Linear Equations Problems

Linear Equations

Example 1

$$\frac{dy}{dx} + 3x^2 y = 6x^2$$

$$I(x) = e^{\int P(x)dx} = e^{\int 3x^2 dx} = e^{x^3}$$

$$e^{x^3}\frac{dy}{dx} + e^{x^3}(3x^2 y) = e^{x^3}(6x^2)$$

$$(e^{x^3}y)' = e^{x^3}(6x^2)$$

$$\int \frac{d}{dx}(e^{x^3}y)\,dx = \int e^{x^3}(6x^2)\,dx$$

$$e^{x^3}y = 2e^{x^3} + c$$

$$y = 2 + ce^{-x^3}$$

let $u = x^3$

$$du = 3x^2 dx$$

$$2du = 6x^2 dx$$

$$\int e^{x^3}(6x^2)\,dx = \int 2e^{u}\,du$$

$$= 2e^{u} + c$$

$$= 2e^{x^3} + c$$

Example 2

$$x^2 y' + xy = 1$$

$$y' + \frac{y}{x} = \frac{1}{x^2}$$

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$xy' + y = \frac{1}{x}$$

$$\frac{d}{dx}xy = \frac{1}{x}$$

$$\int \frac{d}{dx}xy\,dx = \int \frac{1}{x}\,dx$$

$$xy = \ln x + c$$

$$y = \frac{\ln x + c}{x}$$

$$y(1) = 2$$

$$\frac{\ln 1 + c}{1} = 2$$

$$c = 2$$

$$y = \frac{\ln x + 2}{x}$$

Example 3

$$y' + 2xy = 1$$

$$I(x) = e^{\int P(x)dx} = e^{\int 2x\,dx} = e^{x^2}$$

$$(e^{x^2}y)' = e^{x^2}$$

$$\int \frac{d}{dx}e^{x^2}y\,dx = \underline{\int e^{x^2}dx}$$ error function

$$e^{x^2}y = \int e^{x^2}dx$$

$$y = e^{-x^2}\left(\int e^{x^2}dx\right)$$

Example 4

(a) $L\frac{dI}{dt} + RI = E(t)$

$4\frac{dI}{dt} + 12I = 60$

$\frac{dI}{dt} + 3I = 15$

$I(x) = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$

$e^{3x}\frac{dI}{dt} + e^{3x}(3I) = 15e^{3x}$

$(e^{3x}I)' = 15e^{3x}$

$\int \frac{d}{dx} e^{3x}I \, dx = \int 15e^{3x} dx$

$e^{3x}I = 5e^{3x} + C$

$I = 5 + \frac{C}{e^{3x}}$

(b) $I(0) = 0$

$5 + \frac{C}{e^{3(0)}} = 0$

$C = -5$

$\therefore I(t) = 5 - \frac{5}{e^{3x}}$

(c) $I(1) = 5 - \frac{5}{e^3} = 4.75A$

9.6

1. not linear

2. linear

3. linear

4. not linear

5. $y' + 2y = 2e^x$

$I(x) = e^{\int 2dx} = e^{2x}$

$(e^{2x}y)' = 2e^{3x}$

$\int \frac{d}{dx} e^{2x}y \, dx = \int 2e^{3x} dx$

$e^{2x}y = \frac{2}{3}e^{3x} + C$

$y = \frac{2}{3}e^{x} + Ce^{-2x}$

6. $y' = x + 5y$

$y' - 5y = x$

$I(x) = e^{\int -5\,dx} = e^{-5x}$

$(e^{-5x}y)' = xe^{-5x}$

$e^{-5x}y = \int xe^{-5x}\,dx$

let $u = x$ $\quad dv = e^{-5x}\,dx$

$du = dx$ $\quad v = -\frac{1}{5}e^{-5x}$

$\int xe^{-5x} = -\frac{x}{5}e^{-5x} - \int -\frac{1}{5}e^{-5x}\,dx$

$= -\frac{x}{5}e^{-5x} - \frac{1}{25}e^{-5x} - C$

$e^{-5x}y = -\frac{x}{5}e^{-5x} - \frac{1}{25}e^{-5x} - C$

$y = -\frac{x}{5} - \frac{1}{25} - \frac{C}{e^{-5x}} = -\frac{x}{5} - \frac{1}{25} - Ce^{5x}$

