UNIT II

LESSON

2 ASSIGNMENT PROBLEM

LESSON STRUCTURE

Objectives

After Studying this lesson, you should be able to:

- Assignment Problem Formulation
- How to solve the Assignment Problem
- How to solve the unbalanced problem using appropriate method
- Make appropriate modification when some problems are infeasible
- Modify the problem when the objective is to maximize the objective function
- Formulate and solve the crew assignment problems

2.1 Introduction

The Assignment Problem can define as follows:

Given n facilities, n jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness if optimized. Here the optimization means Maximized or Minimized. There are many management problems has a assignment problem structure. For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible. Another example a container company may have an empty container in each of the location 1, 2,3,4,5 and requires an empty container in each of the locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance. The third example here is, a marketing set up by making an estimate of sales performance for different salesmen as well as for different cities one could assign a particular salesman to a particular city with a view to maximize the overall sales.

Note that with n facilities and n jobs there are n! possible assignments. The simplest way of finding an optimum assignment is to write all the n! possible arrangements, evaluate their total cost and select the assignment with minimum cost. Bust this method leads to a calculation problem of formidable size even when the value of n is moderate. For n=10 the possible number of arrangements is 3268800.

2.2 Assignment Problem Structure and Solution

The structure of the Assignment problem is similar to a transportation problem, is as follows:

Jobs

		1	2	…	n	
	1	c_{11}	c_{12}	…	c_{1n}	1
	2	c_{21}	c_{21}	…	c_{2n}	1
		.	.	.	.	.
Workers		.	.	.	.	
		.	.	.	.	
	n	c_{n1}	c_{n2}	…	c_{nn}	1
		1	1	…	1	

The element c_{ij} represents the measure of effectiveness when i^{th} person is assigned j^{th} job. Assume that the overall measure of effectiveness is to be minimized. The element x_{ij} represents the number of i^{th} individuals assigned to the j^{th} job. Since i^{th} person can be assigned only one job and j^{th} job can be assigned to only one person we have the following

$$x_{i1} + x_{i2} + \ldots\ldots\ldots\ldots\ldots + x_{in} = 1, \text{ where } i = 1, 2, \ldots\ldots\ldots, n$$

$$x_{1j} + x_{2j} + \ldots\ldots\ldots\ldots\ldots + x_{nj} = 1, \text{ where } j = 1, 2, \ldots\ldots\ldots, n$$

and the objective function is formulated as

$$\text{Minimize } c_{11}x_{11} + c_{12}x_{12} + \ldots\ldots\ldots\ldots + c_{nn}x_{nn}$$

$$x_{ij} \geq 0$$

The assignment problem is actually a special case of the transportation problem where m = n and $a_i = b_j = 1$. However, it may be easily noted that any basic feasible solution of an assignment problem contains $(2_n – 1)$ variables of which (n – 1) variables are zero. Because of this high degree of degeneracy the usual computation techniques of a transportation problem become very inefficient. So, hat a separate computation technique is necessary for the assignment problem.

The solution of the assignment problem is based on the following results:

“If a constant is added to every element of a row/column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original assignment problem and vice versa”. – This result may be used in two different methods to solve the assignment problem. If in an assignment problem some cost elements $c_{ij\ \text{are}}$ negative, we may have to convert them into an equivalent assignment problem where all the cost elements are non-negative by adding a suitable large constant to the cost elements of the relevant row or column, and then we look for a feasible solution which has zero assignment cost after adding suitable constants to the cost elements of the various rows and columns. Since it has been assumed that all the cost elements are non-negative, this assignment must be optimum. On the basis of this principle a computational technique known as Hungarian Method is developed. The Hungarian Method is discussed as follows.

Hungarian Method:

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

Step 1:

From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

Step 2:
In each row of the table find out the smallest cost element, subtract this smallest cost element from each element in that row. So, that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table.

Step 3:
In each column of the table find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table.

