MODULE - 1: Economic Order Quantity and EOQ Models without Shortage

1.1 Introduction

The word 'inventory' means simply a stock of idle resources of any kind having an economic value. In other words, inventory means a physical stock of goods, which is kept in hand for smooth and efficient running of future affairs of an organization. It may consist of raw materials, work-in-progress, spare parts/consumables, finished goods, human resources such as unutilized labor, financial resources such as working capital, etc. It is not necessary that an organization has all these inventory classes but whatever may be the inventory items, they need efficient management as generally a substantial amount of money is invested in them. The basic inventory decisions include: 1) *How much to order?* 2) *When to order?* 3) *How much safety stock should be kept?* The problems faced by different organizations have necessitated the use of scientific techniques in the management of inventories known as **inventory control**. Inventory control is concerned with the acquisition, storage, and handling of inventories so that the inventory is available whenever needed and the associated total cost is minimized.

1.2 Reasons for Carrying Inventory

Inventories are carried by organisations because of the following major reasons :

1. **Improve customer service**- An inventory policy is designed to respond to individual customer's or organization's request for products and services.

2. **Reduce costs**- Inventory holding or carrying costs are the expenses that are incurred for storage of items. However, holding inventory items in the warehouse can indirectly reduce operating costs such as loss of goodwill and/or loss of potential sale due to shortage of items. It may also encourage economies of production by allowing larger, longer and more production runs.

3. **Maintenance of operational capability**- Inventories of raw materials and work-in-progress items act as buffer between successive production stages so that downtime in one stage does not affect the entire production process.

4. **Irregular supply and demand**- Inventories provide protection against irregular supply and demand; an unexpected change in production and delivery schedule of a product or a service can adversely affect operating costs and customer service level.

5. **Quantity discount**- Large size orders help to take advantage of price-quantity discount. However, such an advantage must keep a balance between the storage cost and costs due to obsolescence, damage, theft, insurance, etc.

6. **Avoiding stockouts (shortages)**- Under situations like labor strikes, natural disasters, variations in demand and delays in supplies, etc., inventories act as buffer against stock out as well as loss of goodwill.

1.3 Costs Associated with Inventories

Various costs associated with inventory control are often classified as follows :

1. *Purchase (or production) cost*: It is the cost at which an item is purchased, or if an item is produced.

2. *Carrying (or holding) cost*: The cost associated with maintaining inventory is known as holding cost. It is directly proportional to the quantity kept in stock and the time for which an item is held in stock. It includes handling cost, maintenance cost, depreciation, insurance, warehouse rent, taxes, etc.

3. *Shortage (or stock out) cost*: It is the cost which arises due to running out of stock. It includes the cost of production stoppage, loss of goodwill, loss of profitability, special orders at higher price, overtime/idle time payments, loss of opportunity to sell, etc.

4. *Ordering (or set up) cost*: The cost incurred in replenishing the inventory is known as ordering cost. It includes all the costs relating to administration (such as salaries of the persons working for purchasing, telephone calls, computer costs, postage, etc.), transportation, receiving and inspection of goods, processing payments, etc. If a firm produces its own goods instead of purchasing the same from an outside source, then it is the cost of resetting the equipment for production.

1.4 Basic Terminologies

The followings are some basic terminologies which are used in inventory theory:

1. **Demand**
   It is an effective desire which is related to particular time, price, and quantity. The demand pattern of a commodity may be either deterministic or probabilistic. In case of deterministic demand, the quantities needed in future are known with certainty. This can be fixed (static) or can vary (dynamic) from time to time. On the contrary, probabilistic demand is uncertain over a certain period of time but its pattern can be described by a known probability distribution.

2. **Ordering cycle**
   An ordering cycle is defined as the time period between two successive replenishments. The order may be placed on the basis of the following two types of inventory review system:

   - *Continuous review*: In this case, the inventory level is monitored continuously until a specified point (known as reorder point) is reached. At this point, a new order is placed.
   - *Periodic review*: In this case, the orders are placed at equally spaced intervals of time. The quantity ordered each time depends on the available inventory level at the time of review.

3. **Planning period**
   This is also known as time horizon over which the inventory level is to be controlled. This can be finite or infinite depending on the nature of demand.

4. **Lead time or delivery lag**
   The time gap between the moment of placing an order and actually receiving it is referred to as lead time. Lead time can be deterministic (constant or variable) or probabilistic.

5. **Buffer (or safety) stock**
   Normally, demand and lead time are uncertain and cannot be predetermined completely. So, to absorb the variation in demand and supply, some extra stock is kept. This extra stock is known as buffer stock.

6. **Re-order level**
   The level between maximum and minimum stocks at which purchasing activity must start for replenishment is known as re-order level.

1.5 Economic Order Quantity (EOQ)

The concept of economic ordering quantity (EOQ) was first developed by F. Harris in 1916. The concept is as follows: Management of inventory is confronted with a set of opposing costs. As the lot size increases, the carrying cost increases while the ordering cost decreases. On the other hand, as the lot size decreases, the carrying cost decreases but the ordering cost increases. The two opposite costs can be shown graphically by plotting them against the order size as shown in Fig. 1.1 below :

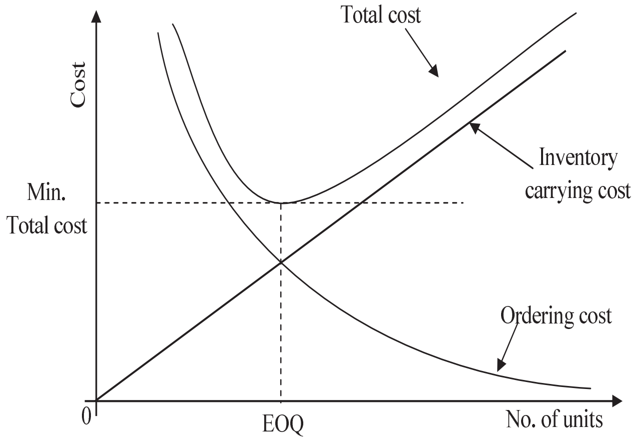


Fig. 1.1: Graph of EOQ

Economic ordering quantity(EOQ) is that size of order which minimizes the average total cost of carrying inventory and ordering under the assumed conditions of certainty and the total demand during a given period of time is known.

1.6 List of Symbols

The following symbols are used in connection with the inventory models presented in this chapter :

$$
\begin{aligned}
c &= \text{purchase (or manufacturing) cost of an item} \\
c_1 &= \text{holding cost per unit per unit time} \\
c_2 &= \text{shortage cost per unit per unit item} \\
c_3 &= \text{ordering (set up) cost per order (set up)} \\
R &= \text{demand rate} \\
P &= \text{production rate} \\
t &= \text{scheduling period which is variable} \\
t_p &= \text{prescribe scheduling period} \\
D &= \text{total demand or annual demand} \\
q &= \text{lot (order) size} \\
L &= \text{lead time} \\
x &= \text{random demand} \\
f(x) &= \text{probability density function for demand } x. \\
z &= \text{order level}
\end{aligned}
$$

1.7 Deterministic Inventory Models

1.7.1 Model I(a): EOQ model without shortage

The basic assumptions of the model are as follows:

- Demand rate R is known and uniform.
- Lead time is zero or a known constant.
- Replenishment rate is infinite, i.e., replenishments are instantaneous.
- Shortages are not permitted.
- Inventory holding cost is c_1 per unit per unit time.
- Ordering cost is c_3 per order.

Our objective is to determine the economic order quantity q^* which minimizes the average total cost of the inventory system. An inventory-time diagram with inventory level on the vertical axis and time on the horizontal axis is shown in Fig. 1.2. Since the actual consumption of inventory varies constantly, the concept of average inventory is applicable here. $Average\ Inventory = 1/2[maximum\ level + minimum\ level] = (q + 0)/2 = q/2$.

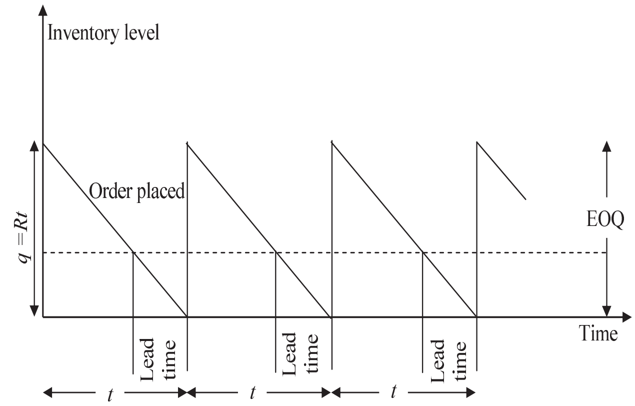


Fig. 1.2: Inventory-time diagram when lead time is a known constant

Thus, the average inventory carrying cost is $= average\ inventory \times holding\ cost = \dfrac{1}{2}qc_1$.

The average ordering cost is $(R/q)c_3$. Therefore, the average total cost of the inventory system is given by

$$C(q) = \frac{1}{2}c_1 q + \frac{c_3 R}{q}. \tag{1.1}$$

Since the minimum average total cost occurs at a point when average ordering cost and average inventory carrying cost are equal, therefore, we have $\frac{1}{2}c_1 q = \frac{c_3 R}{q}$ which gives the optimal order quantity

$$q^* = \sqrt{\frac{2c_3 R}{c_1}}. \tag{1.2}$$

This result was derived independently by F.W. Harris and R.H. Wilson in the year 1915. Thats why the model is called *Harris-Wilson model*.

Characteristics of Model I(a):

(i) Optimal ordering interval $t^* = q^*/R = \sqrt{\dfrac{2c_3}{c_1 R}}$

(ii) Minimum average total cost $C_{\min} = C(q^*) = \sqrt{2c_1 c_3 R}$

If in Model I(a), the ordering cost is taken as $(c_3 + kq)$ where k is the ordering cost per unit item ordered then there will be no change in the optimal order quantity q^*.
In this case, the average total cost is

$$C(q) = \frac{1}{2}c_1 q + \frac{c_3 R}{q} + kR. \tag{1.3}$$

Example 1.1: A manufacturing company purchases 9000 parts of a machine for its annual requirements, ordering one month usage at a time. Each part costs Rs.20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year. Suggest a more economic purchasing policy for the company. How much would it save the company per year ?