7. $xy' - 2y = x^2$

$y' - \frac{2}{x}y = x$

$I(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2}$

$(x^{-2}y)' = x^{-1}$

$x^{-2}y = \int \frac{1}{x}\,dx$

$x^{-2}y = \ln|x| + C$

$y = \frac{\ln|x| + C}{x^{-2}} = x^2(\ln|x| + C)$

8. $x^2y' + 2xy = \cos^2 x$

$y' + \frac{2y}{x} = \frac{\cos^2 x}{x^2}$

$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$

$(x^2y)' = \cos^2 x$

$x^2y = \int \cos^2 x\,dx$

$= \int \frac{1 + \cos 2x}{2}\,dx$

$= \int \frac{1}{2}\,dx + \int \frac{\cos 2x}{2}\,dx$

$= \frac{1}{2}x + \frac{\sin 2x}{4} + C$

$y = 2x^{-1} + \frac{(\sin 2x)x^{-2}}{4} + \frac{C}{x}$

Made with Goodnotes

9. $xy' + y = \sqrt{x}$

$y' + \frac{1}{x}y = \frac{1}{x^{-\frac{1}{2}}}$

$P(x) = \frac{1}{x}$

$I(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$xy' + y = x^{\frac{1}{2}}$

$(xy)' = x^{\frac{1}{2}}$

$\int \frac{d}{dx} xy \, dx = \int x^{\frac{1}{2}} \, dx$

$xy = \frac{2}{3}x^{\frac{3}{2}} + c$

$y = \frac{2}{3}\sqrt{x} + \frac{c}{x}$

10. $1 + xy = xy'$

$xy' - xy = 1$

$y' - y = \frac{1}{x}$

$P(x) = -1$

$I(x) = e^{\int P(x)dx} = e^{\int -1 dx} = e^{-x}$

$e^{-x}y' - e^{-x}y = \frac{e^{-x}}{x}$

$(e^{-x}y)' = e^{-x}x^{-1}$

$\int \frac{d}{dx}(e^{-x}y)\,dx = \underline{\int e^{-x}x^{-1}}\,dx$ no solution

$e^{-x}y = \int e^{-x}x^{-1}\,dx + c$

$y = \frac{(\int e^{-x}x^{-1}dx)}{e^{-x}} + ce^{x}$

11. $\frac{dy}{dx} + 2xy = x^2$

$y' + 2xy = x^2$

$P(x) = 2x$

$I(x) = e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$

$(e^{x^2}y)' = e^{x^2}x^2$

$\int \frac{d}{dx}(e^{x^2}y)\,dx = \int e^{x^2}x^2\,dx$

$e^{x^2}y = \int e^{x^2}x^2\,dx + c$

$y = e^{-x^2}(\int e^{x^2}x^2dx) + ce^{-x^2}$

Skip 12

13. $(1+t)\frac{du}{dt} + u = 1 + t$

$$\frac{du}{dt} + \frac{1}{1+t}u = 1$$

$$P(t) = \frac{1}{1+t}$$

$$I(t) = e^{\int P(t)dt} = e^{\int \frac{1}{1+t}dt} = e^{\ln|t|} = t \quad = e^{\ln|1+t|} = 1+t$$