Step 4:
Now determine an assignment as follows:

1. For each row or column with a single zero element cell that has not be assigned or eliminated, box that zero element as an assigned cell.
2. For every zero that becomes assigned, cross out all other zeros in the same row and for column.
3. If for a row and for a column there are two or more zero and one can’t be chosen by inspection, choose the assigned zero cell arbitrarily.
4. The above procedures may be repeated until every zero element cell is either assigned (boxed) or crossed out.

Step 5:
An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum. If no optimum solution is found i.e. some rows or columns without an assignment then go to **Step 6**.

Step 6:
Draw a set of lines equal to the number of assignments which has been made in **Step 4**, covering all the zeros in the following manner

1. Mark check (√) to those rows where no assignment has been made.

2. Examine the checked (√) rows. If any zero element cell occurs in those rows, check (√) the respective columns that contains those zeros.

3. Examine the checked (√) columns. If any assigned zero element occurs in those columns, check (√) the respective rows that contain those assigned zeros.

4. The process may be repeated until now more rows or column can be checked.

5. Draw lines through all unchecked rows and through all checked columns.

Step 7:
Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them.

Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table.

Example 2.1:

Problem

A work shop contains four persons available for work on the four jobs. Only one person can work on any one job. The following table shows the cost of assigning each person to each job. The objective is to assign person to jobs such that the total assignment cost is a minimum.

		Jobs			
		1	**2**	**3**	**4**
	A	20	25	22	28
Persons	**B**	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Solution

As per the Hungarian Method

Step 1: The cost Table

		Jobs			
		1	**2**	**3**	**4**
	A	20	25	22	28
Persons	**B**	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Step 2: Find the First Reduced Cost Table

		Jobs 1	2	3	4
Persons	A	0	5	2	8
	B	0	3	8	2
	C	2	0	4	7
	D	2	0	1	1

Step 3: Find the Second Reduced Cost Table

		Jobs 1	2	3	4
Persons	A	0	5	1	7
	B	0	3	7	1
	C	2	0	3	6
	D	2	0	0	0

Step 4: Determine an Assignment

By examine row A of the table in Step 3, we find that it has only one zero (cell A1) box this zero and cross out all other zeros in the boxed column. In this way we can eliminate cell B1.

Now examine row C, we find that it has one zero (cell C2) box this zero and cross out (eliminate) the zeros in the boxed column. This is how cell D2 gets eliminated.

There is one zero in the column 3. Therefore, cell D3 gets boxed and this enables us to eliminate cell D4.

Therefore, we can box (assign) or cross out (eliminate) all zeros.

The resultant table is shown below:

Jobs

		1	2	3	4
	A	$\boxed{0}$	5	1	7
Persons	**B**	0	3	7	1
	C	~~2~~		3	6
	D	2	0		0

Step 5:

$\boxed{0}$

The solution obtained in Step 4 is not optimal. Because we were able to make three assignments when four were required.

Step 6:

$\boxed{0}$

Cover all the zeros of the table shown in the Step 4 with three lines (since already we made three assignments).

Check row B since it has no assignment. Note that row B has a zero in column 1, therefore check column1. Then we check row A since it has a zero in column 1. Note that no other rows and columns are checked. Now we may draw three lines through unchecked rows (row C and D) and the checked column (column 1). This is shown in the table given below:

Jobs

		1	2	3	4
	A	$\boxed{0}$	5	1	7
Persons	**B**	0	3	7	1
	C	~~2~~		3	6
	D	2	0		0

$\boxed{0}$

Step 7:
Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (C1 and D1) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		Jobs 1	2	3	4
	A	0	4	0	6
Persons	B	0	2	6	0
	C	3	0	3	6
	D	3	0	0	0

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:
Determine an assignment

Examine each of the four rows in the table given in Step 7, we may find that it is only row C which has only one zero box this cell C2 and cross out D2.

Note that all the remaining rows and columns have two zeros. Choose a zero arbitrarily, say A1 and box this cell so that the cells A3 and B1 get eliminated.

Now row B (cell B4) and column 3 (cell D4) has one zero box these cells so that cell D4 is eliminated.