Solution: Given that $R = 9000$ parts/year, $c_1 = (15/100) \times 20 = Rs.3$ per part/year, $c_3 = Rs.15$ per order. Using Harris-Wilson formula,

$$\begin{aligned} q^* &= \sqrt{2c_3R/c_1} = 300\ units \\ t^* &= q^*/R = 1/30\ year = 12\ days\ (approx.) \\ C_{min} &= \sqrt{2c_1c_3R} = Rs.900 \end{aligned}$$

If the company follows the policy of ordering every month, then
lot size of inventory each month $q = 9000/12 = 750\ parts,$
annual storage cost = $c_1(q/2)$ = Rs. 1125,
annual ordering cost = 15×12 = Rs. 180.
The total cost per year = 1125 + 180 = Rs. 1305.
Therefore, the company should purchase 300 parts at time interval of 12 days instead of ordering 750 parts each month. Then there will be a net saving of Rs. 405.

Lot Size in Discrete Units

Let the lot size q be constrained to values $u, 2u, 3u, \cdots$ Then the necessary conditions for optimal q, i.e., q^* are

$$C(q^*) \leq C(q^* + u) \tag{1.4}$$

$$C(q^*) \leq C(q^* - u) \tag{1.5}$$

From equations (1.4) and (1.5), we get by simplifying

$$q^*(q^* - u) \leq \frac{2Rc_3}{c_1} \leq q^*(q^* + u) \tag{1.6}$$

Example 1.2: Demand in an inventory system is at a constant and uniform rate of 2400 kg. per year. The carrying cost is Rs. 5 per kg. per year. No shortage is allowed. The replenishment cost is Rs. 22 per order. The lot size can only be in 100 kg. unit. What is the optimal lot size of the system ?

Solution: Given that $R = 2400 kg/year,\ c_1 = Rs.5/kg/year,\ c_3 = Rs.22\ per\ order,\ u = 100\ kg$. The optimal lot size q^* has to satisfy the inequality

$$q^*(q^* - u) \leq \frac{2c_3R}{c_1} \leq q^*(q^* + u)$$
$$i.e., \quad q^*(q^* - 100) \leq 21120 \leq q^*(q^* + 100)$$

By trial and error, we find the optimal lot size $q^* = 200$ kg.

Sensitivity of Lot Size System

The average cost $C(q)$ of the lot size system is a function of the controllable variable q where $C(q) = \frac{1}{2}c_1q + \frac{c_3R}{q}$. The optimal results are $q^* = \sqrt{\frac{2c_3R}{c_1}}$ and $C^* = C(q^*) = \sqrt{2c_1c_3R}$.

Suppose that instead of the optimal lot size q^*, the decision maker uses another lot size q' which is related to q^* by the relation $q' = bq^*,\ b > 0$.

Let C' designate the average total cost of the system then. We use the ratio C'/C^* as a measure of sensitivity. It can be shown that $C'/C^* = (1 + b^2)/(2b)$.
So, the measure of sensitivity is a function of b and is independent of the other parameters c_1, c_3 and R.

1.7.2 Model I(b): EOQ Model with Different Rates of Demand

This inventory system operates on the assumptions of Model I(a) except that the demand rates are different in different cycles but order quantity is fixed in each cycle. The objective is to determine the order size in each reorder cycle that will minimize the total inventory cost. Suppose that the total demand D is specified over the planning period T. If $t_1, t_2, ..., t_n$ denote the lengths of successive n inventory cycles and $D_1, D_2, ..., D_n$ are the demand rates in these cycles, respectively, then the total period T is given by $T = t_1 + t_2 + ... + t_n$. Fig. 1.3 depicts the inventory system under consideration.

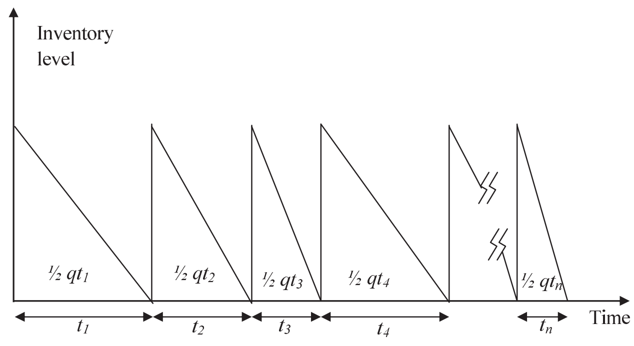


Fig. 1.3: Inventory-time diagram for different cycles

Suppose that each time a fixed quantity q is ordered. Then the number of orders in the time period T is $n = D/q$. Thus, the inventory carrying for the time period T is

$$\frac{1}{2}qt_1c_1 + \frac{1}{2}qt_2c_1 + ... + \frac{1}{2}qt_nc_1 = \frac{1}{2}qc_1(t_1 + t_2 + ... + t_n) = \frac{1}{2}qc_1T$$

Total ordering cost = (Number of orders) $\times\ c_3 = \frac{D}{q}c_3$
Hence, the total inventory cost is $C(q) = \frac{1}{2}c_1qT + \frac{c_3D}{q}$
The optimal ordering quantity (q^*) is then determined by the first order condition as

$$q^* = \sqrt{\frac{2c_3(D/T)}{c_1}}$$

The minimum total inventory cost is obtained by substituting the value of q^* in the cost equation, i.e.,

$$C_{\min} = \sqrt{2c_1c_3(D/T)}$$

Here we observe from the optimal results that the fixed demand rate R in Model I(a) is replaced by the average demand rate (D/T) in this model.

MODULE - 2: EOQ Models with Shortage and EPQ Models with and without Shortages

2.1 Model II. EOQ Model with Shortage

(a) The Case of Constant Scheduling Period :

This model is based on the assumption of Model I(a), except shortages are allowed. The cost of a shortage is assumed to be directly proportional to the average number of units short. We have to determine optimal order level z aiming to minimize the average cost with the assumptions which are as follows:

- z is the order level to which the inventory is raised at the beginning of each scheduling period. Shortages, if any, have to be made up.
- Production rate is infinite.
- Lead time is zero.
- c_1 is the holding cost per unit per unit time.
- c_2 is the shortage cost per unit per unit time.
- R demand rate.
- t_p is the prescribed scheduling time period.
- q_p is the fixed lot size ($q_p = Rt_p$).

In this model, we observe that the inventory carrying cost as well as shortage cost will be involved only when $0 \le z \le q_p$. Figure 2.1 represents a schematic diagram of this model. Since q_p is the lot size sufficient to meet the demand for time t_p, but $z(< q_p)$

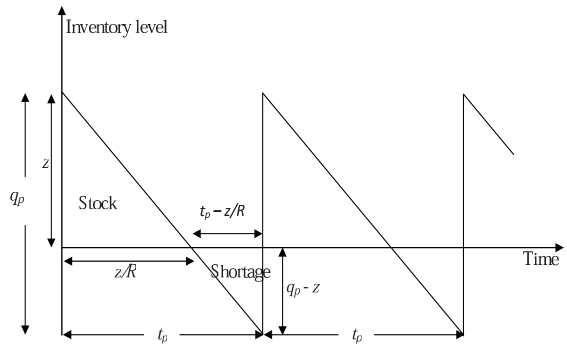


Fig. 2.1: Inventory level variation with time

amount of stock is planned in order to meet the demand for time z/R which is less than t_p. Shortage amount $(q_p - z)$ will arise for the remaining period $(t_p - z/R)$.
The average holding cost is $c_1(\frac{1}{2}\frac{z^2}{R})/t_p = \frac{c_1 z^2}{2q_p}$
The average shortage cost is $c_2[\frac{1}{2}(q_p - z)(t_p - \frac{z}{R})]/t_p = \frac{c_2}{2q_p}(q_p - z)^2$
Therefore, the average total cost is $C(z) = \frac{c_1 z^2}{2q_p} + \frac{c_2}{2q_p}(q_p - z)^2$. Here the set up cost is ignored as t_p is constant.
From the first order optimality condition $\frac{dC(z)}{dz} = 0$, we get

$$z^* = \frac{c_2}{c_1 + c_2} q_p = \frac{c_2}{c_1 + c_2} R t_p. \tag{2.1}$$

Also, $\frac{d^2C(z)}{dz^2} = \frac{c_1 + c_2}{q_p} > 0$. Hence the optimal order level z^* given by (2.1) minimizes $C(z)$ and the minimum average cost is $C_{\min} = \frac{1}{2}\frac{c_1 c_2}{c_1 + c_2} \cdot q_p = \frac{c_1 c_2}{2(c_1 + c_2)} \cdot R t_p$.

Example 2.1: A sub-contractor undertakes to supply diesel engines to a truck manufacturer at the rate of 25 engines per day. He will be penalized Rs.10 per engine per day late for missing the scheduled delivery date. The cost of holding a completed engine in stock is Rs. 16 per month. At the beginning of each month (30 days), he starts a batch of engines and all the engines are available for delivery any time after the end of the month. What should his inventory level be at the beginning of each month to stock the engines made in the previous month and then shipping engines to fulfill unsatisfied demand from previous month ?

Solution: We are given that $R = 25$ engines/day; $c_1 =$ Rs. $16/30$ per engine/day, $c_2 =$ Rs. 10 per engine per day, $t_p = 30$ days.
Then, using the result (2.1), we determine the optimal order level

$$z^* = \frac{c_2}{c_1 + c_2} R t_p = 712 \text{ engines}$$

(b) The Case of Variable Scheduling Period :
Here the only difference from Model II(a) is that the scheduling period (t_p) is not constant, it is a variable and we denote it by t. Because of variable scheduling period, it now becomes important to consider the average set up cost c_3/t in the cost function which is given by

$$C(t,z) = \frac{1}{t}\left[\frac{c_1 z^2}{2R} + \frac{c_2}{2R}(Rt - z)^2 + c_3\right]$$

The necessary conditions for optimum of $C(t,z)$ are $\frac{\partial C}{\partial z} = 0$ and $\frac{\partial C}{\partial t} = 0$. Further, for minimum of $C(t,z)$, the conditions must hold:

$$\frac{\partial^2 C}{\partial t^2} \cdot \frac{\partial^2 C}{\partial z^2} - \left(\frac{\partial^2 C}{\partial z \partial t}\right)^2 > 0, \text{ and } \frac{\partial^2 C}{\partial t^2} > 0, \frac{\partial^2 C}{\partial z^2} > 0.$$

Solving the first order conditions simultaneously, we get $t^* = \sqrt{\frac{2c_3(c_1+c_2)}{Rc_1c_2}}$
Then the optimal order quantity $q^* = Rt^* = \sqrt{\frac{2Rc_3(c_1+c_2)}{c_1c_2}}$
and the corresponding minimum average cost is $C_{\min} = \sqrt{\frac{2c_1c_2c_3R}{(c_1+c_2)}}$.

