Discrete Mathematics

Mathematical Induction

Proving “For All . . . ”

The Well-Ordering Principle

A set S is “well-ordered” if every non-empty subset of S has a least element.

Example: Well-ordered sets

- Set of natural numbers
- Set of nonnegative integers
- Set of positive integers

Prove the following statement by contradiction:

For all positive integers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof: Assume that this statement is false for at least one positive integral value of n. Let S be the set of all positive integers n for which the statement above is false. Using our assumption, S is non-empty. By the Well-Ordering Principle, S must contain a smallest element. Let this smallest element be k.

Prove the following statement by contradiction:

For all positive integers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof (Cont.):

We know that k can NOT be 1 because the statement above is true for n=1. (To verify, check that the LHS = 1 and the RHS = 1(2)/2 = 1.)

Thus, k > 1. Furthermore, it follows that k-1>0. Since k-1 is a positive integer less than k, we can deduce that the formula is true for k-1. This means that

$$\sum_{i=1}^{k-1} i = \frac{(k-1)((k-1)+1)}{2}$$

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Proof (Cont.): Now, consider computing the following sum:

$$\sum_{i=1}^{k} i = \sum_{i=1}^{k-1} i + k \quad = \frac{k(k+1)}{2}$$

But, this contradicts our assumption that k was the smallest positive integer for which the formula was false. Thus, our assumption, that such an integer exists is false, and the formula must be true for all positive integral values of n. ■

We just prove the statement:

For all positive integers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

It is of the form $\forall n \in S\ P(n)$ where S is an infinite set of positive integers and P(n) :: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Induction:

1. If a statement $P(n_0)$ is true for some nonnegative integer say $n_0=1$
2. Suppose that we are able to prove that if P(k) is true for $k \geq n_0$, then P(k+1) is also true

$$P(k) \Rightarrow P(k+1)$$

It follows from these two statement that P(n) is true for all $n \geq n_0$, that is

$$\forall n \geq n_0\ P(n)$$

Prove the following statement:

For all positive integers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Induction Proof:

Base Case: Show P(1): $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$ => 1 = 1 => P(1) is true

Inductive Step: P(k) ⇒ P(k+1)

Assume P(k): $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Show P(k+1): $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Inductive Step: $P(k) \Rightarrow P(k+1)$

Assume P(k): $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Show P(k+1): $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

Induction

$$(P(n_0) \wedge (\forall\ k \geq n_0\ P(k) \rightarrow P(k+1)) \rightarrow \forall n \geq n_0\ P(n)$$

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2. Suppose that we are able to prove that if $P(k)$ is true for $k \geq n_0$, then $P(k+1)$ is also true

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Proof by contradiction:

Assume (1) $P(n_0)$ and
(2) $\forall k\ P(k) \rightarrow P(k+1)$ and
(3) $\neg\ \forall n \geq n_0\ P(n)$.

From (3) We get $\exists n\ \neg P(n)$.

Let $T = \{\ n : \neg P(n)\ \}$.

Since S is well ordered, T (subset of S) has a least element. Call it k.

P(k) is false because it's in S.

$k \neq n_0$ because $P(n_0)$ is true.

P(k-1) is true because k is the least element in T.

But by (2), P(k) is true. Thus, we have the contradiction P(k) is false and P(k) is true. ■

$$(P(n_0) \wedge (\forall\, k \geq n_0\; P(k) \rightarrow P(k+1)) \rightarrow \forall n \geq n_0\; P(n)$$

Induction for $n_0 = 0$

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Summations

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \qquad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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Therefore, $P(k + 1)$ is true, completing the induction. ■

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- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.

The Counterfeit Coin Problem

If we have n weighings on the scale, what is the largest number of coins out of which we can find the counterfeit?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One coin**, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

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Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

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Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is;
  - that $P(0)$ is true; and that
  - whenever $P(k)$ is true, $P(k+1)$ is true,

  the proof is usually valid.

Variations on Induction: ***Starting Later***

Induction Starting at 0

- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

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④ **Inductive step:**

We want to prove that $P(k+1)$ is true, i.e., $3^{(k+1)} \geq (k+1)^2 + 3 = k^2 + 2k + 4$. Note that $3^{(k+1)} = 3(3^k) \geq 3(k^2 + 3)$ by the inductive hypothesis. From this we have $3(k^2 + 3) = 2k^2 + k^2 + 9 \geq k^2 + 2k + 4 = (k+1)^2 + 3$ since $k \geq 2$. Therefore $P(k+1)$ is true.

