Date - 13/10/22

Linear system of equations:

Method of solving system of linear equations are mainly divided into two groups

① Exact method ② Iterative method

Exact method: Exact method are those in which there are finite number of algorithms for solving the system of linear equations.

Iterative method: Iterative method are those which permits the solution of a system of linear equations to a given accuracy by means of convergent infinite process.

Gauss Elimination method (Exact method):—

Given that $A^{(1)}x = b^{(1)}$, where $A^{(1)} = \left(a_{ij}^{(1)}\right)_{n\times n}$

$$b^{(1)} = \begin{pmatrix} b_1^{(1)} \\ \vdots \\ b_n^{(1)} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

i.e.
$$\begin{aligned}
a_{11}^{(1)} x_1 + a_{12}^{(1)} x_2 + \cdots + a_{1n}^{(1)} x_n &= b_1^{(1)} \\
a_{21}^{(1)} x_1 + a_{22}^{(1)} x_2 + \cdots + a_{2n}^{(1)} x_n &= b_2^{(1)} \\
\vdots \qquad \qquad \vdots \qquad \qquad \qquad \vdots \quad &\quad \vdots \\
a_{n1}^{(1)} x_1 + a_{n2}^{(1)} x_2 + \cdots + a_{nn}^{(1)} x_n &= b_n^{(1)}
\end{aligned}$$

Step 1: If $a_{11}^{(1)} \neq 0$, then $\exists$ one equation where the co-efficient of x_1 is non-zero & we call this as $a_{11}^{(1)}$.

Define some multipliers $m_{i1} = \dfrac{a_{i1}^{(1)}}{a_{11}^{(1)}}$, $i = 2, \ldots, n$.

we use this multipliers to eliminate x_1 from 2nd, 3rd, .. n-th equation:

Define
$$a_{ij}^{(2)} = a_{ij}^{(1)} - m_{i1}\, a_{1j}^{(1)}$$
$$b_i^{(2)} = b_i^{(1)} - m_{i1}\, b_1^{(1)}, \quad i, j = 2, 3, \ldots, n$$

Then $A^{(1)}x = b^{(1)}$ is reduced to $A^{(2)}x = b^{(2)}$, where

$$A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix} \quad \& \quad b^{(2)} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(2)} \end{pmatrix}$$

We continue to eliminates unknowns until step k, $1 \le k \le n-1$

step k: Assume that $A^{(k)}x = b^{(k)}$ has constructed with $x_1, x_2, \cdots, x_{k-1}$ eliminated at successive stages.

Then $A^{(k)}$ has a form
$$A^{(k)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1k}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2k}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & & & & & \\ 0 & 0 & \cdots & a_{kk}^{(k)} & \cdots & a_{kn}^{(k)} \\ 0 & 0 & \cdots & a_{k+1\,k}^{(1)} & \cdots & a_{k+1\,n}^{(1)} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & & a_{nk}^{(1)} & \cdots & a_{nn}^{(k)} \end{pmatrix}$$

Let $a_{kk}^{(k)} \neq 0$. Define $m_{ik} = \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$, $i = k+1, \cdots, n$

we use this to eliminate x_k from $(k+1)$-th, $\cdots$, n-th eq^{n}s, by defining
$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)}$$
$$b_i^{(k+1)} = b_i^{(k)} - m_{ik} b_k^{(k)} \quad ; \; i, j = (k+1), \cdots, n$$

continuing in this way we finally obtain $A^{(n)}x = b^{(n)}$, where
$$A^{(n)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn}^{(n)} \end{pmatrix} \quad \& \quad b^{(n)} = \begin{pmatrix} b_1^{(1)} \\ b_2^{(2)} \\ \vdots \\ b_n^{(n)} \end{pmatrix}$$

so, the system reduces to $Ux = g$; where

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}, \quad g = \begin{pmatrix} b_1^{(1)} \\ \vdots \\ b_n^{(n)} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$$

$= A^{(n)}$ $\qquad$ $= b$

Thus we have, $u_{nn} x_n = g_n$

$$\Rightarrow x_n = \frac{g_n}{u_{nn}}$$

& $u_{n-1\,n-1} x_{n-1} + u_{n-1\,n} x_n = g_{n-1}$

$$\Rightarrow x_{n-1} = \frac{g_{n-1} - u_{n-1\,n} x_n}{u_{n-1\,n-1}}$$

& so on.

Thus we have, $x_k = \frac{1}{u_{kk}} \left[g_k - \sum_{j=k+1}^{n} u_{kj} x_j \right]$,

$k = n-1, n-2, \ldots, 1$

<u>Ex:</u> Solve

$$2x_1 - 2x_2 + 4x_3 = -12$$
$$2x_1 + 3x_2 + 2x_3 = 8$$
$$-x_1 + x_2 - x_3 = \frac{7}{2}$$

Ans: $x_1 = 2$, $x_2 = 3$, $x_3 = -2.5$

$\Rightarrow$ The augmented matrix of the given system of equation is

$$\begin{bmatrix} 2 & -2 & 4 & -12 \\ 2 & 3 & 2 & 8 \\ -1 & 1 & -1 & \frac{7}{2} \end{bmatrix} \xrightarrow[R_{31}\left(\frac{1}{2}\right)]{R_{21}(-1)} \begin{bmatrix} 2 & -2 & 4 & -12 \\ 0 & 5 & -2 & 20 \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$$

The reduced augmented matrix can be rewritten as

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 20 \\ -5/2 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 2x_2 + 4x_3 = -12 \quad \text{—— (1)}$$

$$5x_2 - 2x_3 = 20 \quad \text{—— (2)}$$

$$x_3 = -\frac{5}{2} = -2.5$$

From, (2) we get,

$$5x_2 = 20 - 5 = 15$$

$$\Rightarrow x_2 = 3$$

From (1), $2x_1 = -12 + 6 + 10$

$$\Rightarrow 2x_1 = 4$$

$$\Rightarrow x_1 = 2$$

∴ The required solution of the given system of equation is, $x_1 = 2$

$x_2 = 3$

& $x_3 = -2.5$

Operation count in the Gaussian Elimination method:

To analyze the number of operations necessary to solve $Ax=b$ using Gaussian elimination, we will consider sperately the creation of ~~L and~~ U from A, the modification of b to g, and finally the solution of x.

1. Calculation of U: At step 1, $(n-1)$ divisions are used to calculate the multipliers m_{i1}, $i=2,\dots,n$. Then $(n-1)^2$ multiplications and $(n-1)^2$ additions are used to create $a_{ij}^{(2)}$. we can continue in this way for each step.

The results for there are summarized in the following table.

step k	Additions	Multiplications	Division
1	$(n-1)^2$	$(n-1)^2$	$(n-1)$
2	$(n-2)^2$	$(n-2)^2$	$(n-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	1	1
Total	$\frac{n(n-1)(2n-1)}{6}$	$\frac{n(n-1)(2n-1)}{6}$	$\frac{n(n-1)}{2}$

For a convenient notation let ~~multiplicati~~ $MD(\cdot)$ and $AS(\cdot)$ denote the number of multiplications and divisions and the number of additions and subtraction respectively for the computation of the quantity in the parentheses.

$\therefore MD(U) = \frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2}$

Upper triangular matrix

$= \frac{n(n-1)(n+1)}{3}$

& $AS(U) = \frac{n(n-1)(2n-1)}{6}$

$b_i^{(2)} = b_i^{(1)} - m_{i1} b_1^{(1)}$

$i = 2, 3, \dots, n$

$b_i^{(3)} = b_i^{(2)} - m_{i2} b_2^{(2)}$

$i = 3, 4, \dots, n$

$MD = (n-2)$

$MD = (n-1)$

2. Modification of b to $g = b^{(n)}$:

Here, $MD(g) = (n-1) + (n-2) + \dots + 1$

$= \frac{n(n-1)}{2}$

& $AS(g) = (n-1) + (n-2) + \dots + 1$

$= \frac{n(n-1)}{2}$

3. solution of $Ux = g$:

$MD(x) = \frac{n(n+1)}{2}$

& $AS(x) = 0 + 1 + \dots + (n-1)$

$= \frac{n(n-1)}{2}$

$x_n = \frac{g_n}{u_{nn}}$

$x_k = \frac{1}{u_{kk}} [g_k - \sum_{j=k+1}^{n} U_{kj} x_j$

Step	Multi	Divi	Total
x_n	0	1	1
x_{n-1}	1	1	2
⋮	⋮	⋮	⋮
x_1	$n-1$	1	n

Thus to obtain the solution of $Ax = b$ using Gaussian elemination method, we require the following number of operations.

