Differential Equations

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Multivariable Calculus

Partial Derivatives

Definition

If $z = f(x, y)$, the ***partial derivative of f with respect to x***, denoted f_x, is the function, of two variables, given by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided that the limit exists.

The ***partial derivative of f with respect to y***, denoted f_y, is the function, of two variables, given by

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided that the limit exists.

Procedure to Find $f_x(x, y)$ and $f_y(x, y)$

To find f_x, treat y as a constant, and differentiate f with respect to x in the usual way.

To find f_y, treat x as a constant, and differentiate f with respect to y in the usual way.

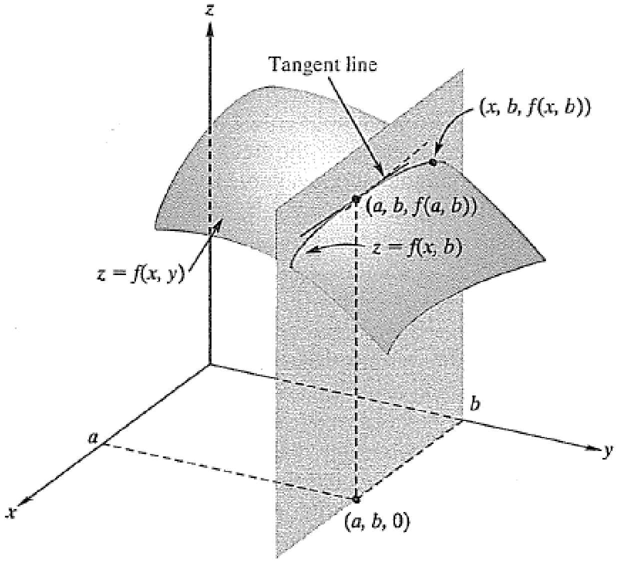


FIGURE 17.1 Geometric interpretation of $f_x(a, b)$.

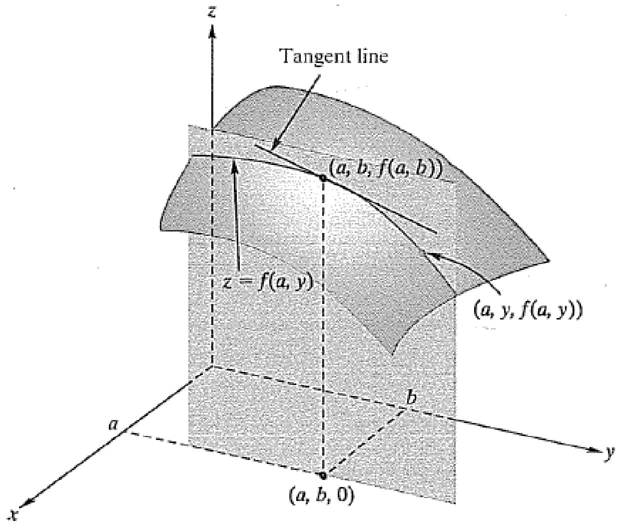


FIGURE 17.2 Geometric interpretation of $f_y(a, b)$.

If $f(x, y) = xy^2 + x^2y$, find $f_x(x, y)$ and $f_y(x, y)$. Also, find $f_x(3, 4)$ and $f_y(3, 4)$.

If $z = 3x^3y^3 - 9x^2y + xy^2 + 4y$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}, \left.\dfrac{\partial z}{\partial x}\right|_{(1,0)}$ and $\left.\dfrac{\partial z}{\partial y}\right|_{(1.0)}$.

If $w = x^2 e^{2x+3y}$, find $\partial w/\partial x$ and $\partial w/\partial y$.

If $f(x, y, z) = x^2 + y^2 z + z^3$, find $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$.

If $p = g(r, s, t, u) = \dfrac{rsu}{rt^2 + s^2t}$, find $\dfrac{\partial p}{\partial s}$, $\dfrac{\partial p}{\partial t}$, and $\left.\dfrac{\partial p}{\partial t}\right|_{(0,1,1,1)}$.

In Problems 1–26, a function of two or more variables is given. Find the partial derivative of the function with respect to each of the variables.

1. $f(x, y) = 2x^2 + 3xy + 4y^2 + 5x + 6y - 7$
2. $f(x, y) = 2x^2 + 3xy$
3. $f(x, y) = 2y + 1$
4. $f(x, y) = \ln 2$
5. $g(x, y) = 3x^4y + 2xy^2 - 5xy + 8x - 9y$
6. $g(x, y) = (x^2 + 1)^2 + (y^3 - 3)^3 + 5xy^3 - 2x^2y^2$
7. $g(p, q) = \sqrt{pq}$
8. $g(w, z) = \sqrt[3]{w^2 + z^2}$
9. $h(s, t) = \dfrac{s^2 + 4}{t - 3}$
10. $h(u, v) = \dfrac{8uv^2}{u^2 + v^2}$
11. $u(q_1, q_2) = \ln\sqrt{q_1 + 2} + \ln\sqrt[3]{q_2 + 5}$
12. $Q(l, k) = 2l^{0.38}k^{1.79} - 3l^{1.03} + 2k^{0.13}$
13. $h(x, y) = \dfrac{x^2 + 3xy + y^2}{\sqrt{x^2 + y^2}}$
14. $h(x, y) = \dfrac{\sqrt{x + 9}}{x^2y + y^2x}$
15. $z = e^{5xy}$
16. $z = (x^3 + y^3)e^{xy+3x+3y}$
17. $z = 5x\ln(x^2 + y)$
18. $z = \ln(5x^3y^2 + 2y^4)^4$
19. $f(r, s) = \sqrt{r + 2s}(r^3 - 2rs + s^2)$
20. $f(r, s) = \sqrt{rs}\, e^{2+r}$
21. $f(r, s) = e^{3-r}\ln(7 - s)$
22. $f(r, s) = (5r^2 + 3s^3)(2r - 5s)$
23. $g(x, y, z) = 2x^3y^2 + 2xy^3z + 4z^2$
24. $g(x, y, z) = 2xy^2z^6 - 4x^2y^3z^2 + 3xyz$
25. $g(r, s, t) = e^{s+t}(r^2 + 7s^3)$
26. $g(r, s, t, u) = rs\ln(t)e^u$

In Problems 27–34, evaluate the given partial derivatives.

27. $f(x, y) = x^3y + 7x^2y^2$; $\quad f_x(1, -2)$
28. $z = \sqrt{2x^3 + 5xy + 2y^2}$; $\quad \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1}}$
29. $g(x, y, z) = e^x\sqrt{y + 2z}$; $\quad g_z(0, 6, 4)$
30. $g(x, y, z) = \dfrac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$, $\quad g_y(1, 1, 5)$
31. $h(r, s, t, u) = (rst^2u)\ln(1 + rstu)$; $\quad h_t(1, 1, 0, 1)$
32. $h(r, s, t, u) = \dfrac{7r + 3s^2u^2}{s}$; $\quad h_t(4, 3, 2, 1)$

Applications of Partial Derivatives

$$\frac{\partial z}{\partial x} \text{ is the rate of change of } z \text{ with respect to } x \text{ when } y \text{ is held fixed.}$$

Similarly,

$$\frac{\partial z}{\partial y} \text{ is the rate of change of } z \text{ with respect to } y \text{ when } x \text{ is held fixed.}$$

We will now look at some applications in which the "rate of change" notion of a partial derivative is very useful.

Suppose a manufacturer produces x units of product X and y units of product Y. Then the total cost c of these units is a function of x and y and is called a **joint-cost function.** If such a function is $c = f(x, y)$, then $\partial c/\partial x$ is called the **(partial) marginal cost with respect to** x and is the rate of change of c with respect to x when y is held fixed. Similarly, $\partial c/\partial y$ is the **(partial) marginal cost with respect to** y and is the rate of change of c with respect to y when x is held fixed. It also follows that $\partial c/\partial x(x, y)$ is approximately the cost of producing one more unit of X when x units of X and y units of Y are produced. Similarly, $\partial c/\partial y(x, y)$ is approximately the cost of producing one more unit of Y when x units of X and y units of Y are produced.

For example, if c is expressed in dollars and $\partial c/\partial y = 2$, then the cost of producing an extra unit of Y when the level of production of X is fixed is approximately two dollars.

If a manufacturer produces n products, the joint-cost function is a function of n variables, and there are n (partial) marginal-cost functions.

A company manufactures two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing x pairs of the Lightning model and y pairs of the Alpine model per week is

$$c = f(x, y) = 0.07x^2 + 75x + 85y + 6000$$

where c is expressed in dollars. Determine the marginal costs $\partial c/\partial x$ and $\partial c/\partial y$ when $x = 100$ and $y = 50$, and interpret the results.

On a cold day, a person may feel colder when the wind is blowing than when the wind is calm because the rate of heat loss is a function of both temperature and wind speed. The equation

$$H = (10.45 + 10\sqrt{w} - w)(33 - t)$$

indicates the rate of heat loss H (in kilocalories per square meter per hour) when the air temperature is t (in degrees Celsius) and the wind speed is w (in meters per second). For $H = 2000$, exposed flesh will freeze in one minute.[5]

a. Evaluate H when $t = 0$ and $w = 4$.

b. Evaluate $\partial H/\partial w$ and $\partial H/\partial t$ when $t = 0$ and $w = 4$, and interpret the results.

c. When $t = 0$ and $w = 4$, which has a greater effect on H: a change in wind speed of 1 m/s or a change in temperature of 1°C?

