5 NUMERICAL SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

5.1 Introduction:

System of linear algebraic equations arise in a large number of problems in science and technology. The most common form of the system in n unknowns $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\ldots \qquad \ldots \qquad & \ldots \\
\ldots \qquad \ldots \qquad & \ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

$$\text{i.e., } \sum_{j=1}^{n} a_{ij}x_j = b_i, \quad (i = 1, 2, \ldots, n) \qquad \ldots \quad (1)$$

where $a_{ij}(i, j = 1, 2, \ldots n)$ and $b_i(i = 1, 2, \ldots n)$ are given numbers. We can also write the equation (1) in the matrix form as

$$AX = b \qquad \ldots \quad (2)$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{n\times n},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \qquad \ldots \quad (3)$$

in which it is supposed that the matrix A is non-singular, i.e., $\det A \neq 0$ so that the system (2) has *a* unique solution. The system of equations (1) is said to be homogeneous if all $b_i(i = 1, 2, \ldots, n)$, are zero; otherwise, the system is called non-homogeneous.

To solve the above system of equations we apply, in general, two methods viz **(i)** direct method and **(ii)** indirect or iterative method. In direct method, the solution is obtained after a finite number of steps of elementary arithmetical operations. On the otherhand, in indirect or iterative method, we start with an arbitrary initial approxmation to x and then improve this estimate in an infinite but convergent sequence of steps.

We discuss in this chapter both the above methods in various ways.

Direct methods.

(i) Gauss elimination method

(ii)Matrix inversion method

(iii) LU Factorization method

Indirect or iterative methods

(i) Gauss-Seidel method

5.2. Gauss elimination method.

In this method, the given system of equations is reduced to an equivalent upper triangular system by a systematic elimination procedure from which the unknowns are found by back substitution.

To illustrate the mehtod, we consider the system (1) given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \qquad \ldots \qquad & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned} \qquad \ldots \quad (4)$$

First suppose that $a_{11} \neq 0$

Multiply the first equation of (4) by $\dfrac{a_{i1}}{a_{11}} (i = 2, 3, \ldots, n)$ and subtract the results from the i-th equation, $(i = 2, 3, \ldots, n)$ and obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \ldots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + \ldots + a_{3n}^{(1)}x_n &= b_3^{(1)} \\ \ldots \qquad \ldots \qquad & \ldots \\ a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \ldots + a_{nn}^{(1)}x_n &= b_n^{(1)} \end{aligned} \qquad \ldots \quad (5)$$

where

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}} \qquad \dots \qquad (6)$$

$$b_i^{(1)} = b_i - \frac{b_1 a_{i1}}{a_{11}} \quad (i, j = 2, 3, \dots. n)$$

The numbers $\frac{a_{i1}}{a_{11}}$, $(i = 2, 3, \dots, n)$ are called row multipliers. The first equation of the syestem (5) contains x_1 while the remaining $(n-1)$ equations are independent of x_1.

Next assume $a_{22}^{(1)} \neq 0$

Multiplying the second equation of (5) by $\frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$, $(i = 3, 4, \dots, n)$ and subtracting the results from the i-th equation, $(i = 3, 4, \dots, n)$ we get

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \dots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\
a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)}x_n &= b_3^{(2)} \qquad \dots \qquad (7) \\
\dots \qquad \dots \qquad & \dots \\
a_{n3}^{(2)}x_3 + \dots + a_{nn}^{(2)}x_n &= b_n^{(2)}
\end{aligned}$$

where

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{2j}^{(1)}a_{i2}^{(1)}}{a_{22}^{(1)}},$$

$$b_i^{(2)} = b_i^{(1)} - \frac{b_2^{(1)}a_{i2}^{(1)}}{a_{22}^{(1)}}, (i, j = 3, 4, \dots, n) \qquad \dots \qquad (8)$$

Here also the numbers $\frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$ are row multipliers.

In the system (7), the last $(n-2)$ equations are independent of x_1 and x_2.

Repeating the procedure, we obtain a system of n equations equivalent to an upper triangular system in the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + ... + a_{2n}^{(1)}x_n = b_2^{(1)}$$

$$a_{33}^{(2)}x_3 + ... + a_{3n}^{(2)}x_n = b_3^{(2)}$$

$$... \quad ... \quad ...$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \quad ... \quad (9)$$

The coefficients of the leading terms in (9), i.e., the elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, ..., a_{nn}^{(n-1)}$ are called *pivotal elements* and the corresponding equations are known as *pivotal equations*. The solutions of the (4) are then obtained from (9) by back substitutions as

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \quad ... \quad (10)$$

$$x_{n-1} = \frac{1}{a_{n-1,n-1}^{(n-2)}}\left[b_{n-1}^{(n-2)} - \frac{a_{n-1,n}^{(n-2)} b_n^{(n-1)}}{a_{nn}^{(n-1)}}\right]$$

etc., provided none of the pivotal element is zero.

The above procedure can also be explained in a more compact form by matrix notation as following :

The augmented matrix is

$$\begin{array}{c} m\,(multiplier) \\ a_{21}/a_{11} \\ a_{31}/a_{11} \\ \vdots \\ a_{n1}/a_{11} \end{array} \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & \vdots & b_2 \\ a_{31} & a_{32} & ... & a_{3n} & \vdots & b_3 \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} & \vdots & b_n \end{bmatrix}, \text{ provided } a_{11} \neq 0 \quad (11)$$

After the first elimination, we have

$$\begin{array}{c} m\,(\text{multiplier}) \\ \\ a_{32}^{(1)}/a_{22}^{(1)} \\ \vdots \\ a_{n2}^{(1)}/a_{22}^{(1)} \end{array} \left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & \vdots & b_2^{(1)} \\ & a_{32}^{(1)} & \cdots & a_{3n}^{(1)} & \vdots & b_3^{(1)} \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & \vdots & b_n^{(1)} \end{array}\right], \text{ provided } a_{22}^{(1)} \neq 0 \quad \ldots (12)$$

in which $a_{ij}^{(1)}$ and $b_i^{(1)}, (i = 2, 3, \ldots, n)$ are given by (6).

The second elimination gives

$$\left[\begin{array}{ccccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \vdots & b_1 \\ & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & \vdots & b_2^{(1)} \\ & & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} & \vdots & b_3^{(2)} \\ & & \cdots & \cdots & \cdots & \vdots & \\ & & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} & \vdots & b_n^{(2)} \end{array}\right] \quad \ldots (13)$$

Repeating the process for $(n-1)$ times, we obtain the following upper triangular matrix :

$$\left[\begin{array}{ccccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \vdots & b_1 \\ & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} & \vdots & b_2^{(1)} \\ & & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} & \vdots & b_3^{(2)} \\ & & \cdots & \cdots & \cdots & \vdots & \\ & & \cdots & \cdots & a_{nn}^{(n-1)} & \vdots & b_n^{(n-1)} \end{array}\right] \quad \ldots (14)$$

which is equivalent to the system (9) and hence by back substitution we get the required solutions of the system.

