Discrete Mathematics

Basic Counting Techniques

Counting with Repetitions

How many ways to rearrange the letters in the word "SYSTEMS"?

SYSTEMS

__ / __ / __ / __ / __ / __ / __

7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced

$$7 \times 6 \times 5 \times 4 = 840$$

SYSTEMS

Let’s pretend that the S’s are distinct:

$S_1YS_2TEMS_3$

SYSTEMS

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There are 7! permutations of $S_1YS_2TEMS_3$

SYSTEMS

Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$

$$\frac{7!}{3!} = 840$$

SYSTEMS

1 Y / 1 T/ 1 E/ 1 M/ 3 S

SYSTEMS

1 Y / 1 T/ 1 E/ 1 M/ 3 S

7 x 6 x 5 x 4 x 1

$$\binom{7}{1}\binom{7-1}{1}\binom{7-2}{1}\binom{7-3}{1}\binom{7-4}{3}$$

SYSTEMS

1 Y / 1 T/ 1 E/ 1 M/ 3 S

$$\binom{7}{1}\binom{7-1}{1}\binom{7-2}{1}\binom{7-3}{1}\binom{7-4}{3}$$

(7!/(6!1!))(6!/(5!1!))(5!/(4!1!))(4!/(3!1!))(3!/3!)

SYSTEMS

1 Y / 1 T/ 1 E/ 1 M/ 3 S

$\binom{7}{1} \binom{7-1}{1} \binom{7-2}{1} \binom{7-3}{1} \binom{7-4}{3}$

(7!/(6!1!))(6!/(5!1!))(5!/(4!1!))(4!/(3!1!))(3!/3!)

SYSTEMS

1 Y / 1 T/ 1 E/ 1 M/ 3 S

$\binom{7}{1}\binom{7-1}{1}\binom{7-2}{1}\binom{7-3}{1}\binom{7-4}{3}$

(7!/(6!1!))(6!/(5!1!))(5!/(4!1!))(4!/(3!1!))(3!/3!)

7!/1!1!1!1!3!

Subset Split Rule :

Arrange n symbols: r_1 of type 1, r_2 of type 2, …, r_k of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!\,r_1!}\;\frac{(n-r_1)!}{(n-r_1-r_2)!\,r_2!}\;\cdots$$

$$= \frac{n!}{r_1!\,r_2!\,\ldots\,r_k!}$$

How many ways to rearrange the letters in the word "CARNEGIEMELLON"?

How many ways to rearrange the letters in the word "CARNEGIEMELLON"?

$$\frac{14!}{2!3!2!} = 3{,}632{,}428{,}800$$

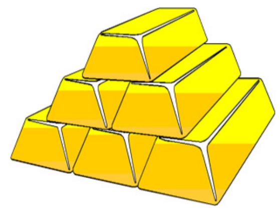
Bookkeeper Rule

Let $l_1, \ldots, l_m$ be distinct elements. The number of sequences with k_1 occurrences of l_1, and k_2 occurrences of $l_2, \ldots,$ and k_m occurrences of l_m is

$$\frac{(k_1 + k_2 + \ldots + k_m)!}{k_1! k_2! \ldots k_m!}$$

Pirates and Bars of Gold

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



Sequences with 20 G’s and 4 /’s

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGG/G

represents the following division among the pirates

1st pirate gets 2

2nd pirate gets 1

3rd gets nothing

4th gets 16

5th gets 1

Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/G

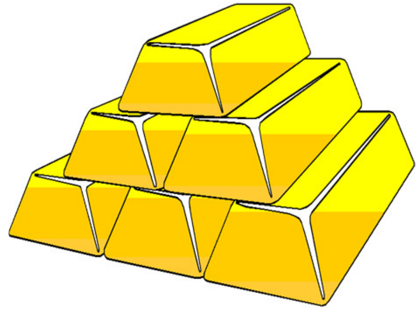
In general, the k^{th} pirate gets the number of G's after the "**k-1**$^{\mathbf{st}}$ **/**" and before the "**k**$^{\mathbf{th}}$ **/**".

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

How many sequences with 20 G's and 4 /'s?

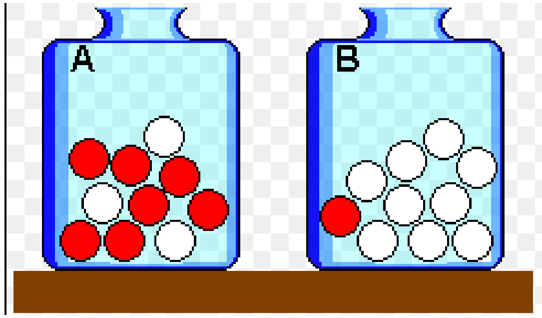
$$\binom{24}{4} = \binom{20+5-1}{5-1}$$

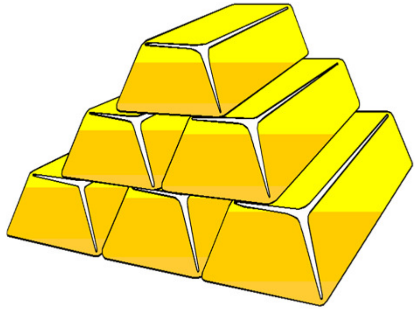


How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Another interpretation





How many different ways to put k indistinguishable balls into n distinguishable urns.

