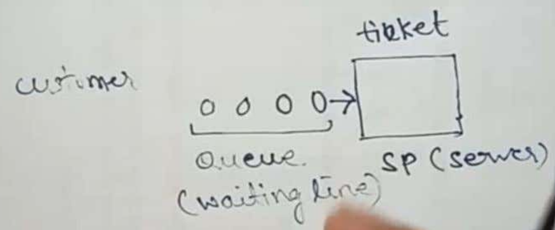
Queuing model (Queuing Theory)

Queuing theory is the mathematical study of waiting lines or queues. In queuing theory a model is constructed so that queue lengths and waiting time can be predicted.



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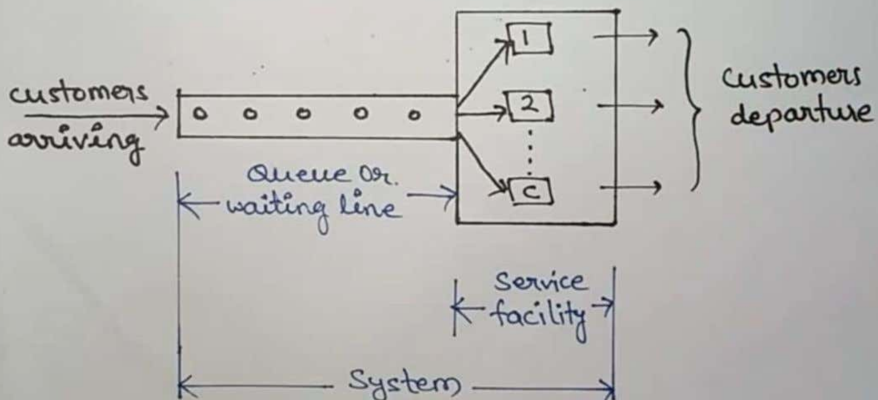

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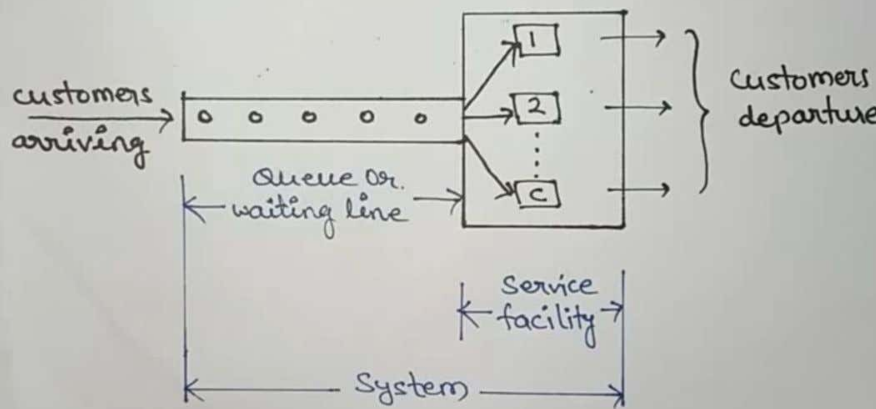
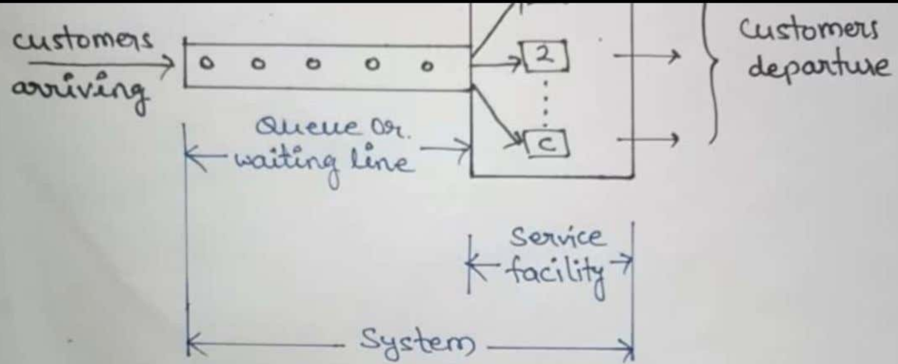


(1) **Customer** -

The arriving unit that requires some service to be performed. The customers may be persons, machines, vehicles, parts etc.