Example: prove $3^n \geq n^2 + 3$ for all $n \geq 2$

① **Let $P(n)$ be $3^n \geq n^2 + 3$.**

We will show that $P(n)$ is true for every integer $n \geq 2$ by induction.

② **Base case ($n = 2$):**

$3^2 = 9 \geq 7 = 4 + 3 = 2^2 + 3$ so $P(2)$ is true.

③ **Inductive hypothesis:**

Suppose that $P(k)$ is true for an arbitrary integer $k \geq 2$.

④ **Inductive step:**

We want to prove that $P(k + 1)$ is true, i.e., $3^{(k+1)} \geq (k + 1)^2 + 3 = k^2 + 2k + 4$. Note that $3^{(k+1)} = 3(3^k) \geq 3(k^2 + 3)$ by the inductive hypothesis. From this we have $3(k^2 + 3) = 2k^2 + k^2 + 9 \geq k^2 + 2k + 4 = (k + 1)^2 + 3$ since $k \geq 2$. Therefore $P(k + 1)$ is true.

⑤ **The result follows for all $n \geq 2$ by induction.** ■

Theorem: If n is a natural number and $n \geq 5$, then $n^2 < 2^n$.

Proof: Let $P(n)$ be the statement "$n^2 < 2^n$." We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$ where $n \geq 5$, from which the theorem follows.

As our base case, we prove $P(5)$, that $5^2 < 2^5$. Note that $5^2 = 25$ and that $2^5 = 32$, so $5^2 < 2^5$.

For the inductive step, assume that for some $k \geq 5$ that $P(k)$ is true, meaning that $k^2 < 2^k$. We will prove $P(k+1)$, that $(k+1)^2 < 2^{k+1}$.

First, recall that $(k + 1)^2 = k^2 + 2k + 1$. Starting with this equality and using the fact that $k \geq 5$ (that is, that $5 \leq k$), we derive the following:

$$\begin{aligned}
(k + 1)^2 &= k^2 + 2k + 1 \\
&< k^2 + 2k + k && \textit{(since } 1 < 5 \leq k\textit{)} \\
&= k^2 + 3k \\
&< k^2 + k \cdot k && \textit{(since } 3 < 5 \leq k\textit{)} \\
&= k^2 + k^2 \\
&= 2k^2
\end{aligned}$$

So we see that $(k + 1)^2 < 2k^2$. Now, by our inductive hypothesis, we know $k^2 < 2^k$. Some further algebra gives us the following:

$$\begin{aligned}
(k + 1)^2 &< 2k^2 && \textit{(from above)} \\
&< 2(2^k) && \textit{(from the inductive hypothesis)} \\
&= 2^{k+1}
\end{aligned}$$

So $(k+1)^2 < 2^{k+1}$. Therefore, $P(k+1)$ is true, completing the induction. ■

Complete Induction

Let P be some predicate. The ***principle of complete induction*** states that if

$P(0)$ is true

If it starts true…

and

…and it stays true…

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

Mathematical Induction

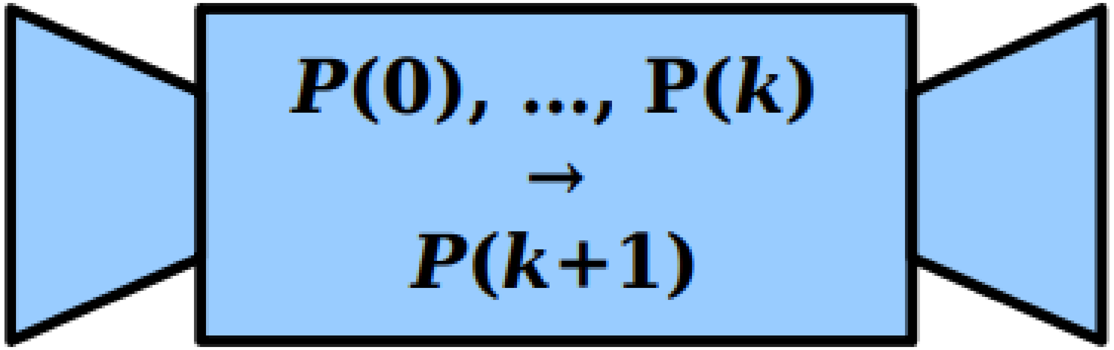
- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Complete Induction

(Strong Induction)

- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that **$P(0)$, $P(1)$, $P(2)$, …, and $P(k)$** are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