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Another interpretation

How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Think of x_k as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer nonnegative solutions to the following equations?

$$x_1 + x_2 + \ldots + x_n = k$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + \dots + x_n = k$$

$$x_1, x_2, x_3, \dots, x_n > 0$$

Think of $x_k \geq 1$, we may replace x_k with y_k+1

bijection with solutions to

$$y_1 + y_2 + y_3 + \dots + y_n = k-n$$

$$y_1, y_2, y_3, \dots, y_n \geq 0$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 > 0$$

bijection with solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 20\text{-}5$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 > 0$$

bijection with solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 15$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

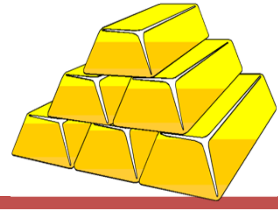
$$x_1, x_2, x_3, x_4, x_5 > 0$$

bijection with solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 15 \qquad \binom{19}{4}$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot? No pirate gets 0 bar.



$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 > 0$$

bijection with solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 15$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

$$\binom{19}{4}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

bijection with solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + n = 20$$

$$x_1, x_2, x_3, x_4, x_5 , n \geq 0$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

bijection with solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + n = 20$$

$$x_1, x_2, x_3, x_4, x_5 , n \geq 0$$

$$\binom{25}{5}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \geq 1,\ x_2 \geq 2,\ x_3 \geq 3,\ x_4 \geq 4,\ x_5 \geq 5$$

bijection with solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20-1-2-3-4-5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \geq 1,\ x_2 \geq 2,\ x_3 \geq 3,\ x_4 \geq 4,\ x_5 \geq 5$$

bijection with solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20\text{-}1\text{-}2\text{-}3\text{-}4\text{-}5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\binom{9}{4}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \geq 1,\ x_2 \geq 2,\ x_3 \geq 3,\ x_4 \geq 4,\ x_5 \geq 5$$

bijection with solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20\text{-}1\text{-}2\text{-}3\text{-}4\text{-}5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\binom{9}{4}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and } x_1 \leq 9$$

its complement is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and } x_1 \geq 10$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and } x_1 \leq 9$$

its complement is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20 - 10$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\binom{14}{4}$$

How many integer positive solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and } x_1 \leq 9$$

