3.2 Error Detection and Correction

As we saw, the telephone system has three parts: the switches, the interoffice trunks, and the local loops. The first two are now almost entirely digital in most developed countries. The local loops are still analog twisted copper pairs and will continue to be so for years due to the enormous expense of replacing them. While errors are rare on the digital part, they are still common on the local loops. Furthermore, wireless communication is becoming more common, and the error rates here are orders of magnitude worse than on the interoffice fiber trunks. The conclusion is: transmission errors are going to be with us for many years to come. We have to learn how to deal with them.

As a result of the physical processes that generate them, errors on some media (e.g., radio) tend to come in bursts rather than singly. Having the errors come in bursts has both advantages and disadvantages over isolated single-bit errors. On the advantage side, computer data are always sent in blocks of bits. Suppose that the block size is 1000 bits and the error rate is 0.001 per bit. If errors were independent, most blocks would contain an error. If the errors came in bursts of 100 however, only one or two blocks in 100 would be affected, on average. The disadvantage of burst errors is that they are much harder to correct than are isolated errors.

3.2.1 Error-Correcting Codes

Network designers have developed two basic strategies for dealing with errors. One way is to include enough redundant information along with each block of data sent, to enable the receiver to deduce what the transmitted data must have been. The other way is to include only enough redundancy to allow the receiver to deduce that an error occurred, but not which error, and have it request a retransmission. The former strategy uses **error-correcting codes** and the latter uses **error-detecting codes**. The use of error-correcting codes is often referred to as **forward error correction**.

Each of these techniques occupies a different ecological niche. On channels that are highly reliable, such as fiber, it is cheaper to use an error detecting code and just retransmit the occasional block found to be faulty. However, on channels such as wireless links that make many errors, it is better to add enough redundancy to each block for the receiver to be able to figure out what the original block was, rather than relying on a retransmission, which itself may be in error.

To understand how errors can be handled, it is necessary to look closely at what an error really is. Normally, a frame consists of m data (i.e., message) bits and r redundant, or check, bits. Let the total length be n (i.e., $n = m + r$). An n-bit unit containing data and check bits is often referred to as an n-bit **codeword**.

Given any two codewords, say, 10001001 and 10110001, it is possible to determine how many corresponding bits differ. In this case, 3 bits differ. To determine how many bits differ, just exclusive OR the two codewords and count the number of 1 bits in the result, for example:

```
10001001
10110001
00111000
```

The number of bit positions in which two codewords differ is called the **Hamming distance** (Hamming, 1950). Its significance is that if two codewords are a Hamming distance d apart, it will

require *d* single-bit errors to convert one into the other.

In most data transmission applications, all 2^m possible data messages are legal, but due to the way the check bits are computed, not all of the 2^n possible codewords are used. Given the algorithm for computing the check bits, it is possible to construct a complete list of the legal codewords, and from this list find the two codewords whose Hamming distance is minimum. This distance is the Hamming distance of the complete code.

The error-detecting and error-correcting properties of a code depend on its Hamming distance. To detect *d* errors, you need a distance *d* + 1 code because with such a code there is no way that *d* single-bit errors can change a valid codeword into another valid codeword. When the receiver sees an invalid codeword, it can tell that a transmission error has occurred. Similarly, to correct *d* errors, you need a distance 2*d* + 1 code because that way the legal codewords are so far apart that even with *d* changes, the original codeword is still closer than any other codeword, so it can be uniquely determined.

As a simple example of an error-detecting code, consider a code in which a single **parity bit** is appended to the data. The parity bit is chosen so that the number of 1 bits in the codeword is even (or odd). For example, when 1011010 is sent in even parity, a bit is added to the end to make it 10110100. With odd parity 1011010 becomes 10110101. A code with a single parity bit has a distance 2, since any single-bit error produces a codeword with the wrong parity. It can be used to detect single errors.

