Bayesian Estimation for Population Parameters

We take that the observed samples are $x = x_1, \ldots, x_n$ which are n i.i.d's from a normal distribution.

We are going to assume that the mean μ of the distribution is unknown while its variance σ^2 is known.

The prior:

The prior is $p(\mu) = \frac{1}{\sqrt{2\pi\tau_0^2}}\exp(-\frac{(\mu - \mu_0)^2}{2\tau_0^2})$

That is μ has a normal distribution with mean μ_0 and variance σ_0^2. The prior is used to express the belief that the unknown parameter μ is most likely equal to μ_0 and that values of μ very far from μ_0 are quite unlikely.

The likelihood:

The p.d.f of a generic draw x_i is

$$p(x_i|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x_i - \mu)^2}{2\sigma^2})$$

where we use the notation $p(x_i|\mu)$ to highlight the fact that the density depends upon the unknown parameter μ.

Since $x_1,\ldots,x_n$ are independent, the likelihood is

$$p(x|\mu) = \prod_{i=1}^{n} p(x_i|\mu) = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n}\exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2)$$

The posterior:

Given the prior and the likelihood as above, the posterior is

$$p(\mu|x) = \frac{1}{\sqrt{2\pi\tau_n^2}}\exp\left(-\frac{1}{2\tau_n^2}(\mu - \mu_n)^2\right)$$

where $\mu_n = (\frac{n}{\sigma^2} + \frac{1}{\tau_0^2})^{-1}\left[\frac{n}{\sigma^2}(\frac{1}{n}\sum_{i=1}^{n} x_i) + \frac{1}{\tau_0^2}\mu_0\right]$

and $\tau_n^2 = \left(\frac{n}{\sigma_2} + \frac{1}{\tau_0^2}\right)^{-1}$

Thus, the posterior distribution of μ is a normal distribution with mean μ_n and variance τ_n^2.