Maximum Likelihood Estimation of Population Parameters

Maximum Likelihood estimation (MLE) is a method used to estimate the parameters of a probability distribution. The basic idea is to find the set of parameter values that maximize the likelihood function, which measures the probability of observing the given data under the assumed statistical model.

Here is a simple example to illustrate MLE:

Let's say we have a set of data points that are assumed to be drawn from a normal distribution with an unknown mean, μ and known s.d. σ. We want to estimate the value of μ using MLE.

Setup: Given a set of observed data points (x_1, x_2,…,x_n), where each x_i is assumed to be normally distributed with mean μ and s.d, σ, we want to find the value of μ that maximizes the likelihood function.

Likelihood Function: The likelihood function for a set of independent and identically distributed (i.i.d) normal random variables is given by the product of the individual probability density functions (PDFs):

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

Log-Likelihood Function: Its often easier to work with the log of the likelihood function, especially for computational reasons:

$$\begin{aligned} l(\mu) = \ logL(\mu) &= \sum_{i=1}^{n} [-\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_i - \mu)^2}{2\sigma^2}] \\ &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 \end{aligned}$$

Maximization: To find the value of μ that maximizes the likelihood function, we differentiate the log-likelihood function w.r.t μ, set the derivative equal to zero and solve for μ:

$$\frac{dl}{d\mu} = \sum_{i=1}^{n} \frac{x_i - \mu}{\sigma^2} = 0$$

$$\sum_{i=1}^{n} (x_i - \mu) = 0$$

$$\sum_{i=1}^{n} x_i - n\mu = 0$$

$$\hat{\mu}MLE = \frac{1}{n}\sum i = 1^n x_i = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Interpretation: The maximum likelihood estimate for the mean μ is simply the sample mean of the observed data points.

In summary, the maximum likelihood estimate for the mean of a normal distribution with known variance is equal to the sample mean of the observed data points. This is a simple example of MLE but the same principles can be applied to estimate parameters in more complex statistical models.

In case, we have to consider the s.d. σ also an unknown, then we can estimate it as follows:

Differentiating the log-likelihood function w.r.t σ^2 and setting it equal to zero, we get

$$\frac{\partial \log L}{\partial(\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$$

Solving for σ^2, we obtain

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})\text{^}2$$

This is the sample variance , which is also the Maximum Likelihood Estimator of the population variance σ^2.