

**UNIT II****LESSON****1 TRANSPORTATION PROBLEM****LESSON STRUCTURE**

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**Objectives**

After Studying this lesson, you should be able to:

- ❖ Formulation of a Transportation Problem
- ❖ Determine basic feasible solution using various methods
- ❖ Understand the MODI, Stepping Stone Methods for cost minimization
- ❖ Make unbalanced Transportation Problem into balanced one using appropriate method
- ❖ Solve Degenerate Problem
- ❖ Formulate and Solve Transshipment Problem
- ❖ Describe suitable method for maximizing the objective function instead of minimizing

## 1.1 Introduction

A special class of linear programming problem is **Transportation Problem**, where the objective is to minimize the cost of distributing a product from a number of **sources** (e.g. factories) to a number of **destinations** (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The transportation problem special feature is illustrated here with the help of following Example 1.1.

### Example 1.1:

Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unit  
 from factory 1 to retail agency 1 is 1,  
 from factory 1 to retail agency 2 is 9,  
 from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as linear programming problem using variables with two subscripts.

Let

$x_{11}$ =Amount to be transported from factory 1 to retail agency 1  
 $x_{12}$ = Amount to be transported from factory 1 to retail agency 2  
 .....  
 .....  
 .....  
 .....  
 $x_{35}$ = Amount to be transported from factory 3 to retail agency 5.

Let the transportation cost per unit be represented by  $C_{11}, C_{12}, \dots, C_{35}$  that is  $C_{11}=1, C_{12}=9$ , and so on.

Let the capacities of the three factories be represented by  $a_1=50, a_2=100, a_3=150$ .

Let the requirement of the retail agencies are  $b_1=100, b_2=60, b_3=50, b_4=50$ , and  $b_5=40$ .

Thus, the problem can be formulated as

Minimize

$$C_{11}x_{11} + C_{12}x_{12} + \dots + C_{35}x_{35}$$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{15} + x_{25} + x_{35} = b_5$$

$$x_{11}, x_{12}, \dots, x_{35} \geq 0.$$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

## 1.2 Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

**Step 1:** Determine a starting basic feasible solution, using any one of the following three methods

1. North West Corner Method
2. Least Cost Method
3. Vogel Approximation Method

**Step 2:** Determine the optimal solution using the following method

1. MODI (Modified Distribution Method) or UV Method.

## 1.3 Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example 1.1 discussed in the previous section. The problem in the example 1.1 has 8 constraints and 15 variables we can eliminate one of the constraints since  $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$ . Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero  $X_{ij}$ . Generally, any basic feasible solution with  $m$  sources (such as factories) and  $n$  destination (such as retail agency) has at most  $m + n - 1$  non-zero  $X_{ij}$ .

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

4. North West Corner Method
5. Least Cost Method
6. Vogel Approximation Method

The difference among these three methods is the **quality** of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the **best** initial basic feasible solution, and the North West Corner Method produces the **worst**, but the North West Corner Method involves least computations.

#### North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable  $x_{11}$ ).

**Step -1:** Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

**Step -2:** Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

**Step -3:** If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step -1.

#### Example 1.2:

Consider the problem discussed in Example 1.1 to illustrate the North West Corner Method of determining basic feasible solution.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

The allocation is shown in the following tableau:

	Capacity					
	1	9	13	36	51	<del>50</del>
<del>50</del>	24	12	16	20	1	<del>100 50</del>
<del>50</del>	14	33	1	23	26	<del>150 140 90 40</del>
Requirement	<del>100</del>	<del>60</del>	<del>50</del>	<del>50</del>	<del>40</del>	
	<del>50</del>	<del>10</del>				

The arrows show the order in which the allocated (**bolded**) amounts are generated. The starting basic solution is given as

$$\begin{aligned}
 x_{11} &= 50, \\
 x_{21} &= 50, x_{22} = 50 \\
 x_{32} &= 10, x_{33} = 50, x_{34} = 50, x_{35} = 40
 \end{aligned}$$

The corresponding transportation cost is

$$50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$$

It is clear that as soon as a value of  $X_{ij}$  is determined, a row (column) is eliminated from further consideration. The last value of  $X_{ij}$  eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most  $m + n - 1$  positive  $X_{ij}$  if the transportation problem has  $m$  sources and  $n$  destinations.

### Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which  $C_{ij}$  is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

**Example 1.3:**

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the section 1.1.

						<b>Capacity</b>
	1	9	13	36	51	<del>50</del>
<b>50</b>						
	24	12	16	20	1	<del>100</del> <del>60</del>
		<b>60</b>			<b>40</b>	
	14	33	1	23	26	
<b>50</b>			<b>50</b>	<b>50</b>		<del>150</del> <del>100</del> <del>50</del>
<b>Requirement</b>	<b>100</b>	<del><b>60</b></del>	<del><b>50</b></del>	<del><b>50</b></del>	<del><b>40</b></del>	
	<del><b>50</b></del>					

The Least Cost method is applied in the following manner:

We observe that  $C_{11}=1$  is the minimum unit cost in the table. Hence  $X_{11}=50$  and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is  $C_{25}=1$ , hence  $X_{25}=40$  and the fifth column is crossed out. Next  $C_{33}=1$  is the minimum unit cost, hence  $X_{33}=50$  and the third column is crossed out. Next  $C_{22}=12$  is the minimum unit cost, hence  $X_{22}=60$  and the second column is crossed out. Next we look for the uncrossed-out row and column now  $C_{31}=14$  is the minimum unit cost, hence  $X_{31}=50$  and crossed out the first column since it was satisfied. Finally  $C_{34}=23$  is the minimum unit cost, hence  $X_{34}=50$  and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is

$$1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710$$

**Note** that the minimum transportation cost obtained by the least cost method is much lower than the corresponding cost of the solution developed by using the north-west corner method.

**Vogel Approximation Method (VAM):**

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

**Step 1:** For each row (column) with strictly positive capacity (requirement), determine a **penalty** by subtracting the **smallest** unit cost element in the row (column) from the next **smallest** unit cost element in the same row (column).

**Step 2:** Identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

**Step 3:** We select  $X_{ij}$  as a basic variable if  $C_{ij}$  is the **minimum cost** in the row or column with **largest penalty**. We choose the numerical value of  $X_{ij}$  as high as possible subject to the row and the column constraints. Depending upon whether  $a_i$  or  $b_j$  is the smaller of the two  $i^{\text{th}}$  row or  $j^{\text{th}}$  column is crossed out.

**Step 4:** The Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

### Example 1.4:

Consider the following transportation problem

Origin	Destination				$a_i$
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
$b_j$	60	40	30	110	240

Note:  $a_i$ =capacity (supply)

$b_j$ =requirement (demand)

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the **second column** and the minimum unit cost i.e.  $c_{ij}$  in this column is  $c_{12}=22$ . Hence assign 40 to this cell i.e.  $x_{12}=40$  and cross out the second column (since second column was satisfied. This is shown in the following table:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	20	22 <b>40</b>	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	

Row Penalty	4	15	8	3	
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The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the **first row** and the minimum unit cost in this row is  $c_{14}=4$ , hence  $x_{14}=80$  and cross out the first row. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del>	17	<del>4</del>	0	13
		<b>40</b>		<b>80</b>		
2	24	37	9	7	70	2
3	32	37	20	15	50	5
$b_j$	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the **third column** and the **minimum cost** in this column is  $c_{23}=9$ , hence  $x_{23}=30$  and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del>	<del>17</del>	<del>4</del>	0	13
		<b>40</b>		<b>80</b>		
2	24	37	<del>9</del>	7	40	17
			<b>30</b>			
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is  $c_{24}=15$ , hence  $x_{24}=30$  and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del>	<del>17</del>	<del>4</del>	0	13
		<b>40</b>		<b>80</b>		
2	24	37	<del>9</del>	<del>7</del>	10	17
			<b>30</b>	<b>30</b>		
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	