MD(Gauss Elimination) $= n(n-1)\left[\frac{n+1}{3} + \frac{1}{2} + \frac{1}{2}\right]$

$= \frac{n(n-1)(n+4)}{3}$

$$= \frac{n(n-1)(n+1)}{3} + \frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n(n-1)(n+1)}{3} + \frac{2n^2}{2}$$

$$= \frac{n(n^2-1)}{3} + n^2$$

$$= \frac{n^3 - n + n^2}{3} = \frac{n^3}{3} - \frac{n}{3} + n^2$$

$$= o\left(\frac{n^3}{3}\right)$$

As (Gauss Elimination)

$$= \frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$$

$$= \frac{n(n-1)(2n-1)}{6} + n(n-1)$$

$$= n(n-1)\left[\frac{2n-1+6}{6}\right]$$

$$= \frac{n(n-1)(2n+5)}{6} = \frac{2n^3 + 5n^2 - 2n^2 - 5n}{6}$$

$$= \frac{n^3}{3} + \frac{\cancel{5}\,\cancel{6}}{2}\,\frac{n^2}{2} - \frac{5}{6}n$$

$$= o\left(\frac{n^3}{3}\right)$$

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Special metrices:

Defn: A matrix $A = a_{ij} \in M_n(\mathbb{R})$ is said to be strictly diagonally dominant, if for each $i = 1, 2, \dots, n$,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

(i.e. if for each row, the magnitude of the diagonal element is strictly larger than the sum of the magnitudes of the other elements on that row).

<u>Th</u>: Let $A \in M_n(\mathbb{R})$ be strictly diagonally dominant.

Then ① A is non-singular i.e $Ax = b$ has unique solution for any column vector b in $\mathbb{R}^n$.

② Gaussian elimination and direct factorization can be performed on A without row interchanges.

(see, Bradie - A friendly introduction to Numerical Analysis p-212)

<u>Note</u>: Diagonally dominant matrices can be similarly defined. $A = (a_{ij}) \in M_n(\mathbb{R})$ is called <u>diagonally dominant</u> if $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$, for $i = 1, 2, \ldots, n$.

<u>Def^n</u>:- A matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is called <u>positive definite</u> if $x^t A x > 0$, for any (column vector) $x \neq \theta$.

<u>Th</u>: Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be symmetric positive definite.

Then ① $a_{ii} > 0$, $\forall\, i = 1, 2, \ldots, n$

② $\max\limits_{1 \leq k, j \leq n} |a_{kj}| \leq \max\limits_{1 \leq i \leq n} |a_{ii}|$

③ $a_{ij}^2 < a_{ii}\, a_{jj}$, $\forall\, i \neq j$, $i, j = 1, 2, \ldots, n$

⇒ (See, Bradie - A Friendly intro. to Nu. Ana, p-213)

Ex: Determine which of the following are symmetric positive definite

(1) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ (2) $\begin{pmatrix} 8 & -1 & 1 \\ -1 & 8 & 9 \\ 1 & 9 & 7 \end{pmatrix}$ (3) $\begin{pmatrix} 3 & 1 & 5 \\ 1 & 4 & 2 \\ 5 & 2 & 8 \end{pmatrix}$

soln: ① If the matrix is symmetric positive definite then it must satisfy $a_{ii} > 0$ but here $a_{22} < 0$ so ~~2 1 0~~ the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ is not symmetric +ve definite.

② If the matrix $\begin{pmatrix} 8 & -1 & 1 \\ -1 & 8 & 9 \\ 1 & 9 & 7 \end{pmatrix}$ is symmetric positive definite then it must satisfy the condition

$\max\limits_{1 \le k,j \le n} |a_{kj}| \le \max\limits_{1 \le i \le n} |a_{ii}|$ but here ~~$\max|a_{23}|$~~ one non-diagonal element is greater than of any diagonal element, so the matrix is not symmetric positive definite.

③ If the given matrix is symmetric positive definite then it must satisfy the condition $a_{ij}^2 < a_{ii} a_{jj}$ $\forall i \neq j$ but here $(a_{13})^2 \nless a_{11} \times a_{33}$ i.e $25 \nless 24$. so, the matrix is not symmetric positive definite.

Th: If $A \in M_n(\mathbb{R})$ is symmetric and all eigenvalues of A are +ve, then A is symmetric positive definite.

corollary: If $A = (a_{ij}) \in M_n(\mathbb{R})$ is symmetric strictly diagonally dominant and $a_{ii} > 0$, $\forall\, i = 1, 2, \ldots, n$, then A is symmetric +ve definite.

$\Rightarrow$ (see - Bradie)

Def^n: Let $A = (a_{ij}) \in M_n(\mathbb{R})$. For each $1 \le k \le n$, the k-th leading principal sub matrix of A is the matrix formed by the 1st k-rows and k-columns of A.

$k=1$ $\quad$ $k=2$ $\quad$ $\to k=3$

$$\begin{pmatrix} \boxed{a_{11}} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{41} \end{pmatrix} \to k=4$$

Th: A symmetric matrix A is positive definite $\Leftrightarrow$ each of its leading principal submatrices has positive determinant.

$\Rightarrow$ (see - Bradie)

Ex: Determine which of the following are symmetric +ve definite.

① $\begin{pmatrix} 6 & -2 & 3 \\ -2 & 8 & 1 \\ 3 & 1 & 7 \end{pmatrix}$ $\qquad$ ② $\begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

$\Rightarrow$ ① Given matrix is $\begin{pmatrix} 6 & -2 & 3 \\ -2 & 8 & 1 \\ 3 & 1 & 7 \end{pmatrix}$.

Now, $a_{11} > a_{12} + a_{13}$ i.e. $6 > 1$

$a_{22} > a_{21} + a_{23}$ i.e. $8 > 1$

$a_{33} > a_{31} + a_{32}$ i.e. $7 > 4$

$\therefore$ The given matrix is symmetric strictly diagonally dominant and $a_{ii} > 0, \forall$ $i = 1, 2, 3$ so, by corollary the given matrix is symmetric +ve definite.

② Let $A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

Now, 1st leading principal submatrix of A is $3 > 0$.

Determinant of 2nd leading principal submatrix of A is $\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 > 0$

Determinant of 3rd leading principal sub-matrix of A is

$$\begin{vmatrix} 3 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 3 \times 8 - 1 \times 5 + 2 \times 7$$
$$= 24 - 5 + 14 = 33 > 0$$

$\therefore$ A is symmetric +ve definite.

Th: Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be symmetric +ve definite. Then

① A is non-singular i.e. $Ax = b$ has a unique solution for any column vector b in $\mathbb{R}^n$.

② Gaussian elimination and direct factorization can be performed without row interchanges.

$\Rightarrow$ (See - Bradie, P-215)

Cholesky decomposition:

Theorem: (Cholesky): Let $A \in M_n(\mathbb{R})$ be symmetric positive definite. Then ① A is non-singular.

② $\exists$ a lower triangular matrix $L = (l_{ij}) \in M_n(\mathbb{R})$ with $l_{ii} > 0, \forall i = 1, 2, \ldots, n$ such that $A = LL^t$.

⇒ See Epperson, An introduction to Numerical methods and analysis, p-458, see also Atkinson - Numerical analysis (p-524).

Note: ① There are a number of different ways of actually constructing the cholesky decomposition. It can be shown that the cholesky factorization/decomposition is unique.

so all the different constructions are just different ways of organising the computation. we use the following formula to construct the lower triangular matrix $L = (l_{ij})$ of the cholesky decomposition.

$$l_{ij} = \frac{1}{l_{jj}} \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right], \quad 0 \leq j \leq i-1 \quad \forall i = 1, 2, \ldots, n$$

& $l_{11} = \sqrt{a_{11}}$

& $l_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right]^{1/2}, \quad i = 2, 3, \ldots, n$

② Cholesky decomposition is a very efficient algorithm. The cholesky decomposition requires $\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$ arithmatic operations plus n square root. This is roughly half the number of operations used by Gaussian elimination and direct factorization.

③ Computation of square roots in the cholesky decomposition can be avoided using modification of the above method. (We can consider the following decomposition). This can be achieved by considering the following decomposition of the form $A = LDL^t$, where L is a lower triangular matrix with 1's on the diagonal and D is a diagonal matrix.

Ex 1: obtain the cholesky decomposition for

① $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$ (Hilbert matrix of order 3)

② $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}$

soln: ① Here ① is symmetric.

Now, we calculate,

$$l_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2\right]^{1/2} \quad \& \quad l_{ij} = \frac{1}{l_{jj}}\left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}\right]$$

Here, $l_{11} = \sqrt{a_{11}} = 1$

$$l_{22} = \left(a_{22} - l_{21}^2\right)^{1/2} = \left(\frac{1}{3} - \frac{1}{4}\right)^{1/2} = \left(\frac{1}{12}\right)^{1/2} = \frac{1}{\sqrt{12}}$$

$$l_{21} = \frac{1}{l_{11}}[a_{21} - 0] = \frac{a_{21}}{1} = a_{21} = \frac{1}{2}$$

$$l_{33} = \left[a_{33} - \left(l_{31}^2 + l_{32}^2\right)\right]^{1/2}$$

$$= \left[\frac{1}{5} - \left(\frac{1}{9} + \frac{1}{12}\right)\right]^{1/2}$$

$$= \left(\frac{1}{5} - \frac{4+3}{36}\right)^{1/2}$$

$$= \left(\frac{1}{5} - \frac{7}{36}\right)^{1/2}$$

$$= \left(\frac{36-35}{180}\right)^{1/2} = \frac{1}{\sqrt{180}}$$

$$l_{31} = \frac{1}{l_{11}}[a_{31} - 0]$$

$$= \frac{1}{3}$$

$$l_{32} = \frac{1}{l_{22}}[a_{32} - l_{31}l_{21}]$$

$$= \sqrt{12}\left[\frac{1}{4} - \frac{1}{3} \times \frac{1}{2}\right]$$

$$= \sqrt{12}\left[\frac{3-2}{12}\right]$$

$$= \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & 0 \\ \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{180}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{6\sqrt{5}} \end{bmatrix} \quad \text{(Ans)}$$

