OPERATIONS RESEARCH

INTRODUCTION TO OPERATION RESEARCH

Multiple Choice Type Questions

1. **Operations Research deals with problems associated with** **[WBUT 2013]**
   a) scarce resources b) goods under transportation
   c) waiting line d) all of these

Answer: (d)

Short Answer Type Questions

1. a) Narrate briefly the history of development of Operations Research. **[WBUT 2013]**

Answer:
Along the history, is frequent to find collaboration among scientists and army officers with the same objective, ruling the optimal decision in battle. In fact that many experts consider the start of Operational Research in the III century B.C., during the II Punic War, with analysis and solution that Archimedes named for the defense of the city of Syracuse, besieged for Romans. Enter his inventions would find the catapult, and a system of mirrors that was setting to fire the enemy boats by focusing them with the Sun's rays.

Leonardo DaVinci took part, in 1503, like engineer in the war against Prisa due to he knew techniques to accomplish bombardments, to construct ships, armored vehicles, cannons, catapults, and another warlike machine.

Another antecedent of use of Operational Research obeys to F.W. Lanchester, who made a mathematical study about the ballistic potency of opponents and he developed, from a system of equations differential, Lanchester's Square Law, with that can be available to determine the outcome of a military battle.

Thomas Edison made use of Operational Research, contributing in the antisubmarine war, with his greats ideas, like shields against torpedo for the ships.

From the mathematical point of view mathematical, in centuries XVII and XVIII, Newton, Leibnitz, Bernoulli and Lagrange, worked in obtaining maximums and minimums conditioned of certain functions. Mathematical French Jean Baptiste Joseph Fourier sketched methods of present-day Linear Programming. And at late years of the century XVIII, Gaspar Monge laid down the precedents of the Graphical Method thanks to his development of Descriptive Geometry.

Janos Von Neumann published his work called "Theory of Games", that provided the basics Mathematicians to Linear Programming. At a later time, in 1947, he viewed the similitude among them Programming linear problems and the matrix theory that developed himself.

In 1939, mathematical Russian L. Kantorovich, in association with the mathematical Dutchman T. Koopmans, developed the mathematical theory called "Linear programming", thanks to that went rewarded with the Nobel.

In the late years 30, George Joseph Stigler presented a particular problem known as special diet optimal or more commonly known as problem of the diet, which came about by the worry of the USA army to guarantee some nutritional requests for his troops to the lowest price. It was solved with a heuristic method which solution only differs in some centimes against the solution contributed years later by the Simplex Method.

During the years 1941 and 1942, Kantorovich and Koopmans studied in independent ways the Transport Problem for first time, knowing this type of problems like problem of Koopmans-Kantorovich. For his solution, they used geometric methods that are related to Minkowski's theorem of convexity.

It is believed that Charles Babbage is the father of the Operational Research due to his research about the transportation's costs and sorting of mail realized for the Uniform Penny Post in in England in 1840.

But it does not considered than has been born a new science called Operations Research until the II World War, during battle of England, where Deutsche Air Force, that is the Luftwaffe, was submitting the Britishers to a hard Air raid, since these had an little aerial capability, although experimented in the Combat. The British government, looking for some method to defend his country, convoked several scientists of various disciplines for try to resolve the problem to get the peak of benefit of radars that they had. Thanks to his work determining the optimal localization of antennas and they got the best distribution of signals to double the effectiveness of the system of aerial defense.

To notice the range of this new discipline, England created another groups of the same nature in order to obtain optimal results in the dispute. Just like United States (USA), when joined the War in 1942, creating the project SCOOP (Scientific Computation Of Optimum Programs), where was working George Bernard Dantzig, who developed in 1947 the Simplex algorithm.

During the Cold War, the old Soviet Union (URRS), excluded of the Plan Marshall, wanted to control the terrestrial communications, including routes fluvial, from Berlin. In order to avoid the rendition of the city, and his submission to be a part of the deutsche communist zone, England and United States decided supplying the city, or else by means of escorted convoys (that would be able to give rise to new confrontations) or by means of airlift, breaking or avoiding in any event the blockage from Berlin. Second option was chosen, starting the Luftbrücke (airlift) at June 25, 1948. This went another from the problems in which worked the SCOOP group, in December of that same year, could carry 4500 daily tons, and after studies of Research Operations optimized the supplying to get to the 8000~9000 daily tons in March of 1949. This cipher was the same that would have been transported for terrestrial means, for that the Soviet decided to suspend the blockage at May 12, 1949.

After Second World War, the order of United States' resources (USA) (energy, armaments, and all kind of supplies) took opportune to accomplish it by models of optimization, resolved intervening linear programming.

At the same time, that the doctrine of Operations Research is being developed, the techniques of computation and computers are also developing, thanks them the time of resolution of the problems decreased.

The first result of these techniques was given at the year 1952, when a SEAC computer from was used National Bureau of Standards in way to obtain the problem's solution. The success at the resolution time was so encouraging that was immediately used for all kind of military problems, like determining the optimal height which should fly the planes to locate the enemy submarines, monetary founds management for logistics and armament, including to determine the depth which should send the charges to reach the enemy submarines in order to cause the biggest number of casualties, that was translated in a increase in five times in Air Force's efficacy.

During the 50's and 60's decade, grows the interest and developing of Operational Research, due to its application in the space of commerce and the industry. Take for example, the problem of the calculation of the optimal transporting plan of sand of construction to the works of edification of the city of Moscow, which had 10 origins points and 230 destinies. To resolve it, was used and Sterna computer, that took 10 days in the month of June of 1958, and such solution contributed a reduction of the 11 % of the expenses in relation to original costs.

Previously, this problems were presented in a discipline knew as Research Companies or Analysis Companies, that did not have so effective methods like the developed during Second World War (for example the Simplex method). No war applications of Operations Research there are so as you want imagine, with problems like nutrition of cattle raising, distribution of fields of cultivation in agriculture, goods transportation, location, personnel's distribution, and nets, queue problems, graphs, etc.

b) Give a few live examples of application of OR, with brief illustrations.

[WBUT 2013]

Answer:

***Production Planning at Harris Corporation - Semiconductor Section*:** For our first application [1], we look at an area that is readily appreciated by every industrial engineer-production planning and due date quotation. The semiconductor section of Harris Corporation was for a number of years a fairly small business catering to a niche market in the aerospace and defense industries where the competition was minimal. However, in 1988 a strategic decision was made to acquire General Electric's semiconductor product lines and manufacturing facilities. This immediately increased the size of Harris Semiconductor's operations and product lines by roughly three times, and more importantly, catapulted Harris into commercial market areas such as automobiles and telecommunications where the competition was stiff. Given the new diversity of product lines and the tremendous increase in the complexity of production planning, Harris was having a hard time meeting delivery schedules and in staying competitive from a financial perspective; clearly, a better system was required.

In the orientation phase it was determined that the MRP type systems used by a number of its competitors would not be a satisfactory answer and a decision was made to develop a planning system that would meet Harris' unique needs - the final result was IMPReSS, an automated production planning and delivery quotation system for the entire production network. The system is an impressive combination of heuristics as well as optimization-based techniques. It works by breaking up the overall problem into smaller, more manageable problems by using a heuristic decomposition approach. Mathematical models within the problem are solved using linear programming along with concepts from material requirements planning. The entire system interfaces with sophisticated databases allowing for forecasting, quotation and order entry, materials and dynamic information on capacities. Harris estimates that this system has increased on-time deliveries from 75% to 95% with no increase in inventories, helped it move from $75 million in losses to $40 million in profits annually, and allowed it to plan its capital investments more efficiently.

Gasoline Blending at Texaco: For another application to production planning, but this time in a continuous as opposed to discrete production environment, we look at a system in use at Texaco [2]. One of the major applications of O.R. is in the area of gasoline blending at petroleum refineries, and virtually all major oil companies use sophisticated optimization models in this area. At Texaco the system is called StarBlend and runs on networked microcomputers. As some background, the distillation of crude petroleum produces a number of different products at different distillation temperatures. Each of these may be further refined through cracking (where complex hydrocarbons are broken into simpler ones) and recombination. These various output streams are then blended together to form end-products such as different grades of gasoline (leaded, unleaded, super-unleaded etc.), jet fuel, diesel and heating oil. The planning problem is very complex, since different grades of crude yield different concentrations of output streams and incur different costs, and since different end-products fetch different revenues and use different amounts of refinery resources. Considering just one product - gasoline - there are various properties that constrain the blends produced. These include the octane number, lead and sulfur content, volatilities and Reid vapor pressure, to name a few. In addition, regulatory constraints impose certain restrictions as well.

As an initial response to this complex problem, in the early to mid 1980's Texaco developed a system called OMEGA. At the heart of this was a nonlinear optimization model which supported an interactive decision support system for optimally blending gasoline; this system alone was estimated to have saved Texaco about $30 million annually. StarBlend is an extension of OMEGA to a multi-period planning environment where optimal decisions could be made over a longer planning horizon as opposed to a single period. In addition to blend quality constraints, the optimization model also incorporates inventory and material balance constraints for each period in the planning horizon. The optimizer uses an algebraic modeling language called GAMS and a nonlinear solver called MINOS, along with a relational database system for managing data. The whole system resides within a user-friendly interface and in addition to immediate blend planning it can also be used to analyze various "what-if" scenarios for the future and for long-term planning.

FMS Scheduling at Caterpillar: For our third application we look at the use of a simulation model. This model was applied to derive schedules for a Flexible Manufacturing System (FMS) at Caterpillar, Inc. [3]. The interested reader may refer to any text on computer integrated manufacturing for details about FMSs; typically, they are systems of general purpose CNC machines linked together by an automated material handling system and completely controlled by computers. The FMS in question at

Caterpillar had seven CNC milling machines, a fixturing station and a tool station, with material and tool handling being performed by four automated guided vehicles (AGVs) traveling along a one-way guided wire path. FMSs can provide tremendous increases in capacity and productivity because of the high levels of automation inherent in them and their potential to manufacture a wide variety of parts. On the other hand, this comes with a price; these systems are also very complex and the process of planning and scheduling production on an FMS and then controlling its operation can be a very difficult one. The efficiency of the scheduling procedure used can have a profound effect on the magnitude of the benefits realized.

At Caterpillar, a preliminary analysis showed that the FMS was being underutilized and the objective of the project was to define a good production schedule that would improve utilization and free up more time to produce additional parts. In the orientation phase it was determined that the environment was much too complex to represent it accurately through a mathematical model, and therefore simulation was selected as an alternative modeling approach. It was also determined that minimizing the make span (which is the time required to produce all daily requirements) would be the best objective since this would also maximize as well as balance machine utilization. A detailed simulation model was then constructed using a specialized language called SLAM. In addition to the process plans required to specify the actual machining of the various part types, this model also accounted for a number of factors such as material handling, tool handling and fixturing. Several alternatives were then simulated to observe how the system would perform and it was determined that a fairly simple set of heuristic scheduling rules could yield near optimal schedules for which the machine utilizations were almost 85%. However, what was more interesting was that this study also showed that the stability of the schedule was strongly dependent on the efficiency with which the cutting tools used by the machines could be managed. In fact, as tool quality starts to deteriorate the system starts to get more and more unstable and the schedule starts to fall behind due dates. In order to avoid this problem, the company had to suspend production over the weekends and replace worn-out tools or occasionally use overtime to get back on schedule. The key point to note from this application is that a simulation model could be used to analyze a highly complex system for a number of what-if scenarios and to gain a better understanding of the dynamics of the system.

Fleet Assignment at Delta Airlines: One of the most challenging as well as rewarding application areas of O.R. has been the airline industry. We briefly describe here one such application at Delta Airlines [4]. The problem solved is often referred to as the fleet assignment problem. Delta flies over 2500 domestic flight legs each day and uses about 450 aircraft from 10 different fleets, and the objective was to assign aircraft to flight legs in such a way that revenues from seats are maximized. The tradeoff is quite simple - if a plane is too small then the airline loses potential revenue from passengers who cannot be accommodated on board, and if it is too large then the unoccupied seats represent lost revenue (in addition to the fact that larger aircraft are also more expensive to operate). Thus the objective is to ensure that an aircraft of the "correct" size be available when required and where required. Unfortunately, ensuring that this can happen is

tremendously complicated since there are a number of logistical issues that constrain the availability of aircraft at different times and locations.
The problem is modeled by a very large mixed-integer linear program - a typical formulation could result in about 60,000 variables and 40,000 constraints. The planning horizon for each problem is one day since the assumption is made that the same schedule is repeated each day (exceptions such as weekend schedules are handled separately). The primary objective of the problem is to minimize the sum of operating costs (including such things as crew cost, fuel cost and landing fees) and costs from lost passenger revenues. The bulk of the constraints are structural in nature and result from modeling the conservation of flow of aircraft from the different fleets to different locations around the system at different scheduled arrival and departure times. In addition there are constraints governing the assignment of specific fleets to specific legs in the flight schedule. There are also constraints relating to the availability of aircraft in the different fleets, regulations governing crew assignments, scheduled maintenance requirements, and airport restrictions. As the reader can imagine, the task of gathering and maintaining the information required to mathematically specify all of these is in itself a tremendous task. While building such a model is difficult but not impossible, the ability to solve it to optimality was impossible until the very recent past. However, computational O.R. has developed to the point that it is now feasible to solve such complex models; the system at Delta is called Cold start and uses highly sophisticated implementations of linear and integer programming solvers. The financial benefits from this project were tremendous; for example, according to Delta the savings during the period from June 1 to August 31, 1993 were estimated at about $220,000 per day over the old schedule.

***KeyCorp Service Excellence Management System*:** For our final application we turn to the service sector and an industry that employs many industrial engineers - banking. This application [5] demonstrates how operations research was used to enhance productivity and quality of service at KeyCorp, a bank holding company headquartered in Cleveland, Ohio. Faced with increasing competition from nontraditional sources and rapid consolidation within the banking industry, KeyCorp's aim was to provide a suite of world-class financial products and services as opposed to being a traditional bank. The key element in being able to do this effectively is high-quality customer service and a natural trade-off faced by a manager was in terms of staffing and service - better service in the form of shorter waiting times required additional staffing which came at a higher cost. The objective of the project was to provide managers with a complete decision support system which was dubbed SEMS (Service Excellence Management System).
The first step was the development of a computerized system to capture data on performance. The system captured the beginning and ending time of all components of a teller transaction including host response time, network response time, teller controlled time, and customer controlled time and branch hardware time. The data gathered could then be analyzed to identify areas for improvement. Queuing theory was used to determine staffing needs for a prespecified level of service. This analysis yielded a required increase in staffing that was infeasible from a cost standpoint, and therefore an estimate was made of the reductions in processing times that would be required to meet the service objective with the maximum staffing levels that were feasible. Using the

performance capture system, KeyCorp was then able to identify strategies for reducing various components of the service times. Some of these involved upgrades in technology while others focused on procedural enhancements, and the result was a 27% reduction in transaction processing time. Once the operating environment was stabilized, KeyCorp introduced the two major components of SEMS to help branch managers improve productivity. The first, a Teller Productivity system, provided the manager with summary statistics and reports to help with staffing, scheduling and identifying tellers who required further training. The second, a Customer Wait Time system, provided information on customer waiting times by branch, by time of day and by half-hour intervals at each branch. This system used concepts from statistics and queuing theory to develop algorithms for generating the required information. Using SEMS, a branch manager could thus autonomously decide on strategies for further improving service. The system was gradually rolled out to all of KeyCorp's branches and the results were very impressive. For example, on average, customer processing times were reduced by 53% and customer wait times dropped significantly with only four percent of customers waiting more than five minutes. The resulting savings over a five year period were estimated at $98 million.

Long Answer Type Questions

1. a) Briefly describe ABC analysis. **[WBUT 2014]**

Answer:

In materials management, the **ABC analysis** (or **Selective Inventory Control**) is an inventory categorization technique. ABC analysis divides an inventory into three categories- "A items" with very tight control and accurate records, "B items" with less tightly controlled and good records, and "C items" with the simplest controls possible and minimal records.

The ABC analysis provides a mechanism for identifying items that will have a significant impact on overall inventory cost, while also providing a mechanism for identifying different categories of stock that will require different management and controls.

The ABC analysis suggests that inventories of an organization are not of equal value. Thus, the inventory is grouped into three categories (**A, B, and C**) in order of their estimated importance.

'A' items are very important for an organization. Because of the high value of these 'A' items, frequent value analysis is required. In addition to that, an organization needs to choose an appropriate order pattern (e.g. 'Just- in- time') to avoid excess capacity. 'B' items are important, but of course less important than 'A' items and more important than 'C' items. Therefore 'B' items are intergroup items. 'C' items are marginally important.

b) Write note on Bin Card. **[WBUT 2014]**

Answer:

A bin card is a common element in a perpetual inventory system. "Perpetual" simply means the inventory is always in flux. A bin card is particularly common in a retail stockroom. The card includes a number of data points about a particular product, but its most important function is to show how many units of a particular product are in stock.

c) What are the functions of purchase department? **[WBUT 2014]**

Answer:

Most major companies and even some government organizations have a purchasing or procurement department as part of everyday operations. These departments provide a service that is the backbone of many manufacturing, retail, military and other industrial organizations. Many individuals, even some who work for these companies, are unaware of what the purchasing department does, why it exists or what purposes it serves. To understand better what the role of the purchasing department is, consider some functions it performs.

d) Explain the difference between marketing and selling. **[WBUT 2014]**

Answer:

Marketing and **sales** are both aimed at increasing revenue. They are so closely intertwined that people often don't realize the difference between the two. Indeed, in small organizations, the same people typically perform both sales and marketing tasks. Nevertheless, marketing is different from sales and as the organization grows, the roles and responsibilities become more specialized.

Comparison chart

	Marketing	Sales
Approach	Broader range of activities to sell product/service, client relationship etc.; determine future needs and has a strategy in place to meet those needs for the long term relationship.	makes customer demand match the products the company currently offers.
Focus	Overall picture to promote, distribute, price products/services; fulfill customer's wants and needs through products and/or services the company can offer.	fulfill sales volume objectives
Process	Analysis of market, distribution channels, competitive products and services; Pricing strategies; Sales tracking and market share analysis; Budget	Usually one to one
Scope	Market research; Advertising; Sales; Public relations; Customer service and satisfaction.	Once a product has been created for a customer need, persuade the customer to purchase the product to fulfill her needs
Horizon	Longer term	Short term
Strategy	Pull	push
Priority	Marketing shows how to reach to the Customers and build long lasting relationship	Selling is the ultimate result of marketing.
Identity	Marketing targets the construction of a brand identity so that it becomes easily associated with need fulfillment.	Sales is the strategy of meeting needs in an opportunistic, individual method, driven by human interaction. There's no premise of brand identity, longevity or continuity. It's simply the ability to meet a need at the right time.

FORMULATION OF LINEAR PROGRAMMING PROBLEM

Multiple Choice Type Questions

1. For an LPP [WBUT 2013]
   a) only the objective function needs to be linear
   b) only the constraints' equation/inequality need to be linear
   c) only the non-negativity conditions need to be linear
   d) all these three must be linear

Answer: (d)

2. Total number of allocations in a basic feasible solution of transportation problem of $m \times n$ size is equal to [WBUT 2014]
   a) $m \times n$ b) $(m/n)-1$ c) $m+n-1$ d) $m-n-1$

Answer: (c)

3. Which method is used to solve the LPP, if constrains having Artificial variables? [WBUT 2015]
   a) VAM b) Big M-Method
   c) Simplex Method d) Stepping stone Method

Answer: (b)

4. A basic feasible solution is called .... if the value of at least one basic variable is zero. [WBUT 2017]
   a) degenerate b) non-degenerate c) optimum d) none of these

Answer: (a)

5. Which of the following is not a major requirement of a Linear Programming Problem? [WBUT 2017]
   a) There must be alternative courses of action among which to decide
   b) An objective for the firm must exist
   c) The problem must be of maximization type
   d) Resources must be limited

Answer: (c)

6. The number of methods used in solving Integer programming problem are [MODEL QUESTION]
   a) 2 b) 3 c) 4 d) 1

Answer: (a)

7. What is the method used to solve an LPP involving artificial variables? [MODEL QUESTION]

a) Simplex method
b) Charnes-M-method
c) VAM
d) None of these

Answer: (b)

8. The basic feasible solutions of the system of equations [MODEL QUESTION]

$x_1 + x_2 + x_3 = 8$

$3x_1 + 2x_2 = 18$ are

a) no basic solution
b) $(2,6,0)\ (6,0,2)$
c) $(1,7,0)\ (7,1,0)$
d) $(2,2,0)\ (6,6,2)$

Answer: (b)

Short Answer Type Questions

1. Formulate the following LPP: [WBUT 2013]

A manufacturer makes 2 products A & B. 2 resources R_1 & R_2 are required to make these products. 1 unit of R_1 & 3 units of R_2 are required to produce one place of product B. The maximum available resources R_1 & R_2 are 5 & 12 units respectively. The profits expected to be earned by the manufacturer are Rs. 6 per piece & Rs. 5 per piece for products A & B respectively. The manufacturer wants to have the best product mix for A & B, so that he can earn the maximum profit.

Answer:

Let the manufacturer be produce x units of product A and y units of product B.

Therefore, according to the given conditions

$$\begin{aligned} &\text{Maximize } z = 6x + 5y \\ &\text{subject to} \quad y \le 5 \\ &\qquad\qquad\quad 3y \le 12 \\ &\text{and} \quad x, y \ge 0 \end{aligned}$$

2. The manager of an oil refinery must decide on the mixture of two possible blending processes of which the input and output production runs are as follows: The maximum amount available of crude A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from Process 1 and Process 2 are Rs. 4.00 and Rs. 6.00 respectively. Formulate the LPP for maximizing the profit of the above problem. [WBUT 2014]

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	6	4	6	9
2	5	6	5	5

Answer:

Let x_1 and x_2 production runs must be undertaken for the processes 1 and 2 respectively, so that we are to

Maximize $z = 4x_1 + 6x_2$

Subject to the constraints:

$$6x_1 + 5x_2 \le 250$$
$$4x_1 + 6x_2 \le 200$$
$$6x_1 + 5x_2 \ge 150$$
$$9x_1 + 5x_2 \ge 130$$

and $x_1, x_2 \ge 0$.

3. Four different metals namely Fe, Cu, Zn, Mn are required to produce three commodities A, B and C. To produce one unit of A 40 kg Fe, 30 kg Cu, 7 kg Zn, 4 kg Mn are needed. Similarly to produce one unit of B 70 kg Fe, 14 kg Cu, 9 kg Mn are required and for producing one unit of C 50 kg Fe, 18 kg Cu, 8 kg Zn are required. The total no. quantities of metals are 1 ton Fe, 5 Quintals Cu, 2 Quintals Zn and Mn each. The profits are Rs 300, Rs 200, and Rs 100 in selling per unit of A, B and C respectively. Formulate the Linear programming Problem. **[WBUT 2015]**

Answer:

Let x, y and z units of A, B and C respectively to be produced.

Therefore,
$$40x + 70y + 50z \le 1000$$
$$30x + 14y + 18z \le 500$$
$$7x + 8z \le 200$$
$$4x + 9y \le 200$$

where $x, y, z \ge 0$

In which $\text{Max } z = 300x + 200y + 100z$

4. Show that $X = \left[(x, y) / x^2 + y^2 \le 4\right]$ is a convex set. **[WBUT 2015]**

Answer:

Given $S = \{(x, y) : x^2 + y^2 \le 4\}$

Let $\vec{x}_1 = (x_1, y_1)$ and $\vec{x}_2 = (x_2, y_2)$ be any two points in S.

Now the convex combination of $\vec{x}_1$ and $\vec{x}_2$ is $\vec{x}_3 = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2,\ 0 \le \lambda \le 1$

$$= \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = \left(\lambda x_1 + (1-\lambda)x_2,\ \lambda y_1 + (1-\lambda)y_2\right)$$

Now,
$$\{\lambda x_1 + (1-\lambda)x_2\}^2 + \{\lambda y_1 + (1-\lambda)y_2\}^2$$
$$\le \lambda^2 x_1^2 + (1-\lambda)^2 x_2^2 + \lambda^2 y_1^2 + (1-\lambda)^2 y_2^2$$
$$= \lambda^2(x_1^2 + y_1^2) + (1-\lambda)^2(x_2^2 + y_2^2)$$
$$\le \left\{\lambda^2 + (1-\lambda)^2\right\}4$$

≤ 4

$\therefore \quad \vec{x} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \in S$

Hence 'S' is a convex set.

5. Solve the following linear programming problem using graphical approach: **[WBUT 2016]**

Minimize $Z = 200x_1 + 300x_2$

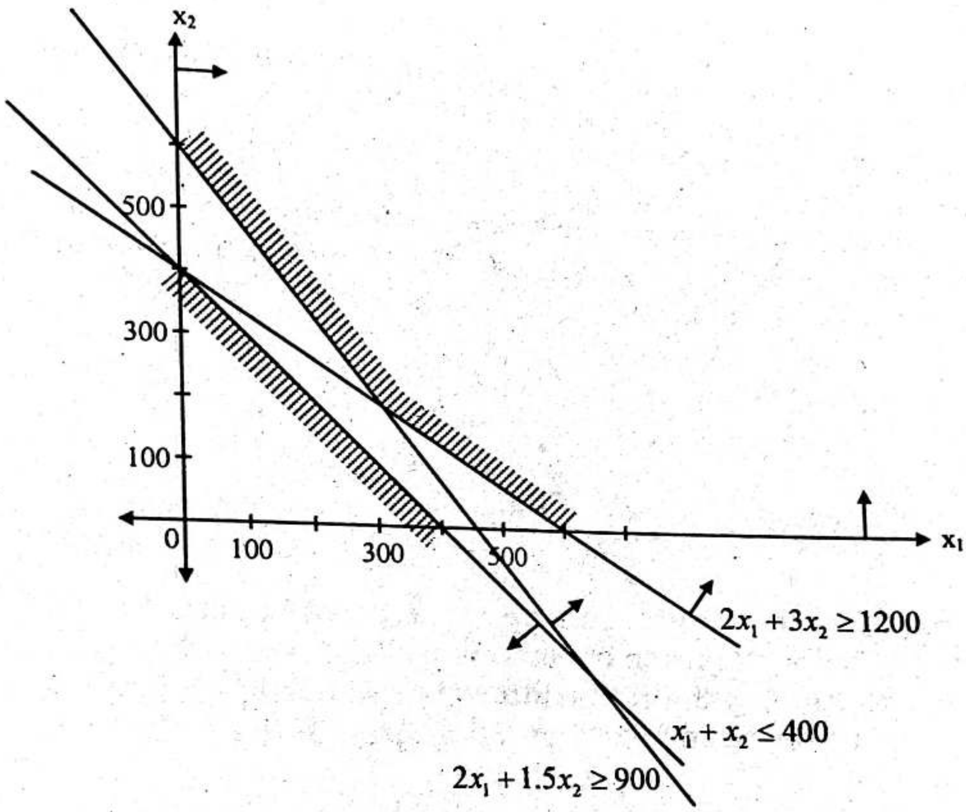
subject to $2x_1 + 3x_2 \geq 1200$

$x_1 + x_2 \leq 400$

$2x_1 + 1.5x_2 \geq 900$

$x_1,\ x_2 \geq 0$

Answer:



This LPP has no feasible region.

6. Find the maximum value of $Z = 5x_1 + 7x_2$ **[WBUT 2016]**

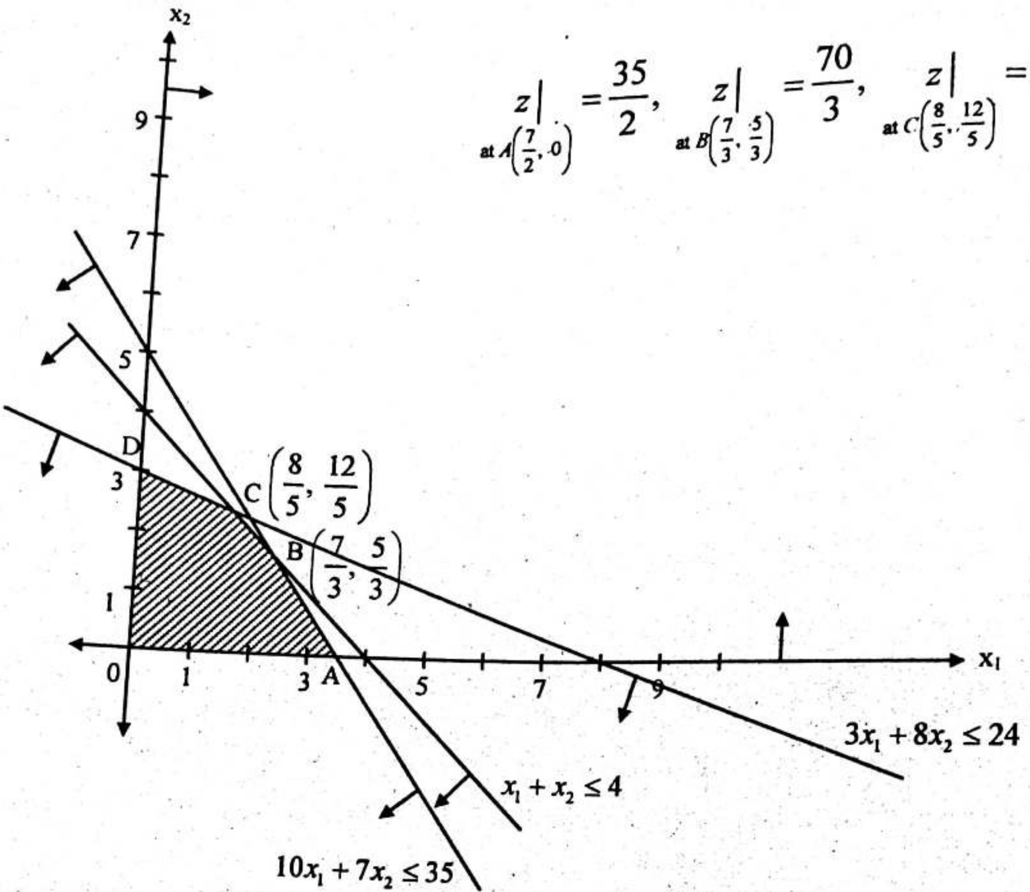
subject to, $x_1 + x_2 \leq 4$

$3x_1 + 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

$x_1,\ x_2 \geq 0$

Answer:



$$z\Big|_{\text{at } A\left(\frac{7}{2},0\right)}=\frac{35}{2},\quad z\Big|_{\text{at } B\left(\frac{7}{3},\frac{5}{3}\right)}=\frac{70}{3},\quad z\Big|_{\text{at } C\left(\frac{8}{5},\frac{12}{5}\right)}=\frac{124}{5},\quad z\Big|_{\text{at } D(0,3)}=21$$

Therefore, $z_{\max}=\dfrac{124}{5}$ and $x_1=\dfrac{8}{5},\ x_2=\dfrac{12}{5}$

7. A small ink manufacturer produces a certain type of ink at a total average cost of Rs. 3 per bottle and sells at a price of Rs. 5 per bottle. The ink is produced over the week-end and is sold during the following week. According to past experience the weekly demand has never been less than 78 or greater than 80 bottles in his place. Formulate the loss table.

Answer: **[WBUT 2016]**

Demand →

Production ↓	78	79	80
78	0	0	0
79	3	0	0
80	6	3	0

Loss Table

8. Explain the purpose of Slack variables and Artificial variables used in LPP problem. What is degeneracy in Linear Programming Problem? **[WBUT 2017]**

Answer:
In Simplex algorithm, in a maximization problem to convert the standard form, less than type constraints are converted to equality type constraints after introducing slack variables. Slack variables help to form the initial basis of simplex algorithm.
When we could not find initial basis in a standard form of LPP after incorporating slack and surplus variables, then to form initial basis we have to introduce artificial variables.
In a basic feasible solution, if any one of the basic variables occurs in zero level then the solution is called degenerate.

9. Write the LPP in its standard form: **[MODEL QUESTION]**

$$\text{Max}(Z) = x_1 - 3x_2 + 5x_3$$

subject to

$$x_1 + x_2 + x_3 \le 7$$
$$x_1 - x_2 + x_3 \ge 2$$
$$3x_1 - x_2 + 2x_3 = -5$$
$x_1, x_2 \ge 0$ and x_3 **is unrestricted.**

Answer:
In the 3rd constrain R.H.S. is –5, multiply by (–1) (on both sides of this equation) such that R.H.S. should be (+)ve, i.e., $-3x_1 + x_2 + 2x_3 = 5$.
The variable x_3 is unrestricted, we will replace it as the difference of two (+)ve variables x_4 and x_5 i.e., $x_3 = x_4 - x_5$
The 1st constrain needs a slack variable x_6 (say) and the 2nd constrain needs a surplus variable x_7 (say). Now, writing the problem in its standard form, we get

$$\text{Maximize } Z = x_1 - 3x_2 + 5x_4 - 5x_5 + 0.x_6 + 0.x_7$$
subject to
$$x_1 + x_2 + x_4 - x_5 + x_6 = 7$$
$$x_1 - x_2 + x_4 - x_5 - x_7 = 2$$
$$-3x_1 + x_2 + 2x_4 - 2x_5 = 5,$$
$$x_1, x_2, x_4, x_5, x_6, x_7 \ge 0.$$

10. Define convex set. **[MODEL QUESTION]**

Answer:
Convex Set: A set X is said to be a convex set if, for any two points $x_1, x_2 \in X$, the line segment joining those two points be also lie in the set.

Long Answer Type Questions

1. a) Explain the major assumptions in Linear programming. **[MODEL QUESTION]**

Answer:
The linear programming can be used for optimisation if the following major assumptions are satisfied

(i) There must be only one well-defined objective function.

(ii) Decision variables of the L.P.P. should be non-negative (≥ 0).

(iii) Each constraint and objective function should be in linear form.

(iv) Constraints should be such that they can produce a feasible region of solutions.

(v) Cost or profit vector decides whether it is a minimization or maximization type of problem.

b) A home decorator manufactures two types of lamps, Alpha and Delta. Both this lamps require the services of a cutter and a finisher. Alpha requires 3 hours of cutter's time and 2 hours of finisher's time. Delta requires 2 hours of cutter's time and 1 hour of finisher's time. The cutter has 180 hours finisher has 110 hours time each month. An Alpha type gives a profit of Rs.10 and a Delta a profit of Rs. 7. Formulate the problem of how many lamps of each type should be produced each month as a LPP. Solve this LPP and obtain the optimal solution.