Row Penalty	8	15	8	8	
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The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **second row** and the **smallest cost** in this row is  $c_{21}=24$ , hence  $x_{21}=10$  and cross out the second row with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del> <b>40</b>	<del>17</del>	<del>4</del> <b>80</b>	0	13
2	<del>24</del> <b>10</b>	<del>37</del>	<del>9</del> <b>30</b>	<del>7</del> <b>30</b>	0	17
3	32	37	20	15	50	17
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the **third row** and the **smallest cost** in this row is  $c_{31}=32$ , hence  $x_{31}=50$  and cross out the third row or first column. The modified table is as follows:

Origin	Destination				$a_i$	Column Penalty
	1	2	3	4		
1	<del>20</del>	<del>22</del> <b>40</b>	<del>17</del>	<del>4</del> <b>80</b>	0	13
2	<del>24</del> <b>10</b>	<del>37</del>	<del>9</del> <b>30</b>	<del>7</del> <b>30</b>	0	17
3	<del>32</del> <b>50</b>	<del>37</del>	<del>20</del>	<del>15</del>	0	17
$b_j$	60	40	30	110	240	
Row Penalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is

$$22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520$$

#### 1.4 Modified Distribution Method

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

**Step 1:** Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

**Step 2:** Introduce dual variables corresponding to the row constraints and the column constraints. If there are  $m$  origins and  $n$  destinations then there will be  $m+n$  dual variables. The dual variables corresponding to the row constraints are represented by  $u_i$ ,  $i=1,2,\dots,m$  where as the dual variables corresponding to the column constraints are represented by  $v_j$ ,  $j=1,2,\dots,n$ . The values of the dual variables are calculated from the equation given below

$$u_i + v_j = c_{ij} \quad \text{if } x_{ij} > 0$$

**Step 3:** Any basic feasible solution has  $m + n - 1$   $x_{ij} > 0$ . Thus, there will be  $m + n - 1$  equation to determine  $m + n$  dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

**Step 4:** If  $x_{ij}=0$ , the dual variables calculated in Step 3 are compared with the  $c_{ij}$  values of this allocation as  $c_{ij} - u_i - v_j$ . If all  $c_{ij} - u_i - v_j \geq 0$ , then by the *theorem of complementary slackness* it can be shown that the corresponding solution of the transportation problem is optimum. If one or more  $c_{ij} - u_i - v_j < 0$ , we select the cell with the least value of  $c_{ij} - u_i - v_j$  and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

**Step 5:** A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

#### Example 1.5:

For example consider the transportation problem given below:

	<b>Supply</b>					
	1	9	13	36	51	<b>50</b>
	24	12	16	20	1	<b>100</b>
	14	33	1	23	26	<b>150</b>
<b>Demand</b>	<b>100</b>	<b>70</b>	<b>50</b>	<b>40</b>	<b>40</b>	<b>300</b>

**Step 1:** First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

$$x_{11}=50, x_{22}=60, x_{25}=40, x_{31}=50, x_{32}=10, x_{33}=50 \text{ and } x_{34}=40$$

**Step 2:** The dual variables  $u_1, u_2, u_3$  and  $v_1, v_2, v_3, v_4, v_5$  can be calculated from the corresponding  $c_{ij}$  values, that is

$$\begin{array}{llll} u_1 + v_1 = 1 & u_2 + v_2 = 12 & u_2 + v_5 = 1 & u_3 + v_1 = 14 \\ u_3 + v_2 = 33 & u_3 + v_3 = 1 & u_3 + v_4 = 23 & \end{array}$$

