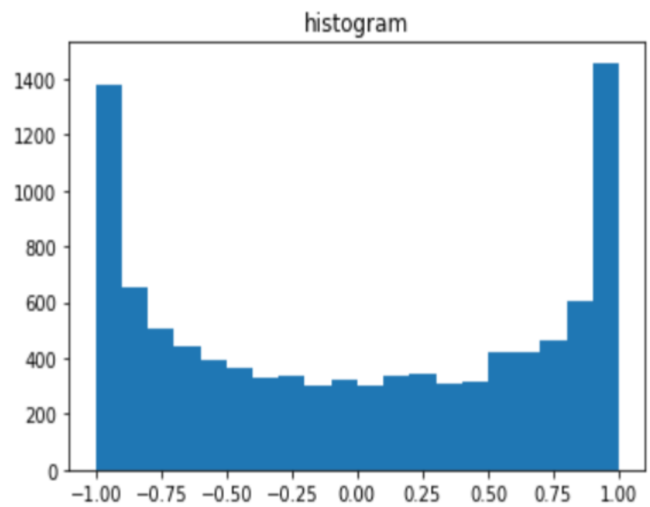
1. For any process to be classified as a Stationary process in a Wide-Sense it needs to satisfy two primary conditions :

   - The mean for every Random Variable should remain constant throughout.
   - $R_x(t1, t2) = R_x(t1 - t2), \quad \nabla t1, t2 \in I$

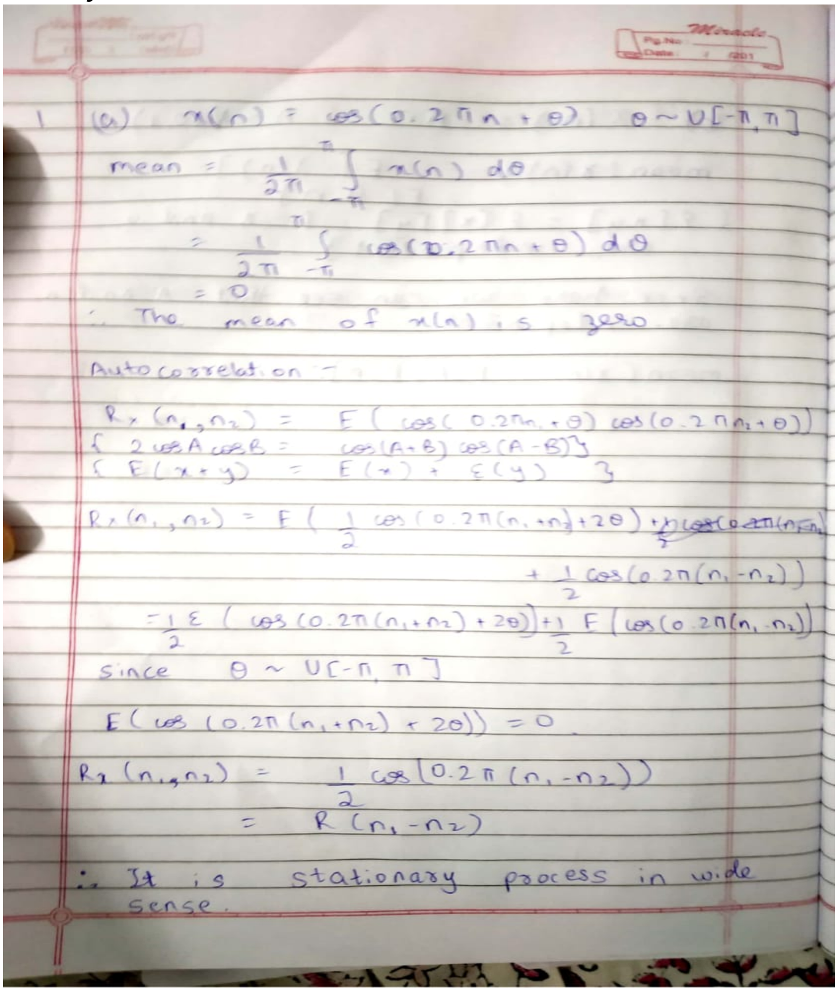
   (a) $X(n) = cos(0.2\pi n + \theta)$

      i. Observations

      - On declaring the number of iterations as 10000 we find that the mean of every Random Variable almost tends to 0. On increasing the Number of iterations we obtain an even more precise result.
      - On analyzing the auto-correlation matrix we observe that the values with equal time difference tend towards the same value clearly verifying that the auto-correlation depends only on the time difference.
      - To obtain a better sense of visualization we modified the auto-correlation matrix such that each row now contains values with the same time difference.In the traditional matrix we needed to check values in a diagonal sense.This modification aims to simplify the process.



