
# Differential Equations

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# Differential Equations

Perhaps the most important of all the applications of calculus is to differential equations. When physical scientists or social scientists use calculus, more often than not it is to analyze a differential equation that has arisen in the process of modeling some phenomenon that they are studying.

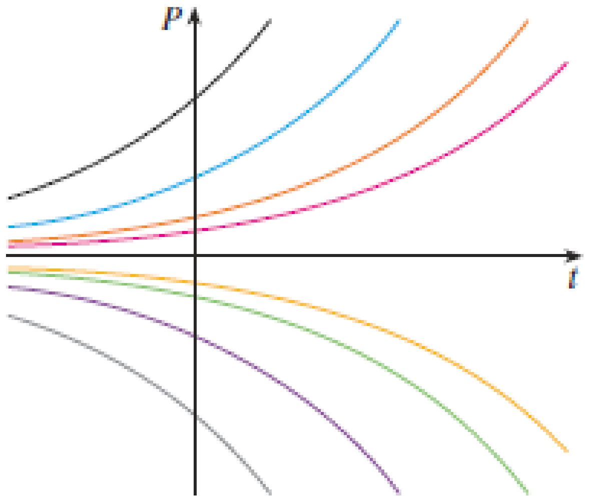



**FIGURE 1**
The family of solutions of $dP/dt = kP$

## Models of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Let's identify and name the variables in this model:

$t$ = time (the independent variable)

$P$ = the number of individuals in the population (the dependent variable)

The rate of growth of the population is the derivative $dP/dt$. So our assumption that the rate of growth of the population is proportional to the population size is written as the equation

$$\boxed{1} \qquad \frac{dP}{dt} = kP$$

where $k$ is the proportionality constant. Equation 1 is our first model for population growth; it is a differential equation because it contains an unknown function $P$ and its derivative $dP/dt$.

Having formulated a model, let's look at its consequences. If we rule out a population of 0, then $P(t) > 0$ for all $t$. So, if $k > 0$, then Equation 1 shows that $P'(t) > 0$ for all $t$. This means that the population is always increasing. In fact, as $P(t)$ increases, Equation 1 shows that $dP/dt$ becomes larger. In other words, the growth rate increases as the population increases.

Let's try to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We know from Chapter 3 that exponential functions have that property. In fact, if we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$$

Thus any exponential function of the form $P(t) = Ce^{kt}$ is a solution of Equation 1. In Section 9.4, we will see that there is no other solution.

Allowing $C$ to vary through all the real numbers, we get the *family* of solutions $P(t) = Ce^{kt}$ whose graphs are shown in Figure 1. But populations have only positive values and so we are interested only in the solutions with $C > 0$. And we are probably concerned only with values of $t$ greater than the initial time $t = 0$. Figure 2 shows the physically meaningful solutions. Putting $t = 0$, we get $P(0) = Ce^{k(0)} = C$, so the constant $C$ turns out to be the initial population, $P(0)$.

Equation 1 is appropriate for modeling population growth under ideal conditions, but we have to recognize that a more realistic model must reflect the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* $M$ (or decreases toward $M$ if it ever exceeds $M$). For a model to take into account both trends, we make two assumptions:

- $\dfrac{dP}{dt} \approx kP$ if $P$ is small (Initially, the growth rate is proportional to $P$.)
- $\dfrac{dP}{dt} < 0$ if $P > M$ ($P$ decreases if it ever exceeds $M$.)

A simple expression that incorporates both assumptions is given by the equation

$$\boxed{2} \qquad \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Notice that if $P$ is small compared with $M$, then $P/M$ is close to 0 and so $dP/dt \approx kP$. If $P > M$, then $1 - P/M$ is negative and so $dP/dt < 0$.

Equation 2 is called the *logistic differential equation* and was proposed by the Dutch mathematical biologist Pierre-François Verhulst in the 1840s as a model for world population growth. We will develop techniques that enable us to find explicit solutions of the logistic equation in Section 9.4, but for now we can deduce qualitative characteristics of the solutions directly from Equation 2. We first observe that the constant functions $P(t) = 0$ and $P(t) = M$ are solutions because, in either case, one of the factors on the right side of Equation 2 is zero. (This certainly makes physical sense: If the population is ever either 0 or at the carrying capacity, it stays that way.) These two constant solutions are called *equilibrium solutions*.