$$(tu)' = t \qquad ((1+t)u)' = 1+t$$

$$\int \frac{d}{dt}(tu)\,dt = \int t\,dt$$

$$tu = \frac{t^2}{2} + c \qquad (1+t)u = \int (1+t)\,dt$$

$$u = \frac{t}{2} + \frac{c}{t} \qquad (1+t)u = t + \frac{t^2}{2} + C$$

14. * $t\ln t\frac{dr}{dt} + r = te^t$

$$\frac{dr}{dt} + \frac{1}{t\ln t}t = \frac{e^t}{\ln t}$$

$$P(t) = \frac{1}{t\ln t}$$

$$I(t) = e^{\int \frac{1}{t\ln t}dt}$$

let $u = \ln t$

$du = \frac{1}{t}dt$

$$= e^{\ln|\ln t|} = \ln t$$

$$\int \frac{1}{t\ln t}dt = \int \frac{1}{u}du$$

$$((\ln t)r)' = e^t$$

$$= \ln|u|$$

$$= \ln|\ln t|$$

$$\int \frac{d}{dt}(\ln t)r\,dt = \int e^t\,dt$$

$$(\ln t)r = e^t + c$$

$$r = \frac{e^t}{\ln t} + \frac{c}{\ln t}$$

15. * $y' = x + y$

$y' - y = x$

$P(x) = -1$

$I(x) = e^{\int P(x)dx} = e^{\int -1dx} = e^{-x}$

$(e^{-x}y)' = xe^{-x}$

$\int \frac{d}{dx} e^{-x}y\,dx = \int xe^{-x}dx$

let $u = x$

$du = dx$

$dv = e^{-x}dx$

$v = -e^{-x}$

$\int xe^{-x}dx$

$= uv - \int v\,du$

$= -xe^{-x} - \int -e^{-x}dx$

$= -xe^{-x} - e^{-x}$

$= -e^{-x}(x+1)$

$e^{-x}y = -e^{-x}(x+1) + C$

$y = -x - 1 + ce^{x}$

$y(0) = 2$

$2 = -1 + c$

$c = 3$

$y = -x - 1 + 3e^{x}$

16. $t\frac{dy}{dt} + 2y = t^3$

$$\frac{dy}{dt} + \frac{2}{t}y = t^2$$

$$P(t) = \frac{2}{t}$$

$$I(t) = e^{\int P(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$$

$$(t^2 y)' = t^4$$

$$\int \frac{d}{dt}(t^2 y)\,dt = \int t^4\,dt$$

$$t^2 y = \frac{t^5}{5} + C$$

$$y = \frac{t^3}{5} + \frac{c}{t^2}$$

$$y(1) = 0$$

$$0 = \frac{1}{5} + C$$

$$c = -\frac{1}{5}$$

$$y = \frac{t^3}{5} - \frac{1}{5t^2}$$

17. $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}$

$$P(t) = -2t$$

$$I(t) = e^{\int P(t)dt} = e^{\int -2t\,dt} = e^{-t^2}$$

$$(e^{-t^2}v)' = 3t^2$$

$$\int \frac{d}{dt} e^{-t^2}v\,dt = \int 3t^2\,dt$$

$$e^{-t^2}v = t^3 + C$$

$$v = (t^3 + c)e^{t^2}$$

$$v(0) = 5$$

$$5 = 0 + c$$

$$c = 5$$

$$v = (t^3 + 5)e^{t^2}$$

27.

$$L\frac{dI}{dt} + RI = E(t)$$

$$2\frac{dI}{dt} + 10I = 40$$

$$\frac{dI}{dt} + 5I = 20$$

$$P(t) = 5$$

Integral factor $i(t) = e^{\int P(t)\,dt} = e^{\int 5dt} = e^{5t}$

$$(e^{5t} I)' = 20e^{5t}$$

$$\int \frac{d}{dt} e^{5t} I\,dt = \int 20 e^{5t}\,dt$$

$$e^{5t} I = 4e^{5t} + C$$

$$I = 4 + Ce^{-5t}$$

$$I(0) = 0$$

$$0 = 4 + C$$

$$C = -4$$

(a) $I = 4 - 4e^{-5t}$

(b) $I(0.1) = 4 - 4e^{-\frac{1}{2}}$

$$= 1.57\ A$$

28. $\frac{dI}{dt} + 20I = 40\sin 60t$

$$P(t) = 20$$

$$i(t) = e^{\int P(t)dt} = e^{\int 20dt} = e^{20t}$$

$$\int e^{at}\sin(bt)\,dt = \frac{e^{at}(a\sin bt - b\cos bt)}{a^2+b^2}$$

$$(e^{20t} I)' = 40\sin 60t\, e^{20t}$$

$a = 20,\ b = 60$

$$\int \frac{d}{dt} e^{20t} I\,dt = \int 40\sin 60t\, e^{20t}\,dt$$

$$\int 40\sin 60t\, e^{20t}dt = 40\left(\frac{e^{20t}(20\sin 60t - 60\cos 60t)}{20^2+60^2}\right)$$

$$= 40\left(\frac{20 \times e^{20t}(\sin 60t - 3\cos 60t)}{4000}\right)$$

$$= \frac{e^{20t}(\sin 60t - 3\cos 60t)}{5}$$

$$e^{20t} I = \frac{e^{20t}(\sin 60t - 3\cos 60t)}{5} + C$$

$$I = \frac{\sin 60t - 3\cos 60t}{5} + Ce^{-20t}$$

$$I(0) = 1$$

$$1 = \frac{0 - 3}{5} + C$$

$$C = \frac{8}{5}$$

(a) $I = \frac{\sin 60t - 3\cos 60t}{5} + \frac{8}{5}e^{-20t}$ (b) $I(0.1) = \frac{\sin 6 - 3\cos 6}{5} + \frac{8}{5}e^{-2} = -0.36A$

$$\boxed{\begin{aligned} b^2-4ac>0 &\Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ b^2-4ac=0 &\Rightarrow y = c_1 e^{rx} + c_2 x e^{rx} \\ b^2-4ac<0 &\Rightarrow y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) \end{aligned}}$$

Example 1

$$r^2 + r - 6 = 0$$
$$(r+3)(r-2) = 0$$
$$r = -3, 2$$

$$y = c_1 e^{-3x} + c_2 e^{2x}$$

Example 2

$$3r^2 + r - 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1+12}}{6}$$
$$= \frac{-1 \pm \sqrt{13}}{6}$$

$$y = c_1 e^{\left(\frac{-1+\sqrt{13}}{6}\right)x} + c_2 e^{\left(\frac{-1-\sqrt{13}}{6}\right)x}$$

Example 3

$$4r^2 + 12r + 9 = 0$$

$$(2r+3)^2 = 0$$
$$r = -\frac{3}{2}$$
$$y = c_1 e^{-\frac{3}{2}x} + c_2 x e^{-\frac{3}{2}x}$$

