1 + y_2 = 0$$

$$y_1 - 1.9375 y_2 + y_3 = 0$$

$$y_2 - 1.9375 y_3 + y_4 = 0$$

where $y_0 = 0, y_4 = 1$

Solving the system we get

$$y_1 = 0.2943, y_2 = 0.5701, y_3 = 0.8108$$

i.e., $y(0.25) = 0.2943, y(0.5) = 0.5701, y(0.75) = 0.8108$

ILLUSTRATIVE EXAMPLES

Ex. 1. Find the solution of the differential equation

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1$$

for $x = 0.3$ taking $h = 0.1$ and using Euler's method. Compare the result with the exact solution.

Solution. Here $f(x, y) = x^2 - y, x_0 = 0, y_0 = 1, h = 0.1$ so that $x_i = x_0 + ih (i = 0, 1, 2, ...)$ gives

$$x_1 = 0.1, x_2 = 0.2 \text{ etc}$$

Thus the recursion formula

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, ...$$

yields

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(0 - 1) = 0.9$$

$$y_2 = y_1 + hf(x_1, y_1) = 0.9 + 0.1\{(0.1)^2 - 0.9\} = 0.811$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.811 + 0.1\{(0.2)^2 - 0.811\} = 0.7339$$

Hence $y(0.3) \approx 0.7339$

The given equation can be written as

$$\frac{dy}{dx} + y = x^2$$

which is a linear equation in y.

$$\therefore I.F. = e^{\int 1.dx} = e^x$$

Multiplying both sides of the equation by e^x and then integrating we get

$$ye^x = \int x^2 e^x dx + c$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

$$\therefore y = x^2 - 2x + 2 + ce^{-x}$$

Also given $y(0) = 1$

$\therefore 1 = 0 - 2 \times 0 + 2 + c$

$\therefore c = -1$

$\therefore y = x^2 - 2x + 2 - e^{-x}$

$\therefore y(0.3) = (0.3)^2 - 2 \times 0.3 + 2 - e^{-0.3} = 0.7492$

Hence the error is $0.7492 - 0.7339 = 0.0153$

Ex.2. Using Euler's method, find an approximate value of y at $= 0.5$ given that

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$

Solution. We have $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1$

Taking $h = 0.1$, we have from recursion formula of Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$= y_n + 0.1(x_n + y_n),\ n = 0, 1, 2, \ldots$$

$$\therefore y_1 = y(0.1) = 1 + 0.1(0 + 1) = 1.10$$

$$y_2 = y(0.2) = y_1 + 0.1(x_1 + y_1)$$

$$= 1.10 + 0.1(0.1 + 1.10)$$

$$= 1.22$$

$$y_3 = y(0.3) = y_2 + 0.1(x_2 + y_2)$$

$$= 1.22 + 0.1(0.2 + 1.22)$$

$$= 1.36$$

$$y_4 = y(0.4) = y_3 + 0.1(x_3 + y_3)$$

$$= 1.36 + 0.1(0.3 + 1.36)$$

$$= 1.53$$

$$y_5 = y(0.5) = y_4 + 0.1(x_4 + y_4)$$

$$= 1.53 + 0.1(0.4 + 1.53)$$

$$= 1.72$$

Thus $y(0.5) = 1.72$

Ex.3. Solve the equation

$$5x\frac{dy}{dx}+y^2-2=0;\ y(4)=1$$

for $y(4.1)$, taking $h=0.1$ and using modified Euler's method.

Solution. The given equation can be written as

$$\frac{dy}{dx}=\frac{2-y^2}{5x}$$

$$\therefore f(x,y)=\frac{2-y^2}{5x}$$

Here $x_0=4,\ y_0=1,\ h=0.1$

So the recursion formula of modified Euler's method gives

$$y_1^{(0)}=y_0+hf(x_0,y_0)=1+0.1f(4,1)$$

$$=1+0.1\times 0.05\quad\left[\because f(4,1)=\frac{2-1}{5\times 4}=0.05\right]$$

$$=1.005$$

$$y_1^{(1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(0)}\right)\right]$$

$$=1+\frac{0.1}{2}[f(4,1)+f(4.1,1.005)]$$

$$=1+0.05\left[0.05+\frac{2-(1.005)^2}{5\times 4.1}\right]$$

$$=1.0049$$

Similarly, $y_1^{(2)}=y_0+\dfrac{h}{2}\left[f(x_0,y_0)+f\left(x_1,y_1^{(1)}\right)\right]$

$$=1+0.05[f(4,1)+f(4.1,1.0049)]$$

$$=1+0.05\left[0.05+\frac{2-(1.0049)^2}{5\times 4.1}\right]$$

$$=1.0049$$

$\therefore y(4.1)\simeq 1.005$, correct upto three decimal places

Ex.4. Find $y(0.10)$ and $y(0.15)$ by Euler's method, from the differential equation $\frac{dy}{dx} = x^2 + y^2$, with $y(0) = 0$, correct to four decimal places, taking step length $h = 0.05$.

