Discrete Mathematics

Introduction to Set Theory

A **set** is an unordered collection of distinct objects, which may be anything (including other sets).

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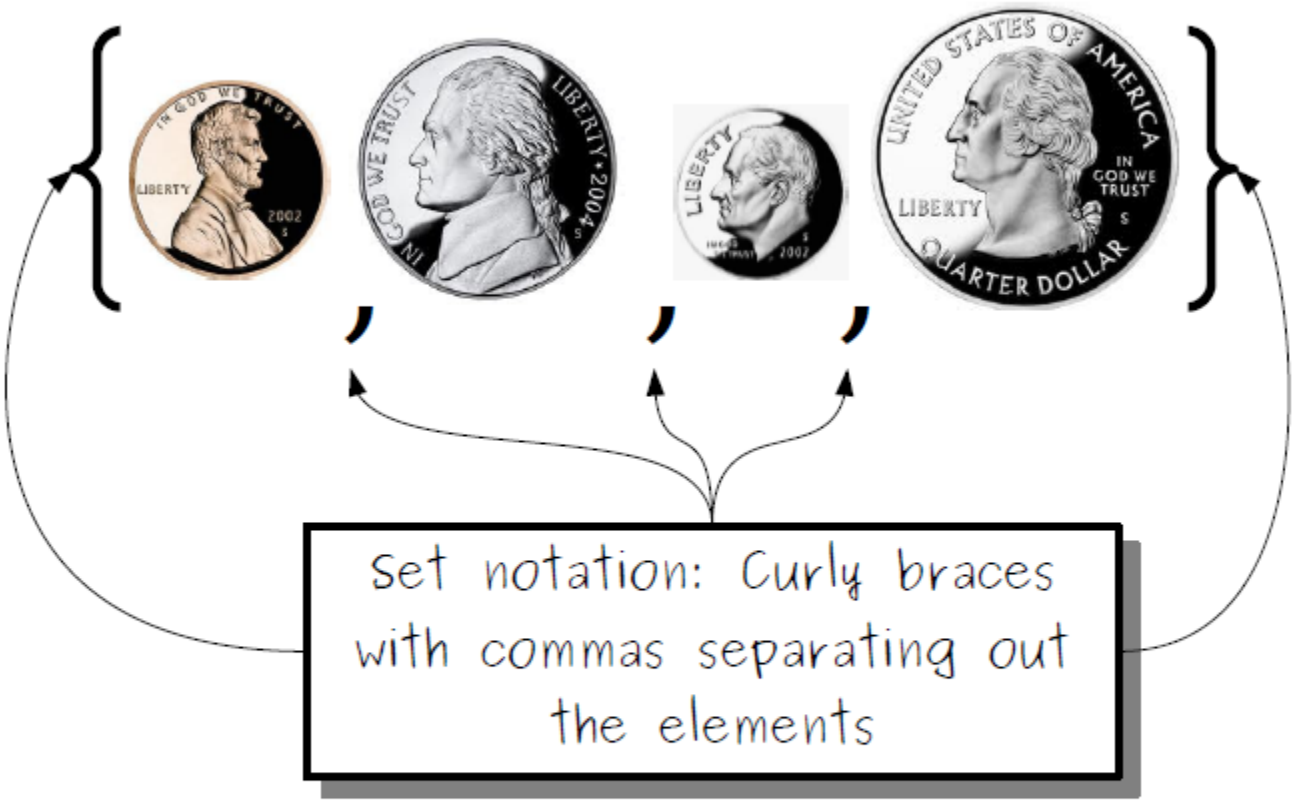

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A **set** is an *unordered* collection of distinct objects, which may be anything (including other sets).

A **set** is an unordered collection of *distinct* objects, which may be anything (including other sets).

$$\{\ \} = \emptyset$$

The **empty set** contains no elements.

We use this symbol to denote the empty set.

A **set** is an unordered collection of distinct objects, which may be anything (including other sets).

This set contains nothing at all.

This set has one element, which happens to be the empty set.

$$\emptyset \stackrel{?}{=} \{\emptyset\}$$

Are these equal to one another?

This set contains nothing at all.

This set has one element, which happens to be the empty set.

$$\emptyset \neq \{\emptyset\}$$

Are these equal to one another?

This is a number.

This is a set. It contains a number.

$$1 \stackrel{?}{=} \{1\}$$

Are these equal to one another?

This is a number.

This is a set. It contains a number.

$$1 \neq \{1\}$$

Are these equal to one another?

Membership

Is in this set?

Membership

Is in this set?

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Membership

Is in this set?

Set Membership

- Given a set S and an object x, we write

$$\boldsymbol{x \in S}$$

  if x is contained in S, and

$$\boldsymbol{x \notin S}$$

  otherwise.
- If $x \in S$, we say that x is an ***element*** of S.
- Given any object x and any set S, either $x \in S$ or $x \notin S$.

Infinite Sets

- Some sets contain *infinitely many* elements!
- The set $\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$ is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ is the set of all the ***integers***.
  - Z is from German "Zahlen."
- The set $\mathbb{R}$ is the set of all ***real numbers***.
  - $e \in \mathbb{R}$, $\pi \in \mathbb{R}$, $4 \in \mathbb{R}$, etc.

Describing Complex Sets

- Here are some English descriptions of infinite sets:

  "The set of all even numbers."

  "The set of all real numbers less than 137."

  "The set of all negative integers."

- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

Even Natural Numbers

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Even Natural Numbers

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The set of all n

Even Natural Numbers

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The set of all n

where

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where

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Even Natural Numbers

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The set of all n

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and n is even

Even Natural Numbers

$$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

The set of all n

where

n is a natural number

and n is even

$$\{\ 0, 2, 4, 6, 8, 10, 12, 14, 16, \ldots\ \}$$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\ x \mid \textit{some property } x \textit{ satisfies}\ \}$$

- For example:

$\{\ r \mid r \in \mathbb{R} \text{ and } r < 137\ \}$

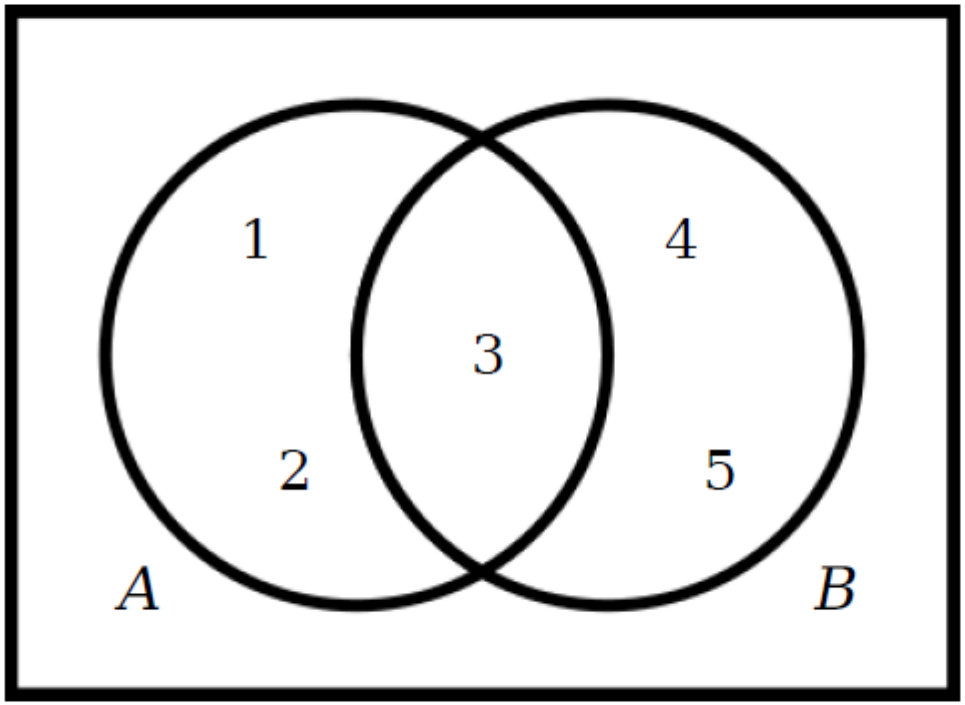
$\{\ n \mid n \text{ is an even natural number}\ \}$

