4 NUMERICAL INTEGRATION

4.1 Introduction:

In this chapter we derive and analyse numerical methods to evaluate definite integrals of the form

$$I = \int_a^b f(x)dx$$

for any finite interval $[a, b]$ by replacing the function $f(x)$ with a suitable polynomial $p(x)$ such that $\int_a^b p(x)dx$ is taken to be an approximation of the integral I. The approximation of I is usually known as *numerical integration or quadrature.*

[*W.B.U.T., CS-312, 2006*]

Let $y = f(x)$ be a real valued function defined in $[a, b]$ such that the values of $f(x)$ are known for $x = x_i$ $(i = 0, 1, 2, \ldots n)$ whose all x_i lies in $[a, b]$ and $y_i = f(x_i)$, $(i = 0, 1, 2, \ldots n)$. Also let $p(x)$ be the interpolating polynomial of degree at most n such that

$$p(x_i) = f(x_i) = y_i \qquad \ldots \quad (1)$$

Thus $\quad p(x) \simeq f(x)$ and so

$$I = \int_a^b f(x)dx \simeq \int_a^b p(x)dx \qquad \ldots \quad (2)$$

Then the expression

$$E(x) = \int_a^b f(x)dx - \int_a^b p(x)dx \qquad \ldots \quad (3)$$

is known as the error of integration.

4.2. The Important Concepts.

(a) Degree of precision.

A quadrature formula is said to have a degree of precision m (m being a positive integer) if it is exact i.e. the error is zero for an arbitrary polynomial of degree $m \le n$ but there exists a polynomial of degree $m+1$ for which it is not exact i.e. the error is not zero.

(b) Composite rule.

Sometimes it is more convenient to divide the interval $[a, b]$, $(a = a_0 < a_1 < a_2 < \ldots < a_{m-1} < a_m = b)$ of integration into finite number of sub-intervals, say $m(< n)$, $[a_{i-1}, a_i]$, $(i = 1, 2, \ldots m)$. Then we apply a quadrature formula separately to each of these sub-intervals and add the results. The formula thus obtained is called *composite rule* corresponding to that *quadrature formula*.

4.3. A general quadrature formula.

Consider the definite integral

$$I = \int_a^b f(x)dx \qquad \ldots \quad (4)$$

and divide the interval $[a, b]$ of integration into n equal sub-intervals such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ and $x_i = x_0 + ih$ $(i = 0, 1, 2, \ldots n)$; h is the space length.

Also let the function $f(x)$ be known at the nodes x_i, i.e., the values $y_i = f(x_i)$, $i = 0, 1, 2, \ldots n$ are given. Then by putting $s = \dfrac{x - x_0}{h}$ i.e. $x = x_0 + sh$ in (4), we get

$$I = \int_{x_0}^{x_0+nh} f(x)dx$$

$$= h\int_0^n f(x_0 + sh)ds$$

$$= h\int_0^n E^s f(x_0)ds \quad \left[\because E^n f(x) = f(x + nh)\right]$$

$$= h\int_0^n (1+\Delta)^s y_0 \, ds \qquad [\because E = 1 + \Delta]$$

$$= h\int_0^n \left[1 + s\Delta + \frac{s(s-1)}{2!}\Delta^2 + \frac{s(s-1)(s-2)}{3!}\Delta^3 + \ldots\right] y_0 \, ds$$

so that

$$I=\int_{x_0}^{x_n} f(x)dx = h\left[ny_0+\frac{n^2}{2}\Delta y_0+\frac{2n^3-3n^2}{12}\Delta^2 y_0\right.$$

$$\left.+\frac{n^4-4n^3+4n^2}{24}\Delta^3 y_0+\dots\right] \qquad \dots \quad (5)$$

The formula (5) is known as general integration formula when the interval of integration is divided into n equal sub-intervals.

We can derive some integration formulae from (5) as particular cases by putting $n=1,2,3,\dots$

4.4. Trapezoidal rule.

Putting $n=1$ in (5), we obtain

$$\int_{x_0}^{x_1} f(x)dx \simeq h\left[y_0+\frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0+\frac{1}{2}(y_1-y_0)\right]$$

$$=\frac{1}{2}h(y_0+y_1) \qquad \dots \quad (6)$$

This is called Trapezoidal rule for numerical integration for two points.

Similarly we have

$$\int_{x_1}^{x_2} f(x)dx \simeq \frac{1}{2}h(y_1+y_2)$$

$$\dots \qquad \dots \qquad \dots$$

$$\int_{x_{n-1}}^{x_n} f(x)dx \simeq \frac{1}{2}h(y_{n-1}+y_n)$$

Adding the above integrals, we obtain

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{2}\left[y_0+y_n+2(y_1+y_2+\dots+y_{n-1})\right] \qquad \dots \quad (7)$$

which is known as composite Trapezoidal rule.

Error in Trapezoidal rule

The error committed in (6) is given by

$$E = \int_{x_0}^{x_1} f(x)dx - \frac{1}{2}h(y_0 + y_1)$$

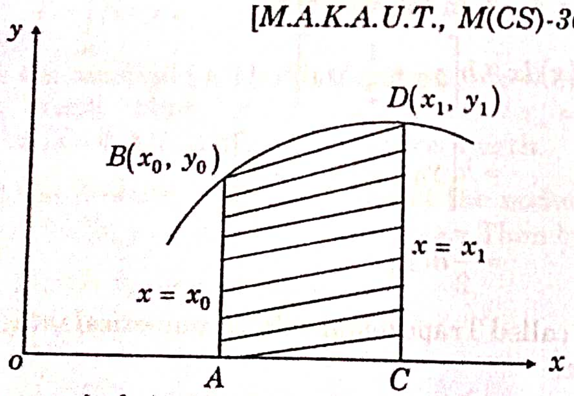
$$= -\frac{1}{12}h^3 f''(x_0) \quad \dots \quad (8)$$

The total error committed in composite Trapezoidal rule (7) is

$$E_T = -\frac{1}{12}h^3 n f''(\xi), x_0 < \xi < x_n \quad \dots \quad (9)$$

Geometrical Significance of Trapezoidal rule.

[M.A.K.A.U.T., M(CS)-301, 2014]



The geometrical significance of Trapezoidal rule lies on the fact that the curve $y = f(x)$ is replaced by straight lines joining the points $B(x_0, y_0)$ and $D(x_1, y_1)$. The area bounded by the curve $y = f(x)$, the ordinates $x = x_0, x = x_1$ and the x-axis is approximately equal to the area of the trapezium $BACD$. For this reason, Trapezoidal rule is also known as *trapezium rule*.

Note. (i) The Trapezoidal rule can be applied for any (even or odd) number of equal sub-intervals.

(ii) In Trapezoidal rule, the error involves second order derivatives of the function $f(x)$ and so it gives the exact value of the integral if $f(x)$ is constant or a first degree polynomial. Hence the degree of precision of this formula is one.