[*W.B.U.T., MCS-301, 2007*]

Solution. Here $f(x, y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.05$

$\therefore x_1 = 0.05,\ x_2 = 0.10,\ x_3 = 0.15$

$\therefore$ The recursion formula

$$y_{n+1} = y_n + h\ f(x_n, y_n),\ x = 0,1,2.....$$

gives

$$y_1 = y_0 + h\ f(x_0, y_0)$$

$$= 0 + 0.05\ f(0, 0) = 0 + 0.05 \times 0 = 0$$

$$y_2 = y_1 + h\ f(x_1, y_1)$$

$$= 0 + 0.05\ f(0.05, 0)$$

$$= 0.05\left\{(0.05)^2 + 0^2\right\}$$

$$= 1.25 \times 10^{-4}$$

$$y_3 = y_2 + h\ f(x_2, y_2)$$

$$= 0.000125 + 0.05\left\{(0.1)^2 + (0.000125)^2\right\}$$

$$= 6.25 \times 10^{-4}$$

Ex.5. Solve by using Euler's method the following differential equation for $x = 1$ by taking $h = 0.2$:

$$\frac{dy}{dx} = xy,\ y = 1 \text{ when } x = 0$$ [*W.B.U.T., MCS-301, 2008*]

Solution. Here $f(x, y) = xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$

$\therefore$ By Euler's iterative formula

$$y_{n+1} = y_n + h\ f(x_n, y_n)\ n = 0,1,2,....$$

we get

$$y_1 = y(0.2) = y_0 + h\,f(x_0, y_0)$$

$$= 1 + 0.2(0 \times 1) = 1$$

$$y_2 = y(0.4) = y_1 + h\,f(x_1, y_1)$$

$$= 1 + 0.2(0.2 \times 1) = 1.04$$

$$y_3 = y(0.6) = y_2 + h\,f(x_2, y_2)$$

$$= 1.04 + 0.2(0.4 \times 1.04) = 1.1232$$

$$y_4 = y(0.8) = 1.1232 + 0.2(0.6 \times 1.1232$$

$$= 1.25798$$

$$y_5 = y(1.0) = 1.25798 + 0.2(0.8 \times 1.25798)$$

$$= 1.45926$$

$\therefore\ y(1.0) \simeq 1.4593$; correct upto four decimal places

Ex.6. Use Runge-Kutta method of order two to find $y(0.2)$ and $y(0.4)$ given that

$$y\frac{dy}{dx} = y^2 - x,\ y(0) = 2, \text{ taking } h = 0.2$$

Solution. The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x}{y}$$

$\therefore$ Here $f(x, y) = \dfrac{y^2 - x}{y}$, $x_0 = 0$, $y_0 = 2$, $h = 0.2$

$\therefore$ By Runge-Kutta method of order 2, we have

$$y(x_0 + h) = y_0 + k$$

where $k = \frac{1}{2}(k_1 + k_2)$,

$$k_1 = hf(x_0, y_0) = 0.2 \times \frac{2^2 - 0}{2} = 0.4$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$
$$= 0.2\ f(0.2,\ 2.4)$$
$$= 0.2 \times \frac{(2.4)^2 - 0.2}{2.4}$$
$$= 0.46333$$

Thus $y(0+0.2) = 2 + \frac{1}{2}(0.4 + 0.46333)$

$$= 2.43166$$

$\therefore y(0.2) \simeq 2.43166$

To compute $y(0.4)$ we have $x_1 = 0.2, y_1 = 2.43166$

$\therefore y(x_1 + h) = y(x_1) + k$

where $k = \frac{1}{2}(k_1 + k_2)$,

$$k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.432) = -0.00633$$
$$k_2 = hf(x_1 + h, y_1 + k_1)$$
$$= 0.2f(0.4, 0.42533)$$
$$= -0.10302$$

$\therefore y(0.2 + 0.2) = 2.43166 + \frac{1}{2}(-0.00633 - 0.10302)$

$\therefore y(0.4) = 2.37698$

Hence $y(0.2) = 2.432$, $y(0.4) = 2.377$ correct upto three decimal places.

Ex.7. Use the fourth order RK-method to find the value of y when $x = 0.2$ given that $y = 0$ when $x = 0$ and $\frac{dy}{dx} = 1 + y^2$.