A manufacturer of a popular toy has determined that the production function is $P = \sqrt{lk}$, where l is the number of labor-hours per week and k is the capital (expressed in hundreds of dollars per week) required for a weekly production of P gross of the toy. (One gross is 144 units.) Determine the marginal productivity functions, and evaluate them when $l = 400$ and $k = 16$. Interpret the results.

Competitive or Complimentary Products

Sometimes two products may be related such that changes in the price of one of them affect the demand for the other. A typical example is that of butter and margarine. If such a relationship exists between products A and B, then the demand for each product is dependent on the prices of both. Suppose q_A and q_B are the quantities demanded for A and B, respectively, and p_A and p_B are their respective prices. Then both q_A and q_B are functions of p_A and p_B:

$$q_A = f(p_A, p_B) \qquad \text{demand function for A}$$

$$q_B = g(p_A, p_B) \qquad \text{demand function for B}$$

We can find four partial derivatives:

$\dfrac{\partial q_A}{\partial p_A}$ *the marginal demand for* A *with respect to* p_A

$\dfrac{\partial q_A}{\partial p_B}$ *the marginal demand for* A *with respect to* p_B

$\dfrac{\partial q_B}{\partial p_A}$ *the marginal demand for* B *with respect to* p_A

$\dfrac{\partial q_B}{\partial p_B}$ *the marginal demand for* B *with respect to* p_B

Under typical conditions, if the price of B is fixed and the price of A increases, then the quantity of A demanded will decrease. Thus, $\partial q_A/\partial p_A < 0$. Similarly, $\partial q_B/\partial p_B < 0$. However, $\partial q_A/\partial p_B$ and $\partial q_B/\partial p_A$ may be either positive or negative. If

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} > 0$$

then A and B are said to be **competitive products** or **substitutes.** In this situation, an increase in the price of B causes an increase in the demand for A, if it is assumed that the price of A does not change. Similarly, an increase in the price of A causes an increase in the demand for B when the price of B is held fixed. Butter and margarine are examples of substitutes.

Proceeding to a different situation, we say that if

$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0$$

then A and B are **complementary products.** In this case, an increase in the price of B causes a decrease in the demand for A if the price of A does not change. Similarly, an increase in the price of A causes a decrease in the demand for B when the price of B is held fixed. For example, cars and gasoline are complementary products. An increase in the price of gasoline will make driving more expensive. Hence, the demand for cars will decrease. And an increase in the price of cars will reduce the demand for gasoline.

The demand functions for products A and B are each a function of the prices of A and B and are given by

$$q_{\mathrm{A}} = \frac{50\sqrt[3]{p_{\mathrm{B}}}}{\sqrt{p_{\mathrm{A}}}} \quad \text{and} \quad q_{\mathrm{B}} = \frac{75p_{\mathrm{A}}}{\sqrt[3]{p_{\mathrm{B}}^2}}$$

respectively. Find the four marginal-demand functions, and determine whether A and B are competitive products, complementary products, or neither.

For the joint-cost functions in Problems 1–3, find the indicated marginal cost at the given production level.

1. $c = 7x + 0.3y^2 + 2y + 900; \quad \frac{\partial c}{\partial y}, x = 20, y = 30$

2. $c = 2x\sqrt{x+y} + 6000; \quad \frac{\partial c}{\partial x}, x = 70, y = 74$

3. $c = 0.03(x+y)^3 - 0.6(x+y)^2 + 9.5(x+y) + 7700;$
$\frac{\partial c}{\partial x}, x = 50, y = 80$

For the production functions in Problems 4 and 5, find the marginal productivity functions $\partial P/\partial k$ and $\partial P/\partial l$.

4. $P = 15lk - 3l^2 + 5k^2 + 500$

5. $P = 2.314l^{0.357}k^{0.643}$

6. Cobb–Douglas Production Function In economics, a Cobb–Douglas production function is a production function of the form $P = Al^{\alpha}k^{\beta}$, where A, α, and β are constants and $\alpha + \beta = 1$. For such a function, show that

(a) $\partial P/\partial l = \alpha P/l$ **(b)** $\partial P/\partial k = \beta P/k$

(c) $l\frac{\partial P}{\partial l} + k\frac{\partial P}{\partial k} = P$. This means that summing the products of the marginal productivity of each factor and the amount of that factor results in the total product P.

In Problems 7–9, q_A and q_B are demand functions for products A and B, respectively. In each case, find $\partial q_A/\partial p_A$, $\partial q_A/\partial p_B$, $\partial q_B/\partial p_A$, $\partial q_B/\partial p_B$ and determine whether A and B are competitive, complementary, or neither.

7. $q_A = 1500 - 40p_A + 3p_B; \quad q_B = 900 + 5p_A - 20p_B$

8. $q_A = 20 - p_A - 2p_B; \quad q_B = 50 - 2p_A - 3p_B$

9. $q_A = \frac{100}{p_A\sqrt{p_B}}; \quad q_B = \frac{500}{p_B\sqrt[3]{p_A}}$

10. Canadian Manufacturing The production function for the Canadian manufacturing industries for 1927 is estimated by[6] $P = 33.0l^{0.46}k^{0.52}$, where P is product, l is labor, and k is capital. Find the marginal productivities for labor and capital, and evaluate when $l = 1$ and $k = 1$.

11. Dairy Farming An estimate of the production function for dairy farming in Iowa (1939) is given by[7]

$$P = A^{0.27}B^{0.01}C^{0.01}D^{0.23}E^{0.09}F^{0.27}$$

where P is product, A is land, B is labor, C is improvements, D is liquid assets, E is working assets, and F is cash operating expenses. Find the marginal productivities for labor and improvements.

12. Production Function Suppose a production function is given by $P = \dfrac{kl}{3k+5l}$.

(a) Determine the marginal productivity functions.

(b) Show that when $k = l$, the marginal productivities sum to $\dfrac{1}{8}$.

13. M.B.A. Compensation In a study of success among graduates with master of business administration (M.B.A.) degrees, it was estimated that for staff managers (which include accountants, analysts, etc.), current annual compensation (in dollars) was given by

$$z = 43{,}960 + 4480x + 3492y$$

where x and y are the number of years of work experience before and after receiving the M.B.A. degree, respectively.[8] Find $\partial z/\partial x$ and interpret your result.

19. Demand The demand equations for related products A and B are given by

$$q_A = 10\sqrt{\frac{p_B}{p_A}} \quad \text{and} \quad q_B = 3\sqrt[3]{\frac{p_A}{p_B}}$$

where q_A and q_B are the quantities of A and B demanded and p_A and p_B are the corresponding prices (in dollars) per unit.

(a) Find the values of the two marginal demands for product A when $p_A = 9$ and $p_B = 16$.

(b) If p_B were reduced to 14 from 16, with p_A fixed at 9, use part (a) to estimate the corresponding change in demand for product A.

20. Joint-Cost Function A manufacturer's joint-cost function for producing q_A units of product A and q_B units of product B is given by

$$c = \frac{q_A^2(q_B^3 + q_A)^{1/2}}{17} + q_A q_B^{1/3} + 600$$

where c is in dollars.

(a) Find the marginal-cost functions with respect to q_A and q_B.

(b) Evaluate the marginal-cost function with respect to q_A when $q_A = 17$ and $q_B = 8$. Round your answer to two decimal places.

(c) Use your answer to part (a) to estimate the change in cost if production of product A is decreased from 17 to 16 units, while production of product B is held constant at 8 units.

Implicit Partial Differentiation

An equation in x, y, and z does not necessarily define z as a function of x and y. For example, in the equation

$$z^2 - x^2 - y^2 = 0 \tag{1}$$

if $x = 1$ and $y = 1$, then $z^2 - 1 - 1 = 0$, so $z = \pm\sqrt{2}$. Thus, Equation (1) does not define z as a function of x and y. However, solving Equation (1) for z gives

$$z = \sqrt{x^2 + y^2} \quad \text{or} \quad z = -\sqrt{x^2 + y^2}$$

each of which defines z as a function of x and y. Although Equation (1) does not explicitly express z as a function of x and y, it can be thought of as expressing z *implicitly* as one of two different functions of x and y. Note that the equation $z^2 - x^2 - y^2 = 0$ has the form $F(x, y, z) = 0$, where F is a function of three variables. Any equation of the form $F(x, y, z) = 0$ can be thought of as expressing z implicitly as one of a set of possible functions of x and y. Moreover, we can find $\partial z/\partial x$ and $\partial z/\partial y$ directly from the form $F(x, y, z) = 0$.