[MODEL QUESTION]

Answer:

Let n_1, n_2 be the no. of alpha and delta type lamp respectively

Formulation: Max. $Z = 10x_1 + 7x_2$

Subject to $3x_1 + 2x_2 \leq 180,\ 2x_1 + x_2 \leq 110;\ x_1,\ x_2 \geq 0$

Optimal solution by graphical method:

$$x_1 = 40,\ x_2 = 30 \text{ and Max. } Z = \text{Rs.}710.$$

2. Solve the following integer programming problem using Gomory's cutting plane algorithm:

Max $Z = x_1 + x_2$

subject to $3x_1 + 2x_2 \leq 5$

$x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0$ **and are integers.**

Answer:

[MODEL QUESTION]

Max $Z = x_1 + x_2 + 0 \cdot x_3 + 0 \cdot x_4$

subject to
$$3x_1 + 2x_1 + x_3 = 5$$
$$x_2 + x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

We solve this problem by simplex algorithm (Table – 1)

Table-1

		$c_j \rightarrow$		1	1	0	0	
$\bar{C}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	Mini. ratio
0	$\bar{a}_3$	x_3	5	$\boxed{3}$	2	1	0	5/3
0	$\bar{a}_4$	x_4	2	0	1	0	1	
—	—	—	$z_j - c_j \rightarrow$	$\boxed{-1}$	–1	0	0	
1	$\bar{a}_1$	x_1	5/3	1	2/3	1/3	0	
0	$\bar{a}_4$	x_4	2	0	$\boxed{1}$	0	1	2
—	—	—	$z_j - c_j \rightarrow$	0	$\boxed{-1/3}$	1/3	0	
1	$\bar{a}_1$	x_1	1/3	1	0	1/3	–2/3	…. (I)
1	$\bar{a}_2$	x_2	2	0	1	0	1	
—	—	—	$z_j - c_j \rightarrow$	0	0	1/3	1/3	

Hence all the $z_j - c_j \geq 0$, the optimal solution by simplex algorithm is $x_1 = 1/3$, $x_2 = 2$ which is not a integer solution (since $x_1 = 1/3$).

From the 1st row of (I), one get using Gomary's cutting plane method,

$$x_1 + 0 \cdot x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = \frac{1}{3}$$

or, $$x_1 + 0 \cdot x_2 + 0 \cdot x_3 - x_4 = \frac{1}{3} - \frac{1}{3}x_3 + \frac{1}{3}x_4 \leq 0$$

$$\Rightarrow \quad \frac{-1}{3}x_3 + \frac{1}{3}x_4 \leq \frac{-1}{3}$$

$$\Rightarrow \quad -x_3 + x_4 \leq -1 \qquad \text{(Gomory's constraint)} \qquad \text{.... (II)}$$

$$\Rightarrow \quad -x_3 + x_4 + x_g = -1$$

[x_g is the slack variable used in the Gomory's constraints]

Now, we will add the constraint (II) with the constraints (I) and solve the problem by **Dual Simplex algorithm.**

(Table-2 for dual simplex algorithm for the solution of the problem).

Table-2

$\bar{C}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_g$	$\text{Min}\left\{\left\lvert\frac{z_j - c_j}{a_{rj}}\right\rvert\right\}, a_{rj} < 0$
1	$\bar{a}_1$	x_1	1/3	1	0	1/3	–2/3	0	1/3
1	$\bar{a}_2$	x_2	2	0	1	0	1	0	
0	$\bar{a}_g$	x_g	$\boxed{-1}$	0	0	$\boxed{-1}$	1	1	
—	—	—	$z_j - c_j \rightarrow$	0	0	1/3	1/3	0	—
1	$\bar{a}_1$	x_1	0	1	0	0	–1/3	1/3	
1	$\bar{a}_2$	x_2	2	0	1	0	1	0	
0	$\bar{a}_3$	x_3	1	0	0	1	–1	–1	
			$z_j - c_j \rightarrow$	0	0	0	2/3	1/3	

Since all of x_1, x_2 and $x_3 \geq 0$ and integers, we get optimal solution by **Dual Simple method.**

$$x_1 = 0,\ x_2 = 2$$
$$\text{Max } Z = 0 + 2 = 2$$

GRAPHICAL SOLUTION OF LPP

Short Answer Type Questions

1. **Solve the following LPP by Graphical Method:** **[WBUT 2013]**

Minimize $Z = 20X_1 + 10X_2$

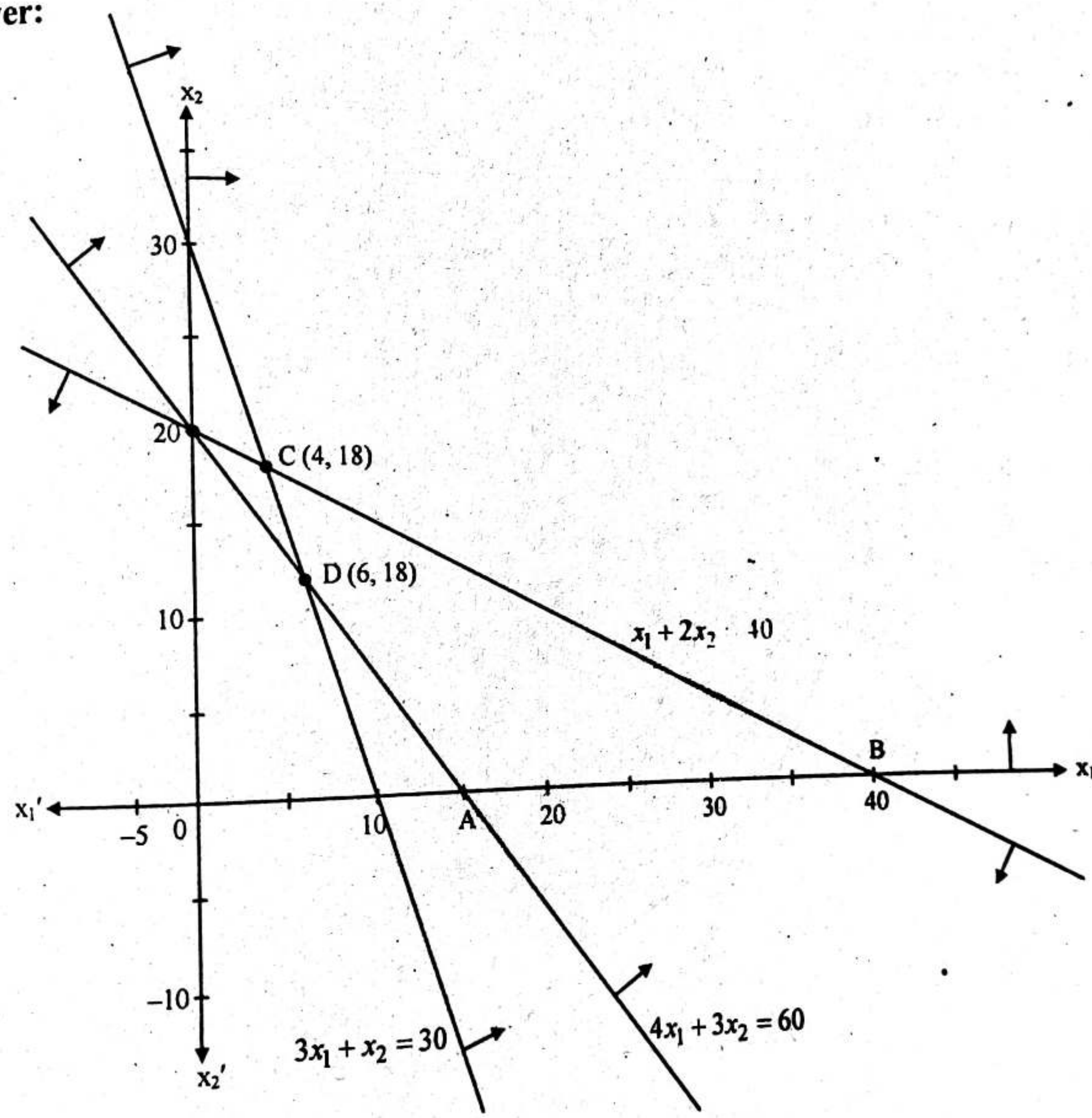
Subject to $X_1 + 2X_2 \le 40$

$3X_1 + X_2 \ge 30$

$4X_1 + 3X_2 \ge 60$

$X_1 + X_2 \ge 0$

Answer:



Here ABCDA is the feasible region.

Now $z\big|_{A(15,0)} = 300$, $z\big|_{B(40,0)} = 800$, $z\big|_{C(4,18)} = 260$, $z\big|_{D(6,2)} = 240$

Hence the required solution is $x_1 = 6, x_2 = 12$ and $z_{\min} = 240$.

2. A company manufactures two products X and Y. The profit contributions of X and Y are Rs. 3 and Rs. 4 respectively. The products X and Y require the services of four facilities. The capacities of four facilities A, B, C and D are limited and the available capacities in hours are 200 hrs, 150 hrs, 100 hrs and 80 hrs respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly the requirements of products Y are 4, 5, 5 and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximize the profit. Use graphical method. **[WBUT 2017]**

Answer:

Let the company be manufacture x_1 and x_2 units of X and Y products respectively. Therefore, the constraints due to the facilities A, B, C, D are respectively.

$$5x_1 + 4x_2 \le 200$$
$$3x_1 + 5x_2 \le 150$$
$$5x_1 + 5x_2 \le 100$$
$$8x_1 + 4x_2 \le 80$$

The optimisation function is Maximize $Z = 3x_1 + 4x_2$

So the LPP corresponding to the given problem is

Max $Z = 3x_1 + 4x_2$

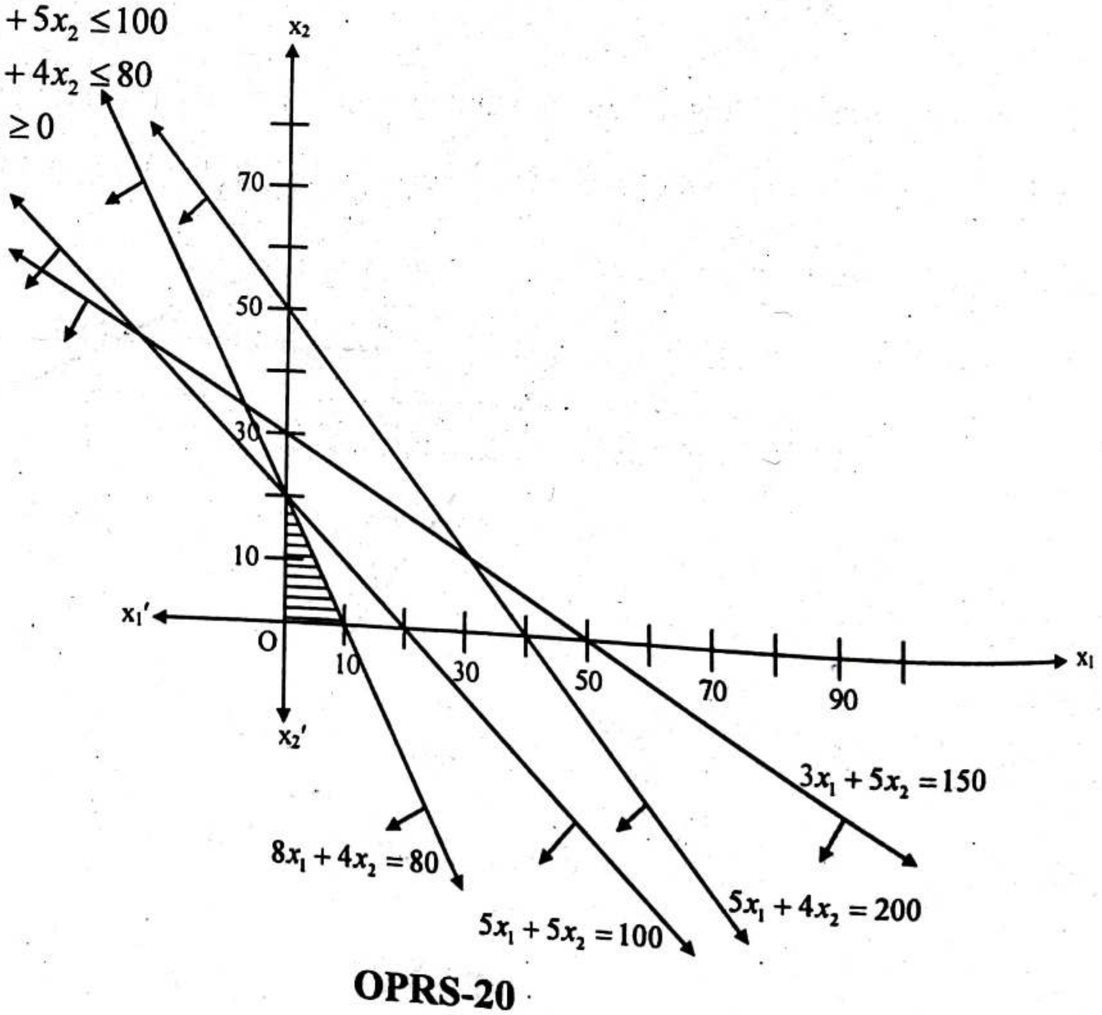
subject to $5x_1 + 4x_2 \le 200$

$3x_1 + 5x_2 \le 150$

$5x_1 + 5x_2 \le 100$

$8x_1 + 4x_2 \le 80$

and $x_1, x_2 \ge 0$



Here OACO is the feasible region.

$Z\Big|_{\text{At } A} = 30$, $\quad Z\Big|_{\text{At } C} = 80$, $\quad Z\Big|_{\text{At } O} = 0$

So the optical solution is $x_1 = 0,\ x_2 = 20$

3. Solve graphically:

Minimize $Z = x_1 + x_2$

Subject to $5x_1 + 9x_2 \le 45$

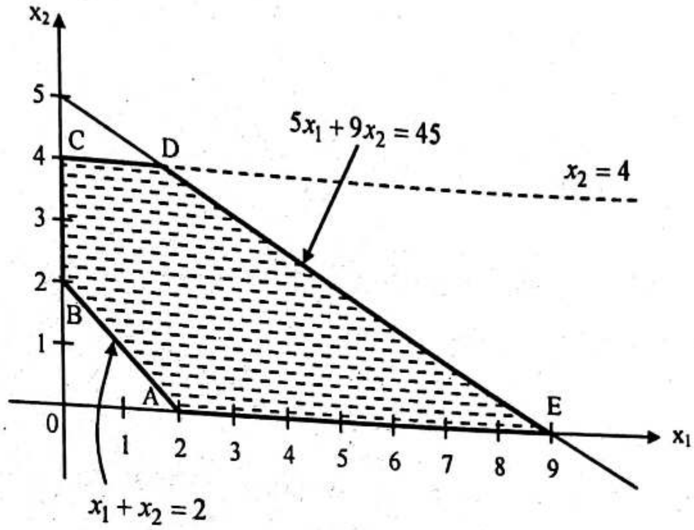
$x_1 + x_2 \ge 2$

$x_2 \le 4$

and $x_1, x_2 \ge 0$.

[MODEL QUESTION]

Answer:



The shaded region ABCDEA is the convex set of feasible solution. Now, $A \equiv (2,0)$, $B \equiv (0,2)$, $C \equiv (0,4)$, $D \equiv \left(\frac{9}{5}, 4\right)$ and $E \equiv (9,0)$ are the vertices of this convex set. Minimum Z attains both at $A \equiv (2,0)$, $B \equiv (0,2)$.

Min. $Z = 2$. Note that every point on the line $\overline{AB}$ is an optimal solution.

Long Answer Type Questions

1. Solve the following all integer programming problem using branch and bound method:

Maximize $\quad Z = 6x_1 + 8x_2$

subject to, $\quad 4x_1 + 16x_2 \le 32$

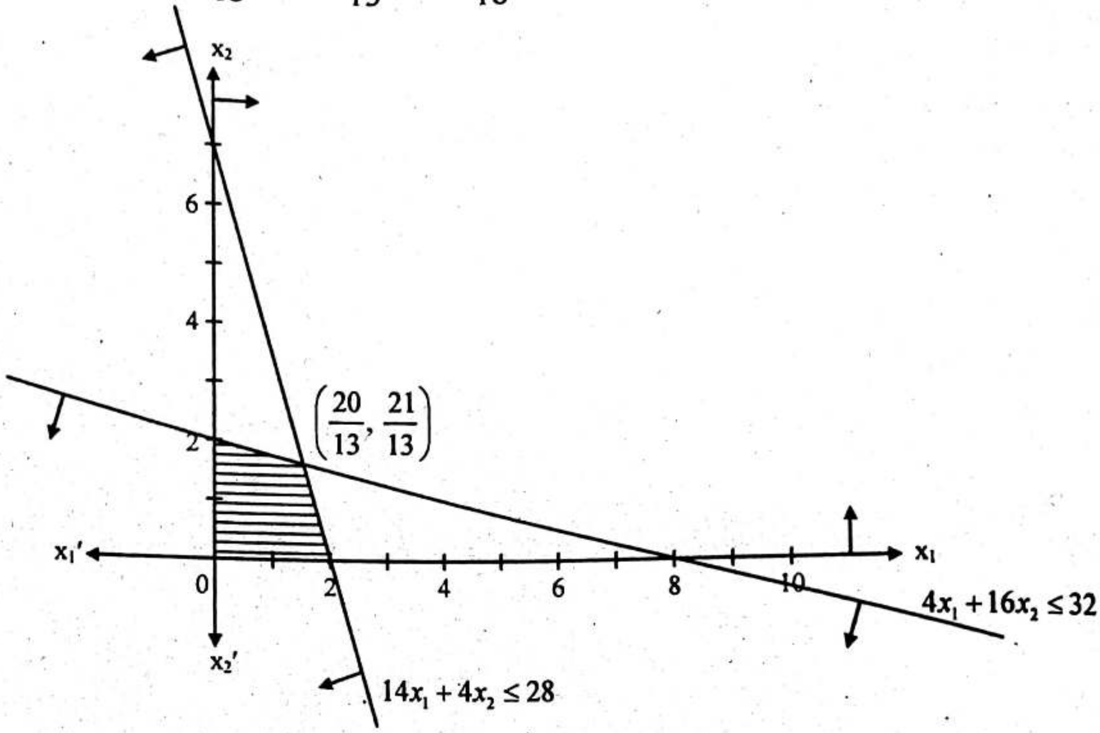
$14x_1 + 4x_2 \le 28$

$x_1, x_2 \ge 0$

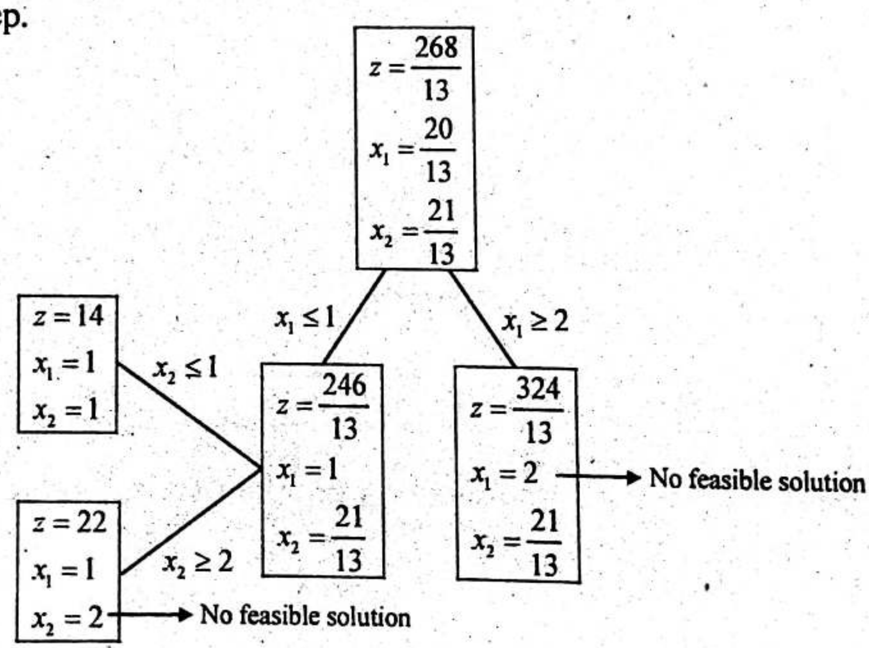
[WBUT 2016]

Answer:

We begin by solving graphically the related LPP, ignoring the integer value constraint, which yields $z_{max} = \frac{268}{13}$, $x_1 = \frac{20}{13}$, $x_2 = \frac{21}{18}$



Since neither x_1 nor x_2 is integer we can choose either variable to begin implementation of the next step.



Hence, the solution is $z_{max} = 14$, $x_1 = 1$, $x_2 = 1$

2. A firm makes two types of furniture: chairs and tables. The profits for each product as calculated by the accounting department are Rs. 20 per chair and Rs. 30 per table. Both products are to be processed on three machines M_1, M_2, M_3.

The time required in hours by each product and total time available in hours per week on each machine is as follows:

Machine	Chair	Table	Available time (hrs)
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

i) Give a mathematical formulation to this linear programming problem.

ii) Use the graphical method to solve this problem. **[WBUT 2017]**

Answer:

Let the company be produce x and y units of chairs and tables respectively.

So the mathematical formulation of the problem is

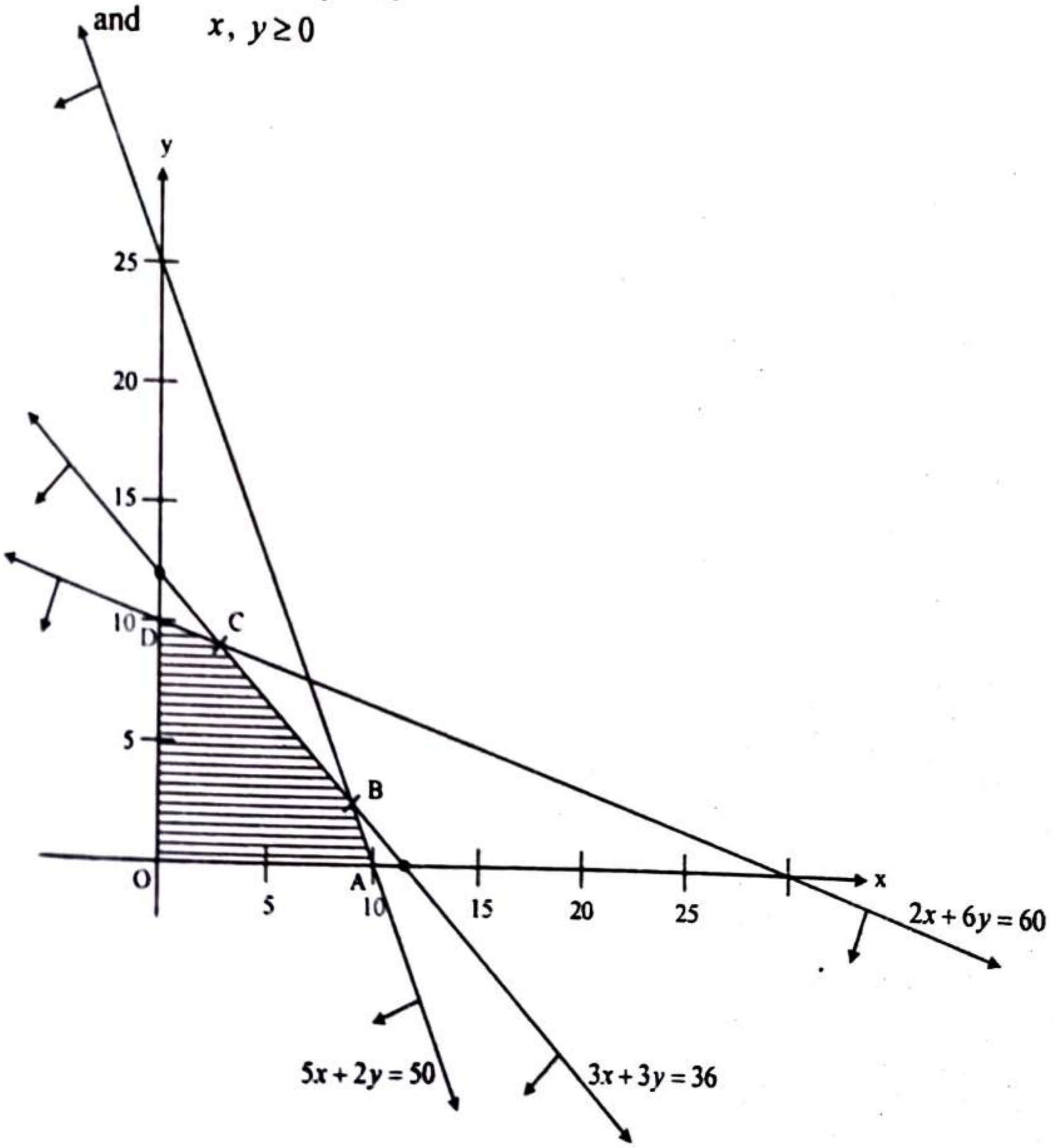
$$\text{Max } Z = 20x + 30y$$

$$\text{subject to } 3x + 3y \le 36$$

$$5x - 2y \le 50$$

$$2x + 6y \le 60$$

$$\text{and } \quad x,\ y \ge 0$$



OABCDO is the feasible region.

$$Z\Big|_{\text{At }A}=200,\qquad Z\Big|_{\text{At }B}=240,\qquad Z\Big|_{\text{At }C}=330,\qquad Z\Big|_{\text{At }D}=300$$

So the optical solution is $x=3,\ y=9$

3. A manufacturer produces TVs and VCRs in a factory. Each TV requires 4 hours to assemble in workstation 1 and 2 hours in workstation 2. Each VCR requires 2 hours to assemble in workstation 1 and 5 hours in workstation 2. Workstation 1 and 2 function for 40 hours and 60 hours respectively in a week. The profit on each TV is Rs. 3000 and on each VCR is Rs.4000. Assuming that all the TVs, VCRs are sold, what should be the strategy of the manufacturer so as to earn maximum profit using LPP? **[MODEL QUESTION]**

Answer:

Let x_1 and x_2 be the numbers of TV and VCR to be produced to maximize profit.

	W.S. – I	W.S. – II	
TV	4 h	2 h	x_1
VCR	2 h	5 h	x_2
	40 h	60 h	

The L.P.P. is

$$\text{Max } Z = 3000x_1 + 4000x_2$$

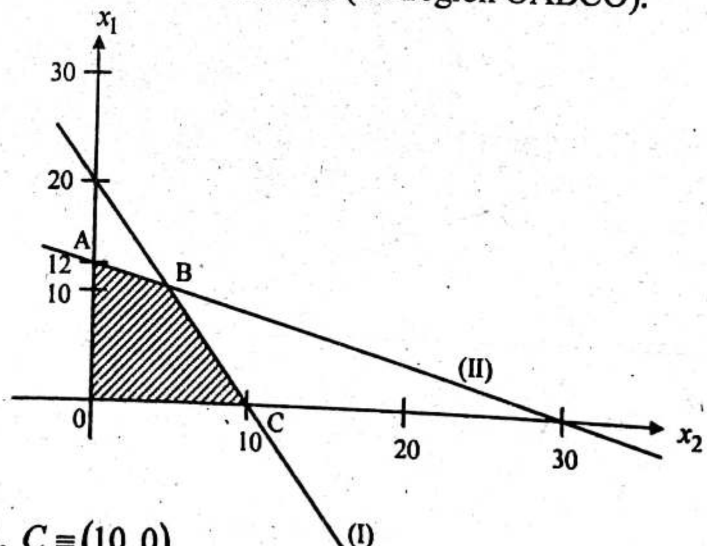
subject to

$$4x_1 + 2x_2 \le 40 \quad (\text{for W.S.} - \text{I})$$
$$2x_1 + 5x_2 \le 60 \quad (\text{for W.S.} - \text{II})$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

Consider $\quad 2x_1 + x_2 = 20 \qquad \text{.... (I)}$

and $\quad 2x_1 + 5x_2 = 60 \qquad \text{.... (II)}$

to form the convex set of feasible solutions (the region OABCO).



$A \equiv (0,12),\ C \equiv (10,0)$

Optimal solution: $x_1 = 5, x_2 = 10$ and Max $Z =$ Rs.55,000/-

SIMPLEX METHOD

Multiple Choice Type Questions

1. Some Linear Programming Problems may have [WBUT 2013]
   a) a unique optimal solution
   b) an infinite number of optimal solutions
   c) no solution
   d) all of these

Answer: (d)

2. Which method is used to solve a LPP problem using artificial variables
   a) simplex method
   b) Charne's Big-M [WBUT 2014]
   c) VAM
   d) none of these

Answer: (b)

3. How many Artificial variables can be introduced in these equations?
   $2X_1 + X_2 = 4$
   $4X_1 + 6X_2 \geq 6$ [WBUT 2015]
   $X_1 + 6X_2 \leq 4$
   a) 0
   b) 1
   c) 2
   d) 3

Answer: (b)

4. The dual of the dual in a LPP will remain [WBUT 2015]
   a) dual
   b) primal
   c) optimal
   d) none of these

Answer: (b)

5. In LPP maximization case $C_j - Z_j <= 0$, then [WBUT 2017]
   a) It is basic feasible solution
   b) basic feasible solution is optimum
   c) cannot say
   d) none of these

Answer: (d)

6. The role of artificial variable in simplex method is [MODEL QUESTION]
   a) to aid in finding the initial basic feasible solution
   b) to start phases of simplex method
   c) to find the shadow principle for the simplex method
   d) none of these

Answer: (a)

7. What is the method used to solve an LPP involving artificial variables? [MODEL QUESTION]
   a) Simple method
   b) Charnes M method
   c) VAM
   d) None of these

Answer: (b)

8. In Simplex method, there will be multiple solutions, if all $z_i - c_j \geq 0$ with some $z_i - c_j = 0$ corresponding to [MODEL QUESTION]

a) all vectors
b) only basis vectors
c) basis vectors and non-basis vectors
d) none of these

Answer: (c)

Short Answer Type Questions

1. Find the basic feasible solutions of the set of equations: [WBUT 2015]

$$2x - y + 3z + t = 6;\ 4x - 2y - z + 2t = 16$$

Answer: We have

$$2x - y + 3z + t = 6$$
$$4x - 2y - z + 2t = 16$$

This system of equations can be expressed in matrix form:

$$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}\begin{Bmatrix} x \\ y \\ z \\ t \end{Bmatrix} = \{\vec{b}\}$$

where $\vec{a}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{a}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ 16 \end{bmatrix}$

Let $B_1 = \{\vec{a}_1, \vec{a}_2\}$, $B_2 = \{\vec{a}_1, \vec{a}_3\}$, $B_3 = \{\vec{a}_1, \vec{a}_4\}$, $B_4 = \{\vec{a}_2, \vec{a}_3\}$, $B_5 = \{\vec{a}_2, \vec{a}_4\}$, $B_6 = \{\vec{a}_3, \vec{a}_4\}$

Now, $|B_1| = 0$, $|B_2| = -14$, $|B_3| = 0$, $|B_4| = 7$, $|B_5| = 0$, $|B_6| = 7$

Therefore,

$$B_2^{-1}\vec{b} = \frac{-1}{14}\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 16 \end{bmatrix} = \frac{-1}{14}\begin{bmatrix} -54 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{27}{7} \\ \frac{-4}{7} \end{bmatrix}$$

So, B.F.S. is $\left(\frac{27}{7}, 0, \frac{-4}{7}, 0\right)$

$$B_4^{-1}\vec{b} = \frac{1}{7}\begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 16 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -54 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{-54}{7} \\ \frac{-4}{7} \end{bmatrix}$$

$\therefore$ B.F.S. is $\left(0, \frac{-54}{7}, \frac{-4}{7}, 0\right)$

and $B_6^{-1}\vec{b} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 16 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -4 \\ 54 \end{bmatrix} = \begin{bmatrix} \frac{-4}{7} \\ \frac{54}{7} \end{bmatrix}$

$\therefore$ B.F.S. is $\left(0,\ 0,\ \frac{-4}{7},\ \frac{54}{7}\right)$

2. Following is the starting tableau of an LPP by the simplex method in incomplete form:

		c_j	3	2	0	0	–M
C_B	B	b	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$
		2	2	1	1	0	0
		12	3	4	0	–1	–1

Write down the actual linear programming problem. **[MODEL QUESTION]**

Answer:

Maximize $z = 3x_1 + 2x_2$

Subject to

$2x_1 + 2_2 \le 2$

$3x_1 + 4x_2 \ge 12$,

$w_1, w_2 \ge 0$

3. Use simplex algorithm to solve the following LPP: **[MODEL QUESTION]**

max $= x_1 + 5x_2$

Subject to

$3x_1 + 4x_2 \le 6$

$x_1 + 3x_2 \ge 3$

where $x_1, x_2 \ge 0$.

Answer:

Optimal solution $x_1 = 0, x_2 = 3/2$; and Max. $Z = 15/2$

4. Following is the starting tableau of an LPP by the simplex method in incomplete form: **[MODEL QUESTION]**

		c_j	3	2	0	0	– M
C_B	B	b	a_1	a_2	a_3	a_4	a_5
		2	2	1	1	0	0
		12	3	4	0	-1	1

Write down the actual linear programming problem.

Answer:

From the column $\overline{a}_3$, it is clear that the first constraint is of '$\leq$' form and also from columns $\overline{a}_4$ and $\overline{a}_5$ it is clear that the 2[nd] constraint is of '$\geq$' form.