[W.B.U.T., MCS-301, 2010]

Solution. Here $f(x, y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$

We take $h = 0.2$

$\therefore k_1 = h\ f(x_0, y_0) = 0.2\ f(0, 0) = 0.2$

$$\therefore\ k_2 = h\ f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$= 0.2\ f(0.1,\ 0.1) = 0.202$$

$$\therefore\ k_3 = h\ f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= 0.2\ f(0.1,\ 0.202) = 0.2020$$

$$\therefore\ k_4 = h\ f(x_0 + h,\ y_0 + k_3)$$

$$= 0.2\ f(0.2,\ 0.20204) = 0.2082$$

$$\therefore\ y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6}(0.2 + 2\times 0.202 + 2\times 0.202 + 0.2082)$$

$$= 0.2027$$

$$\therefore\ y(0.2) = 0.2024$$

Ex.8. Comute $y(0.2)$ from $\frac{dy}{dx} = x + y,\ y(0) = 1$ taking step $h = 0.1$ by 4th order RK method. [*W.B.U.T., MCS-301, 2007*]

Solution. Here $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$\therefore y(0.1) = y(0) + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1 \times (0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.05) = 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.055)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1105) = 0.1(0.1 + 1.1105)$$
$$= 0.12105$$

$$\therefore y(0.1) = y_0 + k$$
$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.11 + 2 \times 0.1105 + 0.12105)$$
$$= 1.0901667$$

$$\therefore y(0.1) \simeq 1.0902$$

To compute $y(0.2)$, we have $x_1 = 0.1, y_1 = 1.0902, h = 0.1$

$$\therefore k_1 = hf(x_1, y_1) = 0.1f(0.1, 1.0902) = 0.11902$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15, 1.149708) = 0.12997$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15, 1.155185) = 0.13052$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.220718) = 0.142072$$

$$\therefore y(0.2) = 1.0902 + \frac{1}{6}(0.11902 + 2 \times 0.12997 + 2 \times 0.13052 + 0.142072)$$
$$= 1.220545$$

$\therefore y(0.2) \simeq 1.2205$, correct upto four decimal places.

Ex.9. Find the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$ using Runge-Kutta method of fourth order, given that

$$\frac{dy}{dx} = xy + y^2, \quad y(0) = 1 \qquad \text{[W.B.U.T., MCS-301, 2009]}$$

Solution. Here $f(x, y) = xy + y^2$, $x_0 = 0$, $y_0 = 1$

Taking $h = 0.1$, we have

$$y(0.1) = y(0) + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_0, y_0) = 0.1\left(0 \times 1 + 1^2\right) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05, 1.05)$$

$$= 0.1\left\{0.05 \times 1.05 + (1.05)^2\right\} \quad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05, 1.05775)$$

$$= 0.1\left\{0.05 \times 1.05775 + (1.05775)^2\right\} \quad = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1172)$$

$$= 0.1\left\{0.1 \times 1.1172 + (1.1172)^2\right\}$$

$$= 0.13598$$

$$\therefore y(0.1) = 1 + \frac{1}{0}(0.1 + 2 \times 0.1155 + 2 \times 0.1172 + 0.135598)$$

$$= 0.1168$$

Similarly we can find out

$$y(0.2) = 1.2689$$

$$y(0.3) = 1.4856$$

Ex.10. Solve the equation $\dfrac{dy}{dx} = \dfrac{1}{x+y}$, $y(0) = 1$, for $y(0.1)$ and $y(0.2)$ using Runge-Kutta method of the fourth order.

[*M.A.K.A.U.T., M(CS)-401, 2006, 2013*]

Solution. Here $f(x, y) = \dfrac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$

Taking $h = 0.1$, we have

$$y(x_0) = y(x_0 + h) = y_0 + k$$

where $k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = h\, f(x_0, y_0) = 0.1 \times \frac{1}{0+1} = 0.1$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1 \times f(0.05,\ 1.05)$$

$$= 0.1 \times \frac{1}{0.5 + 1.05} = 0.09091$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.1 \times f(0.05, 1.04545)$$

$$= 0.1 \times \frac{1}{0.05 + 1.04545} = 0.09129$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0.1 \times f(0.1,\ 1.09129)$$

$$= 0.08394$$

$$\therefore\ k = \frac{1}{6}(0.1 + 2 \times 0.09091 + 2 \times 0.09129 + 0.08394)$$

$$= 0.09139$$

$$\therefore\ y(0.1) = 1 + 0.09139 = 1.0914$$

To find $y(0.2)$, we have

$$x_1 = 0.1,\ y_1 = 1.0914$$

$$\therefore\ k_1 = h\,f(x_1,\ y_1) = 0.1 \times f(0.1,\ 1.0914) = 0.08393$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) = 0.1 \times f(0.15,\ 1.3337)$$

$$= 0.07792$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.1 \times f(0.15,\ 1.13036)$$

$$= 0.07810$$

$$k_4 = h\,f(x_1 + h,\ y_1 + k_3) = 0.1 \times f(0.2,\ 1.16950)$$

$$= 0.07302$$

$$\therefore\ y_2 = y(x_1 + h) = y_1 + \frac{1}{6}(k_1 + k_2 + 2k_3 + k_4)$$

$$= 1.16957$$

$$\therefore\ y(0.2) = 1.1696$$

Ex.11. Using Runge-Kutta method of order 4 obtain the solution of $\frac{dy}{dx} = 2x + y^2$, $y(0) = 1$ and $h = 0.1$ at $x = 0.2$.