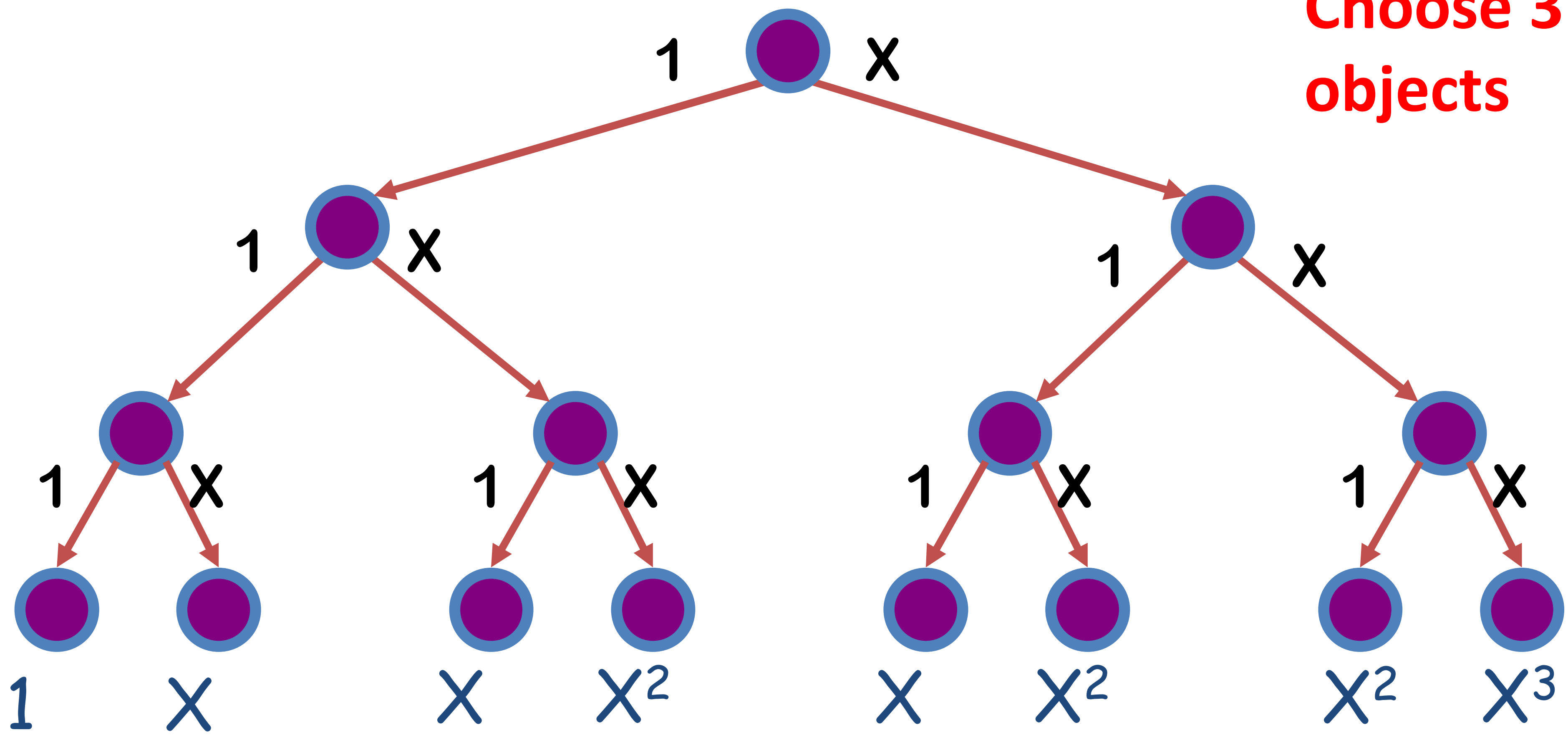
$$\binom{24}{4} - \binom{14}{4}$$

Binomial Theorem

There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice tree for terms of $(1+X)^3$