Example 4

$36-52 = i16$

$$r^2 - 6r + 13 = 0$$

$$\alpha = -\frac{b}{2a} = \frac{6}{2} = 3$$
$$\beta = \frac{i\sqrt{b^2-4ac}}{2a} = \frac{4}{2} = 2$$

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Example 5

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r = -3, 2$$

$$y = c_1 e^{-3x} + c_2 e^{2x}$$

$$y(0) = 1$$

$$c_1 + c_2 = 1 \quad \text{— (1)}$$

$$y' = -3c_1 e^{-3x} + 2c_2 e^{2x}$$

$$y'(0) = 0$$

$$-3c_1 + 2c_2 = 0 \quad \text{— (2)}$$

$$(2) \Rightarrow -3c_1 + 2c_2 = 0$$
$$(1) \times 3 \Rightarrow 3c_1 + 3c_2 = 3$$

$$5c_2 = 3$$
$$c_2 = \frac{3}{5}$$

$$c_1 = 1 - c_2 = \frac{2}{5}$$

$$y = \frac{2}{5} e^{-3x} + \frac{3}{5} e^{2x}$$

Example 6

$$r^2 + 1 = 0 \qquad \alpha = -\frac{b}{2a} = -\frac{0}{2} = 0$$

$$\beta = \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{0 - 4}}{2} = \frac{2}{2} = 1$$

$$y = e^{0 \times x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$$

$$y(0) = 2 \qquad y' = -c_1 \sin x + c_2 \cos x$$
$$c_1 = 2$$

$$y'(0) = 3$$
$$c_2 = 3$$

$$y = 2\cos x + 3\sin x$$

Made with Goodnotes

Example 7

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = -1$$

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

$y(0) = 1$ $\qquad$ $y(1) = 3$

$c_1 e^{-1} + c_2 e^{-1} = 3$

$c_1 = 1$

$1 + c_2 = 3e$

$c_2 = 3e - 1 = 7.15$

$$y = e^{-x} + 7.15 x e^{-x}$$

1. $r^2 - 6r + 8 = 0$

$(r-4)(r-2) = 0$

$r = 4, 2$

$y = c_1 e^{4x} + c_2 e^{2x}$

$16 - 32 = -16$

2. $r^2 - 4r + 8 = 0$

$\alpha = -\frac{b}{2a} = \frac{4}{2} = 2$

$\beta = \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{-16}}{2} = \frac{4}{2} = 2$

$y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$

$64 - 164 = -100$

3. $r^2 + 8r + 41 = 0$

$\alpha = -\frac{8}{2} = -4$

$y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$ $\qquad$ $\beta = \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{-100}}{2} = \frac{10}{2} = 5$

4. $2r^2 - r - 1 = 0$

$(2r+1)(r-1) = 0$

$r = -\frac{1}{2}, 1$

$y = c_1 e^{-\frac{1}{2}x} + c_2 e^{x}$

5. $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$y = c_1 e^{x} + c_2 x e^{x}$

6. $3r^2 - 5r = 0$

$r(3r-5) = 0$

$r = 0, \frac{5}{3}$

$y = c_1 e^{0 \times x} + c_2 e^{\frac{5}{3}x} = c_1 + c_2 e^{\frac{5}{3}x}$

7. $4r^2 + 1 = 0$ $\quad 0 - 16$

$\alpha = -\frac{b}{2a} = 0, \beta = \frac{\sqrt{-16}}{8} = \frac{4}{8} = \frac{1}{2}$

$y = e^{0 \times x}(c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x)$

$= c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x$

8. $16r^2 + 24r + 9 = 0$

$(4r+3)^2 = 0$

$r = -\frac{3}{4}$

$y = c_1 e^{-\frac{3}{4}x} + c_2 x e^{-\frac{3}{4}x}$

9. $4r^2 + r = 0$

$r(4r+1) = 0$

$r = 0, -\frac{1}{4}$

$y = c_1 e^{0 \times x} + c_2 e^{-\frac{1}{4}x} = c_1 + c_2 e^{-\frac{1}{4}x}$

10. $9r^2 + 4 = 0$

$\alpha = -\frac{0}{18} = 0$

$y = c_1 \cos \frac{2}{3}x + c_2 \sin \frac{2}{3}x$ $\quad \beta = \frac{\sqrt{-144}}{18} = \frac{12}{18} = \frac{2}{3}$

