Discrete Mathematics

Relations

A way of modeling connections between objects.

Binary Relations

Binary Relation: “is connected to”

$A = \{a, b, c, d, e\}$

$R_A = ((a, d),(a, c), (b, e))$ where R is a binary relation called “is connected to”.

We can write aRd, aRc, bRe.

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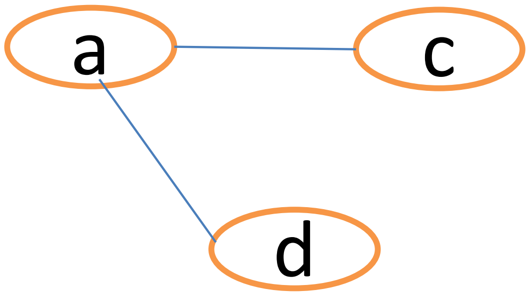

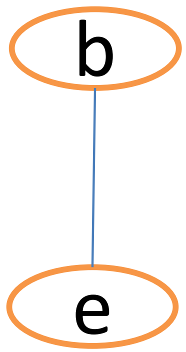

R_A is equivalence (reflexive, symmetric, and transitive).

Equivalence classes (clusters of objects):

$[a]_R = [c]_R = [d]_R = \{a, c, d\}$

$[b]_R = [e]_R = \{b, e\}$

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R_A = ((a, d),(a, c), (b, e)) where R is a binary relation called “is connected to”.

We can write aRd, aRc, bRe.

Observation:

R_A is subset of A x A.

Binary Relations

- A ***binary relation over a set A*** is a predicate R that can be applied to pairs of elements drawn from A.
- If R is a binary relation over A and it holds for the pair (a, b), we write $\boldsymbol{aRb}$.
  - For example: $3 = 3$, $5 < 7$, and $\emptyset \subseteq \mathbb{N}$.
- If R is a binary relation over A and it does not hold for the pair (a, b), we write $\boldsymbol{a\not{R}b}$.
  - For example: $4 \neq 3$, $4 \not< 3$, and $\mathbb{N} \nsubseteq \emptyset$.

What is a Relation?

The Cartesian Product

- The **Cartesian Product** of $A \times B$ of two sets is defined as

$$A \times B \equiv \{\ (a, b) \mid a \in A \text{ and } b \in B\ \}$$

$$\underbrace{\{0, 1, 2\}}_{A} \times \underbrace{\{a, b, c\}}_{B} = \left\{ \begin{array}{l} (0, a),(0, b),(0, c), \\ (1, a),(1, b),(1, c), \\ (2, a),(2, b),(2, c) \end{array} \right\}$$

The Cartesian Product

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- We denote $A^2 \equiv A \times A$

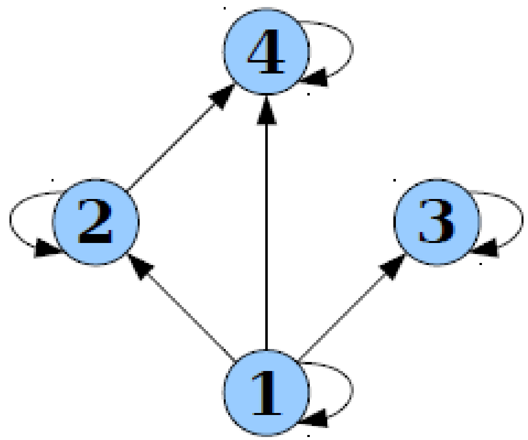
$$\underbrace{\{0, 1, 2\}^2}_{A^2} = \left\{ \begin{matrix} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{matrix} \right\}$$

Relations, Formally

- A binary relation R over a set A is a subset of A^2.
- xRy is shorthand for $(x, y) \in R$.
- A relation doesn't have to be meaningful; *any* subset of A^2 is a relation.

Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set $\{1, 2, 3, 4\}$ looks like this:



Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
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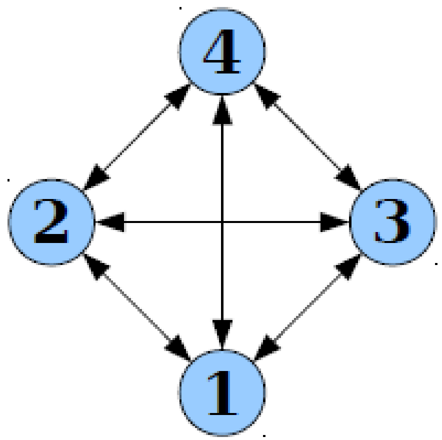
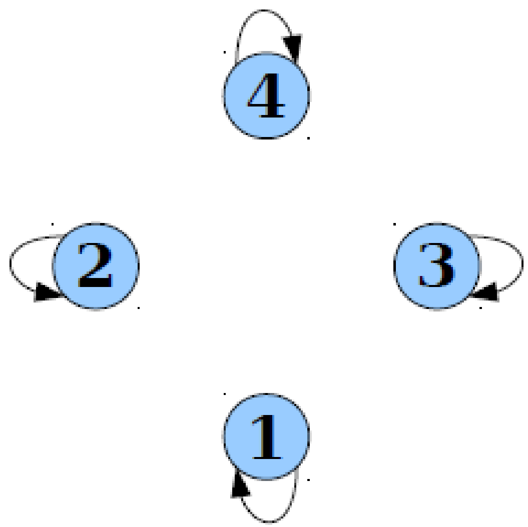


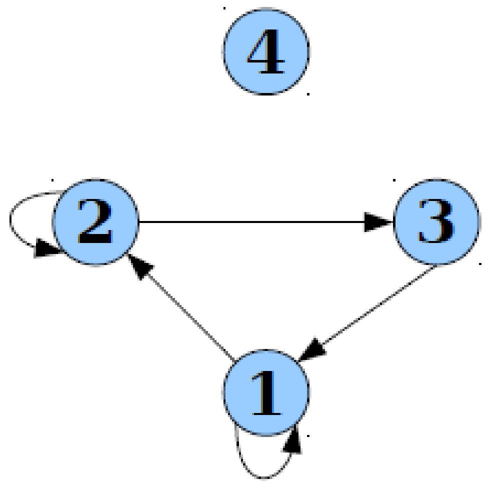
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a = b$ over the set $\{1, 2, 3, 4\}$ looks like this:



Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: below is some relation over {1, 2, 3, 4} that's a totally valid relation even though there doesn't appear to be a simple unifying rule.



Equivalence Relations

$$aRa$$

$$aRb \quad \rightarrow \quad bRa$$

$$aRb \quad \wedge \quad bRc \quad \rightarrow \quad aRc$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

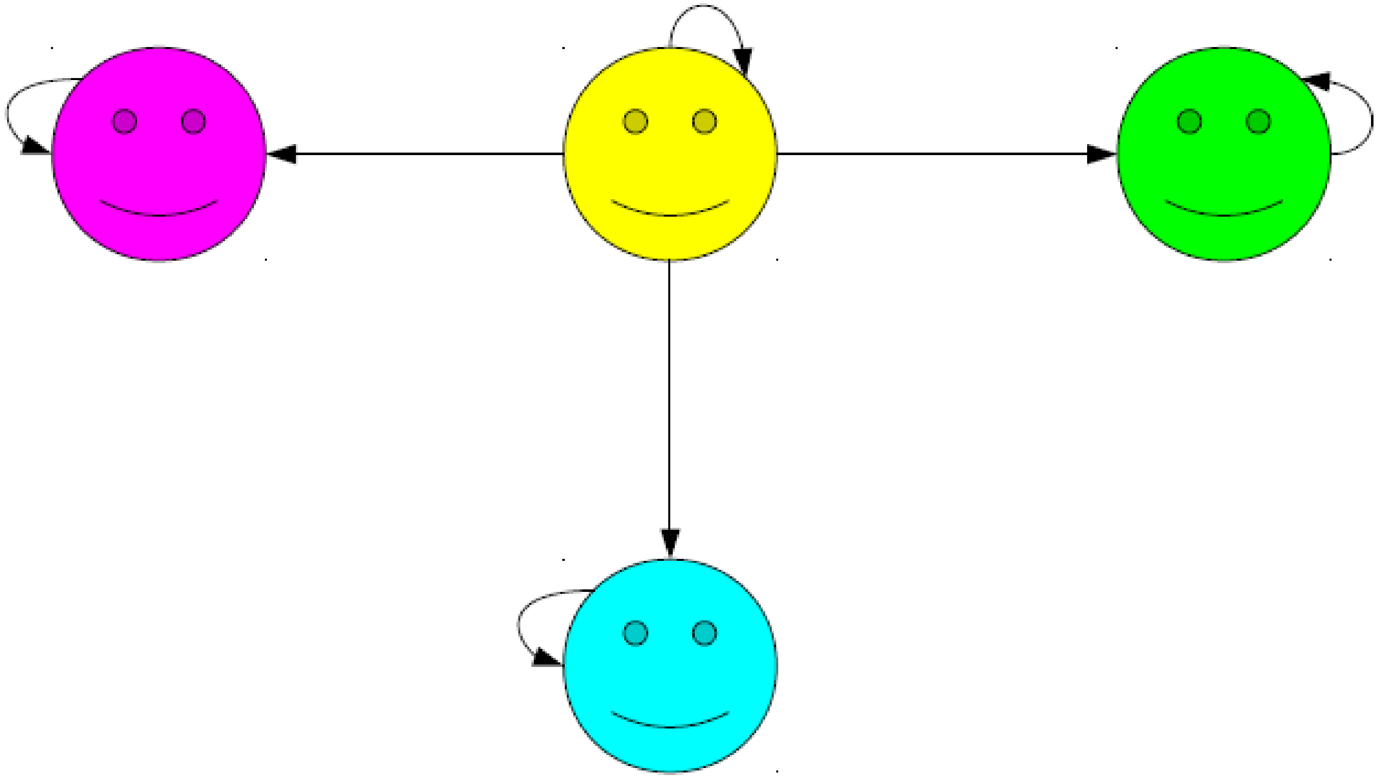
Reflexivity

- Some relations always hold from any element to itself.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation R over a set A is reflexive if the following is true:

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

Reflexivity Visualized



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$$\forall a \in A.\ aRa$$

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$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \wedge bRc \rightarrow aRc)$$

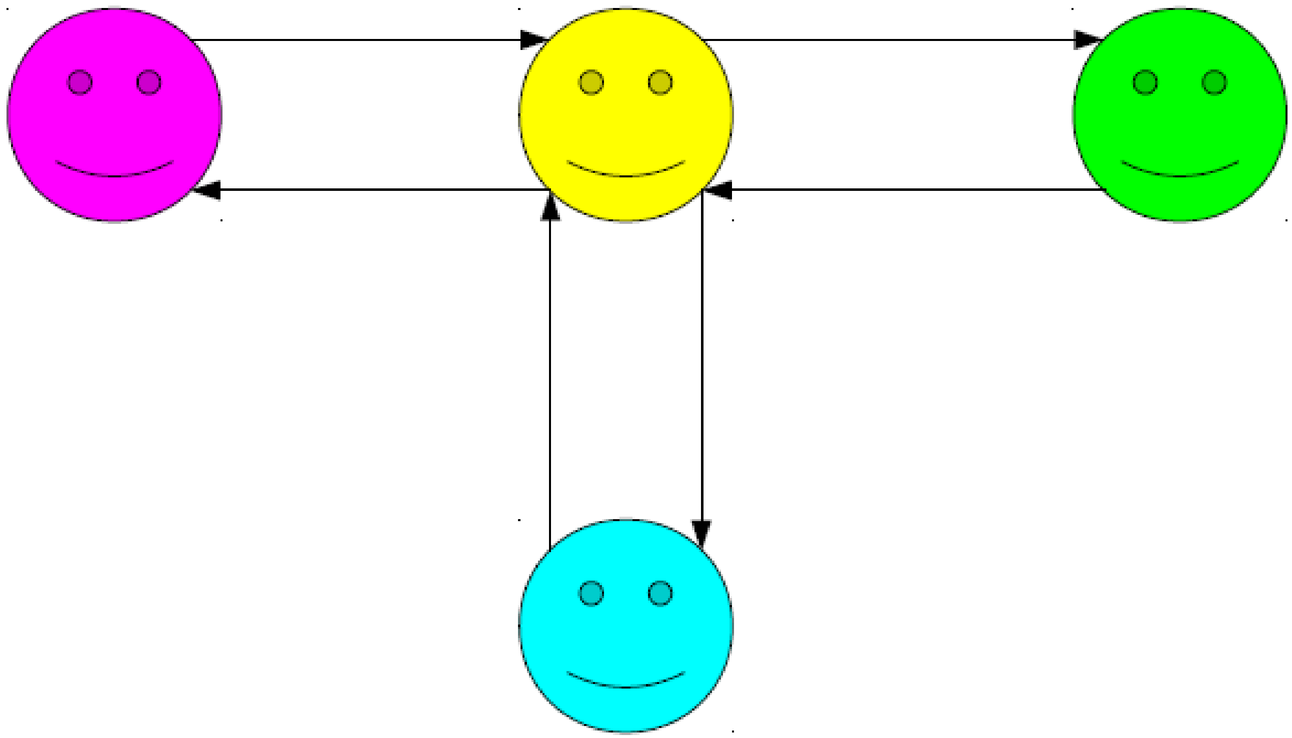
Symmetry

- In some relations, the relative order of the objects doesn't matter.
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called ***symmetric***.
- Formally: a binary relation R over a set A is called *symmetric* if

$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

("If a is related to b, then b is related to a.")

