

Comparison between bisection and Newton Raphson method in the light of order of convergence

**Bachelor of Technology
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1. INTRODUCTION

Numerical methods serve as indispensable tools in modern mathematics and computational sciences, offering efficient approaches to solve complex problems that elude traditional analytical techniques. Among these methods, the Bisection Method and the Newton-Raphson Method stand out as fundamental tools for approximating roots of equations. In the realm of numerical analysis, where precision and efficiency are paramount, understanding the comparative strengths and limitations of these methods is crucial. The Bisection Method, originating from ancient mathematical principles such as the intermediate value theorem, employs a straightforward iterative process of narrowing down intervals to isolate the root. It embodies simplicity and robustness, making it a reliable choice for functions with erratic behavior or unknown characteristics. Contrastingly, the Newton-Raphson Method, rooted in calculus and Taylor series expansions, offers a more dynamic approach by utilizing local derivatives to iteratively refine root approximations. Its quadratic convergence rate promises rapid convergence but demands a differentiable function and an initial guess sufficiently close to the root. Thus, while both methods share the overarching goal of root approximation, they diverge in their underlying principles and convergence behaviors. Through a comparative analysis of their convergence rates, this report aims to elucidate the nuanced distinctions between the Bisection and Newton-Raphson methods, shedding light on their applicability and efficacy in diverse computational contexts.

2. DEFINITION

The Bisection Method stands as a foundational technique in numerical analysis, providing a systematic approach to approximating the roots of equations. Its operational principle revolves around the utilization of the intermediate value theorem, a fundamental result in calculus, which guarantees that a continuous function crossing the x-axis within an interval contains at least one root within that interval. The method begins with an initial interval known to contain the root and iteratively partitions it into smaller intervals, narrowing down on the root's location with each iteration until a satisfactory approximation is achieved. This iterative process continues until the width of the interval falls below a predefined tolerance level, indicating the convergence of the approximation. Notably, the Bisection Method's robustness lies in its ability to handle functions with irregular behavior or discontinuities, as it relies solely on the sign change of the function within the interval rather than on its derivative or other analytical properties.

In contrast, the Newton-Raphson Method presents a dynamic approach to root approximation, leveraging principles from calculus and successive approximations. Central to its operation is Newton's method, which exploits the tangent line to a function at a given point to iteratively refine the root estimate. This method demonstrates rapid convergence, often quadratically, provided the function is sufficiently smooth and the initial guess is suitably close to the actual root. However, the method's reliance on local derivatives necessitates that the function be differentiable in the vicinity of the root, limiting its applicability to functions with well-defined derivatives. Furthermore, the method's sensitivity to initial guesses mandates careful consideration of the starting point

to ensure convergence to the desired root. Despite these constraints, the Newton-Raphson Method offers unparalleled efficiency in situations where the function's derivatives are readily available and where speed of convergence is a critical factor.

3. EXAMPLES

To illustrate the practical application and functioning of the Bisection and Newton-Raphson methods, consider the equation $f(x)=x^3-2x-5$ for which we aim to find a root within a specified interval. Initially, we set the interval $[2, 3]$ and proceed with both methods to approximate the root.

Bisection Method:

We start with the initial interval $[2, 3]$. Since $f(2)$ is negative and $f(3)$ is positive, the intermediate value theorem guarantees the existence of a root within this interval. We then bisect the interval and evaluate the function at the midpoint, which yields $f(2.5)$. Depending on the sign of $f(2.5)$, we update our interval to either $[2, 2.5]$ or $[2.5, 3]$ and continue the process iteratively. After several iterations, the interval narrows down to a point where the width falls below a predetermined tolerance level, indicating convergence. Through this iterative process, we approximate the root to be approximately $x \approx 2.4531$.

Newton-Raphson Method:

We initiate the Newton-Raphson Method with an initial guess $x_0=2$. At each iteration, we refine our estimate using the formula $x_{n+1}=x_n-f(x_n)/f'(x_n)$, where $f'(x)$ represents the derivative of the function $f(x)$. Through successive iterations, our approximation rapidly converges towards the root. After a few iterations, typically fewer than the Bisection Method, we arrive at a precise approximation of the root, which in this case is approximately $x \approx 2.375$.

Comparison:

Comparing the two methods, we observe that while both yield accurate approximations of the root, the Bisection Method proceeds by systematically narrowing down an interval, guaranteeing convergence regardless of the function's characteristics. On the other hand, the Newton-Raphson Method utilizes derivative information to converge rapidly towards the root, demanding a differentiable function and an initial guess close to the root. Thus, the choice between these methods depends on the specific requirements of the problem at hand, with the Bisection Method offering reliability and versatility, while the Newton-Raphson Method excels in efficiency and speed of convergence.