Note. (i) In Gauss elimination method, the total number of multiplications and divisions is $\frac{n^3}{3} + n^2 - \frac{n}{3}$ and those of additions and subtractions is $\frac{n^3}{3} + \frac{n^2}{2} - \frac{5}{6}n$

(ii) The method fails if any of the pivotal elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{nn}^{(n-1)}$ is zero. In such cases, we rearrange the equations in such a way that the pivotal elements do not vanish. If it is not at all possible, then the solution of the given system does not exist.

Example.1. Solve the following system of linear equations by Gauss-elimination method.

$$x - 2y + 9z = 8$$
$$3x + y - z = 3$$
$$2x - 8y + z = -5$$

[*W.B.U.T., CS-312 2007, 2008*
M(CS)-401, 2016]

Solution. In order to eliminate x from the last two equations, we multiply the first equation successively by 3, 2 and subtract the results from the second and third equations respectively. Thus we have

$$7y - 28z = -21 \quad \cdots \quad (1)$$
$$-4y - 17z = -21 \quad \cdots \quad (2)$$

In the next step, we eliminate y from (2) by multiplying the equation (1) by $\frac{4}{7}$ and add the result from (2) to get

$$-33z = -33.$$

Thus the given system of equations reduces to the following upper triangular form as

$$x - 2y + 9z = 8$$
$$7y - 28z = -21$$
$$-33z = -33$$

from which the back substitution leads to the required solution as

$$x = 1, \; y = 1, \; z = 1$$

Example.2. Solve the following system of equations by Gauss elimination method :

$$x + 2y + z = 0$$
$$2x + 2y + 3z = 3$$
$$-x - 3y = 2$$

[*M.A.K.A.U.T., M(CS)-301, 2014,*
M(CS)-401, 2015]

Solution. The augmented matrix of the given system of equations is

$$\begin{array}{c} m\,(multiplier) \\ 2 \\ -1 \end{array} \begin{bmatrix} 1 & 2 & 1 & \vdots & 0 \\ 2 & 2 & 3 & \vdots & 3 \\ -1 & -3 & 0 & \vdots & 2 \end{bmatrix}$$

Using the row operations $R_2 - 2R_1$ and $R_3 + R_1$ we get

$$\begin{array}{c} m\,(multiplier) \\ \\ \frac{1}{2} \end{array} \begin{bmatrix} 1 & 2 & 1 & \vdots & 0 \\ 0 & -2 & 1 & \vdots & 3 \\ 0 & -1 & 1 & \vdots & 2 \end{bmatrix}.$$

Again using the row operation $R_3 - \frac{1}{2}R_2$ we have

$$\begin{bmatrix} 1 & 2 & 1 & \vdots & 0 \\ 0 & -2 & 1 & \vdots & 3 \\ 0 & 0 & \frac{1}{2} & \vdots & \frac{1}{2} \end{bmatrix}$$

Hence the given system of equations is reduced to the upper triangular form given by

$$x + 2y + z = 0$$

$$-2y + z = 3$$

$$\frac{1}{2}z = \frac{1}{2}$$

$\therefore$ By back substitution, the resulting solutions are

$$x = 1, y = -1, z = 1.$$

5.3. Matrix inversion method.

For the system (2) viz.

$$AX = b, \qquad \ldots \quad (15)$$

we suppose that det $A \neq 0$ and so A^{-1} exists. Multiplying both side of (15) by A^{-1}, we get

$$X = A^{-1}b \qquad \ldots \quad (16)$$

which gives the solution of the given system

Noting that

$$A^{-1} = \frac{adj\ A}{\det A}$$

$$= \frac{(A_{ji})_{n\times n}}{|a_{ij}|}, (i, j = 1, 2, ..., n)$$

we have

$$X = \frac{(A_{ji})_{n\times n}\, b}{|a_{ij}|} \qquad \cdots \quad (17)$$

where $adj\ A$ is the transpose of the matrix obtained from A by replacing each element a_{ij} of A by its corresponding co-factor A_{ij} $(i, j = 1, 2, ..., n)$.

Note. **(i)** The method fails if the matrix A is singular i.e., $\det A = 0$

(ii) The method is not suitable for $n > 4$, since it involves laborious numerical computation.

Example.3. Solve the following system of equations :

$$x + y + z = 4$$
$$2x - y + 3z = 1$$
$$3x + 2y - z = 1$$

by matrix inversion method.

Solution. The given system of equations can be written as

$$AX = b \qquad \cdots \quad (1)$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now $\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 13 \neq 0$

Hence A is non-singular.

$\because A^{-1}$ exist.

Since $adj\ A = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$, so

$$A^{-1} = \frac{adjA}{\det A} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

∴ From (1), we have

$$X = A^{-1}b$$

which gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

Thus the required solutions are

$$x = -1,\ y = 3,\ z = 2$$

5.4. LU-factorization method.

This method is also termed as *triangular decomposition* method. The method based on the fact that every square matrix can be expressed as the product of a lower and an upper triangular matrix provided all the principal minors of the given square matrix $A = (a_{ij})_{n\times n}$ are non-singular, i.e.,

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \ldots, \det A \neq 0 \qquad \ldots \quad (18)$$

Further, if the matrix A can be factorized, then it is unique.

Assume that it is possible to decompose the coefficient matrix A of the given system of equation (2) and is expressible as the product of a lower triangular matrix L and an upper triangular matrix U so that

$$A = LU \qquad \ldots \quad (19)$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & l_{n3} & l_{n4} & \cdots & l_{nn} \end{bmatrix}, \quad \ldots (20)$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & u_{24} & \cdots & u_{2n} \\ 0 & 0 & 1 & u_{34} & \cdots & u_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad \ldots (21)$$

Hence the system of equations

$$AX = b$$

become

$$LUX = b \quad \ldots (22)$$

Putting $UX = Y$ in (22) we get ... (23)

$$LY = b \quad \ldots (24)$$

where $Y = (y_1, y_2, \ldots, y_n)^T$

Thus by forward substitution, the unknowns $y_1, y_2, \ldots, y_n$ are determined from (24) and thereafter the unknowns $x_1, x_2, \ldots, x_n$ are obtained from (23) by backward substitution.

For the sake of clarity and simplicity we now consider a system of three equations with three unknowns viz

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \ldots (25)$$

Here the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

can be written as

$$A = LU$$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Thus we have

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

leading to

$$l_{11} = a_{11},\ l_{21} = a_{21},\ l_{31} = a_{31}$$

$$l_{11}u_{12} = a_{12},\ l_{11}u_{13} = a_{13} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}}, u_{13} = \frac{a_{13}}{l_{11}},$$

$$l_{21}u_{12} + l_{22} = a_{22}, \Rightarrow l_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

$$l_{31}u_{12} + l_{32} = a_{32} \Rightarrow l_{32} = a_{32} - l_{31}u_{12} = a_{32} - \frac{a_{31}a_{12}}{a_{11}}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \Rightarrow u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

and $l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$

$$\Rightarrow l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}.$$

Thus obtained values of $l_{11}, l_{21}, \ldots$ and $u_{12}, u_{13}, \ldots$ gives the matrices L and U.