(2) $$l_{ij} = \frac{1}{l_{ii}}\left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}\right]$$

$$l_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2\right]^{1/2}$$

Here, $l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

$$l_{21} = \frac{1}{l_{11}}[a_{21} - 0] = \frac{2}{2} = 1$$

$$l_{22} = \left(a_{22} - l_{21}^2\right)^{1/2} = (4-1)^{1/2} = \sqrt{3}$$

$$l_{31} = \frac{1}{l_{11}}[a_{31} - 0] = -\frac{1}{2}$$

$$l_{32} = \frac{1}{l_{22}}[a_{32} - l_{31} l_{21}]$$

$$= \frac{1}{\sqrt{3}}[1 + \frac{1}{2} \times 1]$$

$$= \frac{1}{\sqrt{3}} \times \frac{3}{2} = \frac{\sqrt{3}}{2}$$

$$l_{33} = [a_{33} - (l_{31}^2 + l_{32}^2)]^{1/2}$$

$$= [4 - (\frac{1}{4} + \frac{3}{4})]^{1/2}$$

$$= (4-1)^{1/2} = \sqrt{3}$$

$$\therefore L = \begin{bmatrix} 4 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \sqrt{3} \end{bmatrix} \quad \text{(Ans.)}$$

Direct Factorization:

Like Gaussian elimination method, direct factorization is an alternative procedure for obtaining an LU decomposition. We need such factorization techniques because (i) there are matrice which are special structure to them. Direct factorization will make it possible to construct schemes take that advantage of that structure. (ii) the formulas associated with direct factorization will allow us, on some computers, to take advantage of architecture to improve both speed and accuracy.

Recall that the factor in an LU decomposition are determined only upto the scaling by a diagonal matrix. Thus different factorizations may be viewed as resulting from different choices for the diagonal entries of either L or U. The two most common choices for the diagonal entries (for $L=(l_{jj})$ & $U=(u_{ij})$) are

$l_{ii}=1$, $\forall i=1,2,\ldots,n$ and

$u_{ii}=1$, $\forall i=1,2,\ldots,n$

which gives rise to what are known as Doolittle decomposition and crout decomposition respectively.

Crout decomposition: Let $A \in M_n(\mathbb{R})$, to obtain the crout decomposition for A, we have to find entries l_{ij} $(i \geq j)$ and u_{ij} $(i<j)$ such that

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

we use the following formulas to calculate such entries:

$$\boxed{\begin{aligned} & l_{ik} = a_{ik} - \sum_{j=1}^{k-1} l_{ij} u_{jk} \ ; \ i=k, k+1, \ldots n \\ & \text{and } u_{kj} = \frac{1}{l_{kk}}\left(a_{kj} - \sum_{i=1}^{k-1} l_{ki} u_{ij}\right) ; \ j=k+1,\ldots,n \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k=1,2,\ldots,n \end{aligned}}$$

NOTE: ① The operation count for a matrix of size n using crout decomposition is, $\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$.

② The formula for Doolittle decomposition can be similarly obtained. The operation count for a matrix of size n using the Doolittle decomposition is

$$\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n.$$

Ex: 1 Obtain the crout decomposition for

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{pmatrix}$$

$$\boxed{\begin{aligned} l_{i1} &= a_{i1}, \; i = 1, 2, \dots, n \\ u_{1j} &= \frac{a_{1j}}{l_{11}}, \; j = 2, 3, \dots, n \end{aligned}}$$

solution: $l_{11} = a_{11} - 0 = 1$

$u_{12} = \frac{1}{l_{11}}(a_{12} - 0) = \frac{4}{1} = 4$

$l_{21} = a_{21} - 0 = 2$

$l_{22} = a_{22} - l_{21}u_{12}$

$= 7 - 2 \times 4$

$= 7 - 8 = -1$

$u_{13} = \frac{a_{13}}{l_{11}} = \frac{3}{1} = 3$.

$u_{23} = \frac{1}{l_{22}}(a_{23} - l_{21}u_{13})$

$= \frac{1}{-1}(9 - 2 \times 3)$

$= -(9 - 6) = -3$

$l_{31} = a_{31} - 0 = 5$

$l_{32} = a_{32} - l_{31}u_{12}$

$= 8 - 5 \times 4$

$= 8 - 20 = -12$

$l_{33} = a_{33} - (l_{31}u_{13} + l_{32}u_{23})$

$= -2 - [5 \times 3 + (-12) \times (-3)]$

$= -2 - (15 + 36)$

$= -2 - 51 = -53$

$\therefore L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -12 & -53 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

Date - 07/4/22

Def: Tridiagonal matrix:

Let $A \in M_n(\mathbb{R})$. A is called tridiagonal if $a_{ij} = 0$ for $|i-j| > 1$. (i.e. on the i-th row of A, the only possible non-zero elements are $a_{i,i-1}$, $a_{i,i}$, $a_{i,i+1}$)

For a given tridiagonal matrix $A \in M_n(\mathbb{R})$, $A = (a_{ij})$, we also obtain the decomposition $A = LU$, where

$$L = \begin{pmatrix} l_{11} & & & & \\ l_{21} & l_{22} & & & \\ & l_{32} & l_{33} & & \\ & \vdots & \vdots & \ddots & \\ & & & l_{n\,n-1} & l_{nn} \end{pmatrix}$$

& $$U = \begin{pmatrix} 1 & u_{12} & & & \\ & 1 & u_{23} & & \\ & & 1 & u_{34} & \\ & & & \ddots & u_{n-1\,n} \\ & & & & 1 \end{pmatrix}$$

(all other entries are zero)

This is a modification of the crout decomposition method.

such L & U can be found using the following formulas:-

$l_{11} = a_{11}, \quad l_{21} = a_{21}, \quad u_{12} = \dfrac{a_{12}}{l_{11}}.$

& $l_{kk} = a_{kk} - l_{k\,k-1}\, u_{k-1\,k}$

$l_{k+1\,k} = a_{k+1\,k}$

& $u_{k\,k+1} = \dfrac{a_{k\,k+1}}{l_{kk}}$, for $k = 2, \ldots, n-1$

& $l_{nn} = a_{nn} - l_{n\,n-1}\, u_{n-1\,n}$

<u>Ex:</u> solve :
$$\begin{bmatrix} 4 & -1 & & \\ 2 & 4 & -1 & \\ & -2 & 4 & -1 \\ & & -2 & 4 \end{bmatrix} x = \begin{bmatrix} 6 \\ -6 \\ 3 \\ 4 \end{bmatrix}$$

$\Rightarrow$ ~~clearly $A \in M_4(\mathbb{R})$~~

$$A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \\ 4 \end{pmatrix}$$

clearly $A \in M_4(\mathbb{R})$ is a tridiagonal matrix.

we write, $A = LU$; where $L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}$ and

$A = (a_{ij})_{4\times 4}$

$$U = \begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we know that $l_{11} = a_{11} = 4$, $l_{21} = a_{21} = 2$, $u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{4}$

$l_{kk} = a_{kk} - l_{k\,k-1} u_{k-1\,k}$, $l_{k+1\,k} = a_{k+1\,k}$

$u_{k\,k+1} = \frac{a_{k\,k+1}}{l_{kk}}$, $k = 2, 3, \ldots$

and $l_{44} = a_{44} - l_{43} u_{34}$

$l_{22} = a_{22} - l_{21} u_{12} = \frac{9}{2}$ $\qquad$ $l_{33} = a_{33} - l_{32} u_{23} = \frac{32}{9}$

$l_{32} = a_{32} = -2$ $\qquad$ $l_{43} = a_{43} = -2$

$u_{23} = \frac{a_{23}}{l_{22}} = -\frac{2}{9}$ $\qquad$ $u_{34} = \frac{a_{34}}{l_{33}} = -\frac{9}{32}$

$$l_{44} = 4 - (-2) \times \left(-\frac{9}{32}\right)$$

$$= 4 - \frac{9}{16} = \frac{64-9}{16} = \frac{55}{16}$$

$$\therefore L = \begin{pmatrix} 4 & & & \\ 2 & \frac{9}{2} & & \\ & -2 & \frac{32}{9} & \\ & & -2 & \frac{55}{16} \end{pmatrix} \quad \& \quad U = \begin{pmatrix} 1 & -\frac{1}{4} & & \\ & 1 & -\frac{2}{9} & \\ & & 1 & -\frac{9}{32} \\ & & & 1 \end{pmatrix}$$

To solve $Ax = b$ i.e. $LUx = b$, we first suppose that $Ux = z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$ and we solve $Lz = b$

or, $4z_1 = 6 \Rightarrow \boxed{z_1 = \frac{3}{2}}$

$2z_1 + \frac{9}{2} z_2 = -6 \Rightarrow \frac{9}{2} z_2 = -6 - 2 \times \frac{3}{2} = -9$

$\Rightarrow \boxed{z_2 = -2}$

$-2z_2 + \frac{32}{9} z_3 = 3 \Rightarrow \frac{32}{9} z_3 = 3 + 2 \times (-2) = -1$

$\Rightarrow \boxed{z_3 = -\frac{9}{32}}$

$-2z_3 + \frac{55}{16} z_4 = 4 \Rightarrow \frac{55}{16} z_4 = 4 \times 2 \times \left(-\frac{9}{32}\right) = \frac{55}{16}$

$\Rightarrow \boxed{z_4 = 1}$

Therefore, $Ux = z = \begin{bmatrix} \frac{3}{2} \\ -2 \\ -\frac{9}{32} \\ 1 \end{bmatrix}$.