To find $\partial z/\partial x$ for

$$z^2 - x^2 - y^2 = 0 \tag{2}$$

we first differentiate both sides of Equation (2) with respect to x while treating z as a function of x and y and treating y as a constant:

$$\frac{\partial}{\partial x}(z^2 - x^2 - y^2) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(y^2) = 0$$

$$2z\frac{\partial z}{\partial x} - 2x - 0 = 0$$

Solving for $\partial z/\partial x$, we obtain

$$2z\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{x}{z}$$

To find $\partial z/\partial y$, we differentiate both sides of Equation (2) with respect to y while treating z as a function of x and y and treating x as a constant:

$$\frac{\partial}{\partial y}(z^2 - x^2 - y^2) = \frac{\partial}{\partial y}(0)$$

$$2z\frac{\partial z}{\partial y} - 0 - 2y = 0 \qquad\qquad \frac{\partial x}{\partial y} = 0$$

$$2z\frac{\partial z}{\partial y} = 2y$$

Hence,

$$\frac{\partial z}{\partial y} = \frac{y}{z}$$

The method we used to find $\partial z/\partial x$ and $\partial z/\partial y$ is called *implicit partial differentiation.*

If $\dfrac{xz^2}{x+y} + y^2 = 0$, evaluate $\dfrac{\partial z}{\partial x}$ when $x = -1, y = 2$, and $z = 2$.

If $se^{r^2+u^2} = u \ln (t^2 + 1)$, determine $\partial t / \partial u$.

In Problems 1–11, find the indicated partial derivatives by the method of implicit partial differentiation.

1. $2x^2 + 3y^2 + 5z^2 = 900; \quad \partial z/\partial x$

2. $z^2 - 5x^2 + y^2 = 0; \quad \partial z/\partial x$

3. $3z^2 - 5x^2 - 7y^2 = 0; \quad \partial z/\partial y$

4. $3x^2 + y^2 + 2z^3 = 9; \quad \partial z/\partial y$

5. $x^2 - 2y - z^2 + x^2yz^2 = 20; \quad \partial z/\partial x$

6. $z^3 + 2x^2z^2 - xy = 0; \quad \partial z/\partial x$

7. $e^x + e^y + e^z = 10; \quad \partial z/\partial y$

8. $xyz + xy^2z^3 - \ln z^4 = 0; \quad \partial z/\partial y$

9. $\ln(z) + 9z - xy = 1; \quad \partial z/\partial x$

10. $\ln x + \ln y - \ln z = e^y; \quad \partial z/\partial x$

11. $(z^2 + 6xy)\sqrt{x^3 + 5} = 2; \quad \partial z/\partial y$

In Problems 12–20, evaluate the indicated partial derivatives for the given values of the variables.

12. $xz + xyz - 5 = 0; \quad \partial z/\partial x, x = 1, y = 4, z = 1$

13. $xz^2 + yz^2 - x^2y = 1; \quad \partial z/\partial x, x = 1, y = 0, z = 1$

14. $e^{zx} = xyz; \quad \partial z/\partial y, x = 1, y = -e^{-1}, z = -1$

15. $e^{yz} = -xyz; \quad \partial z/\partial x, x = -e^2/2, y = 1, z = 2$

16. $\sqrt{xz + y^2} - xy = 0; \quad \partial z/\partial y, x = 2, y = 2, z = 6$

17. $\ln z = 4x + y; \quad \partial z/\partial x, x = 5, y = -20, z = 1$

18. $\dfrac{r^2s^2}{s^2 + t^2} = \dfrac{t^2}{2}; \quad \partial r/\partial t, r = 1, s = 1, t = 1$

19. $\dfrac{s^2 + t^2}{rs} = 10; \quad \partial t/\partial r, r = 1, s = 2, t = 4$

20. $\ln(x + y + z) + xyz = ze^{x+y+z}; \quad \partial z/\partial x, x = 0, y = 1, z = 0$

Higher-Order Partial Derivatives

If $z = f(x, y)$, then not only is z a function of x and y, but also f_x and f_y are each functions of x and y, which may themselves have partial derivatives. If we can differentiate f_x and f_y, we obtain **second-order partial derivatives** of f. Symbolically,

$$f_{xx} \text{ means } (f_x)_x \qquad f_{xy} \text{ means } (f_x)_y$$
$$f_{yx} \text{ means } (f_y)_x \qquad f_{yy} \text{ means } (f_y)_y$$

In terms of ∂-notation,

$$\frac{\partial^2 z}{\partial x^2} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y\,\partial x} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \frac{\partial^2 z}{\partial y^2} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Note that to find f_{xy}, we first differentiate f with respect to x. For $\partial^2 z/\partial x\,\partial y$, we first differentiate with respect to y.

We can extend our notation beyond second-order partial derivatives. For example, $f_{xxy}\ (= \partial^3 z/\partial y\,\partial x^2)$ is a third-order partial derivative of f, namely, the partial derivative of $f_{xx}\ (= \partial^2 z/\partial x^2)$ with respect to y. The generalization of higher-order partial derivatives to functions of more than two variables should be obvious.

Find the four second-order partial derivatives of $f(x, y) = x^2y + x^2y^2$.

Find the value of $\left.\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x}\right|_{(1,2,3)}$ if $w = (2x + 3y + 4z)^3$.

Determine $\dfrac{\partial^2 z}{\partial x^2}$ if $z^2 = xy$.

In Problems 1–10, find the indicated partial derivatives.

1. $f(x,y) = 6xy^2$; $f_x(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$

2. $f(x,y) = 2x^3y^2 + 6x^2y^3 - 3xy$; $f_x(x,y)$, $f_{xx}(x,y)$

3. $f(x,y) = 7x^2 + 3y$; $f_y(x,y)$, $f_{yy}(x,y)$, $f_{yyx}(x,y)$

4. $f(x,y) = (x^2 + xy + y^2)(xy + x + y)$; $f_x(x,y)$, $f_{xy}(x,y)$

5. $f(x,y) = 9e^{2xy}$; $f_y(x,y)$, $f_{yx}(x,y)$, $f_{yxy}(x,y)$

6. $f(x,y) = \ln(x^2 + y^2) + 2$; $f_x(x,y)$, $f_{xx}(x,y)$, $f_{xy}(x,y)$

7. $f(x,y) = (x+y)^2(xy)$; $f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$

8. $f(x,y,z) = x^2y^3z^4$; $f_x(x,y,z)$, $f_{xz}(x,y,z)$, $f_{zx}(x,y,z)$

9. $z = \ln\sqrt{x^2 + y^2}$; $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial y^2}$

10. $z = \dfrac{\ln(x^2+5)}{y}$; $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial^2 z}{\partial y\,\partial x}$

In Problems 11–16, find the indicated value.

11. If $f(x,y,z) = 7$, find $f_{yxx}(4,3,-2)$.

12. If $f(x,y,z) = z^2(3x^2 - 4xy^3)$, find $f_{xyz}(1,2,3)$.

13. If $f(l, k) = 3l^3k^6 - 2l^2k^7$, find $f_{klk}(2, 1)$.

14. If $f(x, y) = x^3y^2 + x^2y - x^2y^2$, find $f_{xxy}(2, 3)$ and $f_{xyx}(2, 3)$.

15. If $f(x, y) = y^2e^x + \ln(xy)$, find $f_{xyy}(1, 1)$.

16. If $f(x, y) = x^3 - 6xy^2 + x^2 - y^3$, find $f_{xy}(1, -1)$.

17. **Cost Function** Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Find the value of

$$\frac{\partial^2 c}{\partial q_A\, \partial q_B}$$

when $p_A = 25$ and $p_B = 4$.

18. For $f(x, y) = x^4y^4 + 3x^3y^2 - 7x + 4$, show that

$$f_{xyx}(x, y) = f_{xxy}(x, y)$$

19. For $f(x, y) = e^{x^2+xy+y^2}$, show that

$$f_{xy}(x, y) = f_{yx}(x, y)$$

20. For $f(x, y) = e^{xy}$, show that

$$f_{xx}(x, y) + f_{xy}(x, y) + f_{yx}(x, y) + f_{yy}(x, y)$$
$$= f(x, y)((x + y)^2 + 2)$$

21. For $z = \ln(x^2 + y^2)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

[17]22. If $z^3 - x^3 - x^2y - xy^2 - y^3 = 0$, find $\dfrac{\partial^2 z}{\partial x^2}$.

[17]23. If $z^2 - 3x^2 + y^2 = 0$, find $\dfrac{\partial^2 z}{\partial y^2}$.

[17]24. If $2z^2 = x^2 + 2xy + xz$, find $\dfrac{\partial^2 z}{\partial x\, \partial y}$.

Suppose a manufacturer of two related products A and B has a joint-cost function given by

$$c = f(q_A, q_B)$$

where c is the total cost of producing quantities q_A and q_B of A and B, respectively. Furthermore, suppose the demand functions for the products are

$$q_A = g(p_A, p_B) \quad \text{and} \quad q_B = h(p_A, p_B)$$

where p_A and p_B are the prices per unit of A and B, respectively. Since c is a function of q_A and q_B, and since both q_A and q_B are themselves functions of p_A and p_B, c can be viewed as a function of p_A and p_B. (Appropriately, the variables q_A and q_B are called *intermediate variables* of c.) Consequently, we should be able to determine $\partial c/\partial p_A$, the rate of change of total cost with respect to the price of A. One way to do this is to substitute the expressions $g(p_A, p_B)$ and $h(p_A, p_B)$ for q_A and q_B, respectively, into $c = f(q_A, q_B)$. Then c is a function of p_A and p_B, and we can differentiate c with respect to p_A directly. This approach has some drawbacks—especially when f, g, or h is given by a complicated expression. Another way to approach the problem would be to use the chain rule (actually *a* chain rule), which we now state without proof.