Symmetry Visualized



$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$

(*"If a is related to b, then b is related to a."*)

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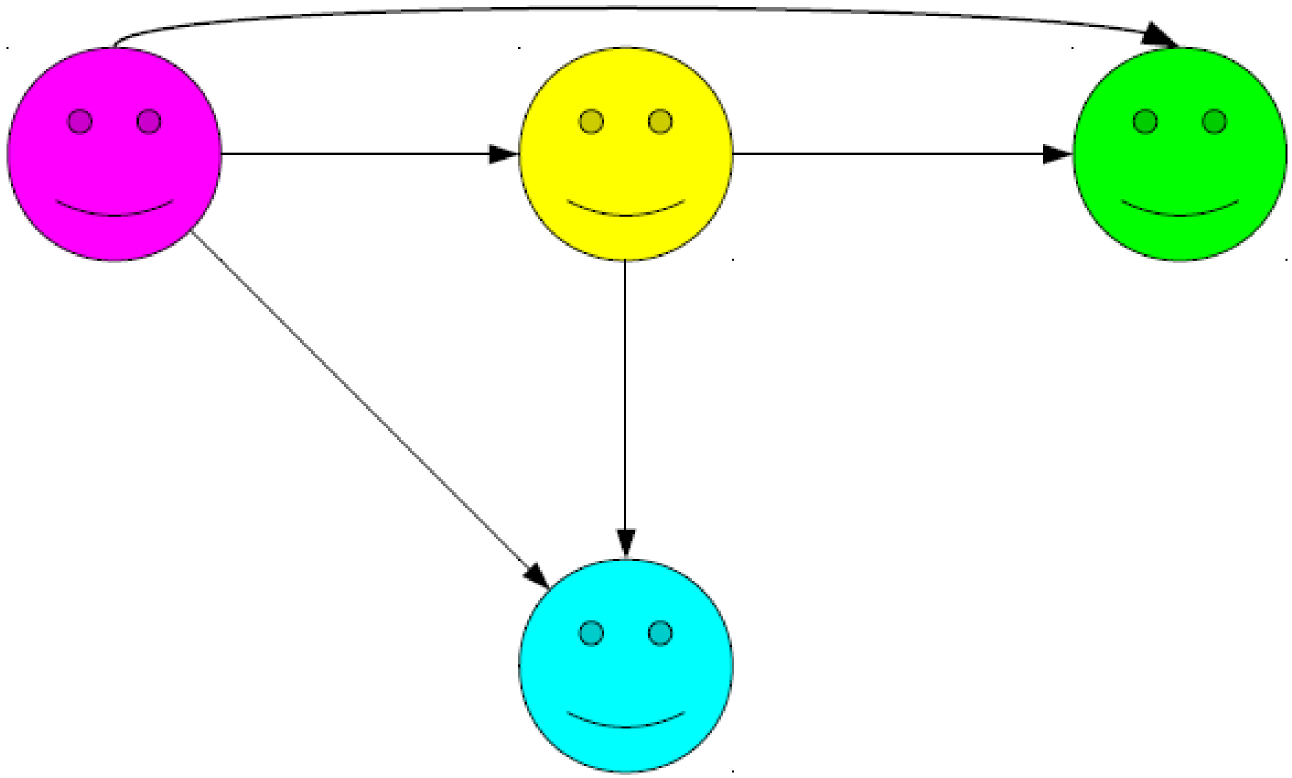
Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation R over a set A is called *transitive* if

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

("Whenever a is related to b and b is related to c, we know a is related to c.)

Transitivity Visualized



$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

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Equivalence Relations

- An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color as y
  - x has the same shape as y.

Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

An Example Relation

- Consider the binary relation ~ defined over the set $\mathbb{Z}$:

$$a \sim b \quad \textbf{if} \quad a+b \textbf{ is even}$$

- Some examples:

$$0 \sim 4 \qquad 1 \sim 9 \qquad 2 \sim 6 \qquad 5 \sim 5$$

- Turns out, this is an equivalence relation! Let's see how to prove it.

What properties must ~ have to be an equivalence relation?

Reflexivity
Symmetry
Transitivity

Let's prove each property independently.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

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What is the formal definition of reflexivity?

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Therefore, we'll choose an arbitrary integer a, then go prove that $a \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$.

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To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even.

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Lemma 2: The binary relation $\sim$ is symmetric.

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What is the formal definition of symmetry?

$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

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$a \sim b$ if $a+b$ is even

Lemma 3: The binary relation $\sim$ is transitive.

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Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

$a \sim b$ if $a+b$ is even

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Proof: Consider arbitrary integers a, b and c where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers k and m where $a+b = 2k$ and $b+c = 2m$.

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$$(a+b) + (b+c) = 2k + 2m.$$

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$$a+c = 2k + 2m - 2b = 2(k+m-b).$$

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So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required.

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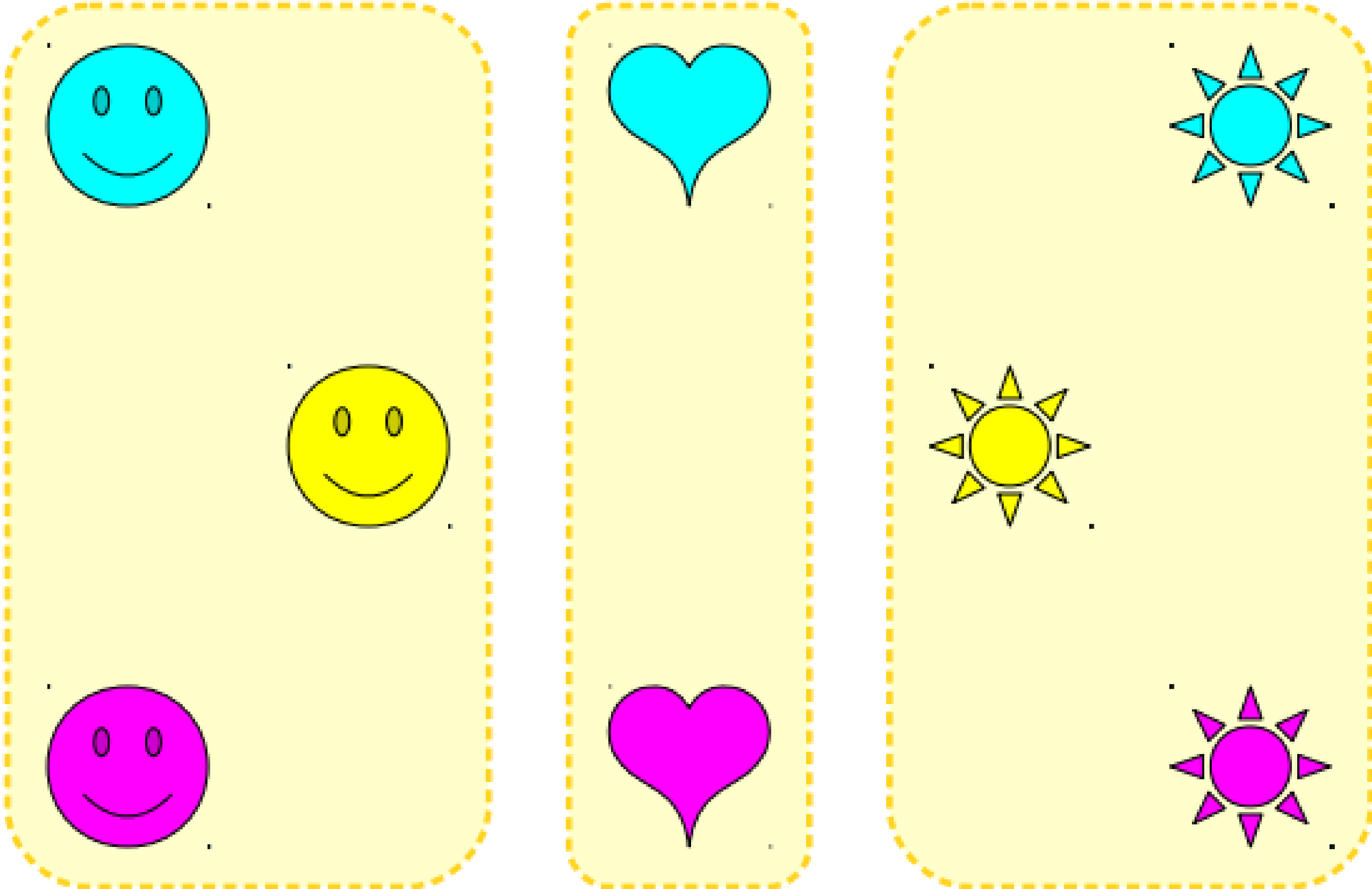
Capturing Structure

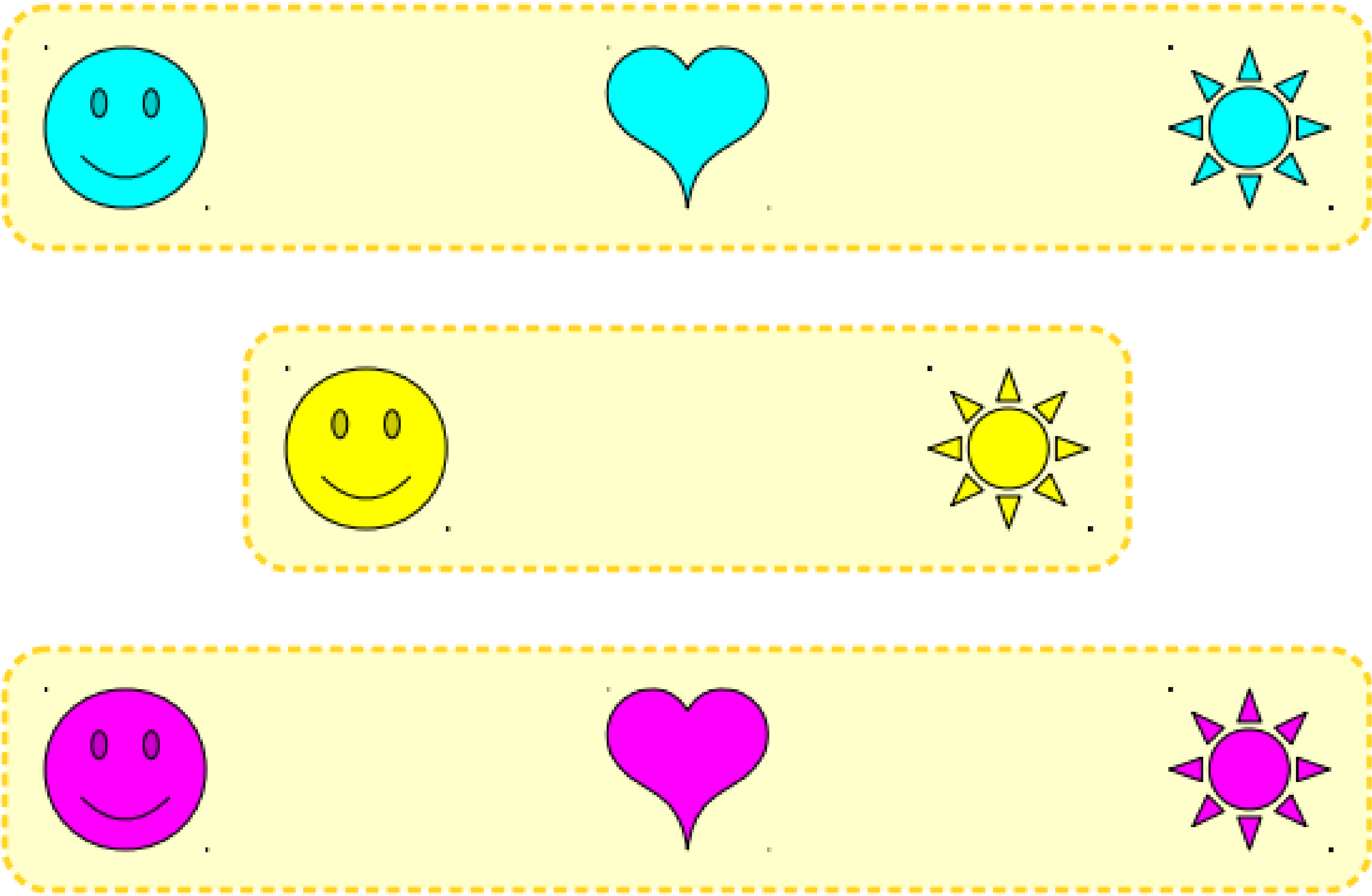
Binary relations are an excellent way for capturing certain structures that appear in computing.

- **Partitions**
- **Prerequisites**

Properties of Equivalence Relations



xRy if x and y have the same shape



xTy if x and y have the same color

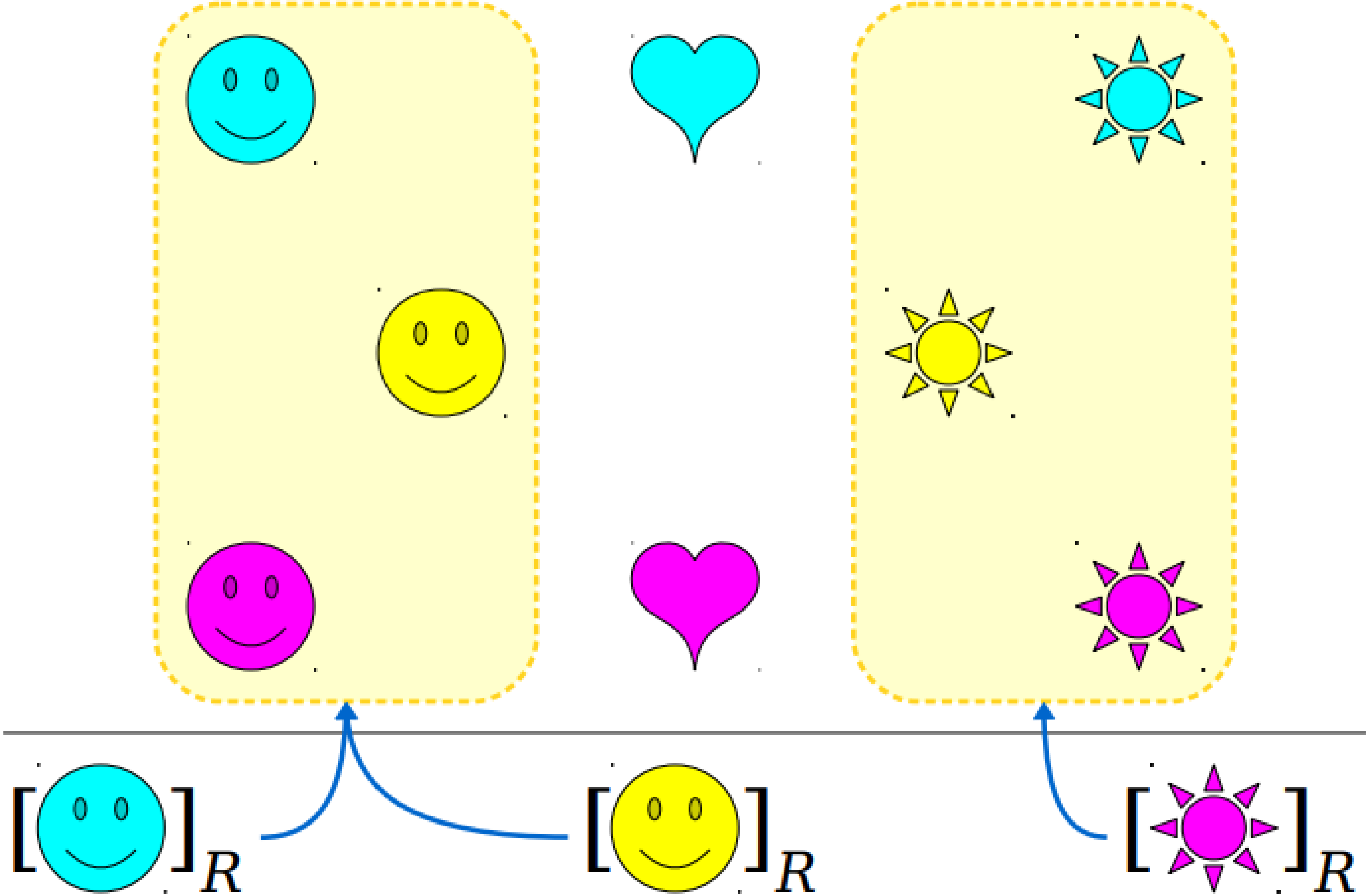
Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ y \in A \mid xRy \}$$

- $[x]_R$ is the set of all elements of A that are related to x by relation R.