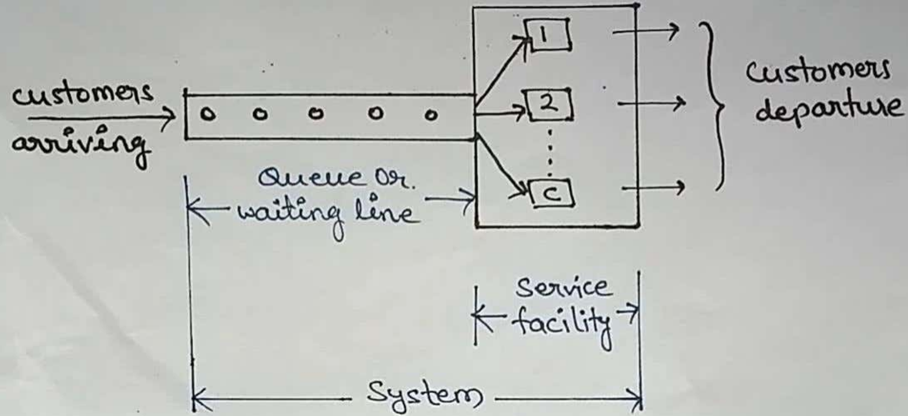


(2) Queue (waiting line) -

The number of customers waiting to be serviced. The queu
does not include the customers being serviced.

(3) Service channel -

The process or facility which is performing the services to the customer. This may be single or multichannel.



Element of a Queuing System (Structure of a Queuing System)-

1. **Arrival Distribution** - It represents the pattern in which the number of customers arrive at the service facility.

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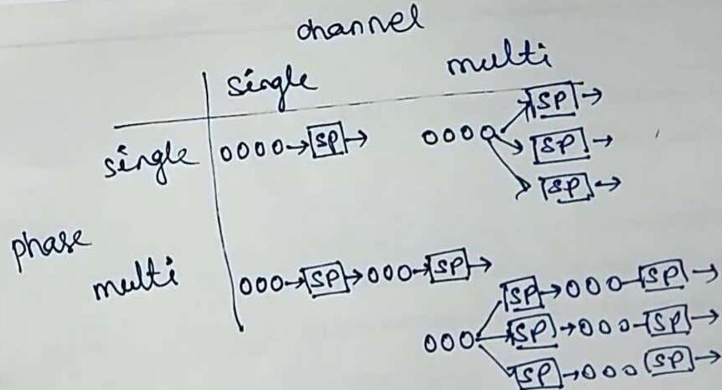

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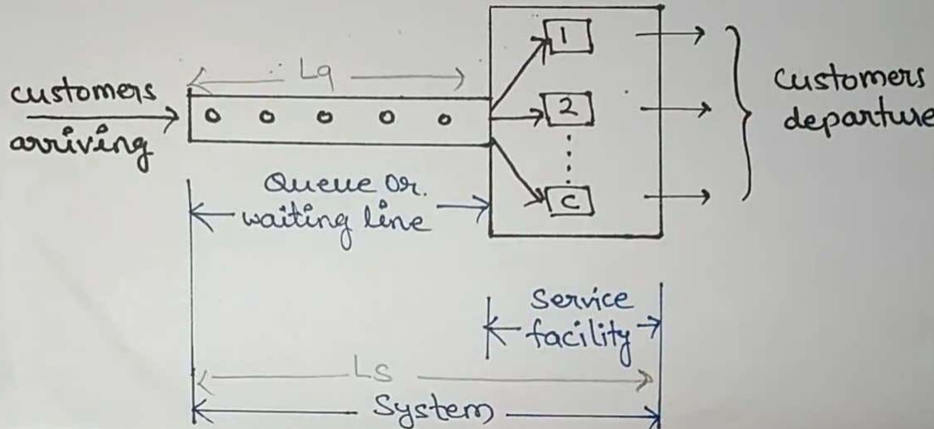
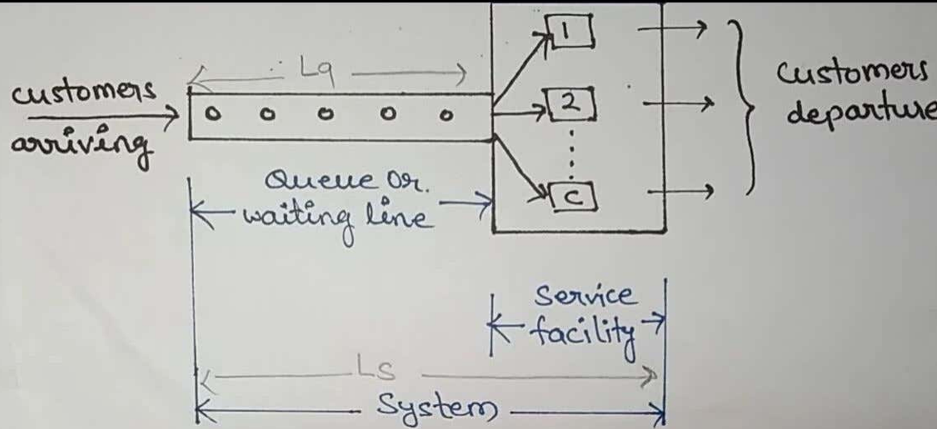
↳ (1) Balking
(2) Reneging
(3) jockeying



Operating characteristics of a queuing system-

1. **Queue length (L_q)**- The average number of customers in the queue waiting to get service. This excludes the customers being served.
2. **System length (L_s)**- The average number of customers in the system including those waiting as well as those being served.
3. **Waiting time in the queue (W_q)**- The average time for which a customer has to wait in the queue to get service.