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M(CS)-301, 2014]

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4.5. Simpson's $\frac{1}{3}$ rd rule [*W.B.U.T, CS-312, 2002*]

Putting $n = 2$ in (5) we obtain

$$\int_{x_0}^{x_2} f(x)dx \simeq 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right],$$ neglecting the third and heigher order differences.

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2) \qquad \dots (10)$$

This is called *Simpson's one-third rule for numerical integration for three points.*

Similarly we have

$$\int_{x_2}^{x_4} f(x)dx \simeq \frac{h}{3}(y_2 + 4y_3 + y_4)$$

... ...

$$\int_{x_{n-2}}^{x_n} f(x)dx \simeq \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

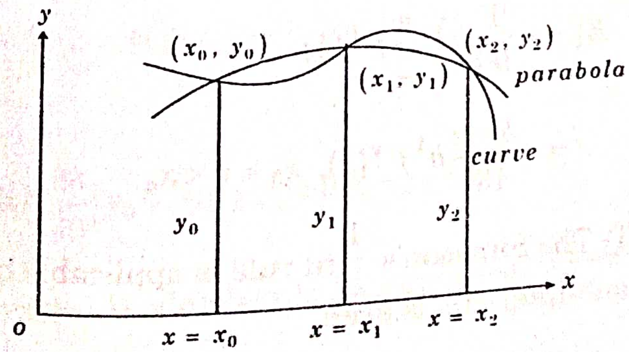
Adding the above integrals, we get

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right] \qquad \dots (11)$$

which is known as *composite Simpson's* $\frac{1}{3}$ *rd rule*

Geometrical Significance of Simpson's $\frac{1}{3}$ rule.

[*W.B.U.T, CS-312, 2008, 2009, 2013, 2014, 2015*]



We have seen that Simpson's one third rule is given by

$$\int_{x_0}^{x_2} f(x)dx \simeq \frac{h}{3}(y_0 + 4y_1 + y_2)$$

The integral $\int_{x_0}^{x_2} f(x)dx$ represents geometrically the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = x_0, x = x_2$. On the otherhand $\frac{h}{3}(y_0 + 4y_1 + y_2)$ represents geometrically the area of the region bounded by a parabola passing through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), x-axis and the lines $x = x_0, x = x_2$. Thus the area bounded by the curve $y = f(x), x = x_0, x = x_2$ and $y = 0$ is approximated to the area bounded by the second degree parabola passing through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), $y = 0$ and the lines $x = x_0, x = x_2$.

Error in Simpson's $\frac{1}{3}$ rd rule.

[W.B.U.T, CS-312, 2002, 2006]

The error committed in formula (10) is given by

$$E_s = \int_{x_0}^{x_2} f(x)dx - \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\simeq -\frac{1}{90}h^5 f^{iv}(x_0) \quad \ldots \quad (12)$$

which is the error in the interval $[x_0, x_2]$.

The total error committed in composite Simpson's $\frac{1}{3}$ rd rule (11) is given by

$$E_s^c \simeq -\frac{1}{90}h^5 \cdot \frac{n}{2} f^{iv}(\xi)$$

$$= -\frac{b-a}{180}h^4 f^{iv}(\xi), \; x_0 < \xi < x_n \quad \ldots \quad (13)$$

Note. (i) The Simpson's $\frac{1}{3}$ rd rule is applicable only if the number of sub-intervals is even.

(ii) Since the error involves fourth order derivatives of $f(x)$, the Simpson's one third rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to three. Hence the degree of precision of this formula is three.

4.6. Weddle's rule

Putting $n = 6$ in (5), we obtain

$$\int_{x_0}^{x_6} f(x)dx = 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0\right]$$

$$= h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right] - \frac{h}{140}\Delta^6 y_0$$

If we now choose h in such a way that the sixth order difference are very small, then we may neglect the small term $\frac{h}{140}\Delta^6 y_0$.

Thus we have

$$\int_{x_0}^{x_6} f(x)dx \simeq \frac{3h}{10}\big[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0)$$
$$+80(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$
$$+11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$
$$+(y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)$$

so that

$$\int_{x_0}^{x_6} f(x)dx \simeq \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \quad \ldots \quad (14)$$

This is called Weddle's rule for numerical integration.

Similarly we have

$$\int_{x_6}^{x_{12}} f(x)dx \simeq \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

... ...

$$\int_{x_{n-6}}^{x_n} f(x)dx \simeq \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{3h}{10}[y_0 + y_n + (y_2 + y_4 + y_8 + ... + y_{n-4})$$

$$+5(y_1 + y_5 + y_7 + ... + y_{n-1}) + 6(y_3 + y_9 + y_{15} + ... + y_{n-3}) \quad ... \quad (15)$$

$$+2(y_6 + y_{12} + y_{18} + ... + y_{n-6})]$$

which is known as composite Weddle's rule.

Error in Weddle's rule

The error committed in formula (14) is given by

$$E_w = \int_{x_0}^{x_n} f(x)dx - \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$\simeq -\frac{h^7}{140} f^{vi}(x_0), \quad ... \quad (16)$$

which is the error in the interval $[x_0, x_n]$

The total error committed in composite Weddle's rule (15) is given by

$$E_w^c \simeq -\frac{nh^7}{840} f^{vi}(\xi)$$

$$= -\frac{(b-a)h^6}{840} f^{vi}(\xi) \quad x_0 < \xi < x_n \quad ... \quad (17)$$

Note. (i) The Weddle's rule is applicable only when the number of sub-intervals is multiple of six.

(ii) Since, the error involves sixth order derivatives of $f(x)$, the Weddle's rule yields an exact value of the integral if $f(x)$ is a polynomial of degree less than or equal to five. Hence the degree of precision of this formula is five.

Example.1. Find the approximate value of

$I=\int\frac{dx}{1+x}$ when the interval is $(0,1)$ and $h=\frac{1}{2}$. Use Trapezaidal rule. *[W.B.U.T.,MCS-301, 2009]*

Solution. Here $f(x)=\frac{1}{1+x}$, $h=\frac{1}{2}=0.5$

$$\therefore\ n=\frac{1-0}{0.5}=2$$

The different values of x and $f(x)$ are given below :

x	:	0	0.5	1
$f(x)$	:	1	0.6667	0.5

$\therefore$ By Trapezoidal rule, we get

$$\int_0^1\frac{dx}{1+x}=\frac{h}{2}(y_0+2y_1+y_2)$$

$$=\frac{0.5}{2}[1+2\times0.6667+0.5]$$

$$=0.70835$$

Example.2. Calculate the area of the function $f(x)=\sin x$ with limits $(0,90^\circ)$ by simpson's $\frac{1}{3}rd$ rule using 11 ordinates.