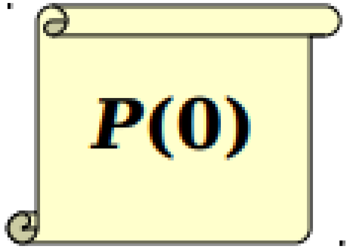
Intuiting Complete Induction



$P(0), \ldots, \mathrm{P}(k)$

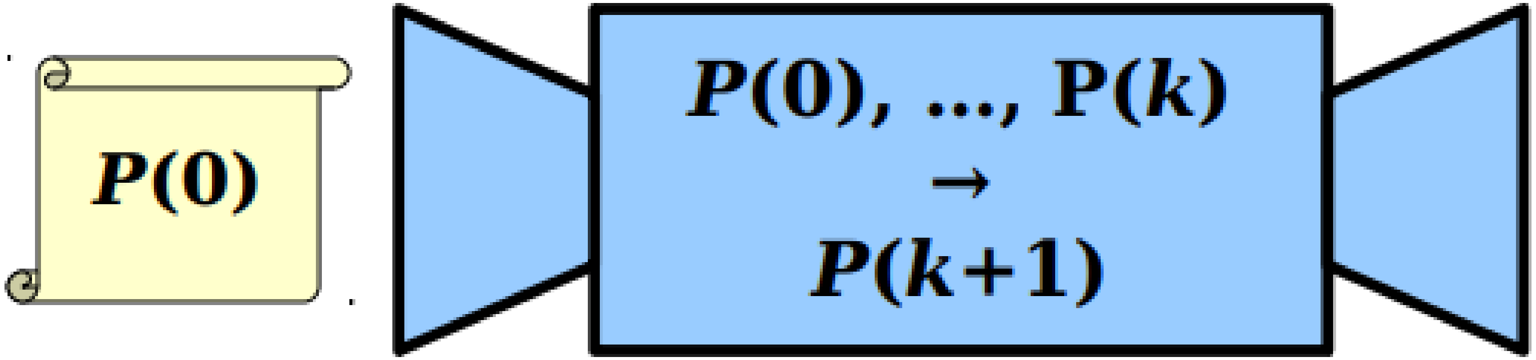
$\rightarrow$

$P(k+1)$



$P(0)$

Intuiting Complete Induction



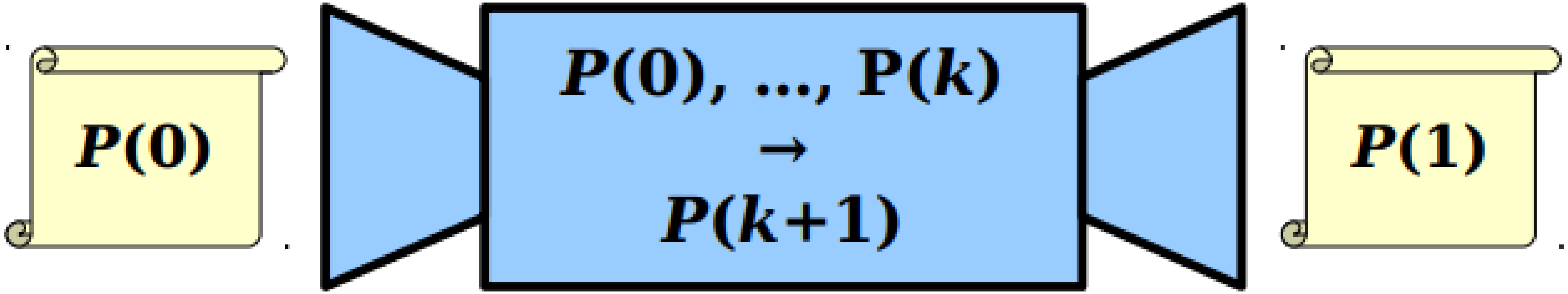
$P(0)$

$P(0), \ldots, \mathrm{P}(k)$

$\rightarrow$

$P(k+1)$

Intuiting Complete Induction



$P(0)$

$P(0), \ldots, \mathrm{P}(k)$

$\rightarrow$

$P(k+1)$

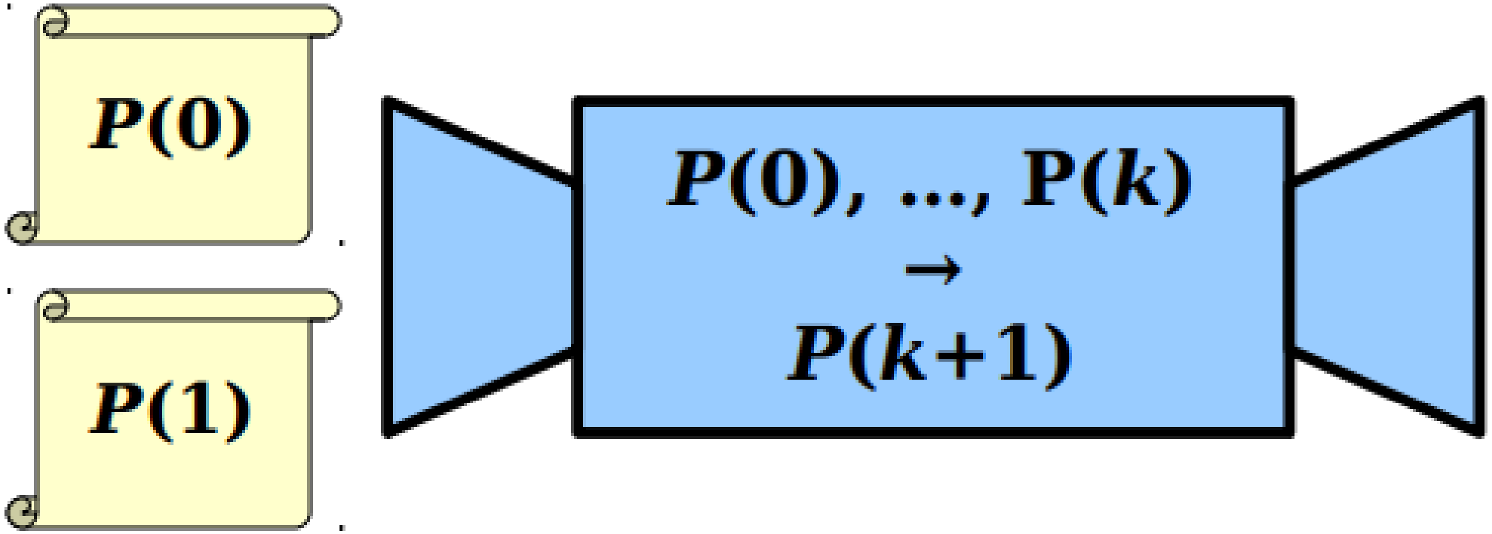
$P(1)$

Intuiting Complete Induction