It is interesting to note that the minimum cost above is less than that of Model I(a). That is, in Model II(b), the cost is further reduced in comparison to Model I(a).

2.2 Model III. Production Lot Size Model

(a) The Case of without Shortage : Here, we shall develop an Economic Production Quantity (EPQ) model in which replenishment is made through production and shortages are no permitted. The assumptions of the model are as follows :

- Demand rate is constant and uniform.
- Replenishment rate is finite and uniform.
- Production rate (P) is greater than the demand rate (R).
- During a production run, the production of the item is continuous and at a constant rate until production of quantity q is completed.

- Shortages are not allowed in inventory.

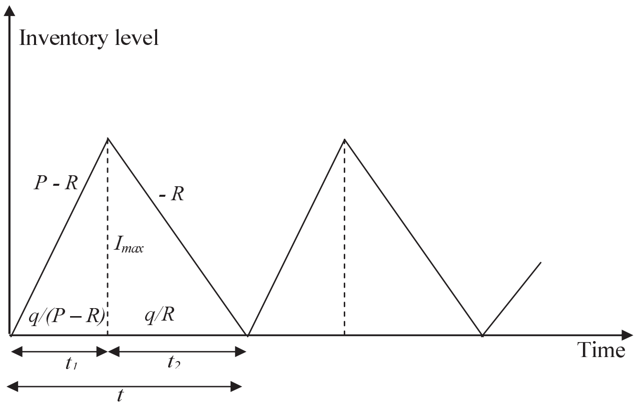


Fig. 2.2: Graphical representation of production-inventory system

As shown in Fig. 2.2, t_1 is the time required to produce one entire batch of amount q. Therefore, we have $t_1 = q/P$. During the production run time t_1, the inventory gradually builds up at the rate $(P-R)$ units and then decreases at the rate R units. Therefore, the maximum inventory level reached at the end of time t_1 is

$$I_{\max} = (P-R)t_1 = (P-R)\frac{q}{P} = \left(1-\frac{R}{P}\right)q$$

Since the minimum inventory level, $I_{\min} = 0$, therefore, the average inventory level is $\frac{q}{2}\left(1-\frac{R}{P}\right)$. Thus, the average inventory carrying cost $= \frac{1}{2}c_1\left(1-\frac{R}{P}\right)q$ and the average production set up cost $= c_3\left(\frac{R}{q}\right)$. Hence the total inventory cost per unit time is given by

$$C(q) = \frac{1}{2}c_1\left(\frac{P-R}{P}\right)q + \frac{c_3R}{q}$$

Then the optimal production lot size can be obtained as

$$q^* = \sqrt{\frac{2c_3R}{c_1(1-R/P)}}$$

Characteristics of Model III(a):

(i) Optimal time interval $t^* = q^*/R = \sqrt{\dfrac{2c_3}{c_1R(1-R/P)}}$

(ii) Optimal inventory cost $C_{\min} = \sqrt{2c_1c_3R(1-R/P)}$

We observe that

- If $P = R$ then $C_{\min} = 0$, which implies that there will be no carrying cost and no set up cost.

- If $P \to \infty$, i.e., the production rate is infinite, then this model becomes exactly the same as Model I(a).

- Although, in this model, c_3 is same as Model I(a) and Model I(b), the carrying cost is reduced in the ratio $(1 - R/P) : 1$ for the minimum cost.

Example 2.2: A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that when he starts a particular run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is Re.0.02 and set up cost of a production run is Rs.18. How frequently should production run remain ?

Solution: We are given that $P = 25,000$ bearings/day; $R = 10,000$ bearings/day;
c_1 = Re. (0.02/365) per bearing/day, c_3 = Rs. 18/set up.
Then $t^* = \sqrt{\frac{2c_3}{c_1 R(1-R/P)}} = 10.5$ days (approx.)
Therefore, production can be planned as an interval of 10.5 days approximately.

(b) The Case when Shortages are Allowed :

In this case, the model is based on the assumptions of Model III(a) except that shortages are allowed. The figure 2.3 shows that inventory starts at zero level and increases

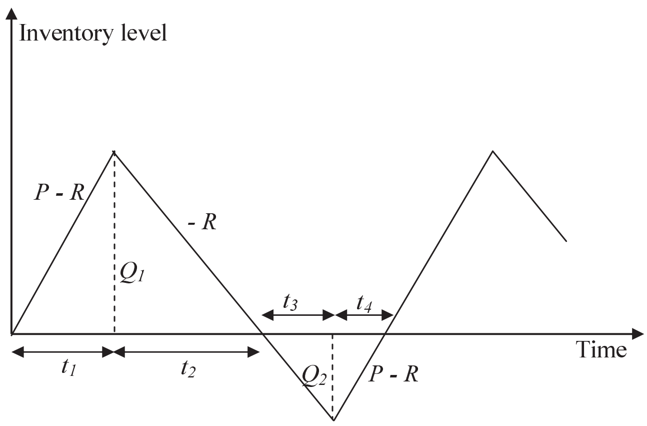


Fig. 2.3: A schematic diagram of production-inventory system with shortage

for a period t_1. Then declines for a period t_2 until it reaches zero at the point where a backlog piles up for the time t_3. At the end of t_3, production starts and backlog is diminished for the time t_4 when it reaches zero. The cycle then repeats itself after total time $(t_1 + t_2 + t_3 + t_4)$. The maximum inventory level is Q_1 and the maximum shortage level is Q_2. Now, the inventory holding cost is $\frac{1}{2}c_1 Q_1(t_1 + t_2)$, shortage cost is

$\frac{1}{2}c_2Q_2(t_3+t_4)$ and set up cost per set up is c_3. Therefore, the total inventory cost per unit time is given by

$$C(t_1,t_2,t_3,t_4) = \frac{\frac{1}{2}[c_1Q_1(t_1+t_2)+c_2Q_2(t_3+t_4)]+c_3}{t_1+t_2+t_3+t_4} \tag{2.2}$$

We have $Q_1 = t_1(P-R)$, $Q_1 = Rt_2$. From these two relations,

$$t_1 = \frac{Q_1}{P-R} = \frac{Rt_2}{P-R}$$

We have $Q_2 = Rt_3$ and $Q_2 = t_4(P-R)$. From these two relations,

$$t_4 = \frac{Q_2}{P-R} = \frac{Rt_3}{P-R}.$$

Production quantity q is just sufficient to meet the demand at the rate R. Therefore, we have $q = R(t_1+t_2+t_3+t_4)$. Substituting the values of t_1 and t_4, we get

$$q = R\left(\frac{Rt_2}{P-R}+t_2+t_3+\frac{Rt_3}{P-R}\right) = \frac{(t_2+t_3)PR}{P-R}$$

Using the above derivations, we have from eqn. (2.2)

$$C(t_2,t_3) = \frac{\frac{1}{2}[c_1t_2^2+c_2t_3^2]RP+c_3(P-R)}{P(t_2+t_3)}$$

To find the optimal values of t_2 and t_3, we differentiate $C(t_2,t_3)$ partially w.r.t. t_2 and t_3 and set equal to zero. Solving these two equations simultaneously, we get

$$t_2^* = \sqrt{\frac{2c_3c_2(1-R/P)}{R(c_1+c_2)c_1}} \quad \text{and} \quad t_3^* = \sqrt{\frac{2c_3c_1(1-R/P)}{R(c_1+c_2)c_2}}$$

The corresponding optimal production lot size and the minimum average cost are

$$\begin{aligned} q^* &= \sqrt{\frac{2c_3R(c_1+c_2)}{c_1c_2(1-R/P)}} \\ C^* &= \sqrt{\frac{2c_1c_2c_3R(1-R/P)}{c_1+c_2}}. \end{aligned}$$

Observations:

- If $P \to \infty$, i.e., the production rate is infinite, then this model becomes exactly the same as Model II(b).
- If $c_2 \to \infty$, then the model becomes exactly the same as Model III(a).

- If $P \to \infty$ and $c_2 \to \infty$, then the model becomes exactly the same as Model I(a).

Example 2.3: The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per month. The cost of one set up is Rs. 500 and the holding cost of one unit per month is Re.0.15. The shortage cost of one unit is Rs.20 per year. Determine the optimal production quantity and the number of shortages. Also, determine the production time and the time between set ups.

Solution: Given that $c_1 = Re.\ 0.15$/unit/month, $c_2 = Rs.\ (20/12)$/unit/month, $c_3 = Rs.\ 500$/set up, $P = 3000$ units/month, $R = (18000/12)$ units/month = 1500 units/-month.

$$\text{Therefore,}\quad q^* \;=\; \sqrt{\left(\frac{2c_3R(c_1+c_2)}{c_1c_2}\right)\left(\frac{P}{P-R}\right)} = 4670 \text{ units.}$$

Number of shortages is given by

$$S = Rt_3^* = \frac{c_1}{c_1+c_2}q^*\left(1-\frac{R}{P}\right) = 193 \text{ units.}$$

Production time $= \frac{q^*}{P} = \frac{4670}{3000\times 12} = 0.13$ year.
Time between set ups $= \frac{q^*}{R} = \frac{4670}{18000} = 0.26$ year.

MODULE - 3: Multi-item Inventory Models, Purchase Inventory Model and Inventory Models with Price Breaks

3.1 Model IV. Multi-Item Inventory Model

When the inventories consist of several items under some limitations such as the availability of warehouse space, the total investment in inventories, the total number of orders to be placed per year for all items, number of deliveries which can be accepted, size of delivery which can be handled, etc., then it is not possible to consider each item separately since there exists a relation among the items. Lagrange multiplier technique can be used to handle the simple cases. To deal with such a problem, we first solve the problem without considering the effect of limitations.
Consider the problem with the following assumptions:

- There are n items with instantaneous production and no lead time.
- R_i is the uniform demand rate for the ith item, $i = 1, 2, ..., n$.
- $c_1^{(i)}$ is the holding cost per unit of the ith item, $i = 1, 2, ..., n$.
- shortages are not allowed.
- $c_3^{(i)}$ is the set-up cost per production run for the ith item, $i = 1, 2, ..., n$.
- q_i is the total quantity of the ith item produced at the beginning of the production run, $i = 1, 2, ..., n$.