Chain Rule

Let $z = f(x, y)$, where both x and y are functions of r and s given by $x = x(r, s)$ and $y = y(r, s)$. If f, x, and y have continuous partial derivatives, then z is a function of r and s, and

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Chain Rule

For a manufacturer of cameras and film, the total cost c of producing q_C cameras and q_F units of film is given by

$$c = 30q_C + 0.015q_Cq_F + q_F + 900$$

The demand functions for the cameras and film are given by

$$q_C = \frac{9000}{p_C\sqrt{p_F}} \quad \text{and} \quad q_F = 2000 - p_C - 400p_F$$

where p_C is the price per camera and p_F is the price per unit of film. Find the rate of change of total cost with respect to the price of the camera when $p_C = 50$ and $p_F = 2$.

The chain rule can be extended. For example, suppose $z = f(v, w, x, y)$ and v, w, x, and y are all functions of r, s, and t. Then, if certain conditions of continuity are assumed, z is a function of r, s, and t, and we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

a. If $w = f(x, y, z) = 3x^2y + xyz - 4y^2z^3$, where

$$x = 2r - 3s \quad y = 6r + s \quad z = r - s$$

determine $\partial w/\partial r$ *and* $\partial w/\partial s$.

b. If $z = \dfrac{x + e^y}{y}$, where $x = rs + se^{rt}$ and $y = 9 + rt$, evaluate $\partial z/\partial s$ when $r = -2$, $s = 5$, and $t = 4$.

a. Determine $\partial y/\partial r$ if $y = x^2 \ln(x^4 + 6)$ and $x = (r + 3s)^6$.

b. Given that $z = e^{xy}, x = r - 4s$, and $y = r - s$, find $\partial z/\partial r$ in terms of r and s.

In Problems 1–12, find the indicated derivatives by using the chain rule.

1. $z = 5x + 3y, x = 2r + 3s, y = r - 2s$; $\partial z/\partial r, \partial z/\partial s$

2. $z = 2x^2 + 3xy + 2y^2, x = r^2 - s^2, y = r^2 + s^2$; $\partial z/\partial r, \partial z/\partial s$

3. $z = e^{x+y}, x = t^2 + 3, y = \sqrt{t^3}$; dz/dt

4. $z = \sqrt{8x + y}, x = t^2 + 3t + 4, y = t^3 + 4$; dz/dt

5. $w = x^2yz + xy^2z + xyz^2, x = e^t, y = te^t, z = t^2e^t$; dw/dt

6. $w = \ln(x^2 + y^2 + z^2), x = 2 - 3t, y = t^2 + 3, z = 4 - t$; dw/dt

7. $z = (x^2 + xy^2)^3, x = r + s + t, y = 2r - 3s + 8t$; $\partial z/\partial t$

8. $z = \sqrt{x^2 + y^2}, x = r^2 + s - t, y = r - s + t$; $\partial z/\partial r$

9. $w = x^2 + xyz + z^2, x = r^2 - s^2, y = rs, z = r^2 + s^2$; $\partial w/\partial s$

10. $w = \ln(xyz), x = r^2s, y = rs, z = rs^2$; $\partial w/\partial r$

11. $y = x^2 - 7x + 5, x = 19rs + 2s^2t^2$; $\partial y/\partial r$

12. $y = 4 - x^2, x = 2r + 3s - 4t$; $\partial y/\partial t$

13. If $z = (4x + 3y)^3$, where $x = r^2s$ and $y = r - 2s$, evaluate $\partial z/\partial r$ when $r = 0$ and $s = 1$.

14. If $z = \sqrt{2x + 3y}$, where $x = 3t + 5$ and $y = t^2 + 2t + 1$, evaluate dz/dt when $t = 1$.

15. If $w = e^{x+y+z}(x^2 + y^2 + z^2)$, where $x = (r - s)^2$, $y = (r + s)^2$, and $z = (s - r)^2$, evaluate $\partial w/\partial s$ when $r = 1$ and $s = 1$.

16. If $y = x/(x - 5)$, where $x = 2t^2 - 3rs - r^2t$, evaluate $\partial y/\partial t$ when $r = 0, s = 2$, and $t = -1$.

17. **Cost Function** Suppose the cost c of producing q_A units of product A and q_B units of product B is given by

$$c = (3q_A^2 + q_B^3 + 4)^{1/3}$$

and the coupled demand functions for the products are given by

$$q_A = 10 - p_A + p_B^2$$

and

$$q_B = 20 + p_A - 11p_B$$

Use a chain rule to evaluate $\dfrac{\partial c}{\partial p_A}$ and $\dfrac{\partial c}{\partial p_B}$ when $p_A = 25$ and $p_B = 4$.

18. Suppose $w = f(x, y)$, where $x = g(t)$ and $y = h(t)$.
(a) State a chain rule that gives dw/dt.
(b) Suppose $h(t) = t$, so that $w = f(x, t)$, where $x = g(t)$. Use part (a) to find dw/dt and simplify your answer.

19. (a) Suppose w is a function of x and y, where both x and y are functions of s and t. State a chain rule that expresses $\partial w/\partial t$ in terms of derivatives of these functions.
(b) Let $w = 2x^2 \ln|3x - 5y|$, where $x = s\sqrt{t^2 + 2}$ and $y = t - 3e^{2-s}$. Use part (a) to evaluate $\partial w/\partial t$ when $s = 1$ and $t = 0$.

20. **Production Function** In considering a production function $P = f(l, k)$, where l is labor input and k is capital input, Fon, Boulier, and Goldfarb[19] assume that $l = Lg(h)$, where L is the number of workers, h is the number of hours per day per worker, and $g(h)$ is a labor effectiveness function. In maximizing profit p given by

$$p = aP - whL$$

where a is the price per unit of output and w is the hourly wage per worker, Fon, Boulier, and Goldfarb determine $\partial p/\partial L$ and $\partial p/\partial h$. Assume that k is independent of L and h, and determine these partial derivatives.

Maxima and Minima for Functions of Two Variables

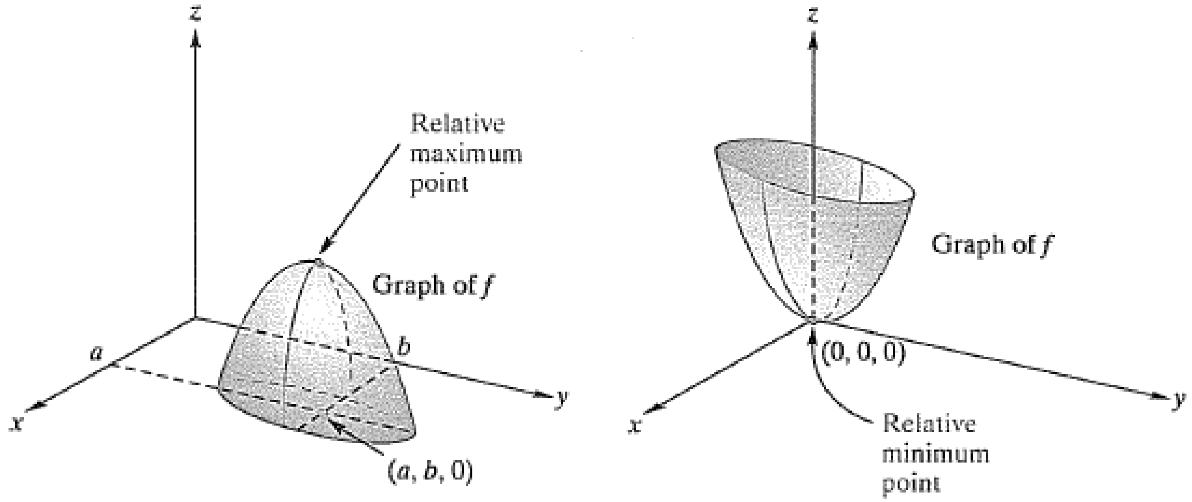
Definition

A function $z = f(x, y)$ is said to have a ***relative maximum*** at the point (a, b) if, for all points (x, y) in the plane that are sufficiently close to (a, b), we have

$$f(a, b) \geq f(x, y) \qquad (1)$$

For a ***relative minimum***, we replace $\geq$ by $\leq$ in Equation (1).

To say that $z = f(x, y)$ has a relative maximum at (a, b) means, geometrically, that the point $(a, b, f(a, b))$ on the graph of f is higher than (or is as high as) all other points on the surface that are "near" $(a, b, f(a, b))$. In Figure 17.4(a), f has a relative maximum at (a, b). Similarly, the function f in Figure 17.4(b) has a relative minimum when $x = y = 0$, which corresponds to a *low* point on the surface.



(a) (b)

Recall that in locating extrema for a function $y = f(x)$ of one variable, we examine those values of x in the domain of f for which $f'(x) = 0$ or $f'(x)$ does not exist. For functions of two (or more) variables, a similar procedure is followed. However, for the functions that concern us, extrema will not occur where a derivative does not exist, and such situations will be excluded from our consideration.