$$P(0), \ldots, \mathrm{P}(k) \rightarrow P(k+1)$$

$P(0)$ $P(1)$

Intuiting Complete Induction



$P(0)$

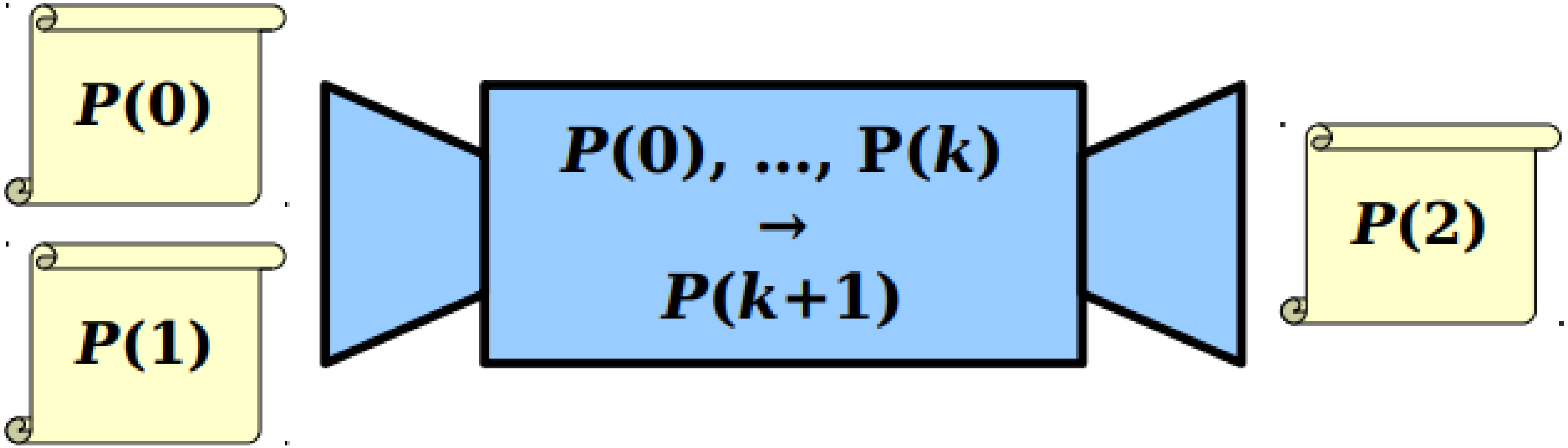
$P(1)$

$P(0), \dots, \mathrm{P}(k)$

$\rightarrow$

$P(k+1)$

Intuiting Complete Induction



$P(0)$

$P(1)$

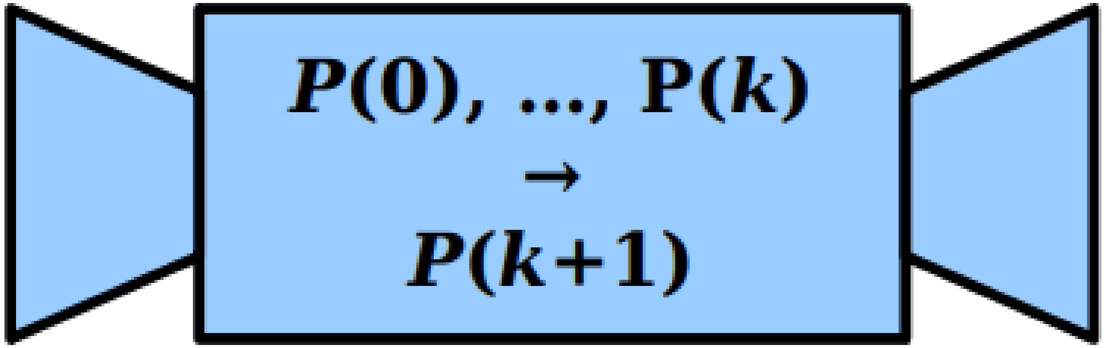
$P(0), \ldots, \mathrm{P}(k)$

$\rightarrow$

$P(k+1)$

$P(2)$

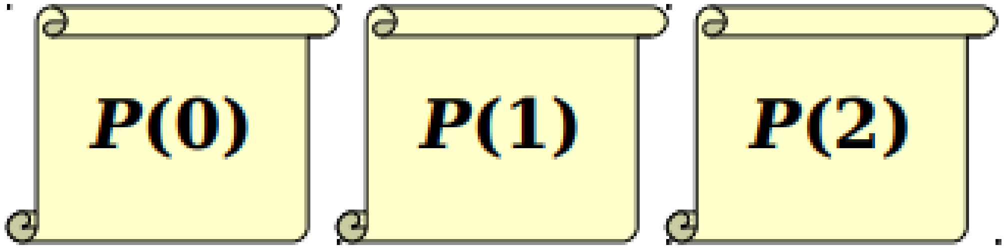
Intuiting Complete Induction



$P(0), \ldots, \mathrm{P}(k)$

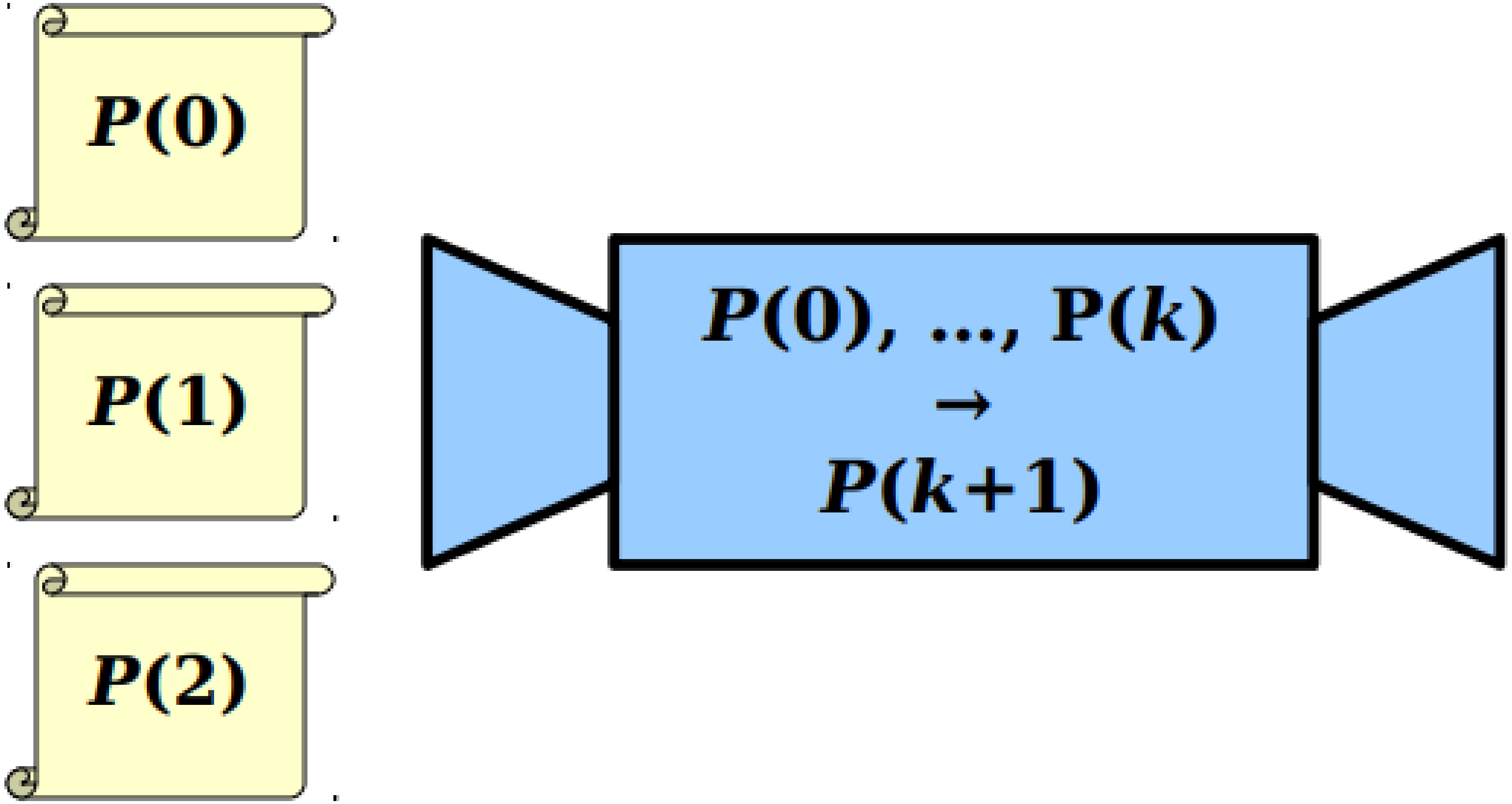