If the initial population $P(0)$ lies between 0 and $M$, then the right side of Equation 2 is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases. Notice that, in either case, if the population approaches the carrying capacity ($P \to M$), then $dP/dt \to 0$, which means the population levels off. So we expect that the solutions of the logistic differential equation have graphs that look something like the ones in Figure 3. Notice that the graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.

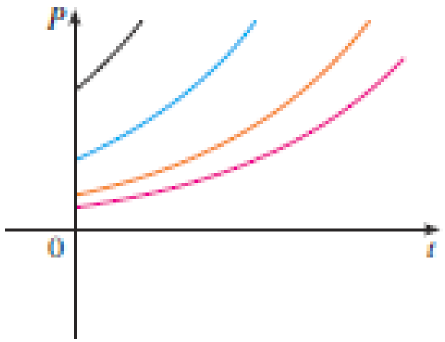



**FIGURE 2**
The family of solutions $P(t) = Ce^{kt}$ with $C > 0$ and $t \geq 0$

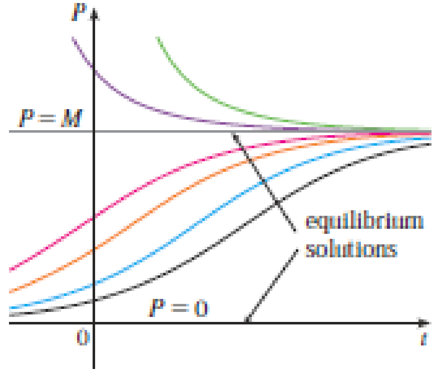



**FIGURE 3**
Solutions of the logistic equation

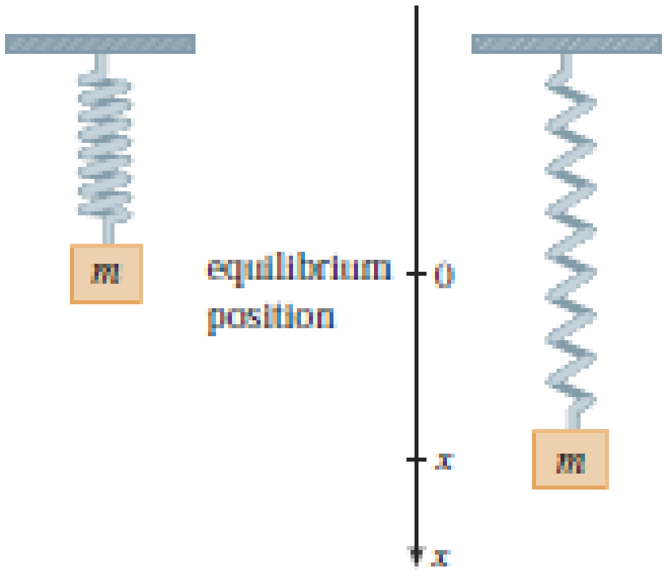



FIGURE 4

## A Model for the Motion of a Spring

Let's now look at an example of a model from the physical sciences. We consider the motion of an object with mass $m$ at the end of a vertical spring (as in Figure 4). In Section 6.4 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) $x$ units from its natural length, then it exerts a force that is proportional to $x$:

$$\text{restoring force} = -kx$$

where $k$ is a positive constant (called the *spring constant*). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx \tag{3}$$

This is an example of what is called a *second-order differential equation* because it involves second derivatives. Let's see what we can guess about the form of the solution directly from the equation. We can rewrite Equation 3 in the form

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

which says that the second derivative of $x$ is proportional to $x$ but has the opposite sign. We know two functions with this property, the sine and cosine functions. In fact, it turns out that all solutions of Equation 3 can be written as combinations of certain sine and cosine functions (see Exercise 4). This is not surprising; we expect the spring to oscillate about its equilibrium position and so it is natural to think that trigonometric functions are involved.

## General Differential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Thus Equations 1 and 2 are first-order equations and Equation 3 is a second-order equation. In all three of those equations the independent variable is called $t$ and represents time, but in general the independent variable doesn't have to represent time. For example, when we consider the differential equation

$$y' = xy \tag{4}$$

it is understood that $y$ is an unknown function of $x$.