ii. Analytical



1 (a) $x(n) = \cos(0.2\pi n + \theta) \quad \theta \sim U[-\pi, \pi]$

$$\text{mean} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(n)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(0.2\pi n + \theta)\, d\theta$$

$$= 0$$

$\therefore$ The mean of $x(n)$ is zero

Auto correlation :-

$$R_x(n_1, n_2) = E\left(\cos(0.2\pi n_1 + \theta)\cos(0.2\pi n_2 + \theta)\right)$$

$\{2\cos A \cos B = \cos(A+B)\cos(A-B)\}$

$\{E(x+y) = E(x) + E(y)\}$

$$R_x(n_1, n_2) = E\left(\frac{1}{2}\cos(0.2\pi(n_1+n_2) + 2\theta) + \frac{1}{2}\cos(0.2\pi(n_1-n_2))\right)$$

$$= \frac{1}{2}E\left(\cos(0.2\pi(n_1+n_2) + 2\theta)\right) + \frac{1}{2}E\left(\cos(0.2\pi(n_1-n_2))\right)$$

Since $\theta \sim U[-\pi, \pi]$

$$E\left(\cos(0.2\pi(n_1+n_2) + 2\theta)\right) = 0$$

$$R_x(n_1, n_2) = \frac{1}{2}\cos(0.2\pi(n_1-n_2))$$

$$= R(n_1 - n_2)$$

$\therefore$ It is stationary process in wide sense.

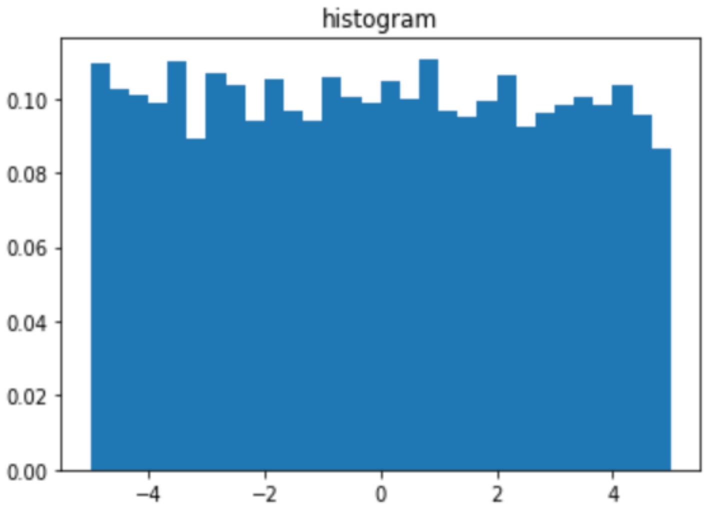
iii. Conclusion

- Since it satisfies all the conditions, hence it can be classified as a stationary process in a wide sense.

(b) $X(n) = Acos(0.25\pi n)$

i. Observations

- On declaring the number of iterations as 10000 we find that the mean of every Random Variable almost tends to 0. On increasing the Number of iterations we obtain an even more precise result.
- On analyzing the auto-correlation matrix we observe that values in with equal time difference are not equal.The modified auto-correlation matrix has been used in that context.



ii. Analytical

(b) $x(n) = A\cos(0.25\pi n)$ $\quad A \sim U[-5,5]$

mean :-

$$\text{mean}(x(n)) = \frac{1}{5-(-5)} \int_{-5}^{5} A\cos(0.25\pi n)\, dA$$

(here A is variable)
(we have used the formula of expected value in a uniform random variable)