11. $r^2 - 2r - 1 = 0$

$r = \frac{2 \pm \sqrt{4-(-4)}}{2} = \frac{2 \pm \sqrt{8}}{2}$

$y = c_1 e^{(\frac{2+\sqrt{8}}{2})x} + c_2 e^{(\frac{2-\sqrt{8}}{2})x}$

12. $r^2 - 6r + 4 = 0$

$r = \frac{6 \pm \sqrt{36-16}}{2} = \frac{6 \pm \sqrt{20}}{2}$

$y = c_1 e^{(\frac{6+\sqrt{20}}{2})x} + c_2 e^{(\frac{6-\sqrt{20}}{2})x}$

13. $r^2+r+1=0$ $\qquad 1-4=-3$

$\alpha = -\frac{1}{2}$

$\beta = \frac{\sqrt{-3}}{2} = \frac{\sqrt{3}}{2}$

$y = e^{-\frac{1}{2}x}(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x)$

17. $2r^2+5r+3=0$

$(2r+3)(r+1)=0$

$r = -\frac{3}{2}, -1$

$y = c_1e^{-\frac{3}{2}x} + c_2e^{-x}$ $\qquad y' = -\frac{3}{2}c_1e^{-\frac{3}{2}x} - c_2e^{-x}$

$y(0)=3$ $\qquad y'(0) = -4$

$c_1+c_2=3$ — (1) $\qquad -\frac{3}{2}c_1 - c_2 = -4$ —(2)

(1) $\Rightarrow c_1+c_2=3$

(2) $\Rightarrow -\frac{3}{2}c_1 - c_2 = -4$

$-\frac{1}{2}c_1 = -1$

$c_1 = 2,\ c_2 = 1$

$\cancel{c_1 = -\frac{1}{2}}$, $c_2 = 3-(-\frac{1}{2}) = \frac{7}{2}$

$\cancel{y = -\frac{1}{2}e^{-\frac{3}{2}x} + \frac{7}{2}e^{-x}}$ $\qquad y = 2e^{-\frac{3}{2}x} + e^{-x}$

18. $r^2+3=0$ $\qquad 0-12 = -12 < 0$

$\alpha = -\frac{b}{2a} = \frac{0}{2} = 0,\ \beta = \frac{\sqrt{0-12}}{2} = \frac{\sqrt{12}}{2}$

$y = c_1\cos\frac{\sqrt{12}}{2}x + c_2\sin\frac{\sqrt{12}}{2}x$

$y(0)=1$

$c_1 = 1$

$y' = -\frac{\sqrt{12}}{2}c_1\sin\frac{\sqrt{12}}{2}x + \frac{\sqrt{12}}{2}c_2\cos\frac{\sqrt{12}}{2}x$

$y'(0) = 3$

$\frac{\sqrt{12}}{2}c_2 = 3$

$c_2 = \frac{6}{\sqrt{12}} = \frac{6\sqrt{12}}{12} = \frac{\sqrt{12}}{2}$

$\therefore y = \cos\frac{\sqrt{12}}{2}x + \frac{\sqrt{12}}{2}\sin\frac{\sqrt{12}}{2}x$

19. $4r^2 - 4r + 1 = 0$

2×-1, 2×-1

$(2r-1)^2 = 0$

$r = \frac{1}{2}$

$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$

$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + c_2 x \left(\frac{1}{2}e^{\frac{1}{2}x}\right) + e^{\frac{1}{2}x} \cdot c_2$

$= \frac{1}{2}c_1 e^{\frac{1}{2}x} + \left(\frac{1}{2}x + 1\right) c_2 e^{\frac{1}{2}x}$

$y(0) = 1$

$c_1 = 1$

$y'(0) = -1.5$

$\frac{1}{2}c_1 + c_2 = -1.5$

$c_2 = -2$

$\therefore y = e^{\frac{1}{2}x} - 2xe^{\frac{1}{2}x}$

20. $2r^2 + 5r - 3 = 0$

2×-1, $1 \times +3$

$(2r-1)(r+3) = 0$

$r = \frac{1}{2}, -3$

$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x}$

$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} - 3c_2 e^{-3x}$

$y(0) = 1$

$c_1 + c_2 = 1$ — (1)

$y'(0) = 4$

$\frac{1}{2}c_1 - 3c_2 = 4$ — (2)

(1) $\Rightarrow$ $c_1 + c_2 = 1$

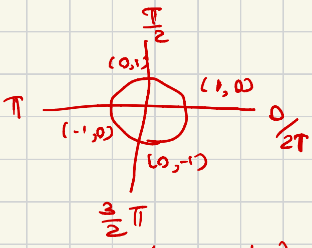
(2) $\times 2 \Rightarrow$ $c_1 - 6c_2 = 8$