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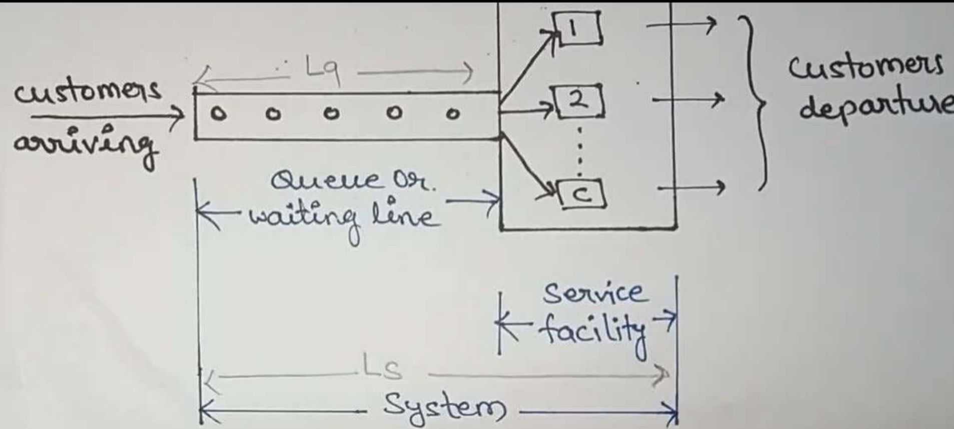
4. <u>Total time in the system (Ws)</u>- The average total time spent by a customer in the system from the moment he arrives till he leaves the system. It is taken to be the waiting time plus the service time.



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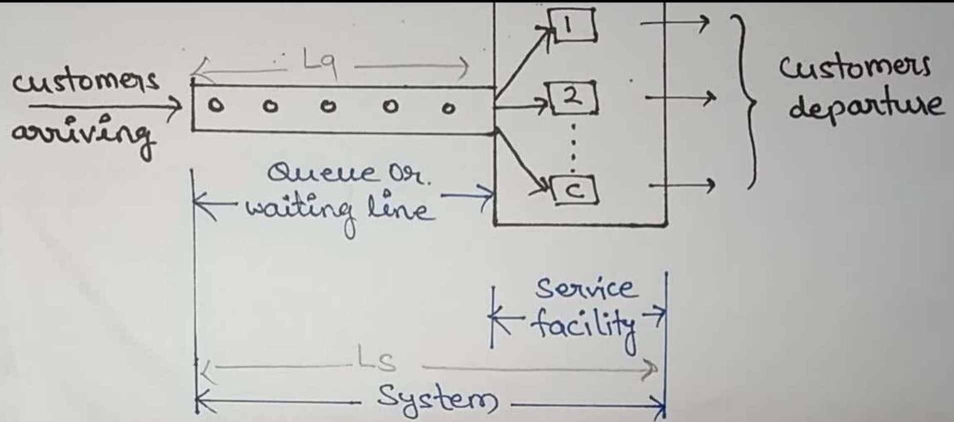

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5. Utilization factor (ρ) - It is the proportion of time a server actually spends with the customers. It is also called "traffic intensity"

Transient and steady states of the system-

Transient state - Queuing theory analysis involves the study of system's behaviour over time. If the operating characteristics (behaviour of the system) vary with time, it is said to be in "transient state".

Steady state - A system is said to be in steady state condition if its behaviour becomes independent of its initial conditions and of the elapsed time.

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arrival rate

service rate

o o o o o → SP

$AR > SR$

Queue length (increase)

Explosive state

Kendall's Notations for representing Queuing models – D.G Kendall 1953, A. Lee 1966

$\boxed{(a/b/c):(d/e)}$

where

$a \rightarrow$ arrival distribution

$b \rightarrow$ service time (or departure) distribution

$M \rightarrow$ Markovian (Poisson) arrival or departure distribution or exponential interarrival or service time distribution

$E_k \rightarrow$ Erlangian or gamma interval or service time distribution with parameter k

$GI \rightarrow$ general independent arrival distribution

$G \rightarrow$ general departure distribution

$D \rightarrow$ deterministic interval or service time

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GI → general independent arrival distribution

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D → deterministic interval or service time

c → Number of parallel service channels in the system

1 → single server

s → fixed number of server

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d → Maximum number of customers allowed in the system

∞ → Infinite number of customers

N → finite number of customers

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e → Queue (or service discipline)

FCFS → First Come First Serve

LCFS → Last come First Serve

SIRO → Service In Random Order

GD → General Discipline

Classification of Queuing Models –

(I) Probabilistic Q.M.