A function $f$ is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus $f$ is a solution of Equation 4 if

$$f'(x) = xf(x)$$

for all values of $x$ in some interval.

When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation. We have already solved some particularly simple differential equations, namely, those of the form

$$y' = f(x)$$

For instance, we know that the general solution of the differential equation

$$y' = x^3$$

is given by

$$y = \frac{x^4}{4} + C$$

where $C$ is an arbitrary constant.

**V** **EXAMPLE 1** Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

Figure 5 shows graphs of seven members of the family in Example 1. The differential equation shows that if $y \approx \pm 1$, then $y' \approx 0$. That is borne out by the flatness of the graphs near $y = 1$ and $y = -1$.

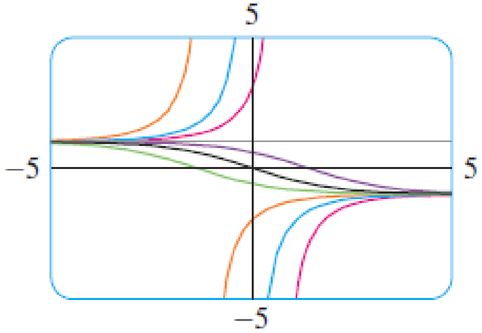



**FIGURE 5**

**V EXAMPLE 2** Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.

## 9.1 Exercises

1. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.

2. Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial-value problem

$$y' + (\tan x)y = \cos^2 x \qquad y(0) = -1$$

on the interval $-\pi/2 < x < \pi/2$.

3. (a) For what nonzero values of $k$ does the function $y = \sin kt$ satisfy the differential equation $y'' + 9y = 0$?
(b) For those values of $k$, verify that every member of the family of functions

$$y = A \sin kt + B \cos kt$$

is also a solution.

4. For what values of $r$ does the function $y = e^{rt}$ satisfy the differential equation $y'' + y' - 6y = 0$?

5. Which of the following functions are solutions of the differential equation $y'' + 2y' + y = 0$?
(a) $y = e^t$ (b) $y = e^{-t}$
(c) $y = te^{-t}$ (d) $y = t^2e^{-t}$

6. (a) Show that every member of the family of functions $y = Ce^{x^2/2}$ is a solution of the differential equation $y' = xy$.

## Direction Fields

Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y \qquad y(0) = 1$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point $(x, y)$ on the graph (called the *solution curve*) is equal to the sum of the $x$- and $y$-coordinates of the point (see Figure 1). In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1. (See Figure 2.)

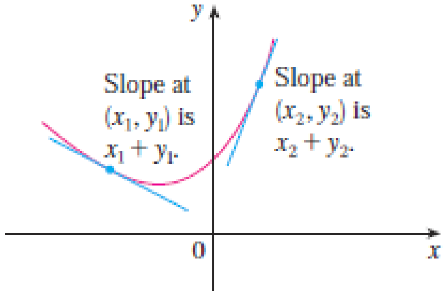



**FIGURE 1**
A solution of $y' = x + y$

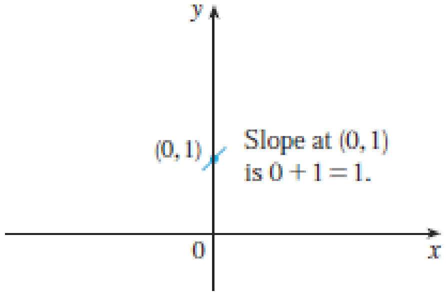



**FIGURE 2**
Beginning of the solution curve through (0, 1)

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points $(x, y)$ with slope $x + y$. The result is called a *direction field* and is shown in Figure 3. For instance, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