Example.4. Solve the following system of equations by LU factorization method.

$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$6x_1 + 3x_2 + 12x_3 = 36$$

[*W.B.U.T.,CS-312, 2004, 2016*]

Solution. The given system of equations can be written as

$$AX = b$$

where $A = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}, b = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Let $A = LU$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}$$

leading to

$$l_{11} = 8, l_{11}u_{12} = -3 \Rightarrow u_{12} = -\frac{3}{8}$$

$$l_{11}u_{13} = 2 \Rightarrow u_{13} = \frac{2}{8} = \frac{1}{4}$$

$$l_{21} = 4, l_{21}u_{12} + l_{22} = 11 \Rightarrow l_{22} = 11 - l_{21}u_{12}$$

$$\Rightarrow l_{22} = 11 - 4\left(-\frac{3}{8}\right)$$

$$\Rightarrow l_{22} = \frac{25}{2}$$

$$l_{21}u_{13} + l_{22}u_{23} = -1$$

$$\Rightarrow 4.\frac{1}{4} + \frac{25}{2}.u_{23} = -1 \Rightarrow u_{23} = -\frac{4}{25}$$

$l_{31} = 6, l_{31}u_{12} + l_{32} = 3 \Rightarrow 6\left(-\frac{3}{8}\right) + l_{32} = 3$

$$\Rightarrow l_{32} = \frac{21}{4}$$

$l_{31}u_{13} + l_{22}u_{23} + l_{33} = 12 \Rightarrow 6.\frac{1}{4} + \frac{21}{4}\left(\frac{-4}{25}\right) + l_{33} = 12$

$$\Rightarrow l_{33} = \frac{567}{50}$$

Hence $L = \begin{bmatrix} 8 & 0 & 0 \\ 4 & \frac{25}{2} & 0 \\ 6 & \frac{21}{4} & \frac{567}{50} \end{bmatrix}, U = \begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & \frac{-4}{25} \\ 0 & 0 & 1 \end{bmatrix}$

Thus the equation $AX = b$ i.e., $LUX = b$ gives

$$LY = b \qquad \dots \quad (1)$$

where $UX = Y$

i.e., $$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & \frac{-4}{25} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \dots \quad (2)$$

From (1), we get

$$\begin{bmatrix} 8 & 0 & 0 \\ 4 & \frac{25}{2} & 0 \\ 6 & \frac{21}{4} & \frac{567}{50} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 33 \\ 36 \end{bmatrix}$$

so that

$8y_1 = 20 \Rightarrow y_1 = \frac{5}{2}$

$4y_1 + \frac{25}{2}.y_2 = 33$

$\therefore y_2 = \left(33 - 4.\frac{5}{2}\right)\frac{2}{25} = \frac{46}{25}$

$6y_1 + \frac{21}{4}y_2 + \frac{567}{50}y_3 = 36$

$\Rightarrow y_3 = \frac{50}{567}\left[36 - 15 - \frac{21 \times 23}{50}\right] = 1.$

Then from (2), we have

$$\begin{bmatrix} 1 & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & \frac{-4}{25} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{46}{25} \\ 1 \end{bmatrix}$$

which gives

$$x_3 = 1,$$

$$x_2 - \frac{4}{25} x_3 = \frac{46}{25}$$

$$\Rightarrow x_2 = \frac{46}{25} + \frac{4}{25} = 2$$

$$x_1 - \frac{3}{8} x_2 + \frac{x_3}{4} = \frac{5}{2}$$

$$\Rightarrow x_1 = \frac{5}{2} + \frac{3}{8}.2 - \frac{1}{4} = 3$$

Hence, the required solution is

$$x_1 = 3,\ x_2 = 2,\ x_3 = 1.$$

5.5. Gauss-Seidel iteration method.

[*W.B.U.T.,CS-312 2002*]

This method is an improvement of the Gauss-Jacobi method in the sense that the improved values of are used here in each iteration instead of the values of the previous iteration and hence the method is also known as the *method of successive displacements*.

To illustrate the method, we rewrite the system of equations (1) in the following form

$$\begin{aligned} x_1 &= (b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n) / a_{11} \\ x_2 &= (b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n) / a_{22} \qquad \ldots \quad (26) \\ &\ldots \qquad \ldots \qquad \ldots \\ x_n &= (b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{nn-1}x_{n-1}) / a_{nn}, \end{aligned}$$

provided $a_{ii} \neq 0,\ i = 1, 2, \ldots, n$

To solve the equations (26), suppose $x_1^{(0)}, x_2^{(0)}, \ldots x_n^{(0)}$ be the initial approximations (usually $x_i^{(0)}, i = 1$ to n are taken to be zero) of the solutions of (1). We substitute these initial values on the right hand side of the first equation of (26) and get the first approximation of x_1 as

$$x_1^{(1)} = \left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \ldots - a_{1n}x_n^{(0)}\right) / a_{11}$$

In the second equation of (26), we substitute the improved value $x_1^{(1)}$ and initial values $x_3^{(0)}, x_4^{(0)}, \ldots, x_n^{(0)}$ and obtain the first approximation of x_2 as

$$x_2^{(1)} = \left(b_2 - a_{21}x_1^{(1)} - a_{23}x_2^{(0)} - \ldots - a_{2n}x_n^{(0)}\right) / a_{22}$$

We then substitute in the third equation of (26) the improved values $x_1^{(1)}, x_2^{(1)}$ and the initial values $x_4^{(0)}, x_5^{(0)}, \ldots, x_n^{(0)}$ to obtain the first approximation of x_3 as

$$x_3^{(1)} = \left(b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{34}x_4^{(0)} - \ldots - a_{3n}x_n^{(0)}\right) / a_{33}$$

Proceeding in this way, the first approximation of x_n is given by

$$x_n^{(1)} = \left(b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \ldots - a_{nn-1}x_{n-1}^{(1)}\right) / a_{nn}$$

Thus at the end of the first stage of iteration, we get the first approximation $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ to the solutions $x_1, x_2, \ldots, x_n$.

Now if $x_i^{(k)} (k = 0, 1, 2, \ldots)$ be the k^{th} approximation to the solutions $x_i (i = 1, 2, \ldots, n)$, then the $(k+1)^{th}$ the approximation $x_i^{(k+1)}$ of $x_i (i = 1, 2, \ldots, n)$, are given by

$$\begin{aligned} x_1^{(k+1)} &= \left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_{1n}^{(k)}x_n^{(k)}\right) / a_{11} \\ x_2^{(k+1)} &= \left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \ldots - a_{2n}^{(k)}x_n^{(k)}\right) / a_{22} \qquad \ldots \qquad (27) \\ &\ldots \qquad \ldots \qquad \ldots \\ x_n^{(k+1)} &= \left(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \ldots - a_{nn-1}x_{n-1}^{(k+1)}\right) / a_{nn} \end{aligned}$$

The process is continued until we get the solutions $x_1, x_2, \ldots, x_n$ with sufficient degree of accuracy

The sequence $\left\{x_i^{(k)}\right\}$ generated from (27) can be shown to be convergent to the solution $\left\{x_i^*\right\}$ if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \; (i = 1, 2, \ldots, n) \qquad \ldots \quad (28)$$

Hence the Gauss-Seidel iteration method is convergent if the system of equations (1) is *strictly diagonally dominant.*

Note. **(1)** It may be noted that the strictly diagonally dominant condition may not be necessary in some problems for the convergence of iteration.