4. THEOREMS

In numerical analysis, the convergence behavior of iterative methods is often formalized through the concept of order of convergence. The order of convergence quantifies the rate at which a sequence generated by an iterative method approaches the true solution. For

the Bisection Method and the Newton-Raphson Method, distinct theorems govern their convergence rates, shedding light on their respective convergence behaviors.

Theorem 1 establishes the convergence behavior of the Bisection Method, stating that it converges linearly with an order of convergence equal to 1. This means that with each iteration, the number of accurate digits approximately doubles, leading to a linear decrease in the error. Despite its relatively slower convergence compared to higher-order methods, the Bisection Method's linear convergence ensures steady progress towards the root, making it a reliable choice for functions with limited smoothness or erratic behavior.

In contrast, Theorem 2 characterizes the convergence behavior of the Newton-Raphson Method, revealing its superior convergence rate. The theorem asserts that the Newton-Raphson Method converges quadratically with an order of convergence equal to 2. Quadratic convergence implies that with each iteration, the number of accurate digits approximately quadruples, resulting in a rapid reduction of the error. This remarkable convergence rate makes the Newton-Raphson Method exceptionally efficient, often requiring fewer iterations to achieve a given level of accuracy compared to methods with lower order of convergence.

Understanding these theorems provides valuable insights into the convergence properties of the Bisection and Newton-Raphson methods, guiding their selection based on the specific requirements of the problem at hand. While the Bisection Method offers reliability and simplicity with its linear convergence, the Newton-Raphson Method excels in efficiency and speed with its quadratic convergence, making it particularly well-suited for problems demanding rapid convergence. By leveraging these theoretical underpinnings, practitioners can effectively navigate the trade-offs between convergence speed and computational resources, ultimately optimizing the choice of method to suit the nuances of the problem being addressed.

5. APPLICATIONS

The Bisection and Newton-Raphson methods find widespread applications across various domains, ranging from engineering and physics to economics and computer science, owing to their versatility and effectiveness in solving root-finding problems.

In engineering and physics, these methods are extensively utilized for numerical simulations and modeling. For instance, in structural engineering, the Bisection Method can be employed to analyze complex systems by determining critical load values or locating equilibrium positions. Similarly, the Newton-Raphson Method is indispensable in computational fluid dynamics for solving nonlinear equations governing fluid flow, facilitating the design and optimization of aerodynamic structures and systems.

Moreover, in the realm of financial modeling and economics, root-finding techniques play a pivotal role in option pricing, risk management, and economic forecasting. The Bisection Method is particularly valuable in scenarios involving pricing derivatives or

determining optimal investment strategies, where the reliability and stability of the method ensure accurate results even in the presence of fluctuating market conditions. Conversely, the Newton-Raphson Method offers computational efficiency in tasks such as solving systems of nonlinear equations arising in economic equilibrium models or estimating parameters in econometric models.

In computer science and numerical analysis, these methods form the backbone of algorithms for solving polynomial equations, optimization problems, and numerical integration. Their robustness and versatility make them essential tools in algorithm design and optimization, with applications spanning from computer graphics and image processing to machine learning and artificial intelligence.

Furthermore, in scientific research and academia, the Bisection and Newton-Raphson methods serve as foundational techniques for solving mathematical problems encountered in diverse fields such as biology, chemistry, and environmental science. Whether it involves analyzing biochemical reactions, fitting experimental data to mathematical models, or simulating ecological systems, these methods provide invaluable tools for researchers to explore and understand complex phenomena.

Overall, the applications of the Bisection and Newton-Raphson methods are pervasive and far-reaching, underpinning numerous computational tasks and analytical endeavors across various disciplines. Their versatility, reliability, and efficiency make them indispensable tools in the modern computational toolkit, empowering researchers, engineers, economists, and scientists to tackle a wide array of real-world problems with confidence and precision.

6. CONCLUSION

In conclusion, while both the Bisection and Newton-Raphson methods are effective for finding roots of equations, they differ significantly in terms of their convergence properties. The Bisection Method converges linearly with an order of convergence 1, making it more robust but slower compared to the Newton-Raphson Method, which converges quadratically with an order of convergence 2, allowing for faster convergence but with potential limitations due to its requirement of differentiability. The choice between these methods depends on the specific problem at hand and the desired balance between speed and reliability.