DISSERTATION

(Mathematical model

of

a

food chain cycle)

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Abstract:

Fishery management is the consideration of the ecological effects of harvesting. Fisherman work to provide fish for a growing human population but because of this some fish populations have been dangerously declining. It is important to balance ecological and economic needs.

In this paper, we take into consideration a logistic growth model with allee and harvesting, and to find the equilibrium points and stability of such a model. We also glance upon the maximum sustainable yield of the system and to minimise effort.

Introduction

Ecological systems (natural systems, prey-predator communities etc.) are dynamic, complex and non-linear in nature. In non-linear systems, the effect is not proportional to the cause. This non-linearity in the presence of dissipation (frictional forces in physics and non-trophic interactions in ecology) gives rise to different attractor types as representations of asymptotic states of these dynamical systems. The attractors which are frequently observed in the model ecological systems are: stable foci, stable limit cycles and chaos [1]. These are obtained in the state space (or phase space) of an ecological system when an intrinsic attribute (control parameter) of the system is varied; e.g., the quantity and quality of food supplied to a chemostat. If the first two are most frequently observed in a model ecological system, the system's dynamics is classified to be regular.

On the other hand, if chaotic attractors are detected in a dense region in two-dimensional parameter space, then the system is termed to be chaotic. Chaotic states are characterized by exponential sensitivity of system's dynamics with respect to initial conditions which can not be fixed or determined without committing a finite amount of error either in the laboratory or in the field.

Interaction networks in natural ecosystems consist of simple units known as food chains. Food chains and webs in the environment are very important systems in many different fields such as ecological science, applied mathematics, economic and engineering science. Food chains and webs can be modeled by systems of differential equations which approximate species or functional feeding group behavior with different functional responses.

The subjects of chaos and chaos control are growing rapidly in many different fields such biological systems, structural engineering, ecological models, aerospace science, and economics. Food chain modeling provides challenges in the fields of both theoretical ecology and applied mathematics.

Modeling efforts of the dynamics of food chains initiated long ago confirm that food chains have a very rich dynamics. In the first place, Lotka and Volterra independently developed a simple model of interacting species that still bears their joint names which can be stated as

$$\frac{dx}{dt} = a_1 x - b_1 xy$$

$$\frac{dy}{dt} = -a_2 y + b_2 xy$$

with all parameters positive, a1, a2, b1, b2 > 0.

Here, x is the prey population and y is the predator population, the predator y preys on x. The parameters a1 is the prey growth rate in the absence of the predators, b1 is the capture rate of prey by per predator, b2 is the rate at which each predator converts captured prey into predator births and a2 is the constant rate at which death in the absence of prey. They showed that ditrophic food chains (i.e. prey-predator systems) permanently oscillate for any initial condition if the prey growth rate is constant and the predator functional response is linear.

A functional response in ecology is the intake rate of a consumer as a function of food density (the amount of food available in a given ecotope). It is associated with the numerical response, which is the reproduction rate of a consumer as a function of food density.

Based on experiments, Holling suggested three different kinds of functional responses for different kinds of species to model the phenomena of predator, which resulted the standard Lotka-Volterra system more realistic. Biologically, it is quite natural for the existence and asymptotical stability of equilibria and limit cycles for autonomous predator-prey systems with these functional responses.

Type II functional response is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food. Type II functional response is often modeled by a rectangular hyperbola, for instance as by Holling's disc equation,[2] which assumes that processing of food and searching for food are mutually exclusive behaviors.

In an example with wolves and caribou, as the number of caribou increases while holding wolves constant, the number of caribou kills increases and then levels off. This is because the proportion of caribou killed per wolf decreases as caribou density increases. The higher the density of caribou, the smaller the proportion of caribou killed per wolf. Explained slightly differently, at very high caribou densities, wolves need very little time to find prey and spend almost all their time handling prey and very little time searching. Wolves are then satiated and the total number of caribou kills reaches a plateau. Type III functional response is similar to type II in that at high levels of prey density, saturation occurs. But now, at low prey density levels, the graphical relationship of number of prey consumed and the density of the prey population is a more than linearly increasing function of prey consumed by predators. This accelerating function is largely descriptive, and often justified by learning time, prey switching, or a combination of both phenomena, but the type III functional response lacks the rigorous theoretical underpinning of the type II functional response.

Learning time is defined as the natural improvement of a predator's searching and attacking efficiency or the natural improvement in their handling efficiency as prey density increases. Imagine a prey density so small that the chance of a predator encountering that prey is extremely low. Because the predator finds prey so infrequently, it has not had enough experience to develop the best ways to capture and subdue that species of prey. Holling identified this mechanism in shrews and deer mice feeding on sawflies. At low numbers of sawfly cocoons per acre, deer mice especially experienced exponential growth in terms of the number of cocoons consumed per individual as the density of cocoons increased. The characteristic saturation point of the type III functional response was also observed in the deer mice. At a certain density of cocoons per acre, the consumption rate of the deer mice reached a saturation amount as the cocoon density continued to increase.

The Allee effect is a phenomenon in biology characterized by a correlation between population size or

density and the mean individual fitness (often measured as per capita population growth rate) of a population or species.

The **component Allee effect** is the positive relationship between any measurable component of individual fitness and population density. The **demographic Allee effect** is the positive relationship between the overall individual fitness and population density.

Allee effects are classified by the nature of density dependence at low densities. If the population shrinks for low densities, there is a strong Allee effect. If the proliferation rate is positive and increasing then there is a weak Allee effect. The null hypothesis is that proliferation rates are positive but decreasing at low densities.