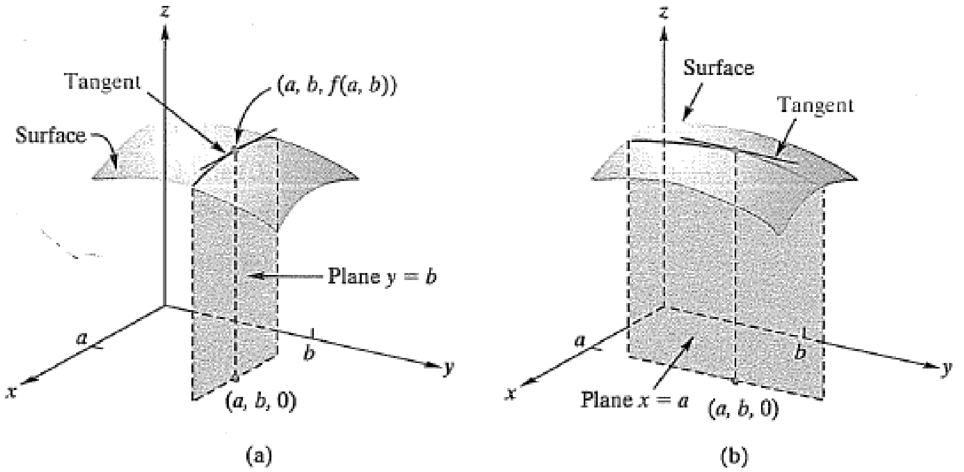


FIGURE 17.5 At relative extremum, $f_x(x, y) = 0$ and $f_y(x, y) = 0$.

Suppose $z = f(x, y)$ has a relative maximum at (a, b), as indicated in Figure 17.5(a). Then the curve where the plane $y = b$ intersects the surface must have a relative maximum when $x = a$. Hence, the slope of the tangent line to the surface in the x-direction must be 0 at (a, b). Equivalently, $f_x(x, y) = 0$ at (a, b). Similarly, on the curve where the plane $x = a$ intersects the surface [Figure 17.5(b)], there must be a relative maximum when $y = b$. Thus, in the y-direction, the slope of the tangent to the surface must be 0 at (a, b). Equivalently, $f_y(x, y) = 0$ at (a, b). Since a similar discussion applies to a relative minimum, we can combine these results as follows:

Rule 1

If $z = f(x, y)$ has a relative maximum or minimum at (a, b), and if both f_x and f_y are defined for all points close to (a, b), it is necessary that (a, b) be a solution of the system

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

A point (a, b) for which $f_x(a, b) = f_y(a, b) = 0$ is called a **critical point** of f. Thus, from Rule 1, we infer that, to locate relative extrema for a function, we should examine its critical points.

Two additional comments are in order: First, Rule 1, as well as the notion of a critical point, can be extended to functions of more than two variables. For example, to locate possible extrema for $w = f(x, y, z)$, we would examine those points for which $w_x = w_y = w_z = 0$. Second, for a function whose domain is restricted, a thorough examination for absolute extrema would include a consideration of boundary points.

Find the critical points of the following functions.

a. $f(x, y) = 2x^2 + y^2 - 2xy + 5x - 3y + 1.$

b. $f(l, k) = l^3 + k^3 - lk.$

c. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 100).$

Find the critical points of

$$f(x, y) = x^2 - 4x + 2y^2 + 4y + 7$$

Rule 2 Second-Derivative Test for Functions of Two Variables

Suppose $z = f(x, y)$ has continuous partial derivatives f_{xx}, f_{yy}, and f_{xy} at all points (x, y) near a critical point (a, b). Let D be the function defined by

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

Then

1. if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b);
2. if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b);
3. if $D(a, b) < 0$, then f has a *saddle point* at (a, b) (see Example 4);
4. if $D(a, b) = 0$, then no conclusion about an extremum at (a, b) can be drawn, and further analysis is required.

We remark that when $D(a, b) > 0$, the sign of $f_{xx}(a, b)$ is necessarily the same as the sign of $f_{yy}(a, b)$. Thus, when $D(a, b) > 0$ we can test either $f_{xx}(a, b)$ or $f_{yy}(a, b)$, whichever is easiest, to make the determination required in parts 1 and 2 of the second derivative test.

Examine $f(x, y) = x^3 + y^3 - xy$ for relative maxima or minima by using the second-derivative test.

Examine $f(x, y) = y^2 - x^2$ for relative extrema.

Solution: Solving

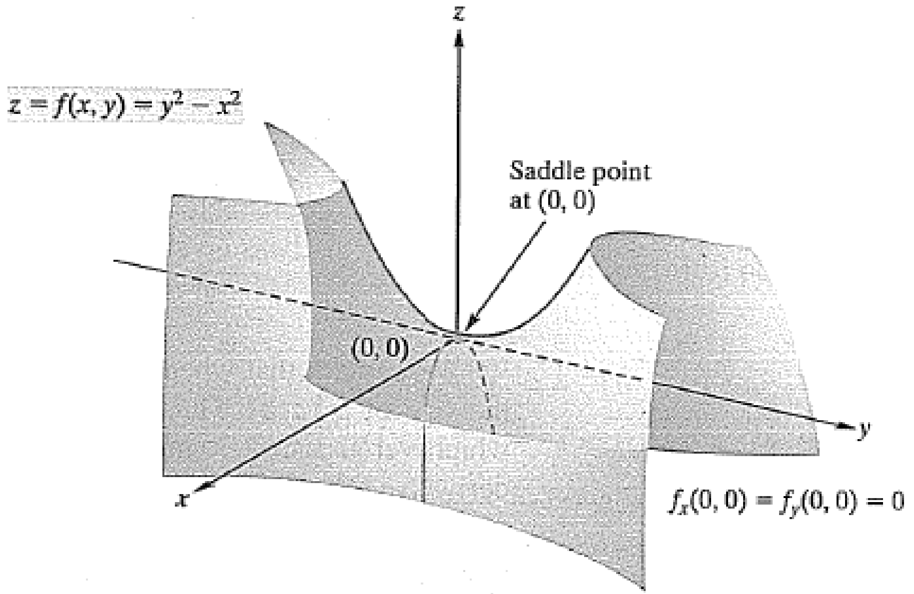
$$f_x(x, y) = -2x = 0 \quad \text{and} \quad f_y(x, y) = 2y = 0$$

we get the critical point (0, 0). Now we apply the second-derivative test. At (0, 0), and indeed at any point,

$$f_{xx}(x, y) = -2 \quad f_{yy}(x, y) = 2 \quad f_{xy}(x, y) = 0$$

Because $D(0, 0) = (-2)(2) - (0)^2 = -4 < 0$, no relative extremum exists at (0, 0). A sketch of $z = f(x, y) = y^2 - x^2$ appears in Figure 17.6. Note that, for the surface curve cut by the plane $y = 0$, there is a *maximum* at (0, 0); but for the surface curve cut by the plane $x = 0$, there is a *minimum* at (0, 0). Thus, on the *surface,* no relative extremum can exist at the origin, although (0, 0) is a critical point. Around the origin the curve is saddle shaped, and (0, 0) is called a *saddle point* of f.

Now Work Problem 11 ◁



Examine $f(x, y) = x^4 + (x - y)^4$ for relative extrema.

Let P be a production function given by

$$P = f(l, k) = 0.54l^2 - 0.02l^3 + 1.89k^2 - 0.09k^3$$

where l and k are the amounts of labor and capital, respectively, and P is the quantity of output produced. Find the values of l and k that maximize P.

A candy company produces two types of candy, A and B, for which the average costs of production are constant at \$2 and \$3 per pound, respectively. The quantities q_A, q_B (in pounds) of A and B that can be sold each week are given by the joint-demand functions

$$q_A = 400(p_B - p_A)$$

and

$$q_B = 400(9 + p_A - 2p_B)$$

where p_A and p_B are the selling prices (in dollars per pound) of A and B, respectively. Determine the selling prices that will maximize the company's profit P.

Profit Maximization for a Monopolist

Suppose a monopolist is practicing price discrimination by selling the same product in two separate markets at different prices. Let q_A be the number of units sold in market A, where the demand function is $p_A = f(q_A)$, and let q_B be the number of units sold in market B, where the demand function is $p_B = g(q_B)$. Then the revenue functions for the two markets are

$$r_A = q_A f(q_A) \quad \text{and} \quad r_B = q_B g(q_B)$$

Assume that all units are produced at one plant, and let the cost function for producing $q = q_A + q_B$ units be $c = c(q)$. Keep in mind that r_A is a function of q_A and r_B is a function of q_B. The monopolist's profit P is

$$P = r_A + r_B - c$$

To maximize P with respect to outputs q_A and q_B, we set its partial derivatives equal to zero. To begin with,

$$\frac{\partial P}{\partial q_A} = \frac{dr_A}{dq_A} + 0 - \frac{\partial c}{\partial q_A}$$

$$= \frac{dr_A}{dq_A} - \frac{dc}{dq}\frac{\partial q}{\partial q_A} = 0 \qquad \text{chain rule}$$

Because

$$\frac{\partial q}{\partial q_A} = \frac{\partial}{\partial q_A}(q_A + q_B) = 1$$

we have

$$\frac{\partial P}{\partial q_A} = \frac{dr_A}{dq_A} - \frac{dc}{dq} = 0 \tag{6}$$

Similarly,

$$\frac{\partial P}{\partial q_B} = \frac{dr_B}{dq_B} - \frac{dc}{dq} = 0 \tag{7}$$

From Equations (6) and (7), we get

$$\frac{dr_A}{dq_A} = \frac{dc}{dq} = \frac{dr_B}{dq_B}$$

But dr_A/dq_A and dr_B/dq_B are marginal revenues, and dc/dq is marginal cost. Hence, to maximize profit, it is necessary to charge prices (and distribute output) so that the marginal revenues in both markets will be the same and, loosely speaking, will also be equal to the cost of the last unit produced in the plant.