1. Erlang model
   $(M/M/1):(\infty/FCFS)$
2. General Erlang
   $(M/M/1):(\infty/FCFS)$
3. $(M/M/1):(N/FCFS)$
4. $(M/M/S):(\infty/FCFS)$
5. $(M/E_K/1):(\infty/FCFS)$
6. Machine Repairing model
   $(M/M/R):(K/GD), K>R$
7. Power Supply model
8. Economic cost profit model
9. $(M/G/1):(\infty/GD)$

(II) Deterministic Q.M.

10. $(D/D/1):(K-1/FCFS)$

(III) Mixed Q.M.

11. $(M/D/1):(\infty/FCFS)$

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Model 1 $(M/M/1):(\infty/FCFS)$

Sandeep Kumar Gour

Qu: A person repairing radios, finds that the time spent on the radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Model I: $(M/M/1):(\infty/FCFS)$

List of symbols and formulae

(i) n = number of customers in the system

(ii) λ = mean arrival rate (number of arrivals per unit of time)

(iii) μ = mean service rate per busy server (number of customers served per unit of time)

(iv) ρ = traffic intensity

$$\rho = \frac{\lambda}{\mu}$$

(v) L_q = Expected (average) number of customers in the queue.

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu - \lambda}$$

(vi) L_s = Expected number of customers in the system (waiting + being served

$$L_s = \frac{\lambda/\mu}{(1 - \lambda/\mu)} = \frac{\lambda}{\mu - \lambda}$$

(vii) W_s = Expected time a customer spends in the system (waiting + being served

$$W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu - \lambda)} = \frac{1}{\mu - \lambda}$$

(viii) W_q = Expected waiting time per customer in the queue

Like & Subscribe

$$L_s = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\lambda}{\mu-\lambda}$$

(vii) W_s = Expected time a customer spends in the system (waiting + being served)

$$W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} = \frac{1}{\mu-\lambda}$$

(viii) W_q = Expected waiting time per customer in the queue

$$W_q = W_s - \frac{1}{\mu} \Rightarrow W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

(ix) L_n = Average length of non-empty queue

$$L_n = \frac{\mu}{\mu-\lambda}$$

(x) W_n = Average waiting time in non-empty queue.

$$W_n = \frac{1}{\mu-\lambda}$$

Qu:- A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming poisson distribution for arrival rate and exponential distribution for service time, find

Solution

1. Average number of customers in the system

2. Average number of customers in the queue or average queue length

3. Average time a customer spends in the system

4. Average time a customer waits before being served.

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Solution

λ = arrival rate $= 9/5 = 1.8$ customer/minute

μ = service rate $= 10/5 = 2$ customer/minute

1. Average number of customers in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{1.8}{2-1.8} = 9$$

2. Average number of customers in the queue or average queue length

$$L_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{(\mu-\lambda)} = \frac{1.8}{2} \times \frac{1.8}{(2-1.8)} = 8.1$$

3. Average time a customer spends in the system

$$W_s = \frac{1}{\mu-\lambda} = \frac{1}{2-1.8} = 5$$

4. Average time a customer waits before being served.

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

Like & Subscribe

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Model 1 $(M/M/1):(\infty/FCFS)$

Sandeep Kumar Gour

Qu: A person repairing radios, finds that the time spent on the radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

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Sol^n:

Arrival rate $\lambda = \frac{15}{8\times 60} = \frac{1}{32}$ units/minute

rate $\Rightarrow \lambda, \mu$

mean $\Rightarrow \frac{1}{\lambda}, \frac{1}{\mu}$

service rate $\mu = \frac{1}{20}$ units/minute

Average number of jobs in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{1/32}{1/20 - 1/32} = \frac{1}{\cancel{32}} \times \frac{20\times \cancel{32}}{32-20} = \frac{20}{12} = \frac{5}{3} \text{ jobs.}$$

Like & Subscribe

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Number of hours for which the repairman busy in an 8-hour day

$$= 8 \cdot \frac{\lambda}{\mu} = \cancel{8} \cdot \frac{1}{\cancel{32}_{\cancel{4}}} \times \frac{\cancel{20}^{5}}{1} = 5 \text{ hours.}$$