As a simple example of an error-correcting code, consider a code with only four valid codewords:

0000000000, 0000011111, 1111100000, and 1111111111

This code has a distance 5, which means that it can correct double errors. If the codeword 0000000111 arrives, the receiver knows that the original must have been 0000011111. If, however, a triple error changes 0000000000 into 0000000111, the error will not be corrected properly.

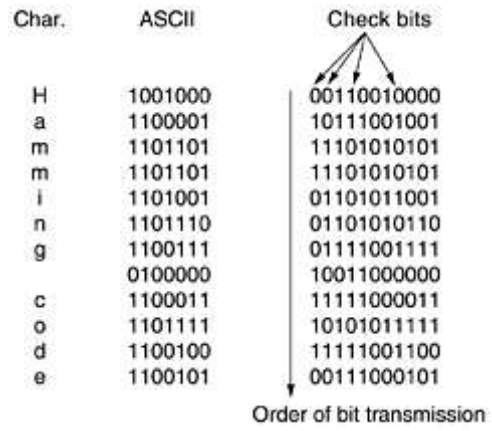
Imagine that we want to design a code with *m* message bits and *r* check bits that will allow all single errors to be corrected. Each of the 2^m legal messages has *n* illegal codewords at a distance 1 from it. These are formed by systematically inverting each of the *n* bits in the *n*-bit codeword formed from it. Thus, each of the 2^m legal messages requires *n* + 1 bit patterns dedicated to it. Since the total number of bit patterns is 2^n, we must have $(n + 1)2^m \leq 2^n$. Using $n = m + r$, this requirement becomes $(m + r + 1) \leq 2^r$. Given *m*, this puts a lower limit on the number of check bits needed to correct single errors.

This theoretical lower limit can, in fact, be achieved using a method due to Hamming (1950). The bits of the codeword are numbered consecutively, starting with bit 1 at the left end, bit 2 to its immediate right, and so on. The bits that are powers of 2 (1, 2, 4, 8, 16, etc.) are check bits. The rest (3, 5, 6, 7, 9, etc.) are filled up with the *m* data bits. Each check bit forces the parity of some collection of bits, including itself, to be even (or odd). A bit may be included in several parity computations. To see which check bits the data bit in position *k* contributes to, rewrite *k* as a sum of powers of 2. For example, 11 = 1 + 2 + 8 and 29 = 1 + 4 + 8 + 16. A bit is checked by just those check bits occurring in its expansion (e.g., bit 11 is checked by bits 1, 2, and 8).

When a codeword arrives, the receiver initializes a counter to zero. It then examines each check bit, *k* (*k* = 1, 2, 4, 8, ...), to see if it has the correct parity. If not, the receiver adds *k* to the counter. If the counter is zero after all the check bits have been examined (i.e., if they were all correct), the codeword is accepted as valid. If the counter is nonzero, it contains the number of the

incorrect bit. For example, if check bits 1, 2, and 8 are in error, the inverted bit is 11, because it is the only one checked by bits 1, 2, and 8. Figure 3-7 shows some 7-bit ASCII characters encoded as 11-bit codewords using a Hamming code. Remember that the data are found in bit positions 3, 5, 6, 7, 9, 10, and 11.

Figure 3-7. Use of a Hamming code to correct burst errors.



Hamming codes can only correct single errors. However, there is a trick that can be used to permit Hamming codes to correct burst errors. A sequence of *k* consecutive codewords are arranged as a matrix, one codeword per row. Normally, the data would be transmitted one codeword at a time, from left to right. To correct burst errors, the data should be transmitted one column at a time, starting with the leftmost column. When all *k* bits have been sent, the second column is sent, and so on, as indicated in Fig. 3-7. When the frame arrives at the receiver, the matrix is reconstructed, one column at a time. If a burst error of length *k* occurs, at most 1 bit in each of the *k* codewords will have been affected, but the Hamming code can correct one error per codeword, so the entire block can be restored. This method uses *kr* check bits to make blocks of *km* data bits immune to a single burst error of length *k* or less.