In Problems 1–6, find the critical points of the functions.

1. $f(x, y) = x^2 - 3y^2 - 8x + 9y + 3xy$

2. $f(x, y) = x^2 + 4y^2 - 6x + 16y$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

4. $f(x, y) = xy - x + y$

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

In Problems 7–20, find the critical points of the functions. For each critical point, determine, by the second-derivative test, whether it corresponds to a relative maximum, to a relative minimum, or to neither, or whether the test gives no information.

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

9. $f(x, y) = y - y^2 - 3x - 6x^2$

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

13. $f(x, y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$

14. $f(x, y) = x^2 + y^2 - xy + x^3$

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

16. $f(l, k) = l^2 + 4k^2 - 4lk$

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

18. $f(x, y) = (x - 3)(y - 3)(x + y - 3)$

19. $f(x, y) = (y^2 - 4)(e^x - 1)$

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

21. Maximizing Output Suppose

$$P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$$

is a production function for a firm. Find the quantities of inputs l and k that maximize output P.

22. Maximizing Output In a certain office, computers C and D are utilized for c and d hours, respectively. If daily output Q is a function of c and d, namely,

$$Q = 18c + 20d - 2c^2 - 4d^2 - cd$$

find the values of c and d that maximize Q.

In Problems 23–35, unless otherwise indicated, the variables p_A and p_B denote selling prices of products A and B, respectively. Similarly, q_A and q_B denote quantities of A and B that are produced and sold during some time period. In all cases, the variables employed will be assumed to be units of output, input, money, and so on.

23. Profit A candy company produces two varieties of candy, A and B, for which the constant average costs of production are 60 and 70 (cents per lb), respectively. The demand functions for A and B are given by

$$q_A = 5(p_B - p_A) \quad \text{and} \quad q_B = 500 + 5(p_A - 2p_B)$$

Find the selling prices p_A and p_B that maximize the company's profit.

24. Profit Repeat Problem 23 if the constant costs of production of A and B are a and b (cents per lb), respectively.

25. Price Discrimination Suppose a monopolist is practicing price discrimination in the sale of a product by charging different prices in two separate markets. In market A the demand function is

$$p_A = 100 - q_A$$

and in B it is

$$p_B = 84 - q_B$$

where q_A and q_B are the quantities sold per week in A and B, and p_A and p_B are the respective prices per unit. If the monopolist's cost function is

$$c = 600 + 4(q_A + q_B)$$

how much should be sold in each market to maximize profit? What selling prices give this maximum profit? Find the maximum profit.

26. Profit A monopolist sells two competitive products, A and B, for which the demand functions are

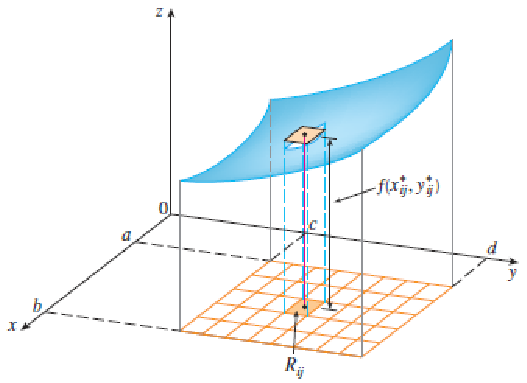
$$q_A = 16 - p_A + p_B \quad \text{and} \quad q_B = 24 + 2p_A - 4p_B$$

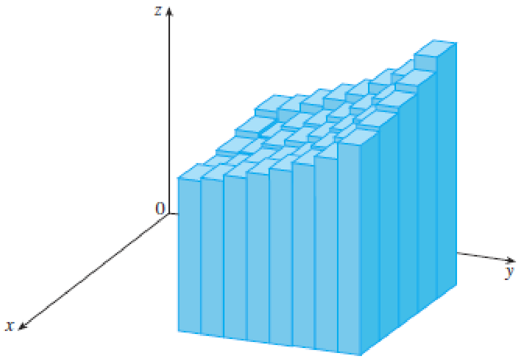
If the constant average cost of producing a unit of A is 2 and a unit of B is 4, how many units of A and B should be sold to maximize the monopolist's profit?

27. Profit For products A and B, the joint-cost function for a manufacturer is

$$c = \frac{3}{2}q_A^2 + 3q_B^2$$

and the demand functions are $p_A = 60 - q_A^2$ and $p_B = 72 - 2q_B^2$. Find the level of production that maximizes profit.





If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Multiple Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\boxed{1} \qquad \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\boxed{2} \qquad \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\boxed{3} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$. Notice that in both Equations 2 and 3 we work *from the inside out*.

EXAMPLE 1 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Find $\int_{-1}^{1} \int_{0}^{1-x} (2x+1)\, dy\, dx.$

Find $\int_{1}^{\ln 2} \int_{e^y}^{2} dx\, dy$.

Find $\int_0^1 \int_0^x \int_0^{x-y} x \, dz \, dy \, dx$.

In Problems 1–22, evaluate the multiple integrals.

1. $\int_0^3 \int_0^4 x\,dy\,dx$

2. $\int_1^4 \int_0^3 y\,dy\,dx$

3. $\int_0^1 \int_0^1 xy\,dx\,dy$

4. $\int_0^1 \int_0^1 x^2y^2\,dy\,dx$

5. $\int_1^3 \int_1^2 (x^2 - y)\,dx\,dy$

6. $\int_{-2}^3 \int_0^2 (y^2 - 2xy)\,dy\,dx$

7. $\int_0^1 \int_0^2 (x + y)\,dy\,dx$

8. $\int_0^3 \int_0^x (x^2 + y^2)\,dy\,dx$

9. $\int_2^3 \int_0^{2x} y\,dy\,dx$

10. $\int_1^2 \int_0^{x-1} 2y\,dy\,dx$

11. $\int_0^1 \int_{3x}^{x^2} 14x^2y\,dy\,dx$

12. $\int_0^2 \int_0^{x^2} xy\,dy\,dx$

13. $\int_0^3 \int_0^{\sqrt{9-x^2}} y\,dy\,dx$

14. $\int_0^1 \int_{y^2}^{y} x\,dx\,dy$

15. $\int_{-1}^1 \int_x^{1-x} 3(x + y)\,dy\,dx$

16. $\int_0^3 \int_{y^2}^{3y} 5x\,dx\,dy$

17. $\int_0^1 \int_0^y e^{x+y}\,dx\,dy$

18. $\int_0^1 \int_0^1 e^{y-x}\,dx\,dy$

19. $\int_0^1 \int_0^2 \int_0^3 xy^2z^3\,dx\,dy\,dz$

20. $\int_0^1 \int_0^x \int_0^{x+y} x^2\,dz\,dy\,dx$

21. $\int_0^1 \int_{x^2}^x \int_0^{xy} dz\,dy\,dx$

22. $\int_1^e \int_{\ln x}^x \int_0^y dz\,dy\,dx$

4 **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

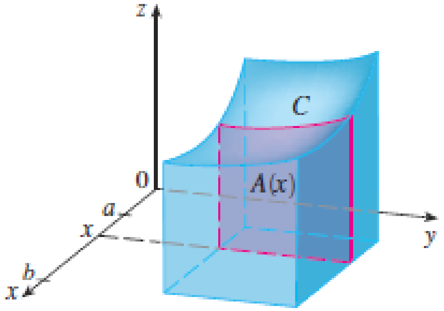


FIGURE 1

TEC Visual 15.2 illustrates Fubini's Theorem by showing an animation of Figures 1 and 2.

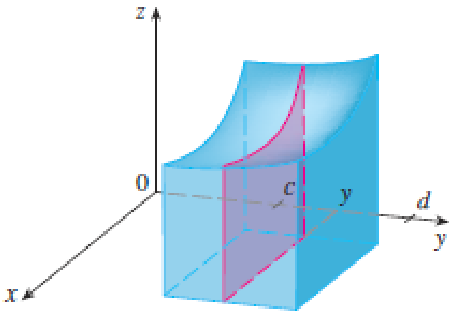


FIGURE 2

The proof of Fubini's Theorem is too difficult to include in this book, but we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geqslant 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y)\, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 6, namely,

$$V = \int_a^b A(x)\, dx$$

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 1 you can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leqslant y \leqslant d$. Therefore

$$A(x) = \int_c^d f(x, y)\, dy$$

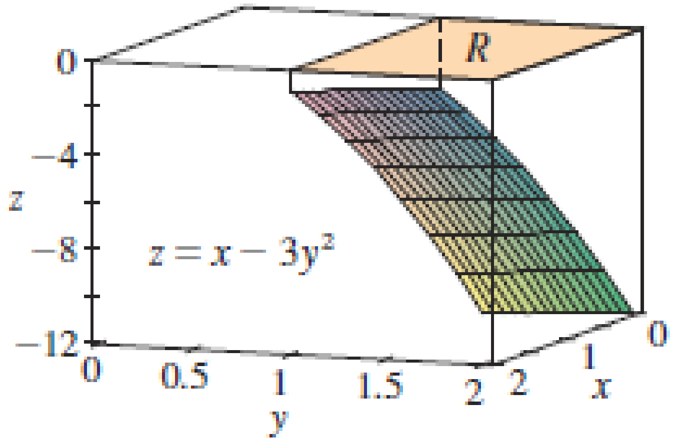
and we have

$$\iint_R f(x, y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

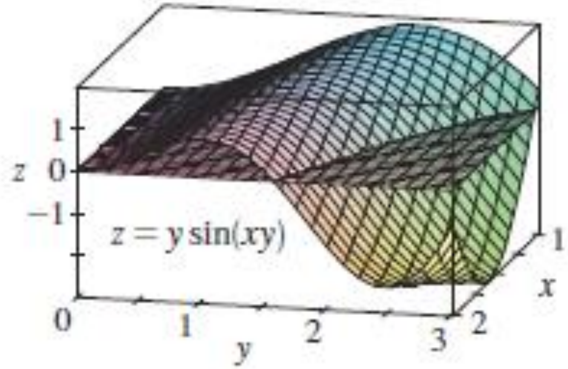
A similar argument, using cross-sections perpendicular to the y-axis as in Figure 2, shows that