$\rightarrow$

$P(k+1)$



$P(0)$ $P(1)$ $P(2)$

Intuiting Complete Induction



$P(0)$

$P(1)$

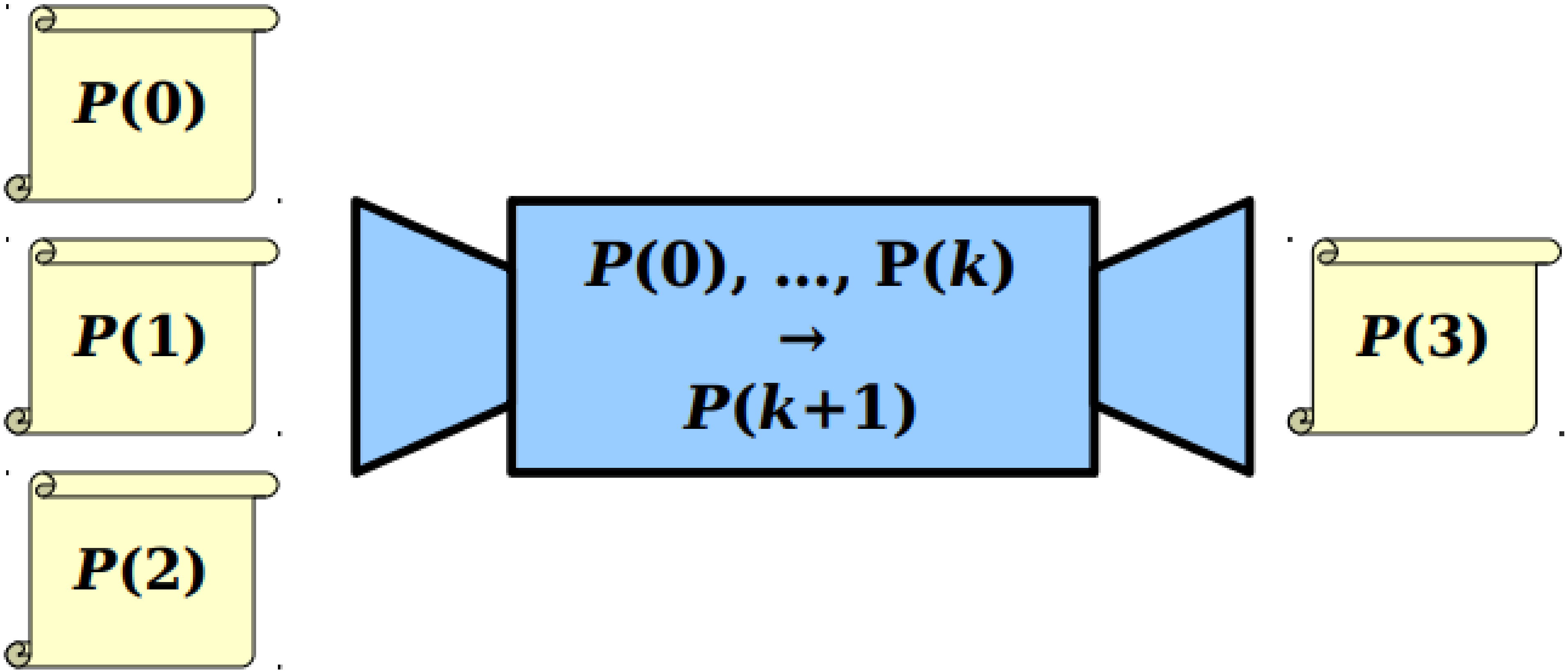
$P(2)$

$P(0), \ldots, \mathrm{P}(k)$

$\rightarrow$

$P(k+1)$

Intuiting Complete Induction



$P(0)$

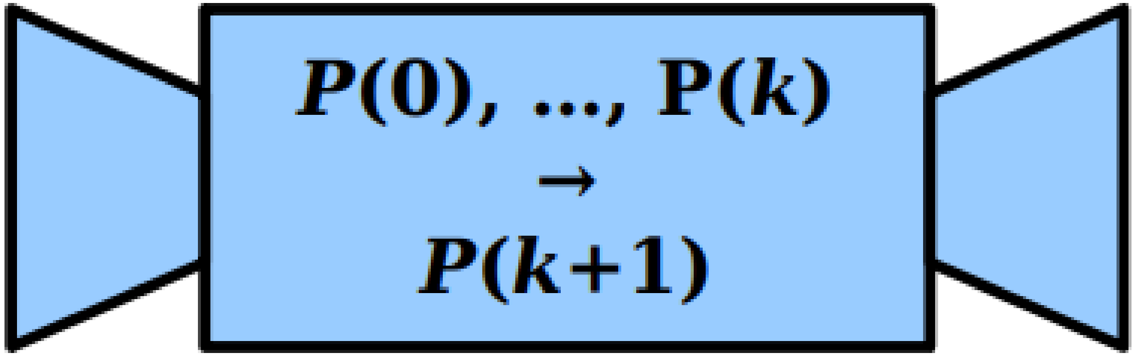
$P(1)$

$P(2)$

$P(0), \ldots, \mathrm{P}(k)$
$\rightarrow$
$P(k+1)$

$P(3)$

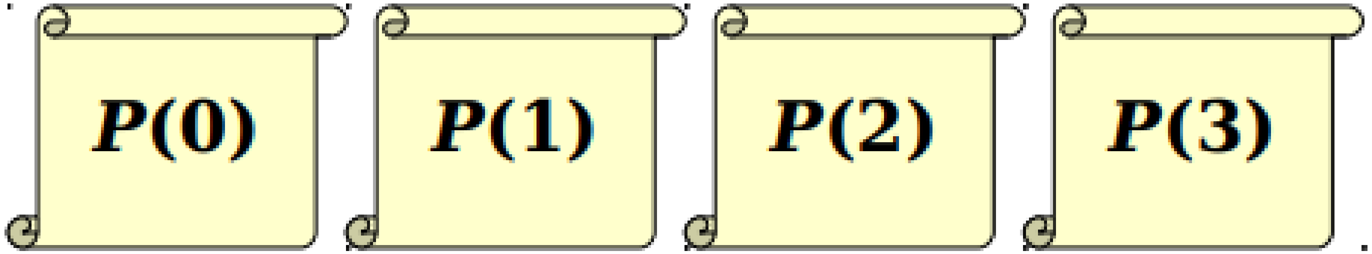
Intuiting Complete Induction



$P(0), \ldots, P(k)$

$\rightarrow$

$P(k+1)$



$P(0)$ $P(1)$ $P(2)$ $P(3)$

Example: the fundamental theorem of arithmetic

Fundamental theorem of arithmetic

Every positive integer greater than 1 has a unique prime factorization.

Examples

$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$

$591 = 3 \cdot 197$

$45,523 = 45,523$

$321,950 = 2 \cdot 5 \cdot 5 \cdot 47 \cdot 137$

$1,234,567,890 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 3,607 \cdot 3,803$

Prove that every integer ≥ 2 is a product of primes

Prove that every integer ≥ 2 is a product of primes

① **Let $\mathbf{P(n)}$ be "$\mathbf{n}$ is a product of one or more primes".**

We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

Prove that every integer ≥ 2 is a product of primes

① **Let $\mathbf{P(n)}$ be "$\mathbf{n}$ is a product of one or more primes".**

We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

② **Base case ($\mathbf{n = 2}$):**

2 is prime, so it is a product of primes. Therefore $P(2)$ is true.