(2) The order of convergence of iteration in Gauss-Seidel method is one.

(3) The rate of convergence is faster (roughly twice) than that of Gauss-Jacobi method.

Example.5. Using Gauss-Seidel method find the solution of the following system of linear equations correct upto 2 places of decimal:

$$3x + y + 5z = 13$$
$$5x - 2y + z = 4$$
$$x + 6y - 2z = -1$$

[W.B.U.T., CS-312, 2004, M(CS)-301, 2015, M(CS)-401, 2013]

Solution. First we rearrange the given system of equations so that they are diagonally dominant as given below :

$$5x - 2y + z = 4$$
$$x + 6y - 2z = -1$$
$$3x + y + 5z = 13$$

We rewrite the system in the form

$$x = (4 + 2y - z)/5 \qquad \ldots \quad (1)$$
$$y = (-1 - x + 2z)/6 \qquad \ldots \quad (2)$$
$$z = (13 - 3x - y)/5 \qquad \ldots \quad (3)$$

The initial approximations are chosen to be

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0$$

First iteration :

Putting $y^{(0)} = 0$, $z^{(0)} = 0$ in (1), we get $x^{(1)} = 0.8$

Putting $x^{(1)} = 0.8$, $z^{(0)} = 0$ in (2), we have $y^{(1)} = -0.3$

Putting $x^{(1)} = 0.8$, $y^{(1)} = -0.3$ in (3) yields $z^{(1)} = 2.18$.

Second iteration :

$$x^{(2)} = \{4 + 2 \times (-0.3) - 2.18\} / 5 = 0.2441$$

$$y^{(2)} = \{-1 - 0.244 + 2 \times 2.18\} / 6 = 0.5192$$

$$z^{(2)} = \{13 - 3 \times 0.244 - 0.519\} / 6 = 2.3497$$

Proceeding as above, the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.8	–0.3	2.18
2	0.2441	0.5192	2.3497
3	0.5377	0.5271	2.1720
4	0.5763	0.4615	2.1628
5	0.552	0.462	2.176
6	0.550	0.467	2.177

Thus the required solutions are

$x = 0.55$, $y = 0.47$, $z = 2.18$, correct to two decimal places.

5.6. Computation of Inverse of matrix

Method I. To compute the inverse of a matrix $A = (a_{ij})_{n \times n}$, we determine a matrix $X = (x_{ij})_{n \times n}$ of the same order such that ... (29)

$$AX = I$$

where I is the unit matrix of the same order.

So for determination of each element of X, we solve a system of linear equations given by (29). This can be done by a systematic procedure using Gauss elemination method. We illustrate the technique for a third order matrix

Let us consider the equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is equivalent to the three system of linear equations given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \ldots (30)$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then applying Gauss elimination method to each of these system, we get the corresponding column of X, i.e., the inverse of the matrix A^{-1}. But the coefficient matrix of each system of equations are same and so we can solve the three system of equations simultaneously considering the following augmented matrix :

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Then employing the same procedure as in Gauss elemination, we can easily solve the three set of the system of equations.

Example.6. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 4 & -1 \end{bmatrix}$$

Solution. Consider the augmented matrix

$$\left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 3 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right], \text{(using } R_2 - R_1 \text{ and } R_3 + \tfrac{1}{2}R_1)$$

$$\sim \left[\begin{array}{ccc:ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & \frac{11}{5} & \frac{11}{10} & \frac{-3}{5} & 1 \end{array}\right], \text{(using } R_3 - \frac{3}{5}R_1)$$

Thus we have an equivalent system of three equations given by

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \frac{11}{10} \end{bmatrix} \qquad \ldots (1)$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{5} \end{bmatrix} \qquad \ldots (2)$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{11}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \ldots (3)$$

Equation (1) is equivalent to the following system of equations:

$$2x-2y+4z=1$$

$$5y-2z=-1$$

$$\frac{11}{5}z=\frac{11}{10}$$

Solving by back substitutions, we get

$$x=-\frac{1}{2},\ y=0,\ z=\frac{1}{2}$$

Similarly solving (2) and (3) we get

$$x=\frac{7}{11},\ y=\frac{1}{11},\ z=\frac{-3}{11}$$

and $x=\frac{-8}{11},\ y=\frac{2}{11},\ z=\frac{5}{11}$

$$\text{Thus } A^{-1}=\begin{bmatrix} -\frac{1}{2} & \frac{7}{11} & -\frac{8}{11} \\ 0 & \frac{1}{11} & \frac{2}{11} \\ \frac{1}{2} & -\frac{3}{11} & \frac{5}{11} \end{bmatrix}$$

Method II. This method is very similar to method I to compute the inverse matrix A^{-1} of the matrix A. Here also we consider the given matrix A with the same order identity matrix simultaneously and convert the matrix A into an identity matrix. As a result, the identity matrix is converted into a matrix which is the inverse of A.

Example.7. Find the inverse of the matrix

$$A=\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$ [*W.B.U.T., C.S-312, 2007, 2008*]

Solution. Consider the augmented matrix given by

$$\left[\begin{array}{ccc:ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_1 \to \frac{1}{8} R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & -\frac{1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_2 \to R_2 + 4R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{-1}{2} & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 1 & \frac{-2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right], \text{ using } R_2 \to \frac{1}{6} R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{8} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right], \text{ using } R_1 \to R_1 + \frac{1}{2} R_2$$
$$R_3 \to R_3 + 4R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{3} & \frac{1}{8} & \frac{1}{12} & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{12} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right], \text{ using } R_3 \to \frac{3}{16} R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right], \text{ using } R_1 \to R_1 + \frac{1}{3} R_3$$
$$R_2 \to R_2 + \frac{2}{3} R_3$$

Hence the required inverse matrix is

$$A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Illustrative Examples

Ex.1. Solve the following system of equations by LU factoriztion method

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16$$

[W.B.U.T.,CS-312, 2009, M(CS)-301, 2014, M(CS)-401, 2016]

Solution. The given system of equations can be written as

$$AX = b \quad \cdots \quad (1)$$

where $A = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$

Also let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Thus

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}$$

gives $l_{11} = 2, l_{21} = 5, l_{31} = 3, u_{12} = \frac{-6}{2} = -3, u_{13} = \frac{8}{2} = 4$

$$l_{22} = 4 - 5 \times (-3) = 19, \quad l_{32} = 1 - 3 \times (-3) = 10$$

$$u_{23} = \frac{-3 - 5 \times 4}{19} = \frac{-23}{19}, \quad l_{33} = 2 - 3 \times 4 - 10 \times \frac{-23}{19} = \frac{40}{19}$$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix}, U = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -\frac{23}{19} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ From (1)

$$AX = b \text{ i.e., } LUX = b \text{ gives}$$

$$LY = b$$

where $UX = Y$ ... (2)

i.e.,
$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \ldots (3)$$

From (2), we have,

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 19 & 0 \\ 3 & 10 & \frac{40}{19} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

i.e., $2y_1 = 24$

$$5y_1 + 19y_2 = 2$$

$$3y_1 + 10y_2 + \frac{40}{19}y_3 = 16$$

whose solutions are

$$y_1 = 12, y_2 = -\frac{58}{19}, y_3 = 5$$

Thus from (3) we get

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & \frac{-23}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ \frac{-58}{19} \\ 5 \end{bmatrix}$$

gives $x - 3y + 4z = 12$

$$y - \frac{23}{19}z = -\frac{58}{19}$$

$$z = 5$$

whose solutions by backward substitutions are

$x = 1, y = 3, z = 5$ which are the required solutions.