- R **partitions** the set A into a set of equivalence classes.



xRy if x and y have the same shape

Equivalence Classes

Example:

- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$

$R= \{(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6) (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)\}$

- **Pick an element** a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R= \{1,4\}$
- Element 2: $[2]_R= \{2,5\}$
- Element 3: $[3]_R= \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R= \{1,4\} = [1]_R$ Element 5: $[5]_R= \{2,5\} = [2]_R$

Equivalence Classes

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- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$
- $R= \{(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6) (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)\}$

Three different equivalence classes all together:

- $[0]_R = [3]_R =[6]_R = \{0,3,6\}$
- $[1]_R= [4]_R= \{1,4\}$
- $[2]_R= [5]_R= \{2,5\}$

Equivalence Classes

- ***Theorem:*** If R is an equivalence relation over A, then every $a \in A$ belongs to exactly one equivalence class.

Closing the Loop

- In any graph $G = (V, E)$, we saw that the connected component containing a node $v \in V$ is given by

$$\{ x \in V \mid v \leftrightarrow x \}$$

- What is the equivalence class for some node $v \in V$ under the relation $\leftrightarrow$?

$$[v]_{\leftrightarrow} = \{ x \in V \mid v \leftrightarrow x \}$$

- *Connected components are just equivalence classes of* $\leftrightarrow$!

Partitions

A **partition of a set** is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.

- Two sets are **disjoint** if their intersection is the empty set; formally, sets S and T are disjoint if $S \cap T = \emptyset$.

Usually, the term **clustering** is used in data analysis rather than **partitioning**.

Relations and Prerequisites

Relations and Prerequisites

- Let's imagine that we have a prerequisite structure with no circular dependencies.
- We can think about a binary relation R where aRb means

“***a* must happen before *b***”

- What properties of R could we deduce just from this?

$$a \not R a$$

$$aRb \land bRc \rightarrow aRc$$

$$aRb \rightarrow b \not R a$$

$$\forall a \in A.\ a \not R a$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not R a)$$

$$\forall a \in A.\ a \not R a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

$$\forall a \in A.\ a \not{R} a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not{R} a)$$

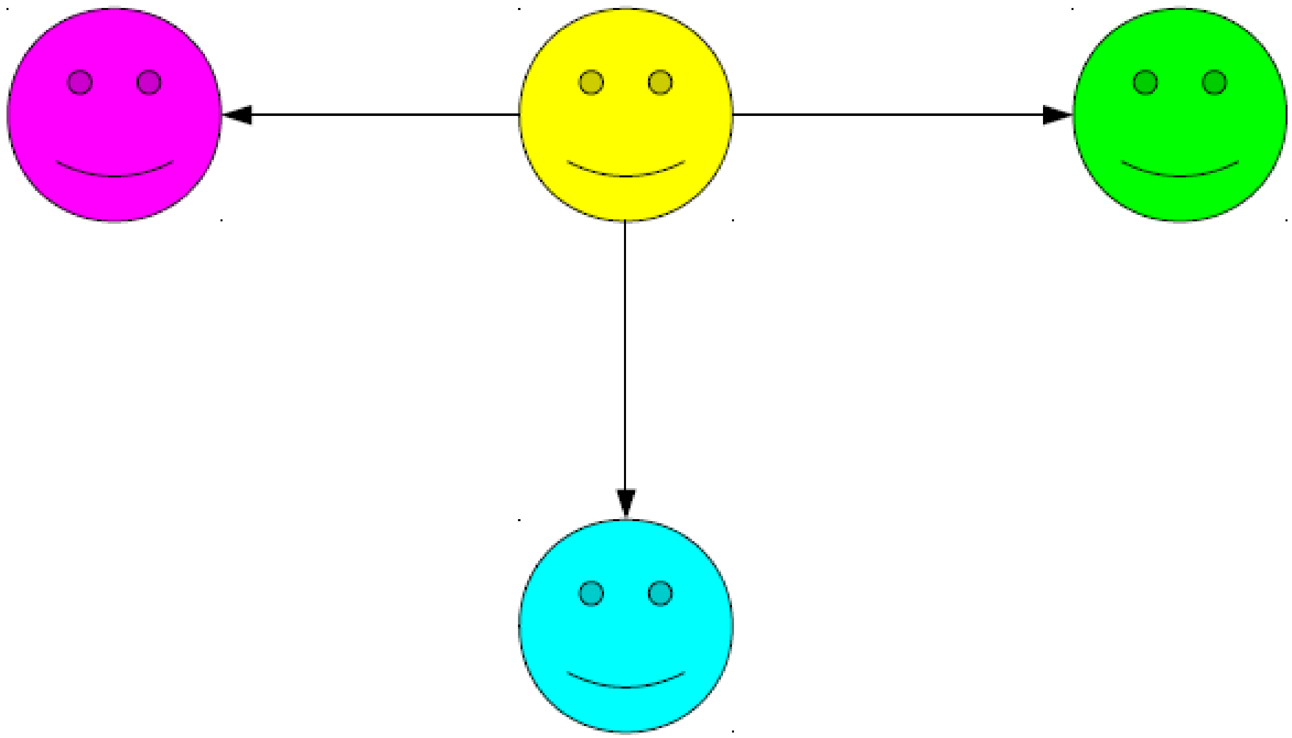
Irreflexivity

- Some relations *never* hold from any element to itself.
- As an example, $x \nless x$ for any x.
- Relations of this sort are called ***irreflexive***.
- Formally speaking, a binary relation R over a set A is ifreflexive if the following is true:

$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")

Irreflexivity Visualized



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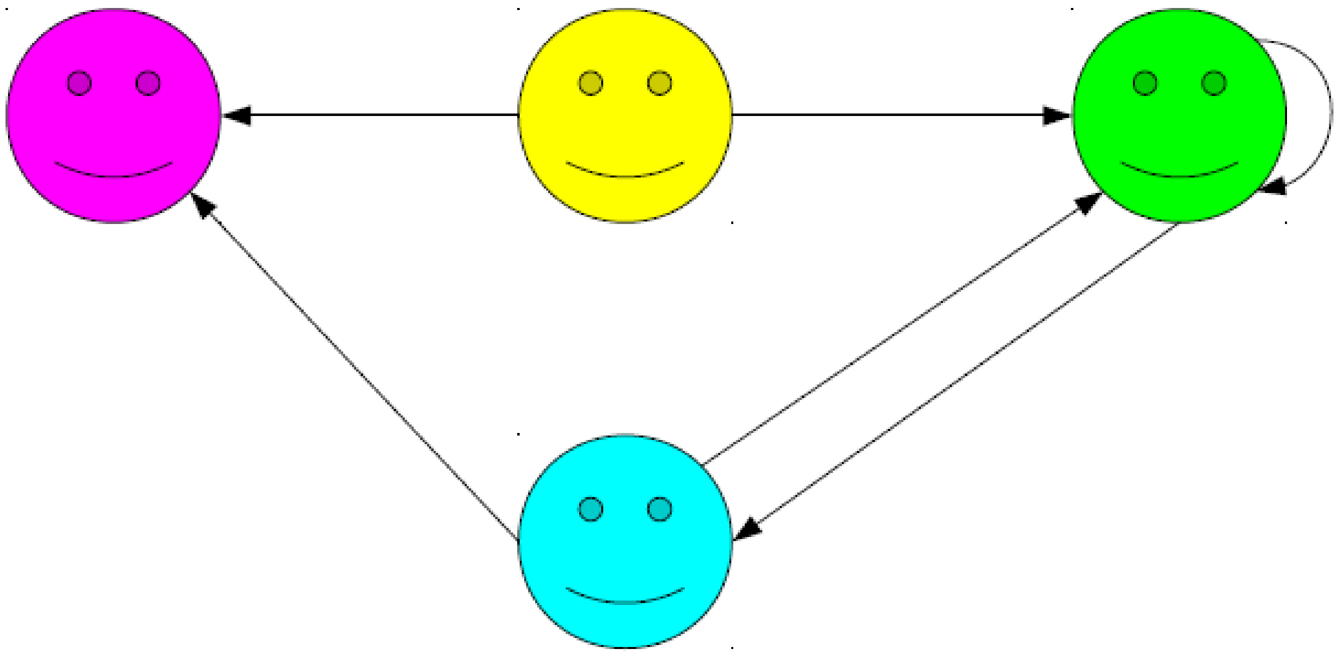
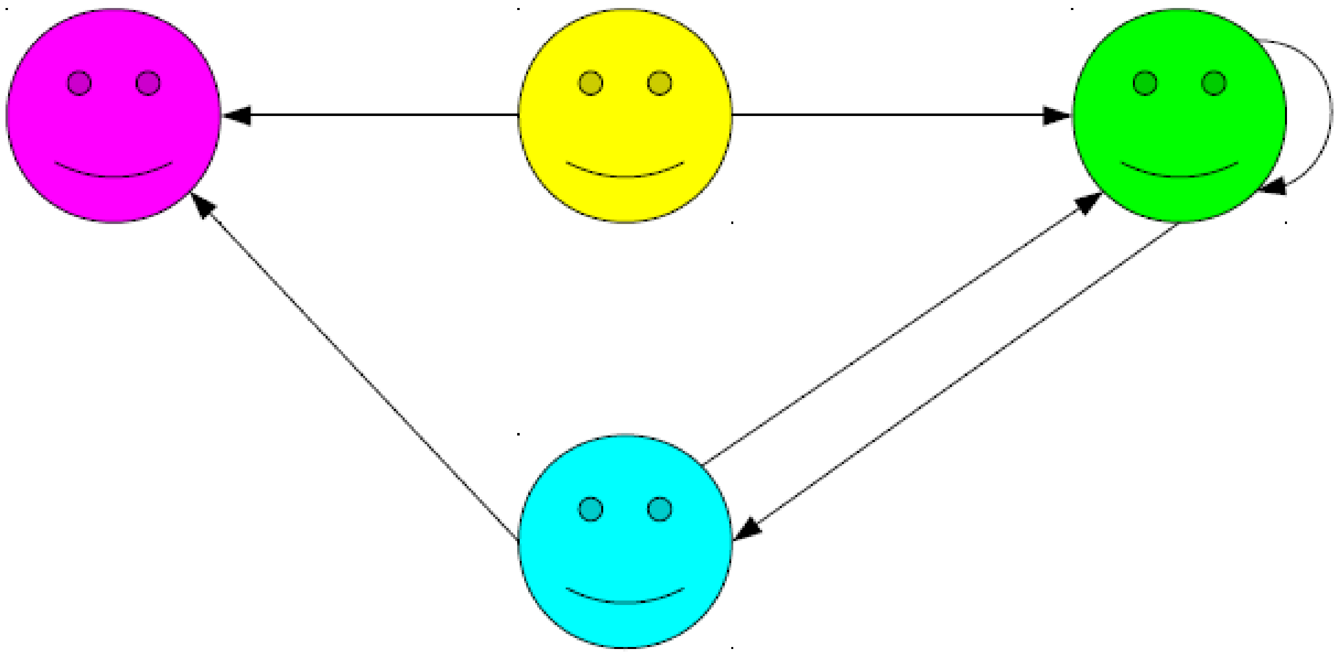
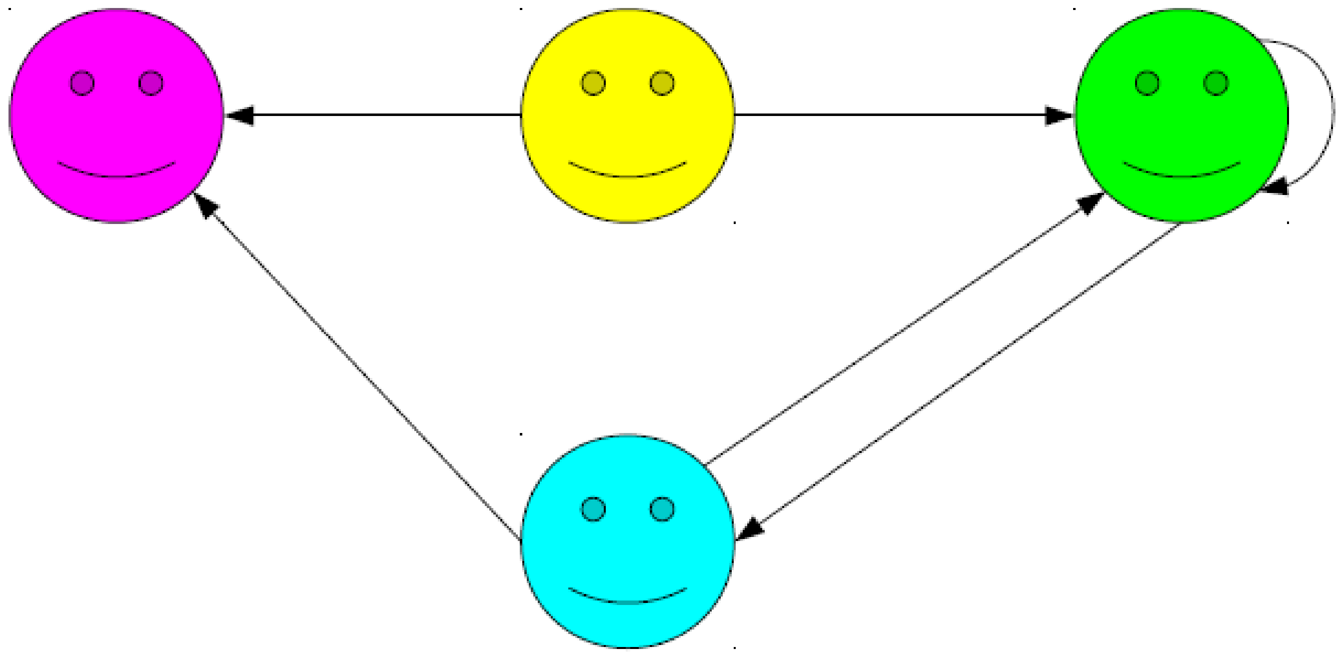