The original L.P.P. is given by Max $Z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$

Long Answer Type Questions

1. a) Solve the following Linear Programming Problem using Two-phase Method: **[WBUT 2013]**

Minimize $Z = 3X_1 + 4X_2$

Subjected to: $2X_1 + 3X_2 \geq 8$

$5X_1 + 2X_2 \geq 12$

$X_1, X_2 \geq 0$

Answer:

Given LPP can be written in the following form:

$$\max z_1 = -3x_1 - 4x_2 + 0.x_3 + 0.x_4$$

Subject to

$$2x_1 + 3x_2 - x_3 = 8$$
$$5x_1 + 2x_2 - x_4 = 12$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Now we introduce artificial variables x_5, x_6 to 1[st] and 2[nd] constraints respectively and the set of constraints becomes,

$$2x_1 + 3x_2 - x_3 + x_5 = 8$$
$$5x_1 + 2x_2 - x_4 + x_6 = 12$$

The auxiliary objective function to be maximized is given by,

$$z_1^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - x_5 - x_6.$$

Phase – 1

		c_j	0	0	0	0	-1	-1	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. Ratio
-1	a_5	8	2	3	-1	0	1	0	4
-1	a_6	12	5	2	0	-1	0	1	12/5 $\rightarrow$
		$z_j - c_j$	-7 $\uparrow$	-5	1	1	0	0	

		c_j	0	0	0	0	-1	-1	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. Ratio

-1	a_5	16/5	0	11/5	-1	2/5	1	-2/5	16/11 →
0	a_1	12/5	1	2/5	0	-1/5	0	1/5	6
		$z_j - c_j$	0	-11/5 ↑	1	-2/5	0	7/5	

		c_j	0	0	0	0	-1	-1	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	
0	a_2	16/11	0	0	-5/11	2/11	5/11	2/11	
0	a_1	20/11	1	1	2/11	-3/11	-2/11	3/11	
		$z_j - c_j$	0	0	0	0	1	1	

Here all $z_j - c_j \geq 0$, so the optimality condition of simplex method is satisfied. Further we see that $Max\, z_1^* = 0$ and artificial variables are not in the basis in the final table. So we start the phase-II table where the objective function to be maximized will be

$$z_1 = -3x_1 - 4x_2$$

Phase – II

		c_j	-3	-4	0	0	
c_B	B	b	a_1	a_2	a_3	a_4	
-4	a_2	16/11	0	0	-5/11	2/11	
-3	a_1	20/11	1	1	2/11	-3/11	
		$z_j - c_j$	0	0	14/11	1/11	

All $z_j - c_j \geq 0$, so the optimal solution is $x_1 = 20/11,\ x_2 = 16/11$ and $z_{Min} = 3 \times 20/11 + 4 \times 16/11 = 124/11$.

b) Solve the following Linear Programming Problem using Charne's Big M Method:

Minimize $Z = 4X_1 + X_2$

Subjected to : $3X_1 + X_2 = 3$

$$4X_1 + 3X_2 \geq 6$$

$$X_1 + 2X_2 \leq 4$$

$$X_1, X_2 \geq 0$$

[WBUT 2013]

Answer:

Introducing slack and surplus variables, the given LPP can be written in the following form:

$$\text{Max } z' = -4x_1 - x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6$$

subject to

$$3x_1 + x_2 + x_3 = 3$$

$$3x_1 + x_2 - x_5 = 3$$

$$4x_1 + 3x_2 - x_6 = 6$$

$$x_1 + 2x_2 + x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Now introducing artificial variables we get,

$$\text{Max } z' = -4x_1 - x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6 - Mx_7 - Mx_8$$

subject to

$$3x_1 + x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_5 + x_7 = 3$$
$$4x_1 + 3x_2 - x_6 + x_8 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

		c_j	-4	-1	0	0	0	0	-M	-M	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	Min. Ratio
0	a_3	3	3	1	1	0	0	0	0	0	1
-M	a_7	3	3	1	0	0	-1	0	1	0	1 →
-M	a_8	6	4	3	0	0	0	-1	0	1	3/2
0	a_4	4	1	2	0	1	0	0	0	0	4
$z_j - c_j$			-7M ↑ +4	-4M +1	0	0	M	M	0	0	

		c_j	-4	-1	0	0	0	0	-M	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_8	Min. Ratio
0	a_3	0	0	0	1	0	1	0	0	-
-4	a_1	1	1	1/3	0	0	-1/3	0	0	3
-M	a_8	2	0	5/3	0	0	4/3	-1	1	6/5 →
0	a_4	3	0	-1/3	0	1	1/3	0	0	-
$z_j - c_j$			0	-5M/3 ↑ -1/3	0	0	-4M/3 +4/3	M	0	

		c_j	-4	-1	0	0	0	0	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. Ratio
0	a_3	0	0	0	1	0	1	0	-
-4	a_1	3/5	1	0	0	0	-3/5	1/5	3 →
-1	a_2	6/5	0	1	0	0	4/5	-3/5	-
0	a_4	17/5	0	0	0	1	3/5	-1/5	-
$z_j - c_j$			0	0	0	0	8/5	-1/5 ↑	

		c_j	-4	-1	0	0	0	0	
c_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	
0	a_3	0	0	0	1	0	1	0	
0	a_6	3	5	0	0	0	-3	0	
-1	a_2	3	3	1	0	0	-1	1	
0	a_4	4	1	0	0	1	0	0	
$z_j - c_j$			1	0	0	0	1	0	

Here all $z_j - c_j \geq 0$, so the optimality condition of simplex method is satisfied. The optimal solution is $x_1 = 0,\ x_2 = 3$ and $z_{Min} = 4\times 0 + 1\times 3 = 3$.

2. Use Big M method to solve the following LP problem: **[WBUT 2014]**

Minimize $z = 5x_1 + 2x_2 + 10x_3$

Subject to $x_1 - x_2 \leq 10$

$x_2 + x_3 \geq 10$

where, $x_1 \geq 0, x_2 \geq 0$ **and** $x_3 \geq 0$

Answer:

Introducing slack and surplus variables the standard form of the LPP is

Max $z' = -5x_1 - 2x_2 - 10x_3 + 0\cdot x_4 + 0\cdot x_5$

s.t. $x_1 - x_2 + x_4 = 10$

$x_2 + x_3 - x_5 = 10$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Now applying Simplex algorithm we obtain,

Table – 1

		c_j		-5	-2	-10	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min Ratio
-5	$\vec{a}_1$	x_1	10	1	-1	0	$\boxed{1}$	0	10 →
-10	$\vec{a}_3$	x_3	10	0	1	1	0	-1	—
			$z_j - c_j$	0	-3	0	-5 ↑	10	

Table – 2

		c_j		-5	-2	-10	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min Ratio
0	$\vec{a}_4$	x_4	10	1	-1	0	1	0	—
-10	a_3	x_3	10	0	$\boxed{1}$	1	0	-1	10 →
			$z_j - c_j$	5	-8 ↑	0	0	10	

Table – 3

		c_j		-5	-2	-10	0	0
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$
0	$\vec{a}_4$	x_4	20	1	0	1	1	-1
-2	$\vec{a}_2$	x_2	10	0	1	1	0	-1
			$z_j - c_j$	15	0	8	0	2

Here all $z_j - c_j \geq 0$, so optimality condition is satisfied.

Hence, optimal solution is $x_1 = 0,\ x_2 = 10,\ x_3 = 0$ and $z_{Min} = 20$.

3. Solve the following by simplex method: [WBUT 2014]

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to $3x_1 + x_2 + 6x_3 \le 600$

$2x_1 + 4x_2 + 3x_3 \ge 480$

$2x_1 + 3x_2 + 3x_3 = 540$

and $x_1 \ge 0, x_2 \ge 0$ and $x_3 \ge 0$

Answer:

Introducing slack variable (x_4), surplus variable and artificial variable (x_6, x_7) the standard form of LPP is

$$\text{Max } z = 2x_1 + 3x_2 + 4x_3 + 0\cdot x_4 + 0\cdot x_5 - Mx_6 - Mx_7$$

$$\begin{aligned} \text{s.t.} \quad & 3x_1 + x_2 + 6x_3 + x_4 &&= 600 \\ & 2x_1 + 4x_2 + 3x_3 - x_5 + x_6 &&= 480 \\ & 2x_1 + 3x_2 + 3x_3 + x_7 &&= 540 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0.$$

Now by Simplex algorithm we get,

			c_j	2	3	4	0	0	−M	−M	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_6$	$\vec{a}_7$	Min Ratio
0	$\vec{a}_4$	x_4	600	3	1	6	1	0	0	0	600
−M	$\vec{a}_6$	x_6	480	2	$\boxed{4}$	3	0	−1	1	0	120 →
−M	$\vec{a}_7$	x_7	540	2	3	3	0	0	0	1	180
			$z_j - c_j$	−4M −2	−7M −3 ↑	−6M −4	0	M	0	0	

			c_j	2	3	4	0	0	−M	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_7$	Min Ratio
0	$\vec{a}_4$	x_4	480	5/2	0	$\boxed{21/4}$	1	1/4	0	$\frac{640}{7}$ →
3	$\vec{a}_2$	x_2	120	1/2	1	3/4	0	−1/4	0	160
−M	$\vec{a}_7$	x_7	180	1/2	0	3/4	0	3/4	1	240
			$z_j - c_j$	$-\frac{M}{2} - \frac{1}{2}$	0	$-\frac{3M}{4} - \frac{7}{4}$ ↑	0	$-\frac{3M}{4} + \frac{3}{4}$	0	

		c_j		2	3	4	0	0	$-M$	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_7$	Min Ratio
4	$\vec{a}_1$	x_3	$\frac{640}{7}$	$\frac{10}{21}$	0	1	$\frac{4}{21}$	$\frac{1}{21}$	0	1920
3	$\vec{a}_2$	x_2	$\frac{480}{7}$	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	$-\frac{2}{7}$	0	—
$-M$	$\vec{a}_7$	x_7	$\frac{780}{7}$	$\frac{1}{7}$	0	0	$-\frac{1}{7}$	$\boxed{\frac{5}{7}}$	1	156 →
			$z_j - c_j$	$-\frac{M}{7}-\frac{12}{21}$	0	0	$\frac{M}{7}+\frac{7}{21}$	$-\frac{5M}{7}-\frac{14}{7}$ ↑	0	

		c_j		2	3	4	0	0
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$
4	$\vec{a}_3$	x_3	$\frac{16}{105}$	$\frac{46}{105}$	0	1	$\frac{16}{105}$	0
3	$\vec{a}_2$	x_2	$\frac{792}{7}$	$\frac{1}{5}$	1	0	$-\frac{1}{5}$	0
0	$\vec{a}_5$	x_5	156	$\frac{1}{5}$	0	0	$-\frac{1}{5}$	1
			$z_j - c_j$	$\frac{37}{105}$	0	0	$\frac{1}{105}$	0

Here all $z_j - c_j \geq 0$. So the optimality condition is satisfied.

Therefore, the optimal solution is

$$x_1 = 0,\ x_2 = \frac{792}{7},\ x_3 = \frac{16}{105}.$$

4. Solve the following LPP by simplex method **[WBUT 2015]**

Maximize, $Z = 6X_1 + 8X_2$

Subject to: $30X_1 + 20X_2 \leq 300$

$5X_1 + 10X_2 \leq 110$

$X_1, X_2 \geq 0.$

Answer:

Introducing slack variables x_3, $x_4 (\geq 0)$ to the respective constraints the standard form of the given LPP is given by

$$\text{Max } z = 6x_1 + 8x_2 + 0 \cdot x_3 + 0 \cdot x_4$$

$$\text{s.t.} \quad 30x_1 + 20x_2 + x_3 = 300$$

$$5x_1 + 10x_2 + x_4 = 100$$

$$\text{and} \quad x_1, x_2, x_3, x_4 \geq 0$$

Now by Simplex algorithm we get

			c_j	6	8	0	0	
$\vec{C}_B$	$\vec{B}$	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	Min Ratio
0	$\vec{a}_3$	x_3	300	30	20	1	0	15
0	$\vec{a}_4$	x_4	100	5	$\boxed{10}$	0	1	10 $\rightarrow$
			$z_j - c_j$	−6	−8	0	0	
					$\uparrow$			

			c_j	6	8	0	0	
$\vec{C}_B$	$\vec{B}$	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	Min Ratio
0	$\vec{a}_3$	x_3	100	$\boxed{20}$	0	1	−2	5 $\rightarrow$
8	$\vec{a}_2$	x_2	10	$\frac{1}{2}$	1	0	$\frac{1}{10}$	20
			$z_j - c_j$	−2	0	0	$\frac{8}{10}$	
				$\uparrow$				

			c_j	6	8	0	0	
$\vec{C}_B$	$\vec{B}$	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	
6	$\vec{a}_1$	x_1	5	1	0	$\frac{1}{20}$	$\frac{-1}{10}$	
8	$\vec{a}_2$	x_2	$\frac{15}{2}$	0	1	$\frac{-1}{40}$	$\frac{3}{20}$	
			$z_j - c_j$	0	0	$\frac{1}{10}$	$\frac{6}{10}$	

Here all $z_j - c_j \geq 0$, so optimality condition of simplex method is satisfied.

Hence the optimal solution is $x_1 = 5$, $x_2 = \frac{15}{2}$ and $z_{\max} = 90$.

5. Use penalty method to solve the following L.P. Problem: **[WBUT 2016]**

Minimize $Z = 6x + 4y$

subject to, $2x + 4y \le 12$

$3x + 2y = 10$

$5x + 3y \ge 15$

$x,\ y \ge 0$.

Answer:

Introducing slack and surplus variables into the given constraints, the standard form of the LPP can be written as follows:

$$\text{Max } W = -6x - 4y + 0\cdot z + 0\cdot s$$

subject to

$$\begin{aligned} 2x + 4y + z \quad &= 12 \\ 3x + 2y \quad &= 10 \\ 5x + 3y \quad -s &= 15 \end{aligned}$$

and $x,\ y,\ z,\ s \ge 0$

Now to form the initial basis, we introduce two artificial variables u and v with 2nd and 3rd constraint and finally the set of constraints becomes:

$$\begin{aligned} 2x + 4y + z \quad &= 12 \\ 3x + 2y \quad +u &= 10 \\ 5x + 3y \quad -s \quad +v &= 15 \end{aligned}$$

and $x,\ y,\ z,\ s,\ u,\ v \ge 0$.

Now according simplex algorithm we get,

		c_j		-6	-4	0	0	$-M$	$-M$	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_6$	Min. Ratio
0	$\vec{a}_3$	z	12	2	4	1	0	0	0	6
$-M$	$\vec{a}_5$	u	10	3	2	0	0	1	0	10/3
$-M$	$\vec{a}_6$	v	15	$\boxed{5}$	3	0	-1	0	1	$3 \rightarrow$
			$z_j - c_j$	$-8M+6$ $\uparrow$	$-5M+4$	0	M	0	0	

		$\vec{c}_j$		-6	-4	0	0	$-M$	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min. Ratio
0	$\vec{a}_3$	z	6	0	14/5	1	2/5	0	15
$-M$	$\vec{a}_5$	u	1	0	1/5	0	$\boxed{3/5}$	1	$5/3 \rightarrow$
-6	$\vec{a}_1$	x	3	1	3/5	0	$-1/5$	0	—
			$z_j - c_j$	0	$-M/5+2/5$	0	$-3M/5+6/5$ $\uparrow$	0	

			$\vec{c}_j$	-6	-4	0	0	$-M$	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min. Ratio
0	$\vec{a}_3$	z	6	0	14/5	1	2/5	0	15
$-M$	$\vec{a}_5$	u	1	0	1/5	0	$\boxed{3/5}$	1	5/3 →
-6	$\vec{a}_1$	x	3	1	3/5	0	$-1/5$	0	—
			$z_j - c_j$	0	$-M/5$ $+2/5$	0	$-3M/5$ $+6/5$ ↑	0	

			$\vec{c}_j$	-6	-4	0	0
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$
0	$\vec{a}_3$	z	16/3	0	8/3	1	0
0	$\vec{a}_4$	s	5/3	0	1/3	0	1
-6	$\vec{a}_1$	x	10/3	1	2/3	0	0
			$z_j - c_j$	0	0	0	0

Here all $z_j - c_j \geq 0$, so the optimality condition is satisfied.

Hence $z_{\min} = 20,\ x = \dfrac{10}{3},\ y = 0$.

6. Solve by Charnes' big-M method the following LPP: **[MODEL QUESTION]**

Minimize $Z = 4x_1 + 2x_2$

subject to

$3x_1 + x_2 \geq 27$

$x_1 + x_2 \geq 21$

$x_1 + 2x_2 \geq 30,\ x_1,\ x_2 \geq 0$.

Answer:

Step I: Standard form of the L.P.P. after introducing surplus variables x_3, x_4, x_5 and artificial variables x_6, x_7, x_8 and after converting to a Maximization probiem.

$$Max\ Z' = -4x_1 - 2x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 + 0\cdot x_5 - Mx_6 - Mx_7 - Mx_8$$

subject to

$$3x_1 + x_2 - x_3 + x_6 = 27$$
$$x_1 + x_2 - x_3 + x_7 = 21$$
$$x_1 + 2x_2 - x_5 + x_8 = 30$$

and $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$; and M is a very large (+)ve integer.

Step II: (Simplex for Big-M method)

			$c_j \rightarrow$	-4	-2	0	0	0	$-M$	$-M$	$-M$	
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	Ratio
$-M$	$\bar{a}_6$	x_6	27	$\boxed{3}$	1	-1	0	0	1	0	0	$\min\{9, 21, 30\} = 9$
$-M$	$\bar{a}_7$	x_7	21	1	1	0	-1	0	0	1	0	
$-M$	$\bar{a}_8$	x_8	30	1	2	0	0	-1	0	0	1	
			$z_j - c_j \rightarrow$	$+5M+4\uparrow$	$-4M+2$	$+M$	$+M$	$+M$	0	0	0	
-4		x_1	9	1	$1/3$	$-1/3$	0	0		0	0	$\min\{27, 18, 63/5\} = 63/5$
$-M$		x_7	12	0	$2/3$	$1/3$	-1	0		1	0	
$-M$		x_8	21	0	$\boxed{5/3}\uparrow$	$1/3$	0	-1		0	1	
			$z_j - c_j \rightarrow$	0	$-7M/3$ $-4/3$	$-2M/3$ $+4/3$	$+M$	$+M$		0	0	
-4	$\bar{a}_1$	x_1	$24/5$	1	0	$-2/5$	0	$1/5$		0		$\min\{24, 9\} = 9$
$-M$	$\bar{a}_7$	x_7	$18/5$	0	0	$+1/5$	-1	$\boxed{2/5}$		1		
-2	$\bar{a}_8$	x_2	$63/5$	0	1	$1/5$	0	$-3/5$ $\uparrow$		0		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	Ratio
			$z_j - c_j \rightarrow$	0	0	$-M/5$ $+6/5$	$+M$	$-2M/5$ $+2/5$		0		
-4	$\bar{a}_1$	x_1	3	1	0	$-1/2$	0	0				
0	$\bar{a}_5$	x_5	9	0	0	½	$-5/2$	1				
-2	$\bar{a}_2$	x_2	18	0	1	$1/2$	$-3/2$	0				
			$z_j - c_j \rightarrow$	0	0	$+1$	$+3$	0		0		

Step III: All $z_j - c_j \geq 0$ the artificial r variables are removed from the basic variables.
Optimal solution is $x_1 = 3; x_2 = 18$ and $Min\ Z = 4 \times 3 + 2 \times 18 = 48$.

7. Using simplex algorithm solve the following LPP:

[MODEL QUESTION]

Minimize $\quad Z = 2x_1 + 3x_2$

Subject to $\quad 3x_1 + 5x_2 \geq 30$

$$5x_1 + 3x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

Answer:

The given problem can be reduced to the standard form by introducing surplus variables. This can also be recast into a maximization problem.

Maximize $\quad (Z') = (-Z) = -2x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4$

S.t. $3x_1 + 5x_2 - x_3 = 30$

$5x_1 + 3x_2 - x_4 = 60$

$x_i \geq 0, \quad i = 1, 2, 3, 4.$

Here x_3, x_4 are surplus variables.

Now to obtain initial basis we introduce artificial variables x_5, x_6 and we get,

Maximize $Z' = -2x_1 - 3x_2 + x_3 + 0 \cdot x_4 - Mx_5 - Mx_6$

s.t. $3x_1 + 5x_2 - x_3 + x_5 = 30$

$5x_1 + 3x_2 - x_4 + x_6 = 60$

$x_i \geq 0, \quad (i = 1, 2, \cdots\cdots, 6)$

			c_j	-2	-3	0	0	$-M$	$-M$	
C_B	B	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_6$	Min. Ratio
$-M$	$\vec{a}_5$	x_5	30	$\boxed{3}$	5	-1	0	1	0	$10 \rightarrow$
$-M$	$\vec{a}_6$	x_6	60	5	3	0	-1	0	1	12
$z_j - c_j$				$-8M+2$	$-8M+3$	M	M	0	0	
				$\uparrow$				$\downarrow$		

			c_j	-2	-3	0	0	$-M$	
C_B	B	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_6$	Min. Ratio
-2	$\vec{a}_1$	x_1	10	1	5/3	$-1/3$	0	0	—
$-M$	$\vec{a}_6$	x_6	10	0	$-16/3$	5/3	-1	1	$6 \rightarrow$
$z_j - c_j$				0	$\frac{16M}{3} - \frac{1}{3}$	$\frac{-5M}{3} + \frac{1}{3}$	M	0	
						$\uparrow$		$\downarrow$	

			c_j	-2	-3	0	0
C_B	$\vec{b}$	X_B	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$
-2	$\vec{a}_1$	x_1	12	1	3/5	0	$-1/5$
0	$\vec{a}_3$	x_3	6	0	$-16/5$	1	$-3/5$
$z_j - c_j$				0	9/5	0	2/5

Here all $z_j - c_j \geq 0$, so optimality condition is satisfied.

$\therefore\ x_1 = 12,\ x_2 = 0$ and $Z_{\min} = 24$.

8. a) Solve the following LPP by simplex method: **[MODEL QUESTION]**

Minimize $Z = -2x_2 - x_3$

subject to $x_1 + x_2 - 2x_3 \le 7$

$-3x_1 + x_2 + 2x_3 \le 3;\ x_1, x_2, x_3 \ge 0.$

Answer:

Converting the given minimization problem to a maximization problem we obtain,

$\text{Max } Z' = 2x_2 + x_3,\ Z' = (-Z)$

s.t. $x_1 + x_2 - 2x_3 \le 7$

$-3x_1 + x_2 + 2x_3 \le 3$

$x_1, x_2, x_3 \ge 0$

Introducing slack variables $x_4, x_5 (\ge 0)$, the initial table of the simplex is given by

			C_j	0	2	1	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min. Ratio
0	$\vec{a}_4$	x_4	7	1	1	–2	1	0	7
0	$\vec{a}_5$	x_5	3	–3	$\boxed{1}$	2	0	1	3 →
		$Z_j - C_j$		0	–2 ↑	–1	0	0	

Here $\vec{a}_5$ is departing vector and $\vec{a}_2$ is entering vector.

			C_j	0	2	1	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min. Ratio
0	$\vec{a}_4$	x_4	4	4	0	–4	1	–1	—
2	$\vec{a}_2$	x_2	3	–3	1	$\boxed{2}$	0	1	3/2 →
		$Z_j - C_j$		0	0	–1 ↑	0	2	

			C_j	0	2	1	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	
0	$\vec{a}_4$	x_4	10	–2	2	0	1	1	
1	$\vec{a}_3$	x_3	3/2	–3/2	1/2	1	0	1/2	
		$Z_j - C_j$		–3/2 ↑	3/2	0	0	1/2	

Here $Z_1 - C_1 = -\frac{3}{2} < 0$, but both the components i.e. y_{11}, y_{21} are **negative. Hence the** solution of this problem is unbounded.

b) Solve the following LPP by Charnes' Big-M method: **[MODEL QUESTION]**

Maximize $Z = 3x_1 + 2x_2$

subject to $x_1 + x_2 \geq 1$

$2x_1 + x_2 \leq 4$

$5x_1 + 8x_2 \leq 15; x_1, x_2, x_3 \geq 0.$

Answer:

Introducing **slack** variables x_4, x_5 and **surplus** variable x_3, the standard form of the given LPP is

$$\text{Max } Z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

s.t.
$$\begin{aligned} x_1 + x_2 - x_3 \quad\quad &= 1 \\ 2x_1 + x_2 \quad + x_4 \quad &= 4 \\ 5x_1 + 8x_2 \quad\quad + x_5 &= 15 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Now to obtain initial basic feasible solution, we introduce an **artificial** variable x_6 with 1st constraint and the resulting LPP is

$$\text{Max } Z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4 + 0.x_5 - Mx_6$$

s.t.
$$\begin{aligned} x_1 + x_2 - x_3 \quad\quad + x_6 &= 1 \\ 2x_1 + x_2 \quad + x_4 \quad\quad &= 4 \\ 5x_1 + 8x_2 \quad\quad + x_5 \quad &= 15 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

According to simplex method (Table-1)

			C_j	3	2	0	0	0	–M	
C_B	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	$\vec{a}_6$	Min. Ratio
–M	$\vec{a}_6$	x_6	1	$\boxed{1}$	1	–1	0	0	1	1 →
0	$\vec{a}_4$	x_4	4	2	1	0	1	0	0	2
0	$\vec{a}_5$	x_5	15	5	8	0	0	1	0	3
		$Z_j - C_j$		–M–3 ↑	–M–2	M	0	0	0	

			C_j	3	2	0	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min. Ratio
3	$\vec{a}_1$	x_1	1	1	1	–1	0	0	—
0	$\vec{a}_4$	x_4	2	0	–1	$\boxed{2}$	1	0	1 →
0	$\vec{a}_5$	x_5	10	0	3	5	0	1	2
		$Z_j - C_j$		0	1	–3 ↑	0	0	

$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	C_j / $\vec{b}$	3 / $\vec{a}_1$	2 / $\vec{a}_2$	0 / $\vec{a}_3$	0 / $\vec{a}_4$	0 / $\vec{a}_5$	Min. Ratio
3	$\vec{a}_1$	x_1	2	1	1	0	1/2	0	—
0	$\vec{a}_3$	x_3	1	0	−1/2	1	1/2	0	—
0	$\vec{a}_5$	x_5	5	0	$\boxed{8}$	0	−5/2	1	5/8 →
		$Z_j - C_j$		0	−1/2 ↑	0	3/2	0	

$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	C_j / $\vec{b}$	3 / $\vec{a}_1$	2 / $\vec{a}_2$	0 / $\vec{a}_3$	0 / $\vec{a}_4$	0 / $\vec{a}_5$
3	$\vec{a}_1$	x_1	27/16	1	0	0	21/32	−1/16
0	$\vec{a}_3$	x_3	21/16	0	0	1	11/32	1/16
2	$\vec{a}_2$	x_2	5/8	0	1	0	−5/16	1/8
		$Z_j - C_j$		0	0	0	43/32	1/16

Here all $Z_j - C_j \geq 0$. So optimality condition is satisfied. Hence the optimal solution is

$$x_1 = \frac{27}{16}, x_2 = \frac{5}{8} \; Z_{\text{Max}} = \frac{101}{16}.$$

9. Use simplex algorithm to solve the following LPP: **[MODEL QUESTION]**

Maximize $Z = 4x_1 + 7x_2$

subject to $2x_1 + x_2 \leq 1000$

$10x_1 + 10x_2 \leq 6000$

$2x_1 + 4x_2 \leq 2000$

$x_1, x_2 \geq 0$

Answer:

Given LPP can be written in the following form:

$$\text{Max } z = 4x_1 + 7x_2$$

$$\text{s.t.} \quad 2x_1 + x_2 \leq 1000$$

$$x_1 + x_2 \leq 600$$

$$x_1 + 2x_2 \leq 1000$$

$$x_1, x_2 \geq 0$$

Now introducing slack variables x_3, x_4, x_5 with the three given constraints we obtain,

$$\text{Max } z = 4x_1 + 7x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5$$

$$2x_1 + x_2 + x_3 = 1000$$

$$x_1 + x_2 + x_4 = 600$$

$$x_1 + 2x_2 \qquad\qquad + x_5 = 1000$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0$$

Simplex Tables:

		c_j		4	7	0	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min Ratio
0	$\vec{a}_3$	x_3	1000	2	1	1	0	0	1000
0	$\vec{a}_4$	x_4	600	1	1	0	1	0	600
0	$\vec{a}_5$	x_5	1000	1	$\boxed{2}$	0	0	1	500 →
$z_j - c_j$				−4	−7 ↑	0	0	0	

		c_j		4	7	0	0	0	
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$	Min Ratio
0	$\vec{a}_3$	x_3	500	$\frac{3}{2}$	0	1	0	$-\frac{1}{2}$	$\frac{1000}{3}$
0	$\vec{a}_4$	x_4	100	$\boxed{\frac{1}{2}}$	0	0	1	$-\frac{1}{2}$	200 →
7	$\vec{a}_2$	x_2	500	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1000
$z_j - c_j$				$-\frac{1}{2}$ ↑	0	0	0	$\frac{7}{2}$	

		c_j		4	7	0	0	0
$\vec{C}_B$	$\vec{B}$	$\vec{X}_B$	$\vec{b}$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{a}_5$
0	$\vec{a}_3$	x_3	200	0	0	1	$-\frac{3}{2}$	1
4	$\vec{a}_1$	x_1	200	1	0	0	1	−1
7	$\vec{a}_2$	x_2	400	0	1	0	$-\frac{1}{2}$	1
$z_j - c_j$				0	0	0	$\frac{1}{2}$	3

Here all $z_j - c_j \geq 0$. So, the optimality condition of simplex method is satisfied. The optimal solution is

$x_1 = 200, x_2 = 400$ and $z_{\text{Max}} = 4 \times 200 + 7 \times 400 = 3600$

10. Using Simplex method solve the following LPP: **[MODEL QUESTION]**

Minimize $Z = 4x_1 + 3x_2,$

subject to $x_1 + 2x_2 \geq 8$

$3x_1 + 2x_2 \geq 12$

and $x_1, x_2 \geq 0$ **by Charnes Big M method.**

Answer:

After introducing surplus and artificial variables, the problem isre-stated as

Maximize $Z^* = \text{Maximize } (-Z) = -4x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4 - Mx_5 - Mx_6$

s.t.

$$x_1 + 2x_2 - x_3 \quad + x_5 \quad = 8$$

$$3x_1 + 2x_2 \quad - x_4 \quad + x_6 = 12$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

where x_3, x_4 are the surplus variables and x_5, x_6 are the artificial variables. M is a very large number.

Now by simplex method:

			C_j	-4	-3	0	0	$-M$	$-M$	
$\vec{C_B}$	$\vec{B}$	$\vec{X_B}$	$\vec{b}$	$\vec{a_1}$	$\vec{a_2}$	$\vec{a_3}$	$\vec{a_4}$	$\vec{a_5}$	$\vec{a_6}$	Min Ratio
$-M$	$\vec{a_5}$	x_5	8	1	$\boxed{2}$	-1	0	1	0	$4 \rightarrow$
$-M$	$\vec{a_6}$	x_6	12	3	2	0	-1	0	1	6
$Z_j - C_j$				$-4M$ $+4$	$-4M$ $+3$ $\uparrow$	M	M	0	0	

			C_j	-4	-3	0	0	$-M$	
$\vec{C_B}$	$\vec{B}$	$\vec{X_B}$	$\vec{b}$	$\vec{a_1}$	$\vec{a_2}$	$\vec{a_3}$	$\vec{a_4}$	$\vec{a_6}$	Min Ratio
-3	$\vec{a_2}$	x_2	4	$1/4$	1	$-1/4$	0	0	16
$-M$	$\vec{a_6}$	x_6	4	$\boxed{5/2}$	0	$1/2$	-1	1	$8/5 \rightarrow$
$Z_j - C_j$				$-\frac{5M}{2} + \frac{13}{4}$ $\uparrow$	0	$-M/2$ $+3/4$	M	0	

			C_j	-4	-3	0	0
$\overline{C_B}$	$\vec{B}$	$\overline{X_B}$	$\vec{b}$	$\overline{a_1}$	$\overline{a_2}$	$\overline{a_3}$	$\overline{a_4}$
-3	$\overline{a_2}$	x_2	18/5	0	1	-3/10	1/10
-4	$\overline{a_1}$	x_1	8/5	1	0	1/5	-2/5
$Z_j - C_j$				0	0	1/10	13/10

Here all $Z_j - C_j \geq 0$. so optimality condition is satisfied.

Therefore. $x_1 = \frac{8}{5}$, $x_2 = \frac{18}{5}$ and $z_{Min} = \frac{86}{5}$.

TRANSPORTATION & ASSIGNMENT PROBLEM

Multiple Choice Type Questions

1. In most of the transportation problems, the nearest optimal solution is given by [WBUT 2013]
   a) North-West Corner Rule (NWCR)
   b) Vogel's Approximation Method (VAM)
   c) Stepping Stone Method
   d) None of these

Answer: (b)

2. Hungarian method is applied for solution of [WBUT 2013, 2016]
   a) Assignment problems
   b) Decision theory related problems
   c) Queuing problems
   d) All of these

Answer: (a)

3. The assignment matrix is always a/an [WBUT 2014, 2016]
   a) rectangular matrix
   b) square matrix
   c) identity matrix
   d) none of these

Answer: (b)

4. For a salesman who has to visit *n* cities which of the following are the ways of his tour plan? [WBUT 2014]
   a) $n!$ b) $(n+1)!$ c) $(n-1)!$ b) n

Answer: (a)

5. Which method in Transportation problems gives the nearest optimal solution [WBUT 2015]
   a) North-West corner rule
   b) Least cost method
   c) Vogal's approximation method
   d) Row minimum method

Answer: (c)

6. The number of Basic variables in a transportation problem is at most [WBUT 2015]
   a) $m+n-1$ b) $m+n+1$ c) $m-n+1$ d) $mn+1$

Answer: (a)

7. Hugarian method is a special type of [WBUT 2015]
   a) LPP
   b) Simplex method
   c) Transportation problem
   d) Inventory

Answer: (b)

8. In balanced assignment problem the cost matrix must be [WBUT 2015]
   a) square b) singular c) non singular d) none of these

Answer: (a)

9. In transportation problem demand and supply should be [WBUT 2015]
a) same
b) not same
c) both (a) and (b)
d) none of these

Answer: (c)

10. In PERT the span of time between the optimistic and pessimistic time estimates of an activity is [WBUT 2017]
a) 3σ
b) 6σ
c) 12σ
d) none of these

Answer: (b)

11. The solution to a transportation problem with m-rows and n-columns is feasible if number of positive allocations is [WBUT 2017]
a) $m+n$
b) $m \times n$
c) $m+n-1$
d) $m+n+1$

Answer: (c)

12. VED analysis is applied for determining [WBUT 2017]
a) slow moving items
b) high value items
c) critical items
d) none of these

Answer: (c)

13. An assignment problem is a special type of [MODEL QUESTION]
a) transportation problem
b) LPP
c) inventory problem
d) none of these

Answer: (a)

14. An assignment problem can be solved by [MODEL QUESTION]
a) Hungarian method
b) VAM
c) Matrix minima method
d) None of these

Answer: (a)

15. In an assignment problem, the minimum number of lines covering all zeros in the reduced cost matrix of order n can be [MODEL QUESTION]
a) at most n
b) n + 1
c) n – 1
d) at least n

Answer: (a)

16. The total number of possible solutions for n×n assignment problem is always [MODEL QUESTION]
a) n
b) n-1
c) 1
d) n!

Answer: (d)

17. Which of the following is not method to obtain the initial basic feasible solution in transportation problem? [MODEL QUESTION]
a) VAM
b) Least cost method
c) North-West corner method
d) MODI Method

Answer: (d)

18. In the case of degeneracy while solving transportation problem, the small allocation is made in [MODEL QUESTION]
a) non-occupied cell
b) occupied cell
c) a non-occupied cell in independent position
d) none of these

Answer: (c)

19. A transportation problem is a balanced transportation problem if - [MODEL QUESTION]
a) total demand and total supply are equal and no. of sources equals to the no. of destinations
b) total demand equals to the total supply irrespective of the no. of sources and destinations
c) number of sources matches with the number of destinations
d) the corresponding basic feasible solution is to be degenerate

Answer: (b)

20. The initial basic feasible solution of a transportation problem becomes non-optimal if the matrix $\lfloor \nabla_{ij} \rfloor = \lfloor C_{ij} \rfloor - \lfloor u_i + v_j \rfloor$ has [MODEL QUESTION]
a) at least one negative element
b) at most one negative element

Answer: (a)

21. The solution of the transportation problem is never unbounded. Is the statement [MODEL QUESTION]
a) True
b) False

Answer: (a)

22. The number of non-basic variables in the balanced TP with 4 rows and 5 columns is [MODEL QUESTION]
a) 4
b) 5
c) 12
d) 20

Answer: (c)

23. The balanced transportation problem is where [MODEL QUESTION]
a) Total supply > Total demand
b) Total supply < Total demand
c) Total supply = Total demand
d) none of these

Answer: (c)

Short Answer Type Questions

1. In a factory there are six machines (of the same type) and five workers. The handling costs for the *i* th worker $(i = 1, 2, 3, ...5)$ *t* handle the *j* th machine (j = 1, 2, 3, ...6) are given below in the form of a matrix. Find the optimal assignment and the minimum cost of handling the machine and find which machine will remain unused: [WBUT 2013]

	M_1	M_2	M_3	M_4	M_5	M_6
W_1	12	8	10	14	11	18
W_2	14	14	8	15	17	12
W_3	9	11	13	15	6	12
W_4	11	9	9	11	8	14
W_5	10	12	15	13	10	12

Answer:

The given problem is an unbalanced problem. To make it balance we add a fiction worker W_6 with zero cost and then apply Hungarian methods to solve it.