CONSTRUCTING THE MATHEMATICAL MODEL:

We consider the following mathematical model for our purposes of a prey predator system with allee and harvesting:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)(x - \theta) - \frac{mx^2y}{a + x^2} - q_1ex$$

$$\frac{dy}{dt} = \beta\frac{mx^2y}{a + x^2} - dy - q_2ey$$

Here, where x and y denote the prey and predator population densities at time t, respectively, and d is the death rate of the predators, β is the conversion rate of prey biomass into the predator biomass, m is the specific predation rate, θ is the threshold of survivance, where q_1 and q_2 is the catchability coefficient of the prey and predator species and e denotes the effort devoted to the harvesting.

Let,

$$P = rx\left(1 - \frac{x}{k}\right)(x - \theta) - \frac{mx^2y}{a + x^2} - q_1ex$$

$$Q = \beta \frac{mx^2 y}{a+x^2} - dy - q_2 ey$$

Equating the RHS of the system to zero, we get the equilibrium points as $(0,0)$, $(\theta,0)$, $(k,0)$, (x^*,y^*).

Then differentiating partially w.r.to x and y, we get,

$$P_x = \left\{ r\left[\left(1 - \frac{x}{k}\right) - \left(\frac{x-\theta}{k}\right) \right] - \frac{my(a+x^2)-2mx^2 y}{(a+x^2)^2} \right\} x$$

$$P_y = -\frac{mx^2}{a+x^2}$$

$$Q_x = \frac{2m\beta y x^3}{(a+x^2)^2}$$

$$Q_y = -d - q_2 e + \frac{m\beta x^2}{a+x^2}$$

The Jacobian of the above system is given by,

$$J = \begin{matrix} P_x & P_y \\ Q_x & Q_y \end{matrix}$$

To find the roots λ_1, λ_2 we consider the characteristic equation: $|J-\lambda|=0$, i.e.,

$$\lambda^2 - (P_x + Q_y)\lambda - P_y Q_x + P_x Q_y = 0$$

For stability, we know that

$$\lambda_1 < 0$$

$$\lambda_2 < 0$$

Hence, we must have,

$\lambda_1 + \lambda_2 < 0$, and

$$\lambda_1 \lambda_2 > 0$$

By properties of roots of quadratic equations, we know,

$$\lambda_1 + \lambda_2 < 0:$$

$$\left\{ r\left[\left(1-\frac{2x}{k}\right)+\left(\frac{\theta}{k}\right)\right]-\frac{mya-mx^2y}{(a+x^2)^2}\right\}<0$$

and,

$$\lambda_1\lambda_2>0:$$

$$\frac{2am^2\beta yx^3}{(a+x^2)^3}>0$$

Equilibrium points	Condition for existence	Stability
E_0	Always	Unstable
E_θ	Always	Unstable
E_k	Always	Unstable
$E(x^*,y^*)$	(i)	Untable
	(ii)	

MAX SUSTAINABLE YIELD:

The equation for maximum sustainable yield is given by: $Y_1 = q_1ex^* + q_2ey^*$

$$\frac{dy_1}{de}=0 \qquad\qquad ; \qquad\qquad \frac{d^2y_1}{de^2}<0$$

We use the values of x^* and y^* using P=0 and then substituting the value in Q=0.

Then differentiating w.r.to e, we have,

$$\frac{dy_1}{de}=q_1ad\left[\frac{m\beta-d}{(m\beta-d-q_2e)^2}\right]+q_1q_2\left[\frac{m\beta-d-q_2e^2}{(m\beta-d-q_2e)^2}\right]-2q_1q_2e\left(\frac{a+x^2}{mx}\right)+q_2r\left(1-\frac{x}{k}\right)(x-\theta)\left(\frac{a+x^2}{mx}\right)$$

Differentiating again,

$$\frac{d^2y_1}{de^2}=q_1ad\left[\frac{-2(m\beta-d)}{(m\beta-d-q_2e)^3}\right]+$$

$$q_1 q_2 \left[\frac{(m\beta - d - q_2 e)^2 (-2q_2 e) - 2(m\beta - d - q_2 e^2)(m\beta - d - q_2 e)(-q_2)}{(m\beta - d - q_2 e)^4} \right] - 2q_1 q_2 \left(\frac{a + x^2}{mx} \right) < 0$$

$$\frac{dy_1}{de} = q_1 a d \left[\frac{m\beta - d}{(m\beta - d - q_2 e)^2} \right] + q_1 q_2 \left[\frac{m\beta - d - q_2 e^2}{(m\beta - d - q_2 e)^2} \right] - 2q_1 q_2 e \left(\frac{a + x^2}{mx} \right)$$

$$+ q_2 r \left(1 - \frac{x}{k} \right) (x - \theta) \left(\frac{a + x^2}{mx} \right) = 0$$

Therefore we have the following equation:

$$e^3 \left(q_2^2 (a + x^2) \right)$$

$$+ e^2 \left[4 q_1 q_2^2 (a + x^2)(m\beta - d) - mx q_1 q_2 + q_2^3 r \left(1 - \frac{x}{k} \right) (x - \theta)(a + x^2) \right]$$

$$- e \left[2 q_2^2 r \left(1 - \frac{x}{k} \right) (x - \theta)(a + x^2)(m\beta - d) + 2 q_1 q_2 (a + x^2)(m\beta - d)^2 \right]$$

$$+ \left[(m\beta - d)(mx q_1 q_2 + mx q_1 a d) + q_2 r \left(1 - \frac{x}{k} \right) (x - \theta)(a + x^2)(m\beta - d)^2 \right] = 0$$

We solve the above equation numerically for any given observation to find the required effort.

CONCLUSION:

MSY exists iff x*>θ, i.e., initial population density is greater than the allee effect.