$- \quad + \quad -$

$7c_2 = -7$

$c_2 = -1$, $c_1 = 1 - c_2 = 2$

$\therefore y = 2e^{\frac{1}{2}x} - e^{-3x}$



21. $r^2 + 16 = 0$

$\alpha = \frac{-b}{2a} = 0$

$y = c_1\cos 4x + c_2 \sin 4x$

$\beta = \frac{\sqrt{b^2 - 4ac}}{2} = \frac{\sqrt{-64}}{2} = \frac{8}{2} = 4$

$y\left(\frac{\pi}{4}\right) = -3$

$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

$y'\left(\frac{\pi}{4}\right) = 4$

$c_1 \cos \pi + c_2 \sin \pi = -3$

$-4c_1 \sin \pi + 4c_2 \cos \pi = 4$

$-4c_2 = 4$

$c_2 = -1$

$-c_1 = -3$

$c_1 = 3$

$y = 3\cos 4x - \sin 4x$

22. $r^2 - 2r + 5 = 0$ $\qquad 4 - 20 = -16$

$$\alpha = -\frac{b}{2a} = \frac{2}{2} = 1$$

$$\beta = \frac{\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{-16}}{2} = \frac{4}{2} = 2$$

$$y = e^{x}(c_1\cos 2x + c_2\sin 2x)$$

$$= c_1e^{x}\cos 2x + c_2e^{x}\sin 2x$$

$$y' = c_1e^{x}(-2\sin 2x) + \cos 2x\,(c_1e^{x}) + c_2e^{x}(2\cos 2x) + \sin 2x\,(c_2e^{x})$$

$$= -2c_1e^{x}\sin 2x + c_1e^{x}\cos 2x + 2c_2e^{x}\cos 2x + c_2e^{x}\sin 2x$$

$$= c_1e^{x}(-2\sin 2x + \cos 2x) + c_2e^{x}(2\cos 2x + \sin 2x)$$

$y(\pi) = 0$ $\qquad y'(\pi) = 2$

$c_1e^{\pi}\cos 2\pi + c_2e^{\pi}\sin 2\pi = 0$ $\qquad c_1e^{\pi}(0+1) + c_2e^{\pi}(2+0) = 2$

$c_1e^{\pi} = 0$ $\qquad 0 + 2c_2e^{\pi} = 2$

$c_1 = 0$ $\qquad c_2e^{\pi} = 1$

$c_2 = e^{-\pi}$

$$\therefore y = e^{-\pi}e^{x}\sin 2x = e^{x-\pi}\sin 2x$$

23. $r^2 + 2r + 2 = 0$ $\qquad 4 - 8$

$$\alpha = \frac{-2}{2} = -1, \quad \beta = \frac{\sqrt{4-8}}{2} = \frac{\sqrt{-4}}{2} = \frac{2}{2} = 1$$

$$y = e^{-x}(c_1\cos x + c_2\sin x)$$

$$= c_1e^{-x}\cos x + c_2e^{-x}\sin x$$

$$y' = -c_1e^{-x}\sin x - c_1e^{-x}\cos x + c_2e^{-x}\cos x - c_2e^{-x}\sin x$$

$$= c_1e^{-x}(-\sin x - \cos x) + c_2e^{-x}(\cos x - \sin x)$$

$y(0) = 2$ $\qquad y'(0) = 1$

$c_1 = 2$ $\qquad c_1(-1) + c_2(1) = 1$

$-2 + c_2 = 1$

$c_2 = 3$

$$y = 2e^{-x}\cos x + 3e^{-x}\sin x$$

24. $r^2 + 12r + 36 = 0$

$(r+6)^2 = 0$

$r = -6$

$y = c_1e^{-6x} + c_2xe^{-6x}$

$y(0) = 0$

$c_1 = 0$

$y' = -6c_1e^{-6x} + c_2x(-6e^{-6x}) + e^{-6x}(c_2)$

$= -6c_1e^{-6x} - 6c_2xe^{-6x} + c_2e^{-6x}$

$y'(1) = 1$

$-6c_1e^{-6} - 6c_2e^{-6} + c_2e^{-6} = 1$

$-5c_2e^{-6} = 1$

$c_2 = -\frac{1}{5}e^{6}$

$$\therefore y = -\frac{1}{5}e^{6}xe^{-6x} = -\frac{1}{5}\frac{e^{6}}{e^{6x}}x$$

25. $4r^2+1=0$ $\quad 0-16$

$$\alpha=-\frac{b}{2a}=\frac{0}{8}=0 \quad \beta=\frac{\sqrt{0-16}}{8}=\frac{4}{8}=\frac{1}{2}$$

$$y=c_1\cos\tfrac{1}{2}x+c_2\sin\tfrac{1}{2}x$$

$y(0)=3$ $\qquad$ $y(\pi)=-4$

$c_1=3$ $\qquad$ $c_2=-4$

$$y=3\cos\frac{x}{2}-4\sin\frac{x}{2}$$

26. $r^2+2r=0$

$r(r+2)=0$

$r=0,-2$

$y=c_1+c_2e^{-2x}$

$y(0)=1$ $\qquad$ $y(1)=2$

$c_1+c_2=1$ $\qquad$ $c_1+c_2e^{-2}=2$

$$\begin{array}{l} c_1+c_2=1 \\ c_1+c_2e^{-2}=2 \\ - \quad - \quad - \\ \hline c_2-c_2e^{-2}=-1 \end{array}$$