Subtracting the minimum element of each row from every element of that row and subtracting the least element of each column of from every element of that column we get. Now we draw minimum number of horizontal and vertical line to cover all the zeros. The number of lines is 5 but the order of the matrix is 6. Now we subtract minimum element from every uncovered element and adding at the of intersection of the lines we get.

Again we draw minimum number lines to cover the zeros.

Here number of lines is equal to the order of the matrix.

Therefore, the optimal assignment is

$$W_1 \to M_2, W_2 \to M_3, W_3 \to M_6, W_4 \to M_5, W_5 \to M_1$$

Minimum cost: $8+8+12+8+10=46$ units

	M_1	M_2	M_3	M_4	M_5	M_6
W_1	12	8	10	14	11	18
W_2	14	14	8	15	17	12
W_3	9	11	13	15	6	12
W_4	11	9	9	11	8	14
W_5	10	12	15	13	10	12
W_6	0	0	0	0	0	0

$$\Downarrow$$

$$\begin{bmatrix} 4 & 0 & 2 & 4 & 3 & 8 \\ 6 & 6 & 0 & 5 & 9 & 2 \\ 3 & 5 & 7 & 7 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 & 4 \\ 0 & 2 & 5 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{c|cccccc|} & M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\ \hline W_1 & 4 & 0 & 2 & 4 & 3 & 7 \\ W_2 & 6 & 6 & 0 & 5 & 9 & 1 \\ W_3 & 3 & 5 & 7 & 7 & 0 & 0 \\ W_4 & 3 & 1 & 1 & 0 & 0 & 3 \\ W_5 & 0 & 2 & 5 & 1 & 0 & 0 \\ W_6 & 2 & 2 & 2 & 0 & 2 & 0 \\ \hline \end{array}$$

2. Five men are available to do five different jobs. From past records the time (in hrs) that each man takes to do each job is known and is given in the following table. Assign each man to a job to get minimum time. [WBUT 2014]

Man	Job				
	1	2	3	4	5
1	8	4	2	6	1
2	0	9	5	5	4
3	3	8	9	2	6
4	4	3	1	0	3
5	9	5	8	9	5

Answer:

Here we apply Hungarian method to solve the given assignment problem.

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 8 & 4 & 2 & 6 & 1 \\
2 & 0 & 9 & 5 & 5 & 4 \\
3 & 3 & 8 & 9 & 2 & 6 \\
4 & 4 & 3 & 1 & 0 & 3 \\
5 & 9 & 5 & 8 & 9 & 5 \\
\hline
\end{array}
\xrightarrow{\text{Row operation}}
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 7 & 3 & 1 & 5 & 0 \\
2 & 0 & 9 & 5 & 5 & 4 \\
3 & 1 & 6 & 7 & 0 & 4 \\
4 & 4 & 3 & 1 & 0 & 3 \\
5 & 4 & 0 & 3 & 4 & 0 \\
\hline
\end{array}$$

$$\xrightarrow{\text{Column operation}}
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 7 & 3 & 0 & 5 & 0 \\
2 & 0 & 9 & 4 & 5 & 4 \\
3 & 9 & 6 & 6 & 0 & 4 \\
4 & 4 & 3 & 0 & 0 & 3 \\
5 & 4 & 0 & 2 & 4 & 0 \\
\hline
\end{array} \Rightarrow$$

No. of straight lines = order of the matrix = 5. So optimality condition is satisfied and we can assign the jobs in the final table as follows:

$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 7 & 3 & \cancel{0} & 5 & 0 \\
2 & 0 & 9 & 4 & 5 & 4 \\
3 & 9 & 6 & 6 & 0 & 4 \\
4 & 4 & 3 & 0 & \cancel{0} & 3 \\
5 & 4 & 0 & 2 & 4 & \cancel{0} \\
\hline
\end{array}$$

So the required assignment is $1 \rightarrow 5, 2 \rightarrow 1, 3 \rightarrow 4, 4 \rightarrow 3, 5 \rightarrow 2$ and minimum time $= 1+0+2+1+5 = 9$ hrs.

3. Solve the Assignment problem: **[WBUT 2015**

	D1	D2	D3	D4
J1	8	26	17	11
J2	13	28	4	26
J3	28	19	18	15
J4	19	26	24	10

Answer:

	D_1	D_2	D_3	D_4
J_1	8	26	17	11
J_2	13	28	4	26
J_3	28	19	18	15
J_4	19	26	24	10

$\xrightarrow{\text{Row operation}}$

0	14	9	3
9	24	0	22
13	4	3	0
9	16	14	0

Column operation →

0	10	9	3
9	20	0	22
13	0	3	0
9	12	14	0

Here no. of straight lines = order of the matrix.
So we can assign in this stage.

	D_1	D_2	D_3	D_4
J_1	0	10	9	3
J_2	9	20	0	22
J_3	13	0	3	0
J_4	9	12	14	0

So the required assignment is

$J_1 \to D_1,\ J_2 \to D_3,\ J_3 \to D_2,\ J_4 \to D_4$

4. Explain how an assignment problem can be solved as a transportation problem. **[WBUT 2016]**

Answer:

The assignment problem becomes a linear programming problem when the 0–1 restriction on the variables is relaxed. In fact, the coefficient matrix of the assignment problem is unimodular and therefore, we can relax the 0–1 restriction and solve it as a linear programming problem to get the optimal solution. But assignment problems are rarely solved using the simplex algorithm but have special algorithms that solve the problem faster and more efficiently.

You may also have observed by now that the assignment problem is similar to the transportation problem since the objective functions are similar and so is the coefficient matrix. The first difference however, is that the assignment problem has a square matrix [equal number of rows (resources) and columns (tasks)] while the transportation problem can have unequal number of supply and demand points. Later in this chapter, we will see what we do if the assignment problem is not square.

The assignment problem is a special case of the transportation problem with $m = n$, all $a_i = 1$ and all $b_j = 1$. We do not solve the assignment problem using the transportation algorithm because it is a degenerate transportation problem. We have already seen that degenerate LP problems and transportation problems may require additional iterations and take long time to reach optimality.

A feasible solution to the $n \times n$ assignment problem has exactly n assignments while the LP formulation has n^2 decision variables and $2n$ constraints. Out of the $2n$ constraints there are $2n-1$ basic variables because the coefficient matrix represents a linearly dependent system of equations. Since the assignment solution has n allocations out of the $2n-1$ basic variables, it means that $n-1$ basic variables take value '0' indicating degeneracy. Hence assignment problem is neither solved as a transportation problem not as a linear programming problem.

5. By North-West corner method, find a basic feasible solution of the given transportation problem: **[WBUT 2017]**

		Destination				
		B_1	B_2	B_3	B_4	a_i
Source	A_1	**2**	**11**	**10**	**3**	**6**
	A_2	**1**	**4**	**7**	**2**	**5**
	A_3	**3**	**9**	**4**	**8**	**4**
	b_j	**3**	**3**	**4**	**5**	

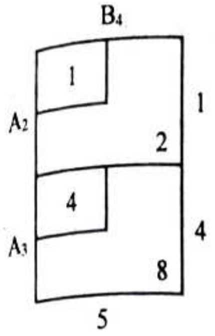
Answer:

According to North-West corner method:

	B_1	B_2	B_3	B_4	a_i
A_1	$\boxed{3}$ 2	11	10	3	6
A_2	1	4	7	2	5
A_3	3	9	4	8	4
b_j	3	3	4	5	

	B_3	B_3	B_4	
A_1	$\boxed{3}$ 11	10	3	3
A_2	4	7	2	5
A_3	9	4	8	4
b_j	3	4	5	

	B_3	B_4	
A_2	$\boxed{4}$ 7	2	5
A_3	4	8	4
	4	5	



So initial basic feasible solution is

$$x_{11} = 3,\ x_{12} = 3,\ x_{23} = 4,\ x_{24} = 1,\ x_{34} = 4$$

and corresponding cost is

$$2\times3 + 11\times3 + 7\times4 + 2\times1 + 8\times4 = 99$$

6. Prove that, the balance transportation problem always has a feasible solution.

Answer: **[MODEL QUESTION]**

Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = T\,(\text{say})$

Consider the solution

$$x_{ij} = \frac{a_i b_j}{T} \geq 0 \text{ for } i = 1,2,\ldots,m \;\&\; j = 1,2,\ldots n.$$

then, $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_i b_j \times \frac{1}{T} = \frac{1}{T} a_i \sum_{j=1}^{n} b_j = \frac{1}{T}.a_i.T = a_i$

$$\Rightarrow \sum_{j=1}^{n} x_{ij} = a_i, \text{ for } i = 1,2,\ldots,m.$$

Similarly, $\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{T} = \frac{b_j}{T} \sum_{i=1}^{m} a_i = \frac{b_j.T}{T} = b_j$

$$\Rightarrow \sum_{i=1}^{m} x_{ij} = b_j, \text{ for } j = 1,2,\ldots,n.$$

Therefore, solution $x_{ij} = \frac{a_i b_j}{T} > 0$, satisfies all the constraints. Hence it is a feasible solution.

7. Solve the assignment problem given below for minimum cost

[MODEL QUESTION]

Machine \ Men	A	B	C	D
M_1	18	26	17	11
M_2	13	28	14	26
M_3	38	19	18	15
M_4	19	26	24	10

Answer:

	A	B	C	D
M_1	18	26	17	11
M_2	13	28	14	26
M_3	38	19	18	15
M_4	19	26	24	10

Here, order of the matrix $n = 4$

Step 1: After Row Operation

	A	B	C	D
M_1	7	15	6	0
M_2	0	15	1	13
M_3	23	4	3	0
M_4	9	16	14	0

Step 2: After Column Operation

	A	B	C	D
M_1	7	11	5	0
M_2	0	11	0	13
M_3	23	0	2	0
M_4	9	12	13	0

The minimum number of lines that covers all the zeros is $N = 3 \Rightarrow N = 3 < n = 4$

Hence optimal condition is not satisfied.

Step 3:

Minimum of the uncovered elements is 5. Subtract 5 from each of the uncovered element and add 5 to each of elements which is covered by both horizontal and vertical lines and other elements left unchanged.

	A	B	C	D
M_1	2	6	$\boxed{0}$	0
M_2	$\boxed{0}$	11	0	18
M_3	23	$\boxed{0}$	2	5
M_4	4	7	8	$\boxed{0}$

Step 4:

Now, N=n= 4, hence optimality condition is satisfied and optimal assignment is $M_1 \to C$; $M \to A$; $M_3 \to B$ and $M_4 \to D$.

Minimum cast = 17+13+19+10 = 59.

8. Prove that the number of basic variables in a transportation problem is $(m+n-1)$. **[MODEL QUESTION]**

Answer:

Here we denote the warehouses as W_i and market by M_j, for $i = 1, 2, \ldots\ldots, m$ and $j = 1, 2, \ldots\ldots, n$. Let x_{ij} be the amount of a certain product supplied from W_i to M_j. Then the total outflow at W_i and the total inflow at M_j are respectively $\sum_{j=1}^{n} x_{ij}$ and $\sum_{i=1}^{m} x_{ij}$ and, therefore, for a balanced T.P.

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1,2,\ldots\ldots,m; \ldots\ldots(1)$$

and $$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1,2,\ldots\ldots,n; \ldots\ldots(2)$$

Also since the supplies x_{ij}, in order to be feasible, should be non-negative i.e., $x_{ij} \geq 0$ for all i and j and the total cost of Transportation (which has to minimize) is

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} x_{ij} \ldots\ldots(3)$$

Moreover, we have $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \ldots\ldots(4)$

Rewriting, we get

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} x_{ij}$$

subject to constraints

$$\left.\begin{aligned} &\sum_{j=1}^{n} x_{ij} = a_i;\ i = 1,2,\ldots\ldots,m \\ &\sum_{i=1}^{m} x_{ij} = b_j;\ j = 1,2,\ldots\ldots,n \end{aligned}\right\}\ldots\ldots(5)$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \ldots\ldots(6)$$

and $$x_{ij} \geq 0$$

In the above formulation of the T.P., we have $m \times n$ cost coefficients C_{ij}, $m \times n$ variables x_{ij} and $(m + n)$ linear equations (as in (5)). Due to the presence of (6), all the $(m + n)$ linear equations in (5) are not linearly independent. Only $(m + n - 1)$ linear equations in (5) are linearly independent. So, this situation dictates us that out of $m \times n$ variables only $(m + n - 1)$ variables x_{ij}, at most should be basic variables and remaining $mn - (m + n - 1)$ variables should be non-basic variable for a T.P.

9. What is an unbalanced transportation problem? How can it be solved? Illustrate. **[MODEL QUESTION]**

Answer:
It is obvious that the market demands can be met if and only if the total supply from the warehouses is at least equal to the total demand at the markets, i.e., $\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$. When the total supply from the warehouses met exactly total demand of all markets i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then the transportation problem is known as ***balance transportation problem.***

Now if, i.e., $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, the T.P. becomes unbalanced.

In physical sense, if $\sum a_i > \sum b_j$, there would be surplus left at the warehouses after all the markets' demand met, and if $\sum a_i < \sum b_j$, there would be deficit at the warehouses to meet all the markets' demand. Problems involving surpluses and deficit are very common and significant in practical life. These type of problems to be posed properly to have feasible solutions after introducing artificial market / warehouse as required to make it balanced T.P.

10. Explain the concept of degeneracy in transportation problem. **[MODEL QUESTION]**

Answer:
Degeneracy
A basic feasible solution (B.F.S.) of a transportation problem, is said to be degenerate if one or more basic variables assume zero value an initial solution could become degenerate whenever the remaining supply and demand are of equal amount for a variable selected as the next basic variable.

11. Find the initial basic feasible solution of the Transportation problem given below: **[MODEL QUESTION]**

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 3 & 2 & 1 \\ 4 & 2 & 5 & 9 \end{bmatrix} \begin{array}{c} \textbf{Availabillity} \\ 30 \\ 50 \\ 20 \end{array}$$

$$\textbf{Demand} \quad 20 \quad 40 \quad 30 \quad 10$$

Using (i) North-West Corner Rule and (ii) Vogel's approximation method and compare the results obtained. Test the optimality of the two solutions.

Answer:
(i) Initial B.F.S by North – West Corner method

	D_1	D_2	D_3	D_4	$a_i\downarrow$
O_1	20 1	10 2	1	4	30
O_2	3	30 3	20 2	1	50
O_3	4	2	10 5	10 9	20
$b_j\rightarrow$	20	40	30	10	100

The T.P. is a balance T.P

Here $m = 3$, $n=4$ $\therefore m+n-1=3+4-1=6$

The I.B.F.S. is $x_{11}=20$, $x_{12}=10$, $x_{22}=30$, $x_{23}=20$, $x_{33}=10$ and $x_{34}=10$

Value of

$Z=1\times 20+2\times 10+3\times 30+2\times 20+5\times 10+9\times 10=310$

Optimality Test: Set $u_1=0$.

	D_1	D_2	D_3	D_4	$u_i\downarrow$
O_1	20 1	10 2	1	4	$u_1=0$
O_2	3	30 3	20 2	1	$u_2=1$
O_3	4	2	10 5	10 9	$u_3=4$
$v_j\rightarrow$	$v_1=1$	$v_2=2$	$v_3=1$	$v_4=5$	

Cell evaluation Δ_{ij} for non-basic cells:

$\Delta_{13}=c_{13}-(u_1+v_3)=1-(0+1)=0$

$\Delta_{14}=c_{14}-(u_1+v_4)=4-(0+5)=-1<0$

$\Delta_{21}=c_{21}-(u_2+v_1)=3-(1+1)=1$

$\Delta_{24}=c_{24}-(u_2+v_4)=1-(1+5)=-5<0$

$\Delta_{31}=c_{31}-(u_3+v_1)=4-(4+1)=-1<0$

$\Delta_{32}=c_{32}-(u_3+v_2)=2-(4+2)=-4<0$

Hence the I.B.F.S (obtained by N-W corner method is not an optimal solution.

I.B.F.S. by VAM-method: Using the idea of penalty, we get the I.B.F.S. as in the table below:

	D1	D_2	D_3	D_4	$a_i\downarrow$
O_1	20 1	2	10 1	4	30
O_2	3	20 3	20 2	10 1	50
O_3	4	2	5	10 9	20
$b_j\rightarrow$	20	40	30	10	100

The I.B.F.S. by VAM is
$x_{11} = 20, x_{13} = 10, x_{22} = 20, x_{23} = 20,$
$x_{24} = 10$ and $x_{32} = 20$ and $Z = 180$.
Optimality Test : Set ; $u_1 = 0$.
Cell evaluation for non- basic cells;

$\Delta_{12} = c_{12} - (u_1 + v_2) = 2 - (0+2) = 0$ $\quad$ $\Delta_{31} = c_{31} - (u_3 + v_1) = 4 - (0+1) = 3$

$\Delta_{14} = c_{14} - (u_1 + v_4) = 4 - (0+0) = 4$ $\quad$ $\Delta_{33} = c_{33} - (u_3 + v_3) = 5 - (1+1) = 4$

$\Delta_{21} = c_{21} - (u_2 + v_1) = 3 - (1+1) = 1$ $\quad$ $\Delta_{34} = c_{34} - (u_3 + v_4) = 9 - (0+0) = 9$

Hence all $\Delta_{ij} \geq 0$ for the non-basic cells, therefore, the I.B.F.S. is an optimal solution and
Min $Z = 20 \times 1 + 10 \times 1 + 20 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 2 = 180$
Comparing the results we conclude that VAM method is a rapid converging method.

12. The following data are given:
The cost of shipment from third source to third destination is not know. How many units should be transported from source to destination so that the total cost of transporting all the units to their destinations is minimum? **[MODEL QUESTION]**

Answer:

Since the cost C_{33} is known, assign a large cost, say $M > 0$, to the cell (3, 3). Now, using VAM, we get initial basic feasible solution as:
$x_{13} = 10, x_{23} = 15, x_{31} = 20, x_{32} = 15, x_{33} = 5,$
Using MODI method, it can be shown that $\Delta_{ij} = C_{ij} - (u_{ij} + v_{ij}) \geq 0$, for all the cells.
Hence the above I.B.F.S is an optimal solution.
Note: It is to be noted that the cell (3, 3) also appears in the above solution for which the cost of shipment is not known. Hence, above optimal solution is known as *pseudo optimal basic feasible solution.*

13. Find the basic feasible solution of the following transportation problem by North-West Corner rule. **[MODEL QUESTION]**

	P	Q	R	Supply
A	5	1	8	12
B	2	4	0	14
C	3	6	7	4
Demand	9	10	11	

Answer:

—	P	Q	R	Supply
A	5 $\boxed{9}$	1 $\boxed{3}$	8	$\cancel{12}$ $\cancel{3}$ 0
B	2	4 $\boxed{7}$	0 $\boxed{7}$	$\cancel{14}$ $\cancel{7}$ 0
C	3	6	7 $\boxed{4}$	$\cancel{4}$ 0
Demand	$\cancel{9}$ 0	$\cancel{10}$ $\cancel{7}$ 0	$\cancel{11}$ $\cancel{4}$ 0	30

A basic feasible solution by North-West corner method (rule):

$$x_{11} = 9, x_{12} = 3, x_{22} = 7, x_{23} = 7 \text{ and } x_{33} = 4.$$

14. There are three X, Y and Z which supply goods to four different dealers D_1, D_2, D_3 and D_4. The production capacities of these factories are 1000, 700 and 900 units per month respectively. The requirements from the dealers are 900, 800, 500 and 400 units per month respectively. The profit matrix at the factories for unit transport to the dealers are given below (in Rs.). Find optimal solution of the problem:
[MODEL QUESTION]

	D_1	D_2	D_3	D_4
X	6	6	6	4
Y	4	2	4	5
Z	5	6	7	8

Answer:

First, we convert maximization problem to a minimization problem by subtracting all the unit profits from the largest unit profit Rs. 8 and solve this minimization problem using VAM

	D_1	D_2	D_3	D_4	Capacity a_i	$u_i \downarrow$
X	$\boxed{200}$ 2	$\boxed{800}$ 2	$\boxed{\varepsilon}$ 2	4	1000	$u_1 = 0$
Y	$\boxed{700}$ 4	6	4	3	700	$u_2 = 2$
Z	3	2	$\boxed{500}$ 1	$\boxed{400}$ 0	900	$u_3 = -1$
Demands b_j	900	800	500	400	2600	
	$v_1 = 2$	$v_2 = 2$	$v_3 = -2$	$v_4 = 1$		

This is a balanced T.P. $\left(\sum a_i = 2600 = \sum b_j\right)$

Using VAM, we find I.B.F.S. as

$$x_{11} = 200, x_{12} = 800, x_{21} = 700, x_{33} = 500 \;\&\; x_{34} = 400$$

which is degenerate solution $(m + n - 1 = 3 + 4 - 1 = 6)$.

Setting $u_1 = 0$, we find values of v_1, v_2, u_2 but not values of v_3, v_4 and u_3. If we allocate ε (> 0) in the cell (1, 3) we find $v_3 = 2$, $u_3 = -1$ and $v_4 = 1$.
Optimality test:

$$\Delta_{14} = 4-(0+1)=3 \qquad \Delta_{24} = 3-(2+1)=0$$
$$\Delta_{22} = 6-(2+2)=2 \qquad \Delta_{31} = 3-(-1+2)=2$$
$$\Delta_{23} = 4-(2+2)=0 \qquad \Delta_{32} = 2-(-1+2)=1$$

Hence, $\Delta_{ij} \geq 0$ for all non-basic cells.;

Optimal supply is

$$X \xrightarrow{200\ units} D_1 \ \& \ X \xrightarrow{800\ units} D_2$$
$$Y \xrightarrow{700\ units} D_1$$
$$Z \xrightarrow{500\ units} D_3 \ \& \ Z \xrightarrow{400\ units} D_4$$

$\text{Max}\, Z = Rs.(6\times200+6\times800+4\times700+7\times500+8\times400) = Rs.15500.$

15. Solve the following assignment problem using Hungarian method:

[MODEL QUESTION]

Machines →

JOBS ↓

	M_1	M_2	M_3	M_4
J_1	1	4	6	3
J_2	9	7	10	9
J_3	4	5	11	7
J_4	8	7	8	5

Answer:

Step I: (Row operation)

We subtract least element of each row from each element of that row (for each row).

Step II: (Column operation)

Performing similar operations for each column, we get the modified matrix as (Table – 1)

	M1	M2	M3	M4
J1	0	3	2	2
J2	2	0	0	2
J3	0	1	4	3
J4	3	2	0	0

Table – 1

Step III: (Optimality test)

Now, we draw minimum numbers of lines to cover all the zeros (see Table – 1). Order of the matrix n = 4. Since, minimum numbers of lines N = 3 < n = 4, optimality criterion is not satisfied. Now, we select the minimum element 1 of the uncovered elements of the Table 1 and subtract it from each of the uncovered elements and add 1 to those elements only which are crossed by both horizontal and vertical lines and construct the (Table – 2) for optimality testing.

	M1	M2	M3	M4
J1	$\boxed{0}$	2	1	1
J2	3	⊗	$\boxed{0}$	2
J3	0	$\boxed{0}$	5	2
J4	4	2	⊗	$\boxed{0}$

Table – 2

Here N = 4; therefore n = N = 4 and the optimality condition is satisfied.

Step IV: (Assignment)

$$J_1 \longrightarrow M_1,\ J_2 \longrightarrow M_3,\ J_3 \longrightarrow M_2 \text{ and } J_4 \longrightarrow M_4.$$

Minimum cost (for this assignment) = 1 + 10 + 5 + 5 = 21

16. Find the initial basic feasible solution of the following transportation problem by North-West Corner method: **[MODEL QUESTION]**

	W_1	W_2	W_3	W_4	*Capacity*
F_1	10	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement:	5	8	7	14	

Answer:

Initial basic feasible solution by North-West Corner method:

$$x_{11} = 5,\ x_{12} = 2,\ x_{22} = 6,\ x_{23} = 3,\ x_{33} = 4,\ x_{34} = 14$$

Long Answer Type Questions

1. Find the initial solution to the following Transportation problem using VAM: **[WBUT 2013]**

		Destination				
		D_1	D_2	D_3	D_4	**Supply**
Factory	F_1	3	3	4	1	100
	F_2	4	2	4	2	125
	F_3	1	5	3	2	75
	Demand	120	80	75	25	300

Answer:
Using VAM we obtain,

	D_1	D_2	D_3	D_4					
F_1	$\boxed{100}$ 3	3	4	1	100 (2)	100 (2)	×		
F_2	4	$\boxed{80}$ 2	$\boxed{45}$ 4	2	125 (2)	45 (2)	45 (2)	45 (2)	×
F_3	$\boxed{20}$ 1	5	$\boxed{30}$ 3	$\boxed{20}$ 2	75 (1)	75 (1)	75 (1)	55 (1)	55 (1)
	120 (2)	80 (1)	75 (1)	25 (1)					
	120 (2)	×	75 (1)	25 (1)					
	20 (3)		75 (1)	25 (0)					
	×		75 (1)	25 (0)					
			30 (3)	25 (0)					

Here the number of allocations is $6(= m+n-1)$

So the initial solution is $x_{11} = 100, x_{22} = 80, x_{23} = 45, x_{31} = 20, x_{33} = 30, x_{34} = 20$.

Corresponding cost $= 100\times3+80\times2+45\times4+20\times1+30\times3+20\times2 = 790$ units.

2. A steel company has three open hearth furnaces and five rolling mills. Transportation cost (Rs. per quintal) for shipping steel from furnaces to rolling mills are shown in the following table. What is the optimal shipping schedule?

[WBUT 2014]

	M_1	M_2	M_3	M_4	M_5	Capacity in quintals
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	3	14
Requirement	4	4	6	8	8	

Answer:
Given problem is an unbalanced problem; at first to balance it we have to add one fictitious column M_6 with zero cost. Then to solve we apply VAM:

	M_1	M_2	M_3	M_4	M_5	M_6	a_i				
	4	[4] 2	3	[4] 2	6	0	8(2)	8(0)	8(0)	4(1)	4(1)
	5	4	5	[4] 2	[8] 1	0	12(1)	12(1)	4(2)	4(3)	×
	[4] 6	5	[6] 4	7	3	[4] 0	14(3)	10(1)	10(1)	10(2)	10(2)
b_j	4 (1)	4 (2)	6 (1)	8 (0)	8 (2)	4 (0)					
	4 (1)	4 (2)	6 (1)	8 (0)	8 (2)	×					
	4 (1)	4 (2)	6 (1)	8 (0)	×						
	4 (1)	×	6 (1)	8 (0)							
	4 (2)		6 (1)	4 (5)							
				×							

So the initial feasible solution is

$$x_{12}=4,\ x_{14}=4,\ x_{24}=4,\ x_{25}=8,\ x_{31}=4,\ x_{33}=6,\ x_{36}=4.$$

We see that no. of allocations is 7 while

$$m+n-1=3+6-1=8.$$

Thus the problem is degenerated.

To complete the basis, we allocate a very small positive quantity ε in $(1,1)$ such that this does not form a loop among some or all of the occupied cells and make them independent. To test the current basic solution for optimality, we calculate u_i and v_j and then the cell evaluation for non-basic cells as in the following table:

	M_1	M_2	M_3	M_4	M_5	M_6	u_i
F_1	[ε] 4	[4] 2	(1) 3	[4] 2	(5) 6	(2) 0	0
F_2	(1) 5	(2) 4	(3) 5	[4] 2	[8] 1	(2) 0	0
F_3	[4] 6	(1) 5	[6] 4	(3) 7	(4) 7	[4] 0	2
v_j	4	2	2	2	1	−2	

All the cell evaluations are positive and hence the optimal solution is

$$x_{12}=4,\ x_{14}=4,\ x_{24}=4,\ x_{25}=8,\ x_{31}=4,\ x_{33}=6$$

and the transportation cost $=4\times2+4\times2+4\times2+8\times1+4\times6+6\times4+4\times0=80$ (Rs.)

3. The warehouse capacities a_i, market demands b_j and the unit cost of shipping c_{ij} are given in the following table **[WBUT 2015]**

	M_1	M_2	M_3	M_4	a_i
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
b_j	4	3	4	4	15

Find the optimum solution for the problem.

Answer:

Here we apply VAM as follows:

	M_1	M_2	M_3	M_4	a_i			
W_1	[3] 2	2	2	1	3(1)	×		
W_2	10	8	[3] 5	[4] 4	7(1)	7(1)	3(3)	×
W_3	[1] 7	[3] 6	[1] 6	8	5(0)	5(0)	5(0)	5(0)
b_j	4 (5)	3 (4)	4 (3)	4 (3)				
	1 (3)	3 (2)	4 (1)	4 (4)				
	1 (3)	3 (2)	4 (1)	×				
	1 (7)	3 (6)	1 (6)					

So initial basic solution is $x_{11}=3,\ x_{23}=3,\ x_{24}=4,\ x_{31}=1,\ x_{32}=3,\ x_{33}=1$

and the corresponding cost $=2\times3+3\times5+4\times4+1\times7+3\times6+1\times6=68$

Optimality Test:

					u_i
	[3] 2	(1) 2	(1) 2	(1) 1	-5
	(4) 10	(3) 8	[3] 5	[4] 4	-1
	[1] 7	[3] 6	[1] 6	(3) 8	$u_3=0$
v_j	7	6	6	5	

Here all $\Delta_{ij}\ge 0$, so optimality condition is satisfied.

Hence the optimal solution is

$$x_{11}=3,\ x_{23}=3,\ x_{24}=4,\ x_{31}=1,\ x_{32}=3,\ x_{33}=1$$

and the optimal shipping charge is 68 units.

4. A company is producing a single product and is selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with the problem of dealing on how to assign the existing agencies to dispatch the product to needy cities in such a way that the travelling distance is minimized (in kms) between the surplus and deficit cities. The details are given in the following distance matrix.

[WBUT 2015]

Deficit Cities / Surplus Cities	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Answer:

To solve the given assignment problem, we apply Hungarian method as follows:

	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

$\xrightarrow{\text{Row operation}}$

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

$\xrightarrow{\text{Column operation}}$

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	05
20	0	25	15	15

Here no. of straight lines = 3 < 5 (order of the matrix)

So optimality condition is not satisfied.

Now subtract minimum most element among all covered elements from each element and add at the point of intersections.

15	0	20	15	0
15	15	0	10	0
15	0	20	15	05
0	15	20	0	05
05	0	10	0	0

Here no. of straight lines = 5 = order of the matrix.
So, we can assign in this matrix as follows:

	I	II	III	IV	V
A	15	0	20	15b	0
B	15	15	0	10	0
C	15	0	20	15	05
D	0	15	20	0	05
E	05	0	10	0	0

Hence the required assignment is

$$A \to V,\ B \to III,\ C \to II,\ D \to I,\ E \to IV$$

and min. travelling distance (is km) is

$$200+130+110+50+80=570$$

5. Four different jobs can be done on four different machines and the take-down time costs are prohibitively high for change over. The matrix below gives the cost in rupees for producing job i on the machine j. **[WBUT 2016]**

Jobs	Machines			
	M_1	M_2	M_3	M_4
J_1	**5**	**7**	**11**	**6**
J_2	**8**	**5**	**9**	**6**
J_3	**4**	**7**	**10**	**7**
J_4	**10**	**4**	**8**	**3**

Answer:

It is an assignment problem. We have to assign 4 jobs among 4 machines optimally.

$$\begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \\ \hline 5 & 7 & 11 & 6 \\ 8 & 5 & 9 & 6 \\ 4 & 7 & 10 & 7 \\ 10 & 4 & 8 & 3 \end{array} \xrightarrow{\text{Row operation}} \begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \\ \hline 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{array}$$

$$\xrightarrow{\text{Column operation}} \begin{array}{c} \\ J_1 \\ J_2 \\ J_3 \\ J_4 \end{array}\begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \\ \hline 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{array}$$

Here the minimum number of straight lines required to cover all the zero is 3 < order of the matrix. So optimality condition is not satisfied. Now we subtract the minimum most element from all the uncovered elements and add with the elements at the points of intersection.

$\downarrow$

$$\begin{array}{c|cccc|}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & 0 & 1 & 1 & 0 \\
J_2 & 4 & 0 & 0 & 1 \\
J_3 & 0 & 2 & 1 & 2 \\
J_4 & 8 & 1 & 1 & 0 \\
\hline
\end{array}$$

Here also no. of straight lines require to cover all the zeros is 3< order of the matrix. So optimality condition is not satisfied. We follow the same procedure again.

$\downarrow$

$$\begin{array}{c|cccc|}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & 0 & 0 & 0 & 0 \\
J_2 & 5 & 0 & 0 & 2 \\
J_3 & 0 & 2 & 1 & 3 \\
J_4 & 8 & 0 & 0 & 0 \\
\hline
\end{array}$$

Here no. of straight lines = 4 = order of the matrix. So optimality condition is satisfied. Now we can assign the jobs among the machines in the last matrix as follows:

$\downarrow$

$$\begin{array}{c|cccc|}
 & M_1 & M_2 & M_3 & M_4 \\
\hline
J_1 & \cancel{0} & 0\checkmark & \cancel{0} & \cancel{0} \\
J_2 & 5 & \cancel{0} & 0\checkmark & \cancel{0} \\
J_3 & 0\checkmark & 2 & 1 & 3 \\
J_4 & 8 & \cancel{0} & \cancel{0} & 0\checkmark \\
\hline
\end{array}$$

Required assignment is

$J_1 \to M_2,\ J_2 \to M_3,\ J_3 \to M_1,\ J_4 \to M_4$

or, $J_1 \to M_3,\ J_2 \to M_2,\ J_3 \to M_1,\ J_4 \to M_4$

or, $J_1 \to M_4,\ J_2 \to M_2,\ J_3 \to M_1,\ J_4 \to M_3$

or, $J_1 \to M_4,\ J_2 \to M_3,\ J_3 \to M_1,\ J_4 \to M_2$

or, $J_1 \to M_3,\ J_2 \to M_4,\ J_3 \to M_1,\ J_4 \to M_2$

or, $J_1 \to M_2,\ J_2 \to M_4,\ J_3 \to M_1,\ J_4 \to M_3$

6. Apply Vogel's approximation method to find the optimal transportation cost for the following problem: [WBUT 2016]

		Destination				
		D_1	D_2	D_3	D_4	Supply
Factory	F_1	19	30	50	10	7
	F_2	70	30	40	60	9
	F_3	40	8	40	20	18
	Demand	5	8	7	14	34

Answer:
Here we use VAM:

	D_1	D_2	D_3	D_4	a_i				
O_1	5 / 19	30	50	2 / 10	7 (9)	7 (9)	2 (40)	2 (40)	×
O_2	70	30	7 / 40	2 / 60	9 (10)	9 (20)	9 (20)	9 (20)	9 (20)
O_3	40	8 / 8	70	10 / 20	18 (12)	10 (20)	10 (50)	×	
b_j	5 (21)	8 (22)	7 (10)	14 (10)					
	5 (21)	×	7 (10)	14 (10)					
	×		7 (10)	14 (10)					
			7 (10)	4 (50)					
			7 (40)	2 (60)					

So, the initial basic solution is $x_{11} = 5, x_{14} = 2, x_{23} = 7, x_{21} = 2, x_{42} = 8, x_{43} = 10$

Corresponding cost $= 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 = 779$

Optimality Test (using $u - v$ method)

					a_i	u_i
	5 / 19	28 / 30	40 / 50	2 / 10	7	10
	1 / 70	−18 / 30	7 / 40	2 / 60	9	60
	11 / 40	8 / 8	70 / 70	10 / 20	18	20
b_j	5	8	7	14		
v_j	+9	−12	−20	0		

Adjusting the allocation from x_{24} is x_{22} we get,

				a_i	u_i
$\boxed{5}$ 19	32 30	42 50	$\boxed{2}$ 10	7	0
19 70	$\boxed{2}$ 30	$\boxed{7}$ 40	18 60	9	32
11 40	$\boxed{8}$ 8	52 70	$\boxed{10}$ 20	18	10
b_j: 5	8	7	14		
v_j: 19	−2	8	10		

Here all $\Delta_{ij} = C_{ij} - (u_i + v_j)$ which are given within the circles in each of the non-allocated cells. Since all $\Delta_{ij} \geq 0$, so the optimality condition is satisfied.