[M.A.K.A.U.T., M(CS)-401, 2015]

Solution. Here $f(x, y) = 2x + y^2$, $x_0 = 0$, $y_0 = 1$

Given $h = 0.1$

$$\therefore k_1 = h\, f(x_0, y_0) = 0.1(0 + 1^2) = 0.1$$

$$k_2 = h\, f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right)$$

$$= 0.1\, f(0.05, 1.05)$$

$$= 0.1\,\{2 \times 0.05 + (1.05)^2\} = 0.12025$$

$$k_3 = h\, f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.1\, f(0.05, 1.06012)$$

$$= 0.1\,\{2 \times 0.05 + (1.060128)^2\} = 0.12239$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3) = 0.1\, f(0.1, 1.12239)$$

$$= 0.1\,\{2 \times 0.1 + (1.2239)^2\} = 0.14598$$

$$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.12025 + 2 \times 0.12239 + 0.14598)$$

$$= 1.12188$$

$$\therefore y(0.1) = 1.12188$$

To compute $y(0.2)$, we have

$x_1 = 0.1$, $y_1 = 1.12188$, $h = 0.1$

$$\therefore k_1 = h\, f(x_1, y_1) = 0.1\, f(0.1, 1.12188)$$

$$= 0.1\,\{(2 \times 0.1 + (1.12188)^2\}$$

$$= 0.14586$$

$\therefore k_1 = h\, f(x_1, y_1) = 0.1\, f(0.1, 1.12188)$

$= 0.1\{2 \times 0.1 + (1.12188)^2\}$

$= 0.14586$

$k_2 = h\, f\left(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_1\right) = 0.1\, f(0.15, 1.19481)$

$= 0.17276$

$k_3 = h\, f\left(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_2\right)$

$= 0.1\, f(0.15, 1.20826)$

$= 0.17599$

$k_4 = h\, f(x_1 + h,\, y_1 + k_3)$

$= 0.1\, f(0.2, 1.29787)$

$= 0.208446$

$\therefore y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.12188 + \frac{1}{6}(0.14586 + 2 \times 0.17278$

$+ 2 \times 0.17599 + 0.208446)$

$= 1.2971877$

$\therefore y(0.2) = 1.2972$ correct up to four decimal places

Ex.12. Find the value of $y(0.4)$ using Runge-Kutta method of fourth order with $h = 0.2$, given that

$$\frac{dy}{dx} = \sqrt{x^2 + y},\ y(0) = 0.8$$

Solution. Here $f(x, y) = \sqrt{x^2 + y}$, $x_0 = 0$, $y_0 = 0.8$, $h = 0.2$

$\therefore k_1 = hf(x_0, y_0) = 0.2f(0, 0.8) = 0.2\sqrt{0^2 + 0.8} = 0.17889$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 0.88944)$$

$$= 0.2\sqrt{(0.1)^2 + 0.88944} = 0.18968$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.89484)$$

$$= 0.2\sqrt{(0.1)^2 + 0.89484}$$

$$= 0.19025$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.99025)$$

$$= 0.2\sqrt{(0.2)^2 + 0.99025}$$

$$= 0.20300$$

$$\therefore y_1 = y(x_0 + h)$$

$$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.8 + \frac{1}{6}(0.17889 + 12 \times 0.18968 + 2 \times 0.19025 + 0.20300)$$

$$= 0.99029$$

$$\therefore y(0.2) = 0.99029$$

To compute $y(0.4)$, we have $x_1 = 0.2, y_1 = 0.99029, h = 0.2$

$$\therefore k_1 = hf(x_1, y_1) = 0.2f(0.2, 0.99029) = 0.20301$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2f(0.3, 1.09180) = 0.21742$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2f(0.3, 1.09901) = 0.21808$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.20838) = 0.23396$$

$\therefore y_2 = y(x_1 + h)$

$= y_1 + \dfrac{1}{6}(k_2 + 2k_2 + 2k_3 + k_4)$

$= 0.99029 + \dfrac{1}{6}(0.20301 + 2 \times 0.21742 + 2 \times 0.21808 + 0.23396)$

$= 1.20832$

$\therefore y(0.4) \simeq 1.2083$, correct upto four decimal places.