Choose 3 objects



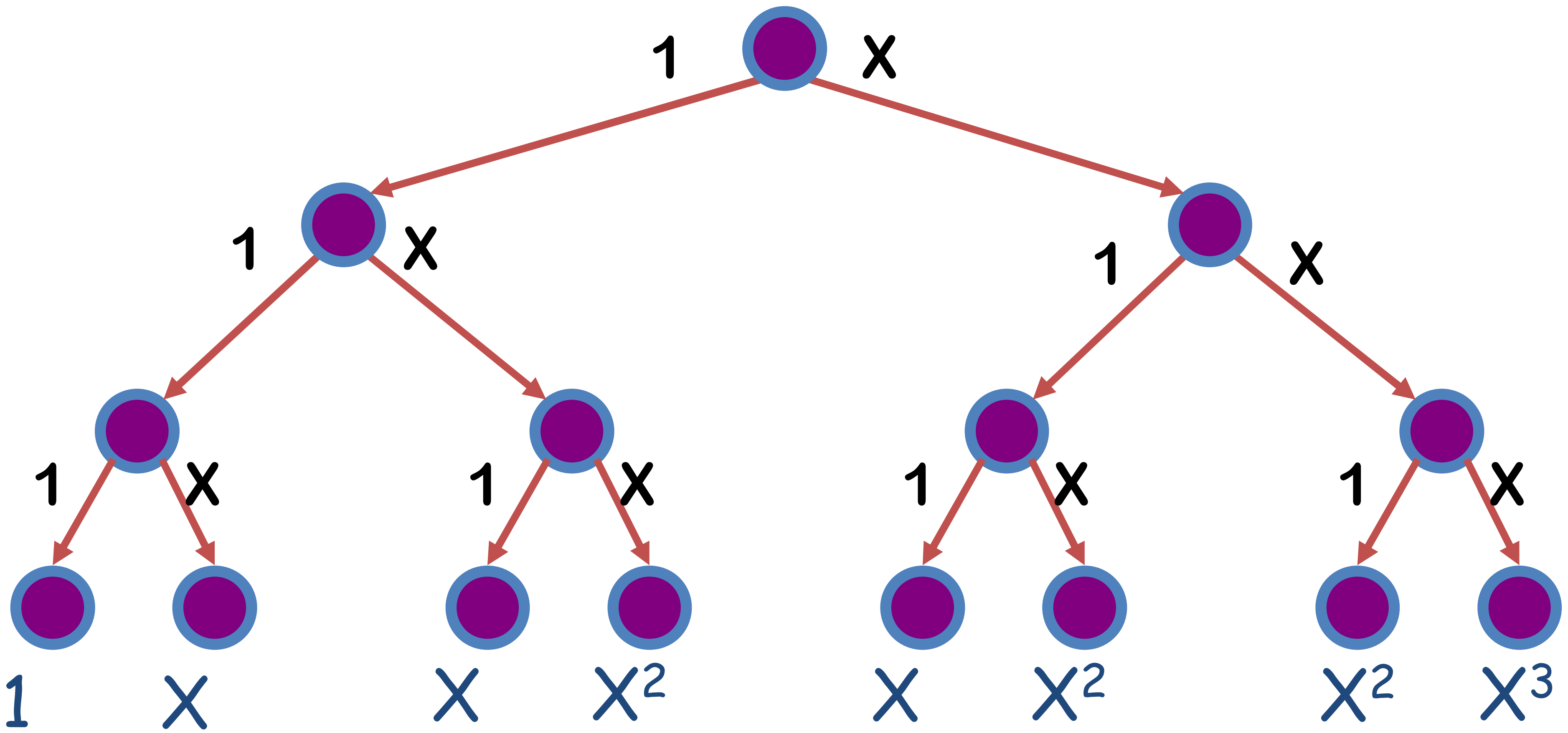
Each Object

1 way for not choosing an object

1 way for choosing an object

$$1+X$$

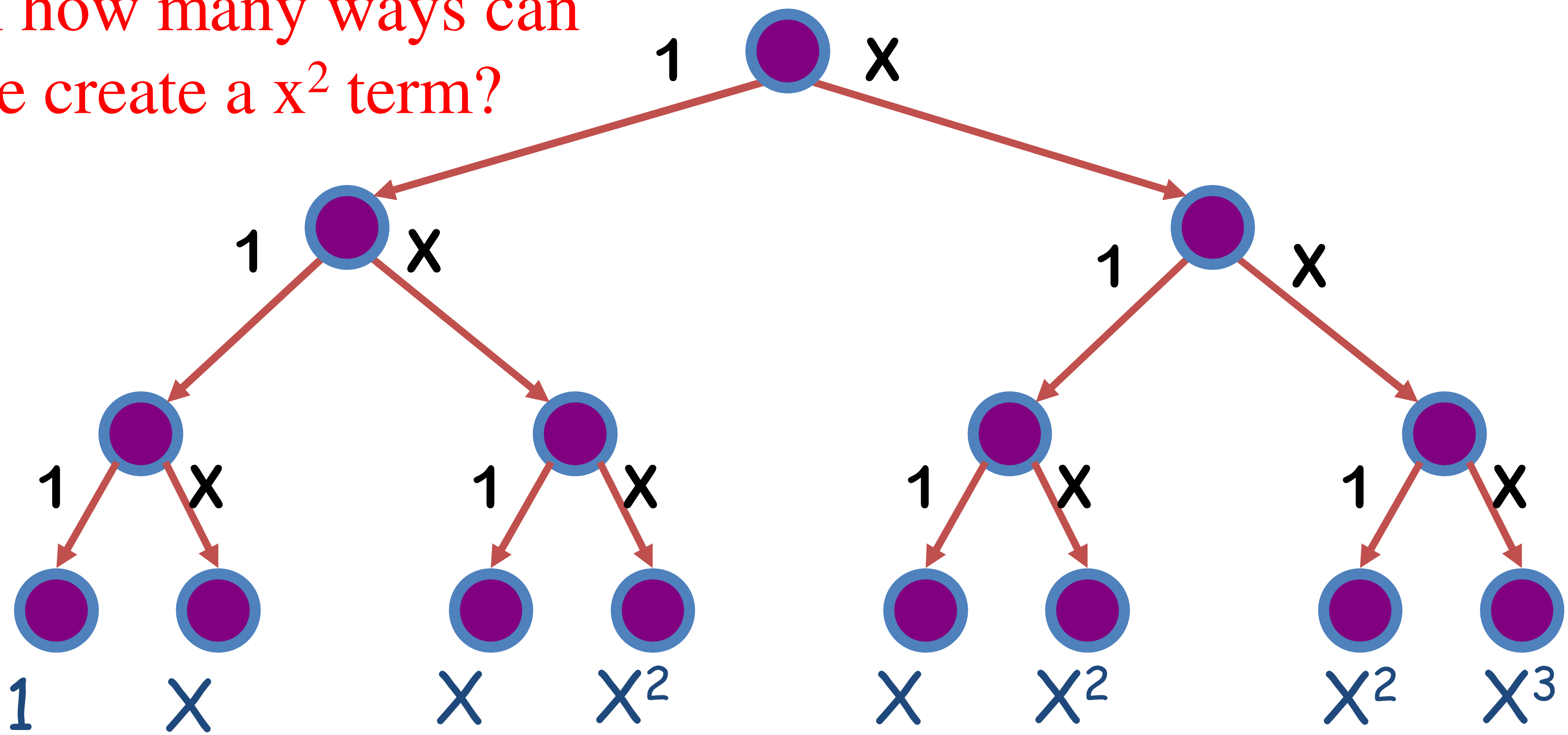
Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