Hence the optimal solution is

$$x_{11} = 5,\ x_{14} = 2,\ x_{23} = 7,\ x_{32} = 8,\ x_{34} = 10$$

and $\quad$ min. cost $= 5\times19 + 2\times10 + 2\times30 + 7\times40 + 8\times8 + 10\times20 = 719$.

7. Find the initial solution for the following transportation problem using VAM and check for optimality: **[MODEL QUESTION]**

	Destination				
		A	**B**	**C**	***Supply***
	I	**2**	**7**	**4**	**5**
Source	**II**	**3**	**3**	**1**	**8**
	III	**5**	**4**	**7**	**7**
	IV	**1**	**6**	**2**	**14**
	Demand	**7**	**9**	**18**	**34**

Answer:

Initial basic feasible solution by VAM

$$\left(x_{11} = 5, x_{23} = 8, x_{32} = 7, x_{41} = 2, x_{42} = 2, x_{43} = 10\right).$$

Optimality test:

	A	B	C	$u_i \downarrow$
I	$\boxed{5}$ 2	7	4	$u_1 = 0$
II	3	3	$\boxed{8}$ 1	$u_2 = -2$
III	5	$\boxed{7}$ 4	7	$u_3 = -3$
IV	1	6	2	$u_4 = -1$
$v_j \rightarrow$	$\boxed{2}$ $v_1 = 2$	$\boxed{2}$ $v_2 = 7$	$\boxed{10}$ $v_3 = 3$	

$\Delta_{12} = C_{12} - (u_1 + v_2) = 7 - (0+7) = 0$

$\Delta_{13} = C_{13} - (u_1 + v_3) = 4 - (0+3) = 1 > 0$

$\Delta_{21} = C_{21} - (u_2 + v_1) = 3 - (-2+2) = 3 > 0$

$\Delta_{22} = C_{22} - (u_2 + v_2) = 3 - (-2+7) = -2 < 0$

$\Delta_{31} = C_{31} - (u_3 + v_1) = 5 - (-3+2) = 6 > 0$

$\Delta_{33} = 7 - (-3+3) = 7 > 0$

Hence the solution

$x_{11} = 5,\ x_{23} = 8,\ x_{32} = 7,\ x_{41} = 2,\ x_{42} = 2,\ x_{43} = 10$, is not an optimal solution.

Value of $z = 5\times2 + 8\times1 + 7\times4 + 2\times1 + 2\times6 + 10\times2$

$= 10 + 8 + 28 + 2 + 12 + 20 = 80$ units.

8. Assign the jobs to the four machinist for the profit matrix given below:

[MODEL QUESTIO

	A	B	C	D	E
I	62	78	50	101	82
II	71	84	61	73	59
III	87	92	111	71	81
IV	48	64	87	77	80

Answer:

$\text{I} \rightarrow D,\ \text{II} \rightarrow B,\ \text{III} \rightarrow C,\ \text{IV} \rightarrow E$

Max. profit = 376 units.

The profit matrix $\lfloor c_{ij} \rfloor$ for this problem is not a square matrix. Here number rows m = 4 and number of column n = 5. This type of assignment problem is known as ***Unbalanced Assignment problem***.

Step I: (Dummy rows)

First of all, we make it balanced by adding two dummy rows (see Table – 1; 5th and 6th rows are dummy rows i.e., X and Y are two dummy machines)

	A	B	C	D	E
A	62	78	50	101	82
B	71	84	61	73	59
C	87	92	111	71	81
D	48	64	87	77	80
X	0	0	0	0	0

Table – 1

Step II: (From maximization problem to minimization problem)

To change this profit matrix to a loss matrix, we subtract each element from the maximum element 111 and rewriting (see Table – 2)

	J1	J2	J3	J4	J5
A	49	33	61	10	29
B	40	27	50	38	52
C	24	19	0	40	30
D	63	47	24	34	31
X	0	0	0	0	0

Table – 2

Step III: (Row operation)

After subtracting least element, for each row from each element of that row.

	A	B	C	D	E
I	39	23	51	0	19
II	13	0	23	11	25
III	24	19	0	40	30
IV	39	23	0	10	7
V	0	0	0	0	0

Table – 3

Step IV: (Column operation)

Similar operations are performed for the entire column, we get Table – 3.

	A	B	C	D	E
I	39	23	51	0	19
II	13	0	23	11	25
III	24	19	0	40	30
IV	39	23	0	10	7
V	0	0	0	0	0

Table – 4

Step V: (Optimality test)

Now, we draw minimum number of lines to cover all the zeros of Table – 3. We get m = 4 < n = 5 (order of the matrix). Hence, optimality condition is not satisfied. The minimum of all the uncrossed elements (see Table – 3) is 1. Subtract 1 from each of the uncrossed elements and add 1 to all the double crossed elements and put the single crossed elements unchanged (see Table– 4)

	A	B	C	D	E
I	32	23	51	0	12
II	6	0	23	11	18
III	17	19	0	40	23
IV	32	23	0	10	0
V	0	0	0	0	0

Since, all the rows and columns have at least one zero, we draw minimum number of lines (both horizontal and vertical) covering all the zeros and test the optimality condition. Since m = 5 and n = 5, i.e., N = n = 6, the optimality condition is satisfied.

	A	B	C	D	E
I	32	23	51	0	12
II	6	0	23	11	18
III	17	19	0	40	23
IV	32	23	0	10	0
V	0	0	0	0	0

Step VI: (Assignment)

$I \to D,\ II \to B,\ III \to C,\ IV \to E;$ **Max. profit = 376 units.**

9. Solve the transportation problem and checking the optimality, find the optimal solution **[MODEL QUESTION]**

	D_1	D_2	D_3	
O_1	4	3	2	10
O_2	1	5	0	13
O_3	3	8	6	12
	8	5	4	

Answer:

Optimal solution: $x_{12} = 5,\ x_{14} = 5,\ x_{23} = 4,\ x_{21} = 8,\ x_{24} = 1$ and $x_{34} = 12$

Minimum cost of transportation = 23.

Here $\sum a_i = 10 + 13 + 12 = 35$ but $\sum b_j = 8 + 5 + 4 = 17 \Rightarrow \sum a_i > \sum b_j$. The problem indicates that to make it a balanced T.P., we have to create an artificial destination D_4 with demand $b_4 = \sum a_i - \sum b_j = 35 - 17 = 18$. So, we also have created the 4th column with heading D_4. We use unit transportation cost as the maximum unit cost 8 (as suggested by Goyel) for the cells of 4th column.

	D_1	D_2	D_3	D_4	$a_i \downarrow$	$u_i \downarrow$
O_1	4	$\boxed{5}$ 3	2	$\boxed{5}$ 8	10	$u_1 = 0$
O_2	$\boxed{8}$ 1	5	$\boxed{4}$ 0	$\boxed{1}$ 8	13	$u_2 = 0$
O_3	3	8	6	$\boxed{12}$ 8	12	$u_3 = 0$
$b_j \rightarrow$	8	5	4	18	35	
$v_j \rightarrow$	$v_1 = 1$	$v_2 = 3$	$v_3 = 0$	$v_4 = 8$		

Table – 1

Assume, $c_{14} = c_{24} = c_{34} = 8$

Using VAM, we get I.B.F.S. as, $x_{12} = 5, x_{14} = 5, x_{21} = 8, x_{23} = 4, x_{24} = 1$ and $x_{34} = 12$

Optimality test: (for all non-basic cells):

After finding all u's and v's, we calculate (see Table 1)

$$\Delta_{11} = c_{11} - (u_1 + v_1) = 4 - (0+1) = 3$$
$$\Delta_{13} = c_{13} - (u_1 + v_3) = 2 - (0+0) = 2$$
$$\Delta_{22} = c_{22} - (u_2 + v_2) = 5 - (0+3) = 2$$
$$\Delta_{31} = c_{31} - (u_3 + v_1) = 3 - (0+1) = 2$$
$$\Delta_{32} = c_{32} - (u_3 + v_2) = 8 - (0+3) = 5$$
$$\Delta_{33} = c_{33} - (u_3 + v_3) = 6 - (0+0) = 6$$

Since all $\Delta_{ij} > 0$ for all non-basic cells, optimality conditions are satisfied and optimal solution is

$$x_{12} = 5,\ x_{21} = 8,\ x_{23} = 4 \text{ and}$$
$$\text{Min.} Z = 3 \times 5 + 1 \times 8 + 0 \times 4 = 23.$$

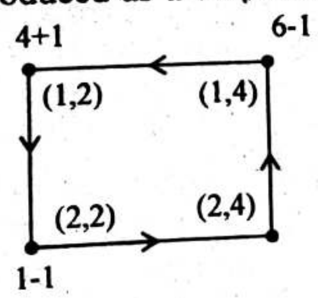
Note (i): We will not consider optimal solution components relating to 4th column, because the 4th column is artificially introduced to make the problem as a balanced T.P.

Note (ii): If we solve the problem with $c_{14} = c_{24} = c_{34} = 0$, and use VAM to find I.B.F.S., we get I.B.F.S. as

$$x_{12} = 4, x_{14} = 6, x_{22} = 1, x_{23} = 4, x_{21} = 8 \text{ and } x_{34} = 12,$$

which is not a optimal solution because $\Delta_{24} = -2 < 0$. (setting $u_1 = 0$).

To find optimal solution, we have to form a loop with the cells (2, 4), (1, 4), (1, 2) and (2, 2). The cell (2, 4) will be introduced as a basic cell replacing the cell (2, 2) from the basic cells.



We get optimal solution as, $x_{12} = 5, x_{14} = 5, x_{23} = 4, x_{21} = 8, x_{24} = 1$ and $x_{34} = 12$ (which is obtained earlier).

Note (iii): So, claim of Prof. S. K. Goyal is verified that dummy cells with largest unit cost provide us a better starting solution with VAM.

When $\sum_i a_i < \sum_j b_j$

The problem in which total demand of the markets (M_j) exceeds the total supply from the warehouses (W_i), may also be posed as following:

$$\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \text{ subject to}$$

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} = a_i; i = 1,2,\ldots\ldots, m \\ \sum_{i=1}^{m} x_{ij} \le b_j; j = 1,2,\ldots\ldots, n \\ \sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \text{ and } x_{ij} \ge 0 \end{array}\right\} \ldots$$

The actual supply from the warehouses (W_i, I = 1, 2, …, m) fails to meet the market demands. This type of problems are solved by introducing a artificial row (i.e., with a artificial/dummy warehouse) with a storage $a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{n} a_i > 0$, to make the problem a balanced T.P.

Now, the number of warehouse becomes (m + 1) but the number of markets remains n.
Unit costs of the (m + 1)th row may be considered the largest unit cost of the actual problem or zero.

Optimal solution: $x_{12} = 5,\ x_{14} = 5,\ x_{23} = 4,\ x_{21} = 8,\ x_{24} = 1$ and $x_{34} = 12$

Minimum cost of transportation = 23.

10. Solve the following assignment problem: **[MODEL QUESTION]**

		Job 1	2	3	4	5
Man	A	6	5	8	11	16
	B	1	13	16	1	10
	C	16	11	8	8	8
	D	9	14	12	10	16
	E	10	13	11	8	16

Answer:

After row operation the matrix takes the form

Order of the matrix $= n = 5 > N = 4$ (minimum number of lines).

Minimum uncovered element is 3. Subtract 3 from each of the uncovered elements and add 3 to each of the crossed elements and put other elements unchanged

	1	2	3	4	5
A	1	0	3	6	11
B	0	12	15	0	9
C	8	3	0	0	0
D	0	5	3	1	7
E	2	5	3	0	8

	1	2	3	4	5
A	4	$\boxed{0}$	3	9	11
B	$\boxed{0}$	9	12	0	6
C	11	3	0	3	$\boxed{0}$
D	0×	2	$\boxed{0}$	1	4
E	2	2	0	$\boxed{0}$	5

Optimal assignments : $A \to 2, B \to 1, C \to 5, D \to 3, E \to 4$

or $A \to 2, B \to 4, C \to 5, D \to 1, E \to 3$

Minimum cost = 34 units.

11. **Solve the following balanced Transformation problem:** **[MODEL QUESTION]**

	D_1	D_2	D_3	D_4	Capacity
F1	**2**	**3**	**11**	**7**	**6**
F2	**1**	**0**	**6**	**1**	**1**
F3	**5**	**8**	**15**	**9**	**10**
Requirement	**7**	**5**	**3**	**2**	**17**

Answer:

This is a balanced transportation problem. Initial B.F.S. by Vogel's approximation method.

	D_1	D_2	D_3	D_4
F_1	$\boxed{1}$ 2	$\boxed{5}$ 3	11	7
F_2	1	0	6	$\boxed{1}$ 1
F_3	$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9

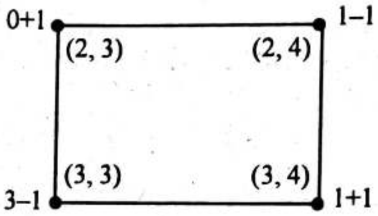
Here $m+n-1 = 3+4-1 = 6$.

Hence the I.B.F.S. is non-degenerate

For the optimality test, let $u_1 = 0$ and write all the cell evaluations $\Delta_{ij} = c_{ij} - (u_i + v_j)$ in the circles (for non-basic cells).

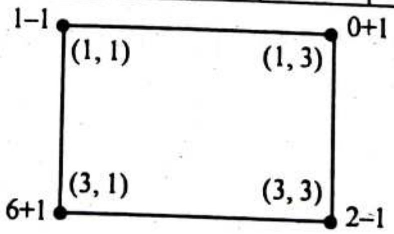
	D_1	D_2	D_3	D_4	$u_i \downarrow$
F_1	$\boxed{1}$ 2	$\boxed{5}$ 3	−1 11	+1 7	$u_1 = 0$
F_2	+4 1	+2 0	−1 6	$\boxed{1}$ 1	$u_2 = -5$
F_3	$\boxed{6}$ 5	+2 8	$\boxed{3}$ 15	$\boxed{1}$ 9	$u_3 = 3$
$V_j \rightarrow$	$V_1=2$	$V_2=3$	$V_3=12$	$V_4=6$	–

We choose $\Delta_{23} = C_{33} - (u_2+v_3) = 6-(-5+12) = 6-7 = -1 < 0$, as the intering cell is the basis. The cell to be removed from the basis is shown below. We form a loop which includes the cell (2, 3).



From the above, it is clear that the cell (2, 3) enter into the new basis and the cell (2, 4) leaves the basis.

	D_1	D_2	D_3	D_4	$u_i \downarrow$
F_1	$\boxed{1}$ 2	$\boxed{5}$ 3	−1 11	+1 7	$u_1=0$
F_2	+5 1	+3 0	$\boxed{1}$ 6	+1 1	$u_2=-6$
F_3	$\boxed{6}$ 5	+2 8	$\boxed{2}$ 15	$\boxed{2}$ 9	$u_3=3$
$V_j \rightarrow$	$V_1=2$	$V_2=3$	$V_3=12$	$V_4=6$	–



Let $u_1 = 0$ $\Delta_{13} = C_{13} - (u_1 + v_3) = 11 - (12 + 0) = -1 < 0$

	D_1	D_2	D_3	D_4	$u_i \downarrow$
F_1	+1 2	$\boxed{5}$ 3	$\boxed{1}$ 11	+2 7	$u_1=0$
F_2	+5 1	+2 0	$\boxed{1}$ 6	+1 1	$u_2=-5$
F_3	$\boxed{7}$ 5	+1 8	$\boxed{1}$ 15	$\boxed{2}$ 9	$u_3=4$
$V_j \rightarrow$	$V_1=1$	$V_2=3$	$V_3=11$	$V_4=5$	–

Let $u_1 = 0$

Since all the $\Delta_{ij} \geq 0$, the optimality conditions are satisfied.

Optimal solution: $x_{12} = 5, x_{13} = 1, x_{23} = 1, x_{31} = 7, x_{33} = 1$ and $x_{34} = 2$

$\min Z = 5\times3 + 1\times11 + 1\times6 + 7\times5 + 1\times15 + 2\times9 = \boxed{100}$

12. Find the optimal assignments to find the minimum cost for the assignment following cost matrix: **[MODEL QUESTION]**

	J1	J2	J3
P1	12	24	15
P2	23	18	24
P3	30	14	28

Answer:

After row and column operations

	J_1	J_2	J_3
P_1	$\boxed{0}$	15	0 ×
P_2	2	0 ×	$\boxed{0}$
P_3	13	$\boxed{0}$	8

Optimal Assignment $P_1 \rightarrow J_1$, $P_2 \rightarrow J_3$ and $P_3 \rightarrow J_2$

Min. assignment cost $= 12 + 24 + 14 = \boxed{50}$

13. Solve the following transportation problem by VAM and find out the optimal solution: **[MODEL QUESTION]**

$n = N = 4$, Optinalitity condition satisfied

$1 \to C, 2 \to A, 3 \to B$

and $4 \to D$. Min cost = 59

15. Find by North-West Corner method the initial basic feasible solution of the following transportation problem. Also find the optimal solution of the resulting degenerate transportation problem. **[MODEL QUESTION]**

	D_1	D_2	D_3	
O_1	7	3	4	2
O_2	2	1	3	3
O_3	3	4	6	5
	4	1	5	

Answer:

Step I: (I.B.F.S.)

Initial basic feasible solution by North-West corner method:

$x_{11} = 2;\ x_{21} = 2;\ x_{22} = 1;\ x_{33} = 5$

Here, $n + m - 1 = 3 + 3 - 1 = 5$ but non-zero components of the I.B.F.S. is 4. Hence, the solution is degenerate.

Step II: (Optimality test)

To make non-degenerate, we put $x_{23} = \in (> 0)$, is very small number.

	D_1	D_2	D_3	$a_i \downarrow$	$u_i \downarrow$
O_1	$\boxed{2}$ 7	3	4	2	$u_1 = 0$
O_2	$\boxed{2}$ 2	$\boxed{1}$ 1	$\boxed{\in}$ 3	3	$u_2 = -5$
O_3	3	4	$\boxed{5}$ 6	5	$u_3 = -2$
$b_j \to$	4	1	5	10	
$v_j \to$	$v_1 = 7$	$v_2 = 6$	$v_3 = 8$		

Set $u_1 = 0$;

Cell evaluation for non-basic cells

$\Delta_{12} = C_{12} - (u_1 + v_2) = 3 - (0 + 6) = -3$

Similarly, $\Delta_{13} = -4$; $\quad \Delta_{31} = -2$ and $\Delta_{32} = 0$;

Though $\Delta_{13} = -4$ is the most negative we use cell (3, 1) to get better result.

Step III: Loop formation

2 − 2	2 + ∈
(2, 1)	(2, 3)
(3, 1)	(3, 3)
0 + 2	5 − 2

Step IV: (new solution of its optimality testing)

	D_1	D_2	D_3	$a_i \downarrow$	$u_i \downarrow$
O_1	$\boxed{2}$ 7	3	4	2	$u_1 = 0$
O_2	2	$\boxed{1}$ 1	$\boxed{2+\epsilon}$ 3	3	$u_2 = -7$
O_3	$\boxed{2}$ 3	4	$\boxed{3}$ 6	5	$u_3 = -4$
$b_j \rightarrow$	4	1	5	10	
$v_j \rightarrow$	$v_1 = 7$	$v_2 = 8$	$v_3 = 10$		

Set $u_1 = 0$;

For non-basic cells:

$\Delta_{12} = -5; \Delta_{13} = -6; \Delta_{21} = 2; \Delta_{32} = 0$

So, the cell now enter in the basic cell.

Step V: (loop formation)

2 − 2	0 + 2
(1, 1)	(1, 3)
(3, 1)	(3, 3)
2 + 2	3 − 1

Step VI: (New solution and optimality testing)

	D_1	D_2	D_3	$a_i \downarrow$	$u_i \downarrow$
O_1	2 7	3	$\boxed{2}$ 4	2	$u_1 = 0$
O_2	2	$\boxed{1}$ 1	$\boxed{2+\epsilon}$ 3	3	$u_2 = -1$
O_3	$\boxed{4}$ 3	4	$\boxed{1}$ 6	5	$u_3 = 2$
$b_j \rightarrow$	4	1	5	10	
$v_j \rightarrow$	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$		

Cell evaluation for non-basic cells:

$$\Delta_{11} = 6 > 0;\ \Delta_{12} = 1 > 0;\ \Delta_{21} = 2 > 0 \text{ and } \Delta_{32} = 0$$

Hence, optimal solution is

$x_{13} = 2; x_{22} = 1; x_{23} = 2; x_{31} = 4$ and $x_{33} = 1$ (making $\epsilon \rightarrow 0$)

Minimum Cost of transportation is 33 (since $\Delta_{32} = 0$, this problem has alternative optimal solution).

16. A company has three plants at locations *A, B* and *C* which supply to warehouses located at *D, E, F, G* and *H*. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in Rupees) are given in the following table. Determine an optimal distribution for the company in order to minimize the total transportation cost. **[MODEL QUESTION]**

To

From	*D*	*E*	*F*	*G*	*H*
A	5	8	6	6	3
B	4	7	7	6	6
C	8	4	6	6	3

Answer:

It is a unbalanced Transportation Problem (T.P.). $\Sigma a_i = 2200 < \Sigma b_j = 2500$. Using an dummy plant *P*, we make this problem a balanced T.P. with all zero transportation costs for thus dummy plant *P*.

I.B.F.S. by Vam:

Ware houses ⟶

Plants ↓

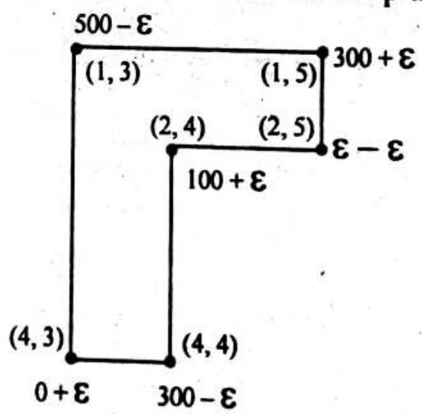
	D	E	F	G	H	
A	5	8	500 / 6	6	300 / 3	$u_1 = -2$
B	400 / 4	7	7	100 / 6	ε / 5	$u_2 = 0$
C	8	400 / 4	6	6	500 / 4	$u_3 = -1$
P	0	0	0	300 / 0	0	$u_4 = -6$
	$v_1 = 4$	$v_2 = 5$	$v_3 = 8$	$v_4 = 6$	$v_5 = 5$	

(Table 1)

Here, $m + n - 1 = 4 + 5 - 1 = 8 >$ no. of basic cells (7).

So, we introduce at the cell $(2, 5)$ a small non-zero amount $\varepsilon > 0$ to remove degeneracy of I.B.F.S. (see table 1).

This solution is not an optimal solution. We make the loop and get the table 2.



	D	E	F	G	H	
A	5	8	500–ε 6	6	300+ε 3	$u_1 = 0$
B	400 4	7	7	100+ε 6	5	$u_2 = 0$
C	8	400 4	6	6	500 4	$u_3 = 1$
P	0	0	0	300–ε 0	0	$u_4 = -6$
	$v_1 = 4$	$v_2 = 5$	$v_3 = 8$	$v_4 = 6$	$v_5 = 5$	

This solution is also not an optimal solution proceeding in this way, we get optimal solution as

$x_{13} = \varepsilon,\ x_{15} = 800 + \varepsilon,\ x_{21} = 400,\ x_{24} = 100 + \varepsilon,$

$x_{32} = 400,\ x_{33} = 200 - \varepsilon,\ x_{34} = 300 - \varepsilon,\ x_{43} = 300$

Now, making $\varepsilon \to 0$, we get optimal solution

$x_{13} = 0,\ x_{15} = 800,\ x_{21} = 400,\ x_{24} = 100,\ x_{32} = 400$

$x_{33} = 200,\ x_{34} = 300$ and $x_{43} = 300$ and minimum transportation cost = Rs.9200/-

17. Determine the initial basic feasible solution of the following:

[MODEL QUESTION]

Destinations

	I	II	III	a_i
Sources 1	8	7	3	60
2	3	8	9	70
3	11	3	5	80
b_j	50	80	80	

Also verify, whether the solution is optimal or not.

Answer:

Using VAM, we get a degenerate basic feasible solution (as a initial basic feasible solution): $x_{13} = 60,\ x_{21} = 50,\ x_{23} = 20$ and $x_{32} = 80$.

Here, $m = 3,\ n = 3 \Rightarrow m + n - 1 = 3 + 3 - 1 = 5$, which indicates the solution is degenerate solution.

	D_1	D_2	D_3	$u_i \downarrow$
O_1	8	ε 7	60 3	$u_1 = 0$
O_2	50 3	8	20 9	$u_2 = 6$
O_3	11	80 3	5	$u_3 = -4$
$v_j \rightarrow$	$v_1 = -3$	$v_2 = 7$	$v_3 = 3$	

Table

Set $u_1 = 0$
Optimality test: $\Delta_{ij} = 0$, for the basic cells.
This gives us, $u_1 + v_3 = 3; u_2 + v_3 = 9; u_2 + v_3 = 3 \& u_3 + v_3 = 3$.
Setting $u_1 = 0$, we cannot find the values of all other 5 unknown u's and v's. This is due to appearance of degeneracy in the I.B.F.S. What cell will represent the 5th component of the basic cell?
To overcome this problem, we tactically allocate a small amount ε (> 0) to a cell such that this cell and other basic cells does not form, in any way, a loop. With this view in mind, we allocate ε amount in the cell (1, 2) (see Table) and this gives $v_2 = 7$ and $u_3 = -4$. So, all the values of u's and v's are now known.
Now, we calculate Δ_{ij} for all non-basic cells.
$\Delta_{11} = c_{11} - (u_1 + v_1) = 11 > 0$
$\Delta_{22} = c_{22} - (u_2 + v_2) = -5 < 0$
$\Delta_{31} = c_{31} - (u_3 + v_1) = 18 > 0$ and
$\Delta_{33} = c_{33} - (u_3 + v_3) = 6 > 0$
Hence, the I.B.F.S.

18. In a textile sales emporium, 4 salesman A, B, C, D are available to 4 counter W, X, Y, Z Each salesman can handle any counter. How the salesman should be allotted the appropriate counter so as to minimize the service time from the following problem? [MODEL QUESTION]

	A	B	C	D
W	41	72	39	52
X	22	29	49	65
Y	27	39	60	51
Z	45	50	48	52

Answer:
Step I: After row operation.

	A	B	C	D
W	2	33	0	13
X	0	7	27	43
Y	0	12	33	24
Z	0	5	3	7

Step II: After column operation.

	A	B	C	D
W	2	28	0	6
X	0	2	27	36
Y	0	7	33	17
Z	0	0	3	0

minimum number of lines $N = 3 < 4 = n$ (order of the matrix)

Step III: Minimum element of the uncovered elements is 2.

	A	B	C	D
W	2	26	$\boxed{0}$	4
X	0	$\boxed{0}$	27	34
Y	$\boxed{0}$	5	33	15
Z	2	0	5	$\boxed{0}$

Therefore optimal assignment is $W \to C, Y \to A, X \to B$ and $Z \to D$

Total min. service time =27+29+39+52=147 units

19. Solve the following transportation problem using Vogel's approximation method: **[MODEL QUESTION]**

	A	B	C	CAPACITY
1	50	40	80	400
2	80	70	40	400
3	60	70	60	500
4	60	60	60	400
5	30	50	40	800
REQUIREMENT	800	600	1100	

Answer:

Using Vogel's approximation method, we get the following degenerate initial Basic feasible solution of the T.P. (It is a balanced T.P.):

	A	B	C	Capacity ↓
1	50	[400] 40	80	400
2	80	70	[400] 40	400
3	60	70	500 60	500
4	60	[200] 60	[200] 60	400
5	[800] 30	50	40	800
Req→	800	600	1100	2500

Hence $n=5$, $m=3$, $\therefore m+n-1=5+3-1=7$

But occupied cells are 6. We allot a small (+) ve amount $\varepsilon>0$ at the cell (5, 3) and test for the optimality of this solution (using u - v method).

	A	B	C	u_i ↓
1	50	[400] 40	80	$u_1=0$
2	80	70	[400] 40	$u_2=0$
3	60	70	[500] 60	$u_3=20$
4	60	[200] 60	[200] 60	$u_4=20$
5	[800] 30	50	[ε] 40	$u_5=0$
v_j →	$v_1=30$	$v_2=40$	$v_3=40$	–

Let $u_1=0$
Cell evaluation
$\Delta_{ij}=C_{ij}-(u_i+v_j)$F
or Basic cells
$\Delta_{ij}=0$

$\Delta_{11}=50-(0+30)=20>0$

$\Delta_{13}=80-(0+40)=40>0$

$\Delta_{21}=80-(0+30)=50>0$

$\Delta_{22}=70-(0+40)=30>0$

$\Delta_{31}=60-(20+30)=10>0$

$\Delta_{32}=70-(20+40)=10>0$

$\Delta_{41}=60-(20+30)=10>0$

$\Delta_{52}=50-(0+40)=10>0$

Hence the solution obtain by VAM is a optimal solution (making $\varepsilon \to 0$)

$\therefore$ Minimum cost of transportation

$= 400\times40+400\times40+500\times60+200\times60+200\times60+800\times30 = 110{,}000$

20. Obtain an Initial Basic Feasible Solution (I.B.F.S) and total transportation cost to the following Transportation Problem using North West Corner Rule.

[MODEL QUESTION]

Destinations

		D_1	D_2	D_3	D_4	Supply
Sources	S_1	10	2	20	11	15
	S_2	12	7	9	20	25
	S_3	4	14	16	18	10
	Demand	15	5	15	15	

Is that Initial Basic Feasible Solution (I.B.F.S) is a Non-Degenerate Basic Feasible Solution? Justify your answer.

Answer:

Initial Basic feasible solution by North-West Corner Rule:

—	D_1	D_2	D_3	D_4	Supply↓
S_1	10 $\boxed{15}$	2	20	11	$\cancel{15}$ 0
S_2	12	7 $\boxed{5}$	9 $\boxed{15}$	20 $\boxed{5}$	$\cancel{25}$ $\cancel{20}$ 5
S_3	4	14	16	18 $\boxed{10}$	10
Demand →	$\cancel{15}$ 0	$\cancel{5}$ 0	$\cancel{15}$ 0	15	50

Here

$m = 3,$

$n = 4;$

$\therefore\ m+n-1 = 3+4-1 = 6$

I.B.F.S. by North-West Corner Rule:

$x_{11} = 15,$

$x_{22} = 5,$

$x_{23} = 15,$

$x_{24} = 5,$

$x_{34} = 10$

and $z = 600$.

Since non-zero components of this solution 5 which is less than 6, this indicates that the solution is a degenerated Basic feasible solution.

21. Solve the transportation problem and checking the optimality, find the optimal solution: **[MODEL QUESTION]**

	D_1	D_2	D_3	Supply
01	4	3	2	10
02	1	5	0	13
03	3	8	5	12
Demand	8	5	4	

Answer:

Here total demand $= 8 + 5 + 4 = 17$

and total supply is $10 + 13 + 12 = 35$.

So this is a unbalanced problems. To balanced it we add a new column (corresponding demand 18) with zero cost and then apply VAM.

	D_1	D_2	D_3	D_4	S				
O_1	4	$\boxed{5}$ 3	2	$\boxed{5}$ 0	10 (2)	10 (2)	10 (2)	10 (3)	10 (3)
O_2	$\boxed{8}$ 1	5	$\boxed{4}$ 0	$\boxed{1}$ 0	13 (0)	13 (0)	5 (0)	1 (5)	
O_3	3	$\boxed{8}$ 8	5	$\boxed{12}$ 0	12 (3)				
D	8 (2)	5 (2)	4 (2)	18 (0)					
	8 (3)	5 (2)	4 (2)	6 (0)					
		5 (2)	4 (2)	6 (0)					
		5 (2)		6 (0)					
		5 (3)		5 (0)					

So, B.F.S. is

$$x_{12} = 5, x_{14} = 5, x_{21} = 8, x_{23} = 4, x_{24} = 1, x_{34} = 12.$$

and minimum cost is

$$5 \times 3 + 8 \times 1 = 23.$$

Now test the optimality of the solution by computing cell evaluation at unoccupied cells.

	D_1	D_2	D_3	D_4	u_i
O_1	4 4	$\boxed{5}$ 3	2 2	$\boxed{5}$ 0	0
O_2	8 1	2 5	$\boxed{4}$ 0	$\boxed{1}$ 0	0
O_3	2 3	5 8	5 5	$\boxed{12}$ 0	0
v_j	1	3	0	0	

Here all the cell evaluation are positive. So the solution is optimal solution.

22. Solve the transportation problem starting with the initial solution obtained by VAM:

[MODEL QUESTION]

	P	Q	R	S	Available
A	21	16	15	3	11
B	17	18	14	23	13
C	32	17	18	41	19
Requirements	6	10	12	15	

Answer:

To obtain B.F.S. of the given transportation problem we use VAM.

	P	Q	R	S	A				
A	21	16	15	$\boxed{11}$ 3	11				
B	$\boxed{6}$ 17	18	$\boxed{3}$ 14	$\boxed{4}$ 23	13 (3)	13 (3)	9 (3)	3 (4)	
C	32	$\boxed{10}$ 17	$\boxed{9}$ 18	41	19 (1)	19 (1)	19 (1)	19 (1)	19 (1)
R	6 (4)	10	12	15 (18)					
	6	10	12	4 (18)					
	6	10	12 (4)						
		10	6 (0)						
		10 (17)	9 (18)						

So the B.F.S. is

$x_{14} = 11,$

$x_{21} = 6,$

$x_{23} = 3,$

$x_{24} = 4,$

$x_{32} = 10,$

$x_{33} = 9$

The minimum cost is $11\times3+6\times17+3\times14+4\times23+10\times17+9\times18=601$.

Now we test the optimality of this solution by computing cell evaluation at unoccupied cells.

	P	Q	R	S	u_i
A	24 21	23 16	21 15	$\boxed{11}$ 3	–20
B	$\boxed{6}$ 17	5 18	$\boxed{3}$ 14	$\boxed{4}$ 23	0
C	11 32	$\boxed{10}$ 17	$\boxed{9}$ 18	14 41	4
v_j	17	13	14	23	

It is seen that all the cell evaluations are positive so the solution is optimal.