Ex.13. Solve initial value problem

$$10\frac{dy}{dx} = x^2 + y^2, \, y(0) = 1$$

for $x = 0.1, 0.2$ by using Runge-Kutta fourth order method and find the solution correct upto 4 places of decimal.

[W.B.U.T., CS-312, 2004]

Solution. Here $f(x, y) = \dfrac{x^2 + y^2}{10}, x_0 = 0, y_0 = 1$

Let $h = 0.1$

$\therefore$ By fourth order Runge-Kutta method,

$$y(x_0 + h) = y_0 + k$$

where $k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

$$k_1 = hf(x_0, y_0) = 0.1\left(\frac{0+1}{10}\right) = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f(0.05, 1.005)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.005)^2}{10} = 0.010125$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05,\ 1.0050625)$$

$$= 0.1 \times \frac{(0.05)^2 + (1.0050625)^2}{10}$$

$$= 0.000025$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1,\ 1.000025)$$

$$= 0.1 \times \frac{(0.1)^2 + (1.000025)^2}{10} = 0.0101$$

$$\therefore k = \frac{1}{6}(0.01 + 2 \times 0.010125 + 2 \times 0.000025 + 0.0101)$$

$$= 0.00673$$

$$\therefore y(x_0 + h) = y(0.1) = 1 + 0.00673 = 1.0067$$

For $y(0.2)$, we have $x_1 = 0.1, y_1 = 1.0067$

$$\therefore k_1 = hf(x_1, y_1) = 0.1 \times \frac{(0.1)^2 + (1.0067)^2}{10} = 0.010235$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1f(0.15,\ 1.0118175)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.0118175)^2}{10}$$

$$= 0.01046$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1f(0.15,\ 1.01193)$$

$$= 0.1 \times \frac{(0.15)^2 + (1.01193)^2}{10}$$

$$= 0.010465$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= 0.1f(0.2, 1.017165)$$
$$= 0.1 \times \frac{(0.2)^2 + (1.017165)^2}{10}$$
$$= 0.010746$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.010235 + 2 \times 0.01046 + 2 \times 0.010465 + 0.010746)$$
$$= 0.01047$$

$$\therefore y(x_1 + h) = y(0.2) = 1.0067 + 0.01047$$
$$\simeq 1.0172$$

Ex.14. Compute $y(0.8)$ by Adams-Moulton predictor-corrector method from

$$\frac{dy}{dx} = 1 + y^2, y(0) = 0$$

given $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6842$

Solution. Here $f(x, y) = 1 + y^2$, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ and $y_0 = 0$, $y_1 = 0.2027$, $y_2 = 0.4228$, $y_3 = 0.6842$

Hence $f_0 = f(x_0, y_0) = 1$, $f_1 = f(x_1, y_1) = 1.0411$, $f_2 = 1.1788$, $f_3 = 1.4680$

Now fourth order Adam's Bashforth formula is

$$y_4^{(p)} = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \quad \ldots \quad (1)$$

and Adam's Moulton formula is

$$y_4^{(c)} = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \quad \ldots \quad (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 0.6842 + \frac{0.2}{24}(80.3960 - 69.5468 + 38.5203 - 9)$$
$$= 1.0235$$

$\therefore f_4 = f\left(x_4, y_4^{(p)}\right) = 2\cdot0475$

$\therefore$ From (2),

$$y_4^{c(1)} = 0.6842 + \frac{0.2}{24}(18.4275 + 27.8945 - 5.8938 + 1.0411)$$

$$= 1.0298$$

$\therefore f_4 = f\left(x_4, y_4^{c(1)}\right) = 2\cdot0604$

$\therefore$ From (2),

$$y_4^{c(2)} = 0.6842 + \frac{0.2}{24}(18.5440 + 27.8945 - 5.8938 + 1.0411)$$

$$= 1.0308$$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(2)}\right) = 2.0624$

$\therefore$ From (2), $y_4^{c(3)} = 1.0309$

$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(3)}\right) = 2.0628$

$\therefore$ From (2), $y_4^{c(4)} = 1.0309$

Hence $y_4^{c(3)} = y_4^{c(4)} = 1.0309$, correct upto four decimal places

$\therefore y(0.8) \simeq 1.0309$

Ex.15. Apply Milne's method to find the solutions of the differential equation

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

at $x = 0.08$,

given $y(0) = 1, y(0.02) = 1.02, y(0.04) = 1.0392$ $y(0.06) = 1.0577$

Solution. Here $x_0 = 0, x_1 = 0.02, x_2 = 0.04, x_3 = 0.06$

and $y_0 = 1, y_1 = 1.02, y_2 = 1.0392$, $y_3 = 1.0577$

Also, $f(x, y) = \dfrac{y - x}{y + x}$

$$\therefore f_1 = f(x_1, y_1) = \frac{1.02 - 0.02}{1.02 + 0.02} = 0.9615$$

Similarly, $f_2 = f(x_2, y_2) = 0.9259$

$$f_3 = f(x_3, y_3) = 0.8926$$

Now Milne's predictor formula of order 4 is

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \ldots (1)$$

and the corrector formula of order 4 is

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \ldots (2)$$

$\therefore$ From (1),

$$y_4^{(p)} = 1 + \frac{4 \times 0.02}{3}(2 \times 0.9615 - 0.9259 + 2 \times 0.8926)$$

$$\simeq 1.0742$$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{(p)}\right) = 0.8614$$