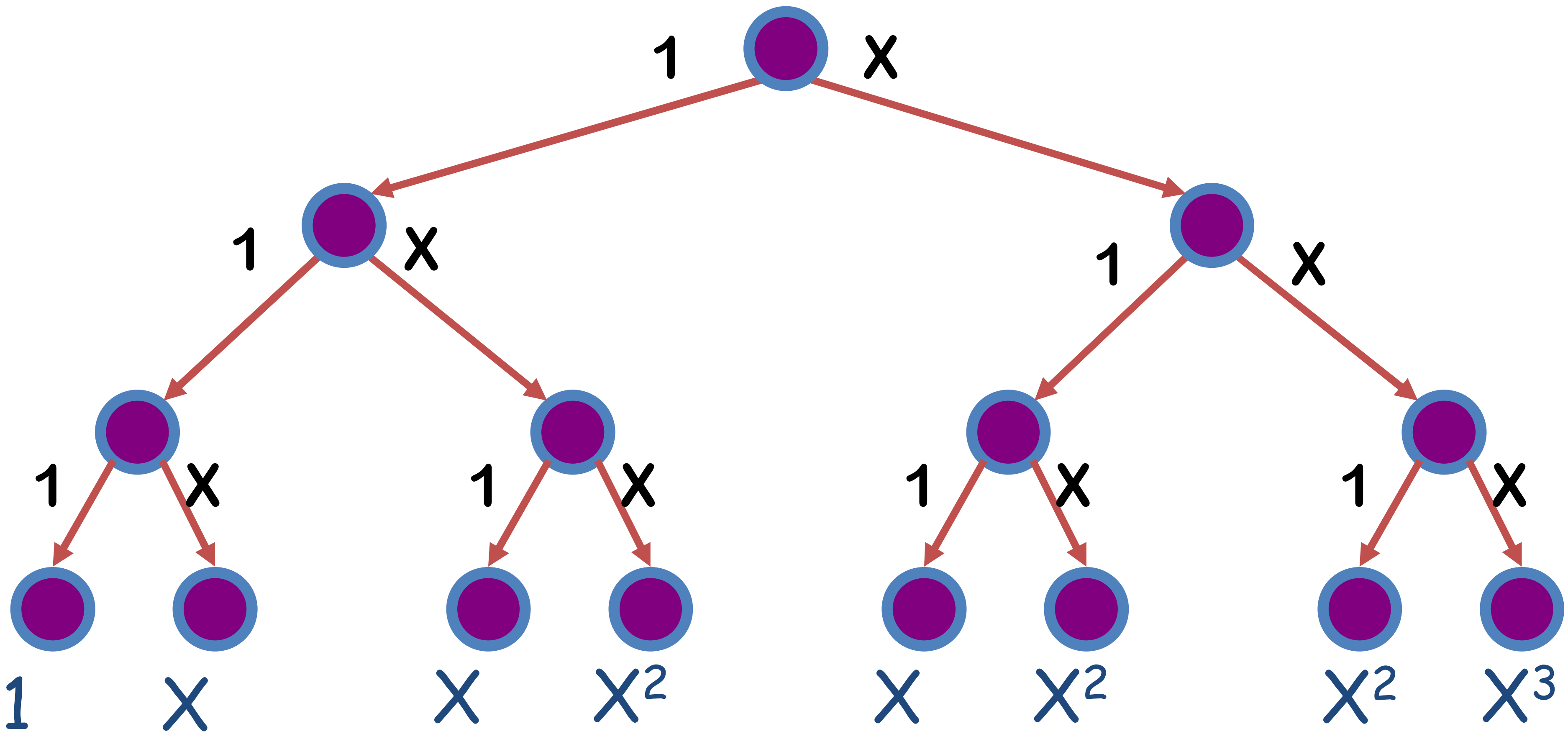
$$(1+X)^3 = 1 + 3X + 3X^2 + X^3$$

In how many ways can we create a x^2 term?



What is the <u>combinatorial meaning</u> of those coefficients?

$$(1+X)^3 = 1 + 3X + 3X^2 + X^3$$



1 way to choose no objects
3 ways to choose one object
3 ways to choose two objects
1 way to choose three objects

What is a closed form expression for c_k?

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$

In how many ways can we create a x^2 term?

What is a closed form expression for c_k?

$$(1+X)^n$$

$$= \overbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}^{\text{n times}}$$

After multiplying things out, but before combining like terms, we get 2^n cross terms, each corresponding to a path in the choice tree.

c_k, the coefficient of X^k, is the number of paths with exactly *k X's*.

$$c_k = \binom{n}{k}$$

The Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

Binomial Coefficients

binomial
expression

The Binomial Formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$
\begin{aligned}
(a+b)^4 = \quad & aaaa + aaab + aaba + aabb \\
+\ & abaa + abab + abba + abbb \\
+\ & baaa + baab + baba + babb \\
+\ & bbaa + bbab + bbba + bbbb
\end{aligned}
$$

All length 4 sequences

$$(a+b)^4 = \binom{4}{0} \cdot a^4 b^0 + \binom{4}{1} \cdot a^3 b^1 + \binom{4}{2} \cdot a^2 b^2 + \binom{4}{3} \cdot a^1 b^3 + \binom{4}{4} \cdot a^0 b^4$$

What is the coefficient of EMPTY in the expansion of $(E + M + P + T + Y)^5$?

$5! = 5!/1!1!1!1!1!$

What is the coefficient of EMP^3TY in the expansion of $(E + M + P + T + Y)^7$?

$7!/1!1!3!1!1!$

What is the coefficient of BA^3N^2 in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA

$6!/1!3!2!$

What is the coefficient of

$$X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$

in the expansion of

$(X_1+X_2+\ldots+X_k)^n$?

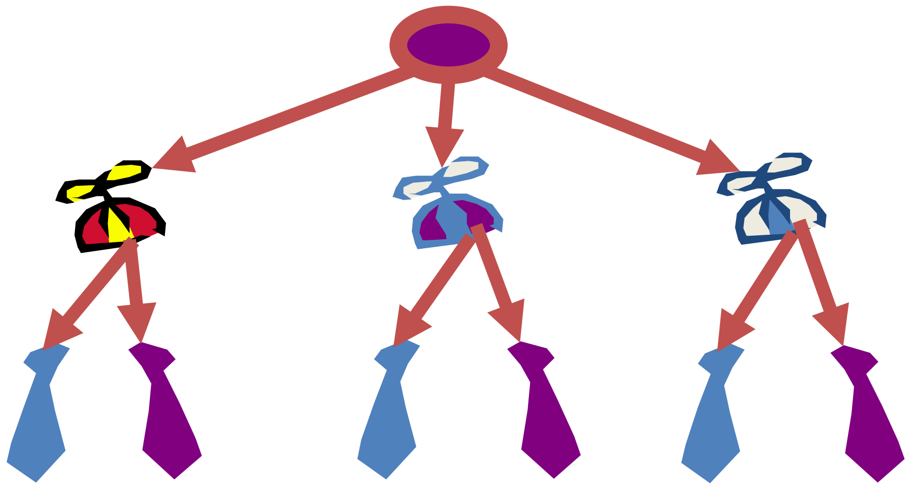
$$\frac{n!}{r_1! r_2! \ldots r_k!}$$

The Multinomial Formula

$$\left(X_1 + X_2 + \ldots + X_k\right)^n = \sum_{\substack{r_1, r_2, \ldots, r_k \\ \sum r_i = n}} \binom{n}{r_1; r_2; \ldots; r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$

Counting with Generating Functions

There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!