23. Find out the initial basic feasible solution and the corresponding transportation cost of the following transportation problem by using VAM: [MODEL QUESTION]

	D_1	D_2	D_3	D_4	Supply
O_1	21	16	25	13	110
O_2	17	18	14	23	130
O_3	32	17	18	41	190
Demand	60	100	120	150	

Answer:

Here we put all the computation (of VAM) in a single table instead of different shrunken matrices, as follows:

	D_1	D_2	D_3	D_4	a_i				
O_1	21	16	25	110 / 13	110 (3)	×			
O_2	60 / 17	18	70 / 14	23	130 (3)	130 (3)	130 (3)	70 (4)	×
O_3	32	100 / 17	50 / 18	40 / 41	190 (1)	190 (1)	150 (1)	150 (1)	150 (1)
b_j	60 (4)	100 (1)	120 (4)	150 (10)					
	60 (15)	100 (1)	120 (4)	40 (18)					
	60 (15)	100 (1)	120 (4)	×					
	×	100 (1)	120 (4)						
		100 (17)	50 (18)						

Here the number of allocated cells is $6(=3+4-1)$.

So, the initial basic feasible solution is

$$x_{14}=110,\ x_{21}=60,\ x_{23}=70,\ x_{32}=100,\ x_{33}=50,\ x_{34}=40.$$

The cost of the corresponding initial basic feasible solution is

$$110\times13+60\times17+70\times14+100\times17+500\times18+40\times41=7650.$$

24. Find out the minimum cost solution for the assignment problem whose cost coefficient are as below: **[MODEL QUESTION]**

	1	2	3	4	5
1	-2	-4	-8	-6	-1
2	0	-9	-5	-5	-4
3	-3	-8	-9	-2	-6
4	-4	-3	-1	0	-3
5	-9	-5	-8	-9	-5

Answer:

Step I: *(Row operation)*

Select minimum element in each row and then subtract it from each of the row. We get the following table.

	B1	B2	B3	B4	B5
A1	6	4	0	2	7
A2	9	0	4	4	5

A3	6	1	0	7	3
A4	0	1	3	4	1
A5	0	4	1	0	4

***Step II:** (Column operation)*
After column operation

	B1	B2	B3	B4	B5
A1	6	4	0	2	6
A2	9	0	4	4	4
A3	6	1	0	7	2
A4	0	1	3	4	0
A5	0	4	1	0	3

***Step III:** (Optimality test)*
Here N < n, optimality condition is not satisfied.
The minimum of the uncovered elements is 2. Subtract 2 from each uncovered element and add 2 to those elements only which are crossed by both horizontal and vertical lines and construct the table below for optimality testing.

	B1	B2	B3	B4	B5
A1	4	4	$\boxed{0}$	0	4
A2	7	$\boxed{0}$	4	2	2
A3	4	1	0	5	$\boxed{0}$
A4	$\boxed{0}$	3	5	4	0
A5	0	6	3	$\boxed{0}$	3

Here N = n = 4.

***Step IV:** (Optimal Assignment)*
So, get the optimal solution.
$A1 \rightarrow B3, A2 \rightarrow B2, A3 \rightarrow B5, A4 \rightarrow B1, A5 \rightarrow B4.$
Minimum cost = –8–9–6–4–9 = –36.

DECISION THEORY

Multiple Choice Type Questions

1. Laplace criteria is applicable for decision making under [WBUT 2014, 2016]
a) certainty b) uncertainty c) risk d) none of these
Answer: (b)

2. The decision makers' knowledge and experience may influence the decision making process when using the criterion of [WBUT 2014]
a) maximax b) minimax regret c) realism d) maximin
Answer: (b)

3. A type of decision making environment is [WBUT 2017]
a) certaintyb) uncertainty c) risk d) all of these
Answer: (d)

4. A type of decision making environment is [MODEL QUESTION]
a) certainty b) uncertainty c) risk d) all of these
Answer: (d)

5. The decision-maker's knowledge and experience may influence the decision-making process when using the criterion of [MODEL QUESTION]
a) maximax b) minimmax regret c) realism d) maximin
Answer: (c)

6. Branch and bound method divides the feasible solution space into smaller parts by [MODEL QUESTION]
a) branching b) bounding c) enumerating d) all of these
Answer: (a)

7. The use of cutting plane method [MODEL QUESTION]
a) requires the use of standard linear programming approach between each cutting plane application
b) yields better value of the adjectives function
c) reduces the number of constraints in the given problem
d) both (b) and (c)
Answer: (a)

8. In decision theory, this of the following criterion is not considered [MODEL QUESTION]
a) Maximax regret criterion b) Laplace criterion
c) Minimin criterion d) EMV criterion
Answer: (c)

Short Answer Type Questions

1. An investor is given the following investment alternatives and percentage rate of return:

	States of Nature (Market Conditions)		
	Low	Medium	High
Regular shares	7%	10%	15%
Risky shares	– 10%	12%	25%
Property	– 12%	18%	30%

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases. On the basis of these data, state the optimum investment strategy for investment. [MODEL QUESTION]

Answer:

By Laplace method:

	Low	Medium	High	Expected pay off
Regular shares	0.07	0.10	0.15	0.327
Risky shares	–.10	0.12	0.25	0.09
Property	–.12	0.18	0.30	0.12

Thus by Laplace method we observe that the Regular shares gives 32.7% return (Expected). Hence the investor should have to invest in Regular shares.

Long Answer Type Questions

1. The R & D wing of an industrial product division has recommended to the industrial marketing department to launch bolt with three different shearing stresses. The manager is to decide the type of shearing stress launch under the following estimated pay-off for the various levels of sales. Examine which types of bolt can be chosen under Laplace and Herwicz $\alpha(=0.6)$ criterion.

← .... Estimated levels of sale in units .... →			
Types of bolts	20,000	10,000	2,000
I	25	15	10
II	40	20	5
III	60	25	3

[WBUT 2013]

Answer:

Laplace Criterion:

We associate equal probability for each event say 1/3. Expected payoff is:

$$I \rightarrow 25\times\frac{1}{3}+15\times\frac{1}{3}+10\times\frac{1}{3}=\frac{50}{3}=16.67$$

$$II \rightarrow 40\times\frac{1}{3}+20\times\frac{1}{3}+5\times\frac{1}{3}=\frac{65}{3}=21.67$$

$III \rightarrow 60 \times \frac{1}{3} + 25 \times \frac{1}{3} + 3 \times \frac{1}{3} = \frac{88}{3} = 29.33$

Since III has maximum expected payoff, III is the bolt that we have to choose under Laplace criterion.

Harwicz alpha Criterion:

$D_1 = 25 \times 0.6 + 10 \times 0.4 = 19$

$D_2 = 40 \times 0.6 + 5 \times 0.4 = 26$

$D_3 = 60 \times 0.6 + 3 \times 0.4 = 37.2$

Since D_3 is maximum, we have to choose the bolt III.

2. A mechanical engineer is seriously considering of drilling a tube well. In the past, only 70% of well drilled were successful at 200 ft of depth in that area. Moreover on finding no water at 200 ft, some persons drilled it further up to 250 ft but only 20% struck water at 250 ft. The prevailing cost of drilling is Rs. 50 per foot. The engineer has estimated that in case he does not get his own tube well he will have to pay Rs.15,000 over the next 10 years, in order to buy water from the neighbourhood. The following decision can be optimal:

i) Do not drill any well

ii) Drill up to 200 ft

iii) If no water is found at 200 ft drill up to 250 ft.

Draw an appropriate decision tree and determine engineer's strategy under EMV approach. **[WBUT 2013]**

Answer:

At D_2 point:

Decision:	(a) Drill up to 250 ft.	(b) Do not drill
Event:	(a) No water	(b) Water
Probability:	(a) 0.2	(b) 0.8

EMV for drill up to 250 ft. $= (12500 \times 0.2) + (27500 \times 0.8) = 24500$.

EMV for do not drill =25000.

EMV is smaller for the act drill up to 250ft. So it is optimal act.

At D_1 point:

Decision:	(a) Drill up to 200 ft.	(b) Do not drill
Event:	(a) No water	(b) Water
Probability:	(a) 0.7	(b) 0.3

EMV for drill up to 200 ft. $= (10000 \times 0.7) + (24500 \times 0.3) = 14300$.

EMV for do not drill =15000.

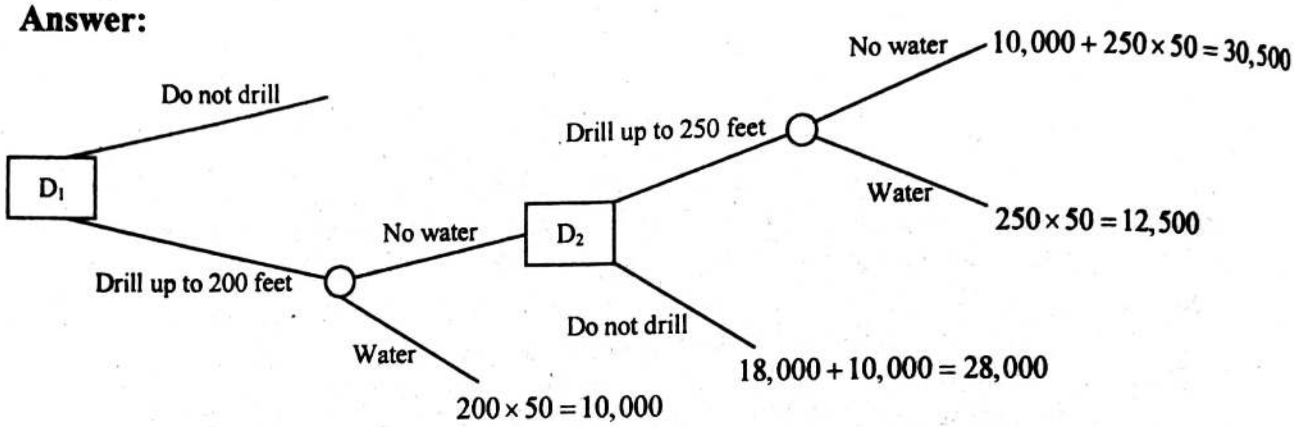
EMV is smaller for the act drill up to 200ft. So it is optimal act.

Therefore, combining D_1 and D_2 the optimal strategy is to drill the well up to 200 ft. and if no water is struck, then further drill it up to 250 ft.

3. Mr. Sinha has to decide whether or not to drill a well on his farm. In his village, only 40% of the wells drilled were successful at 200 feet of depth. Some of the farmers who did not get water at 200 feet, drilled further upto 250 feet but only 20% struck water at 250 feet. Cost of drilling is Rs. 50 per foot. Mr. Sinha estimated that he would pay Rs. 18,000 during a 5-year period in the present value terms. If he continues to buy water from the neighbour rather than go for the well which would have a life of 5 years. Mr. Sinha has three decisions to make:

i) **Should be drill up to 200 feet and**
ii) **If no water is found at 200 feet, should he drill up to 250 feet?**
iii) **Should he continue to buy water from his neighbour?** **[WBUT 2016]**

Answer:



At the point D_1

Decision:	(a) Do not droll	(b) Drill up to 200 feet
Event:	(a) Water	(b) Water
Probability:	(a) 0.7	(b) 0.4

EMV for drill 200 feet $=10,000\times 0.4+18,000\times 0.7=16,600$

EMV for do not drill $=18,000$

At the point D_2

Decision:	(a) Do not drill	(b) Drill up to 250 feet
	(a) Water	(b) Water
	(a) 0.8	(b) 0.2

EMV for drill 250 feet $=12,000\times 0.2+30,000\times 0.8=26,900$

EMV for do not drill $=28,000$

Hence Mr. Sinha will take the decision that if there is not no water at 200 ft. then drill further 250 ft. for water.

4. Under an employment promotion programme, it is proposed to allow sale of newspapers on the buses during off-peak hours. The vendor can purchase the newspapers at a special concessional rate of 25 paise per copy against the selling price of 40 paise. Any unsold copies are however a dead loss. A vendor has estimated the following probability distribution for the number of copies demanded:

Number of copies:	15	16	17	18	19	20
Probability:	0.04	0.19	0.33	0.26	0.11	0.07

i) **How many copies should he order so that his expected profit will be maximum?**

ii) **Compute EPPI**

iii) **The vendor is thinking of spending on a small market survey to obtain additional information regarding the demand levels. How much should he be willing to spend on such a survey?**

[WBUT 2016]

Answer:

Conditional profit table:

Demand (D)	Probability ↓	Possible stock action (in no. of copies) 15	16	17	18	19	20
15	0.04	225	200	175	150	125	100
16	0.19	210	240	215	190	165	140
17	0.33	195	225	255	230	205	180
18	0.26	180	210	240	270	245	220
19	0.11	165	195	225	255	285	260
20	0.07	150	180	210	240	270	300

Expected profit table:

Demand	Profit probability	Possible stock(s) action → 15	16	17	18	19	20
15	0.04	9	8	7	6	5	4
16	0.19	39.9	45.6	40.85	36.1	31.35	26.6
17	0.33	64.35	74.25	84.15	75.9	67.65	59.4
18	0.26	46.8	54.6	62.4	70.2	63.7	57.2
19	0.11	18.26	21.45	24.75	28.05	31.35	28.6
20	0.07	10.5	12.6	14.7	16.8	18.9	21
EMV →		188.81	216.5	$\boxed{233.85}$	233.05	217.95	196.8

$\therefore$ Max (EMV) = 233.85

Now we calculate EPPI from the following table:

Demand	Profit under certainty	Probability	Expected profit under certainty
15	225	0.04	9
16	240	0.19	45.6
17	255	0.33	48.15
18	270	0.26	70.2
19	285	0.11	31.35
20	300	0.07	21
			Total: 225.3

Then $\text{EVPI} = 233.85 - 225.3 = 8.55$

(i) 18 copies should be ordered to maximize the expected profit.

(ii) Total EPPI = 225.3

5. How can you determine the standard cost of product? How is standard costing helpful in budgeting concept? Name the various types of budgets. Analyze the importance of budgetary control in improving company's performance.

[WBTU 2017]

Answer:

Both Standard Costing and Budgetary Control are based on the principle that costs can be controlled along certain lines of supervision and responsibility, that focuses on controlling cost by comparing actual performance with the predefined parameter. However, the two systems are neither similar nor interdependent. **Standard Costing** delineates the variances between actual cost and the standard cost, along with the reasons. In the contrary, **Budgetary Control**, as the name suggest, refers to the creation of budgets, then comparing the actual output with the budgeted one and taking corrective action immediately.

The two systems aim at measuring performance by fixing targets. Nevertheless, The former, forecasts, cost accounts but the later projects detail about financial accounts. Similarly, there are many differences between Standard Costing and Budgetary Control, which has been discussed below.

Standard Costing Vs Budgetary Control

1. Comparison Chart
2. Definition
3. Key Differences
4. Conclusion

By Budgetary Control we mean, a management function in which an organization's activities are directed and regulated in such a way reach the desired objectives. It is a control technique, in which the operations are planned in advance, and then they are compared with the actual results to know whether the expected results are achieved or not. The following are the major characteristics of this system:

- The budgets are designed in accordance with the policy requirements.
- Constant comparisons are made between the actual output and the budgeted targets to review the performance.
- Revisions are made if in case the existing conditions are changed.
- Appropriate actions are taken if the expected results are not achieved.

Here, budget refers to a written financial statement expressed in monetary terms prepared in advance for future periods, containing the details about the economic activities of the business organization.

The Budgetary Control system facilitates the management to fix the responsibilities and coordinate the activities to achieve the desired results. It helps the management to measure the performance of the organization as a whole. Moreover, it helps in the formulation of future policies by reviewing current trends.

6. A dairy firm wants to determine the quantity of butter it should produce to meet the demand. Past records have shown the following demand pattern:

Quantity required (kg)	15	20	25	30	35	40	50
No. of days demand occurred	6	14	20	80	40	30	10

The stock levels are restricted to the range 15 to 50 kg due to inadequate storing facilities. Butter costs Rs. 40 per kg and sold at Rs. 50 per kg.

i) Construct a conditional profit table
ii) Determine the action alternative associated with the maximization of expected profit.
iii) Determine EVPI.

[WBUT 2017]

Answer:

i) Clearly, the diary firm would not produce butter less than 15kg and more than 50kg. From the data given in the problem, we can calculate the conditional profit values for each stock action and event (demand) combination. If CP denotes the conditional profit. S the quantity in stock and D the demand, then

$$CP = \begin{cases} 10S, \text{ when } D \geq S \\ 50D - 40S, \text{ when } D < S \end{cases}$$

The resulting payoff table is given below. Also the quantity of butter required for 6 days out of a total of 200 days is 15kg means that the demand of 15kg has an associated probability of $\frac{6}{200} = 0.03$. Similarly, probabilities associated with other demand levels can be calculated and are shown in column (i) of table 1.

TABLE 1

Conditional profit table

Possible demand (event) (kg)	Probability (i)	Possible stock action (alternative) (kg)						
		15 (ii) ₹	20 (iii) ₹	25 (iv) ₹	30 (v) ₹	35 (vi) ₹	40 (vii) ₹	50 (viii) ₹
15	0.03	150	–50	–250	–450	–650	–850	–1,250
20	0.07	150	200	0	–200	–400	–600	–1,000
25	0.10	150	200	250	50	–150	–350	–750
30	0.40	150	200	250	300	–100	–100	–500
35	0.20	150	200	250	300	350	150	–250
40	0.15	150	200	250	300	350	400	0
50	0.05	150	200	250	300	350	400	500

TABLE 2
Expected profit table

Possible demand (event) (kg)	Probability (i)	15 (i)×(ii) ₹	20 (i)×(iii) ₹	25 (i)×(iv) ₹	30 (i)×(v) ₹	35 (i)×(vi) ₹	40 (i)×(vii) ₹	50 (i)×(viii) ₹
		Possible stock action (alternative) (kg)						
15	0.03	4.50	1.50	7.50	13.50	19.50	25.50	37.50
20	0.07	10.50	14.00	0.00	–14.00	–28.00	–42.00	–70.00
25	0.10	15.00	20.00	25.00	5.00	–15.00	–35.00	–75.00
30	0.40	60.00	80.00	100.00	120.00	40.00	–40.00	–200.00
35	0.20	30.00	40.00	50.00	60.00	70.00	30.00	–50.00
40	0.15	22.50	30.00	37.50	45.00	52.50	60.00	0.00
50	0.05	7.50	10.00	12.50	15.00	17.50	20.00	25.00
EMV (₹)		150.00	192.50	217.50	217.50	117.50	–32.50	–407.50

ii) The calculations for expected payoffs and EMV for each action are shown in Table 2. Since the maximum EMV is ₹ 217.50 for stock of 25 as well as 30 kg of butter, the dairy firm may produce 25 kg or 30 kg of butter and can expect an average daily profit of ₹ 217.50.

iii) The EVPI is calculated from Table 3.

TABLE 3
Expected profit table with perfect information

Market size (event)	Conditional profit under certainty (₹)	Probability of market size	Expected profit with perfect information (₹)
15	150	0.03	4.50
20	200	0.07	14.00
25	250	0.10	25.00
30	300	0.40	120.00
35	350	0.20	70.00
40	400	0.15	60.00
50	500	0.05	25.00
			EPPI = 318.50

The expected value of perfect information is given by

$$\text{EVPI} = \text{EPPI} - \text{max. EMV} = ₹(318.50 - 217.50) = ₹101$$

EMV for items that have a Salvage Value

In the discussion so far, it has been assumed that the product being stocked was completely worthless if not sold on the 'selling' day. This assumption, that the product has no salvage value is not always realistic. If the product does have a salvage value, then it must be considered in calculating the conditional profits for each stock action.

7. Write short notes on the following: **[WBUT 2014]**
i) Training scheme
ii) Budget and budgetary control
iii) Industrial disputes
iv) Industrial safety

Answer:

(i) Training scheme:

The legal definition of a training scheme is:

"*... study or training that —*

(a) leads to an award; but

(b) does not, of itself, lead to an award of a qualification listed on the Qualifications Framework."

A training scheme would normally be a 'short course', with fewer than 40 credits, which is the credit threshold for inclusion on the Job.

A training scheme is likely to be made up of assessment standards listed on the Directory of Assessment Standards, or non-standard-based components, for example courses, modules or papers.

(ii) Budget and budgetary control:

A control technique whereby actual results are compared with budgets.

Any differences (variances) are made the responsibility of key individuals who can either exercise control action or revise the original budgets.

Budgetary control and responsibility centres

These enable managers to monitor organisational functions.

A responsibility centre can be defined as any functional unit headed by a manager who is responsible for the activities of that unit.

There are four types of responsibility centres:

a) **Revenue centres:** Organisational units in which outputs are measured in monetary terms but are not directly compared to input costs.

b) **Expense centres:** Units where inputs are measured in monetary terms but outputs are not.

c) **Profit centres:** Where performance is measured by the difference between revenues (outputs) and expenditure (inputs). Inter-departmental sales are often made using "transfer prices".

d) **Investment centres:** Where outputs are compared with the assets employed in producing them, i.e. ROI.

(iii) Industrial disputes:

According to Sec. 2 of the Industrial Dispute Act, 1947, "Industrial dispute means any dispute or difference between employers and employers or between employers and workmen or between workmen and workmen, which is connected with the employment or non-employment or the terms of employment or with the conditions of labour of any person" Industrial disputes are of symptoms of industrial unrest in the same way that boils are symptoms of a disordered body.

(iv) Industrial safety:
An **industrial safety** system is a countermeasure crucial in any hazardous plants such as oil and gas plants and nuclear plants. They are used to protect human, plant, and environment in case the process goes beyond the control margins. As the name suggests, these systems are not intended for controlling the process itself but rather protection. Process control is performed by means of process control systems (PCS) and is interlocked by the safety systems so that immediate actions are taken should the process control systems fail.

8. A manufacturing company has to choose from several alternative methods to increase its production to meet the increasing market demand. The following pay –off table lists the various alternatives (courses of action) along with the different states of nature **[MODEL QUESTION]**

Table 1

Alternatives	States of Nature (Demand)			
	High	Moderate	Low	Nil
Expand Facility	50,000	25,000	-25,000	-45,000
Construct new facility	70,000	30,000	-40,000	-80,000
Subcontract	30,000	15,000	-1,000	-10,000

Choose the alternative based on Laplace Criterion and Minimax Regret Criterion.

Answer:

(i) By Laplace criterion:

Table 2

Alternative ↓	State of nature (demand)				Expected pay off ↓
	High (Rs)	Moderate (Rs.)	Low (Rs.)	Nil (Rs.)	
Expand facility	50,000	25,000	–25,000	– 45,000	$\frac{(50+25-25-45)}{4}\times 1000$ $=1,250$
Construct new facility	70,000	30,000	– 40,000	– 80,000	$\frac{(70+30-40-80)}{4}\times 1000$ $=-5,000$
Sub contract	30,000	15,000	-1,000	-10,000	$\frac{(30+15-1-10)}{4}\times 1000$ $=\boxed{8,500}$

This Table 2 shows that the alternative 'subcontract' results in maximum average pay off Rs. 8,500 (which is the optimistic return).

(ii) By Minimax regret Criterion

From Table 1, we construct Table 3 to represent amount of regrets of Table 1.

Table 3

Alternative ↓	States of nature (demand)				Maximum of the rows ↓
	High (Rs.)	Moderate (Rs.)	Low (Rs.)	Nil (Rs.)	
Expand facility	20,000	5,000	24,000	35,000	$\boxed{35,000}$
Constant new facility	0	0	39,000	70,000	70,000
Sub contract	40,000	15,000	0	0	40,000

Amount of regrets for Table 1, are shown in the Table 3. This Table 3, tells us the company will minimize its regrets to Rs. 35,000 by selecting the alternative *'Expand facility'*

9. a) Distinguish between decision making under risk and decision making under uncertainty.

[MODEL QUESTION]

Answer:

a) Decision Making Under Condition of Risk

Majority of the decisions that are taken care by the manager in an industry or in business, are under condition of risk. In decision- making under the conditions of uncertainty, the decision- makers does not have sufficient information to assign probability to different states of nature. On the other hand, the decision makers have sufficient information to assign probabilities to each of the states of nature when decision - making under **condition of risk** is considered. Under condition of risk, a number of decision criteria are available which could be of help to the decision – maker. The most popular criterion for evaluating the alternative is the **expected monetary value (EMV) criterion**.

Other important criteria are **expected opportunity loss (EOL), combined expected value and variance** etc.

(i) Expected monetary value (EMV) criterion

This criterion require to calculate the expected value of each of the decision alternatives. **EMV** for a given course of action in the expected value of the conditional pay off for each decision alternative. If A_j denote a decision alternative, then EMV

$(A_j) = E\left(A_j\right) = \sum_{i=1}^{n} D_{ij} p_i$ where $n =$ number of states of nature(of demand).

D_{ij} = pay off for the ith state of nature and to jth the decision alternative.

and p_i = probability of occurrence of the state of nature i

Working rule

Step I: Construct the conditional profit Table

The 1st column should contain state of the nature, 2nd column are their occurrence probability and other column contain pay off elements D_{ij}

Step II: For each decision alternative, calculate the conditional expected profits. i.e. multiply the elements of probability column with the corresponding elements of the column of a decision alternative and add them. This quantity is the EMV for that decision alternative.

Step III: Choose the decision alternative which corresponds to the optimal EMV

b) Decisions under Uncertainty

Decision makers, under this type of uncertainty, very often can know about the states of nature in the future but it is not possible for them to find the probabilities of occurrence of those states. for example, consider a company wishes to launch a new product (viz. DVD player), it knows that the demand of DVD in future is likely to increase but the probability that it will increase its future demands, is not known . This company may also face the problem of price reduction in future because the imported DVD players may be cheaper due to new form of the Government policy.

Under this classification of uncertainty, a few decision criteria are known which could be useful to the decision makers.

Example: ***Table 1***

Alternatives ↓	States of Nature (Market demand)			
	High(Rs)	Moderate(Rs)	Low(Rs)	Nil(Rs)
a) Expand→	250	200	-100	-150
b) Add New facilities→	700	400	-300	-400
c) sub-contract→	300	175	- 50	-100

(i) Maximax criterion (Optimistic criterion):

In **Table 1**, we find maximum element of each row and write them in a column at the end of the table and then find also the maximum element of this new column. This value is the maximum possible pay off for each possible alternatives. For **Table 1**, this value is 700. i.e., max.{250,700,300}=700. By maximax we mean maximum of the maximum elements.

(ii) Maxi-min criterion (pessimistic criterion):

First, find the row minimum for each row and then maximum of all these minimum pay off elements. For **Table 1**, this value is – 100

(iii) Mini-max Regret criterion:

This decision was developed by L.J. Savage. Select column maxima for each column and subtract each element of the corresponding column from this column maxima, to form new columns. This is performed for all columns. After this, we have to perform Mini-Max i.e. after taking maximum element of each new row, find the minimum. In **Table 2**, the last column shows the maximum element of each row.

Table 2

Alternatives ↓	States of nature (Demands)				Max. of each row
	High (Rs)	Moderate (Rs)	Low (Rs)	No demand (Rs)	
a) Expand→	450	200	50	50	450
b) Add new facilities→	0	0	250	300	300
c)Sub-contract→	400	225	0	0	400

From **Table 2**, it is clear that the company will minimize its regret to Rs. 300.00 by selecting the alternative **'Add new facilities'**. Here, 300 = min. {450,300.400}.

(iv) Hurwicz Crietrion (Criterion of Realism):

Under this criterion rational decision-maker should not be completely optimistic nor completely pessimistic. They always choose decision between these two limits with the help of linear combination. Thus, if $\alpha(0<\alpha<1)$ is the coefficient of optimism, then $(1-\alpha)$ will represents the coefficient of pessimism. This method is also known as **weighted average criterion.**

The working principle is as follows:

a) Choose the weight factor $\alpha,(0<\alpha<1)$ and this gives $(1-\alpha)$

b) Determine the maximum as well as minimum pay off for each alternative and calculate $P_i=\alpha\times$maximum pay off + $(1-\alpha)\times$ minimum pay off, for each of the alternatives (i.e., for ith row).

c) Select an alternative with value of P_i as maximum i.e. find Max $\{P_i\}$ for all the rows (i = 1,2,3) and we consider the above example with $\alpha=0.6(0<\alpha<1)$; and construct

Table 3

Alternatives ↓	States of Nature (Demands)				Maxi-mum (Rs)	Minimum (Rs)	$P_i=\alpha\times$Max $+(1-\alpha)$Min (Rs.)
	High (Rs.)	Moderate (Rs.)	Low (Rs)	No Demand (Rs)			
a) Expand→	250	200	−100	−150	250	−150	90
b) Add new facilities→	700	400	−300	-400	700	−400	260
c) sub-contract→	300	175	−50	−100	300	−100	140

We get $\text{Max}\{P_i\}=Rs.260$. It is to be noted that when $\alpha=0\Rightarrow$the criterion is too pessimistic and $\alpha=1\Rightarrow$ the criterion is too optimistic doe α = ½, the value has no influence of any one of the above, and α = ½ seems to be reasonable.

(v) Laplace criterion (Criterion of Rationality)

Due to the lack of knowledge about the probability of the states for each of the alternative, generally equally probable assumption for each states are considered. This criterion is based on the principle known as *'principle of insufficient reason'*

The working rule is as follows:

(a) Assign equal probability to each pay off for each alternative (strategy).

(b) Calculate the expected pay off for each alternative.

(c) Select the alternative having maximum profit value from these averages or minimum value in the case of production prices. For the above example, we construct the Table 4.

Table 4

Alternatives ↓	States of Nature (product demands)				Expected pay off (Rs)
	High (Rs)	Moderate (Rs)	Low (Rs.)	No demand (Rs)	
a) Expand	250	200	-100	-150	$\frac{(250+200-100-150)}{4}=50$
b) Add new alternatives	700	400	-300	- 400	$\frac{700+400-300-400}{4}=100$

c)sub-contract	300	175	-50	-100	$\frac{(300+175-50-100)}{4}=81.25$

Maximum is Rs. 100. Thus the alternative 'add new alternatives' results in maximum average pay off of Rs. 100.

b) A retailer purchases cherries every morning at Rs. 50 per case and sells them for Rs. 80 per case. Any case remaining unsold at the end of the day can be disposed of next day at a salvage value of Rs.20 par case (thereafter they have no value). Past sales have ranged from 15 to 18 cases per day. The following is the record of sales for the past 120 days: **[MODEL QUESTION]**

Table 1

Cases sold	15	16	17	18
Number of days out of 120 days	12	24	48	36

Find how many cases the retailer should purchase per day to maximize his profit.

Answer:

For this problem the Table for pay off matrix is not given. We have to construct the pay offs and from the given Table (Table 1) we can construct probabilities of the sold cases. Let D denotes (sold) and S denotes the stock of charry cases. Then conditional profit,

$C.P=(80-50)S$ when $D\geq S$

$$=80D-50S+20(S-D),\ \text{When}\ D<S$$

$$\Rightarrow C.P=30S \quad \text{When } D\geq S \qquad \text{..........(1)}$$

$$=60D-30S, D<S$$

Now, we construct the conditional profit table and C.P's are the pay off i.e. when $D=15$ and $S=16$, then pay off $=C.P=60\times15-30\times16=420$ etc. Probabilities are obtained from **Table 1** as $\left(\frac{12}{120},\frac{24}{120},\frac{48}{120}\frac{36}{120}\right)=(0.1,0.2,0.4,0.3)$

Table 2

Cases sold ↓	Probability (p_i)	Conditional Profits (Rs) Stock →			
		15	16	17	18
15	0.1	450	420	390	360
16	0.2	450	480	450	420
17	0.4	450	480	510	480
18	0.3	450	480	510	540

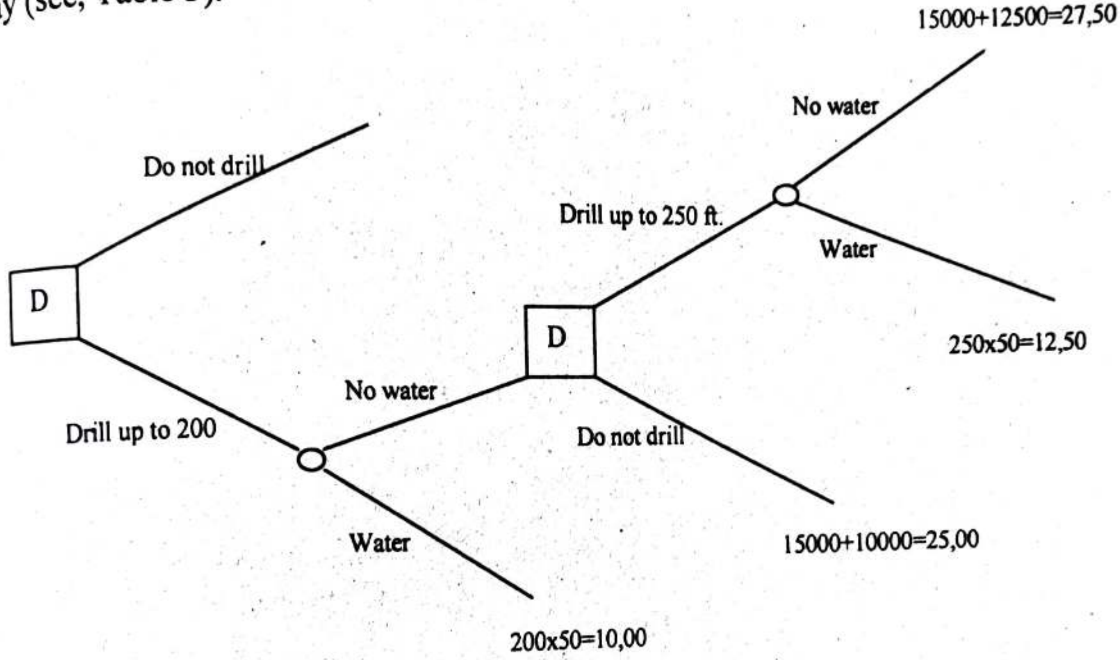
From column three to column seven, each column represents the pay off of the stock alternatives (see, Table 2).

Now we construct the **expected conditional profit** pay off table (Table 3) by multiplying probabilities with these columns.

Table 3

Cases sold ↓	Probability	Possible stock action 15	16	17	18
15	0.1	45.0	42.0	39.0	36.0
16	0.2	90.0	96.0	90.0	84.0
17	0.4	180.0	192.0	204.0	192.0
18	0.3	135.0	144.0	153.0	162.0
	EMV→	450	474	$\boxed{486}$	474

So, EMV for the strategy (when stock is 15 cases) is 475.
Similarly, for the other strategies i.e. when stock is 16, 17 or 18 cases and EMV's are 474,486 and 474 respectively. So, profit becomes maximum when stock is 17 cases of cherries. So, to get maximum (optimum) profit, the retailer should purchase 17 cases per day (see, Table 3).