3.2.2 Error-Detecting Codes

Error-correcting codes are widely used on wireless links, which are notoriously noisy and error prone when compared to copper wire or optical fibers. Without error-correcting codes, it would be hard to get anything through. However, over copper wire or fiber, the error rate is much lower, so error detection and retransmission is usually more efficient there for dealing with the occasional error.

As a simple example, consider a channel on which errors are isolated and the error rate is 10^{-6} per bit. Let the block size be 1000 bits. To provide error correction for 1000-bit blocks, 10 check bits are needed; a megabit of data would require 10,000 check bits. To merely detect a block with a single 1-bit error, one parity bit per block will suffice. Once every 1000 blocks, an extra block (1001 bits) will have to be transmitted. The total overhead for the error detection + retransmission method is only 2001 bits per megabit of data, versus 10,000 bits for a Hamming code.

If a single parity bit is added to a block and the block is badly garbled by a long burst error, the probability that the error will be detected is only 0.5, which is hardly acceptable. The odds can be improved considerably if each block to be sent is regarded as a rectangular matrix *n* bits wide and *k* bits high, as described above. A parity bit is computed separately for each column and affixed to the matrix as the last row. The matrix is then transmitted one row at a time. When the block arrives, the receiver checks all the parity bits. If any one of them is wrong, the receiver requests a retransmission of the block. Additional retransmissions are requested as needed until an entire

block is received without any parity errors.

This method can detect a single burst of length n, since only 1 bit per column will be changed. A burst of length $n + 1$ will pass undetected, however, if the first bit is inverted, the last bit is inverted, and all the other bits are correct. (A burst error does not imply that all the bits are wrong; it just implies that at least the first and last are wrong.) If the block is badly garbled by a long burst or by multiple shorter bursts, the probability that any of the n columns will have the correct parity, by accident, is 0.5, so the probability of a bad block being accepted when it should not be is 2^{-n}.

Although the above scheme may sometimes be adequate, in practice, another method is in widespread use: the **polynomial code**, also known as a **CRC** (**Cyclic Redundancy Check**). Polynomial codes are based upon treating bit strings as representations of polynomials with coefficients of 0 and 1 only. A k-bit frame is regarded as the coefficient list for a polynomial with k terms, ranging from x^{k-1} to x^0. Such a polynomial is said to be of degree $k - 1$. The high-order (leftmost) bit is the coefficient of x^{k-1}; the next bit is the coefficient of x^{k-2}, and so on. For example, 110001 has 6 bits and thus represents a six-term polynomial with coefficients 1, 1, 0, 0, 0, and 1: $x^5 + x^4 + x^0$.

Polynomial arithmetic is done modulo 2, according to the rules of algebraic field theory. There are no carries for addition or borrows for subtraction. Both addition and subtraction are identical to exclusive OR. For example:

```
  10011011     00110011     11110000     01010101
+ 11001010   + 11001101   - 10100110   - 10101111
----------   ----------   ----------   ----------
  01010001     11111110     01010110     11111010
```

Long division is carried out the same way as it is in binary except that the subtraction is done modulo 2, as above. A divisor is said ''to go into'' a dividend if the dividend has as many bits as the divisor.

When the polynomial code method is employed, the sender and receiver must agree upon a **generator polynomial**, $G(x)$, in advance. Both the high- and low-order bits of the generator must be 1. To compute the **checksum** for some frame with m bits, corresponding to the polynomial $M(x)$, the frame must be longer than the generator polynomial. The idea is to append a checksum to the end of the frame in such a way that the polynomial represented by the checksummed frame is divisible by $G(x)$. When the receiver gets the checksummed frame, it tries dividing it by $G(x)$. If there is a remainder, there has been a transmission error.