**Step 3:** Choose one of the dual variables arbitrarily is zero that is  $u_3=0$  as it occurs most often in the above equations. The values of the variables calculated are

$$u_1 = -13, u_2 = -21, u_3 = 0$$

$$v_1 = 14, v_2 = 33, v_3 = 1, v_4 = 23, v_5 = 22$$

**Step 4:** Now we calculate  $c_{ij} - u_i - v_j$  values for all the cells where  $x_{ij}=0$  (.e. unallocated cell by the basic feasible solution)

That is

$$\begin{aligned}\text{Cell}(1,2) &= c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11 \\ \text{Cell}(1,3) &= c_{13} - u_1 - v_3 = 13 + 13 - 1 = 25 \\ \text{Cell}(1,4) &= c_{14} - u_1 - v_4 = 36 + 13 - 23 = 26 \\ \text{Cell}(1,5) &= c_{15} - u_1 - v_5 = 51 + 13 - 22 = 42 \\ \text{Cell}(2,1) &= c_{21} - u_2 - v_1 = 24 + 21 - 14 = 31 \\ \text{Cell}(2,3) &= c_{23} - u_2 - v_3 = 16 + 21 - 1 = 36 \\ \text{Cell}(2,4) &= c_{24} - u_2 - v_4 = 20 + 21 - 23 = 18 \\ \text{Cell}(3,5) &= c_{35} - u_3 - v_5 = 26 - 0 - 22 = 4\end{aligned}$$

Note that in the above calculation all the  $c_{ij} - u_i - v_j \geq 0$  except for cell (1, 2) where  $c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11$ .

Thus in the next iteration  $x_{12}$  will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is

$$-33 - 1 + 9 + 14 = -11$$

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$$x_{11}=40, x_{12}=10, x_{22}=60, x_{25}=40, x_{31}=60, x_{33}=50, x_{34}=40$$

### 1.5 Unbalanced Transportation Problem

In the previous section we discussed about the balanced transportation problem i.e. the total supply (capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as **unbalanced transportation problem**.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

#### Example 1.6:

Consider the following unbalanced transportation problem

#### Warehouses

Plant	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	Supply
X	20	17	25	400
Y	10	10	20	500
<b>Demand</b>	400	400	500	

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

Plant	Warehouses			Supply
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	
X	20	17	25	400
Y	10	10	20	500
<b>Unsatisfied Demand</b>	0	0	0	400
<b>Demand</b>	400	400	500	1300

Now we can solve as balanced problem discussed as in the previous sections.

### 1.6. Degenerate Transportation Problem

In a transportation problem, if a basic feasible solution with  $m$  origins and  $n$  destinations has less than  $m + n - 1$  positive  $X_{ij}$  i.e. occupied cells, then the problem is said to be a **degenerate transportation problem**. The degeneracy problem does not cause any serious difficulty, but it can cause computational problem while determining the optimal minimum solution.

Therefore it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty. The degeneracy can be identified through the following results:

“In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example the following transportation problem is degenerate. Because in this problem

$$a_1 = 400 = b_1$$

$$a_2 + a_3 = 900 = b_2 + b_3$$

Plant	Warehouses			Supply (a <sub>i</sub> )
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	

<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: right; padding-right: 10px;">X</div> <div style="border: 1px solid black; padding: 5px;">20</div> <div style="border: 1px solid black; padding: 5px;">17</div> <div style="border: 1px solid black; padding: 5px;">25</div> <div style="margin-left: 10px;">400</div> </div>				
Y	10	10	20	500
Unsatisfied demand	0	0	0	400
Demand (b <sub>j</sub> )	400	400	500	1300

There is a technique called perturbation, which helps to solve the degenerate problems.