Now, proceeding exactly as in Model I(a), we get the cost per unit time for the ith item as

$$c_i(q_i) = \frac{1}{2}c_1^{(i)}q_i + c_3^{(i)}R_i/q_i, \quad i = 1,2,...,n. \tag{3.1}$$

Hence summing up these costs for $i = 1,2,...,n$, we get the cost per unit time for all items as

$$C = \sum_{i=1}^{n}\left[\frac{1}{2}c_1^{(i)}q_i + c_3^{(i)}R_i/q_i\right] \tag{3.2}$$

We have to determine the optimal value of q_i, $i = 1,2,...,n$ so that the average cost C is minimum. For optimality, we have the necessary conditions $\frac{\partial C}{\partial q_i} = 0$, $i = 1,2,...,n$. Since $\frac{\partial^2 C}{\partial q_i^2} > 0$ for all q_i, the average total cost C is minimum. Hence the optimum value of q_i is given by

$$q_i^* = \sqrt{\frac{2c_3^{(i)}R_i}{c_1^{(i)}}}, \quad i = 1,2,...,n. \tag{3.3}$$

We now proceed to consider the effect of (i) limitation on investment, (ii) limitation on stocked units, and (iii) limitation on warehouse floor space.

3.1.1 Model IV(a): Limitation on Investment

In this case, there is an upper limit, say M (in Rs.), on the amount to be invested on inventory. Let $c_4^{(i)}$ be the unit price of the ith item. Then we have

$$\sum_{i=1}^{n} c_4^{(i)}q_i \leq M \tag{3.4}$$

Our problem is to minimize the average total cost C given in eqn. (3.2) subject to the condition (3.4). In this situation, two cases may arise :

Case 1: $\sum_{i=1}^{n} c_4^{(i)}q_i^* \leq M$.

In this case, q_i^* given in (3.3), is the required optimal value of q_i.

Case 2. $\sum_{i=1}^{n} c_4^{(i)}q_i^* > M$.

In this case, q_i^* given in (3.3) is not the required optimal value. We use Lagrange multiplier technique to find the optimal value. The Lagrangian function (L) is given by

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}c_1^{(i)}q_i + \frac{c_3^{(i)}}{q_i}\right) + \lambda\left(\sum_{i=1}^{n} c_4^{(i)}q_i - M\right)$$

where λ is a Lagrange's multiplier. The necessary conditions for L to be minimum are $\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial \lambda} = 0$ which give

$$q_i^* = \sqrt{\frac{2c_3^{(i)}R_i}{c_1^{(i)} + 2\lambda^* c_4^{(i)}}}, \quad i = 1, 2, \cdots, n \tag{3.5}$$

$$\sum_{i=1}^{n} c_4^{(i)} q_i^* = M \tag{3.6}$$

Eqn.(3.6) implies that q_i^* must satisfy the investment constraint in equality sense. λ^* is usually determined by trial and error method so that q_i^* satisfies (3.6).

Example 3.1: Consider a shop which produces three items. The items are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No backorder is to be allowed. The pertinent data for the items are given in the following table :

Item	1	2	3
Holding cost (Rs.)	20	20	20
Set up cost (Rs.)	50	40	60
Cost per unit (Rs.)	6	7	5
Yearly demand rate	10000	12000	7500

Determine approximately the economic order quantities when the total value of average inventory levels of three items is Rs. 1000.

Solution: Firstly, we compute the optimal value q_i^* without considering the effect of restriction by using the basic formula. Thus, we get

$$\begin{aligned} q_1^* &= \sqrt{\frac{2 \times 50 \times 10000}{20}} = 233 \text{ approx.} \\ q_2^* &= \sqrt{\frac{2 \times 40 \times 12000}{20}} = 216 \text{ approx.} \\ q_3^* &= \sqrt{\frac{2 \times 60 \times 7500}{20}} = 210 \text{ approx.} \end{aligned}$$

Since the average optimal inventory at any time is $q_i^*/2$, the investment over the average inventory is given by

$$\sum_{i=1}^{n} c_4^{(i)}(q_i^*/2) = \text{Rs. } (6 \times 223/2 + 7 \times 216/2 + 5 \times 210/2) = \text{Rs. } 1950 \ > \text{Rs. } 1000$$

We now try to find a suitable value of λ by trial and error method for computing q_i^*. When $\lambda = 4$, we get $q_1^* = 121$, $q_2^* = 112$, $q_3^* = 123$ and the cost of average inventory is Rs. 1112.50 which is greater than Rs. 1000. If we set $\lambda = 5$, we obtain $q_1^* = 111$, $q_2^* = 102$, $q_3^* = 113$ and the corresponding cost of average inventory is Rs. 972.50 which is less than Rs. 1000. From this, we conclude that the most suitable value of λ lies between 4 and 5.
The most suitable value of λ can also be obtained from figure 3.1. Thus we get $\lambda^* =$

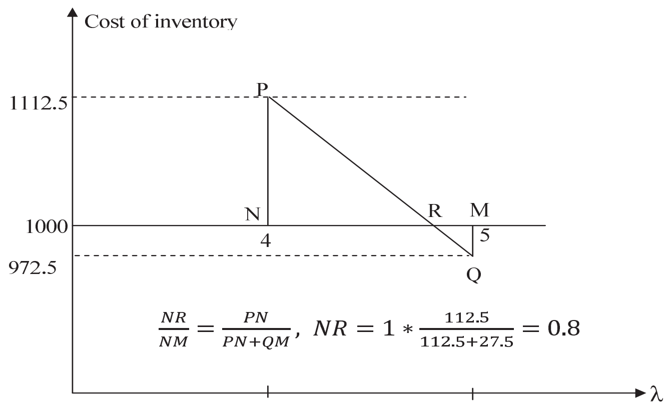


Fig. 3.1:

4.7 (approx.) corresponding to which the cost of inventory is Rs. 999.50 which is sufficiently close to Rs. 1000. For $\lambda = 4.7$, we obtain the required economic order quantities of three items as $q_1^* = 114$, $q_2^* = 105$, and $q_3^* = 116$.

3.1.2 Model IV(b): Limitation on Inventories

Suppose that the upper limit of average number of all items in the stock is N. Since the average number of ith item is $q_i/2$, we have to minimize the average cost C given by (3.2), subject to the condition

$$\frac{1}{2}\sum_{i=1}^{n} q_i \leq N \tag{3.7}$$

If $\frac{1}{2}\sum_{i=1}^{n} q_i^* \leq N$ then the optimum values are q_i^* given by (3.3). If $\frac{1}{2}\sum_{i=1}^{n} q_i^* > N$ then the optimal values can be obtained by Lagrange multiplier technique. We construct the Lagrangian function

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}c_1^{(i)}q_i + \frac{c_3^{(i)}}{q_i}\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} q_i - N\right)$$

From the first order optimality conditions, we get

$$q_i^* = \sqrt{\frac{2c_3^{(i)} R_i}{c_1^{(i)} + \lambda^*}}, \quad i = 1, 2, ..., n. \tag{3.8}$$

$$\sum_{i=1}^{n} q_i^* = 2N. \tag{3.9}$$

To obtain the values of q_i^* from (3.8), we find the optimal value of λ, i.e., λ^* by trial and error method subject to the condition given by eqn.(3.9).

Example 3.2: A company producing three items has a limited stock level averagely 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is given.

Product	1	2	3
Holding cost (Rs.)	0.05	0.02	0.04
Set up cost (Rs.)	50	40	60
Demand rate	100	120	75

Solution: Neglecting the restriction on average stock level of all items, we find $q_1^* = \sqrt{\frac{2\times50\times100}{0.05}} = 447$ approx., $q_2^* = \sqrt{\frac{2\times40\times120}{0.02}} = 693$ approx., $q_3^* = \sqrt{\frac{2\times60\times75}{0.04}} = 464$ approx. Therefore, the average inventory level for three items is $(447+693+464)/2 = 802$ units which is greater than the maximum capacity 750 units. Using Lagrange multiplier technique, for $\lambda = 0.002$, we get $q_1^* = 428$, $q_2^* = 628$, $q_3^* = 444$. The average inventory level becomes $(428+628+444)/2 = 750$ units, which is equivalent to the given amount of average inventory level. Hence, the optimal production quantities for each of the three items are $q_1^* = 428$ units, $q_2^* = 628$ units and $q_3^* = 444$ units.

3.1.3 Model IV(c): Limitation on Storage Space

Suppose that, in an inventory system, $n(> 1)$ items are competing for a limited storage space.

Let A = the maximum storage area available for the n item, a_i = storage area required per unit of the ith item, and q_i = order size of the ith item. Then the storage requirement constraint is given by

$$\sum_{i=1}^{n} a_i q_i \leq A. \tag{3.10}$$

The Lagrange's multiplier method yields the general solution of this problem. However, before applying this method, it is necessary to check whether the unconstraint value of q_i given by (3.3) satisfies the storage constraint. If not, the new optimal value of q_i must be determined which will satisfy the storage constraint in equality sense. The Lagrangian function is given by

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} c_1^{(i)} q_i + \frac{c_3^{(i)}}{q_i} \right) + \lambda \left(\sum_{i=1}^{n} a_i q_i - A \right)$$

Proceeding as in Model III(a), we obtain

$$q_i^* = \sqrt{\frac{2c_3^{(i)} R_i}{c_1^{(i)} + 2\lambda^* a_i}}, \quad i = 1, 2, ..., n. \tag{3.11}$$

$$\sum_{i=1}^{n} a_i q_i^* = A. \tag{3.12}$$

The above shows that q_i^* must satisfy the storage constraint in equality sense. Determination of λ^* by usual trial and error method yields the optimal value for q_i^*, $i = 1, 2, \cdots, n$.

Example 3.3: A small shop produces three machine parts I, II and III in lots. The shop has only 650 sq ft of storage space. The data for three items are given in the following table :

Item	I	II	III
Demand rate (unit/year)	5000	2000	10000
Procurement cost (Rs/order)	100	200	75
Cost per unit (Rs)	10	15	5
Floor space required (sq ft/unit)	0.70	0.80	0.40

The shop uses an inventory carrying charge of 20 per cent of average inventory valuation per year. If no stockout is allowed, determine the optimal lot size for each item under the given storage constraint.

Solution: For $\lambda = 1$, we compute approximately $q_1^* = 542$ units, $q_2^* = 417$ units, and $q_3^* = 913$ units The storage space required then would be

$$\sum_{i=1}^{3} a_i q_i^* = 0.7 \times 542 + 0.8 \times 417 + 0.4 \times 913 = 1078.2 > 650 \text{ sq ft (available storage space)}$$

For $\lambda = 5$, we compute $q_1^* = 333$ units; $q_2^* = 270$ units; and $q_3^* = 548$ units, and the corresponding total storage space required is 668.3 sq ft. This space is slightly more than the available space, 650 sq ft.