$c_2(1-e^{-2})=-1$

$$c_2=-\frac{1}{1-e^{-2}}=\frac{1}{e^{-2}-1}=\frac{1}{\frac{1}{e^2}-1}=\frac{1}{\frac{1-e^2}{e^2}}=\frac{e^2}{1-e^2}$$

$$c_1=1-c_2=1-\frac{e^2}{1-e^2}=\frac{1-e^2-e^2}{1-e^2}=\frac{1-2e^2}{1-e^2}$$

27. $r^2-3r+2=0$

$(r-2)(r-1)=0$

$r=2,1$

$y=c_1e^{2x}+c_2e^{x}$

$y(0)=1$ $\qquad$ $y(3)=0$

$c_1+c_2=1$ $\qquad$ $c_1e^6+c_2e^3=0$

$c_2=1-c_1$ $\qquad$ $c_1e^3+c_2=0$

$c_1e^3+1-c_1=0$

$c_1(e^3-1)=-1$

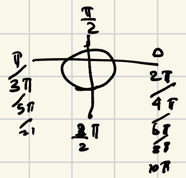
$$c_1=\frac{-1}{e^3-1}=\frac{1}{1-e^3}$$

$$c_2=1-\frac{1}{1-e^3}=\frac{1-e^3-1}{1-e^3}=\frac{-e^3}{1-e^3}=\frac{e^3}{e^3-1}$$

$$y=\frac{e^{2x}}{1-e^3}+\frac{e^{x+3}}{e^3-1}$$

28. $r^2 + 100 = 0$ $\quad \alpha = 0$

$$\beta = \frac{\sqrt{0-400}}{2} = \frac{20}{2} = 10$$



$y = c_1 \cos 10x + c_2 \sin 10x$

$y(\pi)$ is wrong question

$y(0) = 2 \qquad y(\frac{\pi}{20}) = 5$

$c_1 = 2 \qquad c_2 = 5$

$y = 2\cos 10x + 5\sin 10x$

29. $r^2 - 6r + 25 = 0$

$$\alpha = -\frac{b}{2a} = \frac{6}{2} = 3$$

$$\beta = \frac{\sqrt{36-100}}{2} = \frac{\sqrt{-64}}{2} = \frac{8}{2} = 4$$

$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$

$y(\pi)$ is wrong question

$y(0) = 1 \qquad y(\frac{\pi}{8}) = 2$

$c_1 = 1 \qquad c_2 = 2$

$y = e^{3x}(\cos 4x + 2\sin 4x)$

30. $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$

$y = c_1 e^{3x} + c_2 x e^{3x}$

$y(0) = 1 \qquad y(1) = 0$

$c_1 = 1 \qquad c_1 e^3 + c_2 e^3 = 0$

$e^3 + c_2 e^3 = 0$

$1 + c_2 = 0$

$c_2 = -1$

$y = e^{3x} - x e^{3x}$

31. $r^2 + 4r + 13 = 0$ $\quad \alpha = -\frac{b}{2a} = -\frac{4}{2} = -2$

$$\beta = \frac{\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{16-52}}{2} = \frac{\sqrt{-36}}{2} = \frac{6}{2} = 3$$

$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$

$y(0) = 2$

$c_1 = 2$

$y(\frac{\pi}{2}) = 1$

$e^{-\pi}(-c_2) = 1$

$-c_2 = e^{\pi}$

$c_2 = -e^{\pi}$

$y = e^{-2x}(\cos 3x - e^{\pi} \sin 3x)$

32 $9r^2 - 18r + 10 = 0$ $\quad$ 144, 18, 324

$$\alpha = -\frac{b}{2a} = \frac{18}{18} = 1$$

$$\beta = \frac{\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{324-360}}{18} = \frac{\sqrt{-36}}{18} = \frac{6}{18} = \frac{1}{3}$$

$y = e^{x}(c_1 \cos \frac{1}{3}x + c_2 \sin \frac{1}{3}x)$

$y(\pi)$ is wrong question

$y(0) = 0$

$c_1 = 0$

$y(\frac{3\pi}{2}) = 1$

$e^{\frac{3\pi}{2}} c_2 = 1$

$c_2 = e^{-\frac{3\pi}{2}}$

$y = \frac{e^{x}}{e^{\frac{3\pi}{2}}} \sin \frac{1}{3} x$

3. When interest is compounded continuously, the rate of change of the amount x of the investment is proportional to the amount present. In this case, the proportionality constant is the annual interest rate r; that is,

$$\frac{dx}{dt} = rx$$

a) If $2,000 is invested at 8% compounded continuously, find an equation for the future value of the investment as a function of time t, in years. (3 pts)

b) How long will it take for the investment to double? (2 pts)