$$= \frac{1}{10}\cos(0.25\pi n) \int_{-5}^{5} A\, dA$$

$$= \frac{1}{10}\cos(0.25\pi n) \left[\frac{A^2}{2}\right]_{-5}^{5}$$

$$= \frac{1}{10}\cos(0.25\pi n)\left[\frac{25}{2} - \frac{25}{2}\right]$$

$$= 0$$

Autocorrelation :-

$$R_{xx}(n_1, n_2) = E[A\cos(0.25\pi n_1)\, A\cos(0.25\pi n_2)]$$

$$= E[A^2 \cos(0.25\pi n_1)\cos(0.25\pi n_2)]$$

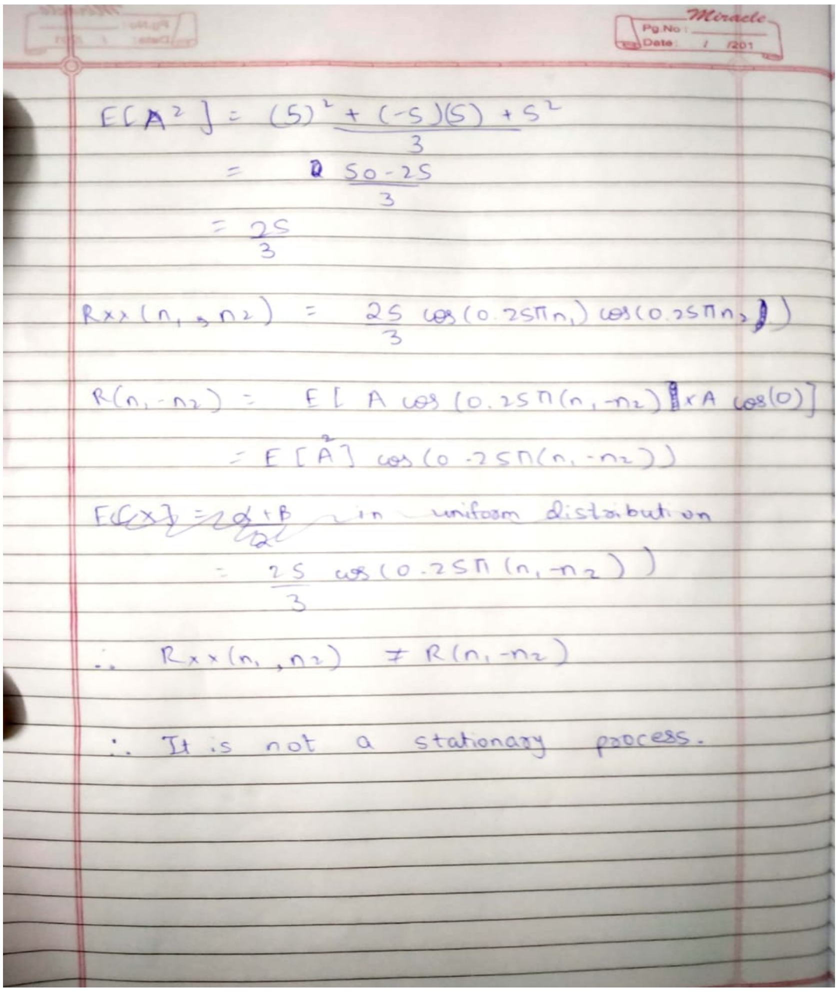
n_1 and n_2 are not random here, they are deterministic.

$$R_{xx}(n_1, n_2) = E[A^2]\cos(0.25\pi n_1)\cos(0.25\pi n_2)$$

In uniform random ~~var~~ distribution

$$E[x^2] = \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

where α, β are the ~~[illegible]~~ $\alpha = -5$, $\beta = 5$



$$E[A^2] = \frac{(5)^2 + (-5)(5) + 5^2}{3}$$

$$= \frac{50-25}{3}$$

$$= \frac{25}{3}$$

$$R_{xx}(n_1, n_2) = \frac{25}{3} \cos(0.25\pi n_1) \cos(0.25\pi n_2)$$

$$R(n_1 - n_2) = E[A \cos(0.25\pi(n_1 - n_2)) \times A \cos(0)]$$

$$= E[A^2] \cos(0.25\pi(n_1 - n_2))$$

~~$E[X] = \frac{\alpha+\beta}{2}$ in uniform distribution~~

$$= \frac{25}{3} \cos(0.25\pi(n_1 - n_2))$$

$$\therefore R_{xx}(n_1, n_2) \neq R(n_1 - n_2)$$

$\therefore$ It is not a stationary process.