$\{\ S \mid S \text{ is a set of US currency}\ \}$

$\{\ a \mid a \text{ is cute animal}\ \}$

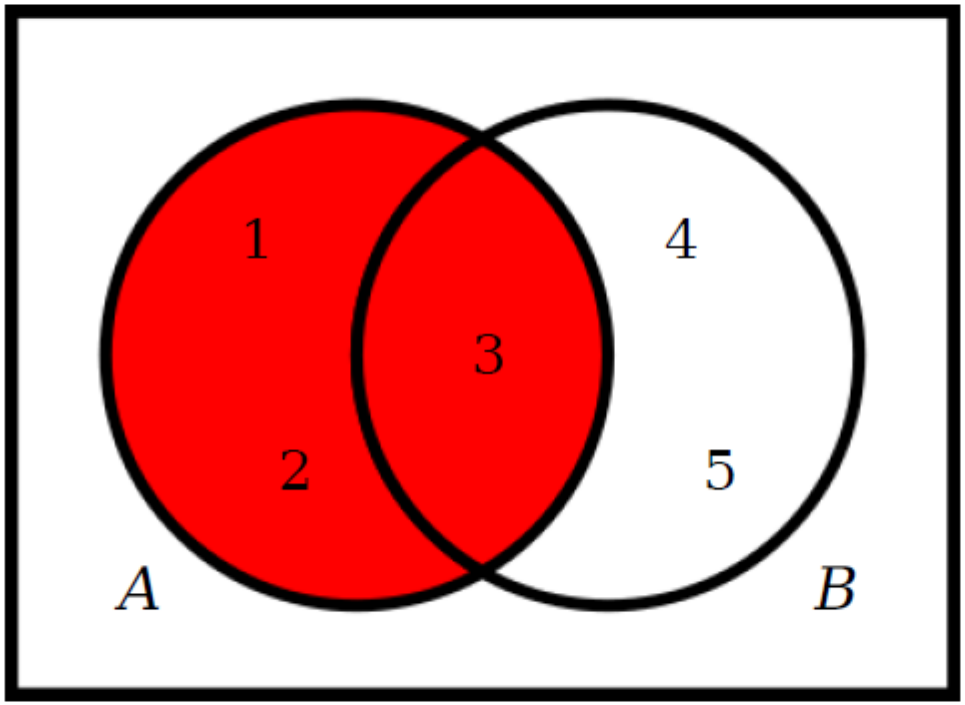
Combining Sets

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



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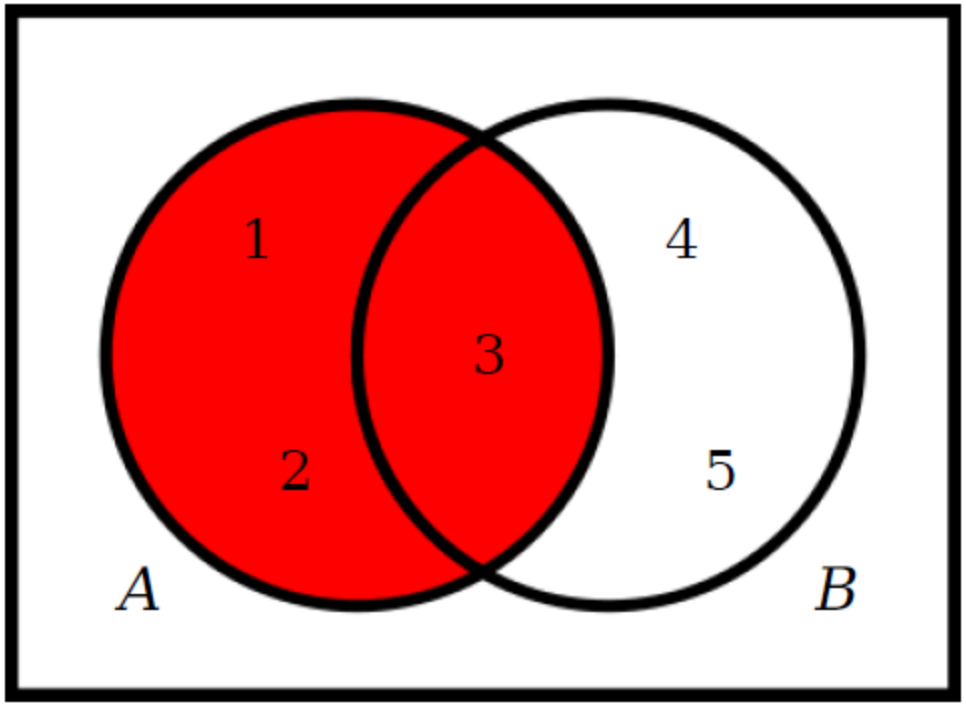

A

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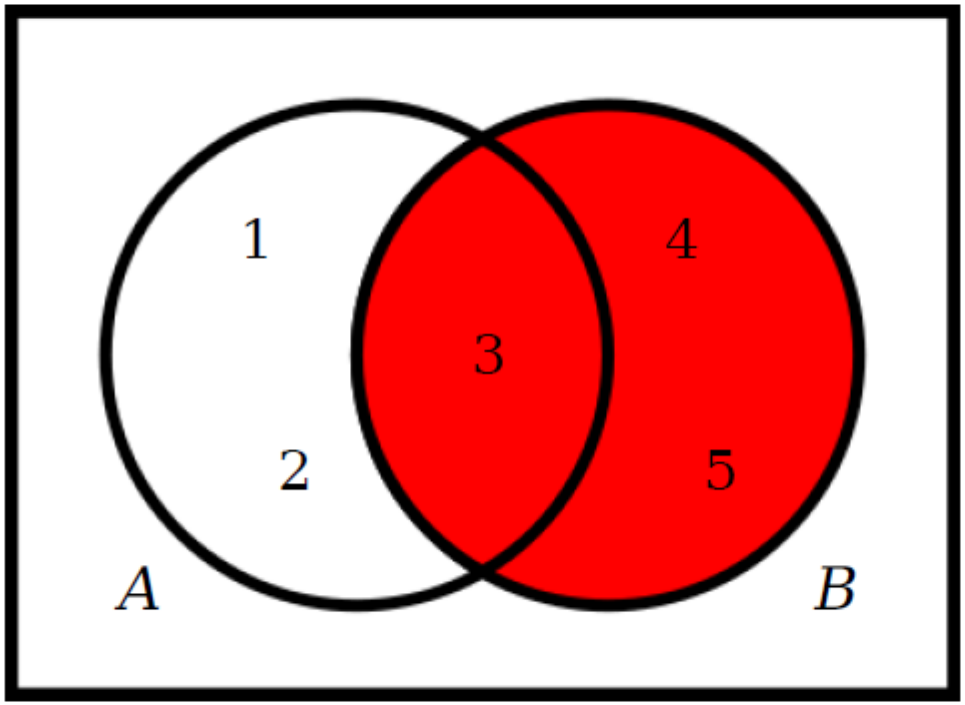