Thus, all the zeros are either boxed or eliminated. This is shown in the following table

Jobs

		1	2	3	4
	A	$\boxed{0}$	4	$\cancel{0}$	6
Persons	B	0	2	6	
	C	$\cancel{3}$		3	$\boxed{0}$
	D	3	0		0

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

The total cost of assignment is: 78 that is A1 + B4 + C2 + D3

$$20 + 17 + 17 + 24 = 78$$

2.3 Unbalanced Assignment Problem

In the previous section we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as **balanced assignment problem**. Suppose if the number of person is different from the number of jobs then the assignment problem is called as **unbalanced**.

If the number of jobs is less than the number of persons, some of them can't be assigned any job. So that we have to introduce on or more dummy jobs of zero duration to make the unbalanced assignment problem into balanced assignment problem. This balanced assignment problem can be solved by using the Hungarian Method as discussed in the previous section. The persons to whom the dummy jobs are assigned are left out of assignment.

Similarly, if the number of persons is less than number of jobs then we have introduce one or more dummy persons with zero duration to modify the unbalanced into balanced and then the problem is solved using the Hungarian Method. Here the jobs assigned to the dummy persons are left out.

Example 2.2:

Solve the following unbalanced assignment problem of minimizing the total time for performing all the jobs.

		Jobs 1	2	3	4	5
Workers	A	5	2	4	2	5
	B	2	4	7	6	6
	C	6	7	5	8	7
	D	5	2	3	3	4
	E	8	3	7	8	6
	F	3	6	3	5	7

Solution

In this problem the number of jobs is less than the number of workers so we have to introduce a dummy job with zero duration.

The revised assignment problem is as follows:

		Jobs 1	2	3	4	5	6
Workers	**A**	5	2	4	2	5	0
	B	2	4	7	6	6	0
	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Now the problem becomes balanced one since the number of workers is equal to the number jobs. So that the problem can be solved using Hungarian Method.

Step 1: The cost Table

		Jobs 1	2	3	4	5	6
Workers	**A**	5	2	4	2	5	0
	B	2	4	7	6	6	0
	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 2: Find the First Reduced Cost Table

		Jobs					
		1	2	3	4	5	6
	A	5	2	4	2	5	0
	B	2	4	7	6	6	0
Workers	C	6	7	5	8	7	0
	D	5	2	3	3	4	0
	E	8	3	7	8	6	0
	F	3	6	3	5	7	0

Step 3: Find the Second Reduced Cost Table

		Jobs					
		1	2	3	4	5	6
	A	3	0	1	0	1	0
	B	0	2	4	4	2	0
Workers	C	4	5	2	6	3	0
	D	3	0	0	1	0	0
	E	6	1	4	6	2	0
	F	1	4	0	3	3	0

Step 4: Determine an Assignment

By using the Hungarian Method the assignment is made as follows:

Jobs

		1	**2**	**3**	**4**	**5**	**6**
	A	3	~~0~~	1	$\boxed{0}$	1	~~0~~
	B		2	4	4	2	0
Workers	**C**	$\boxed{0}$	5	2	6	3	
	D	3	0	0	1		0
	E	6	1	4	6	2	$\boxed{0}$
	F	1	4		3	3	0
						$\boxed{0}$	

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make five assignments when six were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with five lines (since already we made five assignments).

Check row E since it has no assignment $\boxed{0}$ te that row B has a zero in column 6, therefore check column6. Then we check row C since it has a zero in column 6. Note that no other rows and columns are checked. Now we may draw five lines through unchecked rows (row A, B, D and F) and the checked column (column 6). This is shown in the table given below:

		Jobs					
		1	**2**	**3**	**4**	**5**	**6**
	A	3	~~0~~	1	$\boxed{0}$	1	~~0~~
	B		2	4	4	2	0
Workers	**C**	$\boxed{0}$	5	2	6	3	×
	D	3	0	0	1		0
	E	6	1	4	6	2	$\boxed{0}$
	F	1	4		3	3	0
			×	×		$\boxed{0}$	×