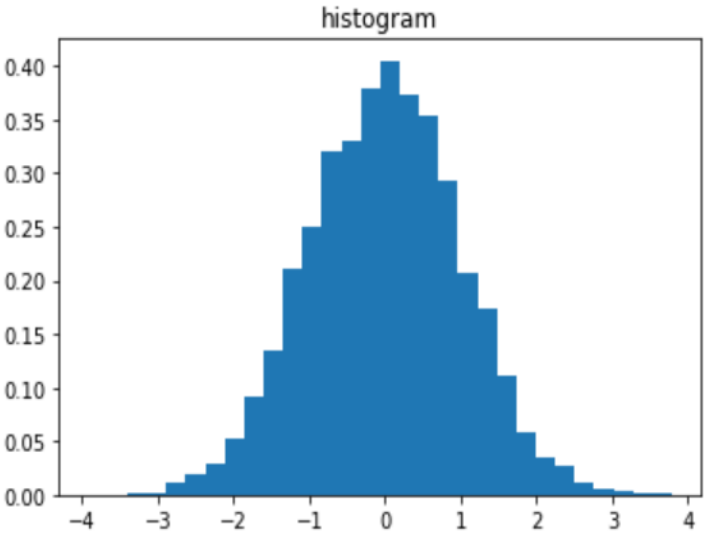
iii. Conclusion

- Although it does satisfy the mean conditions but the auto-correlation values vary a lot, hence it can't be classified as a stationary process in a wide sense.

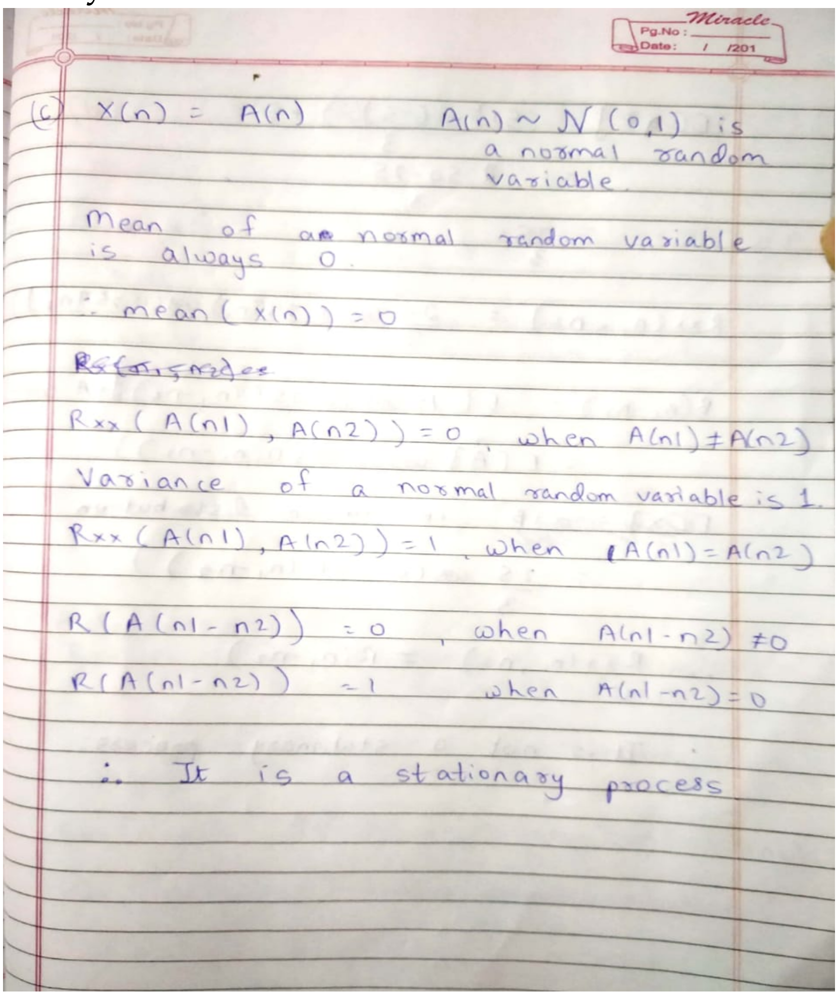
(c) $X(n) = A(n) \quad ; \quad A(n) = \mathcal{N}(0, 1)$

i. Observations

- On declaring the number of iterations as 10000 we find that the mean of every Random Variable almost tends to 0. On increasing the Number of iterations we obtain an even more precise result.
- On analyzing the auto-correlation matrix we observe that the values in with equal time difference tend towards the same value.The modified auto-correlation matrix has been used in that context.



ii. Analytical



(c) $X(n) = A(n)$ $\quad A(n) \sim \mathcal{N}(0,1)$ is a normal random variable.

Mean of a normal random variable is always 0.