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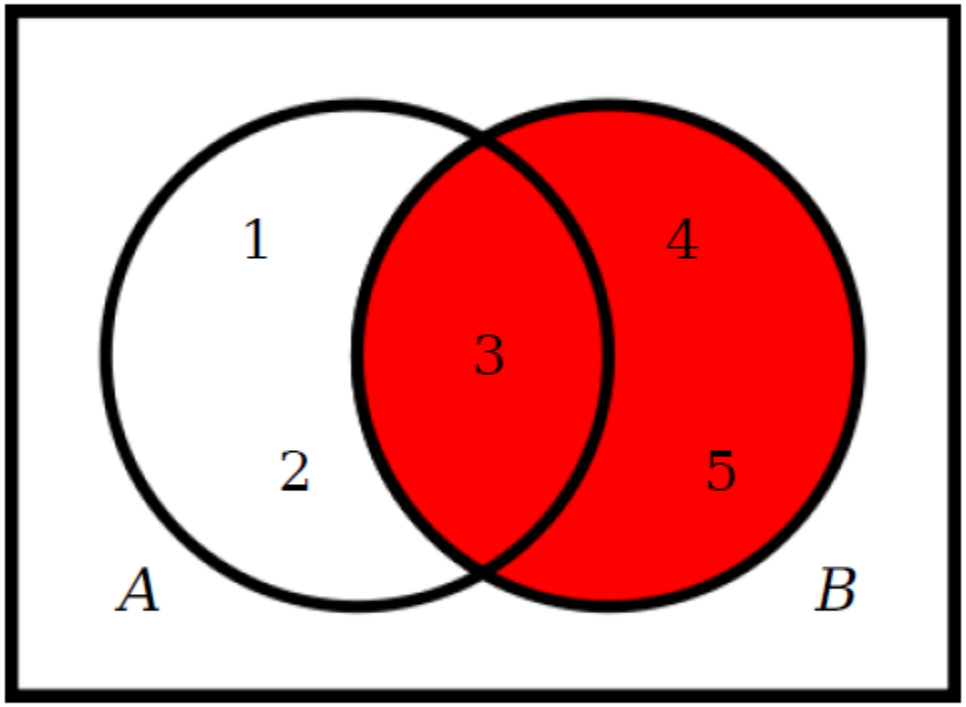
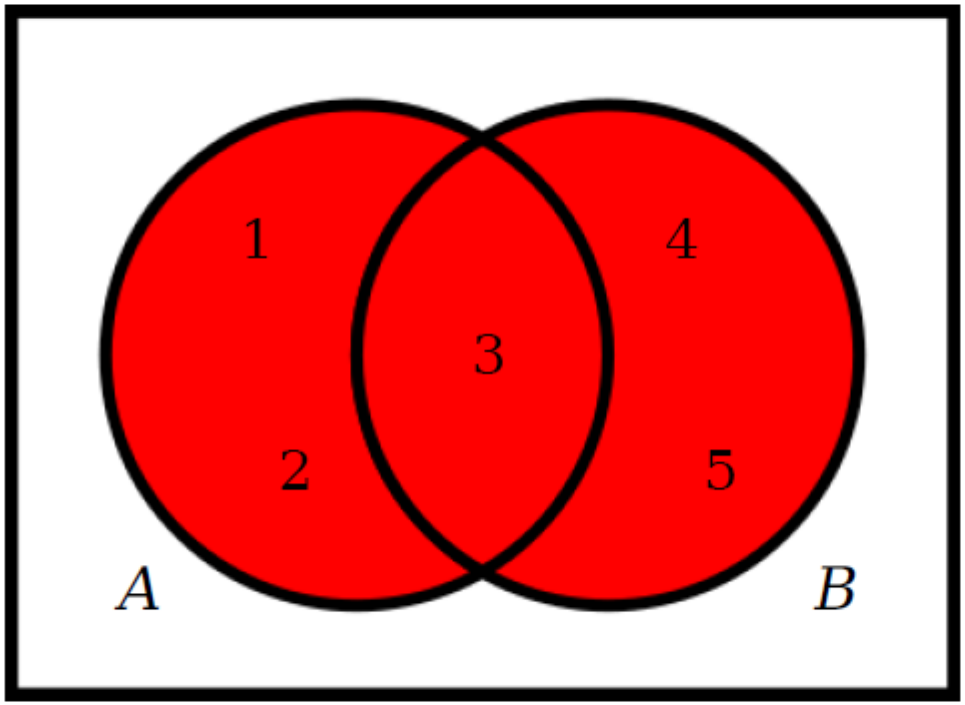


B

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



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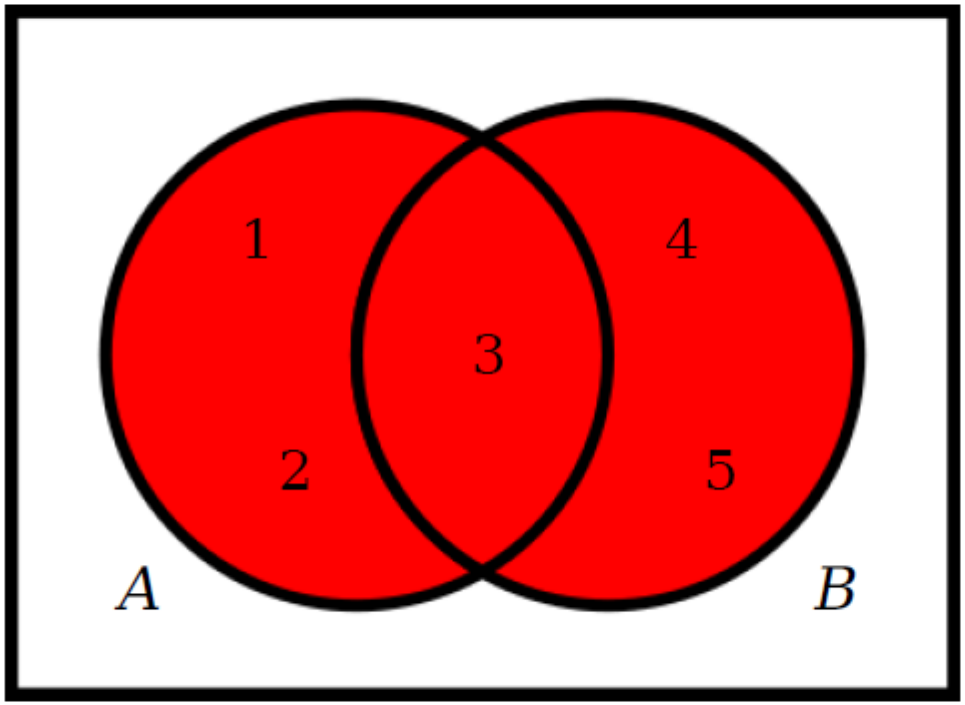
Venn Diagrams