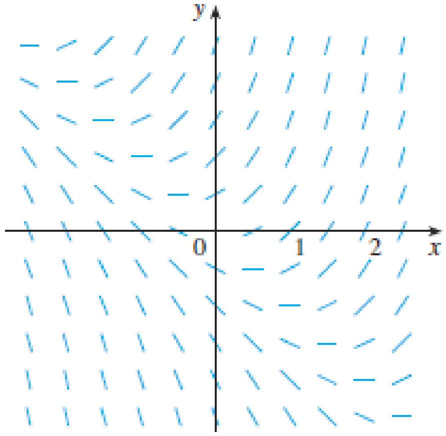



**FIGURE 3**
Direction field for $y' = x + y$

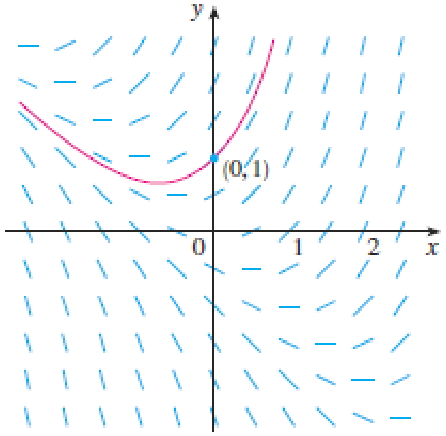



**FIGURE 4**
The solution curve through (0, 1)

Now we can sketch the solution curve through the point $(0, 1)$ by following the direction field as in Figure 4. Notice that we have drawn the curve so that it is parallel to nearby line segments.

In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where $F(x, y)$ is some expression in $x$ and $y$. The differential equation says that the slope of a solution curve at a point $(x, y)$ on the curve is $F(x, y)$. If we draw short line segments with slope $F(x, y)$ at several points $(x, y)$, the result is called a **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

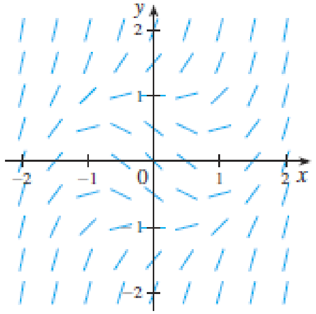



FIGURE 5

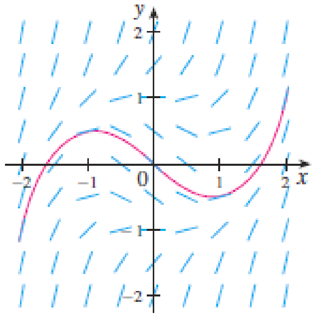



FIGURE 6

**TEC** Module 9.2A shows direction fields and solution curves for a variety of differential equations.

**V EXAMPLE 1**

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.

(b) Use part (a) to sketch the solution curve that passes through the origin.

SOLUTION

(a) We start by computing the slope at several points in the following chart:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | ... |
| $y' = x^2 + y^2 - 1$ | $3$ | $0$ | $-1$ | $0$ | $3$ | $4$ | $1$ | $0$ | $1$ | $4$ | ... |

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

(b) We start at the origin and move to the right in the direction of the line segment (which has slope $-1$). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6. Returning to the origin, we draw the solution curve to the left as well.

The more line segments we draw in a direction field, the clearer the picture becomes. Of course, it's tedious to compute slopes and draw line segments for a huge number of points by hand, but computers are well suited for this task. Figure 7 shows a more detailed, computer-drawn direction field for the differential equation in Example 1. It enables us to draw, with reasonable accuracy, the solution curves shown in Figure 8 with $y$-intercepts $-2, -1, 0, 1,$ and 2.

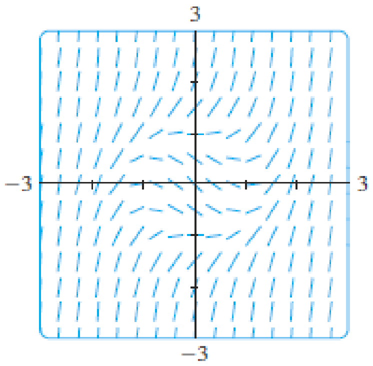



FIGURE 7

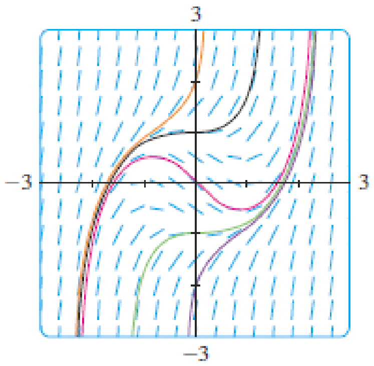



FIGURE 8

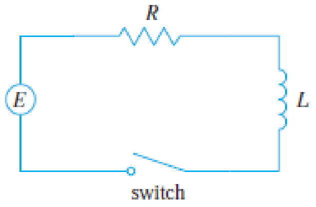



FIGURE 9

Now let's see how direction fields give insight into physical situations. The simple electric circuit shown in Figure 9 contains an electromotive force (usually a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time $t$. The circuit also contains a resistor with a resistance of $R$ ohms ($\Omega$) and an inductor with an inductance of $L$ henries (H).