$\therefore$ From (2),

$$y_4^{c(1)} = 1.0392 + \frac{0.02}{3}(0.9259 + 4 \times 0.8926 + 0.8614)$$

$$\simeq 1.0749$$

$$\therefore \tilde{f}_4 = f\left(x_4, y_4^{c(1)}\right) = 0.8615$$

$\therefore$ From (2),

$$y_4^{c(2)} = 1.0749$$

$\therefore y_4^{c(1)} = y_4^{c(2)} = 1.0749$, correct upto four decimal places.

$\therefore y_4 = 1.0749 \quad$ i.e. $y(0.08) = 1.0749$

Ex.16. Using the method of finite difference find the solution of the boundary value problem

$$x^2 y'' + xy' = 1; \; y(1) = 0, \; y(1.4) = 0.0566$$

Solution. The finite difference form of the given equation is

$$x_i^2 \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + x_i \frac{y_{i+1} - y_{i-1}}{2h} = 1$$

i.e. $\left(2x_i^2 - hx_i\right)y_{i-1} - 4x_i^2 y_i + \left(2x_i^2 + hx_i\right)y_{i+1} = 2h^2$

$$i = 1, 2, \dots n-1$$

with the boundary conditions $y_0 = 0, y_n = 0.0566$

Taking $h = 0.1$ i.e. $n = 4$, the above system becomes

$$2.31y_0 - 4.84y_1 + 2.53y_2 = 0.02$$

$$2.76y_1 - 5.76y_2 + 3y_3 = 0.02$$

$$3.25y_2 - 6.76y_3 + 3.51y_4 = 0.02$$

where $y_0 = 0, y_n = 0.0566$

Solving the system we get

$$y_1 = 0.0046, y_2 = 0.0167, y_3 = 0.0345$$

Hence $y(1.1) = 0.0046$, $y(1.2) = 0.0167$, $y(1.3) = 0.0345$

Ex. 17. Using finite difference method, solve the following BVP:

$$\frac{d^2y}{dx^2} + y + 1 = 0$$

with $y(0) = 0, y(1) = 0$ [*M.A.K.A.U.T., MCS-401, 2014,2016*]

Solution. The finite difference form of the given equation is

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i + 1 = 0$$

i.e. $y_{i-1} + \left(h^2 - 2\right)y_i + y_{i+1} = -h^2$, $i = 1, 2, \dots n-1$

with the boundary conditions at $x_0 = 0$ and at $x_n = 1$ i.e. $y_0 = 0, y_n = 0$.

Taking $h = 0.25$ i.e. $n = 4$, the above system becomes

$$y_0 - 1.9375y_1 + y_2 = -0.0625$$

$$y_1 - 1.9375y_2 + y_3 = -0.0625$$

$$y_2 - 1.9375y_3 + y_4 = -0.0625$$

where $y_0 = 0, y_4 = 0$

Solving the system we obtain,

$$y_1 = 0.10468, y_2 = 0.14031, y_3 = 0.10468$$

Hence $y(0.25) = 0.10468, y(0.5) = 0.14031$

and $y(0.75) = 0.10468$

Exercise

I. Short Answer Questions

1. Describe Euler's method to find the solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

Interpret the error involved in this method geometrically.

[*W.B.U.T., CS-312, 2003*]

2. Evaluate $y(0.1)$ by Euler's method, given

$$\frac{dy}{dx} = 3x + y^2, y(0) = 1$$

3. Solve $\frac{dy}{dx} = xy, y(1) = 1$

by Euler's method to compute $y(1.1)$ and $y(1.2)$

4. Find $y(0.10)$ and $y(0.15)$ by Euler's method from the differential equation

$$\frac{dy}{dx} = x^2 + y^2, \text{ with } y(0) = 0$$

correct to four decimal places, taking step length $h = 0.05$

[*W.B.U.T., CS-312, 2007*]

5. Solve by using Euler's method the following differential equation for $x = 1$ by taking $h = 0.2$.

$$\frac{dy}{dx} = xy, y = 1 \text{ when } x = 0.$$

[*W.B.U.T., CS-312, 2008, 2009, 2013*]

6. What is the truncation error in the Runge-Kutta method of order n? In what way is it related to that of Taylor's series method? [W.B.U.T., CS-312, 2006]

7. Evaluate $y(0.02)$ given

$$\frac{dy}{dx} = x^2 + y,\ y(0) = 1$$

by modified Euler's method.