For $\lambda = 5.4$, the new quantities are $q_1^* = 324$ units, $q_2^* = 263$ units, and $q_3^* = 531$ units. The total storage space required corresponding to these values becomes 649.6 sq ft which is very close to the available storage space, 650 sq ft.

3.2 Model V. Inventory Model with Price Break

In the real world, it is not always true that the unit cost of an item is independent of the quantity procured. Often, discounts are offered for the purchase of large quantities. These discounts take the form of price breaks. Given the chance to purchase large quantities on reduced price, the organization must decide between the economic order quantity ($q_E^* = \sqrt{2Rc_3/c_1}$) and the quantity discount (q_D^*), i.e., to decide which option (q_E^* or q_D^*) minimizes the total cost which includes the purchasing cost. Thus our objective here is to minimize

$$C(q) = (R/q)c_3 + (q/2)pI + Rp \tag{3.13}$$

3.2.1 Purchase Inventory Model

Suppose that the demand rate R is constant and shortages are not allowed. The purchase cost per unit item is p and the cycle length is t. Let I be the cost of carrying one rupee in inventory value for one year.

Since the lot size $q = Rt$, therefore, inventory holding area = $(1/2)qt = \frac{q^2}{2R} = (\frac{q}{2R}) \times q$.

Hence, holding cost

= ordering cost part of the inventory value + purchase cost part inventory value

= $(c_3 + qp) \times \frac{q}{2R} \times I$

Therefore, the total cost for period t is

$$c_3 + qp + c_3(\frac{q}{2R})I + qp(\frac{q}{2R})I.$$

Hence, the total cost per unit time is

$$C(q) = \frac{c_3R}{q} + pR + \frac{c_3I}{2} + \frac{qpI}{2} \quad \text{(since } t = q/R)$$

Here the term $c_3I/2$ being constant throughout the model may be neglected for the purpose of minimization or comparison of total cost. Therefore, we have

$$C(q) = \frac{c_3R}{q} + pR + \frac{qpI}{2} \tag{3.14}$$

For optimum of $C(q)$, the first order necessary condition $\frac{dC(q)}{dq} = 0$ gives

$$q^* = \sqrt{\frac{2c_3R}{pI}} \tag{3.15}$$

Substituting the value of q^* in (3.14), we get the optimal total cost per unit time as

$$C(q^*) = \sqrt{2c_3RpI} + pR \tag{3.16}$$

3.2.2 Purchase Inventory Model with One Price Break

When there is only one price break (or one quantity discount), the situation may be illustrated as follows:

Range of quantity	Purchase cost (p) per unit
$0 \leq q_1 < b$	p_1
$q_2 \geq b$	p_2

where b is the quantity at and beyond which the quantity discount applies. Obviously, $p_2 < p_1$. Procedure for obtaining optimal lot size may be summarized as follows:

Step 1 Compute q_2^* using formula (3.15). If $q_2^* \geq b$ then the optimal lot size is q_2^*.

Step 2 If $q_2^* < b$ then the quantity discount no longer applies to the purchase quantity q_2^*. Furthermore, the minimum cost occurs at a point for which $q_2^* < b$. So, the average total cost will be monotonically increasing over the entire range $(q_2 \geq b)$, and the minimum cost for range $(q_2 \geq b)$ will occur at $q = b$. Hence, to determine the optimum purchase quantity, we only need to compare the average total cost for lot size $q = q_1^*$ with that for lot size $q = b$.

From equation (3.14), we have

$$C(q_1^*) = \frac{c_3 R}{q_1^*} + \frac{p_1 I q_1^*}{2} + p_1 R \tag{3.17}$$

$$C(b) = \frac{c_3 R}{b} + \frac{p_2 I b}{2} + p_2 R \tag{3.18}$$

Now, comparing the sum of first and third terms of $C(b)$ and $C(q_1^*)$

$$\frac{c_3 R}{b} + p_2 R < \frac{c_3 R}{q_1^*} + p_1 R \quad \text{since} \quad q_1^* < b \text{ and } p_2 < p_1$$

The term $\frac{1}{2} p_2 I b$ may or may not be less than the corresponding term $\frac{1}{2} p_1 I q_1^*$. However, if $C(q_1^*) < C(b)$ then q_1^* is the optimum order quantity; otherwise, b is the optimum order quantity.

Example 3.4: Find the optimum order quantity for a product for which the price breaks are as follows:

Range of quantity	*Unit purchase cost(Rs.)*
$0 \leq q_1 < 500$	10.00
$500 \leq q_2$	9.25

The monthly demand for a product is 200 units, the cost of storage is 2% of unit cost and the cost of ordering is Rs. 100.

Solution: Given that $R = 200$ units/month, $I =$ Re. 0.02, $c_3 =$ Rs. 100, $p_1 =$ Rs. 10.00, p_2 =Rs. 9.25. Using the formula for purchase quantity, we have

$$q_2^* = \sqrt{\frac{2c_3R}{p_2I}} = \sqrt{\frac{2 \times 100 \times 200}{9.25 \times 0.02}} = 465 \text{ units} < b_1 = 500$$

So, q_2^* is not the optimal purchase quantity. Now, we compute

$$q_1^* = \sqrt{\frac{2c_3R}{p_1I}} = \sqrt{\frac{2 \times 100 \times 200}{10 \times 0.02}} = 447 \;\text{ units.}$$

We now compare

$$\begin{aligned} C(q_1^*) = C(447) &= \text{Rs. } 2090.42 \\ C(b) = C(500) &= \text{Rs. } 1937.25 \end{aligned}$$

Since $C(b) < C(q_1^*)$, the optimum purchase quantity is $q^* = b = 500$ units.

3.2.3 Purchase Inventory Model with Two Price Breaks

Two price breaks situation may be illustrated as follows:

Range of quantity	Purchase cost (p) per unit
$0 \le q_1 < b_1$	p_1
$b_1 \le q_2 \le b_2$	p_2
$b_2 \le q_3$	p_3

where b_1 and b_2 are the quantities which determine the price breaks, and obviously $p_1 > p_2 > p_3$. The procedure for obtaining optimal order quantity is summarized below:

Step 1 Compute q_3^*. If $q_3^* \ge b_2$, then the optimum purchase quantity is q_3^*; otherwise, go to Step 2.

Step 2 Compute q_2^*. Since $q_3^* < b_2$ and q_2^* is also less than b_2 (because $q_1^* < q_2^* < q_3^* < \ldots < q_n^*$, in general), there are only two possibilities – either $q_2^* \ge b_1$ or $q_2^* < b_1$.

(i) If $q_2^* < b_2$ and $q_2^* \ge b_1$, then proceed as in the case of one price break only. That is, compare the costs $C(q_2^*)$ and $C(b_2)$ to obtain the optimum purchase quantity. The quantity with lower cost will naturally be the optimum.

(ii) If $q_2^* <$ (b_2 and b_1 both), then go to Step 3.

Step 3 Compute q_1^* which will satisfy the inequality $q_1^* < b_1$. Then compare the average total cost $C(q_1^*)$ with both $C(b_1)$ and $C(b_2)$ to determine the optimum purchase quantity.

MODULE - 4: Newsboy Problem and Probabilistic Inventory Model with Instantaneous Demand and No Set up Cost

4.0 Preliminary: Differencing under Summation Sign

Suppose that $C(z) = \sum_{x=a(z)}^{b(z)} f(x,z)$. Then

$$\begin{aligned} C(z+1) &= \sum_{x=a(z+1)}^{b(z+1)} f(x,z+1) \\ &= \sum_{x=a(z)}^{b(z)} f(x,z+1) + \sum_{x=b(z)+1}^{b(z+1)} f(x,z+1) - \sum_{x=a(z)}^{a(z+1)-1} f(x,z+1) \end{aligned}$$

Therefore,

$$\begin{aligned} \triangle C(z) &= C(z+1) - C(z) \\ &= \sum_{x=a(z)}^{b(z)} \triangle f(x,z) + \sum_{x=b(z)+1}^{b(z+1)} f(x,z+1) - \sum_{x=a(z)}^{a(z+1)-1} f(x,z+1) \qquad (*) \end{aligned}$$

Here, we have assumed that $a(z)$ and $b(z)$ are increasing functions of z.
Suppose that $f(x,z)$ is defined by

$$\begin{aligned} f(x,z) &= f_1(x,z), \quad 0 \le x \le b(z) \\ &= f_2(x,z), \quad x > b(z) \end{aligned}$$

We wish to find the difference of $C(z) = \sum_{x=0}^{\infty} f(x,z)$.

We have

$$C(z) = \sum_{x=0}^{b(z)} f_1(x,z) + \sum_{x=b(z)+1}^{\infty} f_2(x,z)$$

Applying formula (*) to both the sums, we obtain

$$\begin{aligned}
\triangle C(z) &= \triangle \sum_{x=0}^{b(z)} f_1(x,z) + \triangle \sum_{x=b(z)+1}^{\infty} f_2(x,z) \\
&= \left[\sum_{x=0}^{b(z)} \triangle f_1(x,z) + \sum_{x=b(z)+1}^{b(z+1)} f_1(x,z+1) - 0 \right] \\
&\quad + \left[\sum_{x=b(z)+1}^{\infty} \triangle f_2(x,z) + 0 - \sum_{x=b(z)+1}^{b(z+1)} f_2(x,z+1) \right] \\
&= \sum_{x=0}^{b(z)} \triangle f_1(x,z) + \sum_{x=b(z)+1}^{\infty} \triangle f_2(x,z) + \sum_{x=b(z)+1}^{b(z+1)} \{ f_1(x,z+1) - f_2(x,z+1) \} \\
&= \sum_{x=0}^{b(z)} \triangle f_1(x,z) + \sum_{x=b(z)+1}^{\infty} \triangle f_2(x,z)
\end{aligned}$$

provided that $f_1(x,z+1) = f_2(x,z+1)$.

4.1 Model VI. Newsboy Model

Single Period Inventory Model for Uncertain Demand

Newsboy problem: A newspaper boy buys newspaper everyday and sells some or all of them. He cannot return unsold newspapers. If he buys papers more than demand, then he is left with unsold papers at the end of the day; if he buys papers less than demand, then he is not able to satisfy demand of his customers and hence a loss to the potential profit and goodwill. The problem of newspaper boy is to decide the number of newspapers that should be procured everyday so as to maximize his expected profit, when the demand is uncertain.