[W.B.U.T., MCS-301, 2008]

Solution. Here $f(x)=\sin x$, $n=11-1=10$

$$\therefore\ h=\frac{\frac{\pi}{2}-0}{10}=\frac{\pi}{20}$$

The different values of x and $f(x)$ are given below :

x :	0	$\frac{\pi}{20}$	$\frac{\pi}{10}$	$\frac{3\pi}{20}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$
$f(x)$:	0	0.15643	0.30902	0.45399	0.58779	0.707107
		$\frac{3\pi}{10}$	$\frac{2\pi}{20}$	$\frac{2\pi}{5}$	$\frac{9\pi}{20}$	$\frac{\pi}{2}$
		0.80902	0.89101	0.95106	0.98769	1

$\therefore$ By Simpson's $\frac{1}{3}$ rd rule, we have

$$\int_0^{\frac{\pi}{2}} \sin x\, dx = \frac{\pi}{60}[0+4+4(0.15643+0.45399+0.707107+0.89101+0.98769)$$
$$+2(0.30902+0.58779+0.80902+0.95106)]$$
$$=1.000005$$
$$\simeq 1.$$

Example.3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using **(i)** Trapezoidal

(ii) Simpson's one-third and [*W.B.U.T.,CS-312, 2006, 2015*]

(iii) Weddle's rules, taking $n=6$. Hence compute an approximate value of π in each case.

[*M.A.K.A.U.T., MCS-401, 2014*]

Solution. Let $f(x)=\frac{1}{1+x^2}$

Since $n=6$, so $h=\frac{1-0}{6}=\frac{1}{6}$

The tabulated values of $f(x)$ for different values of x are given below :

x :	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$y=f(x)$:	1	0.973	0.9	0.8	0.6923	0.5902	0.5

(i) We have by composite Trapezodial rule,

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$= \frac{1}{2\times 6}\left[1 + 0.5 + 2(0.973 + 0.9 + 0.8 + 0.6923 + 0.5902)\right]$$

$$= 0.7842$$

Now, since $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) = \frac{\pi}{4}$,

so we have

$$\frac{\pi}{4} \simeq 0.7842$$

leading to $\pi \simeq 3.1270$

(ii) The composite Simpson's one third rule gives

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

$$= \frac{1}{3\times 6}\left[1 + 0.5 + 4(0.973 + 0.8 + 0.5902) + 2(0.9 + 0.6923)\right]$$

$$\simeq 0.7854$$

and hence $\frac{\pi}{4} \simeq 3.1416$ giving

$$\pi \simeq 3.1416$$

(iii) The composite Weddle's rule yields

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{3h}{10}\left[y_0 + y_n + (y_2 + y_4 + \ldots) + 5(y_1 + y_5 + \ldots) + 6(y_3 + y_9 + \ldots) + 2(y_6 + y_{12} + \ldots)\right]$$

$$= \frac{3}{10\times 6}\left[1 + 0.5 + (0.9 + 0.6923) + 5(0.973 + 0.5902) + 6\times 0.8\right]$$

$$\simeq 0.7854$$

and therefore $\frac{\pi}{4} \simeq 0.7854$ so that

$$\pi \simeq 3.1417$$

4.7. A quadrature formula based on Lagrange's interpolation formula.

Let $y = f(x)$ be a real valued function defined and continuous in $[a, b]$ at $n+1$ nodes x_i $(i = 0, 1, 2, \ldots, n)$. Also let $p(x)$ be the interpolating polynomial of degree less than or equal to n such that $f(x_i) = p(x_i), (i = 0, 1, 2, \ldots, n)$. Then we have

$$f(x) \simeq p(x)$$

so that

$$\int_a^b f(x)\, dx \simeq \int_a^b p(x)\, dx \qquad \ldots \quad (18)$$

Now, if $p(x)$ is the Lagrange's polynomial i.e., if

$$p(x) \simeq L_n(x) = \sum_{i=0}^{n} \frac{p_{n+1}(x)}{(x - x_i)\, p'_{n+1}(x_i)}\, y_i \qquad \ldots \quad (19)$$

where $p_{n+1}(x) = (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n)$,

then

$$\int_a^b f(x)dx \simeq \int_a^b \left[\sum_{i=0}^{n} \frac{p_{n+1}(x)}{(x - x_i)\, p'_{n+1}(x_i)}\, y_i \right] dx.$$

$$= \sum_{i=0}^{n} y_i H_i^{(n)}, \text{ say} \qquad \ldots \quad (20)$$

where $$H_i^{(n)} = \int_a^b \frac{p_{n+1}(x)}{(x - x_i)\, p'_{n+1}(x_i)}\, dx \qquad \ldots \quad (21)$$

The formula (20) is called the quadrature formula based on Lagrange's interpolation formulae.

4.8. Newton- Cote's closed type formula.

Suppose that the interpolating points $x_i (i = 0, 1, 2, \ldots, n)$ are equispaced of step length $h\left(= \dfrac{b-a}{n}\right)$ so that $x_i = x_0 + ih$ $(h > 0, i = 0, 1, 2, \ldots, n)$ and $x_0 = a, x_n = b$. Then from (20) we have

$$\int_{x_0}^{x_n} f(x)dx \simeq \sum_{i=0}^{n} y_i H_i^{(n)} \qquad \ldots \quad (22)$$

$$\text{Now, } H_i^{(n)} = \int_{x_0}^{x_n} \frac{p_{n+1}(x)}{(x-x_i)\,p'_{n+1}(x_i)}\,dx$$

$$= \int_{x_0}^{x_0+nh} \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}\,dx$$

$$= \int_0^n (-1)^{n-i}\,\frac{h}{i!(n-1)!}\,\frac{s(s-1)(s-2)\ldots(s-n)}{s-i}\,ds$$

[putting $x = x_0 + hs$ so that $x - x_i = (s-i)h$]

$$= \frac{b-a}{n}\,\frac{(-1)^{n-i}}{i!(n-i)!}\int_0^n \frac{s(s-1)(s-2)\ldots(s-n)}{s-i}\,ds \quad \left[\because h = \frac{b-a}{n}\right]$$

$$= (b-a)C_i^{(n)}, \text{ say} \qquad \ldots \quad (23)$$

$$\text{where } C_i^{(n)} = \frac{1}{n}\,\frac{(-1)^{n-i}}{i!(n-i)!}\int_0^n \frac{s(s-1)(s-2)\ldots(s-n)}{s-i}\,ds,$$

$$i = 0, 1, 2, \ldots, n \qquad \ldots \quad (24)$$

are called *Cote's coefficients.*

Thus from (22), we have

$$\int_a^b f(x)dx \simeq (b-a)\sum_{i=0}^{n} y_i C_i^{(n)} \qquad \ldots \quad (25)$$

The formulae (25) is called Newton-Cote's integration formula.