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

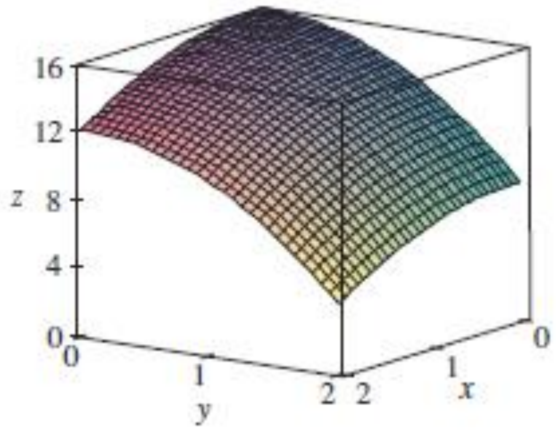
V **EXAMPLE 2** Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$. (Compare with Example 3 in Section 15.1.)



V EXAMPLE 3 Evaluate $\iint_R y\sin(xy)\,dA$, where $R = [1, 2] \times [0, \pi]$.



V EXAMPLE 4 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.



$$\boxed{5} \qquad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

EXAMPLE 5 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 5,

15.2 Exercises

1–2 Find $\int_0^3 f(x, y)\,dx$ and $\int_0^4 f(x, y)\,dy$.

1. $f(x, y) = 2x + 3x^2y$

2. $f(x, y) = \dfrac{y}{x+2}$

3–12 Calculate the iterated integral.

3. $\int_1^3 \int_0^1 (1 + 4xy)\,dx\,dy$

4. $\int_2^4 \int_{-1}^1 (x^2 + y^2)\,dy\,dx$

5. $\int_0^2 \int_0^{\pi/2} x \sin y\,dy\,dx$

6. $\int_1^4 \int_0^2 (x + \sqrt{y})\,dx\,dy$

7. $\int_0^2 \int_0^1 (2x + y)^8\,dx\,dy$

8. $\int_0^1 \int_1^2 \dfrac{xe^x}{y}\,dy\,dx$

9. $\int_1^4 \int_1^2 \left(\dfrac{x}{y} + \dfrac{y}{x}\right)dy\,dx$

10. $\int_1^2 \int_0^1 (x + y)^{-2}\,dx\,dy$

11. $\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,dx\,dy$

12. $\int_0^1 \int_0^1 \dfrac{xy}{\sqrt{x^2 + y^2 + 1}}\,dy\,dx$

13–20 Calculate the double integral.

13. $\iint_R (6x^2y^3 - 5y^4)\,dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 1\}$

14. $\iint_R \cos(x + 2y)\,dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi/2\}$

15. $\iint_R \dfrac{xy^2}{x^2 + 1}\,dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ -3 \leqslant y \leqslant 3\}$

16. $\iint_R \dfrac{1 + x^2}{1 + y^2}\,dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$

17. $\iint_R x \sin(x + y)\,dA$, $R = [0, \pi/6] \times [0, \pi/3]$

18. $\iint_R \dfrac{x}{1 + xy}\,dA$, $R = [0, 1] \times [0, 1]$

19. $\iint_R xye^{x^2y}\,dA$, $R = [0, 1] \times [0, 2]$

20. $\iint_R \dfrac{x}{x^2 + y^2}\,dA$, $R = [1, 2] \times [0, 1]$

21–22 Sketch the solid whose volume is given by the iterated integral.

21. $\int_0^1 \int_0^1 (4 - x - 2y)\,dx\,dy$

22. $\int_0^1 \int_0^1 (2 - x^2 - y^2)\,dy\,dx$

23. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ -2 \leqslant y \leqslant 3\}$.

24. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the square $R = [-1, 1] \times [0, 2]$.

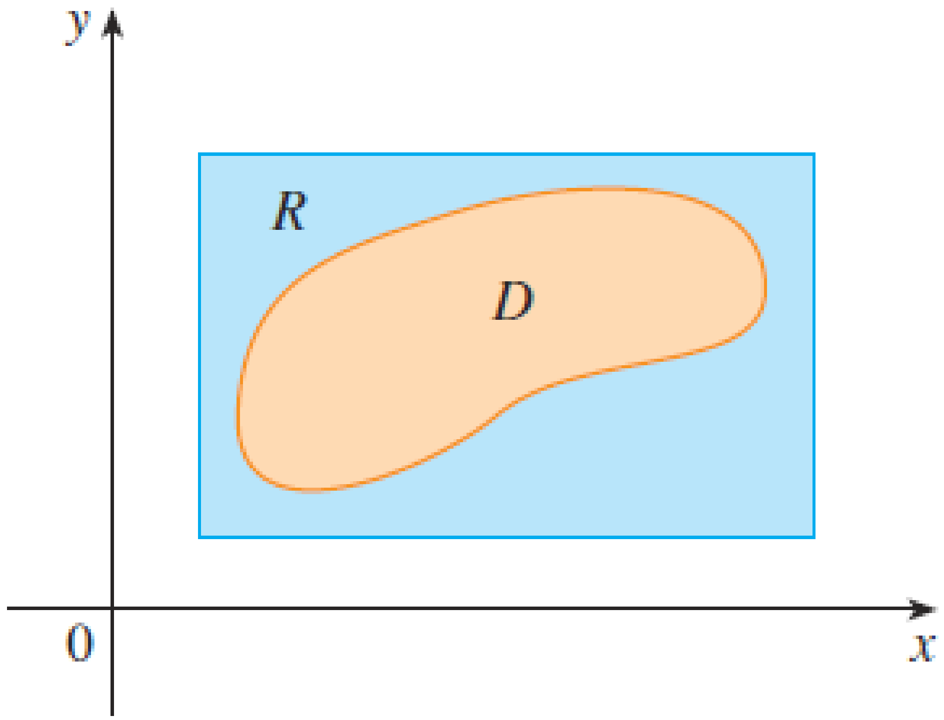
25. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

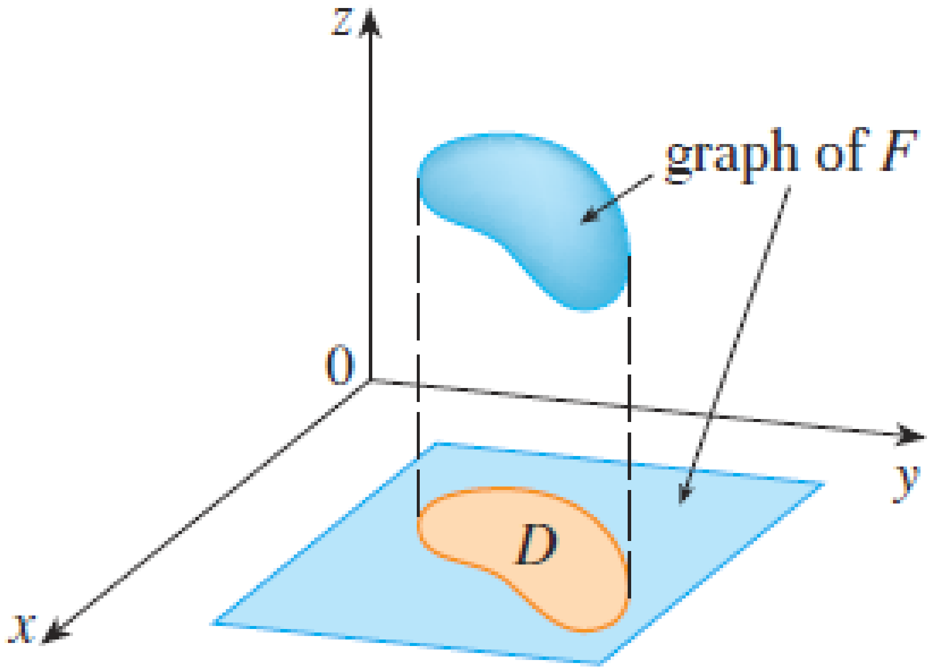
26. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

27. Find the volume of the solid bounded by the surface $z = x\sqrt{x^2 + y}$ and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, and $z = 0$.

28. Find the volume of the solid bounded by the elliptic paraboloid $z = 1 + (x - 1)^2 + 4y^2$, the planes $x = 3$ and $y = 2$, and the coordinate planes.