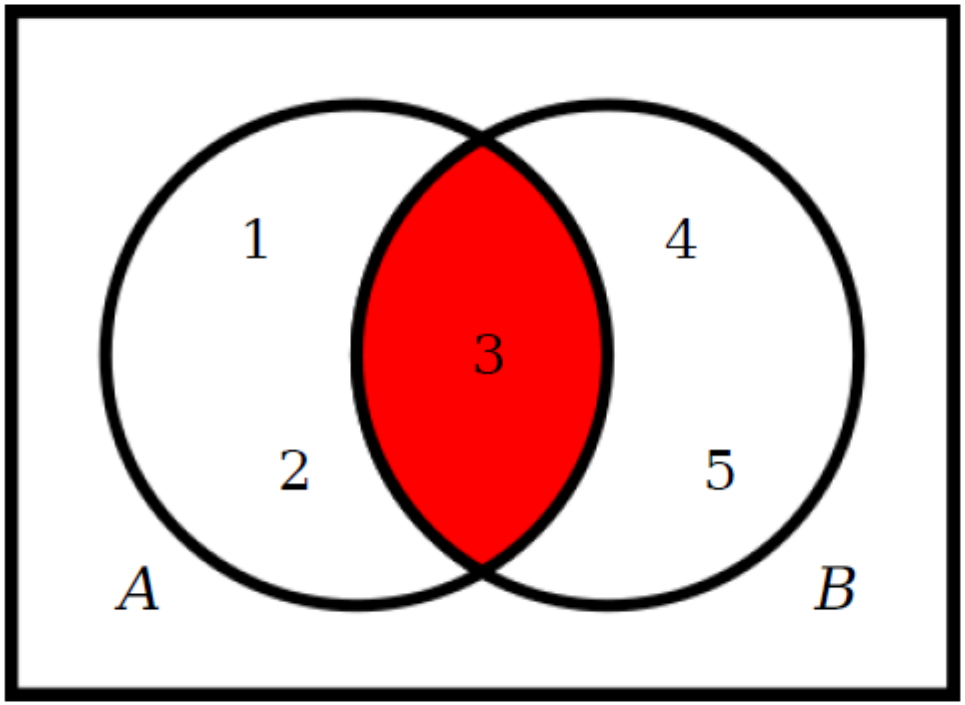
Union

$A \cup B$

$\{ 1, 2, 3, 4, 5 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



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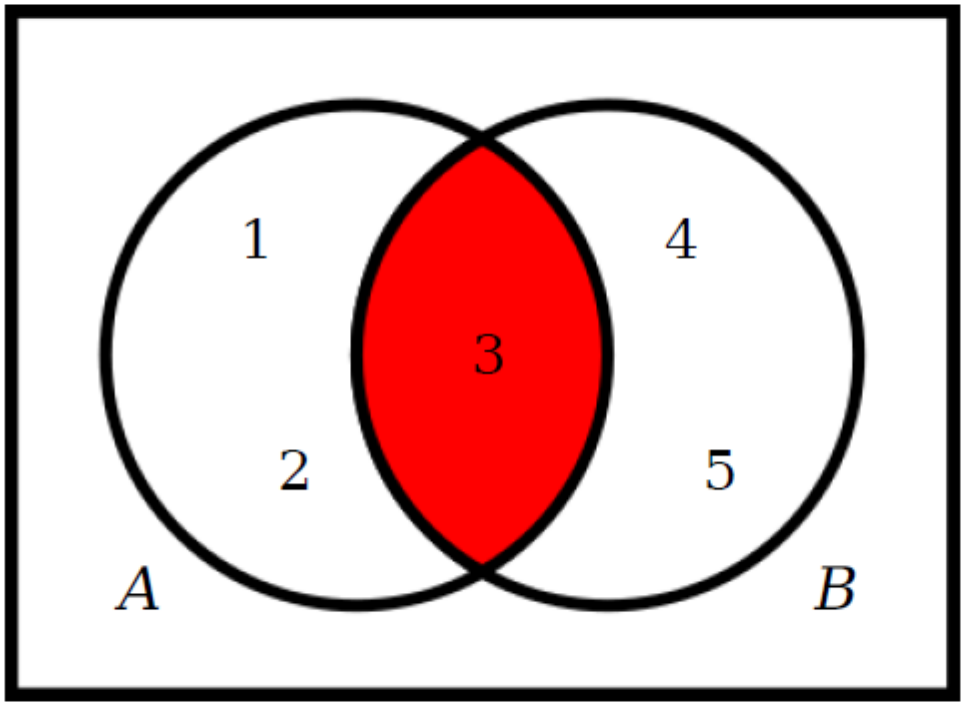


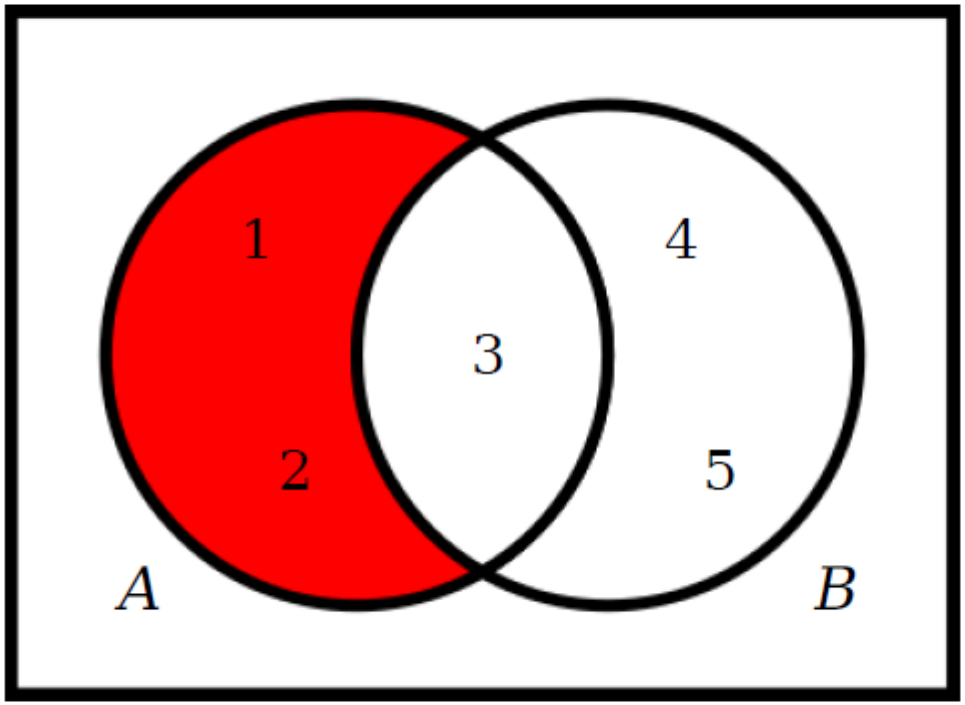
Intersection

$A \cap B$

$\{ 3 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



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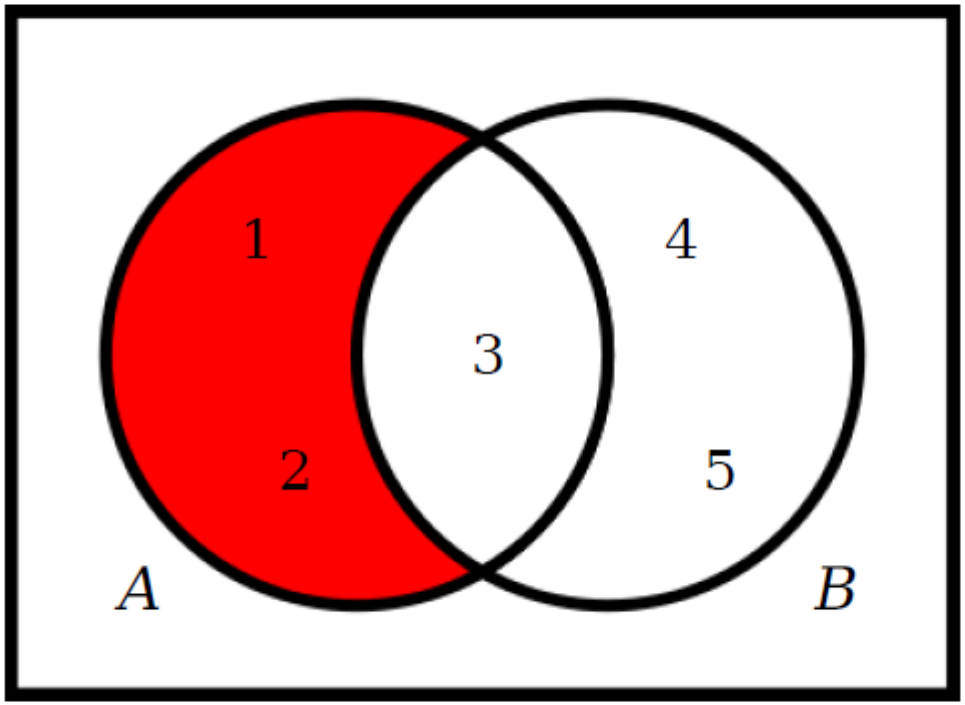
Venn Diagrams