Some properties of Cote's coefficients

1. $\sum_{i=0}^{n} C_i^{(n)} = 1$

Proof. Putting $f(x) = 1$ in (25) we get

$$\int_a^b f(x)dx = (b-a)\sum_{i=0}^{n} C_i^{(n)}$$

$$\text{or, } b-a = (b-a)\sum_{i=0}^{n} C_i^{(n)} \qquad \ldots \quad (26)$$

$$\therefore \sum_{i=0}^{n} C_i^{(n)} = 1$$

2. $C_i^{(n)} = C_{n-i}^{(n)}$ ... (27)

Proof. From (24) we have

$$C_i^{(n)} = \frac{1}{n}\frac{(-)^{n-i}}{i!(n-i)!}\int_0^n \frac{s(s-1)(s-2)\dots(s-n)}{s-i}ds$$

$$= \frac{1}{n}\frac{(-1)^{2n-i}}{i!(n-i)!}\int_0^n \frac{s'(s'-1)(s'-2)\dots(s'-n)}{s'-(n-i)}ds',$$

putting $s = n - s'$

$$= \frac{1}{n}\frac{(-1)^{i}}{i!(n-i)!}\int_0^n \frac{s'(s'-1)\dots(s'-n)}{s'-(n-i)}ds'$$

$$= C_{n-i}^{(n)}$$

$$\therefore C_i^{(n)} = C_{n-i}^{(n)}$$

3. Few values of $C_{n-i}^{(n)}$

(i) For $n=1$, we have from (26) and (27)

$C_0^{(1)} + C_1^{(1)} = 1$ and $C_0^{(1)} = C_1^{(1)}$

leading to $C_0^{(1)} = C_1^{(1)} = \frac{1}{2}$

(ii) For $n=2$, (26) and (27) give

$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1$ and $C_0^{(2)} = C_2^{(2)}$ ... (28)

Also from (24),

$$C_0^{(2)} = \frac{(-1)^{2-0}}{2.0!(2-0)!}\int_0^2 \frac{s(s-1)(s-2)}{s}ds$$

$$= \frac{1}{6}$$

$$\therefore \quad C_0^{(2)} = C_2^{(2)} = \frac{1}{6}$$

$\therefore$ from (28),

$$C_1^{(2)} = \frac{2}{3}$$

(iii) when $n = 3$, we get from (26) and (27),

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} = 1, \quad C_0^{(3)} = C_3^{(3)}, \quad C_1^{(3)} = C_2^{(3)} \qquad \ldots \qquad (29)$$

Also one gets from (24)

$$C_0^{(3)} = \frac{(-1)^3}{3 \cdot 0!(3-0)!} \int_0^3 \frac{s(s-1)(s-2)(s-3)}{s} ds$$

$$= \frac{1}{8}$$

Thus (29) yields

$$C_0^{(3)} = C_3^{(3)} = \frac{1}{8}, \quad C_1^{(3)} = C_2^{(3)} = \frac{3}{8}$$

Proceeding in this way, we can find out the values of $C_i^{(n)}$ for other values of n.

Some deduction from Newton-Cotes formula

1. Trapezoidal rule

[*W.B.U.T., CS-312, 2009,M*(*CS*)-301,2014]

For $n = 1$, we have for the interval $[a, b]$, $h = b - a$. Then from (25), we get

$$\int_{x_0}^{x_1} f(x)dx \simeq (b-a)\sum_{i=0}^{1} y_i C_i^{(1)}$$

$$= h\left(y_0 C_0^{(1)} + y_1 C_1^{(1)}\right)$$

$$= h\left(y_0 \frac{1}{2} + y_1 \frac{1}{2}\right)$$

$$= \frac{1}{2} h(y_0 + y_1)$$

which is the Trapezoidal rule for numerical integration.

2. Simpson's one-third rule

[*W.B.U.T., CS-312, 2006, M(CS)-301, 2014*]

Putting $n = 2, h = \dfrac{b-a}{2}$ in (25) we obtain for the interval $[a, b]$,

$$\int_{x_0}^{x_2} f(x)dx \simeq (b-a)\sum_{i=0}^{2} y_i C_i^{(3)}$$

$$= 2h\left(y_0 C_0^{(2)} + y_1 C_1^{(2)} + y_2 C_2^{(2)}\right)$$

$$= 2h\left(y_0 \cdot \frac{1}{6} + y_1 \cdot \frac{4}{6} + y_2 \cdot \frac{1}{6}\right),$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

which is Simpson's one-third rule.

Illustrative Examples

Ex.1. Evaluate $\int_0^1 \dfrac{dx}{1+x}$ by Simpson's composite rule taking eleven ordinates and hence find the value of $\log_e^2$ correct upto five significant figures. [*W.B.U.T., CS-312, 2004*]

Solution. Let $f(x) = \dfrac{1}{1+x}$

Here $a = 0, b = 1, n = 10$ [$\because$ number of ordinates is 11]

$\therefore h = \dfrac{1-0}{10} = 0.1$

The tabulated values of x and $f(x)$ are given below:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1	0.9091	0.8333	0.7692	0.7143	0.6667	0.6250

x	0.7	0.8	0.9	1
$f(x)$	0.5882	0.5556	0.5263	0.5

Using Simpson's one-third rule, we got

$$\int_0^1 \frac{dx}{1+x} \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

$$= \frac{0.1}{3}\left[1 + 0.5 + 4(0.9091 + 0.7692 + 0.6667 + 0.5882 + 0.5263)\right.$$

$$\left. + 2(0.8333 + 0.7143 + 0.6250 + 0.5556)\right]$$

$$= 0.69314667$$

Now $\int_0^1 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^1 = \log 2$

$\therefore \log 2 \simeq 0.69314667$

$\simeq 0.69315$, correct upto five significant figures.

Ex.2. Use Simpson's one-third rule to evaluate $\int_0^6 \frac{dx}{(1+x)^2}$, taking six equal sub-intervals of $[0, 6]$, correct to three decimal places.

Solution. Let $f(x) = \frac{1}{(1+x)^2}$

Here $x_0 = 0,\ x_n = 6, n = 6$ so that $h = \frac{6-0}{6} = 1$

The tabulated values of $f(x)$ for different values of x are given below:

x :	0	1	2	3	4	5	6
y :	1	0.250	0.111	0.062	0.040	0.028	0.020

$\therefore$ By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^6 \frac{dx}{(1+x)^2} \simeq \frac{1}{3}\left[1 + 0.020 + 4(0.250 + 0.062 + 0.028) + 2(0.111 + 0.040)\right]$$

$= 0.894$, correct upto three decimal places.

Ex.3. Evaluate $\int_0^1 \cos x \, dx$, correct upto three significant figures using the datas:

x :	0	0.2	0.4	0.6	0.8	1.0
$\cos x$:	1	0.98007	0.92106	0.82534	0.69671	0.54030

[M.A.K.A.U.T.,M(CS)-401, 2013]

Solution. As the number of sub-intervals is odd, so we apply Trapezoidal rule to evaluate the integral.

Here $h = 0.2$

Thus using Trapezoidal rule, we get

$$\int_0^1 \cos x \, dx = \frac{0.2}{2}[(1+0.54030)+2(0.98007+0.92106+0.82534$$

$$+0.69671+0.54030)]$$

$$= 2.167928$$

$$= 2.17, \text{ correct upto three significant figures.}$$

Ex.4. Evaluate $\int \frac{x}{\sin x} dx$ where the interval is (0, 2) by using Trapezoidal rule taking $n = 8$.