$\Rightarrow x_1 - \frac{1}{4} x_2 = \frac{3}{2} \quad \Rightarrow x_1 = \frac{3}{2} + \frac{1}{2} = 1$

$x_2 - \frac{2}{9} x_3 = -2 \quad \Rightarrow x_2 = -2$

$x_3 - \frac{9}{32} x_4 = -\frac{9}{32} \quad \Rightarrow x_3 = 0$

$x_4 = 1$

Thus the solution for $Ax = b$ is

$x = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

Date-10/11/22

Ex: Solve (a)

$$\begin{aligned} 3x_1 - x_2 &= +4 \\ x_1 + 4x_2 + 2x_3 &= -7 \\ 3x_2 + 5x_3 - x_4 &= -15 \\ -2x_3 + 7x_4 &= 18 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

(c)

$$\begin{aligned} 4x_1 - x_2 &= 3 \\ -x_1 - 5x_2 + 6x_3 &= 0 \\ x_2 - 3x_3 + 2x_4 &= -4 \\ x_3 + 3x_4 &= -2 \end{aligned}$$

Ex: Find the values of λ for which $A = \begin{pmatrix} \lambda & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$ is

(a) positive definite

(b) strictly diagonally dominant

Ex: Repeat the preceding exercise with

$$A = \begin{pmatrix} 5 & -2 & 2 \\ -2 & 6 & \lambda \\ 2 & \lambda & 7 \end{pmatrix}$$

Matrix Norms: Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. we define the norm of a matrix $A \in M_n(\mathbb{R})$ by $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

$$= \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Note: ① For any $A, B \in M_n(\mathbb{R})$, then $\|AB\| \le \|A\| \|B\|$
Also, for any $x \in \mathbb{R}^n$, $\|Ax\| \le \|A\| \|x\|$

② The supremum norm on $M_n(\mathbb{R})$ is given by

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

(which is also called the maximum row sum)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \vdots \end{pmatrix}$$

$$\sum_{j=1}^{n} a_{ij} x_j \; ; i = 1, 2, \ldots, n$$

$$\|x\|_\infty = \max_{i=1}^{n} |x_i| = \lambda$$

$$\|Ax\|_\infty = \max_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$$

Def^n: The spectral radius of a matrix $A \in M_n(\mathbb{R})$ is given by
$\rho(A) = \max\{|\lambda| : \lambda$ is an eigenvalue of $A\}$

Th: Let $A \in M_n(\mathbb{R})$. Then

① $\|A\|_2 = \sqrt{\rho(A^t A)}$, $\left[\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right]$

② $\rho(A) \le \|A\|$, for any norm $\|\cdot\|$ on $M_n(\mathbb{R})$.

③ For any $\epsilon > 0$, $\exists$ a norm $\|\cdot\|$ on $M_n(\mathbb{R})$ for which $\|A\| \le \rho(A) + \epsilon$.

Note: ~~Here $\|\cdot\|$ are~~ In the above result, the norm $\|\cdot\|$ refers to any of the following norms. $\|\cdot\|_p$, $p = 1, 2, \ldots \infty$.
(see Bradie, p-178)

Def^n: For a matrix $A \in M_n(\mathbb{R})$ and a given norm $\|\cdot\|$ on $M_n(\mathbb{R})$, the condition number of A w.r.t. the given norm is defined as $K(A) = \|A\| \|A^{-1}\|$.

If A is singular, then we take $K(A) = \infty$.

Ex: (a) $3x_1 - x_2 = +4$

$x_1 + 4x_2 + 2x_3 = -7$

$3x_2 + 5x_3 - x_4 = -15$

$-2x_3 + 7x_4 = 18$

$$\Rightarrow A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & +4 & 2 & 0 \\ 0 & 3 & 5 & -1 \\ 0 & 0 & -2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} +4 \\ -7 \\ -15 \\ 18 \end{pmatrix}$$

clearly $A \in M_4(\mathbb{R})$ is a tridiagonal matrix.

we write $A = LU$; where $L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}$ & $U = \begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

we know that, $l_{11} = a_{11} = 3$, $l_{21} = a_{21} = 1$, $u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{3}$

$l_{kk} = a_{kk} - l_{k\,k-1} u_{k-1\,k}$, $l_{k+1\,k} = a_{k+1\,k}$

$u_{k\,k+1} = \frac{a_{k\,k+1}}{l_{kk}}$, $k = 2, 3, \ldots$

and $l_{44} = a_{44} - l_{43} u_{34}$

$l_{22} = a_{22} - l_{21} u_{12}$

$= +4 - \left(1 \times -\frac{1}{3}\right)$

$= +4 + \frac{1}{3} = +\frac{13}{3}$

$l_{33} = a_{33} - l_{32} u_{23}$

$= 5 - \left\{3 \times \left(\frac{6}{13}\right)\right\}$

$= 5 - \frac{18}{13} = \frac{47}{13}$

$l_{32} = a_{32} = 3$

$l_{43} = a_{43} = -2$

$u_{23} = \frac{a_{23}}{l_{22}} = 2 \times +\frac{3}{13}$

$= \frac{6}{13}$

$u_{34} = \frac{a_{34}}{l_{33}} = -\frac{13}{47}$

$$l_{44} = a_{44} - l_{43}\,u_{34}$$

$$= 7 - \left\{(-2) \times -\frac{13}{47}\right\}$$

$$= 7 - \frac{26}{47} = \frac{303}{47}$$

$$\therefore L = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & \frac{13}{3} & 0 & 0 \\ 0 & 3 & \frac{47}{13} & 0 \\ 0 & 0 & -2 & \frac{303}{47} \end{pmatrix} \;\&\; U = \begin{pmatrix} 1 & -\frac{1}{3} & & \\ & 1 & +\frac{6}{13} & \\ & & 1 & -\frac{13}{47} \\ & & & 1 \end{pmatrix}$$

To solve $Ax = b$ i.e $LUx = b$, we first suppose that $Ux = z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$ and we solve $Lz = b$

or, $3z_1 = +4 \Rightarrow z_1 = +\frac{4}{3}$

$$z_1 + \frac{13}{3} z_2 = -7 \Rightarrow +\frac{4}{3} - \frac{13}{3} z_2 = -7$$

$$\Rightarrow -\frac{13}{3} z_2 = -\frac{4}{3} - 7$$

$$\Rightarrow \frac{13}{3} z_2 = -\frac{25}{3}$$

$$\Rightarrow z_2 = -\frac{25}{13}$$

$$3z_2 + \frac{47}{13} z_3 = -15$$

$$\Rightarrow \frac{47}{13} z_3 = -15 - \left(3 \times \frac{-25}{13}\right)$$

$$\Rightarrow z_3 = \cancel{\frac{66}{11} \times \frac{11}{74}} \left(-15 + \frac{75}{13}\right) \times \frac{13}{47}$$

$$\Rightarrow z_3 = -\frac{120}{13} \times \frac{13}{47} \Rightarrow z_3 = -\frac{120}{47}$$

$$-2z_3 + \frac{303}{47} z_4 = 18$$

$$\Rightarrow \frac{303}{47} z_4 = 18 + 2\left(-\frac{120}{47}\right)$$

$$\Rightarrow z_4 = +\frac{606}{47} \times \frac{47}{303}$$

$$\Rightarrow z_4 = +\frac{606}{303} = 2$$

Therefore, $Ux = \begin{pmatrix} +\frac{4}{3} \\ -\frac{25}{13} \\ -\frac{120}{47} \\ 2 \end{pmatrix} = z$

or, $\cancel{x_1 - \frac{1}{3}x_2 = +\frac{4}{3} \Rightarrow x_1 = \frac{4}{3} + \frac{1}{3} \times \frac{5}{11} = \frac{4}{3} + \frac{5}{33} =}$

$$\cancel{x_2 - \frac{6}{11}x_3 = \frac{17}{11} \Rightarrow x_2 = \frac{17}{11} - \frac{12}{11} = \frac{5}{11}}$$

$$x_3 - \frac{13}{47}x_4 = -\frac{120}{47} \Rightarrow x_3 = \frac{26}{47} - \frac{120}{47}$$

$$= -\frac{94}{47} = -2$$

$$x_4 = 2$$

$$x_2 + \frac{6}{13}x_3 = -\frac{25}{13}$$

$$\Rightarrow x_2 = -\frac{25}{13} + \frac{12}{13} = -\frac{13}{13} = -1$$

$$x_1 - \frac{1}{3}x_2 = \frac{4}{3}$$

$$\Rightarrow x_1 = \frac{4}{3} - \frac{1}{3} = 1$$

Thus the solution for $Ax = b$ is $x = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 2 \end{pmatrix}$.