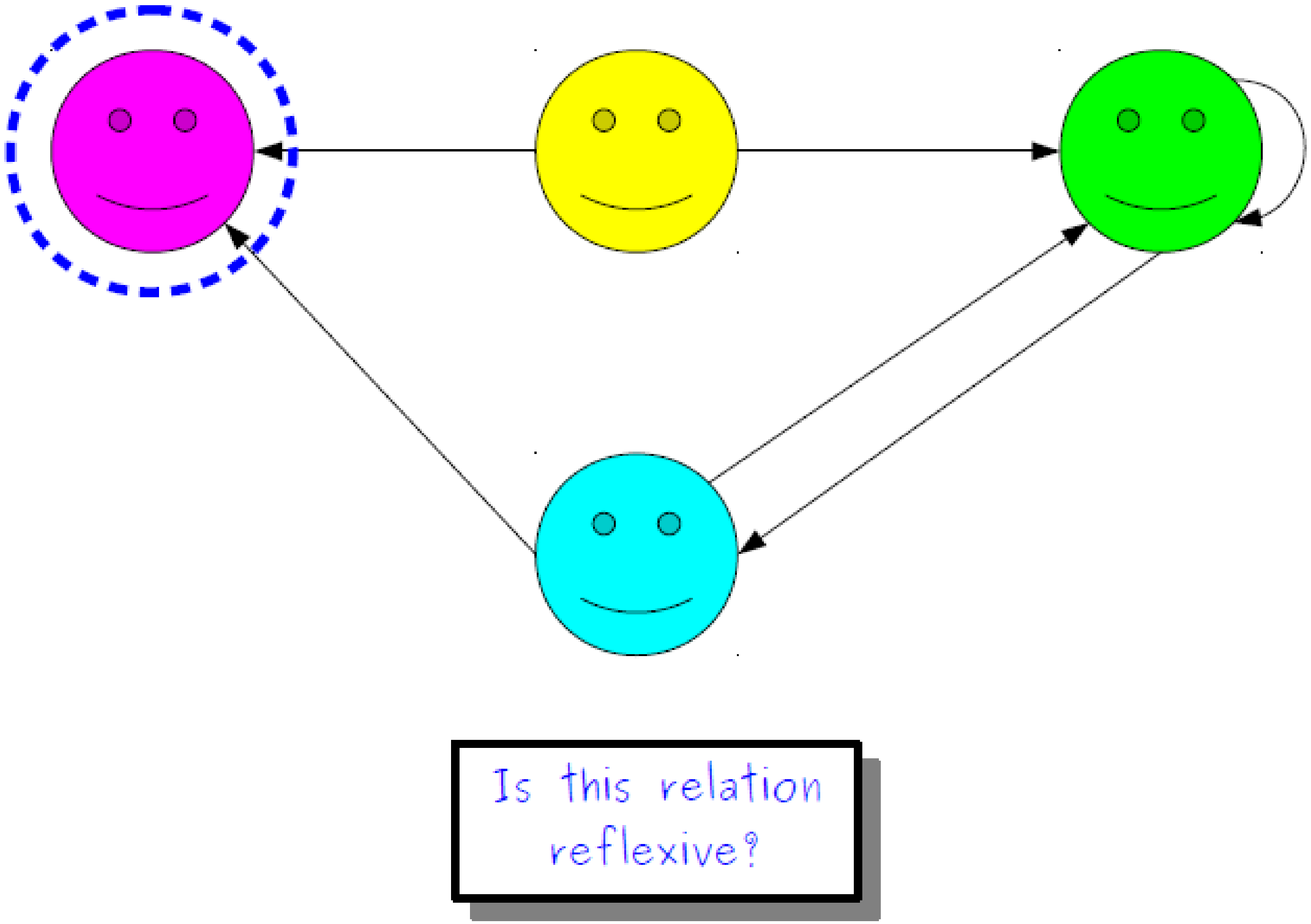
Is this relation reflexive?



Is this relation reflexive?

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")



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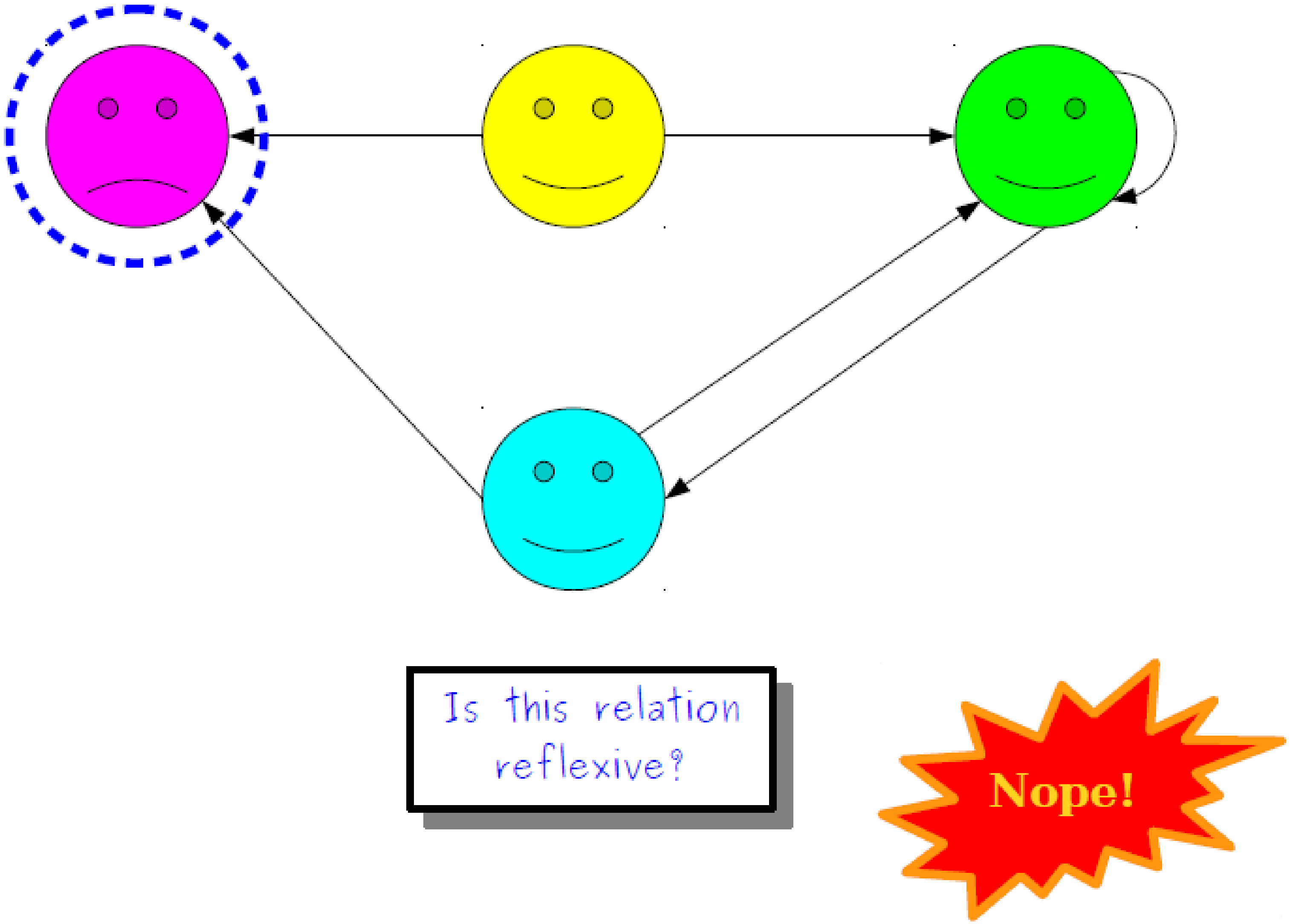

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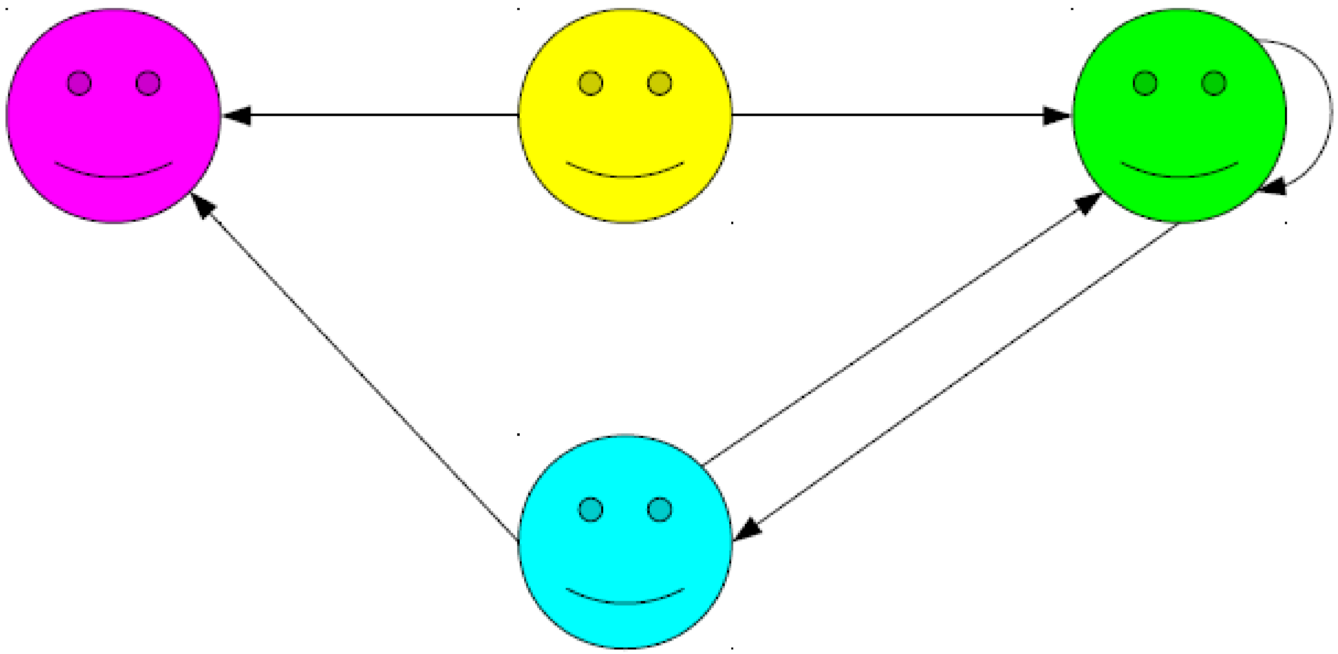
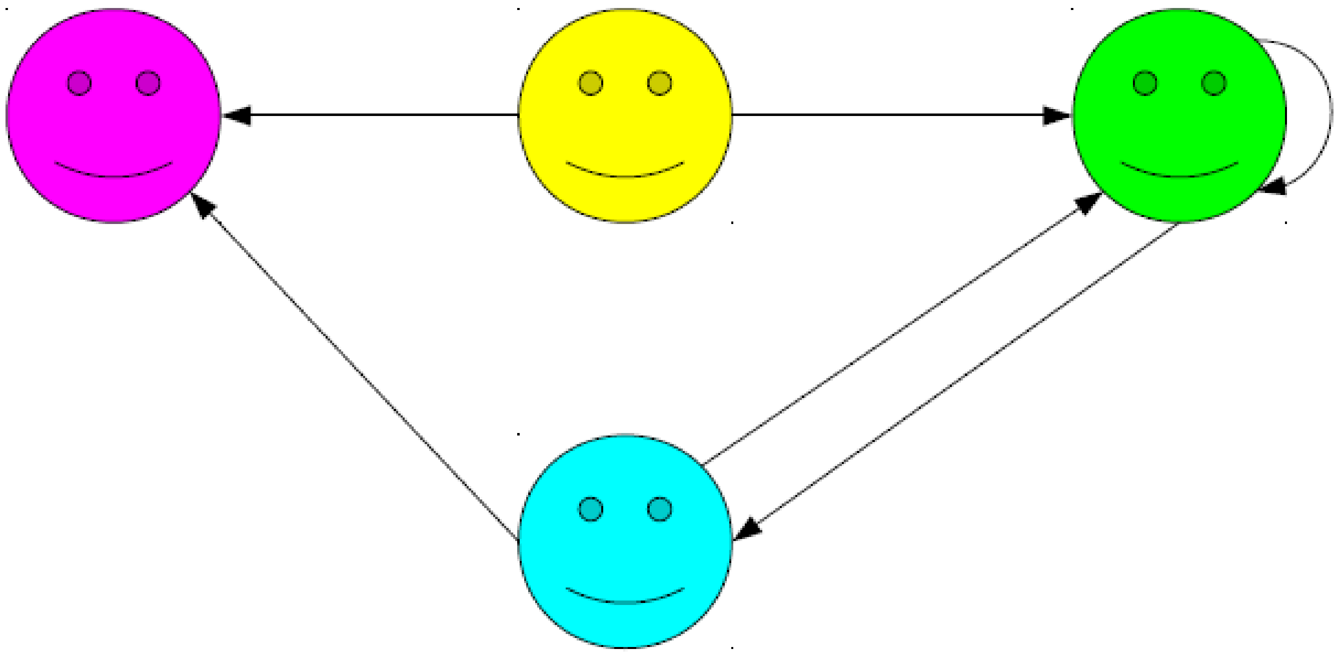
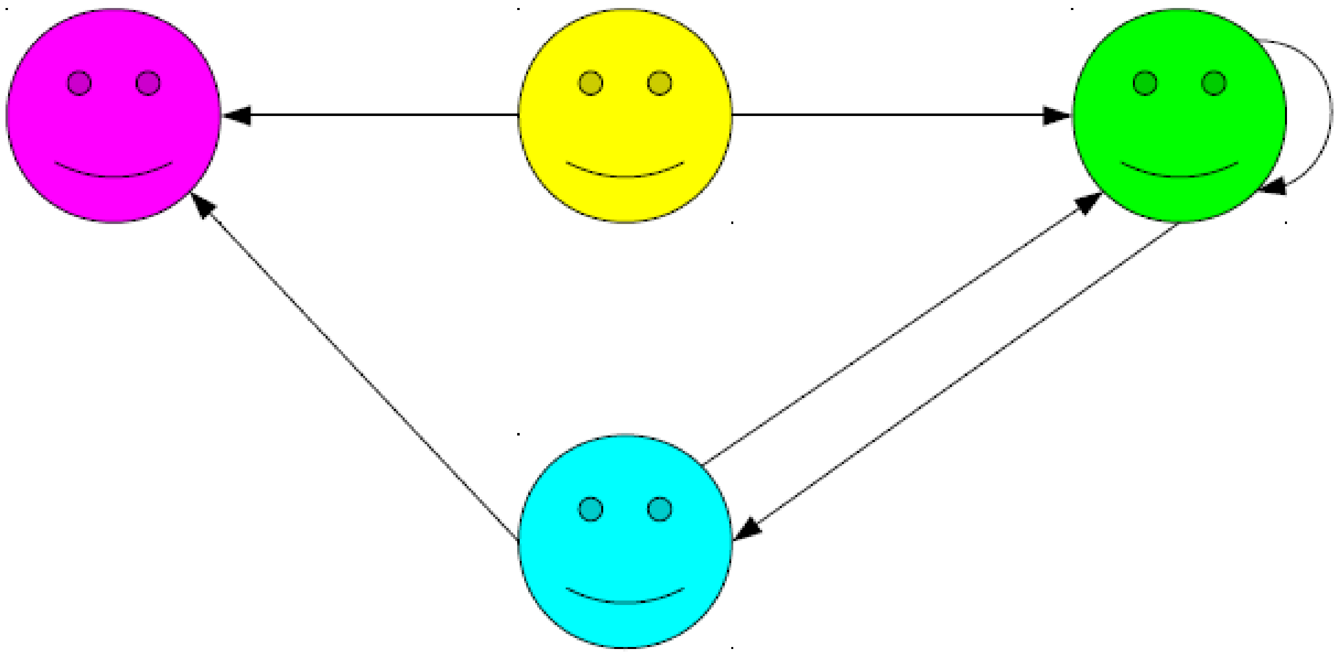