[M.A.K.A.U.T., (M)CS-701, 2014]

Solution. Let $f(x) = \frac{x}{\sin x}$

Here $x_0 = 0$, $x_n = 2$, $n = 8$

$$\therefore h = \frac{2-0}{8} = \frac{1}{4}.$$

The tabulated values of x and $f(x)$ are given below :

x :	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
$f(x)$:	1	1.0105	1.0429	1.1003	1.1884	1.3172	1.5038	1.7785	2.19[illegible]

$\therefore$ By Trapezoidal rule we here

$$\int_0^2 \frac{x}{\sin x}dx = \frac{\frac{1}{4}}{2}[1 + 2.1995 + 2(1.0105 + 1.0429 + 1.1003$$
$$+ 1.1884 + 1.3172 + 1.5038 + 1.7785)]$$
$$= 2.6353375$$
$$= 2.6353, \text{ correct upto 4 decimal places}$$

Ex.5. Calculate by simpson's $\frac{1}{3}rd$ rule the value of the integral $\int_0^1 \frac{x}{1+x}dx$ corrected upto three significant figures.

[*M.A.K.A.U.T., (M)CS-401, 2016*]

Solution. Let $f(x) = \frac{x}{1+x}$

Here $x_0 = 0$, $x_n = 1$. Taking $n = 4$ we get

$h = \frac{1-0}{4} = 0.25.$

We now construct the following table

x	:	0	0.25	0.50	0.75	1.0
$y = f(x)$	:	0	0.2	0.3333	0.4286	0.5

$\therefore$ By using simpson's $\frac{1}{3}rd$ rule we have

$$\int_0^1 \frac{x}{1+x}dx = \frac{0.25}{3}[0 + 0.5 + 4(0.2 + 0.4286) + 2(0.3333)]$$
$$= 0.30675$$
$$= 0.307, \text{ correct upto three significant figure.}$$

Ex.6. compute $I = \int \frac{x}{\sin x}dx$ where the interval is $\left(0, \frac{1}{2}\right)$, using Simpson's rule with $h = \frac{1}{4}$.

[*W.B.U.T., MCS-301, 2009*]

Solution. Here $f(x) = \frac{x}{\sin x}$, $x_0 = 0$, $x_n = \frac{1}{2}$, $h = \frac{1}{4}$

$$\therefore n = \frac{\frac{1}{2} - 0}{\frac{1}{4}} = 2$$

The different values of x and $f(x)$ are given below :

x :	0	$\frac{1}{4}$	$\frac{1}{2}$
$f(x)$:	1	1.0105	1.0429

$\therefore$ By Simpson's $\frac{1}{3}rd$ rule, we have

$$\int_0^{1/2} \frac{x}{\sin x} dx = \frac{\frac{1}{4}}{3}[1 + 4 \times 1.0105 + 1.0429]$$

$$= 0.507075$$

$$= 0.507, \text{ correct upto 3 decimal places}$$

Ex.7. Compute the value of π from the formula $\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with 10 sub-intervals.

[W.B.U.T., MCS-401, 2006]

Solution. Let $f(x) = \frac{1}{1+x^2}$

Since $n = 10$, so $h = \frac{1-0}{10} = 0.1$

The tabulated values of $f(x)$ for different values of x are given below :

x :	0	0.1	0.2	0.3	0.4
$f(x)$:	1	0.990099	0.961538	0.917431	0.862069

0.5	0.6	0.7	0.8	0.9	1.0
0.8	0.735294	0.671141	0.609756	0.552486	0.5

$\therefore$ Using composite Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} \simeq \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$= \frac{0.1}{2}[1 + 0.5 + 2(0.990099 + 0.961538 + 0.917431 + 0.862069 + 0.8 + 0.735294 + 0.671141 + 0.609756 + 0.552486)]$$

$$= 0.784982$$

$$\therefore \frac{\pi}{4} \simeq 0.784982$$

$\therefore \pi \simeq 3.13993$, correct upto 5 decimal places.

Ex.8. Evaluate $\int_0^6 \frac{dx}{1+x^2}$, by Simpson's $\frac{1}{3}rd$ rule, taking 6 equal sub-intervals. [*W.B.U.T., (M)CS-301, 2007*]

Solution. Let $f(x) = \frac{1}{1+x^2}$

Since $n = 6$, so $h = \frac{1-0}{6} = \frac{1}{6}$

The tabulated values of $f(x)$ for different value of x are given below :

x :	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$:	1.000000	0.972972	0.9	0.8	0.692308	0.590164	0.5

Using composite Simpson's $\frac{1}{3}$rd rule, we get

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)\right]$$

$$= \frac{1}{18}\left[1 + 0.5 + 4(0.972972 + 0.8 + 0.590164)\right.$$

$$\left. + 2(0.9 + 0.692308)\right]$$

$$\simeq 0.785398$$

Ex.9. Find the value of the integral $\int_0^1 e^x \, dx$, by Trapezoidal rule with $h = 0.1$.

[*W.B.U.T., MCS-301, 2007*]

Solution. Here $f(x) = e^x$, $x_0 = 0$, $x_n = 1$, $h = 0.1$

$\therefore$ The tabulated values of x and $f(x)$ are given below :

x :	0	0.1	0.2	0.3	0.4
$f(x)$:	1	1.1052	1.2214	1.3498	1.4918

0.5	0.6	0.7	0.8	0.9
1.6487	1.8221	2.0138	2.2255	2.4596

1.0
2.7183

$\therefore$ By Trapezoidal rule we have

$$\int_0^1 e^x dx = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$= \frac{0.1}{2}[1 + 2.7183 + 2(1.1052 + 1.2214 + 1.3498 + 1.4918$$

$$+1.6487 + 1.8221 + 2.0138 + 2.2255 + 2.4596)]$$

$$= 1.7197$$

Ex.10. Evaluate $\int xe^x dx$ where the interval $(-1, 0)$ by using Trapezoidal rule taking $n = 6$. *[W.B.U.T., CS-312, 2009]*

Solution. Let $f(x) = xe^x$

Since $n = 6$, so $h = \frac{0-(-1)}{6} = \frac{1}{6}$

The tabulated values of $f(x)$ for different values of x are given below :

x :	-1	$\frac{-5}{6}$	$\frac{-2}{3}$	$\frac{-1}{2}$	$\frac{-1}{3}$	$\frac{-1}{6}$	0
$f(x)$:	-0.3678	-0.3622	-0.3405	-0.3033	-0.2388	-0.1411	0

Using Trapezoidal rule, we have

$$\int_{-1}^0 xe^x dx = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$= \frac{1}{12}[-0.3678 + 0 + 2(-0.3622 - 0.3405 - 0.3033$$

$$-0.2388 - 0.1411)]$$

$$= -0.261633$$

$= -0.2616$, correct upto four decimal places.