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Number of hours for which the repairman busy in an 8-hour day

$$= 8 \cdot \frac{\lambda}{\mu} = \cancel{8} \cdot \frac{1}{\cancel{32}_{4}} \times \frac{2\cancel{0}\,5}{1} = 5 \text{ hours.}$$

$\therefore$ time for which repairman remains idle in an 8 hour day

$$= 8 - 5 = 3 \text{ hours.}$$

Model (M/M/1) : (∞/FCFS)

Sandeep Kumar Gour

Qu: A branch of a bank has only one typist. Since the typing work varies in length, the typing rate is radomly distributed approximating a poisson distribution with mean service rate of 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8-hou[r] work day. If the typewriter is valued at Rs. 1.50 per hour, determine

Soln:

arrival rate $\lambda = 5$ letter/hour

service rate $\mu = 8$ letter/hour

1. Equipment utilization

$$\rho = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625$$

2. the percent time an arriving letter has to wait

$$= 0.625 \times 100 = 62.5\%$$

3. Average system time

4. Average cost due to wai[tin]g on the part of the typewriter per day

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$\rho = \frac{5}{8}$

$1 - \rho = 1 - \frac{5}{8}$

free

2. the percent time an arriving letter has to wait

$$= 0.625 \times 100 = 62.5\%$$

3. Average system time $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-5} = \frac{1}{3}$ hour $= \frac{1}{\cancel{3}} \times \overset{20}{\cancel{60}} = 20$ minutes

4. Average cost due to waiting on the part of the typewriter per day

Busy time $= \cancel{8} \cdot \frac{5}{\cancel{8}} = 5$ hours,

Remaining idle $= 8 - 5 = \underline{3 \text{ hour}} \times 1.50 = 4.50$ Rs.

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Model II (M/M/1) : (∞/FCFS)

Sandeep Kumar Gour

[Arrival and service rates depend upon the length of the queue]

$$\boxed{P_n = \frac{\rho^n}{n!} e^{-\rho}}$$ ⇒ P_i's in equation are the values of a Poisson distribution with parameter ρ.

Example: A shipping company has a single unloading dock with ships arriving in a Poisson fashion at an average rate of 3/day. The unloading time distribution for a ship with n unloading crews is found to be exponential with average unloading time 1/2 days. The company has a large labour supply without regular working hours, and to avoid long waiting times, the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Find

(a) the average number of unloading crews working at any time, and

(b) the probability that more than 4 crews will be needed

$$\boxed{P_n = \frac{\rho^n}{n!} e^{-\rho}} \Rightarrow$$ P_i's in equation are the values of a Poisson distribution with parameter ρ.

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Sol^n: Arrival rate $\lambda = 3$ ships / day

Service rate $\mu = 2$ ships / day

$$= \frac{1}{\mu} = \frac{1}{1/2} = 2$$

(a) $$L_s = \sum_{n=0}^{\infty} n.P_n = \sum_{n=0}^{\infty} n.\frac{\rho^n}{n!} e^{-\rho} = e^{-\rho}\left[\sum_{n=0}^{\infty} n.\frac{\rho^n}{n!}\right]$$

$$= e^{-\rho}\left[0 + \rho + \frac{2}{2!}\rho^2 + \frac{3}{3!}\rho^3 + \cdots \infty\right]$$

$$= e^{-\rho}\left[\rho + \rho^2 + \frac{1}{2!}\rho^3 + \cdots \infty\right]$$

$$= e^{-\rho}.\rho\left[1 + \rho + \frac{\rho^2}{2!} + \cdots + \infty\right]$$

e^x $x = \rho$

$$= e^{-\rho}.\rho.e^{\rho} = \rho = \frac{\lambda}{\mu} = \frac{3}{2} = 1.5$$

$$\boxed{P_n = \frac{\rho^n}{n!} e^{-\rho}} \Rightarrow P_i\text{'s in equation are the values of a Poisson distribution with parameter } \rho.$$

<u>Example</u>: A shipping company has a single unloading dock with ships ... at an average rate of <u>3/day</u>. The unloading

(b)

= probability that there are at least 5 ships in the system

$$= \sum_{n=5}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{4} P_n$$

$$= 1 - (P_0 + P_1 + P_2 + P_3 + P_4)$$

$$= 1 - e^{-\rho}\left[\frac{\rho^0}{0!} + \frac{\rho^1}{1!} + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \frac{\rho^4}{4!}\right]$$

$$= 1 - e^{-\rho}\left[1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \frac{\rho^4}{4!}\right]$$

$$= 1 - e^{-1.5}\left[1 + 1.5 + \frac{(1.5)^2}{2!} + \frac{(1.5)^3}{3!} + \frac{(1.5)^4}{4!}\right]$$

$$= 0.019$$

Model III (M/M/1):(N/FCFS)