Decision Tree

10. A person is planning to invest Rs. 1 lakh among three companies A, B and C in equity shares. The pay-offs in terms of growth in capital and returns on capital are known for each of the investments under each of the three economic conditions which may prevail, that is, recession, growth and stability. Assuming that the person must take his choice among the three portfolios for a period of one year in advance, his expectations of net earnings (in Rs. 000's) on his 1 lakh portfolio are given below:

Table 1

	Recession	Stability	Growth
Company A	– 15	6	10
Company B	4	7.5	8
Company C	6.5	6	15

Determine the optimal strategy for investment under Laplace's rule and minimax regret criterion. **[MODEL QUESTION]**

Answer:

(i) By Laplace rule

Table 2

Alternative	Recession	Stability	Growth	Expected pay off (Rs)
Company-A→	- 15	6	10	$\frac{-15+6+10}{3}=\frac{1}{3}=0.33$
Company-B→	4	7.5	8	$\frac{4+7.5+8}{2}=\boxed{6.5}$
Company-C→	6.5	6	15	$\frac{6.5+6+15}{3}=4.8$

By Laplace method we get that the alternative 'company B' results in maximum average pay off of Rs.6500.00. So, invest in Company B'.

(ii) By Minimax regret criterion. From Table 1 first we construct Table 3, which is obtained by subtracting each pay off of a column from the maximum pay off of that column.

Table 3

	Recession	Stability	Growth	Max. of the rows (Rs.)
Company – A	21.5	1.5	5	21.5
Company – B	2.5	0	7	7
Company – C	0	1.5	0	$\boxed{1.5}$

Amount of regrets for Table 1 are shown in Table 3. This table shows that the company will minimize its regret to Rs. 1500.00 by selecting the alternative 'Company – C' for investment

Note: In this problem by choosing two different criterion for decision- making, we have selected two different alternatives (companies).i.e. by Laplace rule Company–B is selected where as by Minimax regret criterion it is Company – C.

11. A news agency receives its weekly order for a magazine every Monday and cannot recorder during the week. Each copy costs Rs.1.80 and is sold for Rs. 3.00. Unsold copies may be returned the following week for a Rs.1.20 rebate. When the agency runs out of copies and cannot supply a customer, it estimates its "goodwill" loss at Rs. 2.40 in future profits, as the customer will take his business else where for a couple of weeks at least. Demand has been remarkably constant between 28 and 40 copies a week, as shown below:

Demand (Copies)	**28**	**32**	**36**	**40**
Fraction of time	**0.30**	**0.40**	**0.20**	**0.10**

i) Construct a pay-off table and determine the optimal number of copies to stock.

ii) How much would it be worth to know the exact demand each week? What would be the expected profit if this were possible? [MODEL QUESTION]

Answer:

Conditional profit table: ***Table 1***

Demand (D) ↓	Probability ↓	Possible stock action (in no. of copies)			
		28	32	36	40
28	0.30	33.6	31.2	28.8	26.4
32	0.40	31.2	38.4	36.0	33.6
36	0.20	31.2	36.0	43.2	40.8
40	0.10	31.2	36.0	40.8	48.0

Here, $C.P = 1.2 \times S - 2.4$, when $D > S$
$= (3D - 1.8S) + 1.2(S - D)$
when $D \leq S$

$\Rightarrow C.P = 1.2S - 2.4, \quad D > S$
$= 1.8D - 0.6S, \quad D \leq S$

$D \rightarrow$ demand and
$S \rightarrow$ stock required

Expected Profit table:

Table 2

Demand ↓ D	Probability ↓	Possible stock (S) action →			
		28	32	36	40
28	0.30	10.08	9.2	8.64	12.0
32	0.40	12.48	15.6	14.40	13.44
36	0.20	6.24	7.2	8.64	8.16
40	0.10	3.12	3.6	4.08	4.8
EMV →		31.92	35.60	35.76	$\boxed{38.4}$

From Table 2, we get max {EMV} = Rs. 38.4. Now, we will calculate EPPI from the table below:

Demands ↓ (1)	Profit under certainty (2)	Probability (3)	expected profit under certainty (4)
28	33.6	0.30	10.08
32	38.4	0.40	15.36
36	43.2	0.20	8.64
40	48.0	0.10	4.80
EPPI →			38.88

Then EVPI = Rs.(38.88 − 38.40) = Rs.0.48. Here, the column (2) is constructed from the diagonal elements of Table 1

12. The research department of HLL recommended to the manufacturing department to launch a shampoo of three different types. The marketing manager has to decide one of the types of shampoo to be launched under the following estimated pay-offs for various level of sales.

	Estimated level of sales in units		
Types of shampoo	**15000**	**10000**	**5000**
Egg	**30**	**10**	**10**
Clinic	**40**	**15**	**5**
Deluxe	**55**	**20**	**3**

What will be the marketing manager's decision depending upon
i) Laplace's criterion
ii) Regret criterion
iii) Hurwiez's criterion ($\alpha = 0.6$)? **[MODEL QUESTION]**

Answer:

i) Laplace's Criterion:

Expectation of Egg Shampoo = E (Egg shampoo) $= \frac{1}{3}(30 + 10 + 10) = 16.67$

E (Clinic Shampoo) $= \frac{1}{3}(40+15+5) = 20$

E (Deluxe Shampoo) $= \frac{1}{3}(55+20+3) = 26$

$\therefore$ $\text{Max}\{16.69, 20, 26\} = 26$ which indicates to launch deluxe shampoo.

ii) Regret Criterion: The minimum sale (pay off) 400 each alternative shampoo (Egg, Clinic, Deluxe) are 30, 40, 55 respectively. Now $Max\{30,40,55\} = 55$ which corresponds to deluxe shampoo.

iii) Hurwiez's Criterion: Here given company's degree of optimism$(\alpha) = 0.6$.

$\therefore \quad P_i = \alpha \text{Max} + 1(1-\alpha)\text{Min}$

For egg – shampoo: $P_1 = 0.6 \times 30 + (1-0.6) \times 10$
$= 18 + 4 = 22$

For clinic shampoo: $P_2 = 0.6 \times 40 + (1-0.6) \times 5$
$= 24 + 2 = 28$

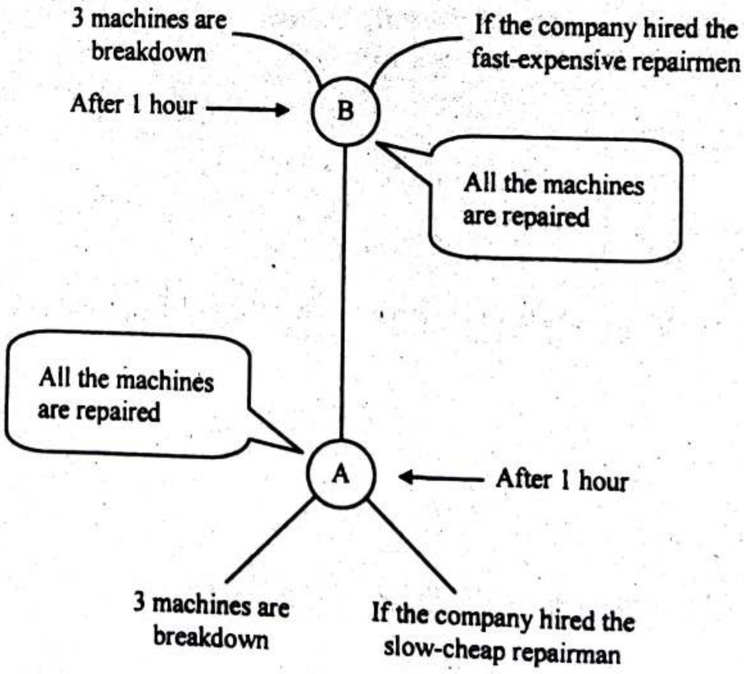
For Deluxe shampoo: $P_3 = 0.6 \times 55 + (1-0.6) \times 3$
$= 33 + 1.2 = 34.2$

$\therefore$ $\text{Max}\{P_1, P_2, P_3\}$ which corresponds deluxe shampoo.

13. A repairman is to be hired to repair machines which break down at a rate of 3 per hour. Non-productive time on any machine will cost the company Rs. 20 per hour. Two repairmen are available. One is slow-cheap who repairs at an average rate of 4/hour and demands Rs. 40/per hour. The other is fast-expensive who repairs at an average rate of 6/hour and demands Rs. 72/hour. Which repairman should be hired? **[MODEL QUESTION]**

Answer:

The problem is shown in the following tree.



Now the expected cost at the node A is

$$₹(3\times 20)+₹40=₹100$$

Expected cost at the node B is

$$₹(3\times 20)+₹72=₹132$$

Hence slow-cheap repairmen should be hired.

14. A newspaper boy sells newspapers on buses during off peak hours. He can purchase at a concession rate of 50 paise and sell it for 60 paise. Any unsold copy is a dead loss. He has estimated the following probabilities for the number of copies demanded:

Number of copies	**20**	**21**	**22**	**23**	**24**	**25**
Probability	**0.1**	**0.15**	**0.25**	**0.3**	**0.12**	**0.08**

Prepare a pay-off table to find out how many copies should be ordered so that expected profit is maximized. **[MODEL QUESTION]**

Answer:

Let S be the number of newspapers ordered per day and 'r' be the demand for it (number of papers sold each day). Given that $C_1=₹0.5$, $C_2=₹(0.6-0.5)=₹0.1$. Now let us work out the cumulative demand for the newspapers (because the demand for units is discrete).

Demand (r)	20	21	22	23	24	25
$P(r)$	0.1	0.15	0.25	0.3	0.12	0.08
$\sum_{r=0}^{S}P(r)$	0.1	0.25	0.50	0.80	0.92	1

Now $\sum_{r=0}^{S_0-1}P(r)<C_2/(C_1+C_2)<\sum_{r=0}^{S_0}P(r)$ and $C_2/(C_1+C_2)=\frac{1}{6}=0.167$

This value lies between demand 20 and 21. Hence the newspaper boy has to order 21 papers per day to maximize his profit.

NETWORK ANALYSIS

Multiple Choice Type Questions

1. The common error(s) that occur(s) in a network construction is/are [WBUT 2013,2016]

   a) Looping (cycling) b) Dangling c) Redundancy d) All of these

Answer: (a)

2. In constructing a network, Forward Pass computations are done for determining. [WBUT 2013]

   a) Earliest Event Time b) Latest allowable Time
   c) Floats and Slack Times d) none of these

Answer: (a)

3. The shortest path between any two nodes in a Network is determined by the following [WBUT 2014]

   a) Dijikastra's Algorithm b) Floyd's Algorithm
   c) Critical path method d) Decision tree

Answer: (a)

4. In PERT the span of times between the optimistic and pessimistic time estimates of an activity is [WBUT 2014, 2017]

   a) 3σ b) 6σ c) 12σ d) σ

Answer: (d)

5. Critical path method (CPM) is [WBUT 2015]

   a) probabilistic b) deterministic c) both (a) & (b) d) none of these

Answer: (b)

6. PERT stands for [WBUT 2015]

   a) Performance Evaluation Report Technique
   b) Programme Evaluation Report Technique
   c) Programme Evaluation and Review Technique
   d) None of these

Answer: (c)

7. The shortest path between any two nodes in a Network is determined by [WBUT 2016]

   a) Dijkstra's Algorithm b) Poisson distribution
   c) Binomial distribution d) Exponential distribution

Answer: (a)

8. In Hurwitz alpha criterion the degree of optimism α varies between [WBUT 2016]

   a) 1 and 2 b) 0 and 1 c) 0 and 0.5 d) 0 and 2

Answer: (b)

9. An activity is said to be critical if [WBUT 2017]
   a) its free float is zero
   b) its total float is zero
   c) its independent float is zero
   d) its time duration is zero

Answer: (b)

10. CPM has which of the following time estimates? [WBUT 2017]
    a) One-time estimate
    b) Two-time estimate
    c) Three-time estimate
    d) Four-time estimate
    e) Nil time estimate

Answer: (a)

11. ABC analysis deals with [WBUT 2017]
    a) flow of material
    b) ordering schedule of raw material
    c) analysis of process chart
    d) inventory management

Answer: (d)

12. The objective of network analysis is to [MODEL QUESTION]
    a) minimize total project duration
    b) minimize total project cost
    c) minimize production delays, interruption and conflicts
    d) all of these

Answer: (a)

13. The process of reducing the activity duration by putting an extra effort is called crashing the activity [MODEL QUESTION]
    a) True
    b) False

Answer: (a)

14. An activity is said to be critical activity is [MODEL QUESTION]
    a) its free float is zero
    b) its total float is zero
    c) its independent float is zero
    d) its time duration is zero

Answer: (b)

15. In a flow pattern $\sum f_{iA} = \sum f_{Ai}$ when the vertex A is [MODEL QUESTION]
    a) arbitrary vertex
    b) any vertex other than source
    c) source
    d) none of these

Answer: (a) (other than source and sink vertex)

16. In PERT analysis, the variance of a job having optimistic time 5, pessimistic time 17, and most likely time 8, is [MODEL QUESTION]
    a) 3
    b) 4
    c) 7
    d) none of these

Answer: (b)

17. CPM is [MODEL QUESTION]
    a) probabilistic
    b) deterministic
    c) event oriented
    d) all of these

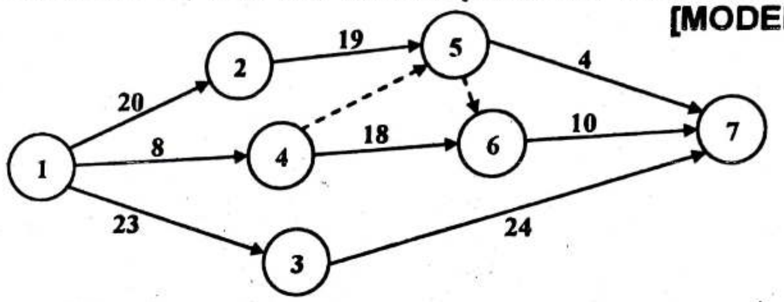
Answer: (b)

18. In a PERT network, the starting vertex is a [MODEL QUESTION]
a) burst node b) merge node c) root d) none of these

Answer: (d)

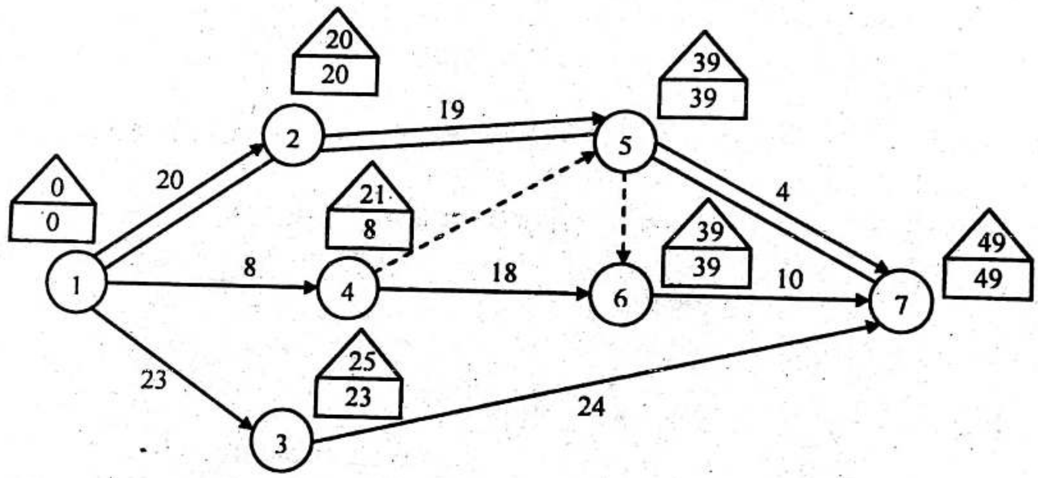
Short Answer Type Questions

1. For the following network, find the critical path and minimum time to complete the project: **[MODEL QUESTION]**



Answer:

The following diagram will give the solution:



The critical path is $1 \to 2 \to 5 \to 7$ and total project duration is $20+19+4=43$ units.

2. A small project consists of seven activities. The details of which are given below: **[MODEL QUESTION]**

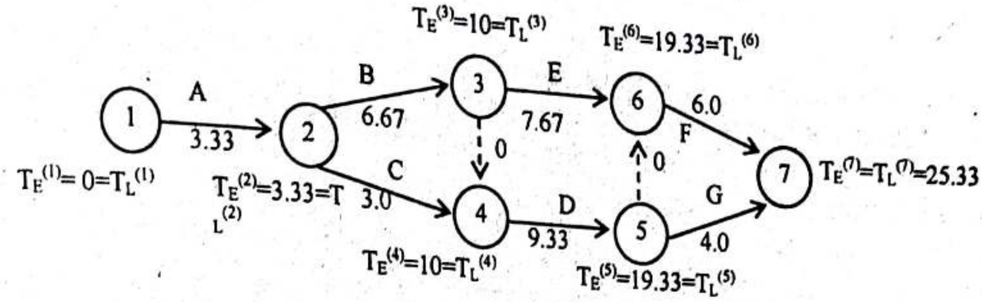
Activity	Duration in days			Expected
	Most likely	Optimistic	Pessimistic	Time
A	3	1	7	–
B	6	2	14	A
C	3	3	3	A
D	10	4	12	B, C
E	7	3	15	B
F	5	2	14	D, E
G	4	4	4	D

Draw the network; find the critical path and the expected project completion time.

Answer:

	Duration in days			
	Most likely(tm)	Optimistic (t_o)	Pessimistic (t_p)	Expected Time (t_e)
A	3	1	7	3.33
B	6	2	14	6.67
C	3	3	3	3.0
D	10	4	12	9.33
E	7	3	15	7.67
F	5	2	14	6.0
G	4	4	4	4.0

Network with expected times (t_e)



To find the critical path, use calculate both forward pass and backward pass:

Forward pass: Let $T_E^{(1)} = 0$; then $T_E^{(2)} = 3.33$

$T_E^{(3)} = 10.0$; $T_E^{(4)} = \max\{3.33 + 3.0, 10.0 + 0\} = 10.0$

$T_E^{(5)} = 10.0 + 9.33 = 19.33$; $T_E^{(6)} = \max\{10.0 + 7.67, 19.33 + 0\} = 19.33$

$T_E^{(7)} = \max\{19.33 + 4.0, 19.33 + 4.0, 19.33 + 6.0\} = 25.33$

Backward pass: Start with $T_L^{(7)} = 25.33$; $T_L^{(6)} = 25.33 - 6.0 = 19.33$

$T_L^{(5)} = \min\{19.33 - 0, 25.33 - 4\} = 19.33$

$T_L^{(4)} = 19.33 - 9.33 = 10.0$; $T_L^{(3)} = \min\{10.0 - 0, 19.33 - 7.67\} = 10.0$

$T_L^{(2)} = \min\{10 - 6.67, 10 - 3\} = 3.33$

$T_L^{(1)} = 3.33 - 3.33 = 0.$

Critical path is $(1)\xrightarrow[3.33]{A}(2)\xrightarrow[6.67]{B}(3)\xrightarrow[0]{\text{dummy}}(4)\xrightarrow[9.33]{D}(5)\xrightarrow[0]{\text{dummy}}(6)\xrightarrow[6.0]{F}(7)$

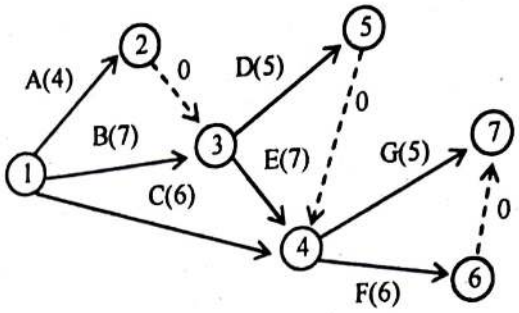
The minimum expected project compilation time = 25.33 days.

3. A small project consists of seven activities for which the relevant data are given below: **[MODEL QUESTION]**

Activity	Preceding Activities	Activity Duration (days)
A	–	4
B	–	7
C	–	6
D	A, B	5
E	A, B	7
F	C, D, E	6
G	C, D, E	5

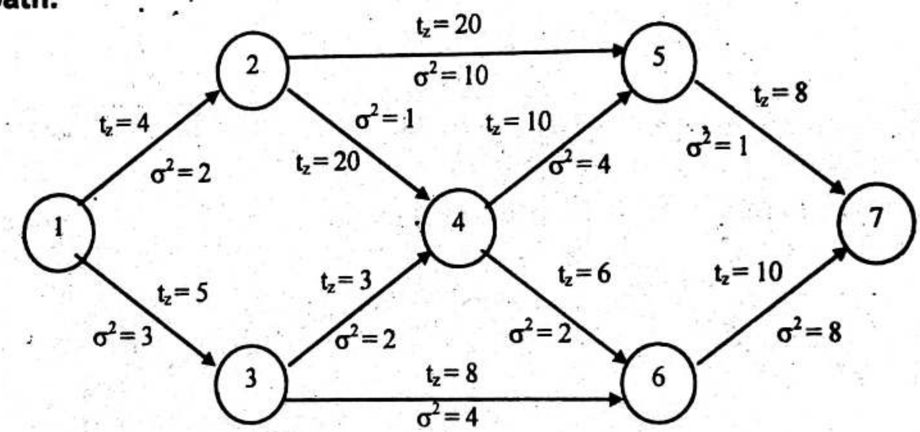
(a) Draw the network
(b) Number of the events by Fulkerson rule.

Answer:

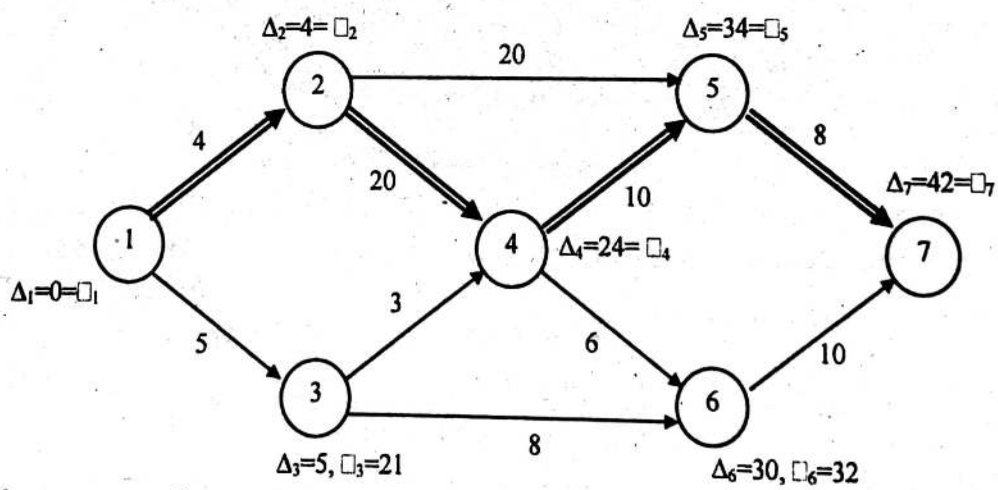


-----> is a dummy activity with duration 0 (zero).
Events are numbered by Fulkerson rule.

4. Given the following PERT network, determine: **[MODEL QUESTION]**
a) earliest expected completion time for such event
b) latest allowable completion time for such event
c) slack time for such event
d) critical path.



Answer:



$\Delta \rightarrow$ earliest expected completion time and
$\square \rightarrow$ latest allowable completion time for such event

Events ↓	EECT = Δ_i	LACT = $\square_i$	Slack time
1	0	0	0
2	4	4	0
3	5	21	16
4	24	24	0
5	34	34	0
6	30	32	2
7	42	42	0

Critical path: $1 \xrightarrow{4} 2 \xrightarrow{20} 4 \xrightarrow{10} 5 \xrightarrow{8} 7$

Total min. time = 42 units.

Long Answer Type Questions

1. A small maintenance project consists of the following jobs, whose predecessor relationships are given below:

Jobs	1 − 2	1 − 3	2 − 3	2 − 5	3 − 4	3 − 6	4 − 5	4 − 6	5 − 6	6 − 7
Duration (Days)	15	15	3	5	8	12	1	14	3	14

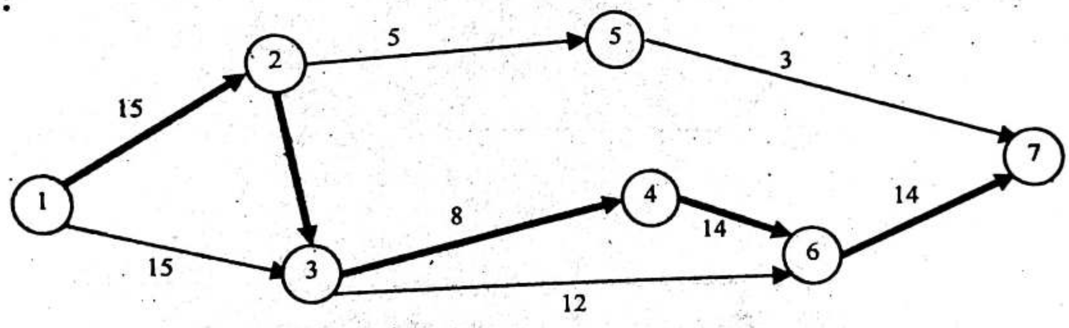
i) Draw the network diagram of the project

ii) Find the total float of each activity

iii) Find the critical path and total project duration. **[WBUT 2013]**

Answer:

(i)



Network Diagram

(ii) Forward pass calculation: In this estimate the earliest start and the earliest finish time ES_j are given by,

$ES_1 = 0$

$ES_2 = ES_1 + t_{13} = 0 + 15 = 15$

$ES_3 = Max(ES_2 + t_{23}, ES_1 + t_{13}) = Max(15+3, 0+15) = 18$

$ES_4 = ES_3 + t_{34} = 18 + 8 = 26$

$ES_5 = Max(ES_2 + t_{25}, ES_4 + t_{45}) = Max(15+5, 26+1) = 27$

$ES_6 = Max(ES_3 + t_{36}, ES_4 + t_{46}, ES_5 + t_{56}) = Max(18+12, 26+14, 27+3) = 40$

$ES_7 = ES_6 + t_{67} = 40 + 14 = 54$.

Backward pass calculation: In this estimate the latest finish and the latest start time LF_j are given by,

$LF_7 = 54$

$LF_6 = LF_7 - t_{67} = 54 - 14 = 40$

$LF_5 = LF_6 - t_{56} = 40 - 3 = 37$

$LF_4 = Min(LF_5 - t_{45}, LF_6 - t_{46}) = Min(37 - 1, 40 - 14) = 26$

$LF_3 = Min(LF_4 - t_{34}, LF_6 - t_{36}) = Min(26 - 8, 40 - 12) = 18$

$LF_2 = Min(LF_5 - t_{25}, LF_3 - t_{23}) = Min(37 - 5, 18 - 3) = 15$

$LF_1 = Min(LF_3 - t_{13}, LF_2 - t_{12}) = Min(18 - 15, 15 - 15) = 0$.

The following table gives the calculation for critical path and total flo

Activity	Normal time	Earliest Start ES_i	Earliest Finish ES_j	Latest Finish LF_i	Latest Start LF_j	To... ..oat $LF_j - ES_j$ or $LF_i - ES_i$
1-2	15	0	15	0	15	0
1-3	15	0	15	3	18	3
2-3	3	15	18	15	18	0
2-5	5	15	20	32	37	17
3-4	8	18	26	18	26	0
3-6	12	18	30	28	40	10
4-5	1	26	27	36	37	10
4-6	14	26	40	26	40	0
5-6	3	27	30	37	40	10
6-7	14	40	54	40	54	0

From the above table we observe that the activities 1-2, 2-3, 3-4, 5-6, 6-7 are the critical activities and the critical path is given by, 1-2-3-4-6-7.

The total project completion is given by 54 days.

2. A small project is composed of 7 activities are listed in the table below. Activities are identified by their beginning (i) and end (j) node numbers.

Activity	Optimistic time (weeks)	Most likely (weeks)	Pessimistic time (weeks)
1-2	1	1	7
1-3	1	4	7
1-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

a) Draw the network diagram of the activities in the project
b) Find the expected duration and variance of each activity.
c) What is the expected project length?

d) **What is the probability that the project will be completed at least 4 weeks earlier than expected time?**

[WBUT 2014]

OR.

A small maintenance project consist of the following jobs whose precedence relationship is given below:

[WBUT 2015]

Activity	Estimated duration (weeks)		
	Optimistic	Most likely	Pessimistic
1-2	1	1	7
1-3	1	4	7
1-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

(a) Draw the project network.
(b) Find the expected duration and variance of each activity.
(c) Calculate the early and late occurrence for each event and the expected project length.
(d) Calculate the variance and standard deviations of project length.
(e) What is the probability that the project will be completed –
  (i) 4 weeks earlier than expected?
  (ii) Not more than 4 weeks later than expected?

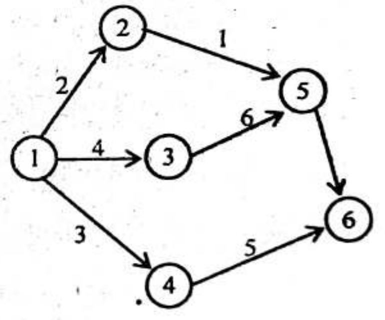
If the project due date is 19 weeks, what is the probability of meeting the due date?

Answer:

Complete the table using,

$$t_e = \frac{t_0 + 4t_m + t_p}{6}; \qquad \sigma^2 = \left(\frac{t_p - t_0}{6}\right)^e$$

Activity	Optimistic (t_0)	Most likely (ten)	Pessimistic (t_p)	t_e	σ^2
(1, 2)	1	1	7	2	1
(1, 3)	1	4	7	4	1
(1, 4)	2	2	8	3	1
(2, 5)	1	1	1	1	0
(3, 5)	2	5	14	6	4
(4, 6)	2	5	8	5	1
(5, 6)	3	6	15	7	4



Let us draw the network and number the nodes and calculate the T_E and T_L values along with S values.

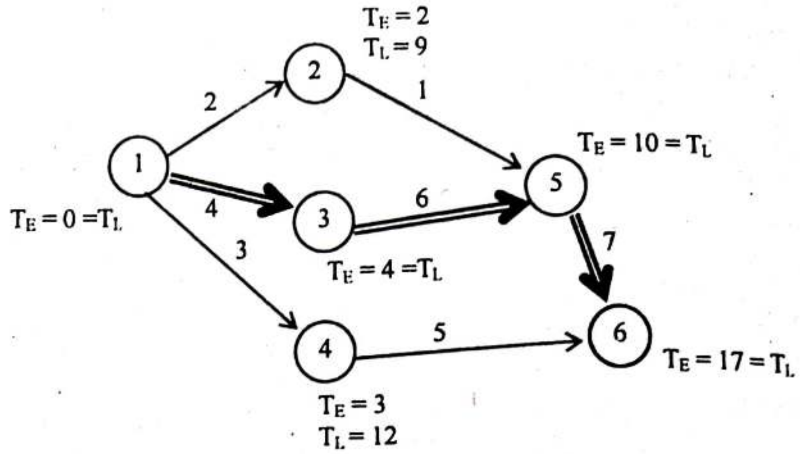
Calculations are shown below:

Forward Pass:

$T_E^{\ 1} = 0$

$T_E^{\ 2} = 2$

$T_E^{\ 3} = 4$



Backward Pass:

$T_L^{6} = 17$

$T_L^{5} = [17 - 7] = 10$

$T_L^{4} = [17 - 5] = 12$

$T_L^{3} = [10 - 6] = 4$

$T_L^{2} = [10 - 1] = 9$

$T_L^{1} = \text{Min}\left[(9-2),(4-4),(12-3)\right] = \text{Min}[7,0,9] = 0$

a) Critical path is 1 – 3 – 5 – 6

b) Expected project length = 4 + 6 + 7 = 17 weeks.

c) $\sigma_{C_r} = \sqrt{\sigma_{1-3}^2 + \sigma_{3-5}^2 + \sigma_{5-6}^2} = \sqrt{1+4+4} = 3$

Standard Normal Deviation (SND) $= \dfrac{T_S - T_E}{\sigma_{C_r}} = \dfrac{18-17}{3} = .33$

Z	0	1	2	3
0.3				0.1293

Probability of completion the project in 18 weeks

$= .5 + 0.1293 = 0.6293 = 62.93\%$

Probability of not completing project in week

$1 - .6293 = .3707 = 37.07\%$

d) i) $P(Z \le Z_1)$ where, $Z_1 = \dfrac{13-17}{3} = -1.33$

$P(Z \le -1.33) = 1 - P(Z \ge 1.33)$

$= 1 - (0.5 + 0.4082)$

$= 1 - 0.9082 = 0.0918$

ii) $P(Z \le Z_2)$ where $Z_2 = \dfrac{21-17}{3} = 4/3 = 1.33$

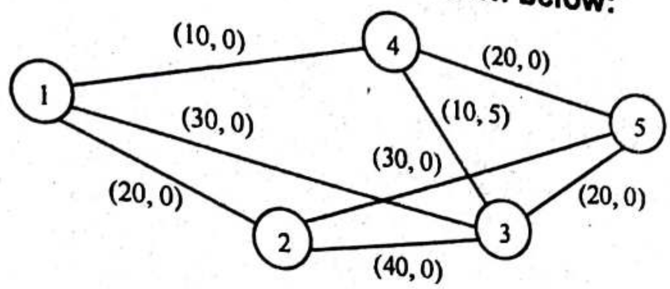
$P(Z \le 1.33) = 0.5 + 0.4082 = 0.9082$

(e) $P\left(Z \le \dfrac{19-17}{2}\right) = P(Z \le 0.666) = 0.5 + 0.2514$

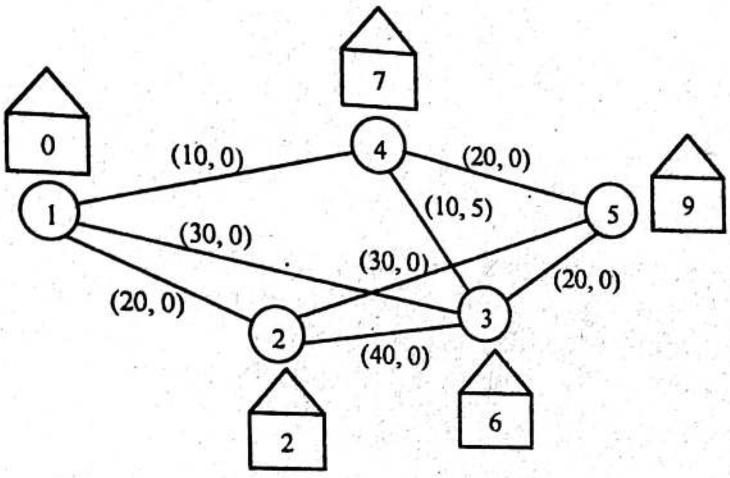
$\therefore P(Z \le 0.666) = 0.7514$

If the project due date is 19 weeks, then 75.14% of the Project will be completed.

3. Determine the maximal flow in the network shown below: **[WBUT 2016]**



Answer:



Forward Pass:

Let $TE(1) = 0$; then $TE(2) = 20$;

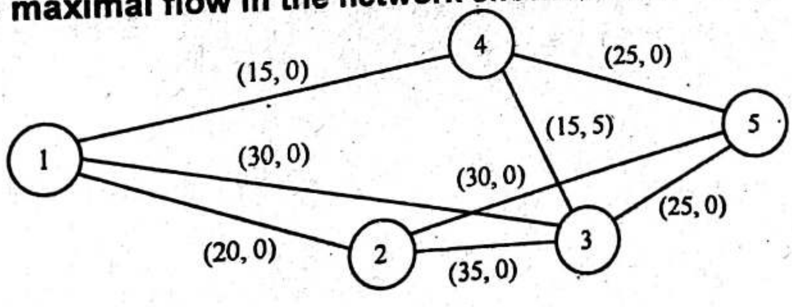
$TE(3) = \text{Max}\{30,\ 20+40,\ 10+5\} = 60$;

$TE(4) = \text{Max}\{10,\ 60+10\} = 70$;

$TE(5) = \text{Max}\{70+20,\ 20+30,\ 60+20\} = 90$.