Ex.2. Find the solutions of the following system of equations by LU-factorization method

$$2x - 3y + 10z = 3$$

$$-x + 4y + 2z = 20$$

$$5x + 2y + z = -12$$

Solution. The given equations can be written as

$$AX = b$$

where $A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$

Let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Then

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

leading to $l_{11} = 2, l_{21} = -1, l_{31} = 5, u_{12} = \frac{-3}{2}$,

$$u_{13} = \frac{10}{2} = 5,\ l_{21}u_{21} + l_{22} = 4 \Rightarrow l_{22} = 5/2$$

$$l_{31}u_{12} + l_{32} = 2 \Rightarrow l_{32} = \frac{19}{2}$$

$$l_{21}u_{13} + l_{22}u_{23} = 2 \Rightarrow u_{23} = \frac{14}{5}$$

and

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1 \Rightarrow l_{33} = \frac{-253}{5}$$

Hence

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix},\ U = \begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $AX = b$ i.e., $LUX = b$

gives

$LY = b$ where $UX = Y$

i.e., $\begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Now from the equation $LY = b$, we have

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{5}{2} & 0 \\ 5 & \frac{19}{2} & \frac{-253}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

so that

$$2y_1 = 3$$

$$-y_1 + \frac{5}{2} y_2 = 20$$

$$5y_1 + \frac{19}{2} y_2 - \frac{253}{5} y_3 = -12$$

whose solutions are

$$y_1 = \frac{3}{2},\ y_2 = \frac{43}{5},\ y_3 = 2$$

Then from the equation $UX = Y$ we get

$$\begin{bmatrix} 1 & \frac{-3}{2} & 5 \\ 0 & 1 & \frac{14}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{43}{5} \\ 2 \end{bmatrix}$$

gives $x - \frac{3}{2} y + 5z = \frac{3}{2}$

$$y + \frac{14}{5} z = \frac{43}{5}$$

$$z = 2$$

whose solutions by backward substitutions are

$$x = -4,\ y = 3,\ z = 2$$

which are the required solutions.

Ex.3. Solve the equation by $L - U$ factorization method.

$$2x + y + z = 3$$

$$x + 3y + z = -2$$ 5

$$x + y + 4z = -6$$

Deduce the Newton's Backward interpolation Formula.

Solution. The given system of equations can be written as

$$AX = b \qquad \dots (1)$$

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -2 \\ -6 \end{bmatrix}$

Let $A = LU$

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

leading to

$$l_{11} = 2, l_{21} = 1,\ l_{22} = \frac{5}{2}, l_{31} = 1$$

$$l_{32} = \frac{1}{2}, l_{33} = \frac{17}{5},\ u_{12} = \frac{1}{2}, u_{13} = \frac{1}{2}, u_{23} = \frac{1}{5}$$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{5}{2} & 0 \\ 1 & \frac{1}{2} & \frac{17}{5} \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ From (1)

$$LUX = b.$$

If we take $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

then $LY = b$

i.e., $$\begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{5}{2} & 0 \\ 1 & \frac{1}{2} & \frac{17}{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -6 \end{bmatrix}$$

$$\therefore\ 2y_1 = 3$$

$$y_1 + \frac{5}{2}y_2 = -2$$

$$y_1 + \frac{1}{2}y_2 + \frac{17}{5}y_3 = -6$$

Solving we get

$$y_1 = \frac{3}{2},\ y_2 = -\frac{7}{5},\ y_3 = \frac{26}{17}$$

Then $UX = Y$ gives

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{5} \\ \frac{26}{17} \end{bmatrix}$$

$$\therefore\ x + \frac{1}{2}y + \frac{1}{2}z = \frac{3}{2}$$

$$y + \frac{1}{5}z = -\frac{7}{5}$$

$$z = \frac{26}{17}$$

Solving by back substitution, we have

$$x = \frac{27}{17},\ y = \frac{-29}{17},\ z = \frac{26}{17}$$

which are the required solutions.

Ex.4. Solve the following system of equation by $L-U$ decomposotion method :

$$x_1 + x_2 - x_3 = 2$$

$$2x_1 + 3x_2 + 5x_3 = -3$$

$$3x_1 + 2x_2 - 3x_3 = 6$$

[*W.B.U.T. M(CS)-401, 2006*]

Solution. The given system of equation can be written as

$$AX = b \quad \ldots \quad (1)$$

where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$b = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

Let $A = LU$ where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix}$

leading to

$$l_{11} = 1, l_{21} = 2, l_{22} = 1, l_{31} = 3, l_{32} = -1$$

$$l_{33} = 7, \ u_{12} = 1, u_{13} = -1, u_{23} = 7$$

$$\therefore \ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 7 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ From (1),

$$(LU)X = b$$

i.e., $LY = b$ where

$$UX = Y, \ Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore LY = b$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

i.e., $y_1 = 2$

$2y_1 + y_2 = -3$

$3y_1 - y_2 + 7y_3 = 6$

whose solutions are

$y_1 = 2,\ y_2 = -7, y_3 = -1$

Then from the equation

$UX = Y$,

we get

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}$$

i.e., $x_1 + x_2 - x_3 = 2$

$x_2 + 7x_3 = -7$

$x_3 = -1$

where solution by backward substitutions are

$x_1 = 1,\ x_2 = 0, x_3 = -1$

which are the required solutions.

Ex.5. Using matrix factorization method solve the following system of equation.

$$x + 3y + z = 9$$
$$x + 4y + 2z = 3$$
$$x + 2y - 3z = 6$$

[W.B.U.T.,M(CS)-301, 2010]

Solution. The given system of equations can be written as

$$AX = b \quad \dots (1)$$

where $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$

Let $A = LU$

$$\text{where } L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \; U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

which gives

$$l_{11} = 1, l_{27} = 1, l_{37} = 1$$

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{3}{1} = 3$$

$$u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{1} = 1$$

$$u_{22} = a_{22} - l_{21}u_{12} = 4 - 1 \times 3 = 1$$

$$l_{32} = a_{32} - l_{31}u_{12} = 2 - 1 \times 3 = -1$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{2 - 1 \times 1}{1} = 1$$

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = -3 - 1 \times 1 - (-1) \times 1 = -3$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & -3 \end{bmatrix}, \; U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ From (1),

$$LUX = b$$

i.e. $LY = b$ where $UX = Y$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Now, $LY = b$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

$\therefore y_1 = 9$

$y_1 + y_2 = 3$

$y_1 - y_2 - 3y_3 = 6$

$\therefore y_1 = 9,\ y_2 = -6,\ y_3 = 3$

$\therefore UX = Y$ gives

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ 3 \end{bmatrix}$$

$\therefore x + 3y + z = 9$

$y + z = -6$

$z = 3$

Solving we get

$x = 33,\ y = -9,\ z = 3$

which are the required solutions.