### Perturbation Technique:

The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of  $a_i$  (supply) and  $b_j$  (demand) is equal. We set up a new problem where

$$\begin{aligned}
 a_i &= a_i + d & i &= 1, 2, \dots, m \\
 b_j &= b_j & j &= 1, 2, \dots, n-1 \\
 b_n &= b_n + m_d & d &> 0
 \end{aligned}$$

This modified problem is constructed in such a way that no partial sum of  $a_i$  is equal to the  $b_j$ . Once the problem is solved, we substitute  $d = 0$  leading to optimum solution of the original problem.

### Example: 1.7

Consider the above problem

Plant	Warehouses			Supply (a <sub>i</sub> )
	w <sub>1</sub>	w <sub>2</sub>	w <sub>3</sub>	
X	20	17	25	400 + d
Y	10	10	20	500 + d
Unsatisfied demand	0	0	0	400 + d
Demand (b <sub>j</sub> )	400	400	500 + 3d	1300 + 3d

Now this modified problem can be solved by using any of the three methods viz. North-west Corner, Least Cost, or VAM.

### 1.7 Transshipment Problem

There could be a situation where it might be more economical to transport consignments in several stages that is initially within certain origins and destinations and finally to the ultimate receipt points, instead of transporting the consignments from an origin to a destination as in the transportation problem.

The movement of consignment involves two different modes of transport viz. road and railways or between stations connected by metre gauge and broad gauge lines. Similarly it is not uncommon to maintain dumps for central storage of certain bulk material. These require transshipment.

Thus for the purpose of transshipment the distinction between an origin and destination is dropped so that from a transportation problem with  $m$  origins and  $n$  destinations we obtain a transshipment problem with  $m + n$  origins and  $m + n$  destinations.

The formulation and solution of a transshipment problem is illustrated with the following Example 1.8.

### Example 1.8:

Consider the following transportation problem where the origins are plants and destinations are depots.

Table 1

Plant	Depot			Supply
	X	Y	Z	
A	\$1	\$3	\$15	150
B	\$3	\$5	\$25	300
<b>Demand</b>	150	150	150	450

When each plant is also considered as a destination and each depot is also considered as an origin, there are altogether five origins and five destinations. So that some additional cost data are necessary, they are as follows:

**Table 2**  
Unit transportation cost From Plant To Plant

From	To	
	Plant A	Plant B
Plant A	0	55
Plant B	2	0

**Table 3**  
Unit transportation cost From Depot To Depot

To		
Depot X	Depot Y	Depot Z

**From**

Depot X

Depot Y

Depot Z

**Table 4**

0	25	2
2	0	3
55	3	0

Unit transportation cost From Depot to Plant

	<b>To</b>	
	Plant A	Plant B
<b>From</b>		
Depot X	3	15
Depot Y	25	3
Depot Z	45	55

Now, from the Table 1, Table 2, Table 3, Table 4 we obtain the transportation formulation of the transshipment problem, which is shown in the Table 5.

**Table 5**  
**Transshipment Table**

						Supply
	A	B	X	Y	Z	
A	0	55	1	3	15	150+450=600
B	2	0	3	5	25	300+450=750
X	3	15	0	25	2	450
Y	25	3	2	0	3	450
Z	45	55	55	3	0	450
Demand	450	450	150+450=600	150+450=600	150+450=600	

A buffer stock of 450 which is the total supply and total demand in the original transportation problem is added to each row and column of the transshipment problem. The resulting transportation problem has  $m + n = 5$  origins and  $m + n = 5$  destinations.

By solving the transportation problem presented in the Table 5, we obtain

$$\begin{array}{llllll}
 x_{11}=150 & x_{13}=300 & x_{14}=150 & x_{21}=300 & x_{22}=450 & x_{33}=300 \\
 x_{35}=150 & x_{44}=450 & x_{55}=450 & & & 
 \end{array}$$

The transshipment problem explanation is as follows:

1. Transport  $x_{21}=300$  from plant B to plant A. This increase the availability at plant A to 450 units including the 150 originally available from A.
2. From plant A transport  $x_{13}=300$  to depot X and  $x_{14}=150$  to depot Y.
3. From depot X transport  $x_{35}=150$  to depot Z.