The following notations are used for this model:

z = number of newspapers ordered per day

d = demand per day

$p(d)$ = probability that the demand is d on a randomly selected day

c_1 = cost per newspaper

c_2 = selling price per newspaper

Now, if the demand d exceeds z, his profit will be $(c_2-c_1)z$ and no newspapers remains unsold. On the other hand, if demand d does not exceed z, his profit becomes $(c_2-c_1)d-(z-d)c_1$, because, out of z number of papers only d papers are sold and $(z-d)$ number of papers remain unsold causing a loss of their purchasing cost $(z-d)c_1$. Therefore, the expected net profit per day is given by

$$P(z) = \underbrace{\sum_{d=0}^{z}(c_2 d - c_1 z)p(d)}_{for\ d\leq z} + \underbrace{\sum_{d=z+1}^{\infty}(c_2-c_1)zp(d)}_{for\ d>z}$$

In order to find the optimal order size, the conditions for maximum of $P(z)$ are $\triangle P(z-1) > 0 > \triangle P(z)$. Also, we can difference under the summation sign if, for $d = z+1$,

$$\begin{aligned}
[c_2 d - c_1(z+1)]p(d) &\equiv (c_2-c_1)(z+1)p(d) \\
\text{i.e. } [c_2(z+1) - c_1(z+1)]p(d) &\equiv (c_2-c_1)(z+1)p(d)
\end{aligned}$$

which is obviously true here. Thus, differencing under the summation sign, we get

$$\begin{aligned}
\triangle P(z) &= \sum_{d=0}^{z}[\{c_2 d - c_1(z+1)\} - (c_2 d - c_1 z)]p(d) + \sum_{d=z+1}^{\infty}(c_2-c_1)[(z+1)-z]p(d) \\
&= -c_1\sum_{d=0}^{z}p(d) + (c_2-c_1)\sum_{d=z+1}^{\infty}p(d) \\
&= -c_1\sum_{d=0}^{z}p(d) + (c_2-c_1)\left[\sum_{d=0}^{\infty}p(d) - \sum_{d=0}^{z}p(d)\right] \\
&= -c_2\sum_{d=0}^{z}p(d) + (c_2-c_1)
\end{aligned}$$

Now, $\triangle P(z) < 0$ gives

$$\sum_{d=0}^{z}p(d) > \frac{c_2-c_1}{c_2} \tag{4.1}$$

Similarly, $P(z-1) < 0$ gives

$$\sum_{d=0}^{z-1}p(d) < \frac{c_2-c_1}{c_2} \tag{4.2}$$

Therefore, from (4.1) and (4.2), the optimum quantity z^* must satisfy the condition

$$\sum_{d=0}^{z-1}p(d) < \frac{c_2-c_1}{c_2} < \sum_{d=0}^{z}p(d) \tag{4.3}$$

Example 4.1: A newspaper boy buys papers for Rs. 2.60 each and sells them for Rs. 3.60 each. He cannot return unsold newspapers. Daily demand has the following distribution:

No. of customers:	23	24	25	26	27	28	29	30	31	32
Probability:	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

If each day's demand is independent of the previous day's, how many papers should he order each day?

Solution: In this problem, we are given c_1 = Rs. 2.60, c_2 = Rs. 3.60, the lower limit for demand d is 23 and upper limit for demand d is 32. Therefore, substituting these numerical values in (4.3), we get

$$\sum_{d=23}^{z-1} p(d) < \frac{3.60-2.60}{3.60} = 0.28 < \sum_{d=23}^{z} p(d)$$

We can easily verify that

$$\sum_{23}^{27} p(d) = p(23)+p(24)+p(25)+p(26)+p(27) = 0.40 > 0.28 \text{ and } \sum_{23}^{26} p(d) < 0.28$$

Hence the newspaper boy should order 27 newspapers everyday.

4.2 Model VII. Instantaneous Demand, No Setup Cost

Here we assume that all demand distributions are stationary and independent over time.

4.2.1 Model $VII(a)$: Discrete case

In this model, it is assumed that the total demand is filled at the beginning of the period. Thus, depending on the amount d demanded, the inventory position just after the demand occurs may be either positive (surplus) or negative (shortage). These two cases are shown in Fig. 4.1.

Case 1. When demand d does not exceed the stock z, i.e. $d \le z$.

In this case, only the cost of holding inventory due to over-supply is involved and there is no shortage as the customer demand is fully satisfied.

Therefore, the holding cost per unit time $= \begin{cases} (z-d)c_1, & \text{for } d \le z \\ 0, & \text{for } d > z \end{cases}$

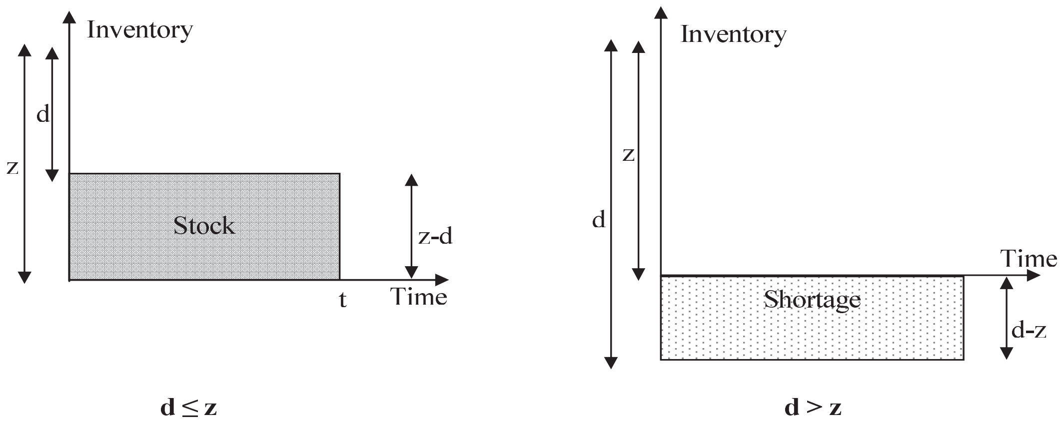


Fig. 4.1: Graphical representation of inventory system

Case 2. When demand d exceeds the stock z, i.e. $d > z$.

In this case, the customer demand is not satisfies at all.

Therefore, the shortage cost per unit time $= \begin{cases} 0, & \text{for } d \leq z \\ (d-z)c_2, & \text{for } d > z \end{cases}$

Hence the total expected cost per unit time is given by

$$C(z) = \sum_{d=0}^{z} (z-d)c_1 p(d) + \sum_{d=z+1}^{\infty} (d-z)c_2 p(d) \tag{4.4}$$

For minimum of $C(z)$, the condition $\triangle C(z-1) < 0 < \triangle C(z)$ must be satisfied. We can find difference $\triangle C(z)$ under the summation sign if, for $d = z+1$, the following condition is satisfied:

$$c_1[(z+1)-d]p(d) \equiv c_2[d-(z+1)]p(d)$$

Since the above condition is obviously satisfied for $d = z+1$, we take the first difference on both sides of (4.4) and obtain

$$\begin{aligned} \triangle C(z) &= c_1 \sum_{d=0}^{z} [\{(z+1)-d\} - (z-d)]p(d) + c_2 \sum_{d=z+1}^{\infty} [\{d-(z+1)\} - (d-z)]p(d) \\ &= c_1 \sum_{d=0}^{z} p(d) - c_2 \sum_{d=z+1}^{\infty} p(d) = c_1 \sum_{d=0}^{z} p(d) - c_2 \left[\sum_{d=0}^{\infty} p(d) - \sum_{d=0}^{z} p(d) \right] \\ &= (c_1 + c_2) \sum_{d=0}^{z} p(d) - c_2 \end{aligned}$$

Now, using the condition $\triangle C(z-1) < 0 < \triangle C(z)$, the optimum value of stock level z can be obtained by the relationship:

$$\sum_{\mathbf{d=0}}^{\mathbf{z-1}} \mathbf{p(d)} < \frac{\mathbf{c_2}}{\mathbf{c_1 + c_2}} < \sum_{\mathbf{d=0}}^{\mathbf{z}} \mathbf{p(d)}$$

Example 4.2: The probability distribution of monthly sale of a certain item is as follows:

Monthly sales	0	1	2	3	4	5	6
Probability	0.01	0.06	0.25	0.35	0.20	0.03	0.10

The cost of carrying inventory is Rs. 30 per unit per month and the cost of unit shortage is Rs 70 per month. Determine the optimum stock level that minimizes the total expected cost.

Solution: Given that c_1 = Rs. 30 per unit per month, c_2 = Rs. 70 per unit per month. Then the critical ratio $= \frac{c_2}{c_1+c_2} = \frac{70}{30+70} = 0.7$. The optimal solution is obtained by developing the cumulative probability distribution of monthly sales as follows:

Monthly sales (d)	0	1	2	3	4	5	6
Probability	0.01	0.06	0.25	0.35	0.20	0.03	0.10
Cumulative probability $P(d \leq z)$	0.01	0.07	0.32	0.67	0.87	0.90	1

Since 0.70 lies between 0.67 and 0.87, the condition for optimality suggests that $z^* = 4$.

4.2.2 Model $VII(b)$: Continuous case

This model is the same as Model VII(a) except that the stock level is now assumed to be continuous (rather than discrete). So, instead of probability $p(d)$ we shall have $f(x)dx$, where $f(x)$ is the probability density function.

Let $\int_{x_1}^{x_2} f(x)dx$ be the probability of an order within the range x_1 to x_2.Then the cost equation for continuous case becomes

$$C(z) = c_1 \int_0^z (z-x)f(x)dx + c_2 \int_z^\infty (x-z)f(x)dx. \tag{4.5}$$

The optimal value of z is obtained by equating to zero the first derivative of $C(z)$, i.e. $dC/dz = 0$ which gives

$$\int_0^{\mathbf{z}} \mathbf{f(x)dx} = \frac{\mathbf{c_2}}{\mathbf{c_1+c_2}} \tag{4.6}$$

Also $\dfrac{d^2C}{dz^2} = (c_1+c_2)f(z) > 0$

Hence, we can get the optimum value of z satisfying (4.6) for which the total expected cost C is minimum.

Example 4.3: A baking company sells cake by kg weight. It makes a profit of Rs. 5.00/kg when sold on the day it is baked. It disposes of all cakes not sold on the date at a loss of Rs. 1.20/kg. If demand is known to be rectangular between 2000 kg. and 3000kg., determine the optimal daily amount baked.