Step 7:

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 1. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

		Jobs			
1	**2**	**3**	**4**	**5**	**6**
		$\boxed{0}$			×

	A	3	0	1	0	1	1
	B	0	2	4	4	2	1
Workers	C	3	4	1	5	2	0
	D	3	0	0	1	0	1
	E	5	0	3	5	1	0
	F	1	4	0	3	3	1

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

Jobs

		1	**2**	**3**	**4**	**5**	**6**
	A	3	$\cancel{0}$	1	$\boxed{0}$	1	1
	B		2	4	4	2	1
Workers	C	$\boxed{0}$	4	1	5	2	
	D	3	0	0	1		1
	E	5		3	5	1	$\boxed{0}$
	F	1	4		3	3	1
						$\boxed{0}$	

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

Thus, the worker A is assigned to Job4, worker B is assigned to job 1, worker C is assigned to job 6, worker D is assigned to job 5, worker E is assigned to job 2, and worker F is assigned to job 3. Since the Job 6 is dummy so that worker C can't be assigned.

The total minimum time is: 14 that is A4 + B1 + D5 + E2 + F3

$$2 + 2 + 4 + 3 + 3 = 14$$

Example 2.3:

A marketing company wants to assign three employees viz. A, B, and C to four offices located at W, X, Y and Z respectively. The assignment cost for this purpose is given in following table.

		Offices			
		W	**X**	**Y**	**Z**
	A	160	220	240	200
Employees	**B**	100	320	260	160
	C	100	200	460	250

Solution

Since the problem has fewer employees than offices so that we have introduce a dummy employee with zero cost of assignment.

The revised problem is as follows:

Offices

W X Y Z

	A	160	220	240	200
Employees	**B**	100	320	260	160
	C	100	200	460	250
	D	0	0	0	0

Now the problem becomes balanced. This can be solved by using Hungarian Method as in the case of Example 2.2. Thus as per the Hungarian Method the assignment made as follows:

Employee A is assigned to Office X, Employee B is assigned to Office Z, Employee C is assigned to Office W and Employee D is assigned to Office Y. Note that D is empty so that no one is assigned to Office Y.

The minimum cost of assignment is: 220 + 160 + 100 = 480

2.4 Infeasible Assignment Problem

Sometimes it is possible a particular person is incapable of performing certain job or a specific job can't be performed on a particular machine. In this case the solution of the problem takes into account of these restrictions so that the infeasible assignment can be avoided.

The infeasible assignment can be avoided by assigning a very high cost to the cells where assignments are restricted or prohibited. This is explained in the following Example 2.4.

Example 2.4:

A computer centre has five jobs to be done and has five computer machines to perform them. The cost of processing of each job on any machine is shown in the table below.

Jobs

1 2 3 4 5

	1	70	30	X	60	30
Computer Machines	**2**	X	70	50	30	30
	3	60	X	50	70	60
	4	60	70	20	40	X
	5	30	30	40	X	70

Because of specific job requirement and machine configurations certain jobs can't be done on certain machines. These have been shown by X in the cost table. The assignment of jobs to the machines must be done on a one to one basis. The objective here is to assign the jobs to the available machines so as to minimize the total cost without violating the restrictions as mentioned above.

Solution

Step 1: The cost Table

Because certain jobs cannot be done on certain machines we assign a high cost say for example 500 to these cells i.e. cells with X and modify the cost table. The revised assignment problem is as follows:

		Jobs				
		1	**2**	**3**	**4**	**5**
	1	70	30	500	60	30
Computer Machines	**2**	500	70	50	30	30
	3	60	500	50	70	60
	4	60	70	20	40	500
	5	30	30	40	500	70

Now we can solve this problem using Hungarian Method as discussed in the previous sections.