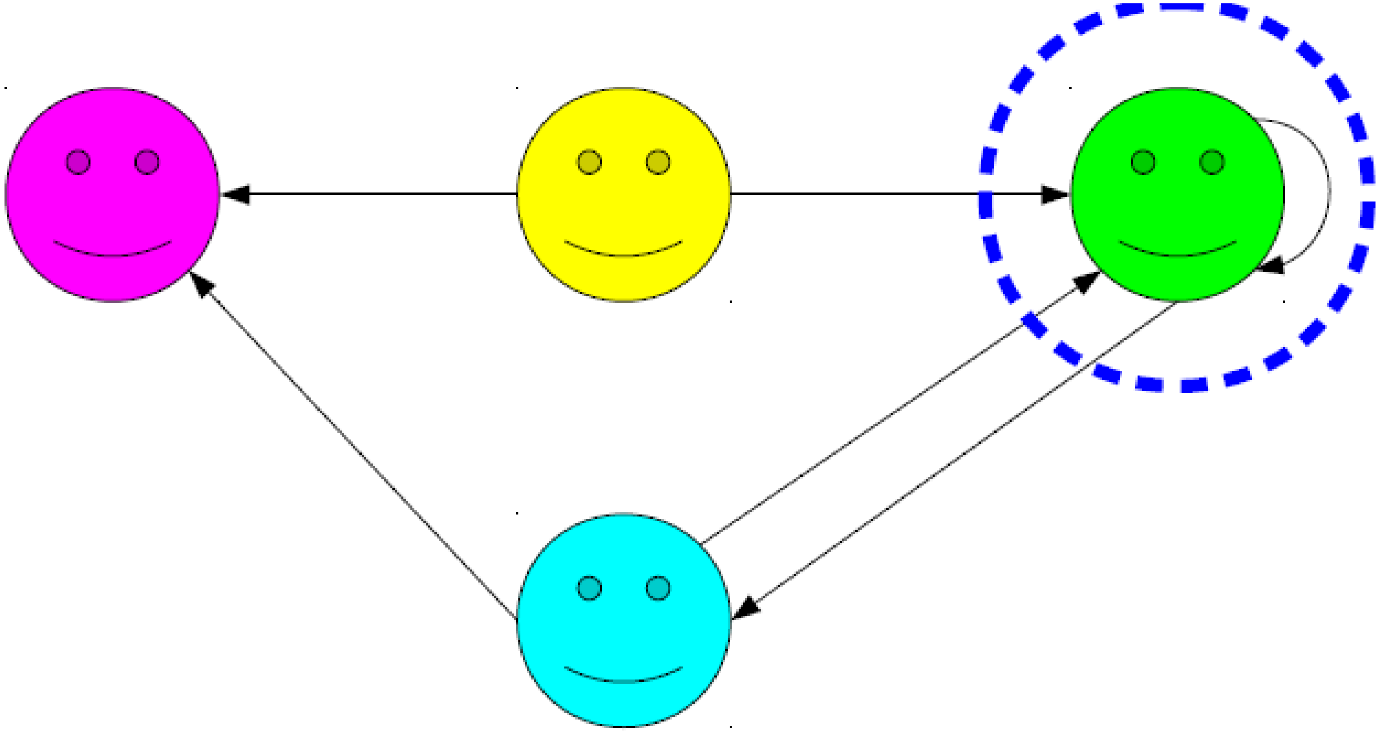


Is this relation irreflexive?



$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")



Is this relation irreflexive?

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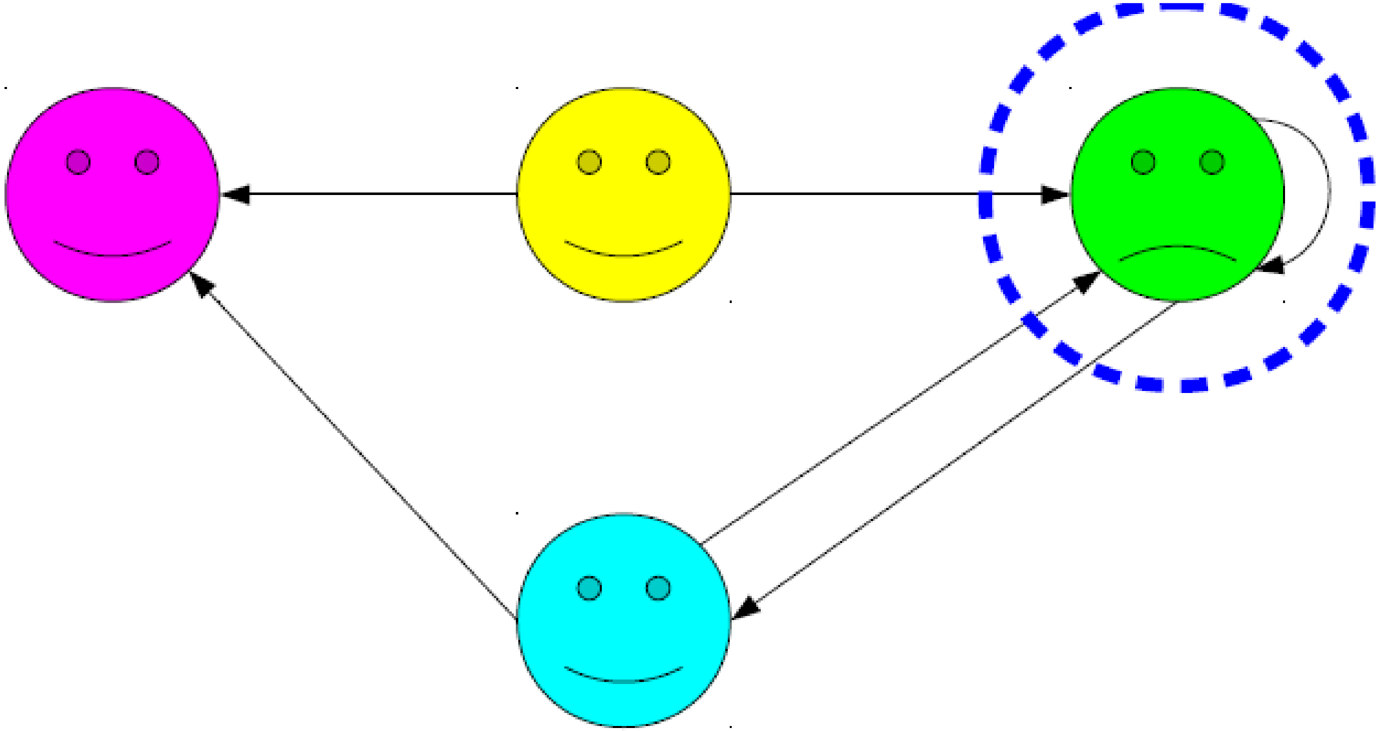
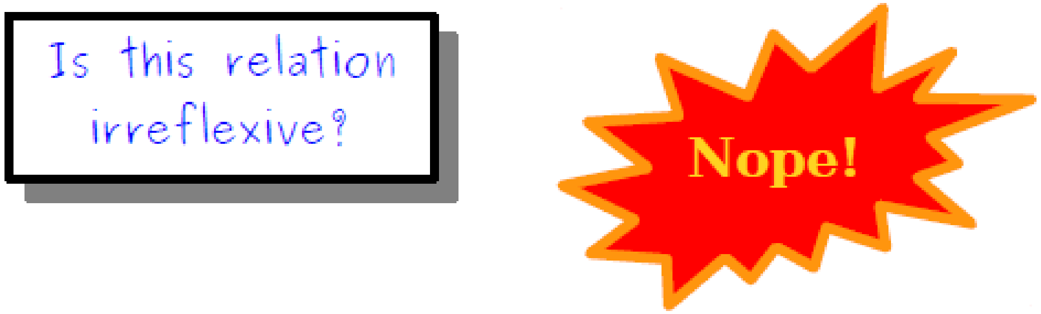




$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")

Reflexivity and Irreflexivity

- Reflexivity and irreflexivity are ***not*** opposites!
- Here's the definition of reflexivity:

$$\forall a \in A.\ aRa$$

- What is the negation of the above statement?

$$\exists a \in A.\ a \not R a$$

- What is the definition of irreflexivity?

$$\forall a \in A.\ a \not R a$$

$$\forall a \in A.\ a \not{R} a$$

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not{R} a)$$

Irreflexivity

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

Irreflexivity

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

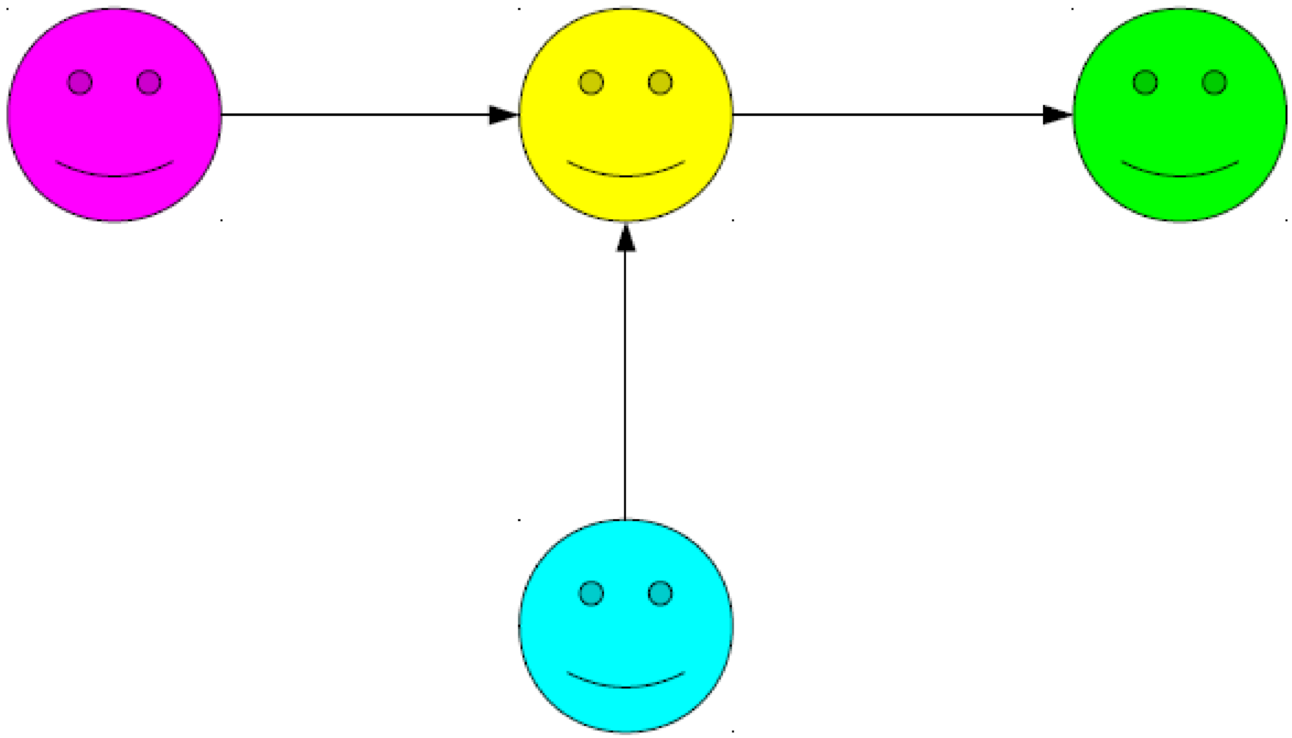
Asymmetry

- In some relations, the relative order of the objects can never be reversed.
- As an example, if $x < y$, then $y \not< x$.
- These relations are called ***asymmetric***.
- Formally: a binary relation R over a set A is called *asymmetric* if

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b \not R a)$$

("If a relates to b, then b does not relate to a.")

Asymmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b\not{R}a)$$

(*"If a relates to b, then b does not relate to a."*)

Irreflexivity

Transitivity

$$\forall a \in A.\ \forall b \in A.\ (aRb \to b\not{R}a)$$

Irreflexivity

Transitivity

Asymmetry

Strict Orders

- A ***strict order*** is a relation that is irreflexive, asymmetric and transitive.
  - We'll refresh those definitions in a second.
- Some examples:
  - $x < y$.
  - a can run faster than b.
  - $A \subset B$ (that is, $A \subseteq B$ and $A \neq B$).

Strict Order Proofs

- Let's suppose that you're asked to prove that a binary relation is a strict order.
- Calling back to the definition, you could prove that the relation is asymmetric, irreflexive, and transitive.
- However, there's a slightly easier approach we can use instead.

Theorem: Let R be a binary relation over a set A. If R is asymmetric, then R is irreflexive.

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$$\forall R. \; (\mathit{Asymmetric}(R) \rightarrow \mathit{Irreflexive}(R))$$

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What's the high-level structure of this proof?

$$\forall R.\ (\mathit{Asymmetric}(R) \to \mathit{Irreflexive}(R))$$

Therefore, we'll choose an arbitrary asymmetric relation R, then go and prove that R is irreflexive.

Theorem: Let R be a binary relation over a set A. If R is asymmetric, then R is irreflexive.

Proof: Let R be an arbitrary asymmetric binary relation over a set A. We will prove that R is irreflexive.

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To do so, we will proceed by contradiction.

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$$\forall x \in A.\ x \not R x$$

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What is the negation of this statement?

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$$\forall x \in A.\ x \not R x$$

What is the negation of this statement?

$$\exists x \in A.\ xRx$$

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What is the definition of irreflexivity?

$$\forall x \in A.\ x \not R x$$

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So let's suppose that there is some element $x \in A$ such that xRx and proceed from there.

Theorem: Let R be a binary relation over a set A. If R is asymmetric, then R is irreflexive.

Proof: Let R be an arbitrary asymmetric binary relation over a set A. We will prove that R is irreflexive.