$\therefore$ mean $(X(n)) = 0$

$R_{xx}(A(n1), A(n2)) = 0$, when $A(n1) \neq A(n2)$

Variance of a normal random variable is 1.

$R_{xx}(A(n1), A(n2)) = 1$, when $A(n1) = A(n2)$

$R(A(n1-n2)) = 0$, when $A(n1-n2) \neq 0$

$R(A(n1-n2)) = 1$ when $A(n1-n2) = 0$

$\therefore$ It is a stationary process

iii. Conclusion

- Since it satisfies all the conditions, hence it can be classified as a stationary process in a wide sense.

2. (a) Observations

    - Month wise mean was calculated for all the years starting from 1901 to 2001 with respect to the given data. The mean values did not come out to be constant and had huge variations and were in the range (0 , 273).
    - Covariance with respect to months was calculated and we observed that the values in same time difference didn't tend to same value.

   (b) Conclusion

    - The data given to us did not satisfy any condition for classifying it into stationary wide-sense process (as the mean of all the Random Variables was not constant and the covariance also is a function of other attributes and not just the time difference).
    - As this stochastic process is not wide sense stationary so it can not be strict-sense stationary.

3. (a) i. Analytical Solution

① $R = E[xx^T]$

$$= \begin{bmatrix} E(x_{t_1}^2) & E(x_{t_1}x_{t_2}) & \cdots & E(x_{t_1}x_{t_n}) \\ E(x_{t_2}x_{t_1}) & E(x_{t_2}^2) & \cdots & E(x_{t_2}x_{t_n}) \\ \vdots & & & \\ E(x_{t_n}x_{t_1}) & E(x_{t_n}x_{t_2}) & \cdots & E(x_{t_n}^2) \end{bmatrix}$$

[mind that $x_{t_1} = x_{t1}$]

⇒ observe one thing value at (i,j) and (j,i) is same

So $E(xx^T) = (E(xx^T))^T$

⇒ From SPECTRAL THEOREM,

⇒ If a matrix is self-adjoint then we can find a basis consisting of its eigen-vectors such that, the matrix can be represented diagonally in that basis and also all eigen-values will be positive

⇒ For now, let $e_1, e_2 \ldots e_n$ be our eigen vectors and therefore $[e_1\ e_2\ e_3 \ldots e_n]_{n\times n \text{ matrix}}$

and $\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \\ \vdots & \vdots & 0 & \ddots \\ \vdots & \vdots & \vdots & & \lambda_n \end{bmatrix}$ be representation of that matrix in our new basis

~~Step~~ None let y be any vector,

we can represent y as, $y = a_1e_1 + a_2e_2 + a_3e_3 + \dots a_ne_n$

or $y = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ ← mind that basis is $[e_1 e_2 \dots e_n]$ not standard

$\Rightarrow$ Now $[a_1\ a_2 \dots a_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \\ \vdots & \vdots & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$= [a_1\ a_2 \dots a_n] \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \vdots \\ \lambda_n a_n \end{bmatrix}$

$= \lambda_1 a_1^2 + \lambda_2 a_2^2 + \dots \lambda_n a_n^2 \geq 0$

(Remember $\lambda_1, \lambda_2 \dots \lambda_n > 0$ according to SPECTRAL THEOREM)

let $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ (calculate mean)

Now $C = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$

Note that x is a $(n \times 1)$ vector (1 instance)

$\therefore$ expression is $y^T C y = y^T \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T \right) y$

$= \frac{1}{n} \sum_{i=1}^{n} y^T (x_i - \bar{x})(x_i - \bar{x})^T y$

$= \frac{1}{n} \sum_{i=1}^{n} ((x_i - \bar{x})^T y)^2 \geq 0$ [Basically sum of squares]

$\therefore$ C is always ~~semi~~ positive semi definite.

Hence proved.

ii. Computational Verification

- Two cases were taken.In the first case the matrices were taken from the second question. In the second one, matrices were taken from the 1st question.
- Two step verification was done. A brute force method with 1000 iterations involving random Y matrices were done in addition to finding the eigen values of the matrix A
- Both of them supported our claim with respect to the analytical derivation. We obtained a value >=0 via the brute force

method for every case. Furthemore, our eigen value matrix also consisted of only non-negative values which establishes our claims.

(b) We observe that autocorrelation/autocovariance matrix of a wide-sense stationary process is a self-adjoint (symmetric when all the values are real) matrix i.e., the matrix is same as its conjugate transpose(only transpose in case of real values).