Ex.11. Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3} dx$. using Simpson's $\frac{1}{3}$*rd* rule with $h = 0.25$.

[W.B.U.T., CS-312, 2003, 2008, M(CS)-401, 2015]

Solution. Let $f(x)=\dfrac{x^2}{1+x^3}$

Here $h=0.25$

So the tabulated values of x and $f(x)$ are given below :

x	:	0	0.25	0.50	0.75	1
$f(x)$	:	0	0.06154	0.22222	0.39560	0.50000

$\therefore$ By Simpson's $\dfrac{1}{3}$rd rule we get

$$\int_0^1 \frac{x^2}{1+x^3}dx \simeq \frac{h}{3}\left[y_0+y_n+4(y_1+y_3+\ldots)+2(y_0+y_4+\ldots)\right]$$

$$=\frac{0.25}{3}\left[0+0.5+4(0.06154+0.39560)+2\times 0.22222\right]$$

$$=0.23108$$

Now $\displaystyle\int_0^1 \frac{x^2}{1+x^3}dx=\frac{1}{3}\left[\log(1+x^3)\right]_0^1$

$$=\frac{1}{3}\log 2=\log 2^{1/3}$$

Hence $\log 2^{1/3} \simeq 0.23108$

Ex.12. Evaluate $\displaystyle\int_0^{0.6}\frac{dx}{\sqrt{1-x^2}}$, using Weddle's rule taking 12 equal sub-intervals. [*W.B.U.T., M(CS)-401, 2013*]

Solution. Here $f(x)=\dfrac{1}{\sqrt{1-x^2}}, n=12$

$$\therefore\ h=\frac{0.6-0}{12}=0.05$$

The tabulated values of $f(x)$ for various values of x are given below:

x	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
$f(x)$	1	1.00125	1.00504	1.01144	1.02062	1.03279	1.04838	1.06752	1.09109	1.11978	1.15470	1.19737	1.25

By using Weddle's rule, we get

$$\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}} \simeq \frac{3\times 0.05}{10}\Big[1+1.25+(1.00504+1.02062+1.09109+1.15470)$$
$$+5(1.00125+1.03279+1.06752+1.19737)+6(1.01144+1.11978)$$
$$+2\times 1.04828\Big]$$

$= 0.64350$

Ex.13. Using Weddle's rule, evaluate $\int_0^3 \sqrt{x}\,dx$, taking 12 equal sub-intervals.

Solution. Let $y=\sqrt{x}$.

Here $x_0=0,\ x_n=3,\ n=12$.

$\therefore\ h=\dfrac{3-0}{12}=0.25$

We now construct the following table:

x :	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
y :	0	0.5	0.707106	0.866025	1	1.118034	1.224745	1.322876	1.414214	1.5	1.581139	1.658312	1.732051

$\therefore$ By using Weddle's rule we get

$$\int_0^3 \sqrt{x}\,dx \simeq \frac{3\times 0.25}{10}\Big[(0+1.732051)+(0.707106+1+1.414214+1.581139)$$
$$+5(0.5+1.118034+1.322876+1.658312)+6(0.866025+1.5)+2\times 1.224745\Big]$$

$= 3.005720$

$\simeq 3.0057$, correct upto four decimal places.

Ex.14. Evaluate the integral $\int_0^{\pi/2}\sqrt{1-0.162\sin^2\theta}\,d\theta$ by using Weddle's rule, taking $n=6$.

Solution. Let $f(\theta)=\sqrt{1-0.162\sin^2\theta}$

Here $\theta_0=0,\ \theta_n=\dfrac{\pi}{2},\ n=6$

$\therefore\ h=\dfrac{\pi/2-0}{6}=\dfrac{\pi}{12}=0.26179$, in radian.

The tabulated values of $f(\theta)$ for different values of θ are given below:

θ :	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(\theta)$:	1	0.99455	0.97954	0.95864	0.93728	0.92133	0.91542

By Weddle rule, we get

$$\int_0^{\pi/2} \sqrt{1-0.162\sin^2\theta}\, d\theta \simeq \frac{3\times 0.26179}{10}[1+0.91542+$$

$$(0.97954+0.93728)+5(0.99455+0.92133)+6\times 0.95864]$$

$$= 1.50504$$

Ex.15. **A curve is drawn to pass through the points given by the following table:**

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Estimate by Simpson's one third rule the area bounded by the curve, the x-axis and the lines $x=1, x=4$.

Solution. **Here $h = 0.5$**

So, by using Simpson's one third rule

$$\int_{x_0}^{x_n} f(x)\,dx \simeq \frac{h}{3}[y_0+y_n+4(y_1+y_3+\ldots)+2(y_2+y_4+\ldots)],$$

we have the area bounded by the curve, the x-axis and the lines $x=1, x=4$ as

$$\frac{0.5}{3}[2+2.1+4(2.4+2.8+2.6)+2(2.7+3)]$$

$$= 7.78333$$

Hence the area bounded by the curve, x-axis, and the lines $x=1$ and $x=4$ is 7.78 sq.units, correct upto three decimal places.

Ex.16. Find from the following table, the area bounded by the curve and x-axis from $x = 7.47$ to $x = 7.52$.

x :	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

[W.B.U.T., CS-312, 2010]

Solution. To find the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$, we evaluate the integral,

$$\int_{7.47}^{7.52} f(x)dx$$

Here the number of sub-interval is odd. So we should use Trapezoidal rule to evaluate the integral. Thus Trapezoidal rule

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots y_{n-1})]$$

gives

$$\int_{7.47}^{7.52} f(x)dx \simeq \frac{0.01}{2}[1.93 + 2.06 + 2(1.95 + 1.98 + 2.01 + 2.03)]$$

$$= 0.09965$$

$$\simeq 0.100, \text{ correct upto three decimal places}$$

Ex.17. The following table gives the values of acceleration (f) of a particle in cm/ sec^2 at equal interval of time (t) in sec. Find the velocity of the particle at $t = 2$ secs.

t	:	0.0	0.5	1.0	1.5	2.0
f	:	0.3989	0.3521	0.2420	0.1295	0.0540

Solution. The velocity (v) at $t = 2$ secs is given by

$$v = \int_0^2 f(t)dt$$

Since the number of sub-intervals is 4, we use Simpson's one-third rule. Hence we get

$$\int_0^2 f(t)dt \simeq \frac{0.5}{3}[0.3989 + 0.0540 + 4(0.3521 + 0.1295) + 2 \times 0.2420)]$$

$$= 0.4772$$

Ex.18. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x=0, x=1$ and a curve through the points with the following co-ordinates:

x	:	0	0.25	0.50	0.75	1
y	:	1	0.9896	0.9589	0.9089	0.8415

Find the volume of the solid of revolution.

