CS374: Practice Sheet 2
Deadline for Submission: 25 Sep. 2020

Prob 1) Evaluate $\sqrt{5}$ using the equation $x^2 - 5 = 0$ by applying the fixed-point iteration. Perform at least 4 iterations.

Prob 2) Consider the equation $x^3 - 5x + 1 = 0$, which has a root in the interval $(0, 1)$. We can write the equation in the form $\phi(x)$ and the corresponding iteration method in the following ways:

(i) $x = \frac{1}{5}(x^3 + 1)$ and $x_{k+1} = \frac{1}{5}(x_k^3 + 1)$.

(ii) $x = (5x - 1)^{1/3}$ and $x_{k+1} = (5x_k - 1)^{1/3}$.

(iii) $x = x^3 - 4x + 1$ and $x_{k+1} = x_k^3 - 4x_k + 1$.

Discuss the convergence of (i), (ii) and (iii).

Prob 3) The equation $x^2 + ax + b = 0$ has two real roots α and β. Show (analytically) that the method

(i) $x_{k+1} = -\frac{1}{x_k}(ax_k + b)$ converges to α if $\mid \alpha \mid > \mid \beta \mid$.

(ii) $x_{k+1} = -\frac{b}{x_k + a}$ converges to α if $\mid \alpha \mid < \mid \beta \mid$.

(iii) $x_{k+1} = -\frac{1}{a}(x_k^2 + a)$ converges to α if $2 \mid \alpha \mid < \mid \alpha + \beta \mid$.

Prob 4) An iteration method is defined by

$$x_{n+1} = \frac{x_n}{2a}(3a - x_n^2), \;\; a > 0, n = 0, 1, 2, ...$$

Find the quantity to which the method converges. Hence, determine the rate of convergence of the method. Also, obtain the asymptotic error constant.

Prob 5) Let $\phi : [a, b] \to [a, b]$ be differentiable on (a, b) and there exists a constant $L \in (0, 1)$ such that

$$\mid \phi'(x) \mid \leq L, \;\text{ for all } x \in (a, b).$$

Let r be a unique fixed point of ϕ in $[a, b]$. Show that, for any $x_0 \in [a, b]$, the sequence generated by $x_{k+1} = \phi(x_k), k \geq 0$ satisfies

$$\mid x_{n+1} - r \mid \leq \frac{L}{1 - L} \mid x_{n+1} - x_n \mid .$$

Prob 6) Let $f : [a, b] \to \mathbb{R}$ be twice continuously differentiable and r is a **simple zero** of f. Then, there exists a neighborhood of r and a constant C such that if Newton-Raphson method is started in that neighborhood, the successive points become steadily closer to r and satisfy

$$\mid x_{n+1} - r \mid \leq C \mid x_n - r \mid^2, \;\; (n \geq 0).$$

Prob 7) If r is a double zero of the function f and $f : [a, b] \to \mathbb{R}$ is twice continuously differentiable, then in Newton's method we shall have $\phi'(r) = 1/2 \neq 0$, where $\phi(x) = x - \frac{f(x)}{f'(x)}$. Determine the value of α such that the $\phi'(r) = 0$ by re-defining ϕ as

$$\phi(x) = x - \alpha \frac{f(x)}{f'(x)}.$$

Prob 8) Show that the formula for the Secant method can be written in the following mathematical form

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, (n \geq 1).$$

This formula is called **Regula-Falsi method**, if you choose x_{n-1} and x_n such that $f(x_n)f(x_{n-1}) < 0$ at each step.

Prob 9) Discuss the rate of convergence for the Secant and Regula-Falsi method.

Prob 10) Solve the following problems using Secant and Regula-Falsi method (at least 3 iterations)

(i) $x^3 - \sinh x + 4x^2 + 6x + 9 = 0$.

(ii) $x^5 + x^3 + 3 = 0$ with $x_0 = -1$ and $x_1 = 1$.

Prob 11) Consider an iteration function of the form

$$F(x) = x + f(x)g(x),$$

where $f(r) = 0$ and $f'(r) \neq 0$. Find the precise conditions on the function g so that the method of functional iteration $x_{n+1} = F(x_n)$ will converge cubically to r if started near r.

Prob 12) Let F be continuously differentiable in an open interval, and suppose that F has a fixed point s in this open interval. Prove that if $\mid F'(s) \mid < 1$, then the sequence defined by $x_{n+1} = F(x_n)$ will converge to s if started sufficiently close to s.

Lab Exercises

Ex 1) Write codes for solving the problems 2,3 and 4 and interpret the results graphically.

Ex 2) Write a code for the method proposed in Problem 7 and use it to get an approximate root of the equation $x^3 - 3x^2 + 4 = 0$ near $x = 2$.

Ex 3) Write codes for solving Problem 10 by using Secant and Regula-Falsi methods.