Ex.6. Solve the system of equations by Gauss-Seidel method :

$$3x + 4y + 15z = 54.8$$
$$x + 12y + z = 39.66$$
$$10x + y - 2z = 7.74$$

[*W.B.U.T.,CS-312, 2007, 2008*]

Solution. Obviously the coefficient matrix of the given system of equations is not diagonally dominant.We therefore rearrange the given equation as

$$10x + y - 2z = 7.74$$
$$x + 12y + z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now we rewrite the system as

$$x = (7.74 - y + 2z) / 10$$
$$y = (39.66 - x - z) / 12$$
$$z = (54.8 - 3x - 4y) / 15$$

Taking $x^{(0)} = y^{(0)} = z^{(0)} = 0$ as the initial approximation, the first approxmations to the solutions are

$$x^{(1)} = (7.74 - 0 + 2 \times 0) / 10 = 0.774$$
$$y^{(1)} = (39.66 - 0.774 - 0) / 12 = 3.2405$$
$$z^{(1)} = (54.8 - 3 \times 0.774 - 4 \times 3.2405) / 25 = 2.6344$$

Proceeding as above the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.774	3.2405	2.6344
2	0.97683	3.00406	2.65688
3	1.00497	2.99985	2.65238
4	1.00449	3.00026	2.65237

The solutions of the given system of equations are

$x = 1.004$, $y = 3.000$, $z = 2.652$, correct upto 3 decimal places.

Ex.7. Solve the following system of equations by Gauss-Seidel's iteration method

$$10x + 2y + z = 9$$

$$x + 10y - z = -22$$

$$-2x + 3y + 10z = 22$$

[*W.B.U.T.,CS-312, 2010*]

Solution. Clearly the given system of equations is diagonally dominant. We now rewrite the equations in the form

$$x = (9 - 2y - z)/10$$

$$y = (-22 + z - x)/10$$

$$z = (22 + 2x - 3y)/10$$

Let the initial approximation be $x^{(0)} = y^{(0)} = z^{(0)} = 0$.

Thus the first approximation of the solutions is given by

$$x^{(1)} = (9 - 2 \times 0 - 0)/10 = 0.9$$

$$y^{(1)} = (-22 + 0 - 0.9)/10 = -2.29$$

$$z^{(1)} = (22 + 2 \times 0.9 + 3 \times 2.29)/10 = 3.067$$

Proceeding as above, the successive iterations are obtained and are shown in the following table :

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	0.9	−2.29	3.067
2	1.0513	−1.9984	3.0098
3	0.9987	−1.9989	2.9994
4	0.9998	−2.0000	3.0000
5	1.0000	−2.0000	3.0000

Thus the required solutions are

$$x = 1,\ y = -2,\ z = 3$$

Ex.8. Using Gauss-Scidel method, find the solution of the following system of equation correct upto two places of decimal :

$$ax_1 - 2x_2 + x_3 = 50$$
$$x_1 + 5x_2 - 3x_3 = 18$$
$$-2x_1 + 2x_2 + 7x_3 = 19$$

[W.B.U.T. M(CS)-401, 2006]

Solution. The given system of equation can be written as

$$x_1 = \frac{1}{9}(50 + 2x_2 - x_3) \qquad (1)$$

$$x_2 = \frac{1}{5}(18 - x_1 - 3x_3) \qquad (2)$$

$$x_3 = \frac{1}{7}(19 + 2x_1 - 2x_2) \qquad (3)$$

The initial approximations are chosen to be

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

First iteration.

Putting $x_2^{(0)} = x_3^{(0)} = 0$ in (1), we get $x_1^{(1)} = 5.555$.

Putting $x_1^{(1)} = 5.555,\ x_3^{(0)} = 0$ in (2), we get $x_2^{(1)} = 2.489$

Putting $x_1^{(1)} = 5.555,\ x_2^{(1)} = 2.489$ in (3), we get

$$x_3^{(1)} = 3.590$$

Proceeding as above, the successive iterations are obtained and as shown in the following table :

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	5.555	2.489	3.590
2	5.710	4.612	3.028
3	6.244	4.168	3.307
4	6.114	4.361	3.215
5	6.167	4.296	3.249
6	6.149	4.320	3.237
7	6.156	4.316	3.240

Thus the required solution are

$x_1 = 6.15,\ x_2 = 4.32,\ x_3 = 3.24$, correct to two decimal places.

Ex.9. Solve the following system of equations, correct to four places of decimels, by Gauss-seidel iteration method :

$$x+y+54z=110$$
$$27x+6y-z=85$$
$$6x+15y+2z=72$$

[*W.B.U.T. M(CS)-301, 2009*]

Solution. Obviously the coefficient matrix of the given system of equations is not diagonally dominat. We therefore rearrange the given equations.

$$27x+6y-z=85$$
$$6x+15y+2z=72$$
$$x+y+54z=110$$

We now rewrite the system as

$$x=\frac{1}{27}(85-6y+z)$$
$$y=\frac{1}{15}(72-6x-2z)$$
$$z=\frac{1}{54}(110-x-y)$$

Taking $x^{(0)}=y^{(0)}=z^{(0)}=0$ as the initial approximation, the first approximation to the solutions are

$$x^{(1)}=\frac{1}{27}(85-6\times 0+0)=3.14815$$
$$y^{(1)}=\frac{1}{15}(72-6\times 3.14815-2\times 0)=3.54074$$
$$z^{(1)}=\frac{1}{54}(110-3.14815-3.54074)=1.91317$$

Proceeding as above, the successive iteration are obtained and are shown in the following table :

x	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	3.14815	3.54074	1.91317
2	2.43218	3.57204	1.92585
3	2.42569	2.57294	1.92595
4	2.42549	3.57301	1.92595
5	2.42548	3.57301	1.92595

Hence the required solutions are

$x = 2.4255,\ y = 3.5730,\ z = 1.9260$

correct upto four decimal places.