Thus, the total cost of transshipment is:

$$2*300 + 3 * 150 + 1*300 + 2*150 = \$1650$$

Note: The consignments are transported from plants A, B to depots X, Y, Z only according to the transportation Table 1, the minimum transportation cost schedule is  $x_{13}=150$   $x_{21}=150$   $x_{22}=150$  with a minimum cost of 3450.

Thus, transshipment **reduces** the cost of consignment movement.

### 1.8 Transportation Problem Maximization

There are certain types of transportation problem where the objective function is to be maximized instead of minimized. These kinds of problems can be solved by converting the maximization problem into minimization problem. The conversion of maximization into minimization is done by subtracting the unit costs from the highest unit cost of the table.

The maximization of transportation problem is illustrated with the following Example 1.9.

#### Example 1.9:

A company has three factories located in three cities viz. X, Y, Z. These factories supply consignments to four dealers viz. A, B, C and D. The dealers are spread all over the country. The production capacity of these factories is 1000, 700 and 900 units per month respectively. The net return per unit product is given in the following table.

Factory	Dealers				capacity
	A	B	C	D	
X	6	6	6	4	1000
Y	4	2	4	5	700
Z	5	6	7	8	900
<b>Requirement</b>	900	800	500	400	2600

Determine a suitable allocation to maximize the total return.



This is a maximization problem. Hence first we have to convert this in to minimization problem. The conversion of maximization into minimization is done by subtracting the unit cost of the table from the highest unit cost.

Look the table, here 8 is the highest unit cost. So, subtract all the unit cost from the 8, and then we get the revised minimization transportation table, which is given below.

Factory	Dealers				capacity
	A	B	C	D	
X	2	2	2	4	1000 = a <sub>1</sub>
Y	4	6	4	3	700 = a <sub>2</sub>
Z	3	2	1	0	900 = a <sub>3</sub>
<b>Requirement</b>	900=b <sub>1</sub>	800=b <sub>2</sub>	500=b <sub>3</sub>	400=b <sub>4</sub>	2600

Now we can solve the problem as a minimization problem.

The problem here is degenerate, since the partial sum of  $a_1=b_2+b_3$  or  $a_3=b_3$ . So consider the corresponding perturbed problem, which is shown below.

Factory	Dealers				capacity
	A	B	C	D	
X	2	2	2	4	1000+d
Y	4	6	4	3	700+d
Z	3	2	1	0	900+d
<b>Requirement</b>	900	800	500	400+3d	2600+3d

First we have to find out the basic feasible solution. The basic feasible solution by lest cost method is  $x_{11}=100+d$ ,  $x_{22}=700-d$ ,  $x_{23}=2d$ ,  $x_{33}=500-2d$  and  $x_{34}=400+3d$ .

Once if the basic feasible solution is found, next we have to determine the optimum solution using MODI (Modified Distribution Method) method. By using this method we obtain

$$\begin{array}{lll} u_1+v_1=2 & u_1+v_2=2 & u_2+v_2=6 \\ u_2+v_3=4 & u_3+v_3=1 & u_3+v_4=0 \end{array}$$

Taking  $u_1=0$  arbitrarily we obtain

$$u_1=0, u_2=4, u_3=1 \text{ and } v_1=2, v_2=3, v_3=0$$

On verifying the condition of optimality, we know that

$$C_{12}-u_1-v_2 < 0 \quad \text{and} \quad C_{32}-u_3-v_2 < 0$$

So, we allocate  $x_{12}=700-d$  and make readjustment in some of the other basic variables.