Solution. The required volume of the solid is

$$V = \pi \int_0^1 y^2\, dx$$

We now construct the following table:

x	:	0	0.25	0.50	0.75	1
y^2	:	1	0.97931	0.91949	0.82610	0.70812

Here $h = 0.25$. So by using Simpson's one-third rule, we have

$$V = \pi \int_0^1 y^2\, dx$$

$$\simeq \pi \times \frac{0.25}{3}\left[1 + 0.7081 + 4(0.97931 + 0.82610) + 2 \times 0.91949\right]$$

$$= 2.82038$$

Exercise

I. Short Answer Questions

1. Integrate numerically $\int_0^4 x^2\, dx$ by Simpson's $\frac{1}{3}$rd rule with 4 sub-intervals.

2. Find the approximate value of $\int \frac{dx}{1+x}$ when the interval is (0, 1) and $h = 1/2$. Use Trapezoidal rule. [*W.B.U.T., CS-312, 2009*]

3. Estimate the value of the following integral by Simpson's $\frac{1}{3}$rd rule taking 4 strips :

$$\int_1^3 \frac{1}{x}\, dx$$ [*W.B.U.T., CS-312, 2010*]

4. Is it possible to evaluate $\int_1^4 f(x)dx$ by Simpson's one-third rule by using the following table? Give reasons.

x :	1	1.6	2.2	2.8	3.4	4
$f(x)$:	0.5	0.1	0.1	0.4	0.9	0.8

5. Simpson's $\frac{1}{3}$rd rule is exact for $\int_a^b x^3 dx$ – comment.

6. Show that the approximate value of $\int_0^1 x\,dx$ calculated by Trapezoidal rule (taking two intervals of equal lengths) is equal to its exact value.

7. Using Trapezoidal rule, evaluate $\int_0^2 \frac{dx}{x^2}$ given :

x :	0	1/2	1	3/2	2
x^2 :	0	1/4	1	9/4	4

8. Find the value of the integral $\int_0^1 e^x dx$ by Trapezoidal rule with $h = 0.1$. [*W.B.U.T., CS-312, 2007*]

9. Calculate the area of the function $f(x) = \sin x$ with limits $(0, 90°)$ by Simpson's $\frac{1}{3}$rd rule using 11 ordinates.

[*W.B.U.T., CS-312, 2008*]

10. What type of function is used in

(i) Simpson's $\frac{1}{3}$rd rule (ii) Trapezoidal rule

11. Using Simpson's $\frac{1}{3}$rd rule, estimate the area, bounded by the curve $y = f(x)$, x-axis and the ordinates $x = 0$ and $x = 2$ from the following data:

x :	0.0	0.5	1.0	1.5	2.0
$f(x)$:	0.3989	0.3521	0.2420	0.3295	0.0540.

11. Evaluate $\int_0^{[illegible]} \frac{dx}{[illegible]}$ taking [illegible] ordinates.

12. Compute $\int_0^{6} f(x)\, dx$ from the following table by Weddle's rule

x :	0	1	2	3	4	5	6
$f(x)$:	0.146	0.165	0.1775	0.1905	0.2035	0.2165	0.2295

13. Use Simpson's one-third rule to estimate the value of $\log_e$ [illegible]

from $\int \frac{dx}{[illegible]}$ where the number of sub-intervals is [illegible]

14. Calculate $\int x(x-1)\, dx$ by (i) Trapezoidal rule

(ii) Simpson's one-third rule dividing the interval into [illegible] equal parts.

15. Evaluate $\int \sqrt{x}\, dx$ using Trapezoidal rule taking $n = 2$

Answers

1. [illegible] 2. 0.7855

[illegible] because the no. of subintervals is odd

[illegible] 9. [illegible] 10. (i) [illegible] (ii) [illegible]

11. [illegible] 12. [illegible] 13. [illegible] 14. [illegible]

15. (i) [illegible] (ii) [illegible] 16. [illegible]

III. Long Answer Questions

1. Compute $I = \int \frac{x}{\sin x}\, dx$ where the interval is [illegible] using [illegible] rule with [illegible]. [illegible]

2. Compute $\int_{1.0}^{1.5} (\ln x)^2 dx$ by taking 10 equal sub-intervals by Trapezoidal rule, correct upto five significant figures.

3. Compute the value of π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

using Trapezoidal rule with 10 sub-intervals.

[*W.B.U.T., CS-312, 2010*]

4. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's $\frac{1}{3}$rd rule, taking 6 equal sub-intervals. What is the geometrical significance of this rule.

[*W.B.U.T., CS-312, 2007*]

5. Find the approximate value of $\int_0^1 \frac{x}{1+x^2} dx$ upto four places of decimal by Simpson's $\frac{1}{3}$rd rule, taking 6 equal sub-intervals of [0, 1] and hence find the approximate value of log2 correct to four places of decimal.

6. Applying Trapezoidal rule, evaluate $\int_0^1 e^{-x^2} dx$ with 10 divisions.

7. Evaluate $\int_0^1 \cos x \, dx$ correct upto three decimal places, by using any suitable formula.

8. Evaluate $\int_0^8 \frac{dx}{1+x}$ by

(i) Trapezoidal rule and (ii) Simpson's $\frac{1}{3}$rd rule, taking 8 equal sub-intervals. Hence estimate the value of $\log_e^3$ in each case.

9. Use Simpson's $\frac{1}{3}$rd rule to evalute $\int_0^6 \frac{dx}{(1+x)^2}$, taking six equal sub-intervals of $[0, 6]$.

10. Evaluate approximately by Trapezoidal rule, the integral $\int_0^1 (4x-3x^2)dx$, by taking $n=10$. Compute the exact integral and hence find the absolute and relative error.

11. Compute $\int_{0.2}^{1.0} x^2($
Simpson's $\frac{1}{3}$rd rule
significant figures.

12. Find the value
places by Simpson

13. Find the valu
significant figures
$\frac{1}{3}$rd rule.

14. Compute the
places by using (i)

15. Evaluate $\int_{-1}^{3} |x| d$
and Simpson's $\frac{1}{3}$

16. Using Weddle

17. Find Lagrang
the set of points

x
y

Use it to evalu

18. Evaluate $\int_{4.0}^{5.2} \text{lo}$

11. Compute $\int_{0.2}^{1.0} x^2(1-x)\,dx$ by taking step length 0.1 by Simpson's $\frac{1}{3}$rd rule, obtaining the result correct to three significant figures.

12. Find the value of $\int_{1}^{5} \log_{10}^{x}\,dx$ correct upto three decimal places by Simpson's one-third rule.

13. Find the value of $\int_{0}^{\pi/2} \sqrt{\sin x}\,dx$, taking $n = 8$, correct to five significant figures, by using **(i)** Trapezoidal rule **(ii)** Simpson's $\frac{1}{3}$rd rule.

14. Compute the value of $\int_{1\cdot2}^{1\cdot6} (x + \frac{1}{x})\,dx$, correct upto five decimal places by using **(i)** Trapezoidal rule **(ii)** Simpson's $\frac{1}{3}$rd rule.