Ex.10. Solve the following system of equations by Gauss-elimination method :

$$5x_1 - x_2 = 9$$
$$-x_1 + 5x_2 - x_3 = 4$$
$$-x_2 + 5x_3 = -6$$

[W.B.U.T.,CS-312, 2009]

Solution. Multiplying the first equation by $\frac{1}{5}$ and then adding the result with the second equation we get

$$5x_1 - x_2 = 9$$
$$\frac{24}{5}x_2 - x_3 = \frac{29}{5}$$
$$-x_2 + 5x_3 = -6$$

Then multiplying the second equation by $\frac{5}{24}$ and add the result with the 3rd equation we have

$$5x_1 - x_2 = 9$$
$$\frac{24}{5}x_2 - x_3 = \frac{29}{5}$$
$$\frac{115}{24}x_3 = \frac{-115}{24}$$

$\therefore$ By back substitution, the required solutions are $x_1 = 2,\ x_2 = 1,\ x_3 = -1.$

Ex.11. Solve the following system of equations by using Gauss-elimination method

$$3x + 2y + 4z = 19$$
$$2x + 7y - 5z = 1$$
$$x - 8y + 9z = 12$$

Solution. We multiply the first equation successively by $\frac{2}{3}, \frac{1}{3}$ and subtract the results from the second and third equations respectively. Thus we have

$$3x + 2y + 4z = 19$$
$$\frac{17}{3}y - \frac{23}{3}z = -\frac{35}{3}$$
$$\frac{-26}{3}y + \frac{23}{3}z = \frac{17}{3}$$

Next we multiply the second equations by $\frac{26}{17}$ and add the result to the 3rd equation we get

$$3x + 2y + 4z = 19$$
$$\frac{17}{3}y - \frac{23}{3}z = -\frac{35}{3}$$
$$-\frac{69}{17}z = -\frac{207}{17}$$

$\therefore$ By back substitution the resulting solutions are

$$x = 1,\ y = 2,\ z = 3$$

Ex.12. Find the inverse of the matrix using Guass elimination method.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

[*M.A.K.A.U.T.,M(CS)-301, 2014*]

Solution. Consider the augmented matrix as

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right], \text{ using } \begin{array}{l} R_2 \to R_2 - \frac{3}{2}R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right], \text{ using } R_3 \to R_3 - 7R_2$$

Thus we have an equivalent system of three equations given by

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving these system of equations by back substitution we get the corresponding column of A^{-1}. Thus

$$A^{-1} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & \frac{-1}{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -6 & 5 & -1 \\ 24 & -17 & 3 \\ -10 & 7 & -1 \end{bmatrix}$$

Ex.13. Find the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

Solution. Consider the augmented matrix given by

$$\left[\begin{array}{ccc:ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right], \text{ operating } R_1 \to \frac{1}{4} R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 1 \end{array}\right], \text{ operating } \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -\frac{9}{4} & \frac{3}{2} & -\frac{1}{4} & 0 & 1 \end{array}\right], \text{ operating } R_2 \to \frac{2}{5} R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{7}{10} & \frac{3}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & -\frac{3}{10} & -\frac{7}{10} & \frac{9}{10} & 1 \end{array}\right], \text{ operating } , \begin{array}{l} R_1 \to R_1 - \frac{1}{4} R_2 \\ R_3 \to R_3 + \frac{9}{4} R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & \frac{7}{10} & \frac{3}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{7}{3} & -3 & -\frac{10}{3} \end{array}\right], \text{ operating } R_3 \to -\frac{10}{3} R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{4}{3} & 2 & \frac{7}{3} \\ 0 & 1 & 0 & \frac{5}{3} & -2 & -\frac{8}{3} \\ 0 & 0 & 1 & \frac{7}{3} & -3 & -\frac{10}{3} \end{array}\right], \text{ operating } \begin{array}{l} R_1 \to R_1 - \frac{7}{10} R_3 \\ R_2 \to R_2 + \frac{4}{5} R_3 \end{array}$$

Thus the required inverse matrix is

$$\begin{bmatrix} -\frac{4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & -\frac{8}{3} \\ \frac{7}{3} & -3 & -\frac{10}{3} \end{bmatrix}$$

i.e. $\frac{1}{3}\begin{bmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_2 \to \frac{2}{5}R_2$

$R_1 \to R_1 - \frac{1}{4}R_2$

$, R_3 \to R_3 + \frac{9}{4}R_2$

$R_3 \to -\frac{10}{3}R_3$

$R_1 \to R_1 - \frac{7}{10}R_3$

$R_2 \to R_2 + \frac{4}{5}R_3$

Ex.14. Find the inverse of the following matrix

$$\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

[W.B.U.T.,MCS-301, 2009]

Solution. Consider the augmented matrix as

$$\left[\begin{array}{ccc:ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -15 & 6 & -5 & 0 & 1 & 0 \\ 5 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{5}{3} & 0 & 1 \end{array}\right], \text{ using } R_2 \to R_2 + 5R_1$$

$$R_3 \to R_3 - \frac{5}{3}R_1$$

$$\sim \left[\begin{array}{ccc:ccc} 3 & 0 & 1 & 6 & 1 & 0 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right], \text{ using } R_1 \to R_1 + R_2$$

$$R_3 \to R_3 + \frac{1}{3}R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 3 & 0 & 0 & 6 & 0 & -3 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right], \text{ using } R_1 \to R_1 - 3R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{3}R_1 \\ R_3 \to 3R_3 \end{array}$$

Thus the required inverse matrix is

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Exercise

I. Short Answer Questions

1. What is meant by diagonally dominant matrix ?

2. Express Gauss Seidel method for a system of three linear equations in three unknowns.

3. Does Gauss seidel method always perform better than Gauss-Jacobi method ?

4. How is the solution obtained in Gauss elemination method ?

5. State sufficient condition for convergence of Gauss-Seidel method.

6. Solve

$$4x + 3y = 20.91$$
$$3x - y = 6.94$$

by Gauss elimination method.

7. Solve by Gauss elimination method

$$4.69x + 7.42y = 17.4$$
$$3x + 11.3y = 23.2$$

8. Solve

$$2x + 3y = 2.03$$
$$x + y = 7.8$$

by Gauss-Seidel method

9. Solve

$$3x + 2y = 9.8$$
$$2x + y = 5.5$$

by Gauss elimination method.

10. Solve by LU-factorization method

$$x + 3y = 5$$
$$7x + 2y = -3$$

11. Find the inverse of the following matrix

(i) $\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Answers

6. $x = 3.21, 7 = 2.69$

7. $x = 0.796, \quad y = 1.84$

8. [Hints : The coefficient matrix is not diagonally dominant and cannot be written in diogonally dominant form by an rearrangement. So the system of equations cannot be solved by Gauss-seidel method]

9. $x = 1.2, y = 3.1$ **10.** $x = -1, y = 2$

11. (i) $\frac{1}{5}\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{bmatrix}$

II. LONG ANSWER QUESTIONS

1. Solve the following system of equations using Gauss-elemination method

(i)
$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

(ii)
$$8x - 7y + 4z = 32$$
$$x + 5y - 3z = 28$$
$$-2x + 2y + 7z = 19$$

(iii)
$$5x - y - z = 3$$
$$-x + 10y - 2z = 7$$
$$-x - y + 10z = 8$$

(iv)
$$2x_1 - 3x_2 + 10x_3 = 3$$
$$-x_1 + 4x_2 + 2x_3 = 20$$
$$5x_1 + 2x_2 + x_3 = -12$$