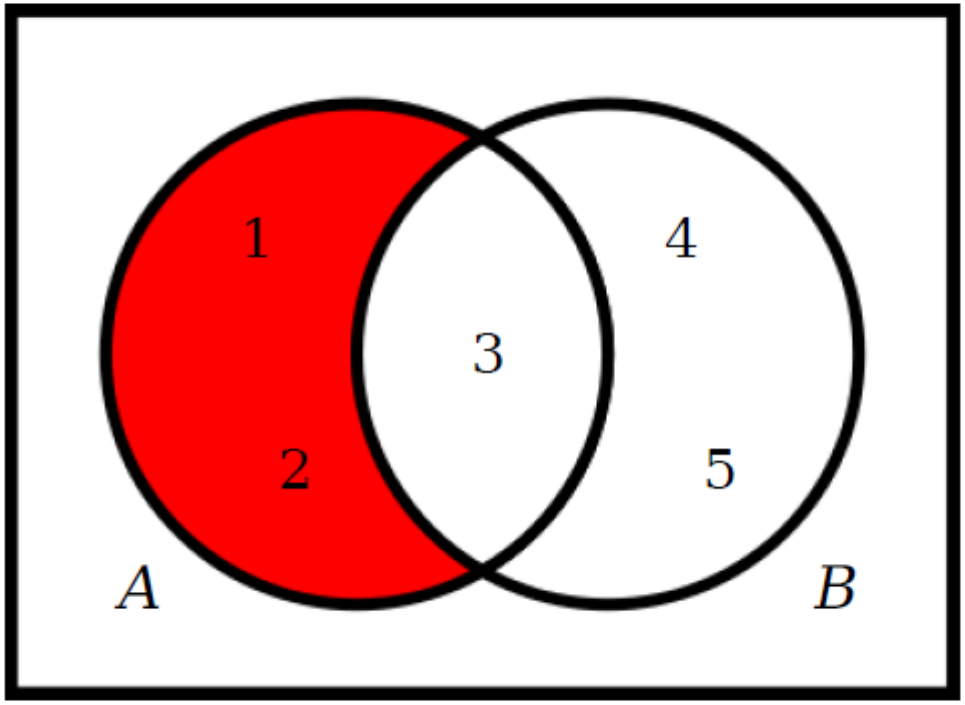
Difference

$A - B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

$A \setminus B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$
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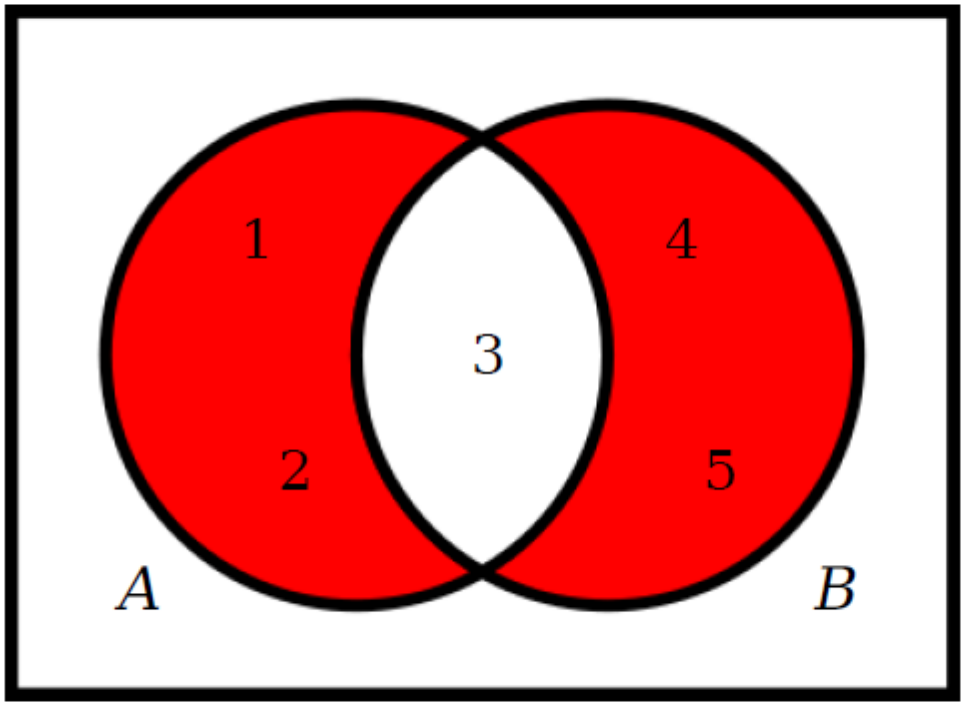
Venn Diagrams



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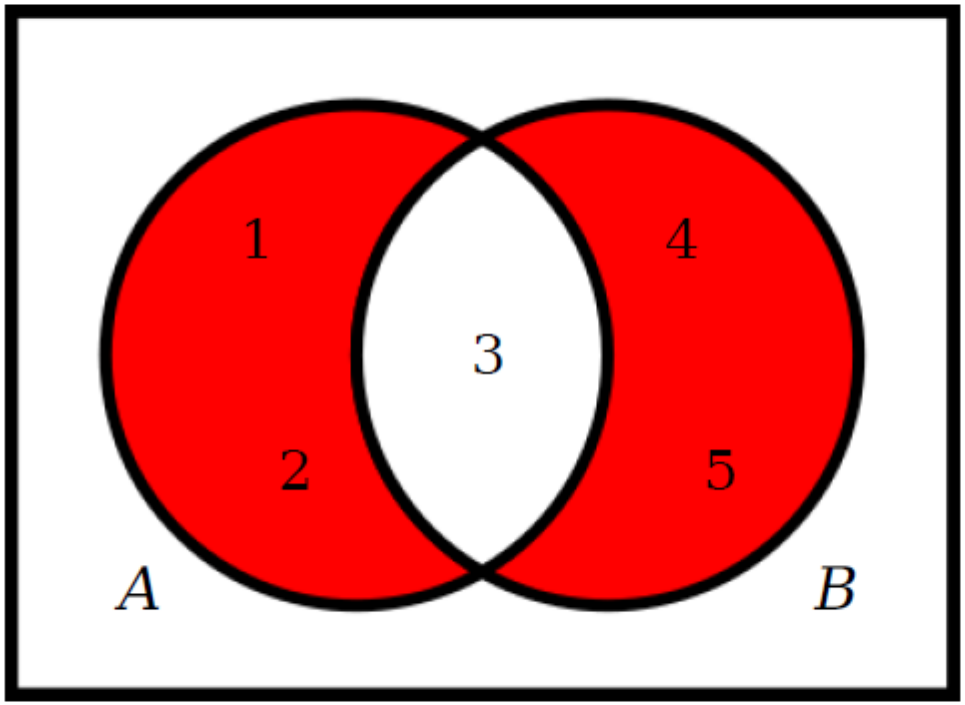