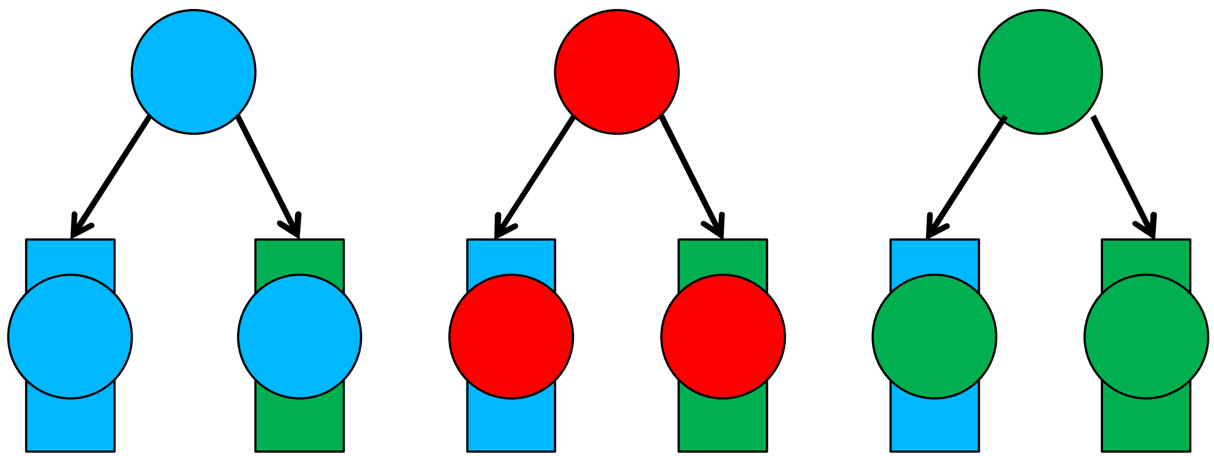
A Particular Counting Problem

Let’s talk about a particular counting problem

Danny owns 3 beanies and 2 ties. How many ways can he dress up in a beanie and a tie? (choose one for each type)

Choice 1

Choice 2



A Particular Counting Problem

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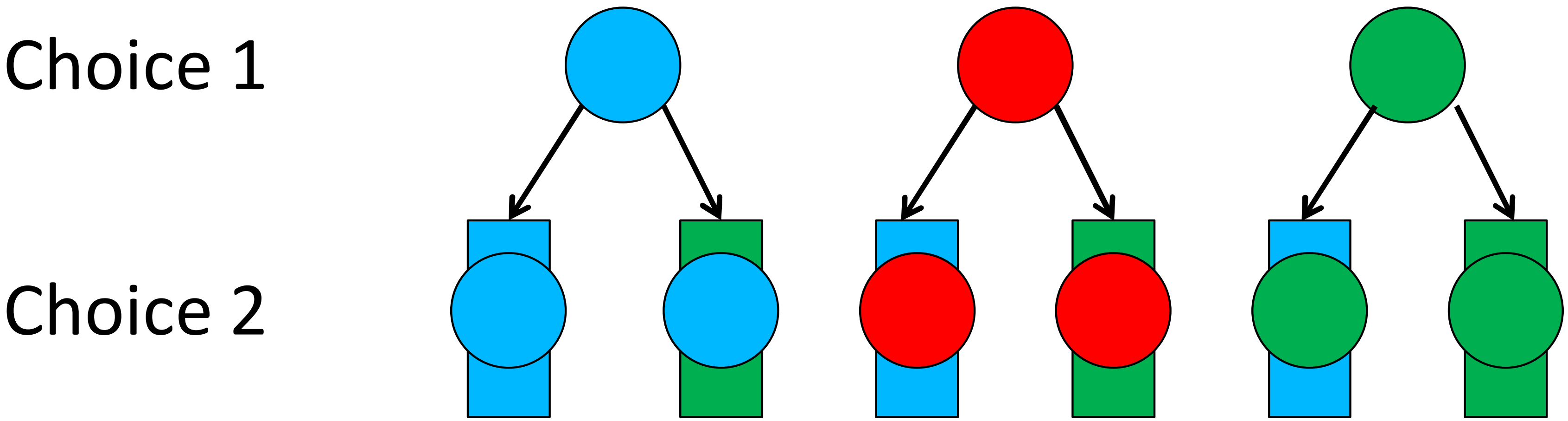
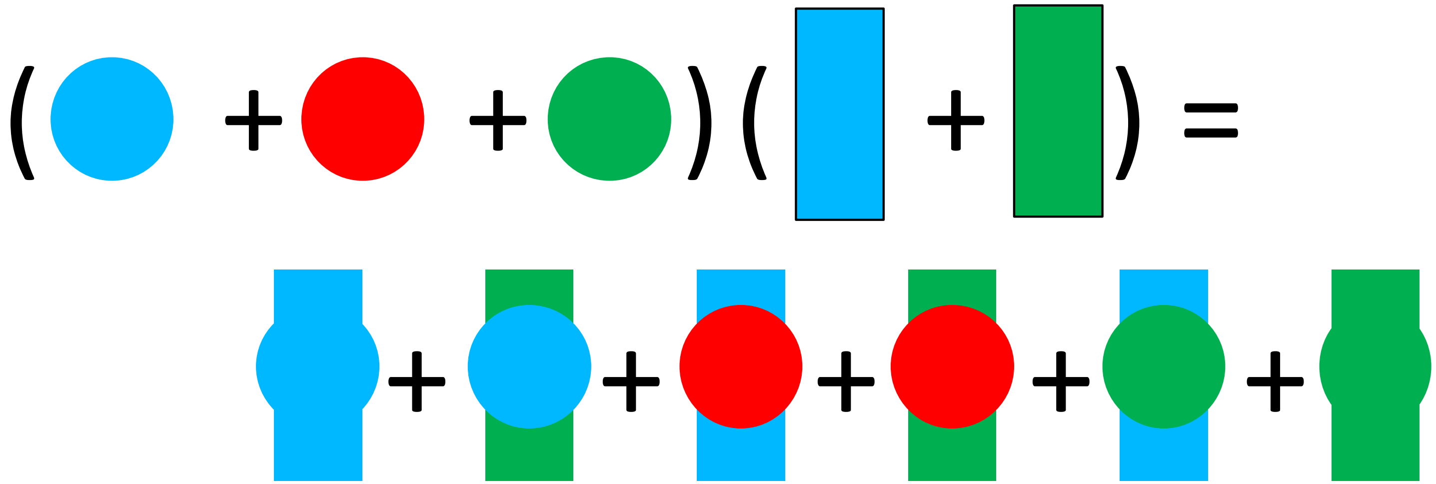