Ex: c solve $4x_1 - x_2 = 3$
$-x_1 - 5x_2 + 6x_3 = 0$
$x_2 - 3x_3 + 2x_4 = -4$
$x_3 + 3x_4 = -2$

$\Rightarrow A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & -5 & 6 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \\ 2 \end{pmatrix}$

clearly $A \in M_4(\mathbb{R})$ is a tridiagonal matrix.
we write $A = LU$; where $L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}$ &

$U = \begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

we know that $l_{11} = a_{11} = 4$, $l_{21} = a_{21} = -1$, $u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{4}$

$l_{kk} = a_{kk} - l_{k\,k-1}\, u_{k-1\,k}$

$l_{k+1\,k} = a_{k+1\,k}$

$u_{k\,k+1} = \frac{a_{k\,k+1}}{l_{kk}}$

$l_{32} = a_{32} = 1$

$l_{43} = a_{43} = 1$

and $l_{22} = a_{22} - l_{21} u_{12}$
$= -5 + 1 \times (-\frac{1}{4})$
$= -\frac{21}{4}$

$u_{23} = \frac{a_{23}}{l_{22}} = \frac{6}{-\frac{21}{4}}$
$= -6 \times \frac{4}{21}$
$= -\frac{24}{21} = -\frac{8}{7}$

$l_{33} = a_{33} - l_{32} u_{23}$
$= -3 + \frac{8}{7} = -\frac{13}{7}$

$$l_{44} = a_{44} - l_{43} u_{34}$$
$$= 3 + \frac{14}{13} = \frac{53}{13}$$

$$u_{34} = \frac{a_{34}}{l_{33}}$$
$$= 2 \times -\frac{7}{13} = -\frac{14}{13}$$

$$\therefore L = \begin{pmatrix} 4 & & & \\ -1 & -\frac{21}{4} & & \\ & 1 & -\frac{13}{7} & \\ & & 1 & \frac{53}{13} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & -\frac{1}{4} & & \\ & 1 & -\frac{8}{7} & \\ & & 1 & -\frac{14}{13} \\ & & & 1 \end{pmatrix}$$

To solve $Ax = b$ i.e $LUx = b$ we first suppose that $Ux = z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$ and we solve $Lz = b$.

or, $4z_1 = 3 \Rightarrow z_1 = \frac{3}{4}$

$$-z_1 - \frac{21}{4} z_2 = 0 \Rightarrow z_2 \frac{21}{4} = -\frac{3}{4}$$
$$\Rightarrow z_2 = -\frac{1}{7}$$

$$z_2 - \frac{13}{7} z_3 = -4 \Rightarrow \frac{13}{7} z_3 = 4 - \frac{1}{7}$$
$$\Rightarrow z_3 = \frac{27}{13}$$

$$z_3 + \frac{53}{13} z_4 = -2 \Rightarrow \frac{53}{13} z_4 = -2 - \frac{27}{13}$$
$$\Rightarrow z_4 = -\frac{53}{13} \times \frac{13}{53}$$
$$\Rightarrow z_4 = -1$$

Therefore, $Ux = z = \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{7} \\ \frac{27}{13} \\ -1 \end{pmatrix}$

or, $\boxed{x_4 = -1}$

$$x_3 - \frac{14}{13}x_4 = \frac{27}{13}$$

$$\Rightarrow x_3 = \frac{27}{13} - \frac{14}{13} \Rightarrow x_3 = \frac{13}{13} \Rightarrow \boxed{x_3 = 1}$$

$$x_2 - \frac{8}{7}x_3 = -\frac{1}{7}$$

$$\Rightarrow x_2 = \frac{8}{7} - \frac{1}{7} \Rightarrow \boxed{x_2 = 1}$$

$$x_1 - \frac{1}{4}x_2 = \frac{3}{4}$$

$$\Rightarrow x_1 = \frac{3}{4} + \frac{1}{4} \Rightarrow \boxed{x_1 = 1}$$

Thus the solution for $Ax = b$ is $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

Date - 14/11/22

Iterative techniques for system of linear equations:

Iterative techniques for the solution of system of linear equations are analogous to the fixed point iterative techniques. The original linear system

$$A = D - L - U$$

$$Ax - b = 0$$

$$\Rightarrow (D - L - U)x - b = 0$$

$$\Rightarrow Dx - Lx - Ux - b = 0$$

$$\Rightarrow x = D^{-1}(L+U)x + D^{-1}b$$

$$= Hx + C$$

$(A \in M_n(\mathbb{R}), b \in \mathbb{R}^n)$

$Ax = b$, which can be interpreated as the root finding problem:

$$\boxed{\text{find } x \in \mathbb{R}^n : Ax - b = 0} \quad — \quad ①$$

is first converted to the fixed point problem $\boxed{\text{find } x \in \mathbb{R}^n : x = Hx + c}$, for some matrix $H \in M_n(\mathbb{R})$ and vector $c \in \mathbb{R}^n$.

Next starting from some initial approximation $x^{(0)}$ to the solution of the fixed point problem, a sequence of vector $(x^{(k)})$ is computed according to the rule

$$\boxed{x^{(k+1)} = H x^{(k)} + c} \quad — \quad (*)$$

$A = D - L - U$

$\overset{\shortparallel}{(a_{ij})}$

$$D = \begin{pmatrix} a_{11} & & \\ & a_{12} & \\ & & \ddots \; a_{1n} \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & & & & \\ -a_{21} & 0 & & & \\ -a_{31} & -a_{31} & 0 & & \\ \vdots & & & & \\ -a_{n1} & -a_{n2} & \cdots & a_{nn-1} & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & 0 & -a_{23} & & -a_{2n} \\ \vdots & \vdots & & & \\ 0 & \cdots & & & -a_{n-1\,n} \\ & & & & 0 \end{pmatrix}$$

In this case H is called the iteration matrix (associated with A).

The functional iteration is terminated when some appropriate measure of the difference between successive vectors $x^{(k)}, x^{(k+1)}$ falls below a specified tolerance.

Now it is natural to consider the following

① Under what conditions the fixed point problem (*) has a unique solution (i.e. when (*) has unique fixed point?)?

② Under what conditions the sequence generated by (*) converge to this unique fixed point?

③ Does the sequence in (*) converge? How quickly?

④ Under what conditions on H and c, the systems ① & (*) have the same solution?

We need the following result.

Th: Let $A \in M_n(\mathbb{R})$. The following conditions are equivalent:

① $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A.

② $A^k \to 0$ as $k \to \infty$ (in $M_n(\mathbb{R})$)

③ $A^k x \to 0$ as $k \to \infty$, for any $x \in \mathbb{R}^n$ (in $\mathbb{R}^n$)

⇒ (See Bradie, p-224)

Th: Let $A \in M_n(\mathbb{R})$. Then $\rho(A) < 1 \Leftrightarrow \exists$ a norm $\|\cdot\|$ on $M_n(\mathbb{R})$ such that $\|A\| < 1$.

⇒ (see, Epperson, p-463)

Splitting Methods:

A broad class of consistent iterative method can be constructed by introducing the notion of a "splitting"

Def: Let $A \in M_n(\mathbb{R})$. A splitting of a matrix A is a pair (M, N) of matrices where $M, N \in M_n(\mathbb{R})$ with M non-singular and $A = M - N$.

- $A = M(I - M^{-1}N)$

- $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 5 & 3 \end{pmatrix}$, $M = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} = D$, $N = L + U$
$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

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Now, suppose that (M, N) forms a spliting of A.
Then $Ax = b \Leftrightarrow (M-N)x = b$

$\therefore Mx = Nx + b \Rightarrow x = M^{-1}Nx + M^{-1}b$

Hence the splitting $A = M-N$ determines the fixed point problem $x = Hx + c$ and the associated iteration scheme is $\boxed{x^{(k+1)} = Hx^{(k)} + c}$ ... (0), where $H = M^{-1}N$ (it is called the iteration matrix) and $c = M^{-1}b$.

Thus, to solve $Ax = b$ with a splitting $A = M-N$, with M non-singular, we can use the above iteration scheme with $x^{(0)}$ as the initial guess. we compute the sequence of vectors according to (0), untill the sequence converges

For the convergence we use the following result.

contraction map
$f:(X, d) \to (Y, \rho)$
$\rho(f(x), f(y)) \le M d(x,y)$

<u>Th:</u> (Convergence for iteration method):
Let $A \in M_n(\mathbb{R})$ with a splitting (M, N). Let $H = M^{-1}N$ and $c = M^{-1}b$.
The iteration on $x^{(k+1)} = Hx^{(k)} + c$ converges for all choice of the initial guess $x^{(0)} \Leftrightarrow \exists$ a norm $\|\cdot\|$ on $M_n(\mathbb{R})$ such that $\|H\| < 1$.
$\Rightarrow$ [see Epperson, p-463]

Since, we know that for any $B \in M_n(\mathbb{R})$ the spectral radius $\rho(B) < 1 \Leftrightarrow \exists$ a norm $\|\cdot\|$ on $M_n(\mathbb{R})$ such that $\|B\| < 1$

Thus combining, we obtain the following.

Th: (Convergence for iteration using spectral radius):

The iteration scheme (0) converges for all choices of the initial guess $x^{(0)}$ $\Leftrightarrow$ the spectral radius (of the iteration matrix) $\rho(H) < 1$.