Step 2: Find the First Reduced Cost Table

		Jobs 1	2	3	4	5
Computer Machines	**1**	40	0	470	30	0
	2	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 3: Find the Second Reduced Cost Table

Jobs

1 2 3 4 5

	1	40	0	470	30	0
Computer Machines	**2**	470	40	20	0	0
	3	10	450	0	20	10
	4	40	50	0	20	480
	5	0	0	10	470	40

Step 4: Determine an Assignment

		Jobs				
		1	**2**	**3**	**4**	**5**
	1	40	$\boxed{0}$	470	30	~~0~~
Computer Machines	**2**	470	40	20		0
	3	10	450		$\boxed{0}$	~~10~~
	4	40	50	0	20	480
	5		0	$\boxed{0}$	470	40

Step 5:

The solution obtained in Step 4 is not optimal. Because we were able to make four assignments when five were required.

Step 6:

Cover all the zeros of the table shown in the Step 4 with four lines (since already we made four assignments).

Check row 4 since it has no assignment. Note that row 4 has a zero in column 3, therefore check column3. Then we check row 3 since it has a zero in column 3. Note that no other rows and columns are checked. Now we may draw four lines through unchecked rows (row 1, 2, 3 and 5) and the checked column (column 3). This is shown in the table given below:

Jobs

		1	**2**	**3**	**4**	**5**
	1	40	$\boxed{0}$	470	30	~~0~~
Computer Machines	**2**	470	40	20		0
	3	10	450		$\boxed{0}$	~~10~~
	4	40	50	0	20	480
	5		0	$\boxed{0}$	470	40

Step 7:

Develop the new revised table.

Examine those elements that are not covered by a line in the table given in Step 6. Take the smallest element in this case the smallest element is 10. Subtract this smallest element from the uncovered cells and add 1 to elements (A6, B6, D6 and F6) that lie at the intersection of two lines. Finally, we get the new revised cost table, which is shown below:

Jobs

		1	**2**	**3**	**4**	**5**
	1	40	0	471	30	0
Computer Machines	**2**	470	40	21	0	0
	3	0	440	0	10	0
	4	30	40	0	10	470
	5	0	0	11	470	40

Step 8:

Now, go to Step 4 and repeat the procedure until we arrive at an optimal solution (assignment).

Step 9:

Determine an assignment

Jobs

		1	2	3	4	5
	1	40	$\cancel{0}$	471	30	$\boxed{0}$
Computer Machines	2	470	40	21		0
	3		440	0	$\boxed{}$	$\cancel{0}$
	4	30	40		10	470
	5	$\boxed{0}$		$\cancel{11}$	470	$\cancel{40}$

Since the number of assignments equal to the number of rows (columns), the assignment shown in the above tale is optimal.

Thus, the Machine1 is assigned to Job5, Machine 2 is assigned to job4, Machine3 is assigned to job1, Machine4 is assigned to job3 and Machine5 is assigned to job2.

The minimum assignment cost is: 170

2.5 Maximization in an Assignment Problem

There are situations where certain facilities have to be assigned to a number of jobs so as to maximize the overall performance of the assignment. In such cases the problem can be converted into a minimization problem and can be solved by using Hungarian Method. Here the conversion of maximization problem into a minimization can be done by subtracting all the elements of the cost table from the highest value of that table.

Example 2.5:

Consider the problem of five different machines can do any of the required five jobs with different profits resulting from each assignment as illustrated below:

Machines

1 2 3 4 5

Jobs					
1	40	47	50	38	50
2	50	34	37	31	46
3	50	42	43	40	45
4	35	48	50	46	46
5	38	72	51	51	49

Find out the maximum profit through optimal assignment.

Solution

This is a maximization problem, so that first we have to find out the highest value in the table and subtract all the values from the highest value. In this case the highest value is 72.

The new revised table is given below:

	Machines				
Jobs	1	2	3	4	5
1	32	35	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	34	0	21	21	23

This can be solved by using the Hungarian Method.