A Particular Counting Problem

Danny owns 3 beanies and 2 ties. How many ways can he dress up in a beanie and a tie?

How many beanies are we choosing?

How many hats?

Since we only care about the NUMBER, we can replaces beanies and hats with ‘x’

$$(x + x + x)(x + x) =$$

$$x^2 + x^2 + x^2 + x^2 + x^2 + x^2 = 6x^2$$

That is, 6 is the number of ways to choose 2 things

A General Counting Problem

Danny owns 3 beanies and 2 ties. How many ways can he dress up if he doesn't always wear a beanie or a tie (and wears at most one of each)?

How many ways for a beanie?

1) 1 way for no beanie
2) 3 ways for one beanie

$$1 + 3x$$

How many ways for a tie?

1) 1 way for no tie
2) 2 ways for one tie

$$1 + 2x$$

$$(1 + 3x)(1 + 2x) = 1 + 5x + 6x^2$$

Problem

You are allowed to choose two items from a tray containing an apple, an orange, a pear, a banana, and a plum. In how many ways can you choose?

$$\binom{5}{2}$$

Counting with Generating Functions

(0 apple + 1 apple)(0 orange + 1 orange)
(0 pear + 1 pear)(0 banana + 1 banana)
(0 plum + 1 plum)

In this notation, apple^2 stand for choosing 2 apples., and + stands for an exclusive or.

$$(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x)$$

Take the coefficient of x^2 which is $\binom{5}{2}$.

Problem

You are allowed to choose two items from a tray containing **TWO apples**, an orange, a pear, a banana, and a plum. In how many ways can you choose?

The two apples are identical.

$$\binom{5}{2} + |\{aa\}| = 11$$

Counting with Generating Functions

(0 apple + 1 apple + 2 apple)
(0 orange + 1 orange)(0 pear + 1 pear)
(0 banana + 1 banana)(0 plum + 1 plum)

$$(1+x+x^2)(1+x)(1+x)(1+x)(1+x)$$

$(1+x+x^2)(1+x)^4 = (1+x+x^2)[\binom{4}{0}+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+\binom{4}{4}x^4]$

Take the coefficient of x^2.

11

The function

$f(x) = (1 + x)^n$

is the generating function for the sequence

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

The function f(x) that has a polynomial expansion

$f(x)=a_0 + a_1 x + \ldots + a_n x^n$

is the generating function for the sequence

$a_0, a_1, \ldots, a_n$

Four Representations

$$\langle 1, 1, 1, \ldots \rangle$$

$$a_k = 1$$

$$a_0 = 1$$
$$a_n = a_{n-1}$$

$$1 + 1x + 1x^2 + \cdots = \frac{1}{1-x}$$

What *IS* a Generating Function?