15. Evaluate $\int_{-1}^{3} |x|\,dx$ analytically and numerically by Trapezoidal and Simpson's $\frac{1}{3}$rd rule , taking four equal sub-intervals.

16. Using Weddle's rule, compute $\int_{0}^{\pi/2} \sqrt{\cos\theta}\,d\theta$.

17. Find Lagrange's interpolation polynomial passing through the set of points

x	:	0	1	2
y	:	4	3	6

Use it to evaluate $\int_{0}^{3} y\,dx$ [*W.B.U.T., CS-312, 2008*]

18. Evaluate $\int_{4.0}^{5.2} \log_e^x\,dx$ by Weddle's rule .

19. Estimate the area bounded by the curve $y = f(x)$, x-axis, and the ordinates $x = 0$ and $x = 5$ from the following data.

x :	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5
$y = f(x)$:	1	2.1	3.2	4.0	5.1	6.2	7.2	8.3	9.0	10.2	12

20. A river is 80 ft wide. The depth d in feet at a distance x ft from one bank is given by the following table:

x :	0	10	20	30	40	50	60	70	80
d :	0	4	7	9	12	15	14	8	3

Estimate the area of the cross-section of the river.

Answers

1. 1.6213 **2.** 0.030321 **3.** 3.14156

5. 0.3466, 0.6932 **6.** 0.74621

7. 0.839 **8.** (i) 2.273, 1.136 (ii) 2.210, 1.105

9. 0.849 **10.** 0.995, 1, 0.005, 0.005

11. 0.0833 **12.** 1.757 **13.** (i) 1.1703 (ii) 1.1873

14. 0.84794, 0.84768 **15.** 4.666, 5

16. 1.1872 **18.** 1.827847 **19.** 30.9

20. 710sq.ft (approx)

III. Multiple Choice Questions

1. In evaluating $\int_a^b f(x)dx$, the error in Trapezoidal rule is of order

(a) h^2 (b) h^3 (c) h^4 (d) h

[W.B.U.T., CS-312, 2003, 2006, 2008, 2009]

2. In Simpson's $\frac{1}{3}$rd rule for finding $\int_a^b f(x)dx$, $f(x)$ is approximate by

(a) line segment (b) parabola

(c) circular sector (d) part of ellipse

[W.B.U.T., CS-312, 2010, CS-401, 2013]

3. The truncution error in composite Simpson's one third rule is of order

(a) h^3 (b) h^4 (c) h^5 (d) none

[W.B.U.T., CS-312, 2004, M(CS)-301, 2015]

4. In Trapezoidal rule for finding $\int_a^b f(x)dx$, $f(x)$ is approximated by

(a) line segment (b) parabola

(c) circular sector (d) none

[M.A.K.A.U.T., M(CS)-301, 2015]

5. In evaluating $\int_a^b f(x)$ the error in Weddle's rule is of order

(a) h^2 (b) h^3 (c) h^6 (d) none

6. Simpson's $\frac{1}{3}$rd rule requires the interval to be divided into an odd number of sub-intervals

(a) True (b) False

[W.B.U.T., CS-312, 2002]

7. Trapezoidal rule can be applied if the number of equal sub-intervals of the interval of integration is

(a) odd (b) even (c) both (d) none

[M.A.K.A.U.T., M(CS)-401, 2016]

8. In Trapezoidal rule for evaluating the approximate value of $\int_a^b f(x)dx$, the area given by this integral is approximated by the sum of area of some

(a) rectangle (b) sectorial figure

(c) trapezium (d) none of these

9. Simpson's one third rule is applicable only if the number of sub-interval is even

(a) Ture (b) False

[W.B.U.T. CS-312, 2008]

10. In Trapezoidal rule for finding the value of $\int_a^b f(x)dx$ there exists no error if $f(x)$ is

(a) parabolic function (b) linear function

(c) logarithemic function (d) none of these

11. The inherent error for Weddle's rule for integration is as (the notation have their usual meanings).

(a) $-\frac{nh^5}{180}f^{iv}(x_0)$ (b) $-\frac{nh^7}{180}f^{vi}(x_0)$

(c) $-\frac{nh^7}{840}f^{vi}(x_0)$ (d) none of these

[W.B.U.T, CS-312, 2008]

12. In Trapezoidal rule for finding the approximate value of $\int_{12}^{24} f(x)dx$, the error is (when number of sub-interval is 12)

(a) $-f''(\xi)$ (b) $-2f''(\xi)$

(c) $f'(\xi)$ (d) none, where $12 < \xi < 24$

13. The degree of precision of Simpson's one third rule is

(a) 1 (b) 2 (c) 3 (d) 5

[W.B.U.T., CS-312, 2007, 2009, M(CS)-401, 2014, M(CS)-301, 2014]

14. The degree of precision of Trapezoidal rule is

(a) 1 (b) 3 (c) 5 (d) 2

15. The degree of precision of Weddle's rule is

(a) 1 (b) 3 (c) 5 (d) 2

16. In Simpson's one third rule for finding the approximate value of $\int_{12}^{24} f(x)dx$, the error is (when the number of sub-interval is 12)

(a) $-\frac{1}{90}f^{iv}(\xi)$ (b) $-\frac{1}{15}f^{iv}(\xi)$

(c) $-\frac{2}{15}f^{iii}(\xi)$ (d) $-\frac{2}{15}f^{ii}(\xi)$

17. The degree of the approxmating polynomial corresponding to Trapezoidal rule and Simpson's $\frac{1}{3}$ rule are respectively

(a) 1.1 (b) 2.1 (c) 10, 2 (d) 2,2

18. The interent error for Trapezoidal rule of integration is as (the notations have their usual meanings)

(a) $-\frac{nh^3}{140}f''(x_0)$ (b) $-\frac{nh^5}{140}f^{iv}(x_0)$

(c) $-\frac{nh^3}{12}f''(x_0)$ (d) none of there.

[*M.A.K.A.U.T., M(CS)-301, 2015*]

19. The interent error for simpson's $\frac{1}{3}rd$ rule of integration is as (the notation have their usual meaning)

(a) $-\frac{nh^5}{180}f^{iv}(x_0)$ (b) $-\frac{nh^5}{180}f''(x_0)$

(c) $-\frac{nh^3}{180}f^{iv}(x_0)$ (d) none of there

20. Simpson's $\frac{1}{3}rd$ formula always requires

(a) even number of ordinates

(b) odd number of ordinates

(c) even or odd number of ordinates

(d) none of these [*M.A.K.A.U.T., M(CS)-401, 2015*]

21. Simpson's one-thrid rule is applicable only when the number of sub-intervals is

(a) even (b) odd

(c) both even and odd (d) none of these

[M.A.K.A.U.T., M(CS)-401, 2013]

Answers

1.a	2.b	3.b	4.a	5.b	6.b	7.c	8.c	9.ia	10.b
11.c	12. a	13.c	14.a	15.c	16.b	17.c	18.c	19.a	20.b
21.a									