Solution: Let
c_1 = profit per kg.
c_2 = loss per kg. for unsold cake
x = demand which is continuous with probability density $f(x)$
$\int_{x_1}^{x_2} f(x)dx$ = probability of an order within the range x_1 to x_2
z = order level.
There are two possible cases:

(i) If $x \leq z$, then clearly the demand x is satisfied leaving $(z-x)$ quantity unsold which is returned with a loss of c_2 per kg. Since profit is c_1x and loss is $c_2(z-x)$, the net profit is $c_1x - c_2(z-x) = (c_1+c_2)x - c_2z$.

(ii) If $x > z$, then nothing remains unsold. Therefore, the net profit is c_1z.

Thus, the total expected profit within the range x_1 to x_2 is given by

$$P(z) = \underbrace{\int_{x_1}^{z} [(c_1+c_2)x - c_2z]f(x)dx}_{x\leq z} + \underbrace{\int_{z}^{x_2} c_1zf(x)dx}_{x>z}$$

From the first order condition $\frac{dP(z)}{dz} = 0$, we get

$$\int_{x_1}^{z} f(x)dx = \frac{c_1}{c_1+c_2}. \tag{4.7}$$

Also, $\frac{d^2P(z)}{dz^2} = -(c_1+c_2)f(z) < 0$. Hence the value of z in (4.7) maximizes $P(z)$.
In our problem, c_1 = Rs. 5, c_2 = Rs. 1.20, $x_1 = 2000$, $x_2 = 3000$, $f(x) = \frac{1}{x_2-x_1} = \frac{1}{1000}$
Therefore, $\int_{2000}^{z} \frac{1}{1000}dx = \frac{5.00}{5.00+1.20}$ gives $z = 2807$. Thus the optimal daily amount baked is 2807 kg.

MODULE - 5: Probabilistic Inventory Model with Uniform Demand and No Set up Cost, and Buffer Stock in Probabilistic Inventory Model

5.1 Model VIII. Uniform Demand, No Setup Cost

The type of problem is similar to that considered under Model VII, except that the withdrawal from stock is continuous (rather than instantaneous) and the rate of withdrawal is assumed to be constant. Here demand distribution is dependent over time.

5.1.1 Model $VIII(a)$: Continuous case

Basic assumptions of the model are as follows:

(i) t is the scheduling period which is a prescribed constant,

(ii) demand rate in a period t is constant,

(iii) lead time is zero,

(iv) replenishments are instantaneous,

(v) z is the stock level to which the stock is raised at the end of every period t,

(vi) $f(x)$ is the probability density function for demand x,

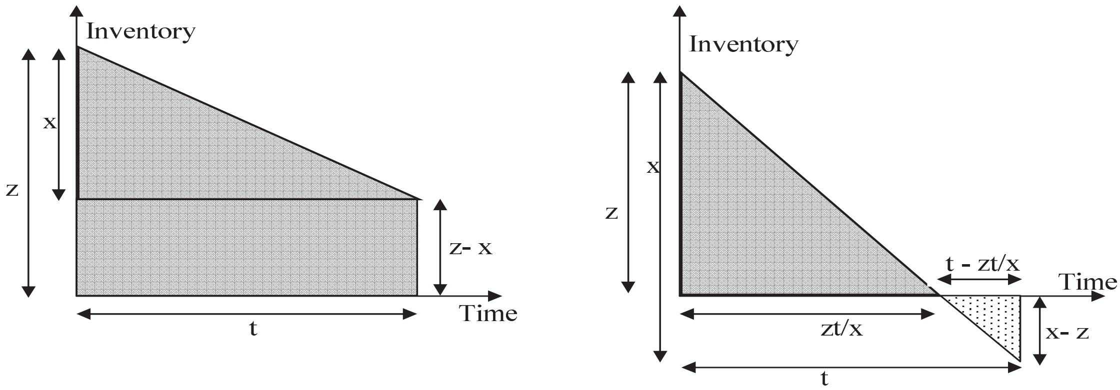
(vii) c_1 is the carrying cost per unit per unit time,

(viii) c_2 is the shortage cost per unit per unit time.

In this model, the demand occurs uniformly rather than instantaneously during period t, see Fig. 5.1. There are two possibilities:

(i) $\mathbf{x \le z}$: In this situation, there is no shortage. So only holding cost is involved. Holding cost $= c_1 \frac{1}{2}(z - x + z)t = \frac{1}{2}c_1[2z - x]t$.

(ii) $\mathbf{x > z}$: In this situation, both holding and shortage costs are involved. Holding cost is $\frac{c_1}{2x}z^2 t$ and shortage cost is $\frac{c_2}{2x}(x-z)^2 t$. Therefore, the total cost is $\frac{c_1}{2x}z^2 t + \frac{c_2}{2x}(x-z)^2 t$.



$\mathbf{x \le z}$ $\mathbf{x > z}$

Fig. 5.1: Graphical representation of inventory system

Thus, the expected average total cost is given by

$$C(z) = \underbrace{\int_0^z \frac{c_1}{2}(2z - x) f(x) dx}_{(\mathbf{x \le z})} + \underbrace{\int_z^\infty \left[\frac{c_1 z^2}{2x} + \frac{c_2 (x-z)^2}{2x}\right] f(x) dx}_{(\mathbf{x > z})}$$

For optimum of $C(z)$, the first order condition $\frac{dC}{dz} = 0$ gives

$$\int_0^{\mathbf{z}} \mathbf{f(x)dx} + \mathbf{z} \int_{\mathbf{z}}^\infty \frac{\mathbf{f(x)dx}}{\mathbf{x}} = \frac{\mathbf{c_2}}{\mathbf{c_1 + c_2}} \tag{5.1}$$

Now

$$\frac{d^2 C}{dz^2} = (c_1 + c_2) \int_z^\infty \frac{f(x)}{x} dx > 0.$$

Hence $C(z)$ is minimum for optimum value of z given by (5.1).

Example 5.1: Let the probability density of demand of an item during a week be

$$f(x) = \begin{cases} 0.1, & 0 \le x \le 10 \\ 0, & otherwise \end{cases}$$

This demand is assumed to occur with a uniform pattern over the week. Let the unit carrying cost of the item in inventory be Rs. 2.00 per week and unit shortage cost be Rs. 8.00 per week. How will you determine the optimal order level of the inventory?

Solution: Given that $f(x) = 0.1, 0 \le x \le 10$, c_1 =Rs. 2.00, c_2 = Rs. 8.00. Using the result (5.1), we get

$$\begin{aligned} \int_0^z (0.1)dx + z\int_z^{10} \frac{0.1}{x}dx &= \frac{8}{10} \\ \text{or, } 3.3z - z\log z - 8 &= 0. \end{aligned}$$

The solution of the above equation is obtained by trial and error method as $z = 4.5$. Hence the optimal order level of the inventory is 4.5 units.

5.1.2 Model $VIII(b)$: Discrete case

Here, the cost function is similar to that derived for Model VIII(a). We simply replace integration by summation and $f(x)dx$ by the probability $p(x)$. Thus we have

$$C(z) = \sum_{x=0}^{z} c_1\left(z - \frac{x}{2}\right)p(x) + \sum_{x=z+1}^{\infty}\left(\frac{c_1 z^2}{2x} + \frac{c_2(x-z)^2}{2x}\right)p(x) \tag{5.2}$$

For minimum of $C(z)$, we require that z should satisfy the relationship $\triangle C(z-1) < 0 < \triangle C(z)$. We can find difference under the summation sign if, for $x = z+1$, the condition

$$c_1\sum_{x=0}^{z}\left(z+1-\frac{x}{2}\right)p(x) \equiv c_1\sum_{x=z+1}^{\infty}\frac{(z+1)^2}{2x}p(x) + c_2\sum_{x=z+1}^{\infty}\frac{[x-(z+1)^2]}{2x}p(x)$$

holds. Since the above condition obviously holds, therefore, we have

$$\begin{aligned} \triangle C(z) &= c_1\sum_{x=0}^{z}\left\{\left((z+1)-\frac{x}{2}\right)-\left(z-\frac{x}{2}\right)\right\} + c_1\sum_{x=z+1}^{\infty}\left[\frac{(z+1)^2}{2x}-\frac{z^2}{2x}\right]p(x) \\ &\quad + c_2\sum_{x=z+1}^{\infty}\left[\frac{(x-z-1)^2}{2x}-\frac{(x-z)^2}{2x}\right]p(x) \\ &= c_1\sum_{x=0}^{z}p(x) + c_1\sum_{x=z+1}^{\infty}\frac{2z+1}{2x}p(x) - c_2\sum_{x=z+1}^{\infty}\frac{(2x-2z-1)}{2x}p(x) \\ &= c_1\sum_{x=0}^{z}p(x) + (c_1+c_2)(z-1/2)\sum_{x=z+1}^{\infty}\frac{p(x)}{x} - c_2\left[\sum_{x=0}^{\infty}p(x) - \sum_{x=0}^{z}p(x)\right] \\ &= (c_1+c_2)\sum_{x=0}^{z}p(x) + (c_1+c_2)(z+1/2)\sum_{x=z+1}^{\infty}\frac{p(x)}{x} - c_2 \end{aligned}$$

Therefore, $\triangle C(z) > 0$ gives

$$\sum_{x=0}^{z}p(x) + (z+1/2)\sum_{x=z+1}^{\infty}\frac{p(x)}{x} > \frac{c_2}{c_1+c_2} \tag{5.3}$$

Similarly, the condition $\triangle C(z-1) < 0$ gives

$$\sum_{x=0}^{z-1}p(x) + (z-1/2)\sum_{x=z}^{\infty}\frac{p(x)}{x} < \frac{c_2}{c_1+c_2} \tag{5.4}$$

Combining the relationships (5.3) and (5.4), we get

$$\sum_{\mathbf{x=0}}^{\mathbf{z-1}} \mathbf{p(x)} + (\mathbf{z} - \frac{1}{2}) \sum_{\mathbf{x=z}}^{\infty} \frac{\mathbf{p(x)}}{\mathbf{x}} < \frac{\mathbf{c_2}}{\mathbf{c_1 + c_2}} < \sum_{\mathbf{x=0}}^{\mathbf{z}} \mathbf{p(x)} + (\mathbf{z} + \frac{1}{2}) \sum_{\mathbf{x=z+1}}^{\infty} \frac{\mathbf{p(x)}}{\mathbf{x}} \tag{5.5}$$

Using the relationship (5.5) above, we can find the range of optimum value of z.