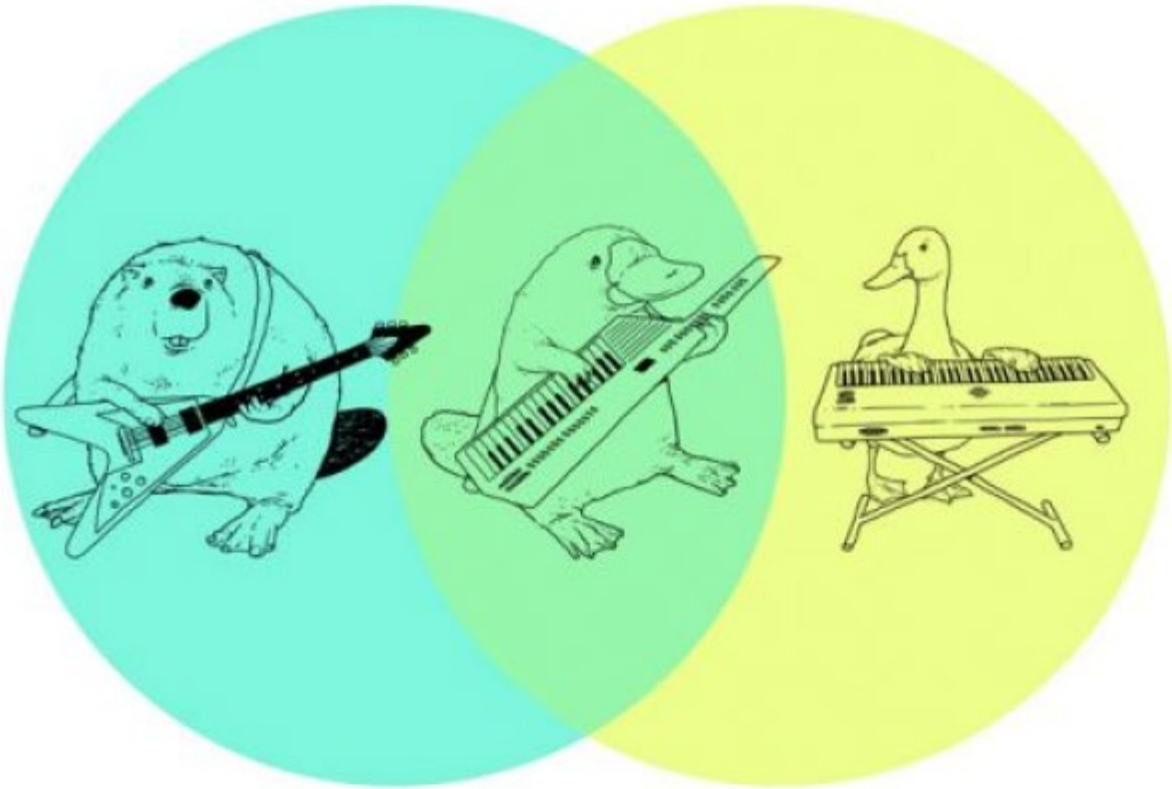
Symmetric Difference

$A \Delta B$

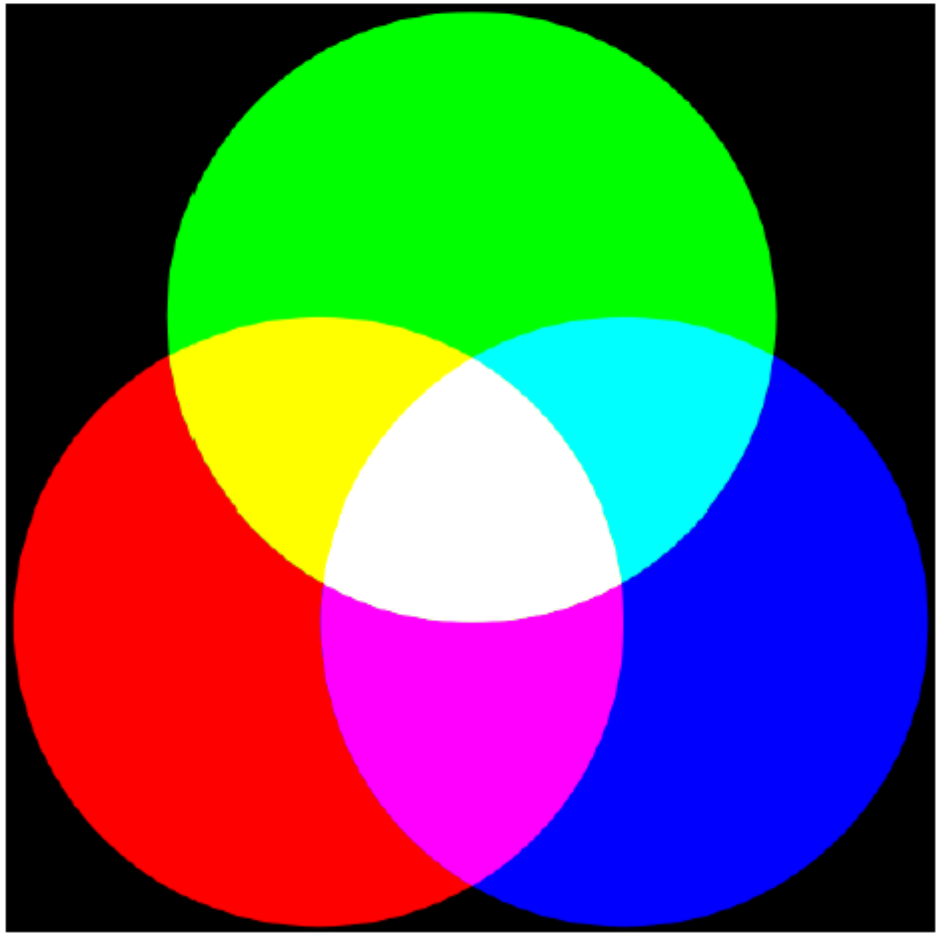
$\{ 1, 2, 4, 5 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

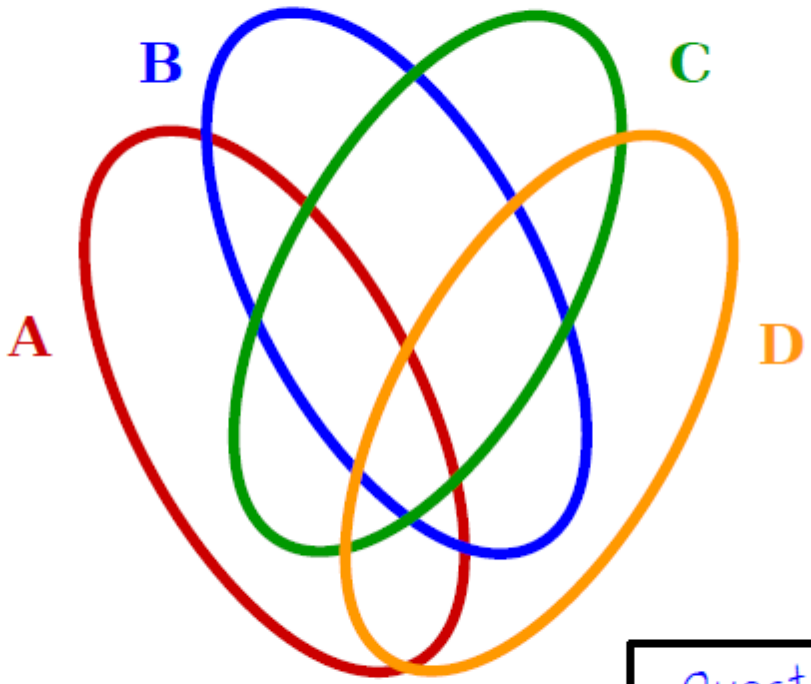
Venn Diagrams



Venn Diagrams for Three Sets



Venn Diagrams for Four Sets



Question to ponder:
why can't we just
draw four circles?

Venn Diagram for Seven Sets

https://www.moebio.com/research/sevensets/

Subsets and Power Sets

Subsets

- A set S is a **subset** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

What About the Empty Set?

- A set S is a **subset** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Are there any sets S where $\emptyset \subseteq S$?
- Equivalently, is there a set S where the following statement is true?

"All elements of $\emptyset$ are also elements of S"

- **Yes!** In fact, this statement is true for every choice of S!

Vacuous Truth

- A statement of the form

  "All objects of type *P* are also of type *Q*"

  is called ***vacuously true*** if there are no objects of type P.
- Vacuously true statements are true *by definition*. This is a convention used throughout mathematics.
- Some examples:
  - All unicorns are pink.
  - All unicorns are blue.
  - Every element of $\emptyset$ is also an element of S.

Trivial and Vacuous Proofs

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Let S be a universal set, P(x) and Q(x) be two open sentences.

Prove: For all $x \in S$, $P(x) \Rightarrow Q(x)$

Trivial Proof

For all $x \in S$, **we show that Q(x) is true** regardless of the truth value of P(x). Then we can conclude that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$.

Example : Show that for all $x \in R$. If $x \geq 0$, then $x^2+2x+1 \geq 0$.

Proof: For all $x \in R$, $x^2+2x+1 = (x+1)(x+1)= (x+1)^2 \geq 0$

Vacuous Proof

For all $x \in S$, **we show that P(x) is false** regardless of the truth value of Q(x). Then we can conclude that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$.