The revised values are:

$$x_{11}=200+d, x_{12}=800, x_{21}=700-d, x_{23}=2d, x_{33}=500-3d, \text{ and } x_{34}=400+3d$$

$$\begin{array}{lll} u_1+v_1=2 & u_1+v_2=2 & u_2+v_1=4 \\ u_2+v_3=4 & u_3+v_3=1 & u_3+v_4=0 \end{array}$$

Taking  $u_1=0$  arbitrarily we obtain

$$\begin{array}{l} u_1=0, u_2=2, u_3=-1 \\ v_1=2, v_2=2, v_3=2, v_4=1 \end{array}$$

Now, the optimality condition is satisfied.

Finally, taking  $d=0$  the optimum solution of the transportation problem is

$$x_{11}=200, x_{12}=800, x_{21}=700, x_{33}=500 \text{ and } x_{34}=400$$

Thus, the maximum return is:

$$6*200 + 6*800 + 4*700 + 7*500 + 8*400 = 15500$$

## 1.9 Summary

Transportation Problem is a special kind of linear programming problem. Because of the transportation problem special structure the simplex method is not suitable. But which may be utilized to make efficient computational techniques for its solution.

Generally transportation problem has a number of origins and destination. A certain amount of consignment is available in each origin. Similarly, each destination has a certain demand/requirements. The transportation problem represents amount of consignment to be transported from different origins to destinations so that the transportation cost is minimized with out violating the supply and demand constraints.

There are two phases in the transportation problem. First is the determination of basic feasible solution and second is the determination of optimum solution.

There are three methods available to determine the basic feasible solution, they are

1. North West Corner Method
2. Least Cost Method or Matrix Minimum Method
3. Vogel's Approximation Method (VAM)

In order to determine optimum solution we can use either one of the following method

1. Modified Distribution (MODI) Method
- Or
2. Stepping Stone Method

Transportation problem can be generalized into a Transshipment Problem where transportation of consignment is possible from origin to origin or destination as well as destination to origin or

destination. The transshipment problem may be result in an economy way of shipping in some situations.

### 1.10 Key Terms

**Origin:** is the location from which the shipments are dispatched.

**Destination:** is the location to which the shipments are transported.

**Unit Transportation Cost:** is the transportation cost per unit from an origin to destination.

**Perturbation Technique:** is a method of modifying a degenerate transportation problem in order to solve the degeneracy.

### 1.11 Self Assessment Questions

Q1. Four companies viz. W, X, Y and Z supply the requirements of three warehouses viz. A, B and C respectively. The companies' availability, warehouses requirements and the unit cost of transportation are given in the following table. Find an initial basic feasible solution using

- North West Corner Method
- Least Cost Method
- Vogel Approximation Method (VAM)

Company	Warehouses			Supply
	A	B	C	
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35
Requirement	25	26	49	100

Q2. Find the optimum Solution of the following Problem using MODI method.

Source	Destination			Capacity
	1	2	3	
A 42	8	9	10	30
B	9	11	11	
C	10	12	9	28
Demand	35	40	25	100

Q3. The ABT transport company ships truckloads of food grains from three sources viz. X, Y, Z to four mills viz. A, B, C, D respectively. The supply and the demand together with the unit transportation cost per truckload on the different routes are described in the following transportation table. Assume that the unit transportation costs are in hundreds of dollars. Determine the optimum minimum shipment cost of transportation using MODI method.

Source	Mill				Supply
	A	B	C	D	
X	10	2	20	11	15
Y	12	7	9	20	25
Z	4	14	16	18	10
<b>Demand</b>	5	15	15	15	

Q4. An organization has three plants at X, Y, Z which supply to warehouses located at A, B, C, D, and E respectively. The capacity of the plants is 800, 500 and 900 per month and the requirement of the warehouses is 400, 400, 500, 400 and 800 units respectively. The following table shows the unit transportation cost.

	A	B	C	D	E
X	\$5	\$8	\$6	\$6	\$3
Y	\$4	\$7	\$7	\$6	\$6
Z	\$8	\$4	\$6	\$6	\$3

Determine an optimum distribution for the organization in order to minimize the total cost of transportation.