Sandeep Kumar Gour

Probability $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$; $P_n = \dfrac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n$, for $n = 0, 1, 2, 3, \ldots N$

Measures of model III -

(i) $L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\left(\dfrac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$ or $L_s = \dfrac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n = P_0 \sum_{n=0}^{N} n\rho^n$

(ii) $L_q = L_s - \dfrac{\lambda}{\mu}$

(iii) $W_s = L_s / \lambda$

(iv) $W_q = W_s - \dfrac{1}{\mu} =$

Que:- Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find

(i) the probability that the yard is empty.

(ii) the average number of trains in the system.

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Model III $(M/M/1):(N/FCFS)$

Probability $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$; $P_n = \dfrac{1-\rho}{1-\rho^{N+1}} \cdot \rho^n$, for $n = 0, 1, 2, 3, \ldots N$

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(iii) $W_s = L_s / \lambda$

(iv) $W_q = W_s - \dfrac{1}{\mu} = \dfrac{L_s}{\lambda} - \dfrac{1}{\mu} = \dfrac{1}{\lambda}\left(L_s - \dfrac{\lambda}{\mu}\right) = L_q/\lambda$

Que: T[illegible]ns arrive at the yard every 15 minutes and the ser[illegible] time is 33 minutes. If the line capacity of the yar[illegible]s limited to 4 trains, find

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Sol^n: Arrival rate $\lambda = \frac{1}{15}$ trains/minute $\qquad 1/\lambda \quad 1/\mu$

Service rate $\mu = \frac{1}{33}$ trains/minute

$N = 4$

$\therefore$ Traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/33} = 2.2$

(i) $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-2.2}{1-(2.2)^{4+1}} = \frac{-1.2}{-50.536} = 0.0237$$

Sandeep Kumar Gour

Model III $(M/M/1):(N/FCFS)$

Sol^n: Arrival rate $\lambda = \frac{1}{15}$ trains/minute

Service rate $\mu = \frac{1}{33}$ trains/minute

$\left(\frac{1}{\lambda}\right) \quad \left(\frac{1}{\mu}\right)$

$$N = 4$$

$\therefore$ Traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/33} = 2.2$

(i) $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-2.2}{1-(2.2)^{4+1}} = \frac{-1.2}{-50.536} = 0.0237$$

(ii) $$L_s = P_0 \sum_{n=0}^{N=4} n\rho^n = P_0\left[0.\rho^0 + 1.\rho^1 + 2.\rho^2 + 3.\rho^3 + 4.\rho^4\right]$$

$$= (0.0237)\left[1(2.2) + 2(2.2)^2 + 3(2.2)^3 + 4(2.2)^4\right]$$

$$= (0.0237)\left[2.2 + 9.68 + 31.944 + 93.7024\right]$$

$$= 3.259$$

$$= 3.26$$

Model IV $(M/M/s):(\infty/FCFS)$

Sandeep Kumar Gour

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)\right]^{-1} \quad \text{where } \rho = \frac{\lambda}{s\mu}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 & , \text{ if } n = 0, 1, 2, \ldots\ldots s-1 \\ \frac{1}{s!} \cdot \frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 & , \text{ if } n = s, s+1, s+2, \ldots \end{cases}$$

Measures of Model IV

(i) $L_q = \frac{\rho (s\rho)^s}{s!(1-\rho)^2} \cdot P_0 = P_s \frac{\rho}{(1-\rho)^2}$

(ii) $L_s = L_q + \frac{\lambda}{\mu}$

(iii) $W_q = \frac{L_q}{\lambda} = P_s \cdot \frac{1}{s\mu(1-\rho)^2}$

(iv) $W_s = \frac{L_s}{\lambda} = W_q + \frac{1}{\mu}$

Model IV $(M/M/s):(\infty/FCFS)$

Sandeep Kumar Gour

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)\right]^{-1} \quad \text{where } \rho = \frac{\lambda}{s\mu}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{if } n = 0, 1, 2, \ldots\ldots s-1 \\ \frac{1}{s!}\cdot\frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, & \text{if } n = s, s+1, s+2, \ldots \end{cases}$$