Prove that every integer ≥ 2 is a product of primes

① **Let $\mathbf{P(n)}$ be “$\mathbf{n}$ is a product of one or more primes”.**

We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

② **Base case ($\mathbf{n = 2}$):**

2 is prime, so it is a product of primes. Therefore $P(2)$ is true.

③ **Inductive hypothesis:**

Suppose that for some arbitrary integer $k \geq 2, P(j)$ is true for every integer $2 \leq j \leq k$.

Prove that every integer ≥ 2 is a product of primes

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We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

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Suppose that for some arbitrary integer $k \geq 2, P(j)$ is true for every integer $2 \leq j \leq k$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true, i.e., $k+1$ is a product of primes.

Prove that every integer ≥ 2 is a product of primes

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④ **Inductive step:**

We want to prove that $P(k + 1)$ is true, i.e., $k + 1$ is a product of primes.

Case: $\mathbf{k + 1}$ is prime. Then by definition, $k + 1$ is a product of primes.

Prove that every integer ≥ 2 is a product of primes

① **Let $\mathbf{P(n)}$ be "$\mathbf{n}$ is a product of one or more primes".**

We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

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Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer $2 \leq j \leq k$.

④ **Inductive step:**

We want to prove that $P(k+1)$ is true, i.e., $k+1$ is a product of primes.

Case: $\mathbf{k + 1}$ is prime. Then by definition, $k+1$ is a product of primes.

Case: $\mathbf{k + 1}$ is composite. Then by $k+1 = ab$ for some integers ab where $2 \leq a, b \leq k$.

By inductive hypothesis, we have $P(a) = p_1 p_2 \ldots p_r$ and $P(b) = q_1 q_2 \ldots q_s$, where $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ are prime. Thus, $k + 1 = ab = p_1 p_2 \ldots p_r q_1 q_2 \ldots q_s$, which is a product of primes.

Prove that every integer ≥ 2 is a product of primes

① **Let $\mathbf{P(n)}$ be “$\mathbf{n}$ is a product of one or more primes”.**

We will show that $P(n)$ is true for every integer $n \geq 2$ by strong induction.

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Suppose that for some arbitrary integer $k \geq 2, P(j)$ is true for every integer $2 \leq j \leq k$.

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We want to prove that $P(k+1)$ is true, i.e., $k+1$ is a product of primes.

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By inductive hypothesis, we have $P(a) = p_1 p_2 \ldots p_r$ and $P(b) = q_1 q_2 \ldots q_s$, where $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ are prime. Thus, $k+1 = ab = p_1 p_2 \ldots p_r q_1 q_2 \ldots q_s$, which is a product of primes.

⑤ **The result follows for all $\mathbf{n} \geq \mathbf{2}$ by strong induction.** ■

Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.

Theorem: Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

I have 4 cent stamps and 5 cent stamps (as many as I want of each). Prove that I can make exactly n cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think…how would the inductive step go?

Stamp Collection (attempt)

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.

Inductive Hypothesis Suppose [maybe some other stuff and] $P(k)$, for an arbitrary $k \geq 12$.

Inductive Step:

We want to make $k + 1$ cents of stamps. By IH we can make $k - 3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k + 1$ cents.

Stamp Collection

Is the proof right?

How do we know $P(13)$

We're not the base case, so our inductive hypothesis assumes $P(12)$, and then we say if $P(9)$ then $P(13)$.

Wait a second….

If you go back s steps every time, you need s base cases.

Or else the first few values aren't proven.

Stamp Collection

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
13 cents can be made with two 4 cent stamps and one 5 cent stamp.
14 cents can be made with one 4 cent stamp and two 5 cent stamps.
15 cents can be made with three 5 cent stamps.

Inductive Hypothesis Suppose $\mathrm{P}(12) \wedge P(13) \wedge \cdots \wedge P(k)$, for an arbitrary $k \geq 15$.

Inductive Step:

We want to make $k+1$ cents of stamps. By IH we can make $k-3$ cents exactly with stamps. Adding another 4 cent stamp gives exactly $k+1$ cents.

Theorem: Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

BASIS STEP: $P(12)$, $P(13)$,$P(14)$, and $P(15)$ hold.

- $P(12)$ uses three 4-cent stamps.
- $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
- $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
- $P(15)$ uses three 5-cent stamps.

INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k+1)$ holds.

Using the inductive hypothesis, $P(k-3)$ holds since $k-3 \geq 12$. To form postage of $k+1$ cents, add a 4-cent stamp to the postage for $k-3$ cents.

Hence, $P(n)$ holds for all $n \geq 12$. ■