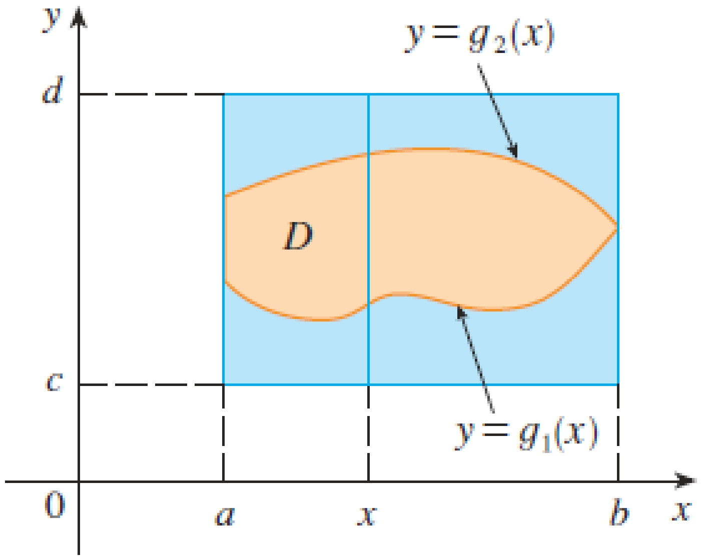
29. Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

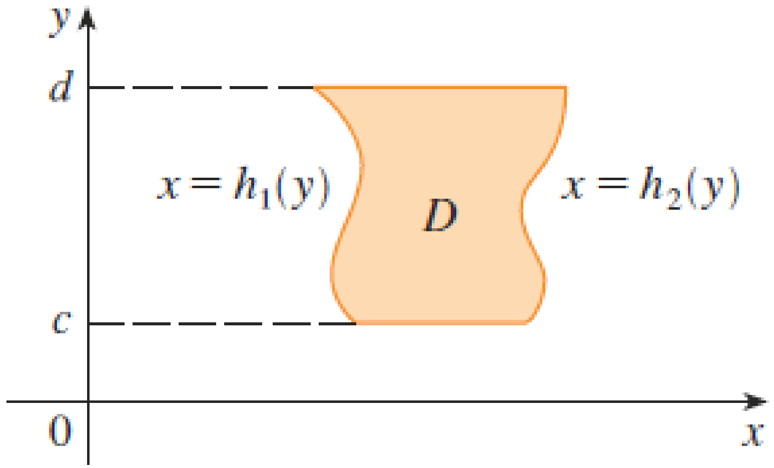




Double Integrals over General Regions

Type 1 and Type 2 Region





3 If f is continuous on a type I region D such that

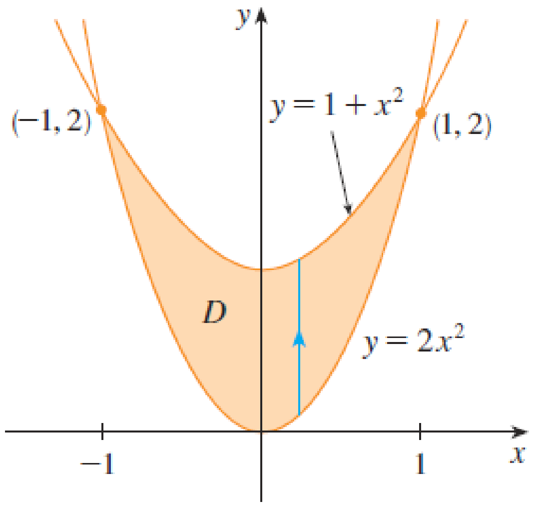
$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

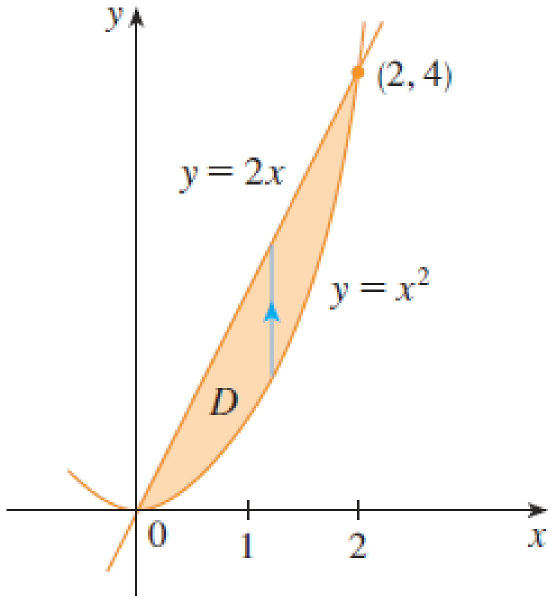
5
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

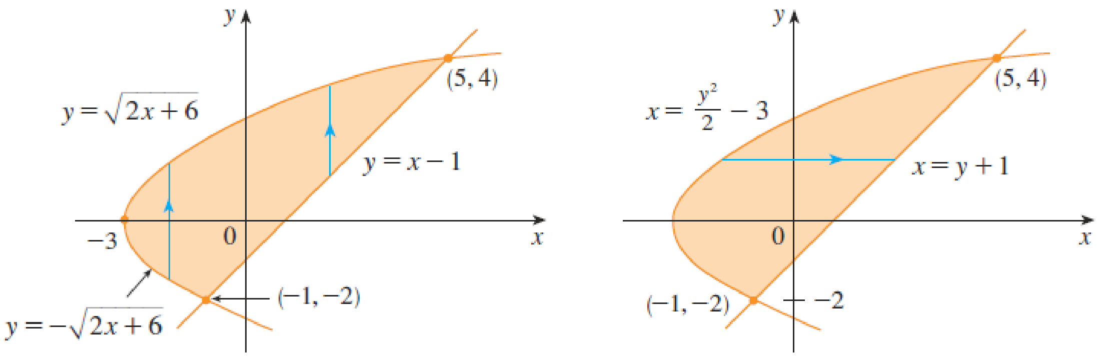
V **EXAMPLE 1** Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



V EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



(a) D as a type I region

(b) D as a type II region

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region D follow immediately from Definition 2 in this section and Properties 7, 8, and 9 in Section 15.1.

$$\boxed{6} \qquad \iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

$$\boxed{7} \qquad \iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

$$\boxed{8} \qquad \iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

$$\boxed{9} \qquad \iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA$$

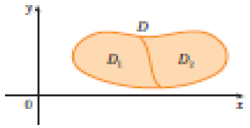


FIGURE 17

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 55 and 56.)

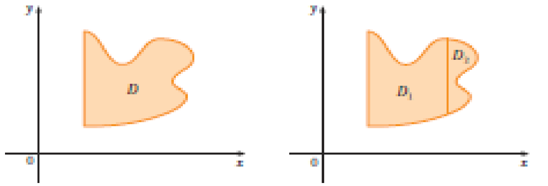


FIGURE 18 (a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

$$\boxed{10} \qquad \iint_D 1\, dA = A(D)$$

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 61.)

$\boxed{11}$ If $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$m A(D) \leq \iint_D f(x, y)\, dA \leq M A(D)$$

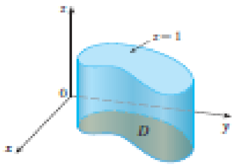


FIGURE 19
Cylinder with base D and height 1

15.3 Exercises

1–6 Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\, dx\, dy$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\, dr\, d\theta$

6. $\int_0^1 \int_0^v \sqrt{1 - v^2}\, du\, dv$

7–18 Evaluate the double integral.

7. $\iint_D x^3y^2\, dA$, $D = \{(x, y) \mid 0 \leq x \leq 2,\ -x \leq y \leq x\}$

8. $\iint_D \frac{4y}{x^3 + 2}\, dA$, $D = \{(x, y) \mid 1 \leq x \leq 2,\ 0 \leq y \leq 2x\}$

9. $\iint_D \frac{2y}{x^2 + 1}\, dA$, $D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$

10. $\iint_D e^{y^2}\, dA$, $D = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$

11. $\iint_D e^{x/y}\, dA$, $D = \{(x, y) \mid 1 \leq y \leq 2,\ y \leq x \leq y^3\}$

12. $\iint_D x\sqrt{y^2 - x^2}\, dA$, $D = \{(x, y) \mid 0 \leq y \leq 1,\ 0 \leq x \leq y\}$

13. $\iint_D x \cos y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

14. $\iint_D (x + y)\, dA$, D is bounded by $y = \sqrt{x}$ and $y = x^2$

15. $\iint_D y^3\, dA$,
D is the triangular region with vertices (0, 2), (1, 1), and (3, 2)

16. $\iint_D xy^2\, dA$, D is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

17. $\iint_D (2x - y)\, dA$,
D is bounded by the circle with center the origin and radius 2

18. $\iint_D 2xy\, dA$, D is the triangular region with vertices (0, 0), (1, 2), and (0, 3)

19–28 Find the volume of the given solid.

19. Under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$

20. Under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$

21. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

22. Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, $z = 0$

23. Bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$

24. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

25. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

26. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

27. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

28. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

29. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

30. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

31–32 Find the volume of the solid by subtracting two volumes.

31. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

32. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

CAS **33–36** Use a computer algebra system to find the exact volume of the solid.

33. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geq 0$

34. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

35. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

36. Enclosed by $z = x^2 + y^2$ and $z = 2y$

37–42 Sketch the region of integration and change the order of integration.

37. $\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

38. $\int_0^1 \int_{4x}^4 f(x, y)\, dy\, dx$