[M.A.K.A.U.T., M(CS)-401, 2016]

8. Using Runge-Kutta method with $h = 0.1$, find $y(0.1)$ given

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$

9. Using Runge-Kutta method with $h = 0.2$ find $y(1)$ given

$$\frac{dy}{dx} = y - x,\ y(0) = 1.5$$

Answers

2. 1.1272 3. 1.111, 1.412 4. $1.25 \times 10^{-4}, 6.25 \times 10^{-4}$

5. 1.4593 7. 1.0202 8. 1.11034 9. 3.36

II. Long Answer Questions

1. Solve numerically the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\left(y^3 - \frac{y}{x}\right)$$

using Euler's method at $x = 1.6$, given that $y = 1$ when $x = 1$. Also find the exact value at $x = 1.6$

2. Solve the following equation by Euler's method

(a) $\frac{dy}{dx} = 1 + x^2 y,\ y(0) = 0.5$ at $x = 0.1$

(b) $\frac{dy}{dx} = 3x + y^2,\ y(0) = 1$ at $x = 0.1$

3. Solve numerically the differential equation

$$\frac{dy}{dx} = 1 - 2xy$$

provided that $y = 0$ at $x = 0$ using Euler's method in $[0, 0.6]$ taking $h = 0.2$

4. Solve the equation

$$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

with the initial condition $y(0) = 0$ by Euler's method to obtain y for $x = 0.25, 0.5$

5. Compute $y(0 \cdot 3)$ from $\frac{dy}{dx} = 1 + xy$ by modified Euler's method, given that

$$y(0) = 2$$

6. Using modified Euler's method, solve the following

(i) $\frac{dy}{dx} = x + \sqrt{y}$, $y(0) = 1$ at $x = 0.6$

(ii) $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ at $x = 0.2$

7. Given $\frac{dy}{dx} = y - x$, $y(0) = 2$. Find $y(0.1)$ and $y(0.2)$ using second order Runge-Kutta method.

8. Using Runge-Kutta method with $h = 0.1$, find $y(0.2)$, $y(0.4)$ given

(i) $\frac{dy}{dx} = x - y^2$, $y(0) = 1$

(ii) $\frac{dy}{dx} = -xy$, $y(0) = 1$

(iii) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$ *[W.B.U.T., CS-312, 2010]*

9. Find the values of $y(0.1)$, $y(0.2)$ and $y(0.3)$, using Runge-Kutta method of the fourth order, given that

$$\frac{dy}{dx} = xy + y^2, \quad y(0) = 1$$

[W.B.U.T., CS-312, 2009]

10. Solve the equation
$$\frac{dy}{dx} = \frac{1}{x+y},\ y(0) = 1,$$ for $y(0.1)$ and $y(0.2)$, using Runge-Kutta method of the fourth order. [*W.B.U.T., CS-312, 2006*]

11. Compute $y(0.2)$ from the equation
$$\frac{dy}{dx} = x + y,\ y(0) = 1$$
taking step length $h = 0.1$ by 4th order Runge-Kutta method correct to three decimal places. [*W.B.U.T., CS-312, 2007*]

12. Solve by Milne's predictor-corrector method
$$\frac{dy}{dx} = \frac{2y}{x}$$ at $x = 2$, given that $y(1) = 2,\ y(1.25) = 3.13,$ $y(1.5) = 4.5$ and $y(1.75) = 6.13$

13. Apply Milne's method to find $y(0.8)$ for the equation
$$\frac{dy}{dx} = x + y^2,$$ given that $y(0) = 0,\ y(0.2) = 0.02,\ y(0.4) = 0.0805,$ $y(0.6) = 0.1839$

14. Find $y(4.4)$ using Adam's Bashforth method from
$$\frac{dy}{dx} = \frac{2-y^2}{5x},\ y(4) = 1,\ y(4.1) = 1.0049,\ y(4.2) = 1.0097,$$ $y(4.3) = 1.0143$

15. Using Adam's Bashforth method, determine $y(1.4)$ given that $\frac{dy}{dx} = x^2(1+y),\ y(1) = 1$ where the starting values are to be obtained from Runge-Kutta method.