■ The following are the two commonly used splitting methods.

Jacobi and Gauss-Seidel Method:

Consider solving the system $Ax = b$, where $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$. We first express the co-efficient matrix A (= (a_{ij})) in the form $A = D - L - U$, where D (= (d_{ij})) is the diagonal part of A (i.e.

$$d_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}\Big) \quad \&$$

L (= (l_{ij})), U (= (u_{ij}))

$-L$ is the strictly lower triangular part of A (with $L = (l_{ij})$)

$$\Big(\text{i.e. } l_{ij} = \begin{cases} 0 & , i \leq j \\ -a_{ij} & , i < j \end{cases}\Big)$$

and $-U$ is the strictly upper triangular part of A

$$\Big(\text{i.e. } u_{ij} = \begin{cases} 0 & \text{if } i \geq j \\ a_{ij} & \text{if } i < j \end{cases}\Big) \quad (\text{with } U = (u_{ij}))$$

Thus,
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -a_{21} & 0 & \cdots & 0 & 0 \\ -a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & & & \vdots & \\ -a_{n1} & -a_{n2} & \cdots & a_{n\,n-1} & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ 0 & 0 & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & -a_{n-1\,n} \\ 0 & 0 & \cdots & & 0 \end{pmatrix}$$

$$= D - L - U$$

Note that, the matrices L & U appeared here are no longer related to the LU- decomposition of the matrix A.

$\underline{M}$	$\underline{N}$
D	$L+U$
$D-L$	U
$\frac{1}{\lambda}D-L$	

Ex: Express $A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 5 & 7 & -5 \\ -8 & 3 & -9 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ into $A = D - L - U$, where D, L & U are defined above.

$\Rightarrow$ Here, $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 8 & -3 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & -7 & 5 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore A = D - L - U$

$= M - N$; where $M = D$ and $N = L + U$

Jacobi Method : The Jacobi method is based on the splitting $M = D$ and $N = L+U$, in order for M to be non-singular, it must be the case that for each i, $d_{ii} = a_{ii} \neq 0$. If the relationship does not hold for a single value of i, then the equations in the system must be reordered befor the Jacobi method can be applied. The iteration scheme for the Jacobi method is given by $\boxed{x^{(k+1)} = H_J x^{(k)} + C_J}$ ------ (1), where

$H_J = D^{-1}(L+U)$ and $C_J = D^{-1} b$.

Taking into account the structure of the iteration matrix H_J and the vector C_J, the individual components of (1) can be written as $x_i^{(k+1)} = \frac{1}{a}$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right]} \quad \text{---} (2)$$

Thus, the Jacobi method is equivalent to solving the i-th equation in the above system for the unknown x_i. Note that the Jacobi method is also called the, " method of simultaneous Relaxation" or the "method of Simultaneous displacement"), as all the components of $x^{(k+1)}$ can be computed in any order and that on a parallel or vector machine, all the components of $x^{(k+1)}$ can be computed simultaneously.

<u>Note:</u> In the Jacobi method, $M = D$, $N = L+U = D-A$,

$H_J = D^{-1}(L+U) = D^{-1}(D-A) = (I - D^{-1}A)$

so, the iteration scheme becomes,

$$\boxed{x^{(k+1)} = (I - D^{-1}A)x^{(k)} + D^{-1}b}$$

<u>Ex:</u> solve
$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$2x_1 + 2x_2 + 10x_3 = 14$$

using Jacobi method.

<u>Soln:</u> Here $A = \begin{pmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 12 \\ 13 \\ 14 \end{pmatrix}$

$\therefore A = \begin{pmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{pmatrix}$

$= \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$D^{-1} = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}, \quad L+U = \begin{pmatrix} 0 & -1 & -1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

$$H_J = D^{-1}(L+U) = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -0.1 & -0.1 \\ -0.2 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix}$$

$$c_J = D^{-1} b$$

$$= \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 12 \\ 13 \\ 14 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

The iteration scheme is,

$$x^{(k+1)} = H_J x^{(k)} + c_J.$$

Let $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ be the initial guess.

Then, $x^{(1)} = c_J$

$$= \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 0 & -0.1 & -0.1 \\ -0.2 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix} + \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.94 \\ 0.92 \\ 0.90 \end{pmatrix}$$

$$x^{(3)} = \begin{pmatrix} 0 & -0.1 & -0.1 \\ -0.2 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 0.94 \\ 0.92 \\ 0.90 \end{pmatrix} + \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.94 \\ 0.92 \\ 0.90 \end{pmatrix}$$

$$x^{(3)} = \begin{pmatrix} 0 & -0.1 & -0.1 \\ -0.2 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 0.94 \\ 0.92 \\ 0.90 \end{pmatrix} + \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix}$$

$$= \begin{pmatrix} 1.018 \\ 1.022 \\ 1.028 \end{pmatrix}$$

$$x^{(4)} = \begin{pmatrix} 0. & -0.1 & -0.1 \\ -0.2 & 0 & -0.1 \\ -0.2 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 1.018 \\ 1.022 \\ 1.028 \end{pmatrix} + \begin{pmatrix} 1.2 \\ 1.3 \\ 1.4 \end{pmatrix} = \begin{pmatrix} 0.9954 \\ 0.9936 \\ 0.9908 \end{pmatrix}$$

$\therefore$ The solution is $x_1 = 1, x_2 = 1, x_3 = 1$

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Gauss-seidel method:

A natural improvement that can be made to the Jacobi method is to use the value of $x_i^{(k+1)}$ as soon as it has been calculated in the computation of all subsequent entries in the vector $x^{(k+1)}$, rather than waiting until the next iteration. Thus in this case $x_i^{(k+1)}$ is suppose to be a better approximation to x_i than $x_i^{(k)}$. This modification results in changing equation ② to the following.

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j<i} a_{ij}\, x_j^{(k+1)} - \sum_{j>i} a_{ij}\, x_j^{(k)}\right.} \quad \cdots ③$$

The only difference between the equations is that subscript on x in the 1st summation is now $(k+1)$. The iteration scheme corrosponding to equation ③ is known as Gauss-seidel method.

Note that the Gauss-seidel method is not vectorizable. The entries $x^{(k+1)}$ must be computed in succession. Hence the Gauss-seidel method is known as "method of successive Relaxation" or the "method of successive displacements".

Note that the splitting for the Gauss-seidel method is $A = M - N$, with $M = D - L$, $N = U$
"
$D - L - U$

Thus the iteration scheme for the G-S method is given by $\boxed{x^{(k+1)} = H_{GS}\, x^{(k)} + C_{GS}}$ ...... (4)

where $H_{GS} = (D-L)^{-1}U$, $C_{GS} = (D-L)^{-1}b$.

The necessary and sufficient condition for the matrix M to be non-singular is that $d_{ii} = a_{ii} \neq 0$, for each i.

Note: In the Gauss-seidel method,

$$M = D - L \cancel{= A+U} \quad , \quad N = U = D - L - A$$

$$\therefore H_{GS} = (D-L)^{-1}(D-L-A)$$
$$= \left[I - (D-L)^{-1}A\right]$$

so, the iteration becomes,

$$x^{(k+1)} = \left[I - (D-L)^{-1}A\right]x^{(k)} + (D-L)^{-1}b.$$

☑ Specific properties for the Jacobi and G-S method:

We obtain certain conditions on the coefficient matrix A, so that the Jacobi, G-S method converge.

Unfortunately there is no general theory (on the coefficient matrix) for these methods, just a collection of special cases.

Th: (Convergence for Jacobi and G-S method):

Let $A \in M_n(\mathbb{R})$.

(I) If A is strictly diagonally dominant, then both the Jacobi and G-S method converge and the G-S method converges faster in the sense that $\rho(H_{GS}) < \rho(H_J)$, where H_J and H_{GS} are the iteration matrix for the Jacobi and G-S method respectively.

② If A is symmetric positive definite, then both the Jacobi and G-S method converge.

⇒ (See Epperson, P-465)

Th: (Convergence for the G-S method)

Let $A \in M_n(\mathbb{R})$.

(1) If A is symmetric with all positive diagonal entries, then the Gauss-seidel method converges iff A is positive definite.

(2) If A is positive definite, then the Gauss-seidel method converges for any choice of the initial guess $x^{(0)}$.

⇒ (See Bradie, P-232)

Also note that, in general cases, we cannot compare the speed of the convergence for the Jacobi and G-S method. There are examples of a co-efficient matrix A, for which the Jacobi method converges but the G-S method does not. There is no general theory to indicate which method, Jacobi or G-S will perfom best on an arbitrary problem. In the following we can list some special cases:

Th: (spectral radius for Jacobi and G-S method)

Let $A \in M_n(\mathbb{R})$ with $a_{ii} > 0$ for each i and $a_{ij} \leq 0$ for each
$\overset{\shortparallel}{(a_{ij})}$

$i \neq j$. Then one and only one of the following statements holds.

(1) $0 \leq \rho(H_{GS}) = \rho(H_J) < 1$

(2) $1 < \rho(H_J) < \rho(H_{GS})$

(3) $\rho(H_J) = \rho(H_{GS}) = 0$

(4) $\rho(H_J) = \rho(H_{GS}) = 1$

Thus under the hypothesis of this theorem, when one method converges, so does the other method, with the G-S method converging faster. On the other hand when one method diverges, so does the other, with the G-S method diverging faster.