$s\rho = \frac{\lambda}{\mu}$

Measures of Model IV

(i) $L_q = \frac{\rho(s\rho)^s}{s!(1-\rho)^2}\cdot P_0 = P_s \frac{\rho}{(1-\rho)^2}$

$$= \frac{\rho}{(1-\rho)^2}\cdot\frac{\left(\frac{\lambda}{\mu}\right)^s}{s!}\cdot P_0$$

$$= P_s \frac{\rho}{(1-\rho)^2}$$

(ii) $L_s = L_q + \frac{\lambda}{\mu}$

(iii) $W_q = \frac{L_q}{\lambda} = P_s \cdot \frac{1}{s\mu(1-\rho)^2}$

$$= P_s \cdot \frac{\cancel{\lambda}}{s\mu}\cdot\frac{1}{(1-\rho)^2}\cdot\frac{1}{\cancel{\lambda}}$$

$$= P_s \frac{1}{s\mu(1-\rho)^2}$$

(iv) $W_s = \frac{L_s}{\lambda} = W_q + \frac{1}{\mu}$

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Waiting time distribution

$$\text{Prob.}(T>t) = e^{-\mu t}\left[1+\frac{P_0(\lambda/\mu)^s}{s!(1-\rho)}\left(\frac{1-e^{-\mu t(s-1-\lambda/\mu)}}{s-1-\lambda/\mu}\right)\right]$$

Probability distribution of busy period

$$\text{Prob.}(W>0) = \frac{P_s}{1-\rho} \quad \text{where } P_s = \frac{(\lambda/\mu)^s P_0}{s!}$$

Efficiency of M/M/s Model

$$= \frac{\text{Average number of customers served}}{\text{Total number of customer served}}$$

Eg:- A telephone exchange has two long distance operators. The telephone company finds that, during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.

(a) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

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$\underline{\underline{Sol^n}}$: Here $s = 2$, arrival rate $\lambda = {}^{15}/_{60} = {}^{1}/_{4}$ call / minute

service rate $\mu = {}^{1}/_{5}$ call / minute

$\rho = \frac{\lambda}{s\mu}$

$\boxed{s\rho = \frac{\lambda}{\mu}}$

$$\therefore \frac{\lambda}{\mu} = \frac{1/4}{1/5} = \frac{5}{4} \quad \Big| \therefore \rho = \frac{\lambda}{s.\mu} = \frac{1}{s} \cdot \frac{\lambda}{\mu} = \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}$$

Therefore, first compute

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!} \left(\frac{\rho^s}{1-\rho} \right) \right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{(5/4)^n}{n!} + \frac{2^2}{2!} \left(\frac{(5/8)^2}{1 - 5/8} \right) \right]^{-1}$$

$$=$$

$(5/4)^0$

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service rate $\mu = 1/5$ call / minute

$\boxed{s\rho = \frac{\lambda}{\mu}}$

$$\therefore \frac{\lambda}{\mu} = \frac{1/4}{1/5} = \frac{5}{4} \quad \Big| \quad \therefore \rho = \frac{\lambda}{s.\mu} = \frac{1}{s}.\frac{\lambda}{\mu} = \frac{1}{2}.\frac{5}{4} = \frac{5}{8}$$

Therefore, first compute

$$\frac{(5/4)^0}{0!} = \frac{1}{1}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{(5/4)^n}{n!} + \frac{2^2}{2!}\left(\frac{(5/8)^2}{1-5/8}\right)\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{4}{2} \times \frac{25/64}{3/8}\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{\cancel{4}^{2}}{\cancel{2}} \times \frac{25}{\cancel{64}_{\cancel{8}\,4}} \times \frac{\cancel{8}}{3}\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + \frac{25}{12}\right]^{-1}$$

$$= \left[\frac{12 + 15 + 25}{12}\right]^{-1} = \left[\frac{52}{12}\right]^{-1} = \left[\frac{13}{3}\right]^{-1} = \frac{3}{13}$$

(a) $$\text{Prob}(W>0) = \frac{(\lambda/\mu)^s}{s!\,(1-\rho)} \cdot P_0$$

$$= \frac{(5/4)^2}{2!\,(1-5/8)} \cdot \frac{3}{13} = \frac{25}{\cancel{16}_{\,2}} \times \frac{\cancel{3}}{13} \times \frac{\cancel{8}}{2 \times \cancel{3}} = \frac{25}{52} = \underline{\underline{0.48}}$$