By solving this, we obtain the solution is as follows:

Jobs	Machines
1	3
2	5
3	1
4	4
5	2

The maximum profit through this assignment is: 264

2.6 Crew Assignment Problem

The crew assignment problem is explained with the help of the following problem

Problem:

A trip from Chennai to Coimbatore takes six hours by bus. A typical time table of the bus service in both the direction is given in the Table 1. The cost of providing this service by the company based on the time spent by the bus crew i.e. driver and conductor away from their places in addition to service times. The company has five crews. The condition here is that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours for the return trip. Also the company has guest house facilities for the crew of Chennai as well as at Coimbatore.

Find which line of service is connected with which other line so as to reduce the waiting time to the minimum.

Table 1

Departure from Chennai	Route Number	Arrival at Coimbatore	Arrival at Chennai	Route Number	Departure from Coimbatore
06.00	1	12.00	11.30	a	05.30
07.30	2	13.30	15.00	b	09.00
11.30	3	17.30	21.00	c	15.00
19.00	4	01.00	00.30	d	18.30
00.30	5	06.30	06.00	e	00.00

Solution

For each line the service time is constant so that it does not include directly in the computation. Suppose if the entire crew resides at Chennai then the waiting times in hours at Coimbatore for different route connections are given below in Table 2.

If route 1 is combined with route a, the crew after arriving at Coimbatore at 12 Noon start at 5.30 next morning. Thus the waiting time is 17.5 hours. Some of the assignments are infeasible. Route c leaves Coimbatore at 15.00 hours. Thus the crew of route 1 reaching Coimbatore at 12 Noon are unable to take the minimum stipulated rest of four hours if they are asked to leave by route c. Hence 1-c is an infeasible assignment. Thus it cost is M (a large positive number).

Table 2

Route	a	b	c	d	e
1	17.5	21	M	6.5	12
2	16	19.5	M	5	10.5
3	12	15.5	21.5	M	6.5
4	4.5	8	4	17.5	23
5	23	M	8.5	12	17.5

Similarly, if the crews are assumed to reside at Coimbatore then the waiting times of the crew in hours at Chennai for different route combinations are given below in Table 3.

Table 3

Route	a	b	c	d	e
1	18.5	15	9	5.5	M
2	20	16.5	10.5	7	M
3	M	20.5	14.5	11	5.5
4	7.5	M	22	18.5	13
5	13	9.5	M	M	18.5

Suppose, if the crew can be instructed to reside either at Chennai or at Coimbatore, minimum waiting time from the above operation can be computed for different route combination by choosing the minimum of the two waiting times (shown in the Table 2 and Table 3). This is given in the following Table 4.

Table 4

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

Note: The asterisk marked waiting times denotes that the crew are based at Chennai; otherwise they are based at Coimbatore.

Now we can solve the assignment problem (presented in Table 4) using Hungarian Method.

Step 1: Cost Table (Table 5)

Table 5

Route	a	b	c	d	e
1	17.5*	15	9	5.5	12*
2	16*	16.5	10.5	5*	10.5*
3	12*	15.5*	14.5	11	5.5
4	4.5*	8*	14*	17.5*	13
5	13	9.5	8.5*	12*	17.5*

Step 2: Find the First Reduced cost table (Table 6)

Table 6

Route	a	b	c	d	e
1	12	9.5	3.5	0	6.5
2	11	11.5	5.5	0	5.5
3	6.5	10	9	5.5	0
4	0	3.5	9.5	13	8.5
5	4.5	1	0	3.5	9

Step 3: Find the Second Reduced cost table (Table 7)

Table 7

Route	a	b	c	d	e
1	12	8.5	3.5	0	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	5.5	0
4	0	2.5	9.5	13	8.5
5	4.5	0	0	3.5	9

Step 4: Determine an Assignment (Table 8)

Table 8

Route	a	b	c	d	e
1	12	8.5	3.5	$\boxed{0}$	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	~~5.5~~	
4		2.5	9.5	13	8.5
5	4.5		0	3.5	$\boxed{0}$

Step 5: The solution obtained in Step 4 is not optimal since the number of assignments are less than the number of rows (columns).