To do so, we will proceed by contradiction. Suppose that R is not irreflexive.

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Since R is asymmetric, we know for any $a, b \in A$ that if aRb holds, then $b\not{R}a$ holds.

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Since R is asymmetric, we know for any $a, b \in A$ that if aRb holds, then $b \not R a$ holds. Plugging in $a = x$ and $b = x$, we see that if xRx holds, then $x \not R x$ holds.

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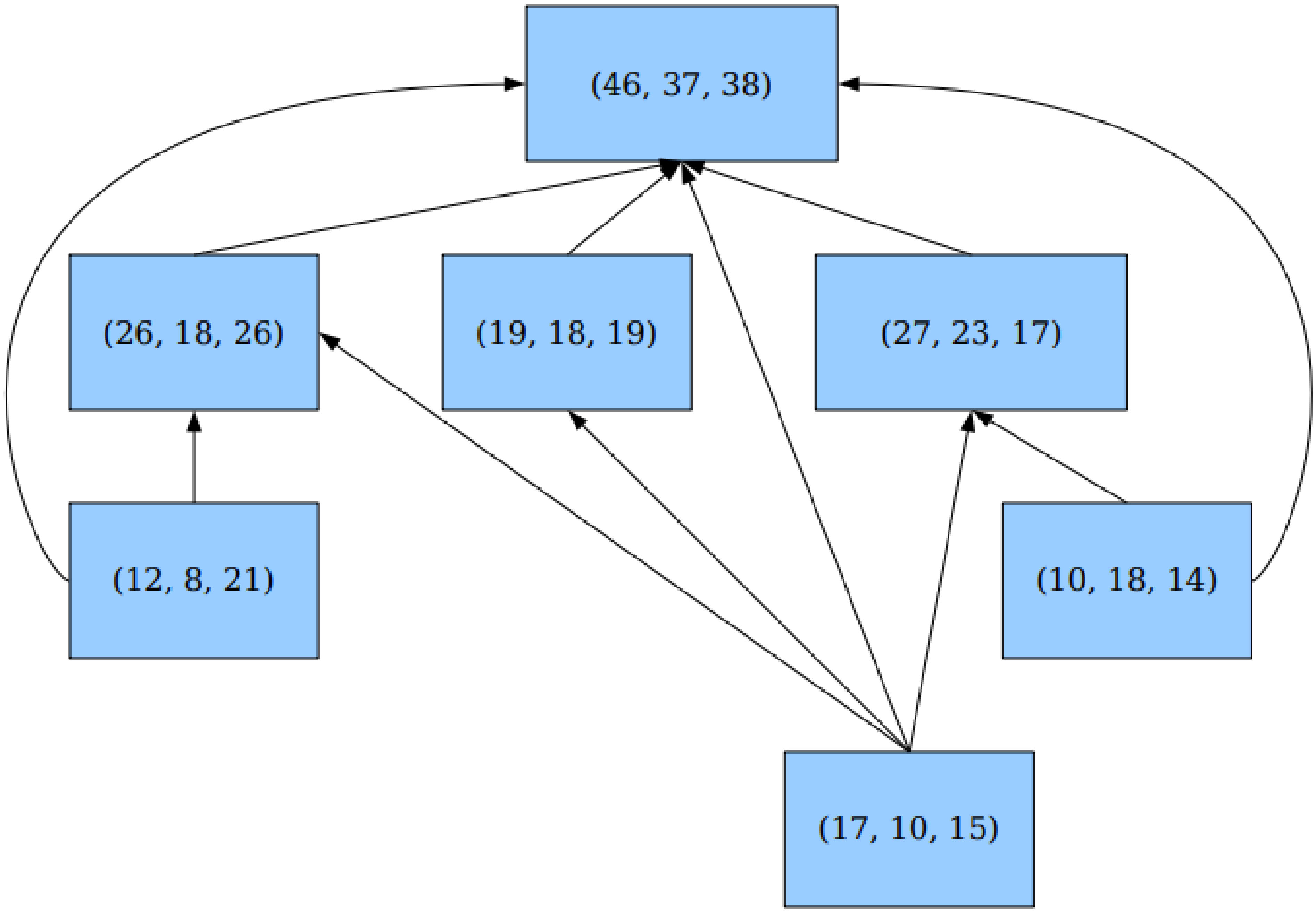
We have reached a contradiction, so our assumption must have been wrong. Thus R must be irreflexive. ■

Theorem: If a binary relation R is asymmetric and transitive, then R is a strict order.

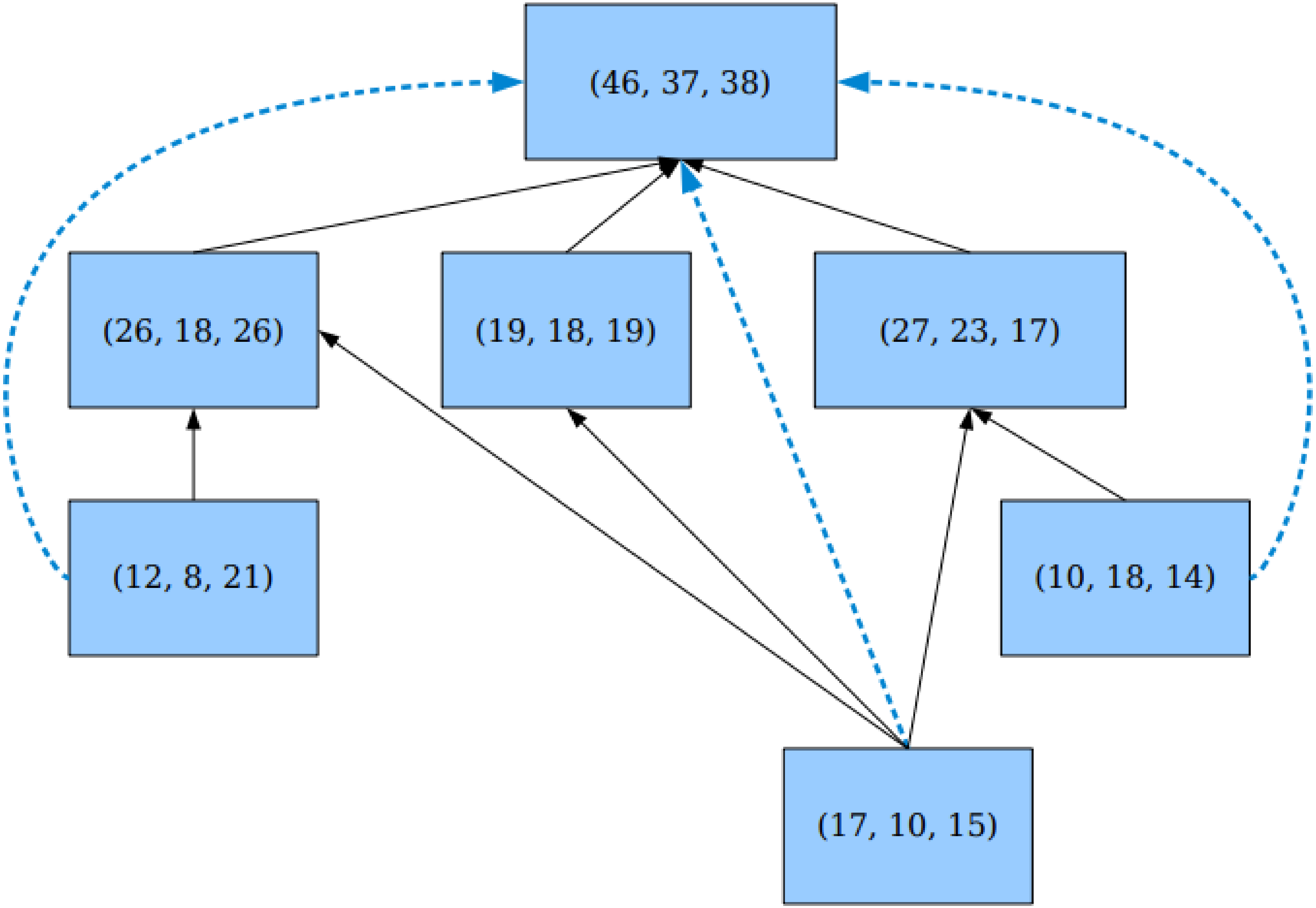
Proof: Let R be a binary relation that is asymmetric and transitive. Since R is asymmetric, by our previous theorem we know that R is also irreflexive. Therefore, R is asymmetric, irreflexive, and transitive, so by definition R is a strict order. ■

Drawing Strict Orders

Gold	Silver	Bronze
46	37	38
27	23	17
26	18	26
19	18	19
17	10	15
12	8	21
10	18	14
9	3	9
8	12	8
8	11	10
8	7	4
8	3	4
7	6	6
7	4	6
6	6	1
6	3	2



$$(g_1, s_1, b_1)\ R\ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \wedge \ s_1 < s_2 \ \wedge \ b_1 < b_2$$



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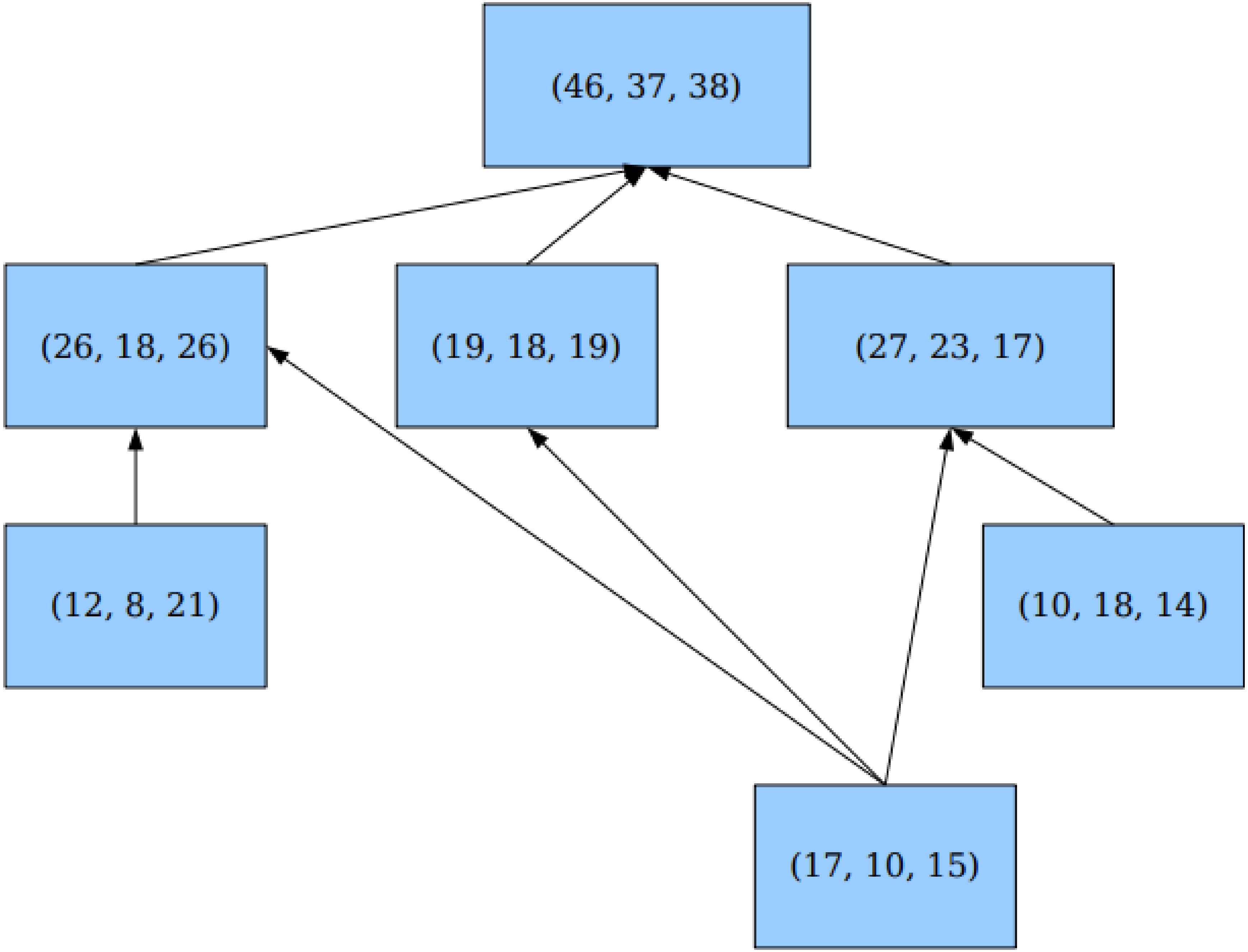

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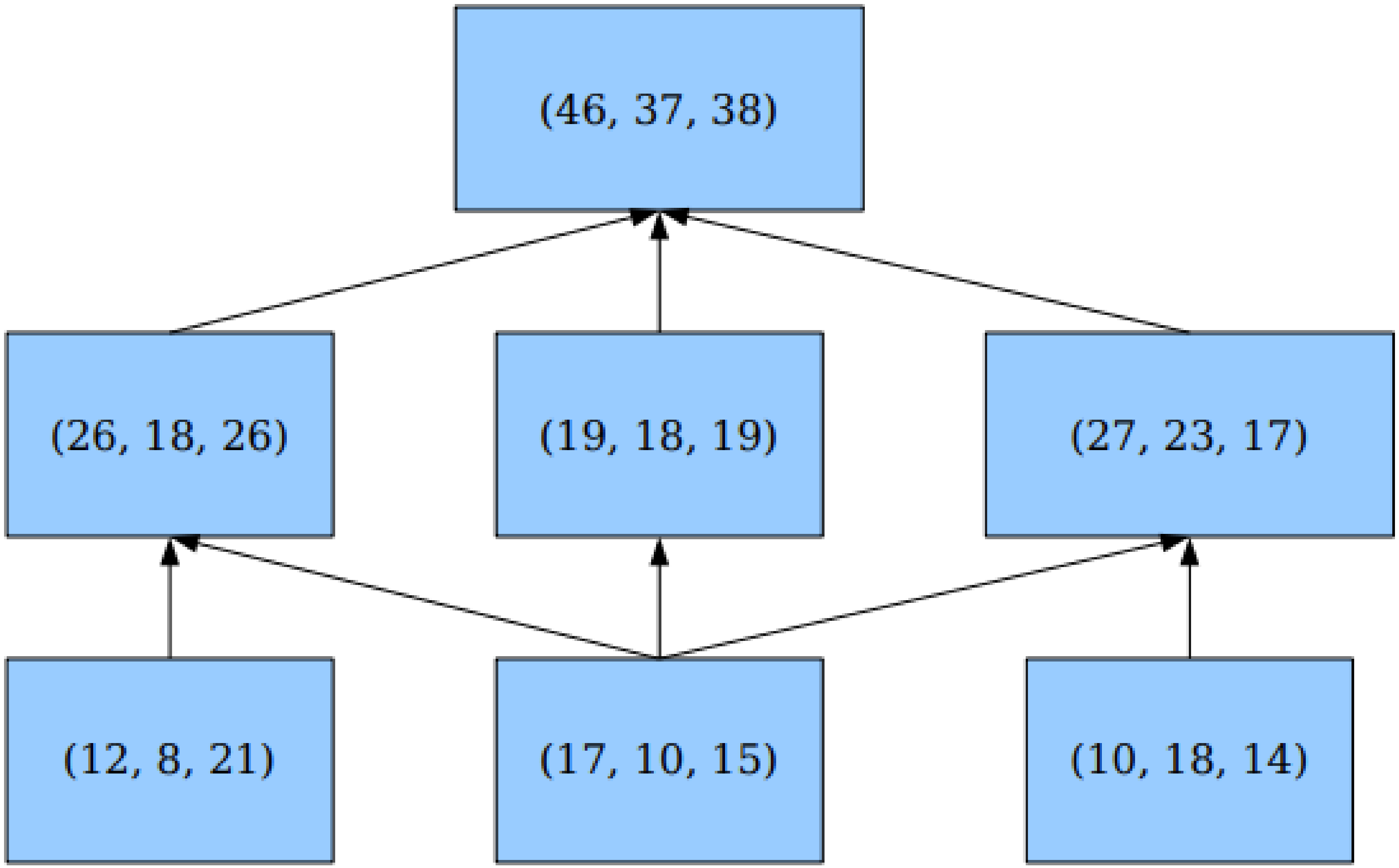