(b) $$W_q = \frac{L_q}{\lambda} = \frac{1}{\lambda} \cdot \frac{\rho\,(s\rho)^s}{s!\,(1-\rho)^2} \cdot P_0$$

$$= 4 \cdot \frac{5/8\,(5/4)^2}{2!\,(1-5/8)^2} \cdot \frac{3}{13}$$

$$= \overset{2}{\cancel{4}} \cdot \frac{5}{\cancel{8}} \times \frac{25}{\cancel{16}_{\,2}} \times \frac{\cancel{3}}{13} \times \frac{\cancel{64}\ \cancel{8}}{\cancel{2} \times \cancel{9}_{\,3}}$$

$$= \frac{125}{39} = 3.2$$

(a) $\text{Prob}(W>0) = \dfrac{(\lambda/\mu)^s}{s!\,(1-\rho)} \cdot P_0$

$$= \frac{(5/4)^2}{2!\,(1-5/8)} \cdot \frac{3}{13} = \frac{25}{\cancel{16}_{\,2}} \times \frac{\cancel{3}}{13} \times \frac{\cancel{8}}{2 \times \cancel{3}} = \frac{25}{52} = \underline{\underline{0.48}}$$

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$$= \overset{2}{\cancel{4}} \cdot \frac{5}{\cancel{8}} \times \frac{25}{\cancel{16}_{\,\cancel{2}}} \times \frac{\cancel{3}}{13} \times \frac{\cancel{64}\ \cancel{8}}{\cancel{2} \times \cancel{9}\ 3}$$

$$= \frac{125}{39} = 3.2 \text{ minutes}$$

Model V $(M/E_K/1):(\infty/FCFS)$

Sandeep Kumar Gour

Measures

(1) Average number of units in the system

$$L_s = \frac{K+1}{2K} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda} + \frac{\lambda}{\mu}$$

(2) Average number of units in the queue

$$L_q = \frac{K+1}{2K} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu-\lambda}$$

(3) Average time spent by a unit in the system

$$W_s = \frac{K+1}{2K} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

(4) Average waiting time of a unit in the queue

$$W_q = \frac{K+1}{2K} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$$

Qu:- Repairing a certain type of machine which breaks down in a given factory consists of 5 basic steps that must be performed sequentially. The time taken to perform each of the 5 steps is

14:- Repairing a certain type of machine which breaks down in a given factory consists of 5 basic steps that must be performed sequentially. The time taken to perform each of the 5 steps is found to have an exponential distibution with mean 5 minutes and is independent of other steps. If these machines break down in a Poisson fashion at an average rate of two per hour and

if there is only one repairman, what is the average idle time for each machine that has broken down?

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✓ ✓ - - 5

000 → $\boxed{SP}$ → 000 – $\boxed{SP}$

Solution Here, no. of phases $K = 5$

service time per phase = 5 minutes

$\therefore$ service time per unit $= 5 \times 5 = 25$ minute

$\therefore$ service rate $\mu = \frac{1}{25}$ units/minute

and arrival rate $\lambda = 2$ units/hour

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000 → SP → 000 → SP ... (5)

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service time per phase $= 5$ minutes

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$\therefore$ service rate $\mu = \frac{1}{25}$ units/minute $= \frac{1}{25} \times 60 = \frac{12}{5}$ units/hour

and arrival rate $\lambda = 2$ units/hour service time per phase = 5 minutes

$\therefore$ service time per unit $= 5 \times 5 = \underline{25}$ minute

$\therefore$ service rate $\mu = \frac{1}{25}$ units/minute $= \frac{1}{\cancel{25}_5} \times \cancel{60}^{12} = \frac{12}{5}$ units/

and arrival rate $\lambda = 2$ units/hour

Now Average idle time for each machine

= Average time spent by the machine in the system

$$W_s = \frac{K+1}{2K} \cdot \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

$$= \frac{5+1}{2\times 5} \cdot \frac{2}{\frac{12}{5}\left(\frac{12}{5}-2\right)} + \frac{1}{12/5}$$

$$= \frac{6}{10} \cdot \frac{2}{12/5\,(2/5)} + \frac{5}{12}$$

$$= \frac{\cancel{6}}{\cancel{10}_2} \times \frac{2 \times \cancel{25}\,5}{\cancel{24}\,\cancel{12}_2} + \frac{5}{12}$$

$$= \frac{5}{4} + \frac{5}{12}$$

$$= \frac{15+5}{12}$$

$$= \frac{20}{12}$$

$$= \frac{5}{3} \text{ hour}$$

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$$= \frac{5+1}{2\times5} \cdot \frac{2}{\frac{12}{5}\left(\frac{12}{5}-2\right)} + \frac{1}{12/5}$$

$$= \frac{6}{10} \cdot \frac{2}{12/5\,(2/5)} + \frac{5}{12}$$

$$= \frac{6}{10} \times \frac{2 \times 25}{24} + \frac{5}{12}$$

$$= \frac{5}{4} + \frac{5}{12}$$

$$= \frac{15+5}{12}$$

$$= \frac{20}{12}$$

$$= \frac{5}{3} \text{ hour} = \frac{5}{3} \times 60 = 100 \text{ minu}$$

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