Q5. Solve the following transshipment problem

Consider a transportation problem has two sources and three depots. The availability, requirements and unit cost are as follows:

Source	Depot			Availability
	D1	D2	D3	
S1	9	8	1	30
S2	1	7	8	30
<b>Requirement</b>	20	20	20	<b>60</b>

In addition to the above, suppose that the unit cost of transportation from source to source and from depot to depot are as:

**Source**

		S1	S2
<b>Source</b>	S1	0	1
	S2	2	0

  

		<b>Depot</b>		
		D1	D2	D3
<b>Depot</b>	D1	0	2	1
	D2	2	0	9
	D3	1	9	0

Find out minimum transshipment cost of the problem and also compare this cost with the corresponding minimum transportation cost.

Q6. Saravana Store, T.Nagar, Chennai interested to purchase the following type and quantities of dresses

Dress Type	V	W	X	Y	Z
Quantity	150	100	75	250	200

Four different dress makers are submitted the tenders, who undertake to supply not more than the quantities indicated below:

Dress Maker	A	B	C	D
Dress Quantity	300	250	150	200

Saravana Store estimates that its profit per dress will vary according to the dress maker as indicates in the following table:

	V	W	X	Y	Z
A	2.75	3.5	4.25	2.25	1.5
B	3	3.25	4.5	1.75	1
C	2.5	3.5	4.75	2	1.25
D	3.25	2.75	4	2.5	1.75

Determine how should the orders to be places for the dresses so as to maximize the profit.

### 1.12 Key Solutions

Q1. a.  $x_{11} = 15$ ,  $x_{21} = 10$ ,  $x_{22} = 10$ ,  $x_{32} = 16$ ,  $x_{33} = 14$ ,  $x_{43} = 35$   
Minimum Cost is: 753

b.  $x_{13} = 15$ ,  $x_{22} = 20$ ,  $x_{33} = 30$ ,  $x_{41} = 25$ ,  $x_{42} = 6$ ,  $x_{43} = 4$   
Minimum Cost is: 542

c.  $x_{13}=15$ ,  $x_{22}=20$ ,  $x_{33}=30$ ,  $x_{41}=25$ ,  $x_{42}=6$ ,  $x_{43}=4$   
Minimum Cost is: 542

Q2.  $x_{11}=2$ ,  $x_{12}=40$ ,  $x_{21}=30$ ,  $x_{31}=3$ ,  $x_{33}=25$   
Minimum Transportation Optimal cost is: 901.

Q3.  $x_{12}=5$ ,  $x_{14}=10$ ,  $x_{22}=10$ ,  $x_{23}=15$ ,  $x_{31}=5$ ,  $x_{34}=5$   
Minimum Optimal Cost is: \$435

Q4.  $x_{15}=800$ ,  $x_{21}=400$ ,  $x_{24}=100$ ,  $x_{32}=400$ ,  $x_{33}=200$ ,  $x_{34}=300$ ,  $x_{43}=300$  (supply shortage)  
Minimum Cost of Transportation is: \$9200

Q5. Transportation Problem  
 $S1-D2=10$ ,  $S1-D3=20$ ,  $S2-D1=20$ ,  $S2-D2=10$  and  
Minimum Transportation Cost is: 100

Transshipment Problem  
 $x_{11}=60$ ,  $x_{12}=10$ ,  $x_{15}=20$ ,  $x_{22}=50$ ,  $x_{23}=40$ ,  $x_{33}=40$ ,  $x_{34}=20$ ,  $x_{44}=60$ ,  $x_{55}=60$  and  
Minimum Transshipment Cost is: 100

Q6. 150 dresses of V and 50 dresses of Z by Dress Maker A  
250 dresses of Y by Dress Maker B  
150 dress of Z by Dress Maker C  
100 dress of W and 75 dresses of X by Dress Maker D

Maximum Profit is: 1687.50

### 1.13 Further References

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