The algorithm for computing the checksum is as follows:

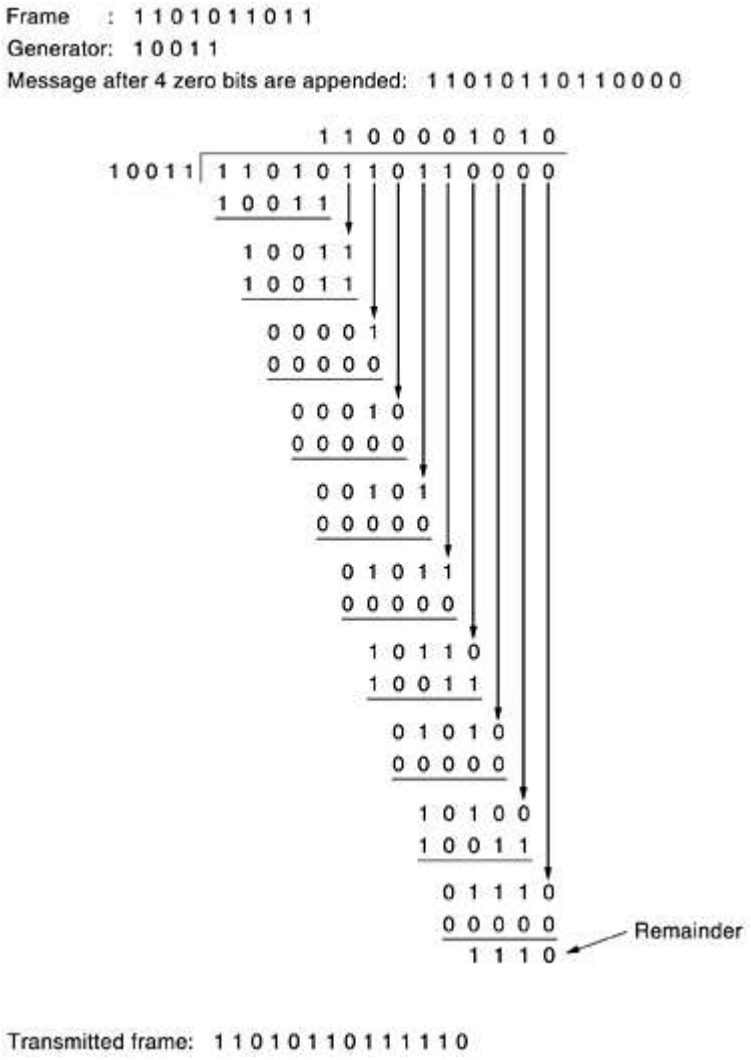
1. Let r be the degree of $G(x)$. Append r zero bits to the low-order end of the frame so it now contains $m + r$ bits and corresponds to the polynomial $x^r M(x)$.

2. Divide the bit string corresponding to $G(x)$ into the bit string corresponding to $x^r M(x)$, using modulo 2 division.

3. Subtract the remainder (which is always r or fewer bits) from the bit string corresponding to $x^r M(x)$ using modulo 2 subtraction. The result is the checksummed frame to be transmitted. Call its polynomial $T(x)$.

Figure 3-8 illustrates the calculation for a frame 1101011011 using the generator $G(x) = x^4 + x + 1$.

Figure 3-8. Calculation of the polynomial code checksum.



It should be clear that $T(x)$ is divisible (modulo 2) by $G(x)$. In any division problem, if you diminish the dividend by the remainder, what is left over is divisible by the divisor. For example, in base 10, if you divide 210,278 by 10,941, the remainder is 2399. By subtracting 2399 from 210,278, what is left over (207,879) is divisible by 10,941.

Now let us analyze the power of this method. What kinds of errors will be detected? Imagine that a transmission error occurs, so that instead of the bit string for $T(x)$ arriving, $T(x) + E(x)$ arrives. Each 1 bit in $E(x)$ corresponds to a bit that has been inverted. If there are k 1 bits in $E(x)$, k single-bit errors have occurred. A single burst error is characterized by an initial 1, a mixture of 0s and 1s, and a final 1, with all other bits being 0.