Step 6: Check (√) row 2 sinc $\boxed{0}$ has no assignment and note that row 2 has a zero in column d, therefore check (√) column d also. Then check row 1 since it has zero in column d. Draw the lines through the unchecked rows and checked column using 4 lines (only 4 assignments are made). This is shown in Table 9.

Table 9

Route	a	$\boxed{0}$	c	d	e
1	12	8.5	3.5	$\boxed{0}$	6.5
2	11	10.5	5.5	0	5.5
3	6.5	9	9	~~5.5~~	
4		2.5	9.5	13	8.5
5	4.5		0	3.5	$\boxed{0}$

$\boxed{0}$

Step 7: Develop a new revised table (Table 10)

Take the smallest element from the elements not covered by the lines in this case 3.5 is the smallest element. Subtract all the uncovered elements from 3.5 and add 3.5 to the elements lie at the intersection of two lines (cells 3d, 4d and 5d). The new revised table is presented in Table 10.

Table 10

Route	a	b	c	d	e
1	8.5	5	0	0	3
2	7.5	7	2	0	2
3	6.5	9	9	9	0
4	0	2.5	9.5	16.5	8.5
5	4.5	0	0	7	9

Step 8: Go to Step 4 and repeat the procedure until an optimal solution if arrived.

Step 9: Determine an Assignment (Table 11)

Table 11

Route	a	b	c	d	e

1	8.5	5	$\boxed{0}$	$\cancel{0}$	3
2	7.5	7	2		2
3	6.5	9	9	$\boxed{0}$	
4		2.5	9.5	16.5	8.5
5	4.5		0	7	$\boxed{0}$

The assignment illustrated in the above Table 11 is **optimal** since the **number of assignments is equal to the number of rows (columns).**

Thus, the routes to be prepared to achieve the minimum waiting time are as follows:

$\boxed{0}$ - c, 2 – d, 3 – e, 4 – a and 5 – b

By referring Table 5, we can obtain the waiting times of these assignments as well as the residence (guest house) of the crews. This is presented in the following Table 12.

Table 12

Routes	Residence of the Crew	Waiting Time
1 – c	Coimbatore	9
2 – d	Chennai	5
3 – e	Coimbatore	5.5
4 – a	Chennai	4.5
5 - b	Coimbatore	9.5

2.7 Summary

The assignment problem is used for the allocation of a number of persons to a number of jobs so that the total time of completion is minimized. The assignment problem is said to be **balanced** if it has equal number of person and jobs to be assigned. If the number of persons (jobs) is different from the number of jobs (persons) then the problem is said to be **unbalanced**. An unbalanced assignment problem can be solved by converting into a balanced assignment problem. The conversion is done by introducing dummy person or a dummy job with zero cost.

Because of the special structure of the assignment problem, it is solved by using a special method known as **Hungarian Method**.

2.8 Key Terms

Cost Table: The completion time or cost corresponding to every assignment is written down in a table form if referred as a cost table.

Hungarian Method: is a technique of solving assignment problems.

Assignment Problem: is a special kind of linear programming problem where the objective is to minimize the assignment cost or time.

Balanced Assignment Problem: is an assignment problem where the number of persons equal to the number of jobs.

Unbalanced Assignment Problem: is an assignment problem where the number of jobs is not equal to the number of persons.

Infeasible Assignment Problem: is an assignment problem where a particular person is unable to perform a particular job or certain job cannot be done by certain machines.

2.9 Self Assessment Questions

Q1. A tourist company owns a one car in each of the five locations viz. L1, L2, L3, L4, L5 and a passengers in each of the five cities C1, C2, C3, C4, C5 respectively. The following table shows the distant between the locations and cities in kilometer. How should be cars be assigned to the passengers so as to minimize the total distance covered.