16. Solve the following B.V.P using finite difference method:

(a) $x^2y'' + xy'^2 - 1 = 0;\ y(1) = 0,\ y(1.4) = 0.0566$

(b) $xy'' + y' = 1$ with $y(1) = 1$ and $y(1.4) = 1.736$

17. Solve $y'' - 64y + 10 = 0,\ y(0) = y(1) = 0$ by finite difference method and compute the value of $y(0.5)$

Answers

1. 1.0859, 1.0659 **2.** (a) 0.100025 (b) 1.127

3. $y(0.2) = 0.1948,\ y(0.4) = 0.3599,\ y(0.6) = 0.4748$

4. 0.0052, 0.0416 **5.** 2.4019

6. (i) 1.88278 (ii) 0.181408 **7.** 2.2050, 0.2421

8. (i) 0.851, 0.780 (ii) 0.9802, 0.9231 (iii) 0.2027, 0.4228

9. 1.1138, 1.2689, 1.4856 **10.** 1.0914, 1.1696

11. 1.2205 **12.** 8 **13.** 0.3364 **14.** 1.0187 **15.** 2.3840

16. (a) $y(1.1) = 0.0046,\ y(1.2) = 0.0167,\ y(1.3) = 0.0345$

(b) $y(1.1) = 1.193,\ y(1.2) = 1.560,\ y(1.3) = 1.378$

17. $y(0.5) = 0.1470$

III. Multiple Choice Questions

1. The recursion formula of Euler's method is

(a) $y_{n+1} = y_n + hf(x_n, y_{n-1})$ (b) $y_{n+1} = y_n + hf(x_n, y_{n+1})$

(c) $y_{n+1} = y_n + hf(x_n, y_n)$ (d) none

2. Error in the 4-th order Runge-Kutta method is of

(a) $0(h^3)$ (b) $0(h^2)$ (c) $0(h^4)$ (d) $0(h^5)$

3. Runge-Kutta formula has a truncation error, which is of the order of

(a) h^2 (b) h^4 (c) h^5 (d) none

[M.A.K.A.U.T., CS-312, 2004, 2006, 2010, M(CS)-401, 2016, 2013]

4. Error in the 2nd order Runge-Kutta method is of

(a) $O(h^3)$ (b) $O(h^2)$ (c) $O(h^4)$ (d) $O(h^5)$

5. The truncation error of Euler's method is

(a) $O(h)$ (b) $O(h^3)$ (c) $O(h^4)$ (d) $O(h^2)$

[M.A.K.A.U.T., MCS-301, 2014, MCS-401, 2014]

6. The predictor-corrector method is

(a) Euler's method

(b) 4-th order Runge-Kutta method

(c) Taylor's series method

(d) Modified Euler's method

7. Which of the following are predictor corrector method?

(a) Milne's method (b) Adams Bash forth method

(c) Both (a) and (b) (d) Newton's formula

[M.A.K.A.U.T., M(CS)-401, 2015]

8. Runge-Kutta method is used to solve

(a) an algebraic equation

(b) a first order ordinary differential equation

(c) a first order partial differential equation

(d) none of these [M.A.K.A.U.T., M(CS)-301, 2014, M(CS)-401, 2014, 2015]

9. Milne's predictor formula of order 4 is

(a) $y_{n-1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$

(b) $y_{n+1} = y_{n-3} + \frac{h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$

(c) $y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$

(d) none of these

10. Milne's corrector's formula of order 4 is

(a) $y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1})$

(b) $y_{n+1} = y_{n-1} + \frac{4h}{3}(f_{n-1} + 4f_n + f_{n+1})$

(c) $y_n = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1})$

(d) none of these

11. 4-th step Adam's Bashforth formula is

(a) $y_{n+1} = y_n - \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$

(b) $y_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$

(c) $y_n = y_{n-1} + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}]$

(d) none

12. Milne's predictor-corrector method is a single step method

(a) False **(b)** True

13. The finite difference method is also known as net method

(a) False **(b)** True

14. The finite difference method is used to solve

(a) a system of ordinary differential equation

(b) a B.V.P

(c) a partial differential equation

(d) a system of transcendental equation

15. In finite difference method, $\left(\dfrac{dy}{dx}\right)_{x=x_i}$ is replaced by

(a) $\dfrac{y_{i-1}-y_{i+1}}{2h}$ (b) $\dfrac{y_{i+1}-y_i}{2h}$

(c) $\dfrac{y_{i+1}-y_{i-1}}{2h}$ (d) $\dfrac{y_{i+1}-y_{i-1}}{2h^2}$

16. In finite difference method, $\left(\dfrac{d^2y}{dx^2}\right)_{x=x_n}$ is replaced by

(a) $\dfrac{y_{n+1}-2y_{n-1}+y_n}{2h^2}$ (b) $\dfrac{y_{n+1}-2y_n+y_{n-1}}{2h^2}$

(c) $\dfrac{y_{n+1}-2y_n+y_{n-1}}{2h}$ (d) $\dfrac{y_{n+1}-2y_{n-1}+y_n}{2h}$

Answers

1.c	2.d	3.c	4.a	5.d	6.d	7.b	8.c	9.c	10.a
11.b	12.a	13.b	14.b, c	15.c	16.b				