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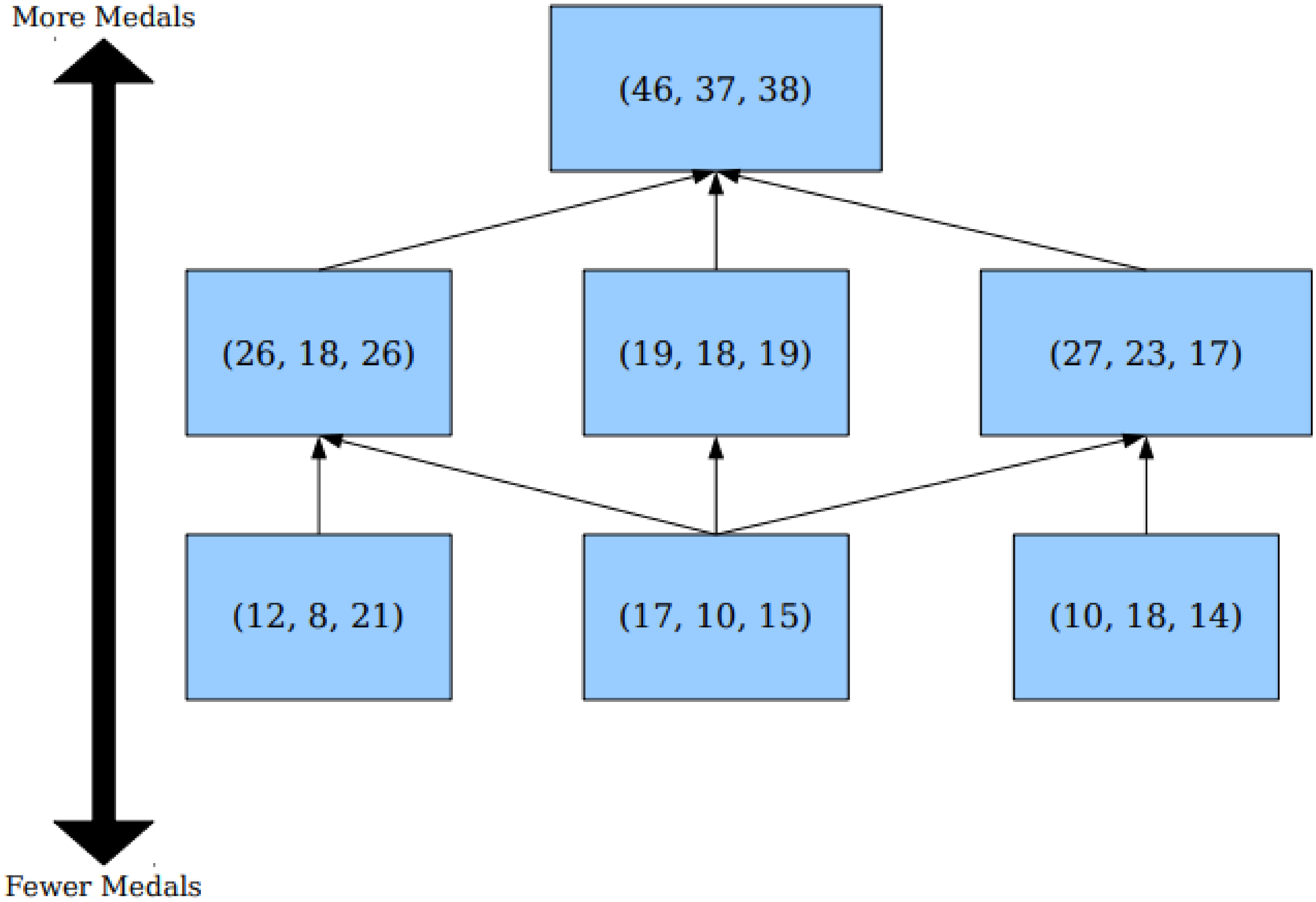

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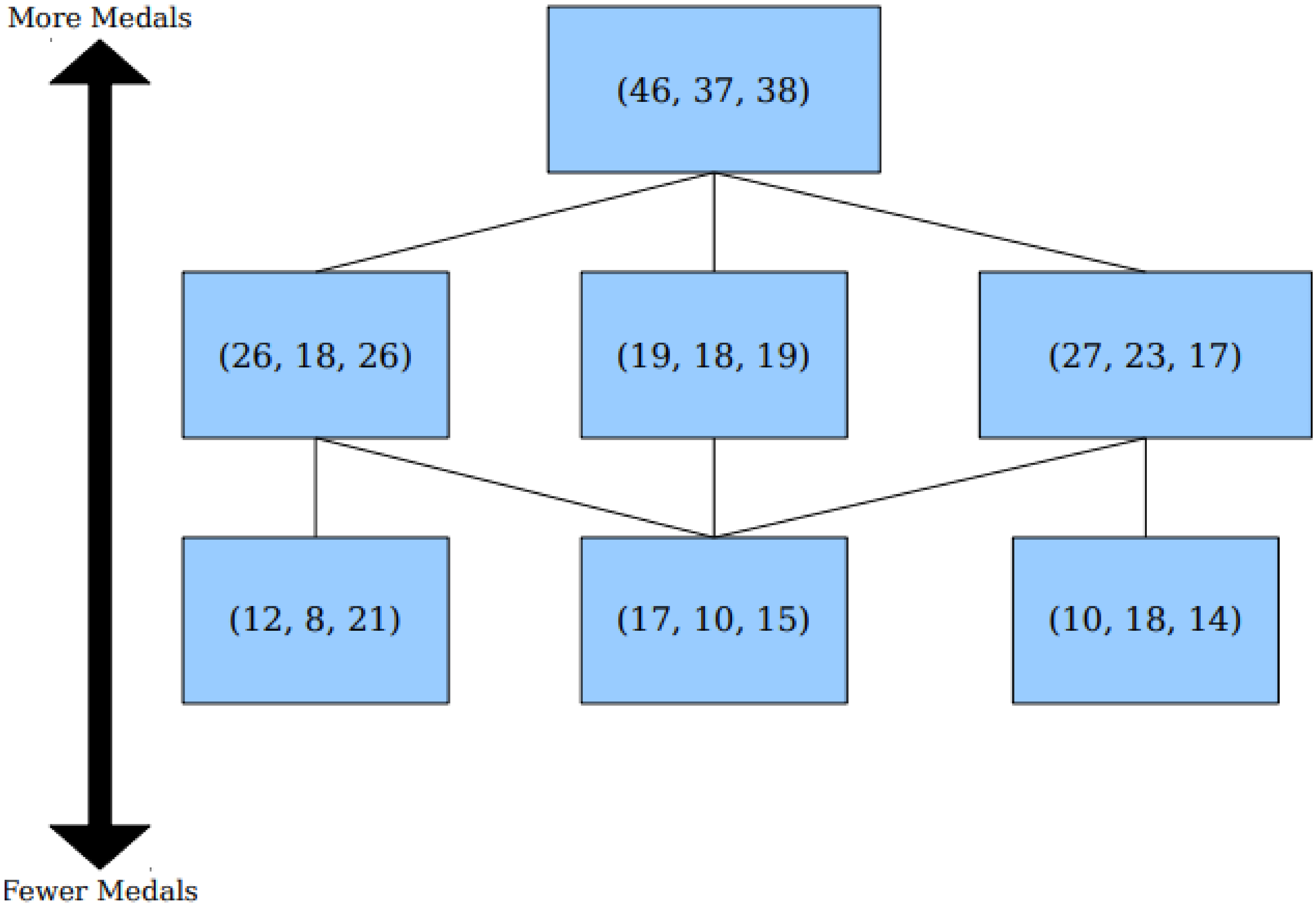

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Hasse Diagrams

- A ***Hasse diagram*** is a graphical representation of a strict order.
- Elements are drawn from bottom-to-top.
- Higher elements are bigger than lower elements: by ***asymmetry***, the edges can only go in one direction.
- No redundant edges: by ***transitivity***, we can infer the missing edges.

(46, 37, 38)

(27, 23, 17)

(26, 18, 26)

(19, 18, 19)

(17, 10, 15)

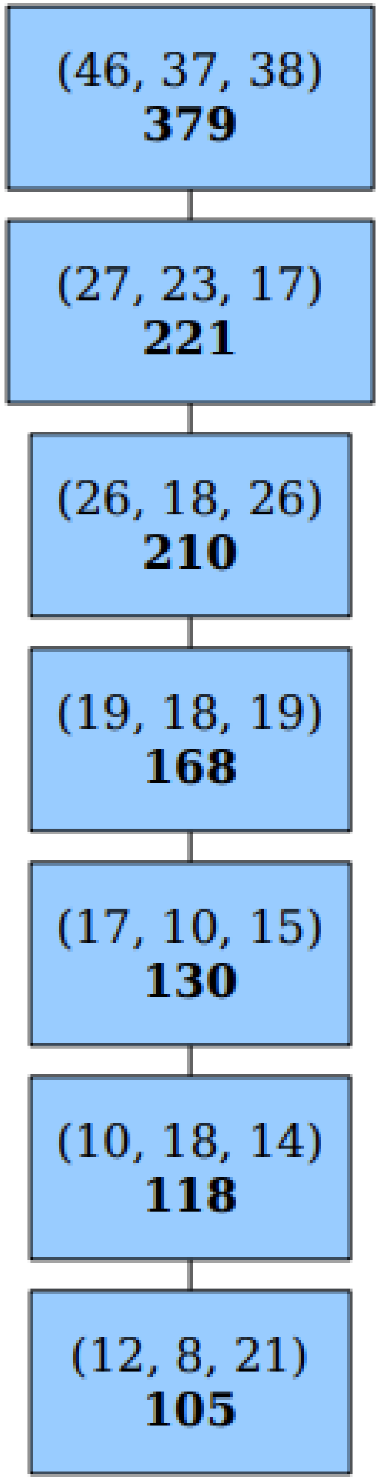
(10, 18, 14)

(12, 8, 21)