Example : Show that for all $x \in R$. $x^2 + 1 < 0$, then $x^5 \geq 4$.

Proof: Observe that $x^2 + 1 > x^2 \geq 0$. Thus $x^2 + 1 < 0$ is false for all $x \in S$, and so the implication is true.

$S = \{$, $\}$

$S = \{$, $\}$

$\emptyset$ $\{$ $\}$ $\{$ $\}$ $\{$, $\}$

$S = \{$, $\}$

$\{\varnothing,$ $,$ $,$ $,$ $\}\}$

$S = \{$, $\}$

$\wp(S) = \{ \varnothing ,$ $\{$ $\}$, $\{$ $\}$, $\{$, $\}\}$

$S = \{$, $\}$

$\wp(S) = \{ \varnothing ,$ $\{$ $\}$, $\{$ $\}$, $\{$, $\}\}$

$\wp(S)$ is the ***power set*** of S (the set of all subsets of S)

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Cardinality

Cardinality

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by writing $|S|$.
- Examples:
  - $|\{a, b, c, d, e\}| = 5$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$

The Cardinality of $\mathbb{N}$

- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.

The Cardinality of $\mathbb{N}$

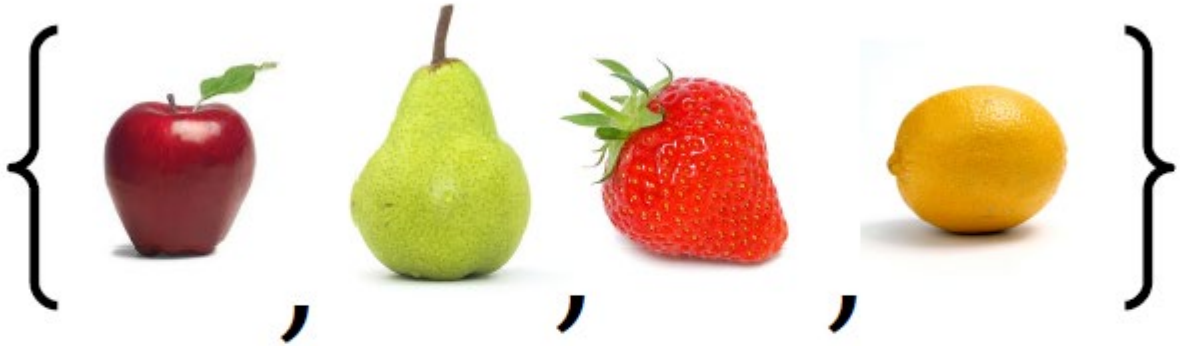
- What is $|\mathbb{N}|$?
  - There are infinitely many natural numbers.
  - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define $\aleph_0 = |\mathbb{N}|$.
  - $\aleph_0$ is pronounced "aleph-zero," "aleph-nought," or "aleph-null."

Consider the set

$$S = \{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

What is $|S|$?

How Big Are These Sets?



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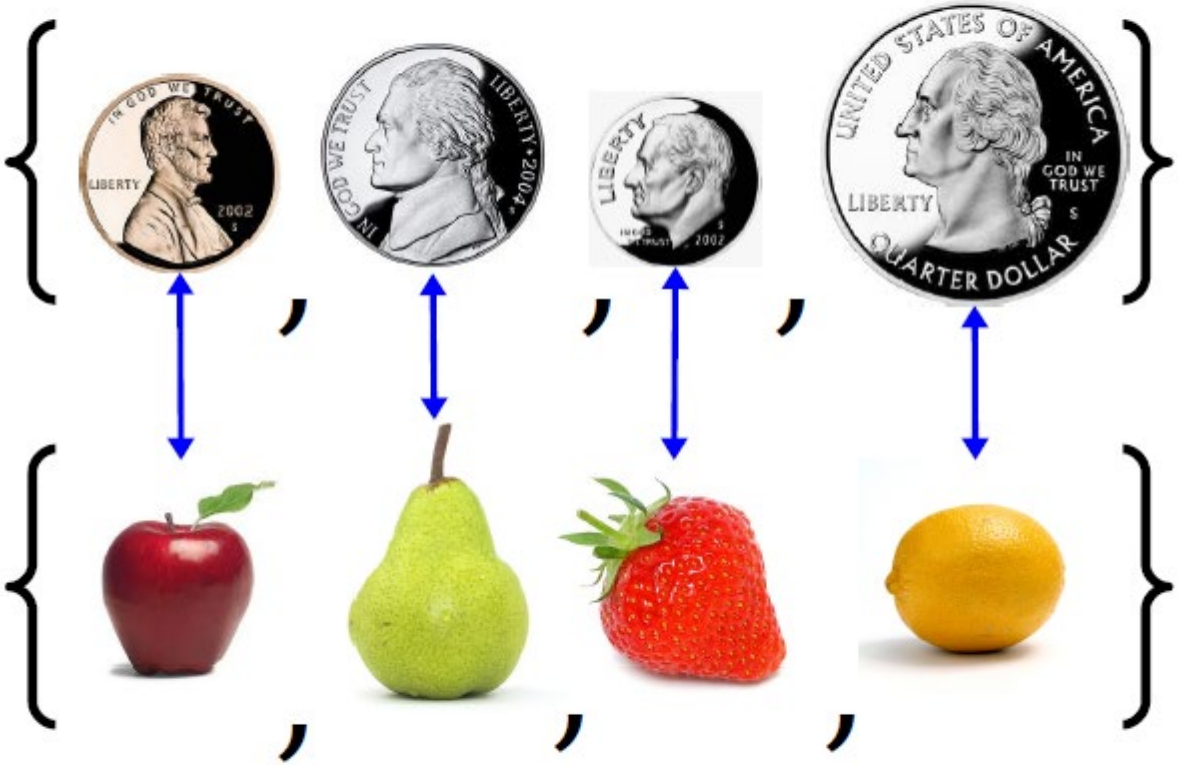
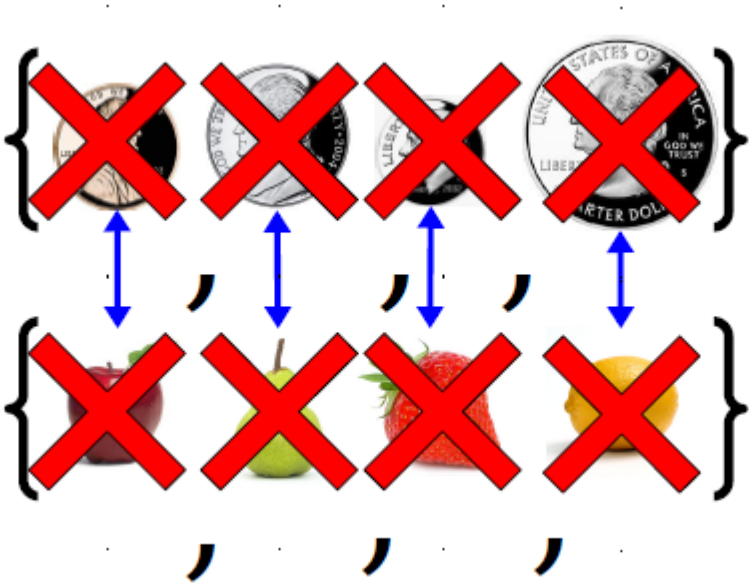


Comparing Cardinalities

- *By definition,* two sets have the same size if their elements can be paired off with no elements remaining.
- The intuition:



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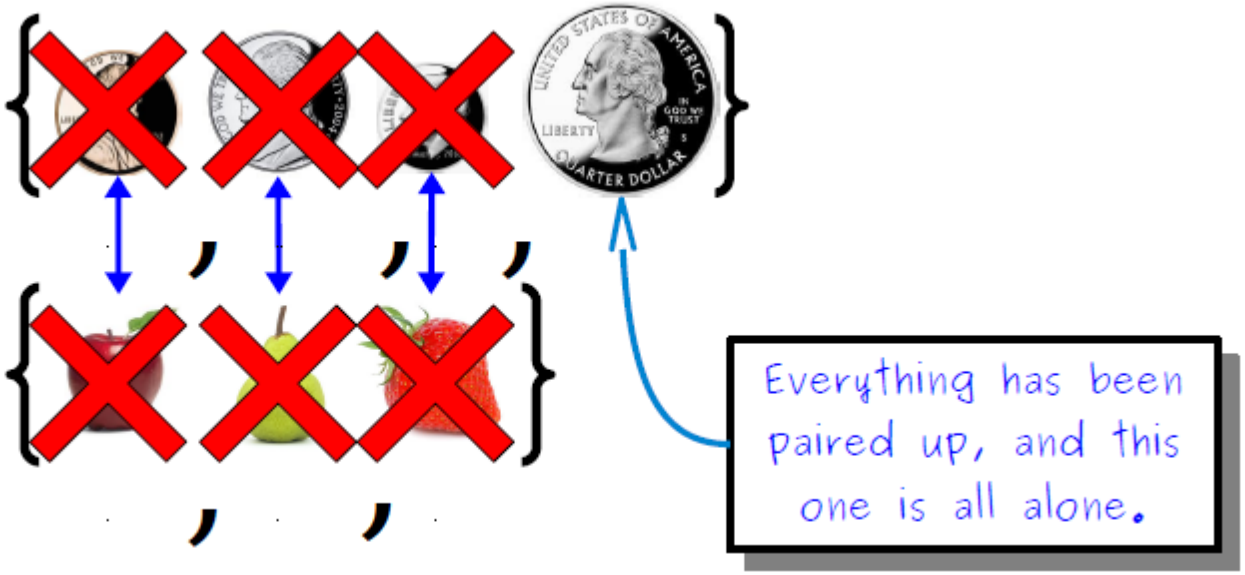

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
S	0		2		4		6		8	...

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
S	0	2	4	6	8	10	12	14	16	...

$$n \leftrightarrow 2n$$

$$S = \{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$$

$$|S| = |\mathbb{N}| = \aleph_0$$

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
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Infinite Cardinalities

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Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$	
$\mathbb{Z}$	0		1		2		3		4	...

... -3 -2 -1

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$	
$\mathbb{Z}$	0		1		2		3		4	...

... -3 -2 -1

Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

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Infinite Cardinalities

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	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

Pair nonnegative integers with even natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	...

$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Important Question

Do all infinite sets have the same cardinality?

$S = \{$, $\}$

$\wp(S) = \{ \emptyset,$ $\{$ $\},$ $\{$ $\},$ $\{$, $\}\}$

$$|S| < |\wp(S)|$$

$S = \{$, , $\}$

$\wp(S) = \{ \emptyset,$ $\{$ $\},$ $\{$ $\},$ $\{$ $\},$

$\{$, $\},$ $\{$, $\},$ $\{$, $\},$

$\{$, , $\}$ $\}$

$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$
$$\emptyset,$$
$$\{a\}, \{b\}, \{c\}, \{d\},$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$
$$\{a, b, c, d\}$$
$$\}$$

$$|S| < |\wp(S)|$$

If S is infinite, what is the relation between $|S|$ and $|\wp(S)|$?

Does $|S| = |\wp(S)|$?

If $|S| = |\wp(S)|$, we can pair up the elements of S and the elements of $\wp(S)$ without leaving anything out.

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If $|S| = |\wp(S)|$, we can pair up the elements of S and the subsets of S without leaving anything out.

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What would that look like?

Complemented Diagonal

Given the set {a, b, c}, the subsets are

a	b	c	Result
Yes	Yes	Yes	{a, b, c}
Yes	Yes	No	{a, b}
Yes	No	Yes	{a, c}
Yes	No	No	{a}
No	Yes	Yes	{b, c}
No	Yes	No	{b}
No	No	Yes	{c}
No	No	No	Ø

Complemented Diagonal

	a?	b?	c?
a paired with	**Y**	**Y**	**N**
b paired with	**N**	**N**	**N**
c paired with	**Y**	**N**	**Y**

Complemented Diagonal

	a?	b?	c?
a paired with	Y	Y	N
b pared with	N	N	N
c paired with	Y	N	Y

Complemented Diagonal

	a?	b?	c?
a paired with	Y	Y	N
b paired with	N	N	N
c paired with	Y	N	Y
Complemented Diagonal	**N**	**Y**	**N**

Complemented Diagonal

	a?	b?	c?
a paired with	Y	N	N
b paired with	N	Y	N
c paired with	Y	Y	N
Complemented Diagonal	**N**	**N**	**Y**

Complemented Diagonal

	a?	b?	c?	d?	e?	f?
a paired with	**N**	Y	Y	Y	N	N
b paired with	N	**N**	N	N	Y	Y
c paired with	Y	Y	**Y**	Y	Y	Y
d paired with	N	N	N	**N**	N	N
e paired with	Y	N	N	N	**N**	Y
f paired with	N	Y	Y	Y	Y	**N**
Complemented Diagonal	**Y**	**Y**	**N**	**Y**	**Y**	**Y**

S is Infinite set

x_0

x_1

x_2

x_3

x_4

x_5

$\ldots$

$$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$$

$$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$$

$$x_2 \longleftrightarrow \{ x_4, \dots \}$$

$$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$$

$$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$$

$$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$$

$$\dots$$

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \ldots \}$

$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \longleftrightarrow \{ x_4, \ldots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**

$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \longleftrightarrow \{ x_4, \ldots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \leftrightarrow \{ x_4, \ldots \}$

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \longleftrightarrow \{ \quad x_4, \quad \ldots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2 ↔	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2 ↔	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \leftrightarrow \{ \quad x_1, \quad x_4, \quad \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \longleftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \longleftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \longleftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**

$x_4 \longleftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2 ↔	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3 ↔	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4 ↔	**Y**	**N**	**N**	**N**	**N**	**Y**	…

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0 ↔	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1 ↔	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2 ↔	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3 ↔	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4 ↔	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5 ↔	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\{ x_0, \quad x_5, \ \ldots \}$$

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**N**	**N**	**N**	**Y**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**N**	**N**	**N**	**Y**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**N**	**N**	**N**	**Y**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Flip all Y's to N's and vice-versa to get a new set

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\{ x_1, x_2, x_3, x_4, \ldots \}$$

Flip all Y's to N's and vice-versa to get a new set

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…
	N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**N**	**N**	**N**	**Y**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…

Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

Complemented Diagonal

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**N**	**N**	**N**	**N**	**N**	**Y**	…
x_1	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
x_2	**N**	**Y**	**N**	**Y**	**Y**	**N**	…
x_3	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_4	**N**	**N**	**Y**	**Y**	**N**	**Y**	…
x_5	**Y**	**N**	**N**	**N**	**N**	**Y**	…
…	…	…	…	…	…	…	…
	Y	**N**	**Y**	**Y**	**Y**	**N**	…

The Diagonalization Proof

- No matter how we pair up elements of S and subsets of S, the complemented diagonal won't appear in the table.
  - In row n, the nth element must be wrong.
- No matter how we pair up elements of S and subsets of S, there is *always* at least one subset left over.
- This result is **Cantor's theorem**: Every set is strictly smaller than its power set:

If S is a set, then $|S| < |\wp(S)|$.

Infinite Cardinalities

- By Cantor's Theorem:

$$
\begin{aligned}
|\mathbb{N}| &< |\wp(\mathbb{N})| \\
|\wp(\mathbb{N})| &< |\wp(\wp(\mathbb{N}))| \\
|\wp(\wp(\mathbb{N}))| &< |\wp(\wp(\wp(\mathbb{N})))| \\
|\wp(\wp(\wp(\mathbb{N})))| &< |\wp(\wp(\wp(\wp(\mathbb{N}))))| \\
&\ldots
\end{aligned}
$$

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***