Upon receiving the checksummed frame, the receiver divides it by $G(x)$; that is, it computes $[T(x) + E(x)]/G(x)$. $T(x)/G(x)$ is 0, so the result of the computation is simply $E(x)/G(x)$. Those errors that happen to correspond to polynomials containing $G(x)$ as a factor will slip by; all other errors will be caught.

If there has been a single-bit error, $E(x) = x^i$, where i determines which bit is in error. If $G(x)$ contains two or more terms, it will never divide $E(x)$, so all single-bit errors will be detected.

If there have been two isolated single-bit errors, $E(x) = x^i + x^j$, where $i > j$. Alternatively, this can be written as $E(x) = x^j(x^{i-j} + 1)$. If we assume that $G(x)$ is not divisible by x, a sufficient condition for all double errors to be detected is that $G(x)$ does not divide $x^k + 1$ for any k up to the maximum value of $i - j$ (i.e., up to the maximum frame length). Simple, low-degree polynomials that give protection to long frames are known. For example, $x^{15} + x^{14} + 1$ will not divide $x^k + 1$ for any value of k below 32,768.

If there are an odd number of bits in error, $E(X)$ contains an odd number of terms (e.g., $x^5 + x^2 + 1$, but not $x^2 + 1$). Interestingly, no polynomial with an odd number of terms has $x + 1$ as a factor in the modulo 2 system. By making $x + 1$a factor of $G(x)$, we can catch all errors consisting of an odd number of inverted bits.

To see that no polynomial with an odd number of terms is divisible by $x + 1$, assume that $E(x)$ has an odd number of terms and is divisible by $x + 1$. Factor $E(x)$ into $(x + 1)\ Q(x)$. Now evaluate $E(1) = (1 + 1)Q(1)$. Since $1 + 1 = 0$ (modulo 2), $E(1)$ must be zero. If $E(x)$ has an odd number of terms, substituting 1 for x everywhere will always yield 1 as the result. Thus, no polynomial with an odd number of terms is divisible by $x + 1$.

Finally, and most importantly, a polynomial code with r check bits will detect all burst errors of length $\leq r$. A burst error of length k can be represented by $x^i(x^{k-1} + \ldots + 1)$, where i determines how far from the right-hand end of the received frame the burst is located. If $G(x)$ contains an x^0 term, it will not have x^i as a factor, so if the degree of the parenthesized expression is less than the degree of $G(x)$, the remainder can never be zero.

If the burst length is $r + 1$, the remainder of the division by $G(x)$ will be zero if and only if the burst is identical to $G(x)$. By definition of a burst, the first and last bits must be 1, so whether it matches depends on the $r - 1$ intermediate bits. If all combinations are regarded as equally likely, the probability of such an incorrect frame being accepted as valid is $½^{r-1}$.

It can also be shown that when an error burst longer than $r + 1$ bits occurs or when several shorter bursts occur, the probability of a bad frame getting through unnoticed is $½^r$, assuming that all bit patterns are equally likely.

Certain polynomials have become international standards. The one used in IEEE 802 is

$$x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^5 + x^4 + x^2 + x^1 + 1$$

Among other desirable properties, it has the property that it detects all bursts of length 32 or less and all bursts affecting an odd number of bits.

Although the calculation required to compute the checksum may seem complicated, Peterson and Brown (1961) have shown that a simple shift register circuit can be constructed to compute and verify the checksums in hardware. In practice, this hardware is nearly always used. Virtually all LANs use it and point-to-point lines do, too, in some cases.

For decades, it has been assumed that frames to be checksummed contain random bits. All analyses of checksum algorithms have been made under this assumption. Inspection of real data has shown this assumption to be quite wrong. As a consequence, under some circumstances, undetected errors are much more common than had been previously thought (Partridge et al., 1995).