$$(g_1, s_1, b_1)\ T\ (g_2, s_2, b_2)$$

if

$$5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2$$



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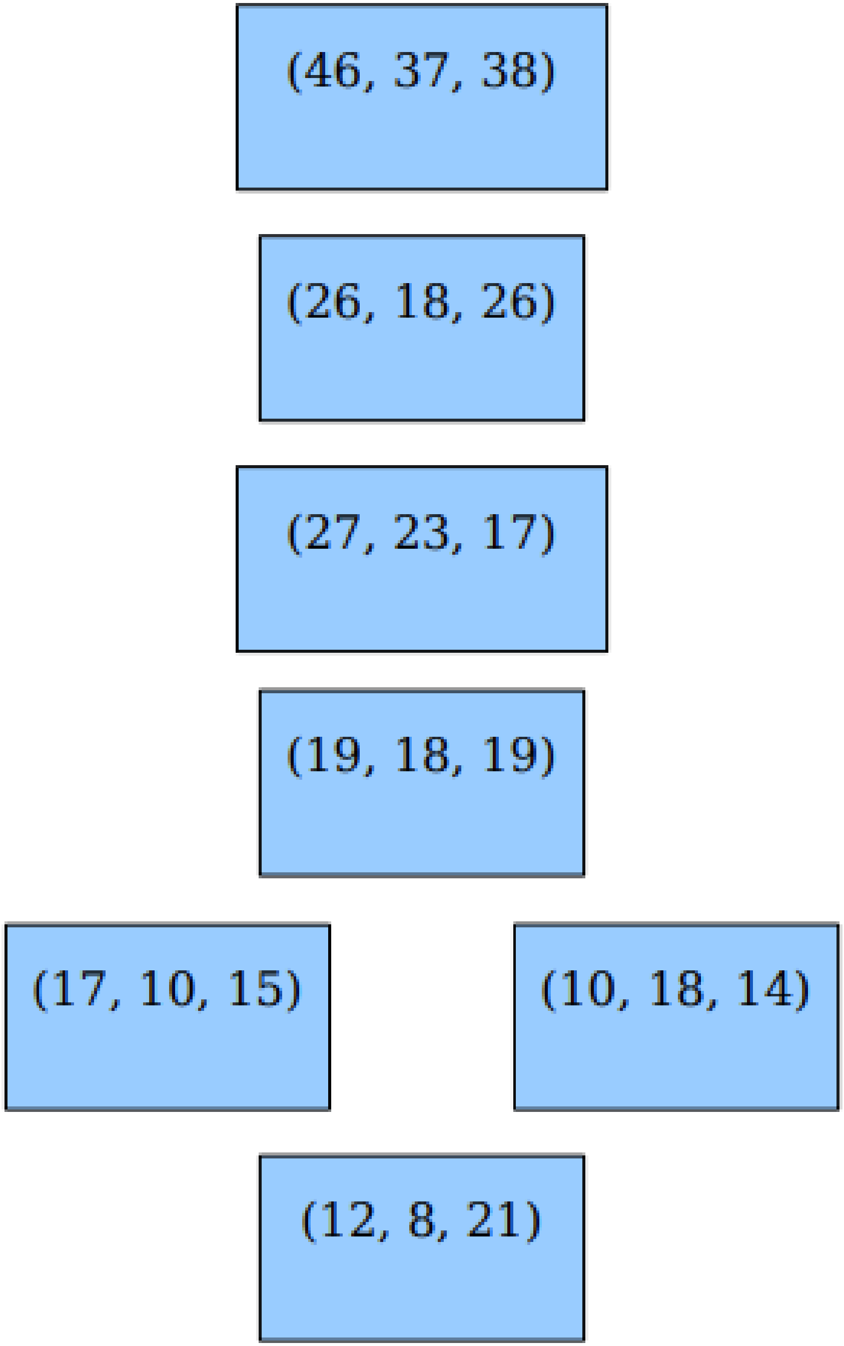
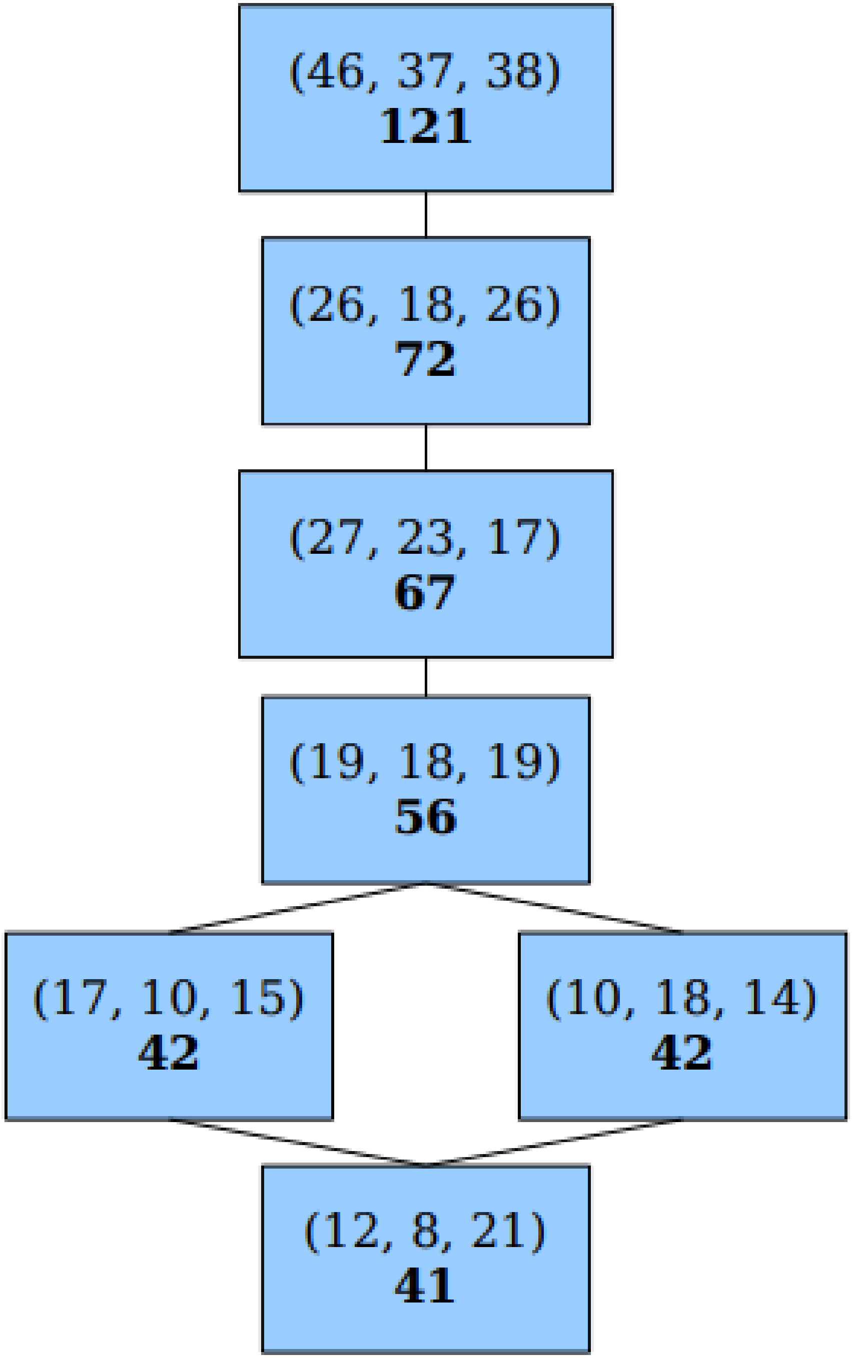


$(g_1, s_1, b_1)\ U\ (g_2, s_2, b_2)$

if

$g_1 + s_1 + b_1 < g_2 + s_2 + b_2$

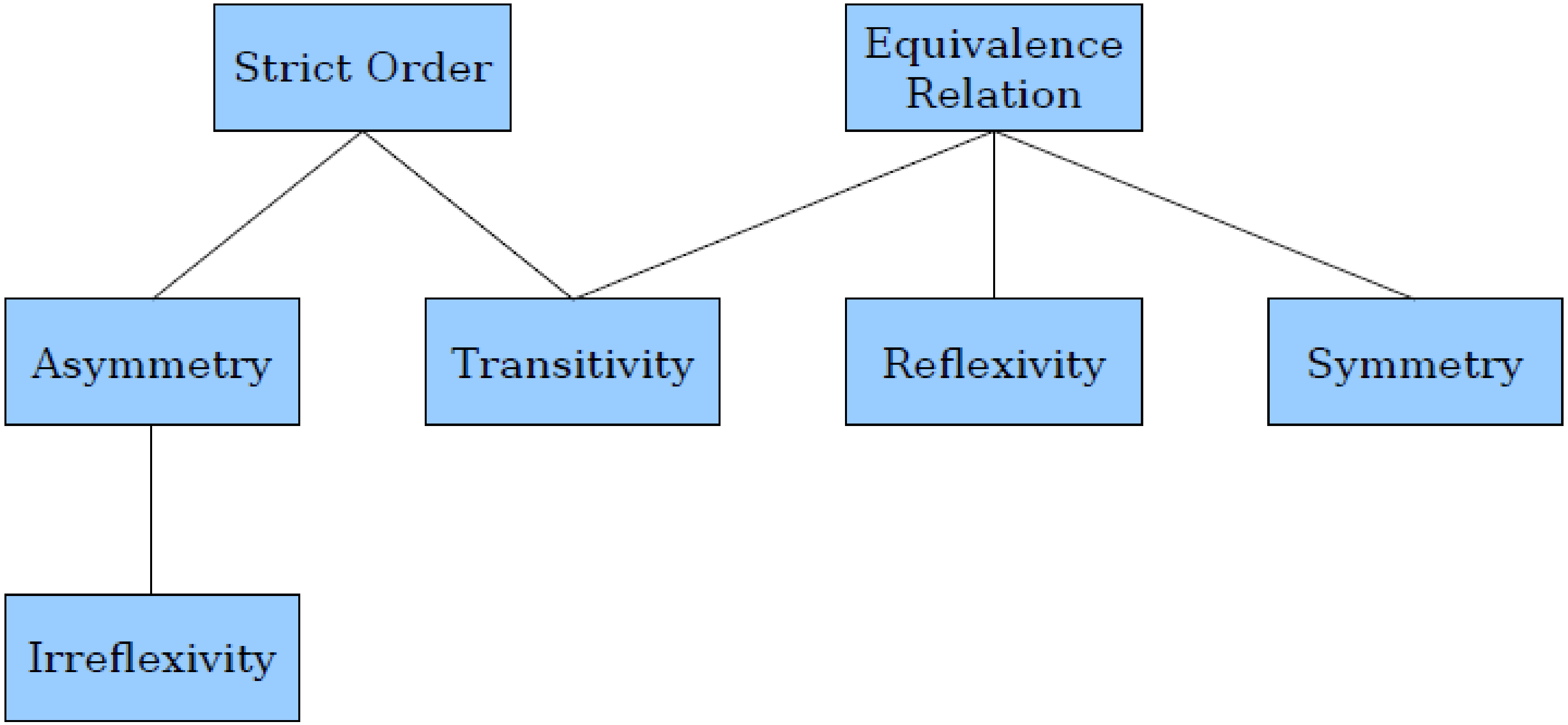


$$(g_1, s_1, b_1) \; U \; (g_2, s_2, b_2)$$

if

$$g_1 + s_1 + b_1 < g_2 + s_2 + b_2$$

The Meta Strict Order



aRb **if** ***a* is less specific than *b***

Antisymmetry

A binary relation R over a set A is called **antisymmetric** iff

For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not R x$.

Equivalently:

For any $x \in A$ and $y \in A$,
if xRy and yRx, then $x = y$.

Antisymmetry

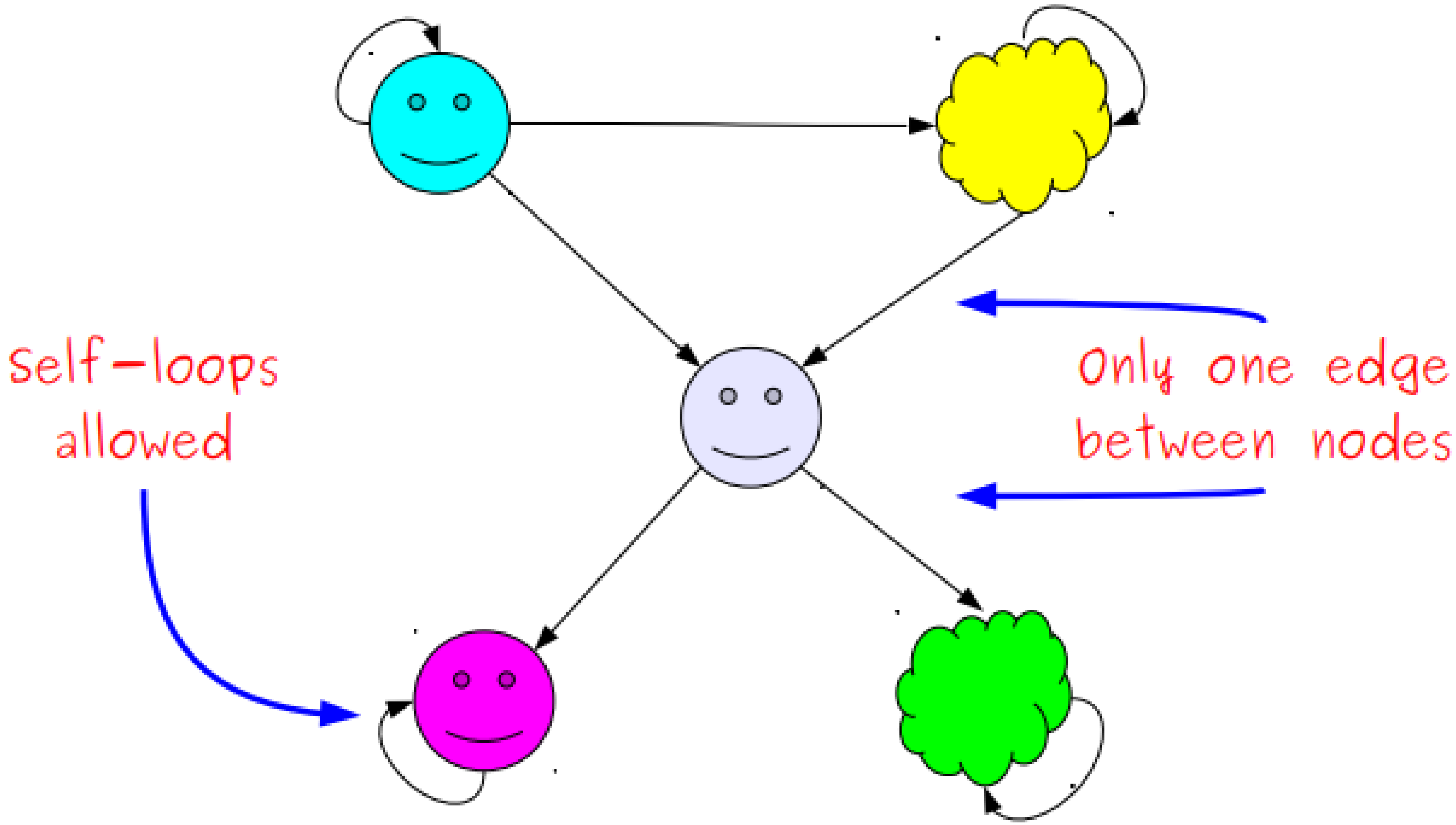
A binary relation R over a set A is called **antisymmetric** iff

For any $x \in A$ and $y \in A$,
if xRy and yRx, then $x = y$.

Example: Let A={1,2,3,6} and R is a relation on A,
where R = {(a, b) : a∈ A, b ∈ A, and a|b}.

R is antisymmetric relation.

An Intuition for Antisymmetry



For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not R x$.

Partial Orders

- A binary relation R is a **partial order** over a set A iff it is
  - **reflexive**,
  - **antisymmetric**, and
  - **transitive**.
- A pair (A, R), where R is a partial order over A, is called a **partially ordered set** or **poset**.

Partial Orders

- A binary relation R is a **partial order** over a set A iff it is
  - **reflexive**,
  - **antisymmetric**, and
  - **transitive**.

Why "partial"?

- A pair (A, R), where R is a partial order over A, is called a **partially ordered set** or **poset**.

Partial Orders

A partial order is a relation that is reflexive, antisymmetric, and transitive.

Example: Let $A=\{1,2,3,6\}$ and R is a relation on A, where $R = \{(a, b) : a \in A,\ b \in A, \text{ and } a|b\}$.

R is antisymmetric relation.
R is reflexive relation.
R is transitive relation.

R is a partial order relation. (A,R) is a poset.

Partial Orders

A partial order is a relation that is reflexive, antisymmetric, and transitive.

Example: Show that the divisibility relation (|) is a partial ordering on Z.

- Reflexivity: a | a for all integers a.
- Antisymmetry: If a and b are positive integers with a | b and b | a, then a = b.
- Transitivity: Suppose that a divides b and b divides c. Then there are positive integers k and l such that b = ak and c = bl. Hence, c = a(kl), so a divides c. Therefore, the relation is transitive. ■

Note that (Z, |) is a poset.

Comparability

The elements a and b of a set that a relation R defined on are comparable if either aRb or bRa.

Example: The “greater than or equal” relation (≥) on the set of integers is comparable because any two integers i and j are either I ≥ j or j ≥ i.

Total Orders

A total order is a relation that is partial order and comparable.

Example: Show that the "greater than or equal" relation (≥) is a total ordering on the set of integers.

- Reflexivity: a ≥ a for every integer a.
- Antisymmetry: If a ≥ b and b ≥ a , then a = b.
- Transitivity: If a ≥ b and b ≥ c , then a ≥ c.
- Comparabilty: a ≥ b or b ≥ a ■

Example: Is a partial order divisibility relation (|) is a total order on Z.

- No: It contains elements that are incomparable, such as 5 and 7.