$\Rightarrow$ (see Bradie, p-233)

Ex: Find the value of λ for which $A = \begin{pmatrix} 5 & -2 & 2 \\ -2 & 6 & \lambda \\ 2 & \lambda & 7 \end{pmatrix}$ is

(a) positive definite

(b) strictly diagonally dominant.

$\Rightarrow$ Given $A = \begin{pmatrix} 5 & -2 & 2 \\ -2 & 6 & \lambda \\ 2 & \lambda & 7 \end{pmatrix}$

This is symmetric.

(a) We know A is positive definite iff each of its leading principle submatrices has +ve determinant.

$$\det A = \begin{vmatrix} 5 & -2 & 2 \\ -2 & 6 & \lambda \\ 2 & \lambda & 7 \end{vmatrix} > 0$$

$\Rightarrow 5(42-\lambda^2)+(-2)(2\lambda+14)+2(-2\lambda-12)>0$

$\Rightarrow -5\lambda^2+210-4\lambda-28-4\lambda-24>0$

$\Rightarrow -5\lambda^2-8\lambda+158>0$

$\Rightarrow 5\lambda^2+8\lambda-158<0$

For $5\lambda^2+8\lambda-158<0$, the matrix A is +ve definite.

(b) For strictly diagonally dominant matrix,

Here, $a_{22}=6>-2+\lambda \Rightarrow \lambda<8$

$a_{33}=7>2+\lambda \Rightarrow \lambda<5$

Combining these two we get, for $\lambda<5$ the matrix A is strictly diagonally dominant.

Ex: Find the values of λ for which $A=\begin{pmatrix}\lambda & -1 & 0\\ -1 & 4 & 1\\ 0 & 1 & 5\end{pmatrix}$ is

(a) Positive definite

(b) strictly diagonally dominant.

$\Rightarrow$ Here $A=\begin{pmatrix}\lambda & -1 & 0\\ -1 & 4 & 1\\ 0 & 1 & 5\end{pmatrix}$

This is a symmetric matrix.

(a) Thus A is positive definite iff each of its leading principle submatrices has positive determinant.

so, ① $\det|\lambda|>0 \Rightarrow \lambda>0$

② $\begin{vmatrix}\lambda & -1\\ -1 & 4\end{vmatrix}>0 \Rightarrow 4\lambda-1>0$

$\Rightarrow \lambda>1/4$

③ $\begin{vmatrix} \lambda & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{vmatrix} > 0 \Rightarrow 19\lambda - 5 > 0$

$\Rightarrow \lambda > \frac{5}{19}$

combining all the cases we have for $\lambda > \frac{5}{19}$ the matrix A is +ve definite.

(b) we have, if $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$ then the matrix (a_{ij}) is strictly diagonally dominant.

Now, Here, $a_{11} = \lambda > a_{12} + a_{13}$

$= -1 + 0$

$= -1$

$\Rightarrow \lambda > -1$

and clearly, $a_{22} > a_{21} + a_{23}$ i.e. $4 > 0$

and $a_{33} > a_{31} + a_{32}$ i.e. $5 > 1$

Thus for $\lambda > -1$, the matrix A is strictly diagonally dominant.

1.(b) solve

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

$\Rightarrow$ we can write $Ax = b$.

where $A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}$

clearly A is tridiagonal matrix.

Thus $A = LU$

where $l_{11} = a_{11} = 2$

$l_{21} = a_{21} = -1$

$u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{2}$

$l_{22} = a_{22} - l_{21} u_{12} = 2 - (-1) \times (-1/2) = 2 - 1/2 = \frac{3}{2}$

$l_{32} = a_{32} = -1$

$u_{34} = \frac{a_{34}}{l_{33}} = -\frac{1}{2}$

$l_{43} = a_{43} = -1$

$l_{44} = a_{44} - l_{43} u_{34} = 2 - (-1) \times (-1/2) = 2 - 1/2 = 3/2$

$\therefore L = \begin{pmatrix} 2 & & & \\ -1 & \frac{3}{2} & & \\ & -1 & 2 & \\ & & -1 & 3/2 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1/2 & & \\ & 1 & 0 & \\ & & 1 & -1/2 \\ & & & 1 \end{pmatrix}$

Then, $Ax = b \Rightarrow LUx = b$

Let $Ux = z$, where $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$

Thus $Lz = b$

$$\Rightarrow \begin{pmatrix} 2 & & & \\ -1 & 3/2 & & \\ & -1 & 2 & \\ & & -1 & 3/2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

$\Rightarrow 2z_1 = 0 \Rightarrow z_1 = 0$

$-z_1 + \frac{3}{2} z_2 = 0 \Rightarrow z_2 = 0$

$-z_2+2z_3=0 \Rightarrow z_3=0$

$-z_3+3/2\, z_4=5 \Rightarrow 3/2\, z_4=5 \Rightarrow z_4=\frac{10}{3}$

Then $Ux=z$

$$\Rightarrow \begin{pmatrix} 1 & -1/2 & & \\ & 1 & 0 & \\ & & 1 & -1/2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 10/3 \end{pmatrix}$$

$\Rightarrow x_4 = 10/3$

$x_3 - \frac{1}{2}x_4 = 0 \Rightarrow x_3 = \frac{1}{2}\times\frac{10}{3} \Rightarrow x_3 = 5/3$

$x_2 = 0$

$x_1 - 1/2\, x_2 = 0 \Rightarrow x_1 = 0$

$\therefore$ The solution of the system is $x = \begin{pmatrix} 0 \\ 0 \\ 5/3 \\ 10/3 \end{pmatrix}$.

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Ex: Jacobi method converges but G-S does not:

Let $A = \begin{pmatrix} 2 & 4 & -4 \\ 3 & 3 & 3 \\ 10 & 10 & 5 \end{pmatrix}$

(a) Compute the iteration matrix H_J corresponding to A and determin $\rho(H_J)$. Does the Jacobi method converge for any choice of $x^{(0)}$?

(b) Compute the iteration matrix H_{GS} corresponding to A and determine $\rho(H_{GS})$. Does the G-S method converge for any choice of $x^{(0)}$?

Sol^n: We write $A = D - L - U$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -10 & -10 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Now for the Jacobi method, the spliting (M, N) for A is $M = D$, $N = L + U$.

$\therefore$ The iteration matrix for the Jacobi matrix is

$$H_J = D^{-1}(J+U) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 0 & -4 & 4 \\ -3 & 0 & -3 \\ -10 & -10 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

The characteristic equation is

$$\det(H_J - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -2 & 2 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 2) - 2(2 - \lambda) + 2(2 - 2\lambda) = 0$$

$$\Rightarrow -\lambda^3 + \cancel{2\lambda} - \cancel{4} + \cancel{2\lambda} + \cancel{4} - \cancel{4\lambda} = 0$$

$$\Rightarrow \lambda^3 = 0$$

$$\Rightarrow \lambda = 0, 0, 0$$

The eigen values of H_J are : 0, 0, 0.

$\therefore \rho(H_J) = 0 < 1$.

Thus, the Jacobi method converges for any choice of $x^{(0)}$.

(b) Now for the Jacobi method, the splitting (M, N) for A is $M = D-L$, $N = U$.

$\therefore$ The iteration matrix for the G-S method is

$$H_{GS} = (D-L)^{-1}U = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 10 & 10 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} 15 & -15 & 0 \\ 0 & 10 & -20 \\ 0 & 0 & 6 \end{pmatrix}^{t} \begin{pmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} 15 & 0 & 0 \\ -15 & 10 & 0 \\ 0 & -20 & 6 \end{pmatrix} \begin{pmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

ch equation:

$\det(H_{GS} - tI) = 0$

$\Rightarrow -t(2-t)(2-t) = 0$

$\Rightarrow t = 0, 2, 2$

The eigen values of H_{GS} are 0, 2, 2

$\therefore \rho(H_{GS}) = 2 \nless 1$

Thus, the G-S method does not converge for any choice of $x^{(0)}$.

Ex: Consider the iteration $x^{(k+1)} = 1 + x^{(k)} + c$, where $H \in M_n(\mathbb{R})$, $c \in \mathbb{R}^n$. Let $\|H\| < 1$ for some natural norm (sub ordinate norm) $\|\cdot\|$ on $M_n(\mathbb{R})$. Then prove that any $x^{(0)} \in \mathbb{R}^n$,

$$\|x^{(k)} - x\| \leq \frac{\|H\|^k}{1 - \|H\|} \|x^{(1)} - x^{(2)}\| \quad \forall\, k \in \mathbb{N},$$

where x is the exact solution of $x = Hx + c$.