(a) $\frac{dx}{dt} = rx$

$\frac{1}{x} dx = r\, dt$

$\ln|x| = rt + c$

$x = e^{rt} \cdot e^{c}$

$x = ke^{rt}$

$2000 = ke^{0}$

$k = 2000$

$x = 2000e^{0.08t}$

(b) $4000 = 2000e^{0.08t}$

$2 = e^{0.08t}$

$\ln 2 = 0.08t$

$t = 8.66$

4. A 300-gal tank initially contains a solution with 100 lb of a chemical. A mixture containing 2 lb/gal of the chemical enters the tank at 3 gal/min, and the well-stirred mixture leaves at the same rate. Find an equation that gives the amount of the chemical in the tank as a function of time. How long will it be before there is 500 lb of chemical in the tank? (5 pts)

$y' = \text{rate-in} - \text{rate out}$

$= (2)(3) - \left(\frac{y(t)}{300}\right)(3)$

$= 6 - \frac{y(t)}{100}$

$\frac{dy}{dt} = 6 - \frac{y(t)}{100} = \frac{600 - y(t)}{100}$

$\frac{1}{600-y(t)} dy = \frac{1}{100} dt$

$-\ln|600-y(t)| = \frac{t}{100}$

$\ln|600-y(t)| = -\frac{t}{100} + c$

$y(0) = 100$

$\ln|600-100| = c$

$c = \ln|500|$

$\ln|600-y(t)| = -\frac{t}{100} + \ln|500|$

$|600-y(t)| = e^{-\frac{t}{100}} \cdot e^{\ln 500}$

$600 - y(t) = 500e^{-\frac{t}{100}}$

$y(t) = 600 - 500e^{-\frac{t}{100}}$

$u = 600 - y$

$du = -dy$

$\ln|u|$

$500 = 600 - 500e^{-\frac{t}{100}}$

$-100 = -500e^{-\frac{t}{100}}$

$0.2 = e^{-\frac{t}{100}}$

$\ln|0.2| = -\frac{t}{100}$

$100(-1.60) = -t$

$160 = t$

$t = 160$

5. $xy' + y = \sqrt{x}$; $y(1) = 2$

$$y' + \frac{y}{x} = x^{-\frac{1}{2}}$$

$$I(x) = e^{\int P(x)\, dx} = e^{\ln|x|} = x$$

$$(xy)' = x^{\frac{1}{2}}$$

$$\int \frac{d}{dx} xy \; dx = \int x^{\frac{1}{2}} \, dx$$

$$xy = \frac{2}{3} x^{\frac{3}{2}} + c$$

$$y = \frac{2}{3} x^{\frac{1}{2}} + \frac{c}{x}$$

$$y(1) = 2$$

$$2 = \frac{2}{3} + c$$

$$c = 2 - \frac{2}{3} = \frac{4}{3}$$

$$y = \frac{2}{3}\sqrt{x} + \frac{4}{3x}$$

$2 \quad -1 \quad 1$
$1 + \quad 3 \quad +6-1$

7. $2y'' + 5y' - 3y = 0$, $y(0) = 1$, $y'(0) = 4$

$$2r^2 + 5r - 3 = 0$$

$$(2r-1)(r+3) = 0$$

$$r = \frac{1}{2}, -3$$

$$y = c_1 e^{\frac{1}{2}x} + c_2 e^{-3x}$$

$$y(0) = 1$$

$$c_1 + c_2 = 1 \quad , \; c_1 = 1 - c_2$$

$$y' = \frac{c_1}{2} e^{\frac{1}{2}x} - 3c_2 e^{-3x}$$

$$y'(0) = 4$$

$$y = 2e^{\frac{1}{2}x} - e^{-3x}$$

$$\frac{c_1}{2} - 3c_2 = 4$$

$$c_1 - 6c_2 = 8$$

$$1 - c_2 - 6c_2 = 8$$

$$-7c_2 = 7$$

$$c_2 = -1$$

$$c_1 = 2$$

8. $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y(3) = 0$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 1, 2$

$y = c_1 e^{x} + c_2 e^{2x}$

$y(0) = 1$

$c_1 + c_2 = 1$

$y(3) = 0$

$c_1 e^3 + c_2 e^6 = 0$

$c_1 + c_2 e^3 = 0$

$c_1 = -c_2 e^3$

$-c_2 e^3 + c_2 = 1$

$c_2(1 - e^3) = 1$

$c_2 = \frac{1}{1-e^3}$

$c_1 = \frac{-e^3}{1-e^3}$

$y = \frac{-e^{x+3}}{1-e^3} + \frac{e^{2x}}{1-e}$