		Cities				
		C1	**C 2**	**C3**	**C4**	**C5**
Locations	**L1**	120	110	115	30	36
	L2	125	100	95	30	16
	L3	155	90	135	60	50
	L4	160	140	150	60	60
	L5	190	155	165	90	85

Q2. Solve the following assignment problem

1 2 3 4 5

1	Rs.3	Rs.8	Rs.2	Rs.10	Rs.3
2	Rs.8	Rs.7	Rs.2	Rs.9	Rs.7
3	Rs.6	Rs.4	Rs.2	Rs.7	Rs.5
4	Rs.8	Rs.4	Rs.2	Rs.3	Rs.5
5	Rs.9	Rs.10	Rs.6	Rs.9	Rs.10

Q3. Work out the various steps of the solution of the Example 2.3.

Q4. A steel company has five jobs to be done and has five softening machines to do them. The cost of softening each job on any machine is given in the following cost matrix. The assignment of jobs to machines must be done on a one to one basis. Here is the objective is to assign the jobs to the machines so as to minimize the total assignment cost without violating the restrictions.

		Jobs				
		1	**2**	**3**	**4**	**5**
Softening Machines	**1**	80	30	X	70	30
	2	70	X	60	40	30
	3	X	80	60	80	70
	4	70	80	30	50	X
	5	30	30	50	X	80

Q5. Work out the various steps of the solution of the problem presented in Example 2.5.

Q6. A marketing manager wants to assign salesman to four cities. He has four salesmen of varying experience. The possible profit for each salesman in each city is given in the following table. Find out an assignment which maximizes the profit.

		Cities 1	2	3	4
	1	25	27	28	38
Salesmen	2	28	34	29	40
	3	35	24	32	33
	4	24	32	25	28

Q7. Shiva's three wife, Rani, Brinda, and Fathima want to earn some money to take care of personal expenses during a school trip to the local beach. Mr. Shiva has chosen three chores for his wife: washing, cooking, sweeping the cars. Mr. Shiva asked them to submit bids for what they feel was a fair pay for each of the three chores. The three wife of Shiva accept his decision. The following table summarizes the bid received.

		Chores Washing 1	Cooking 2	Sweeping 3
	Rani	25	18	17
Wife's	Brinda	17	25	15
	Fathima	18	22	32

Q8. Solve the following problem

Office

O1 O2 O3 O4

	E1	2600	3200	3400	3000
Employees	**E2**	2000	4200	3600	2600
	E3	2000	3000	5600	4000

Q9. The railway operates seven days a week has a time table shown in the following table. Crews (Driver and Guard) must have minimum rest of six hours between trans. Prepare the combination of trains that minimizes waiting time away from the city. Note that for any given combination the crew will be based at the city that results in the smaller waiting time and also find out for each combination the city where the crew should be based at.

Train No.	Departure at Bangalore	Arrival at Chennai	Train No.	Departure at Chennai	Arrival at Bangalore
101	7 AM	9 AM	201	9 AM	11 AM
102	9 AM	11 AM	202	10 AM	12 Noon
103	1.30 PM	3.30 PM	203	3.30 PM	5.30 PM
104	7.30 PM	9.30 PM	204	8 PM	10 PM

2.10 Key Solutions

Q1. L1 – C1, L2 – C3, L3 – C2, L4 _ C4, L5 – C5 and
Minimum Distance is: 450

Q2. 1 – 5, 2 – 3, 3 – 2, 4 – 4, 5 – 1 and
Minimum Cost is: Rs.21

Q4. 1 – 2, 2 – 4, 3 – 3, 4 – 4, 5 – 1 and
Minimum Assignment Cost is:

Q6. 1 - 1, 2 – 4, 3 – 3, 4 – 2 and
Maximum Profit is: 139

Q7. Rani – Cooking, Brinda – Sweeping, Fathima – Washing and
Minimum Bids Rate is: 51

Q8. E1 – O2, E2 – O4, E3 – O1
Since E4 is empty, Office O3 cannot be assigned to any one.
Minimum Cost is: 7800

Q9.

Trains	Cities
201 – 103	Bangalore
202 – 104	Chennai

203 – 101	Bangalore
204 – 102	Bangalore

2.11 Further References

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