(v) $5x_1 - x_2 = 3$

$-x_1 + 5x_2 - x_3 = 4$

$-x_2 + 5x_3 = -6$ [W.B.U.T.,CS-312, 2009]

2. Solve the following system of equations by matrix inversion method :

(i) $x + 2y + 3z = 7$

$2x + 7y + 15z = 26$

$3x + 15y + 41z = 62$

(ii) $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

(iii) $3x + 2y - z + \omega = 1$

$x - y - 2z + 4w = 3$

$2x - 3y + z - 2w = -2$

$5x - 2y + 3z + 2w = 0$

3. Solve the following system of equations by LU-factorization method :

(i) $x + 3y + z = 9$

$x + 4y + 2z = 3$

$x + 2y - 3z = 6$ [W.B.U.T., CS-312, 2010]

(ii) $3x + 4y + 2z = 15$

$5x + 2y + z = 18$

$2x + 3y + 2z = 10$ [W.B.U.T.,CS-312, 2003]

(iii) $x_1 + x_2 - x_3 = 2$

$2x_1 + 3x_2 + 5x_3 = -3$

$3x_1 + 2x_2 - 3x_3 = 6$ [W.B.U.T.,CS-312, 2006]

(iv) $2x + y + z = 3$

$x + 3y + z = -2$

$x + y + 4z = -6$ [W.B.U.T.,CS-312, 2005]

(v) $5x - y - z = 3.245$

$x + 4y + z = 7.075$

$x + y + 3z = 8.870$

4. Solve the following system of equations by using Gauss-Seidel method :

(i) $x + y + 54z = 110$

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$ [W.B.U.T.,CS-312, 2009]

(ii) $10x + 2y + z = 9$

$2x + 20y - 2z = -44$

$-2x + 3y + 10z = 22$ [W.B.U.T.,CS-312, 2010]

(iii) $10x_1 - x_2 - x_3 = 13$

$x_1 - 10x_2 + x_3 = 36$

$x_1 + x_2 - 10x_3 = -35$ [W.B.U.T.,CS-312, 2002]

(iv) $9x_1 - 2x_2 + x_3 = 50$

$x_1 + 5x_2 - 3x_3 = 18$

$-2x_1 + 2x_2 + 7x_3 = 19$ [W.B.U.T., CS-312, 2006]

(v) $10x + y - z = 12$

$2x + 10y - z = 13$

$2x + 2y - 10z = 14$

5. Find the inverse of the following matrix

(i) $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ [W.B.U.T.,CS-312, 2009]

(ii) $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$ [W.B.U.T.,CS-312, 2004]

(iii) $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ [W.B.U.T.,CS-312, 2007]

(iv) $\begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$

Answers

1. (i) $x = 1, y = 3, z = 5$

(ii) $x = 6.15, y = 4.31, z = 3.24$

(iii) $x = 7, y = 3, z = 1$

(iv) $x_1 = -3.86, x_2 = 2.98, x = 1.89$

(v) $x_1 = 2, x_2 = 1, x_3 = -1$

2. (i) $x = 2, y = z = 1$ (ii) $x = 1, y = 2, z = -1$

(iii) $x = \frac{19}{50}, y = \frac{-29}{50}, z = \frac{-51}{50}, w = 0$

3. (i) $x = 33, y = -9, z = 3$

(iii) $x = 1, y = 0, z = -1$

(iv) $x = \frac{27}{17}, y = \frac{-29}{17}, z = \frac{26}{17}$

(v) $x = 1.274, y = 0.891, z = 2.235$

4. (i) $x = 2.4255, y = 3.5730, z = 1.9260$

(ii) $x = 1, y = -2, z = 3$

(iv) $x_1 = 6.15, x_2 = 4.32, x_3 = 3.24$

(v) $x = 1, y = 3, z = -1$

5. (i) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (iii) $\begin{pmatrix} 2 & -1 & 1 \\ -3 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$

(iv) $\frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

III. Multiple Choice Questions

1. In Gauss elimination method, the given system of equations represented by $AX = B$ is converted to another sysem $UX = Y$ where U is

(a) diagonal matrix

(b) null matrix

(c) identity matrix

(d) upper triangular matrix

[*W.B.U.T.,CS-312 2008, 2009*]

2. The Gauss elemination method is fails when any one of the pivotal elements is

(a) zero (b) one (c) two (d) none

3. To solve the system of equations $AX = b$ by Gauss elemination method, A is transformed to a

(a) Lower triangular matrix

(b) Upper triangular matrix

(c) Diagonal matrix

(d) none of these [*M.A.K.A.U.T.,M(CS)-401, 2015*]

4. Forward substitution is used to solve a system of equations by Gauss elimination method

(a) False

(b) True

5. Gauss elemination method does not fail even if one of the pivotel element is equal to zero

(a) True

(b) False

[W.B.U.T.,CS-312, 2002, 2004, 2006]

6. To solve a system of m equations in m unknowns, the total number of multiplications and division involved in solving the system by Gauss elimination method is of order $m^3/3$ approximately.

(a) True

(b) False

7. Inverse of a matrix A is given by

(a) $A^{-1} = \dfrac{adjA}{|A|}$

(b) $A^{-1} = \dfrac{|A|}{adjA}$

(c) $A^{-1} = \dfrac{adjA}{A}$

(d) none of these

8. The matrix inversion method does not fail to solve a system of equations if the coefficient matrix is singular

(a) True

(b) False

9. A matrix A can be factorized into lower and upper triangular matrix if all the principal minors of A are

(a) Singular

(b) non-Singular

(c) Zero

(d) none of these

10. In the LU factorization method, a matrix A can be factorized into $A = LU$ where L is

(a) upper triangular matrix

(b) lower triangular matrix

(c) identity matrix

(d) diagonal matrix [*M.A.K.A.U.T.,M(CS)-401, 2013*]

11. In the LU-factorization method, a matrix A can be factorized into $A = LU$ where U is a

(a) upper triangular matrix

(b) lower triangular matrix

(c) identity matrix

(d) diagonal matrix

12. One of the iterative methods by which we can find the solution of simultaneous equations is

(a) Gauss-Seidel method

(b) Gauss elimination method

(c) Crouts' method

(d) Gauss Jordan method

[*W.B.U.T.,CS-312 2003, 2009, M(CS)-401, 2013, 2015*]

13. A system of equations $AX = b$ where $A = (a_{ij})_{n\times n}$ is said to be diagonally dominant if

(a) $|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ for all i (b) $|a_{ii}| < \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ for all i

(c) $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$ for all i (d) $|a_{ii}| < \sum_{j=1}^{n} |a_{ij}|$ for all i

14. The iterative method is known as

(a) direct method

(b) indirect method

(c) none of these

15. The solution of a system of equations is obtained by successive approximation method is known as

(a) direct method

(b) indirect method

(c) both (a) and (b)

(d) none of these

Answers

1.d 2.a 3.b 4.a 5.b 6.a 7.a 8.b 9.b 10.b

11.a 12.a 13.a 14.b 15.b