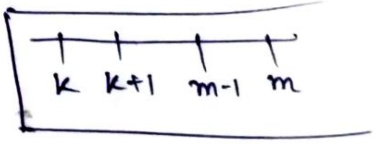
$\Rightarrow$ For any $k \in \mathbb{N} \cup \{0\}$

$$x^{(k+1)} - x^{(k)} = Hx^{(k)} + c - (Hx^{(k-1)} + c)$$
$$= H(x^{(k)} - x^{(k-1)})$$

Thus, $\|x^{(k+1)} - x^{(k)}\| = \|H(x^{(k)} - x^{(k-1)})\|$

$$= \|H^2(x^{(k-1)} - x^{(k-2)})\|$$
$$\vdots$$
$$= \|H^k(x^{(1)} - x^{(0)})\|$$
$$\leq \|H\|^k \|x^{(1)} - x^{(0)}\|, \quad \forall\, k \in \mathbb{N} \cup \{0\}$$

Now, for any $m > k$, we have



$$\|x^{(m)} - x^{(k)}\|$$
$$= \|x^{(m)} - x^{(m-1)} + x^{(m-1)} - x^{(m-2)} + \cdots + x^{(k+1)} - x^{(k)}\|$$
$$\leq \|x^{(m)} - x^{(m+1)}\| + \|x^{(m-1)} - x^{(m-2)}\| + \cdots + \|x^{(k+1)} - x^{(k)}\|$$
$$= \sum_{j=k}^{m-1} \|x^{(j+1)} - x^{(j)}\|$$
$$\leq \sum_{j=k}^{m-1} \|H\|^j \|x^{(1)} - x^{(0)}\|$$

$= \cancel{\|H\|^k (1 + \|H\| + \|H\|^{m-k-1})}$

$= \|H\|^k (1 + \|H\| + \cdots + \|H\|^{m-k-1}) \|x^{(1)} - x^{(0)}\|$

$= \|H\|^k \dfrac{(1 - \|H\|^{m-k})}{1 - \|H\|} \|x^{(1)} - x^{(0)}\|$

$\boxed{(1-x)(1+x+x^2+\cdots+x^p) = 1 - x^{(p+1)} \text{ for } x \neq 0}$

$< \dfrac{\|H\|^k}{(1 - \|H\|)} \|x^{(1)} - x^{(0)}\|$ [as $\|H\| < 1$] ——— (*)

Since $\|H\|^k \to 0$ as $k \to \infty$, the sequence $(x^{(k)})$ is cauchy in $\mathbb{R}^n$ and hence it converges.

$\boxed{\because \|H\| < 1 \text{ so if } k \to \infty \text{ then } \|H\|^k \to 0}$

say, $x^{(k)} \to x$ (in $\mathbb{R}^n$)

From the linear iteration ~~the~~ scheme we have $x = Hx + c$.

Also, taking $m \to \infty$ in (*),

$$\|x^{(k)} - x\| \leq \frac{\|H\|^k}{1 - \|H\|} \|x^{(1)} - x^{(0)}\|, \ \forall k \in \mathbb{N}.$$

Ex: Consider a matrix A (given below).

① Compute the iteration matrix H_J and the spectral ~~the~~ ~~iteration~~ radius $\rho(H_J)$ for the Jacobi method. Does this method converge for any choice of the initial guess $x^{(0)}$?

② Compute the iteration matrix H_{GS} and the spectral radius $\rho(H_{GS})$ for the G-S method. Does this method converge for any choice of the initial guess $x^{(0)}$?

where A = (i) $\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & -1 & -2 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{pmatrix}$

(iv) $\begin{pmatrix} 3 & 2 & -2 \\ -2 & -2 & 1 \\ 5 & -5 & 4 \end{pmatrix}$

Ex: Let A be strictly diagonally dominant and let H_J be the iteration matrix for the Jacobi method. Prove that $\rho(H_J) < 1$.

$\Rightarrow$ Now, if $A \in M_n(\mathbb{R})$, then for the Jacobi method the splitting for A is given by $M = D$, $N = L + U$, where $A = D - L - U$.

Thus $H_J = D^{-1}(L+U)$.

$$= \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \vdots & & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & -\frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & & & \\ & \frac{1}{a_{22}} & & \\ & & \ddots & \\ & & & \frac{1}{a_{nn}} \end{pmatrix}$$

$L+U$

$$= \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 0 & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{pmatrix}$$

i.e. the diagonal entries of H_J are all zero and the off-diagonal entries for the position (i,j) is given by $-\frac{a_{ij}}{a_{ii}}$.

i.e. $H_J = (h_{ij})_{n\times n}$ with $h_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\frac{a_{ij}}{a_{ii}} & \text{if } i \neq j \end{cases}$

$$\therefore \|H_J\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}|$$

$$= \max_{i=1}^{n} \sum_{\substack{j=1,\\ (j\neq i)}}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

$$= \max_{i=1}^{n} \frac{1}{|a_{ii}|} \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{---} \quad (*)$$

Since A is strictly diagonally dominant, $|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$, for each $i = 1, 2, \ldots, n$.

$$\therefore \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} < 1, \text{ for each } i = 1, 2, \ldots, n$$

Therefore by $(*)$ $\|H_J\|_\infty < 1$.

Since the spectral radius of a matrix is less than any ~~method~~ natural matrix norm, so we must have $\rho(H_J) < \|H_J\|_\infty < 1$.

[$A \in M_n(\mathbb{R})$

Let, Eigen values are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding eigen vectors $y_1, y_2, \ldots, y_n$

Then $Ay_j = \lambda_j y_j$, $1 \leq j \leq n$

$$\frac{\|Ay_j\|}{\|y_j\|} = |\lambda_j|$$

$$\rho(A) = \max_{j=1}^{n} \frac{\|Ay_j\|}{\|y_j\|}$$

$$\leq \sup_{\substack{x \in \mathbb{C}^n\\ x \neq 0}} \frac{\|Ax\|}{\|x\|} = \|A\|$$

$\therefore \rho(A) \leq \|A\|$]

Ex: Let $A \in M_2(\mathbb{R})$ be symmetric. Find conditions on A. so that A is +ve definite.

$\Rightarrow$ Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

A is +ve definite $\Leftrightarrow$ $a>0$, $\det(A)>0$

Ex: Let $A = \begin{pmatrix} b & -1 & a \\ -1 & 3 & 0 \\ a & 0 & 4 \end{pmatrix}$

(a) Find conditions on a & b so that A is symmetric +ve definite.

(b) Find conditions on a & b so that A is strictly diagonally dominant.

$\Rightarrow$ Given $A = \begin{pmatrix} b & -1 & a \\ -1 & 3 & 0 \\ a & 0 & 4 \end{pmatrix}$

Pentadiagonal

$|i-j|>2$

$|a_{ij}|>0$

A is symmetric.

(a) we know A is symmetric +ve definite iff each of its leading principle submatrices has +ve determinant.

i.e. $b>0$, $3b-1>0$

$\Rightarrow b > 1/3$

and $\det A > 0$

$$\Rightarrow \begin{vmatrix} b & -1 & a \\ -1 & 3 & 0 \\ a & 0 & 4 \end{vmatrix} > 0$$

$$\Rightarrow 12b \cancel{*} 4 + a(-3a) > 0$$

$$\Rightarrow 12b - 3a^2 - 4 > 0$$

$\Rightarrow b > \frac{1}{3} + \frac{1}{4}a^2$

Hence, A is symmetric positive definite iff $b > 1/3 + \frac{1}{4}a^2$.

(b) For strictly diagonall matrix we have,

$|a_{11}| > |a| + 1$ i.e $\boxed{|b| > |a| + 1}$

$|a_{22}| > |a_{21}| + |a_{23}|$

i.e. $3 > 1$

& $|a_{33}| > |a_{31}| + |a_{32}|$

i.e. $\boxed{4 > |a|}$

Therefore, If $|b| > |a| + 1$ and $|a| < 4$ then A is strictly diagonally dominant.

<u>Note</u>: If $A = D - L - U$ and the splitting for A is $M = \frac{1}{\omega} D - L$ and $N = \left(\frac{1}{\omega} - 1\right) D + U$, then iteration method is known as SOR (Successive over Relaxation) method. The iteration matrix is $H_{SOR} = M^{-1}N = \left(\frac{1}{\omega} D - L\right)^{-1} \left\{\left(\frac{1}{\omega} - 1\right) D + U\right\}$

& $C_{SOR} = \left(\frac{1}{\omega} D - L\right)^{-1} b$. (i.e. $x^{(k+1)} = H_{SOR}\, x^{(k)} + C_{SOR}$)

The iteration scheme is given by

$$x_i^{(k+1)} = (1-\omega) x_i^{(k)} + \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

For $a=1$, the SOR methods reduces to the Gauss-seidel method.

It can be shown that (1) If A ~~is~~ has all non-zero diagonal entries (i.e. $a_{ii} > 0$, $\forall i = 1, 2, \ldots, n$), then $\rho(H_{SOR}) \geq |\omega - 1|$.
so, the SOR method converges $\Leftrightarrow 0 < \omega < 2$.

(2) If A is +ve definite and $0 < \omega < 2$, then the SOR method will converge for any choice of initial guess $x^{(0)}$.

- Such ω is called the relaxation parameter for the iteration method.

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Interpolation:

we consider the following mathematical problem:

Given a set of point (x_i, f_i) for $i = 0, 1, \ldots, n$ where x_i's are distinct values of the independent variable and f_i's are corresponding values of some function f, either, (a) approximate the value of f at some value of x not listed among the x_i's. ~~or~~

or, (b) determinant of function g that (in some sense) apporoximate the ~~date~~ data.

Note that problem data can also include derivative values in addition to function values. In some intances, the problem data will be specified ~~and~~ as the function f itself, rather than as a discrete set of points form the graph of f.