Hence, the maximum flow in the network is 90.

4. Determine the maximal flow in the network shown below: **[WBUT 2017]**



Answer:

Similar type Question No. 3 of Long Answer type Questions.

QUEUING MODEL

Multiple Choice Type Questions

1. When customers move from one queue to another hoping to receive quicker service, it is termed as [WBUT 2013, 2016]
a) Reneging b) Balking c) Jockeying d) none of these

Answer: (b)

2. "Balking" is a term associate with [WBUT 2013]
a) Transportation problem b) LPP
c) Queuing problem d) Decision theory

Answer: (a)

3. Railways Reservation Centre is a typical example of [WBUT 2013]
a) Multi-server, Finite Queue Length, Infinite Population Queuing problem
b) Single server, Infinite Queue Length, Infinite Population Queuing Problem
c) Multi-server, Infinite Queue Length, Infinite Population Queuing Problem
d) None of these

Answer: (c)

4. If the inter-arrival time between two consecutive arrivals follows exponential distribution, then the distribution of the total number of persons could be present after a certain period of time will follow [WBUT 2014]
a) normal distribution b) Poisson distribution
c) binomial distribution d) exponential distribution

Answer: (b)

5. Queueing system comprising [WBUT 2015]
a) the arrival and service pattern b) the queue discipline
c) customer's behaviour d) all of these

Answer: (d)

6. In (M/M/1):(FCFS/∞/∞) queue system formula for expected number of customer in the system is [WBUT 2015]
a) $\frac{1}{\mu-\lambda}$ b) $\frac{\lambda}{\mu-\lambda}$ c) $\frac{\lambda^2}{\mu(\mu-\lambda)}$ d) none of these

Answer: (b)

7. When customers move from one queue to another hoping to receive quicker service, it is termed as [WBUT 2016]
a) Reneging b) Balking c) Jockeying d) None of these

Answer: (c)

8. In a M/M/1 queue, the service rate is [WBUT 2016]
   a) Poisson b) linear c) exponential d) none of these

Answer: (a)

9. Queuing theory deals with the problem of [WBUT 2017]
   a) material handling b) effective use of machines
   c) reduce the waiting time or idle time d) better utilization of man services

Answer: (c, d)

10. If everything else remains constant, including the mean arrival rate and service rate, except that the service time becomes constant instead of exponential, [MODEL QUESTION]
    a) The average queue length will be halved
    b) The average waiting time will be doubled
    c) The average queue length will increase
    d) None of these

Answer: (d)

11. Which of the following is not an assumption in common queuing mathematical models? [MODEL QUESTION]
    a) Arrivals come form an infinite or very large population
    b) Arrivals are Poisson distributed
    c) Arrivals are treated on a first-in, first-out basis and do not balk or renege
    d) The average arrival rate is faster than the average service rate

Answer: (d)

12. Monte Carlo is a [MODEL QUESTION]
    a) Inventory technique b) Queuing technique
    c) Markov process d) Simulation technique

Answer: (d)

13. Which of the following relationship is not true? [MODEL QUESTION]
    a) $W_s = W_q + 1/\mu$ b) $L_s = \lambda W_s$ c) $L_s = L_q + 1/\lambda$ d) $L_q = \lambda W_q$

Answer: (c)

14. The general purpose system simulation language [MODEL QUESTION]
    a) requires programme writing
    b) doesn't require programme writing
    c) requires predefined coding form
    d) needs a set of equations to describe a system

Answer: (a)

15. In an $(M / M / 1)$: (∞ / FIFO) model, the average number of customers $E(n)$ is given by [MODEL QUESTION]
    a) $\dfrac{\rho^2}{1+\rho}$ b) $\dfrac{\rho}{(1-\rho)}$ c) $\dfrac{\rho^2}{(1-\rho)}$ d) $\dfrac{\rho}{1+\rho}$

Answer: (b)

16. For Queuing model $M/M/1:\infty/FIFO$, which of the following is not true? [MODEL QUESTION]
    a) Input follows Poisson distribution
    b) output process follows exponential distribution
    c) capacity is finite
    d) it has only one service provider.

Answer: (c)

17. A queue-system will remain unchanged if [MODEL QUESTION]
    a) $\rho<1$ b) $\rho=1$ c) $\rho>1$ d) none of these

Answer: (b)

Short Answer Type Questions

1. A TV repairman find that the time spent on repairing has an exponential distribution with mean 30 min per unit. The arrival of TV set is Poisson with an average of 10 sets per day of 8 hr. What is the expected idle time per day? How many sets are there on average? [WBUT 2013]

Answer:

Here $\mu=\frac{1}{30}$, $\lambda=\frac{10}{8\times 60}=\frac{1}{48}$

Expected no. of jobs: $L_S=\frac{P}{1-P}=\frac{\lambda}{\mu-\lambda}=\frac{\frac{1}{48}}{\frac{1}{30}-\frac{1}{48}}=1\frac{2}{3}$ jobs

No. of hours in a days for which the repairman remains busy (in a 8 hour day) is $\left(8\times\frac{\lambda}{\mu}\right)=8\times\frac{30}{48}=5$ hours.

Hence, the required idle time is $(8-5)$ hours $=3$ hours.

2. In a Supermarket, the average arrival rate of customers is 10 every 30 minutes following poisson distribution. The average time taken by a cashier to list and calculate the customers purchase is 2.5 minutes, following exponential distribution. What is the probability that the queue length exceeds 7? What is the expected time spent by a customer in the system? [WBUT 2015]

Answer

Mean arrival rate of customer $\lambda=10/30=1/3$ per minute.

Mean service rate of customer $\mu=\frac{1}{2.5}$ per minute.

The utilization factor $=\rho=\lambda/\mu=\frac{1/2}{1/2.5}=\frac{5}{6}<1$.

(i) Probability that queue length exceeds 6

$= \text{Prob. } [\text{queue length} \geq 7] = \left(\frac{\lambda}{\mu}\right)^7 = \left(\frac{5}{6}\right)^7 = 0.279$

(ii) Expected waiting time of a customer in the system

$$(W_s) = \frac{1}{\mu - \lambda} = \frac{1}{3 - 2.5} = \frac{1}{0.5} = 2 \text{ minutes.}$$

3. A self service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service time. Find

i) **average number of customers in the system**
ii) **average number of customers in the queue or average queue length**
iii) **average time a customer spends in the system.**
iv) **average time a customer waits before being served.** **[WBUT 2016, 2017]**

Answer:

Here, $\lambda = \frac{9}{5}$ per minute

$\mu = \frac{10}{5}$ per minute

(i) Average no. of customer in the system

$$L_s = \frac{P}{1-\rho} = \frac{\lambda}{1-\rho} = \frac{915}{\frac{10}{5} - \frac{9}{5}} = 9$$

(ii) Average queue length $= L_s - \frac{\lambda}{\mu} = 9 - \frac{9}{10} = \frac{81}{10} \approx 8$

(iii) Expected waiting time in the system $= \frac{1}{\mu - \lambda} = 5$ minutes.

(iv) Expected average waiting time for the customer before being served $= \frac{\lambda}{\mu(\mu - \lambda)} = \frac{9}{10} \times 5 = 4.5$ minutes.

4. Customers arrive at a sales counter manned by a single person according to a Poisson's process with a mean rate of 20 per hour. The time required to serve customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer. **[MODEL QUESTION]**

Answer:

Average arrival rate, $\lambda = 20$/hour

μ = Average service rate $= \frac{60 \times 60}{100}$/hour $= 36$/hour

Expected waiting time of a customer in the queue

$$=W_q=\frac{\lambda}{\mu(\mu-\lambda)}=\frac{20}{36(36-20)}=\frac{5}{144}\text{hour}=2\frac{1}{12}\text{minutes}=\text{ 2 minutes 5 sec.}$$

5. A self service store employs one cashier at its counter. Nine customers arrive on an average of every fiva minutes while the cashier can serve 10 customers in every 3. minuets. Assuming Poisson distribution for arrival rate and exponential distribution for service rate, find **[MODEL QUESTION]**
i) average number of customer in the system
ii) average number of customers in the queue
iii) average time a customer spends in the system

Answer:
Arrival rate, $\lambda=9/5=1.8$ customers/minutes
Service rate, $\mu=10/5=2$ customers/minutes

i) Average no. of customers in the system $(L_s)=\frac{\lambda}{\mu-\lambda}=\frac{1.8}{2-1.8}=9$

ii) Average no. of customers in the queue $(L_q)=\frac{\lambda}{\mu}\times\frac{\lambda}{\mu-\lambda}=8.1$

iii) Average time of customer spends in the system $(W_s)=\frac{1}{\mu-\lambda}=\frac{1}{2-1.8}=5\text{minutes}$

iv) Average time a customer spends in the queue $=\frac{\lambda}{\mu}\times\left(\frac{1}{\mu-\lambda}\right)=4.5\text{minutes}$.

6. If the arrival rate is λ and service rate is μ, then prove that the expected queue length is $\frac{\lambda^2}{\mu(\mu-\lambda)}$. **[MODEL QUESTION]**

Answer:
Since there are (n – 1) units in the queue excluding one being serviced,

$$L_q=\sum_{n=1}^{\infty}(n-1)P_n=\sum_{n=1}^{\infty}nP_n-\sum_{n=1}^{\infty}P_n$$

$$=\sum_{n=1}^{\infty}nP_n-\left[\sum_{n=0}^{\infty}P_n-P_0\right]$$

$$=\sum_{n=1}^{\infty}nP_n-(1-P_0)=L_s-1+\left(1-\frac{\lambda}{\mu}\right)=L_s-\frac{\lambda}{\mu}$$

$$=\frac{\rho}{1-\rho}-\rho=\frac{\rho^2}{1-\rho}=\frac{\lambda^2}{\mu(\mu-\lambda)}$$

7. What is meant by traffic intensity? **[MODEL QUESTION]**

Answer:

If λ is the mean arrival rate and μ is the mean departure rate of a queue then ***traffic intensity*** (or utilization factor) is defined by $\rho = \lambda/\mu$ and $\rho < 1$.

Let, $\lambda = 8$ customers/hr. and $\mu = 10$ customers / hour of a queue-system.

Then $\rho = \frac{8}{10}$, which means that utilization of the service facility is 80%.

8. With reference to Queuing Theory, what do you mean by $(M/M/1):(\infty/FIFO)$? What do you mean by "transient state" and "steady state" of queuing model? Write the formula for L_s and L_b.

[MODEL QUESTION]

Answer:

1st Part:

(M/M/1/∞-FIFO) is the simplest queue system where the arrivals follow a Poisson process, service times are exponentially distributed and there is only one server. In other words, it is a system with Poisson input, exponential waiting time and Poisson output with single channel.

Queue capacity of the system is infinite with first in first out mode. The first M in the notation stands for Poisson input, second M for Poisson output, 1 for the number of servers and μ for infinite capacity of the system.

2nd Part:

A system is said to be in ***transient state*** when its operating characteristics are dependent on ***time***.

A ***steady state*** condition is said to prevail when the behaviour of the system becomes independent of time.

3rd Part:

Formulas

Average number of units in the system (L_s) = $\frac{\lambda}{\mu - \lambda}$

Average queue length (L_b) = $\frac{\lambda^2}{\mu(\mu - \lambda)}$

9. Assuming for a period of 2 hours in a day (8 A.M. – 10 A.M.), trains arrive at the yard every 20 minutes; the service time is 36 minutes per train, then calculate mentioning the queuing model:

(i) The probability that the yard is empty

(ii) Average number of trains in the queue assuming that the capacity of the yard is 4 trains only.

[MODEL QUESTION]

Answer:

Here, $\rho = \text{traffic intensity} = \frac{\lambda}{\mu} = \frac{20}{36} = \frac{5}{9}$

$\therefore$ i) Probability that the yard is empty $= (1-\rho) = \frac{4}{9}$

ii) Average no. of trains in the queue is $\frac{\rho^2}{1-\rho} = \frac{\left(\frac{5}{9}\right)^2}{\left(1-\frac{5}{9}\right)} = \frac{25}{9(9-5)} = \frac{25}{36}$.

10. Customer arrives at a sales counter managed by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve customer has an exponential distribution with a mean 100 seconds. Find the average waiting time of a customer. [MODEL QUESTION]

Answer:

Here $\lambda = 20$ per hour

$$= \frac{20}{3600} \text{ per sec}$$

$$\mu = \frac{1}{100} \text{ per sec.}$$

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{20}{3600} \times 100 = \frac{5}{9}$$

Average waiting time of a customer is

$$\frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\frac{1}{100}\left(1-\frac{5}{9}\right)} = \frac{100\times 9}{4} = 225 \text{ sec.}$$

11. Briefly discuss the Kendall notation of queuing model. [MODEL QUESTION]

Answer:

Kendall's Notation for Representing Queuing Models

Generally, queuing model may be completely specified in the following symbolic form (a/b/c) : (d/e), where

- a = Probability law for the arrival (or inter-arrival) time
- b = Probability law according to which the customers are being served
- c = Number of channels (or service stations)
- d = Capacity of the system, i.e., the maximum number allowed in the system (in service and waiting)
- e = Queue discipline

12. Consider a self-service store with one cashier. Assume Poisson arrivals and exponential service time. Suppose that nine customers arrives on an average every 5 minutes and the cashier can serve 10 in 5 minutes. Find
i) The average number of customers queuing for service
ii) The probability of having more that 10 customers in the system and
iii) The probability that a customer has to queue for more than 2 minutes.

[MODEL QUESTION]

Answer:

Average arrival rate $(\lambda)=\frac{9}{5}$ per min.

Average service rate $(\mu)=\frac{10}{5}=2$ per min.

Traffic intensity rate $(\rho)=\frac{\lambda}{\mu}=\frac{9}{10}=0.9<1$

(i) Average no. of customers in the queue

$$L_q=\frac{\rho^2}{1-\rho}=\frac{(0.9)^2}{0.1}=8.1$$

(ii) Probability (queue size ≥ 11) $=\rho^{11}=(0.9)^{11}=0.3138$

(iii) $\rho\left(T_q>t\right)=\rho.e^{-(\mu-\lambda)t}$

Here $t=2$, $\rho=0.9$ and $\mu-\lambda=2-1.8=0.2$

$$\therefore \rho\left(T_q>t\right)=\rho\left(T_q>2\right)=0.9\cdot e^{-0.2\times 2}=0.9e^{-0.4}=0.6033$$

Long Answer Type Questions

1. A car servicing station has 3 stalls where servicing can be offered simultaneously.
The cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most four cars waiting (seven in station) at one time. The arrival patterns are Poisson with a mean of one car per minute during the peak hours. The service time is exponential with mean 6 minutes. Find the average number of cars in the service station during peak hours, the average waiting time and the average number of cars per hour that cannot enter the station because of full capacity. **[WBUT 2013]**

Answer:

Here, $S=3, N=7, \lambda=1\,per\,\text{min.}, \mu=1/6\,per\,\text{min}.$

$$P_0=\left[\sum_{n=0}^{S-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n+\sum_{n=0}^{N}\frac{1}{S^{n-S}}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}=\left[\sum_{n=0}^{2}\frac{1}{n!}(6)^n+\sum_{n=0}^{7}\frac{1}{3^{n-3}}(6)^n\right]^{-1}=0.000145$$

$$P_n = \begin{cases} \frac{1}{n!}(6)^n \times 0.000145, & 0 \le n \le 3 \\ \frac{1}{3^{n-3}3!}(6)^n \times 0.000145, & 3 \le n \le 7 \end{cases}$$

Now, $L_s = L_q + S - P_0 \sum_{n=0}^{S-1} \frac{(S-n)(S\rho)^n}{n!} = \sum_{n=3}^{N}(n-3)P_n + 3 - P_0 \sum_{n=0}^{3-1} \frac{(S-n)(18)^n}{n!}$

$= (P_4 + 2P_5 + 3P_6 + 4P_7) + 3 - 0.000145 \times \left[3 + 2 \times 18 + 18^2\right]$

$= 3.459$

Therefore, average number of cars in the service station during peak hours is 3.459 (approx.).

Average waiting time in the system is

$$W_s = \frac{L_s}{\lambda(1-P_N)} = \frac{3.459}{1(1-0.8352)} = 20.99 \approx 21\,\text{min.}$$

2. Arrivals at telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of the phone calls is assumed to be distributed exponentially, with a mean of 3 minutes.

a) What is the probability that a person arriving at the booth will have to wait?

b) The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

c) What is the average length of the queue that forms from time to time?

d) What is the probability that it will take customer more than 10 minutes altogether to wait for the phone & complete his call? [WBUT 2013]

Answer:

Given: $\lambda = \frac{1}{10}$, $\mu = \frac{1}{3}$.

(a) Prob. $(w > 0) = 1 - P_0 = \frac{\lambda}{\mu} = \frac{3}{10} = 0.3$

(b) Average waiting time in the queue $W_q = \frac{\lambda}{\mu(\lambda - \mu)}$.

Since $W_q = 3, \mu = \frac{1}{3}, \lambda = \lambda_1$ for second booth, $3 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)} \Rightarrow \lambda_1 = 0.16$.

Hence, increase in the arrival rate $= 0.16 - 0.10 = 0.06$ arrival per minute.

(c) Average length of the queue $= \frac{\mu}{(\mu - \lambda)} = \frac{1}{3\left(\frac{1}{3} - \frac{1}{10}\right)} = 1.43$ persons.

(d) Prob. (waiting time in the system ≥ 10)

$$\int_{10}^{\infty}\lambda\left(1-\frac{\mu}{\lambda}\right)e^{-(\mu-\lambda)x}dx=\int_{10}^{\infty}\frac{1}{10}\left(1-\frac{10}{3}\right)e^{-\frac{7x}{30}}dx=0.03.$$

3. Customers arrive at a one-window drive-in bank according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The space in front of the window, including that for the serviced car can accommodate a maximum of 3 cars. Other cars can wait outside this space.

i) What is the probability that an arriving customer can drive directly to the space in front of the window?

ii) What is the probability that an arriving customer will have to wait outside the indicated space?

iii) How long is an arriving customer expected to wait before starting service?

iv) How many spaces should be provided in front of the window so that all the arriving customers can wait in front of the window at least 20% of the time?

[WBUT 2015]

Answer:

Here, $\lambda=10$ cars / hour and $\mu=12$ cars / hour

i) $$p_0+p_1+p_2=1-\left(\frac{\lambda}{\mu}\right)^3=1-\left(\frac{10}{12}\right)^3=0.42$$

ii) $$1-(p_0+p_1+p_2)=1-0.42=0.58$$

iii) $$\frac{\lambda}{\mu(\mu-\lambda)}=\frac{10}{12(12-10)}=0.417 \text{ hours.}$$

iv) $$P_0+P_1=0.30$$

There should be at least one car space for waiting at least 20% of the time.

4. Arrival rate of telephone calls at a telephone booth is according to Poisson distribution, with an average time of 9 minutes between two consecutive arrivals. The length of telephone call is assumed to be exponentially distributed, with mean 3 minutes.

i) Determine the probability that a person arriving at the booth will have to wait.

ii) Find the average queue length that is formed from time to time.

iii) The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least 4 minutes for the phone. Find the increase in flow rate of arrivals which will justify a second booth.

iv) What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free?

[WBUT 2016]

Answer:

Given $\lambda=\frac{1}{9},\ \mu=\frac{1}{3}$

(i) Probability $(w>0)=1-P_0=\frac{\lambda}{\mu}=\frac{3}{9}=\frac{1}{3}$.

(ii) Expected queue length $=\frac{\mu}{(\mu-\lambda)}=\frac{\frac{1}{3}}{\left(\frac{1}{3}-\frac{1}{9}\right)}=\frac{1}{3}\times\frac{9}{2}=\frac{3}{2}$.

(iii) Expected waiting time in the queue

$$W_q=\frac{\lambda}{\mu(\mu-\lambda)}$$

Since $W_q=3,\ \mu=\frac{1}{3},\ \lambda=\lambda'$ for second booth.

$$\therefore \qquad 3=\frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3}-\lambda'\right)} \Rightarrow \lambda'=0.16$$

Hence, increase in the arrival rate $=0.16-0.11=0.05$ per minute.
(iv) Probability (waiting time is the system ≥ 10)

$$=\int_{10}^{\infty} f(\mu-\lambda)e^{-(\mu-\lambda)w}dw=e^{-10(\mu-\lambda)}=e^{-\frac{20}{9}}=0$$

NON LINEAR PROGRAMMING PROBLEM

Multiple Choice Type Questions

1. "Kuhn-tucker" conditions pertain to [WBUT 2013]
   a) Non-linear Programming Problems
   b) $\{(M/M/1) : (\infty / FCFS)\}$ Problems
   c) LPP with degeneracy
   d) None of these.

Answer: (a)

2. Dynamic problem deals with the [WBUT 2014]
   a) multi-stage decision making problems
   b) single stage decision making problems
   c) time dependent decision making problems
   d) problems which fix the levels of different decisions

Answer: (c)

Long Answer Type Questions

1. Use dynamic program to solve the following problem: [WBUT 2013, 2017]

$$\text{Maximize } Z = y_1^2 + y_2^2 + y_3^2$$

$$\text{Subject to: } y_1 + y_2 + y_3 \geq 15$$

$$y_1, y_2, y_3 \geq 0$$

Answer:

Since the number of decision variable is three, the given problem is a three stage problem defined as follows:

$s_3 = y_1 + y_2 + y_3 \geq 15$

$s_2 = y_1 + y_2 \quad = s_3 + y_3$

$s_1 = y_1 \quad = s_2 + y_2$

and $f_3(s_3) = \min\limits_{y_3}\left(y_3^2 + f_2(s_2)\right)$

$f_2(s_2) = \min\limits_{y_2}\left(y_2^2 + f_1(s_1)\right)$

$f_1(s_1) = y_1^2 = (s_2 - y_2)^2$

$y_2^2 + (s_2 - y_2)^2$ is minimum, if its derivative w.r.t. y_2 is zero.

i.e. $2y_2 + 2(s_2 - y_2) = 0$

or, $y_2 = \dfrac{s_2}{2}$.

Hence, $f_2(s_2) = \frac{s_2^2}{2}$.

Now, $f_3(s_3) = \min_{y_3}\left(y_3^2 + \frac{1}{2}(s_3 - y_3)^2\right)$

or, $f_3(15) = \min_{y_3 \le 15}\left(y_3^2 + \frac{1}{2}(15 - y_3)^2\right)$ $[\because y_1 + y_2 + y_3 \ge 15]$

Since the minimum value of the function $y_3^2 + \frac{1}{2}(15 - y_3)^2$ occur at $y_3 = 5$,

$f_3(15) = 5^2 + \frac{1}{2}(15-5)^2 = 75$.

Thus $s_3 = 15 \Rightarrow y_3^* = 5$,

$s_2 = s_3 - y_3 = 15 - 5 = 10 \Rightarrow y_2^* = \frac{s_2}{2} = 5$,

$s_1 = s_2 - y_2 = 10 - 5 = 5 \Rightarrow y_1^* = 5$.

Hence, the optimal solution is $(5,5,5)$ and $f_3^*(s_3) = 75$.

2. Use the method of Lagrangian multipliers to solve the following non-linear programming problem:

Minimize: $Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 > 0$. **[WBUT 2016]**

Answer:

Lagrangian function can be formulated as:

$L(x,\lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 10 - \lambda(x_1 + x_2 + x_3 - 20)$

The necessary conditions for maximum or minimum are

$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0$; $\frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0$

$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0$; $\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20) = 0$

Putting the value of x_1, x_2 and x_3 in the last equation $\frac{\partial L}{\partial \lambda} = 0$ and solving for λ, we get $\lambda = 30$. Substituting the value of λ in the other three equations, we get an extreme point: $(x_1, x_2, x_3) = 5, 11, 4$.

To prove the sufficient condition whether the extreme point solution gives maximum or minimum value of the objective function we evaluate $(n-1)$ principal minors as follows:

$$\Delta_3 = \begin{vmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 g}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 g}{\partial x_1 \partial x_2} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 g}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 g}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

Since sign of Δ_3 and Δ_4 are alternative, therefore extreme point: $(x_1, x_2, x_3) = (5, 11, 4)$ is a local maximum. At this point the value of objective function is, $Z = 281$.

GAME THEORY

Multiple Choice Type Questions

1. In a game theory problem if the payoff matrix is formed in favour of player A (all the elements of the matrix will be profit of A), then he should follow [WBUT 2014]
   a) max-max principle b) min-max principle
   c) max-min principle d) min-min principle

Answer: (c)

Short Answer Type Questions

1. Solve the following game whose pay off matrix is given by: **[WBUT 2014]**

		Player B			
		B1	B2	B3	B4
Player A	A1	18	4	6	4
	A2	6	2	13	7
	A3	11	5	17	3
	A4	7	6	12	2

Answer:
We have,

	B_1	B_2	B_3	B_4
A_1	18	4	6	4
A_2	6	2	13	7
A_3	11	5	17	3
A_4	7	6	12	2

By dominance property, B_2 is dominated by B_3. Therefore we can discard B_3 and the reduced matrix is

	B_1	B_2	B_4
A_1	18	4	4
A_2	6	2	7
A_3	11	5	3
A_4	7	6	2

Again the linear combination of B_2 and B_4 dominated by B_1, so we can discard B_1 and we get,

	B_2	B_4
A_1	4	4
A_2	2	7
A_3	5	3
A_4	6	2

Linear combination of A_1 and A_4 dominates A_3. So discarding A_3 we get,

	B_2	B_4
A_1	4	4
A_2	2	7
A_4	6	2

Again, linear combination of A_2 and A_4 dominates A_1, so we discard A_1 and the following matrix is

	B_2	B_4
A_2	2	7
A_4	6	2

Now if x_2, x_4 be the probabilities with which A_2 and A_4 played and y_2, y_4 be the probabilities with which B_2, B_4 played for the optimal solution.

$$\therefore \quad x_2 = \frac{2-6}{4-13} = \frac{4}{9},\ x_4 = \frac{7-2}{9} = \frac{5}{9}.$$

$$y_2 = \frac{2-7}{4-13} = \frac{6}{9},\ y_4 = \frac{6-2}{9} = \frac{4}{9}.$$

and the value of the game is $\dfrac{4-42}{4-13} = \dfrac{38}{9}$.

Long Answer Type Questions

1. A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in table represent wage increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union? What is the game value? [WBUT 2017]

		Union Strategies			
		U_1	U_2	U_3	U_4
Company Strategies	C_1	**0.25**	**0.27**	**0.35**	**–0.02**
	C_2	**0.20**	**0.06**	**0.08**	**0.08**
	C_3	**0.14**	**0.12**	**0.05**	**0.03**
	C_4	**0.30**	**0.14**	**0.19**	**0.00**

Answer:

Given, the payoff matrix is

	U_1	U_2	U_3	U_4	Row minima
C_1	0.25	0.27	0.35	–0.02	–0.02
C_2	0.20	$\boxed{0.06}$	0.08	$\boxed{0.08}$	0.06
C_3	0.14	0.12	0.05	0.03	0.03
C_4	0.30	0.14	0.19	0.00	0.00
Column maxima	0.30	0.27	0.35	0.08	

$$\text{Max (Row minima)} = \text{Max}\{-0.02,\ 0.06,\ 0.03,\ 0.00\} = 0.06$$

$$\text{Min (Column maxima)} = \text{Min}\{0.30,\ 0.27,\ 0.35,\ 0.08\} = 0.08$$

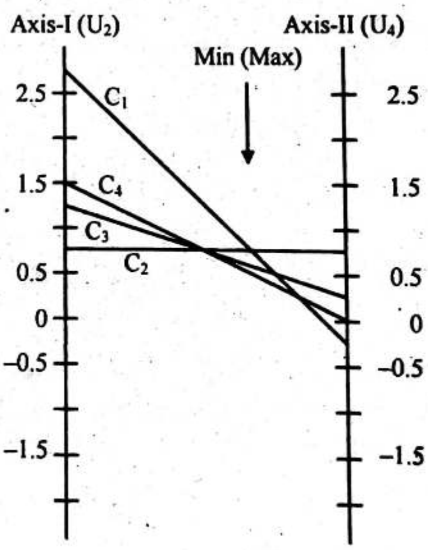
Therefore, this game has no pure strategy.
So for mixed strategies, now we apply dominance properties and the resulting matrix is:
(1) U_3 is dominated by U_4; therefore,

	U_1	U_2	U_4
C_1	0.25	0.27	–0.02
C_2	0.20	0.06	0.08
C_3	0.14	0.12	0.03
C_4	0.30	0.14	0.00

(2) U_4 dominates U_1; therefore,

	U_2	U_4
C_1	0.27	–0.02
C_2	0.06	0.08
C_3	0.12	0.03
C_4	0.14	0.00

Now we apply graphical method, in order to obtain the mixed strategies:



So the resulting payoff is

	U_2	U_4
C_1	0.27	–0.02
C_2	0.06	0.08

The optional strategies are:

$$S_C = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ p_1 & p_2 & 0 & 0 \end{pmatrix},\ p_1 + p_2 = 1$$

and $$S_U = \begin{pmatrix} U_1 & U_2 & U_3 & U_4 \\ 0 & q_1 & 0 & q_2 \end{pmatrix},\ q_1 + q_2 = 1$$

$$p_1 = \frac{0.08 - 0.06}{(0.27 + 0.08) - (0.06 - 0.02)} = \frac{0.02}{0.31} = \frac{2}{31}$$

$$p_2 = 1 - \frac{2}{31} = \frac{29}{31}$$

$$q_1 = \frac{0.08 + 0.02}{0.31} = \frac{1}{31}$$

and $$q_2 = \frac{30}{31}$$

The value of the game is $\dfrac{0.08 \times 0.27 + 0.06 \times 0.02}{0.31} = 0.07355$

DUALITY THEORY

Multiple Choice Type Questions

1. If Dual has unbounded solution, primal has **[WBUT 2014, 2016]**
a) no feasible solution
b) unbounded solution
c) feasible solution
d) none of these

Answer: (a)

Short Answer Type Questions

1. Obtain the dual of the following primal LPP: **[WBUT 2014]**

Maximize $z = -2x_1 - 2x_2 - 4x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

where, $x_1, x_2, x_3 \geq 0$

Answer:

At first the standard form of the given LPP is

Max $z = -2x_1 - 2x_2 - 4x_3$

s.t. $-2x_1 - 3x_2 - 5x_3 \leq -2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

So the dual problem of the given primal LPP is

Min $w = -2x_1 + 3x_2 + 5x_3$

s.t. $-2x_1 + 3x_2 + x_3 \geq -2$

$-3x_1 + x_2 + 4x_3 \geq -2$

$-5x_1 + 7x_2 + 6x_3 \geq -4$

and $x_1, x_2, x_3 \geq 0$

2. Find the dual of the LPP: **[WBUT 2017]**

Maximize $Z = 2x_1 - x_2 + x_3$

Subject to $x_1 + x_2 + x_3 \leq 50$

$x_1 + 2x_2 - 5x_3 \leq 80$

$2x_1 + x_2 + 4x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

Answer:
Dual of the given primal problem is given by

$$\text{Min } W = 50y_1 + 80y_2 + 20y_3$$

subject to $y_1 + y_2 + 2y_3 \geq 2$

$y_1 + 2y_2 + y_3 \geq -1$

$y_1 - 5y_2 + 4y_3 \geq 1$

and $y_1, y_2, y_3 \geq 0$

Long Answer Type Questions

1. Write down the dual of the following L.P. Problem: **[WBUT 2016]**

Minimize $Z = 2x_1 + 5x_2 + x_3$

subject to, $2x_1 + x_2 - x_3 \leq 2$

$-3x_1 + 2x_2 - 5x_3 \geq -6$

$4x_1 + 3x_2 + x_3 \geq 10$

$x_1, x_2, x_3 \geq 0$

Answer:
The standard form of the given LPP is

$$\text{Max } W = -(2x_1 + 5x_2 + x_3)$$

subject to $2x_1 + x_2 - x_3 \leq 2$

$3x_1 - 2x_2 + 5x_2 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -10$

and $x_1, x_2, x_3 \geq 0$.

So the dual of the given primal is

$$\text{Min } W' = 2u + 6v - 10w$$

subject to $2u + 3v - 4w \geq -2$

$u - 2v - 3w \geq -5$

$-u + 5v - w \geq -1$

and $u, v, w \geq 0$.

2. Find out the dual of the following problem: **[WBUT 2017]**

Maximize $Z = 2x_1 + 3x_2 - 4x_3$

subject to $3x_1 + x_2 + x_3 \leq 2$

$-4x_1 + 3x_3 \geq 4$

$x_1 - 5x_2 + x_3 = 5$

$x_1, x_2 \geq 0$ **and** x_3 **is unrestricted.**

Answer:

Since the primal problem has 3rd constraint is equality in sign, so the 3rd variable of the dual problem will be unrestricted in sign.

Similarly, as the 3rd variable of the primal LPP is unrestricted in sign, therefore, 3rd constraint of the dual LPP will be equality in sign.

The standard form of the given LPP is

$$\text{Max } Z = 2x_1 + 3x_2 - 4x_3$$

$$\begin{aligned} \text{subject to } \quad & 3x_1 + x_2 + x_3 \le 2 \\ & 4x_1 - 3x_3 \le -4 \\ & x_1 - 5x_2 + x_3 = 5 \end{aligned}$$

and x_1, $x_2 \ge 0$ and x_3 is unrestricted in sign.

Therefore, the required dual problem is

$$\text{Min } W = 2y_1 - 4y_2 + 5y_3$$

$$\begin{aligned} \text{subject to } \quad & 3y_1 + 4y_2 + y_3 \ge 2 \\ & y_1 - 5y_3 \ge 3 \\ & y_1 - 3y_2 + y_3 = -4 \end{aligned}$$

and y_1, $y_2 \ge 0$ and y_3 is unrestricted in sign.

INVENTORY CONTROL

Multiple Choice Type Questions

1. In EOQ inventory problem with no shortage, in which demand is assumed to be fixed and completely pre-determined, the economic lot size is [WBUT 2017]

a) $\sqrt{\frac{2DC_0}{C_h}}$ b) $\sqrt{\frac{2DC_h}{C_0}}$ c) $\sqrt{2DC_0C_h}$ d) none of these

Answer: (a)

Short Answer Type Questions

1. The rate of use of particular raw material from stores is 20 units per year. The cost of placing and receiving an order is Rs. 40. The cost of each unit is Rs. 100. The cost of carrying inventory in percent per year is 0.16 and it depends upon the average stock. Determine the economic order quantity. [WBUT 2017]

Answer:

Given $R = 20$ units per year

$C_1 = C \times I = 100 \times 0.16 = ₹16$ per unit per year

$C_3 = ₹40$ per order

$\therefore$ Optimal lot size $q^* = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 40 \times 20}{16}} = 10$ units.

$\therefore$ No. of orders per year $= 2$