We'll just looking at a particular representation of sequences…

$$1 + 1x + 1x^2 + \ldots \ = \sum_{n=0}^{\infty} x^n$$

In general, when $\boxed{a_n}$ is a sequence…

$$\sum_{n=0}^{\infty} a_n x^n$$

If the polynomial $(1+x+x^2)^4$

is the generating polynomial

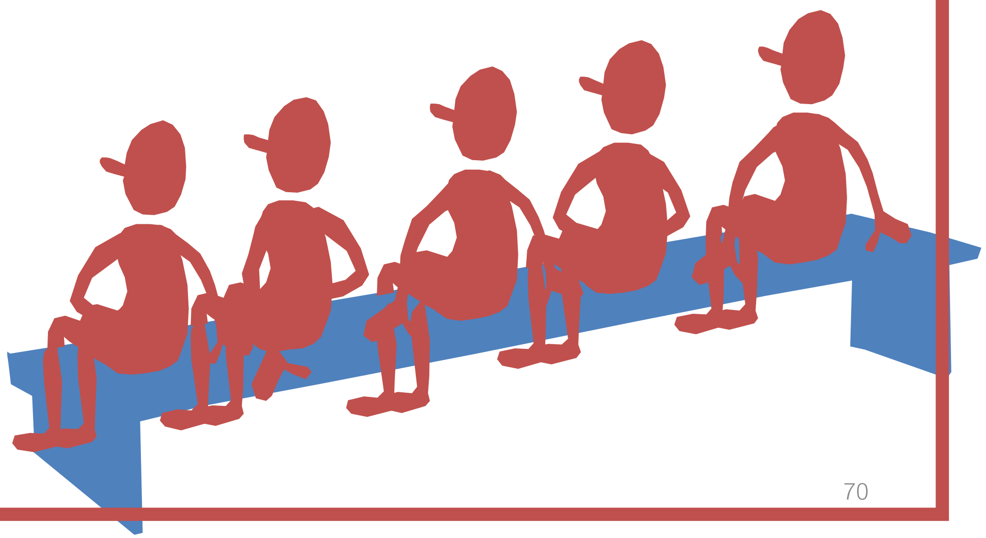
for a_k, what is the combinatorial meaning of a_k?

The number of ways to select k

object from 4 types with at most 2 of each type.

The shop only has 2 apple, 3 cheese, and 4 raspberry pastries left.

What the number of possible orders?



Counting with Generating Functions

$$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x+x^2+x^3+x^4)$$

apple *cheese* *raspberry*

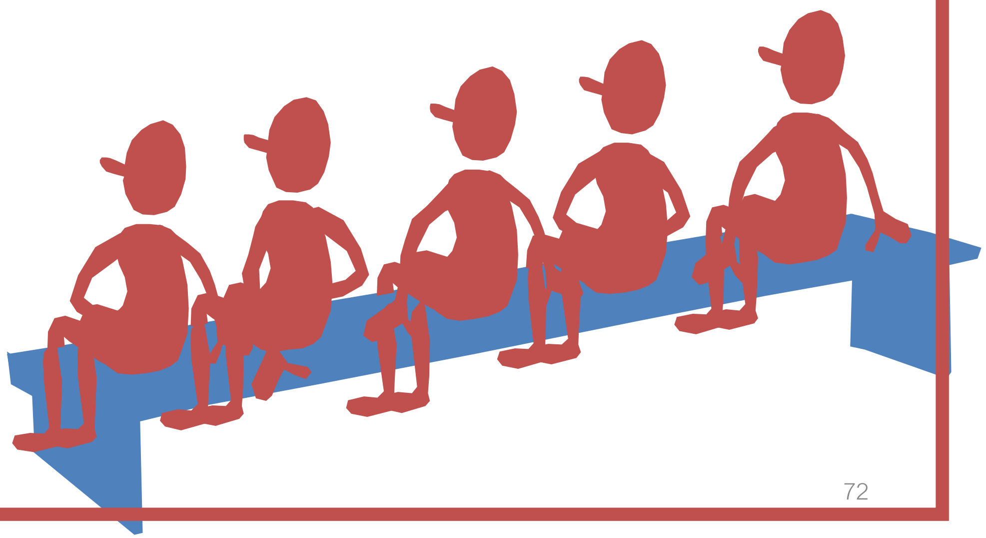
$$= 1+3x+6x^2+9x^3+11x^4+11x^5+9x^6+6x^7+3x^8+x^9$$

The coefficient by x^8 shows that there are only 3 orders for 8 pastries.

Note, we solve the whole parameterized family of problems!!!

The shop only has 2 apple, 3 cheese and 4 raspberry pastries left.

What the number of possible orders? Raspberry pastries come in multiples of two.



Counting with Generating Functions

$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x^2+x^4)$

apple *cheese* *raspberry*

$= 1+2x+4x^2+5x^3+6x^4+6x^5+5x^6+4x^7+2x^8+x^9$

The coefficient by x^8 shows that there are only 2 possible orders.

Exercise

Find the number of integer solutions to

$x_1+x_2+x_3+x_4=21$

$0 \leq x_k \leq 7$

Take the coefficient by x^{21} in $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^4$

Exercise

Find the number of integer solutions to

$x_1+x_2+x_3+x_4 = 21$

$0 < x_k < 7$

$$x_1+x_2+x_3+x_4 = 21$$
$$0 < x_k < 7$$

Observe

$$\mathbf{0 < x_k < 7}$$
$$-1 < x_k-1 < 6$$
$$0 \leq x_k-1 \leq 5$$

Then

$$(x_1-1)+(x_2-1)+(x_3-1)+(x_4-1) = 21-4$$

$$x_1+x_2+x_3+x_4=21$$
$$0<x_k<7$$

The problem is reduced to solving

$$y_1 + y_2 + y_3 + y_4 = 17, \qquad 0 \le y_k \le 5$$

Solution: take the coefficient by x^{17} in

$$(1+x+x^2+x^3+x^4+x^5)^4$$

which is 20

Problem

$x_1+x_2+x_3=9$

$1 \leq x_1 \leq 2$

$2 \leq x_2 \leq 4$

$1 \leq x_3 \leq 4$

Solution

$$x_1+x_2+x_3=9$$

$$1 \le x_1 \le 2 \qquad (x+x^2)$$

$$2 \le x_2 \le 4 \qquad (x^2+x^3+x^4)$$

$$1 \le x_3 \le 4 \qquad (x+x^2+x^3+x^4)$$

Take the coefficient by x^9 in the product of generating functions.