Example 5.2: The probability distribution of monthly sales of a certain item is as follows :

Monthly sales :	0	1	2	3	4	5	6
Probability :	0.02	0.05	0.30	0.27	0.20	0.10	0.06

The cost of carrying inventory is Rs. 10.00 per unit per month. The current policy is to maintain a stock of four items at the beginning of each month. Assuming that the cost of storage is proportional to both time and quantity short, obtain the cost of shortage of one item for one time unit.

Solution: In this problem, we are given that optimal stock $z = 4$ items, carrying cost c_1 = Rs 10.00 per item per month, and the probability $p(x)$ for sale x in each month is as follows:

$p(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$	$p(5)$	$p(6)$
0.02	0.05	0.30	0.27	0.20	0.10	0.06

The shortage cost c_2 is to be determined.
We have the relationship (5.5):

$$\sum_{\mathbf{x=0}}^{\mathbf{z-1}} \mathbf{p(x)} + (\mathbf{z} - \frac{1}{2}) \sum_{\mathbf{x=z}}^{\infty} \frac{\mathbf{p(x)}}{\mathbf{x}} < \frac{\mathbf{c_2}}{\mathbf{c_1 + c_2}} < \sum_{\mathbf{x=0}}^{\mathbf{z}} \mathbf{p(x)} + (\mathbf{z} + \frac{1}{2}) \sum_{\mathbf{x=z+1}}^{\infty} \frac{\mathbf{p(x)}}{\mathbf{x}}$$

Now, the least value of c_2 can be determined by letting

$$\begin{aligned}
\frac{c_2}{c_1 + c_2} &= \sum_{x=0}^{z-1} p(x) + (z - 1/2) \sum_{x=z}^{\infty} \frac{p(x)}{x} \\
\text{or, } \frac{c_2}{10 + c_2} &= \sum_{x=0}^{3} p(x) + (4 - 1/2) \sum_{x=4}^{6} \frac{p(x)}{x} \\
&= [p(0) + p(1) + p(2) + p(3)] + \frac{7}{2}\left[\frac{p(4)}{4} + \frac{p(5)}{5} + \frac{p(6)}{6}\right] = 0.92
\end{aligned}$$

Therefore, the least value of $c_2 = 9.2/0.08 =$ Rs. 115.

Similarly, the greatest value of c_2 can be determined by letting

$$\begin{aligned}
\frac{c_2}{c_1+c_2} &= \sum_{x=0}^{z-1} p(x) + (z+\frac{1}{2})\sum_{x=z}^{\infty}\frac{p(x)}{x} \\
\text{or, } \frac{c_2}{10+c_2} &= \sum_{x=0}^{3} p(x) + (4+1/2)\sum_{x=4}^{6}\frac{p(x)}{x} \\
&= [p(0)+p(1)+p(2)+p(3)] + \frac{9}{2}\left[\frac{p(4)}{4}+\frac{p(5)}{5}+\frac{p(6)}{6}\right] = 0.975
\end{aligned}$$

The greatest value of $c_2 = 9.75/0.025 =$ Rs. 390. Hence, the required range of c_2 is Rs. $115 < c_2 <$ Rs. 390.

5.2 Equivalence of Models VII and VIII

Equivalence of Probabilistic Inventory Models with Instantaneous and Uniform Demands

The expected total cost of the probabilistic inventory model with instantaneous demand is given by

$$C(z) = c_1\int_0^z (z-x)f(x)\,dx + c_2\int_z^{\infty}(x-z)f(x)\,dx$$

The optimal order level z^* of the system can be obtained from the relation

$$\int_0^{z^*} f(x)\,dx = \frac{c_2}{c_1+c_2}$$

$$\text{or,}\qquad F(z^*) = \frac{c_2}{c_1+c_2} \tag{5.6}$$

Now, the expected total cost of the probabilistic inventory model with uniform demand is given by

$$\begin{aligned}
C(z) &= c_1\int_0^z (z-\frac{x}{2})f(x)\,dx + c_1\int_z^{\infty}\frac{z^2}{2x}f(x)\,dx \\
&\quad + c_2\int_z^{\infty}\frac{(x-z)^2}{2x}f(x)\,dx
\end{aligned} \tag{5.7}$$

and the optimal order level z^* can be obtained from the relation

$$\int_0^{z^*} f(x)\,dx + \int_{z^*}^{\infty}\left(\frac{z^*}{x}\right)f(x)\,dx = \frac{c_2}{c_1+c_2}$$

The above equation can be written as

$$M(z^*) = \frac{c_2}{c_1+c_2} \tag{5.8}$$

$$\text{where}\qquad M(z) = \int_0^z f(x)\,dx + \int_z^{\infty}\frac{z}{x}f(x)\,dx \tag{5.9}$$

An examination of the function $M(z)$ reveals the following properties:

(i) $M(0) = 0$

(ii) M is non-decreasing

(iii) $M(\infty) = 1$.

Thus $M(z)$ satisfies all the properties of cumulative distribution function. The similarity of equation (5.6) and (5.8) where both F and M are cumulative distribution functions suggest that equation (5.7) can also be expressed in the form

$$C(z) = c_1 \int_0^z (z-y)m(y)\,dy + c_2 \int_z^\infty (y-z)m(y)\,dy \tag{5.10}$$

where $m(y) = \frac{d}{dy} M(y)$ with $M(y)$ as given in equation (5.9). This gives

$$m(y) = \int_y^\infty \frac{f(x)}{x}\,dx.$$

Hence, the probabilistic inventory models with instantaneous and uniform demands are equivalent.

Example 5.3: In a probabilistic order level system with uniform pattern of demand, the density of demand is the following:

$$f(x) = \frac{1}{b^2} x e^{-x/b} \quad \text{when} \quad x \geq 0.$$

Find the optimal order level of the above system considering the equivalent system of instantaneous demand.

Solution: We have

$$m(y) = \int_y^\infty \frac{f(x)}{x}\,dx = \int_y^\infty \frac{1}{b^2} \cdot e^{-x/b}\,dx = \frac{e^{-y/b}}{b}.$$

Then, from the following relation, optimal z is obtained:

$$\int_0^{z^*} m(y)\,dy = \frac{c_2}{c_1+c_2} \Rightarrow \int_0^{z^*} \frac{e^{-y/b}}{b}\,dy = \frac{c_2}{c_1+c_2}$$

$$\Rightarrow \quad z^* = b \log\left[\frac{c_1+c_2}{c_1}\right]$$

5.3 Buffer Stock in Probabilistic Inventory Model

To face uncertainties in demand as well as in lead time, an additional stock is maintained, which is called '*safety stock*' or '*buffer stock*'. For an average demand and an

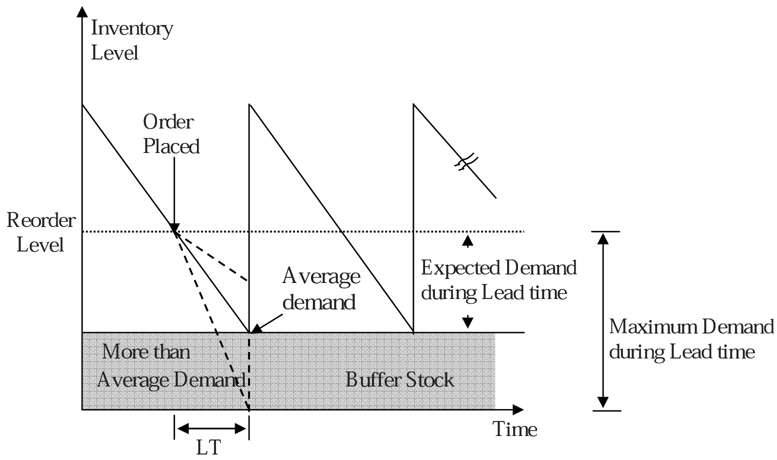


Fig. 5.2: Buffer stock

average lead time, the Buffer Stock (BS) is given by

$$Buffer\ stock = Average\ demand \times Average\ lead\ time$$

When no stockouts are desired,

$$Buffer\ stock \quad = \quad Maximum\ DDLT - Average\ DDLT$$

where DDLT stands for demand during lead time.

When demand varies and lead time is constant, and an order for fixed quantity is ordered, the re-order level (ROL) is set equal to the level of inventory required to satisfy the average demand during lead time plus buffer stock, i.e., $ROL = Average\ DDLT + BS$.

When demand rate varies about the average demand during a constant lead time (LT) period, prediction of exact demand during lead time period becomes difficult, and therefore, the reorder level is defined as $ROL = Average\ DDLT \times LT$.

However, this policy of setting reorder level results in stockouts for about 50% time during lead time period. Thus, to avoid the chance of stockout to occur, a buffer stock would be needed and reorder level is determined as follows:

$$ROL = Average\ DDLT \times LT + BS$$

5.3.1 Buffer Sock under Normal Distribution of Demand

Let the lead time (L) be fixed and

d = average demand during lead time

B = buffer stock (safety stock)

x = random demand during lead time

Suppose that the demand during lead time follows a normal distribution with mean $\bar{x}$ and s.d. σ_x. Buffer stock depends on the service level required by the organization. The 95% service level means that the chances of demand during lead time exceeding reorder point quantity will be 5% only. The value of B on the normal curve can be determined using normal distribution table. For 95% service level, the value of B on the curve will be 1.645 σ_x.

Therefore, buffer stock $B = 1.645\ \sigma_x$

Reorder point $= \bar{x} + 1.645\ \sigma_x = Ld + 1.645\ \sigma_x$.

Example 5.4: A company wants to provide 95% service level to its customers. Daily demand follows normal distribution with average daily demand of 20 units and s.d. of 5 units. The lead time is 4 days. The ordering cost is Rs. 10 per order and carrying cost is Re. 1 per unit per year. There is no stockout cost and unfilled demands are made after the items are received. What should be the inventory policy for the comapany?

Solution: The annual demand $D = 20 \times 365 = 7300$ units. Order quantity $= \sqrt{\frac{2c_3 D}{c_1}} =$ 381 units.

Expected demand during lead time is $\mu = d \times L = 20 \times 4 = 80$ units.

Variance of the demand during lead time (DDLT) is

σ^2 = sum of the variances of demand for each day with lead time $= \sum_{i=1}^{4} \sigma_i^2 = 4(5^2) =$ 100.

Therefore, s.d. of demand during lead time is $\sigma = 10$.

For 95% service level, the buffer stock $B = 1.645\sigma = 16$ units.

Reorder point = average demand during lead time + buffer stock = 80+16 = 96 units.

Hence, the reorder level R corresponding to 95% service level can be determined from the following:

$z = \frac{R-\mu}{\sigma}$ or, $1.645 = \frac{R-\mu}{\sigma}$ which gives $R = 96$ units.