Ohm's Law gives the drop in voltage due to the resistor as $RI$. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$\boxed{1} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order differential equation that models the current $I$ at time $t$.

**V EXAMPLE 2** Suppose that in the simple circuit of Figure 9 the resistance is 12 $\Omega$, the inductance is 4 H, and a battery gives a constant voltage of 60 V.

(a) Draw a direction field for Equation 1 with these values.
(b) What can you say about the limiting value of the current?
(c) Identify any equilibrium solutions.
(d) If the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, use the direction field to sketch the solution curve.

SOLUTION

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 1, we get

$$4\frac{dI}{dt} + 12I = 60 \qquad \text{or} \qquad \frac{dI}{dt} = 15 - 3I$$

The direction field for this differential equation is shown in Figure 10.

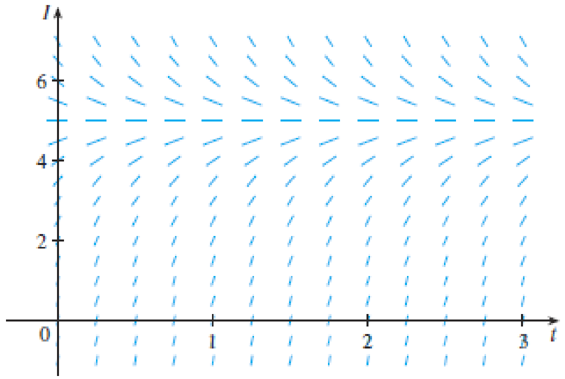



FIGURE 10

(b) It appears from the direction field that all solutions approach the value 5 A, that is,

$$\lim_{t\to\infty} I(t) = 5$$

(c) It appears that the constant function $I(t) = 5$ is an equilibrium solution. Indeed, we can verify this directly from the differential equation $dI/dt = 15 - 3I$. If $I(t) = 5$, then the left side is $dI/dt = 0$ and the right side is $15 - 3(5) = 0$.

(d) We use the direction field to sketch the solution curve that passes through (0, 0), as shown in red in Figure 11.

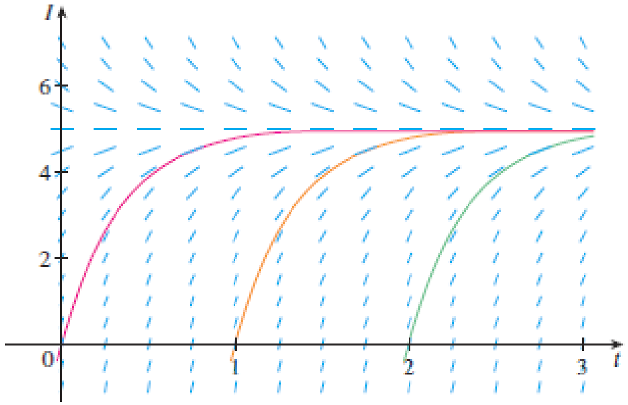



FIGURE 11

# Separable Equations

A **separable equation** is a first-order differential equation in which the expression for $dy/dx$ can be factored as a function of $x$ times a function of $y$. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)\,f(y)$$

The name *separable* comes from the fact that the expression on the right side can be "separated" into a function of $x$ and a function of $y$. Equivalently, if $f(y) \neq 0$, we could write

$$\boxed{1} \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that all $y$'s are on one side of the equation and all $x$'s are on the other side. Then we integrate both sides of the equation:

$$\boxed{2} \qquad \int h(y)\,dy = \int g(x)\,dx$$

Equation 2 defines $y$ implicitly as a function of $x$. In some cases we may be able to solve for $y$ in terms of $x$.

We use the Chain Rule to justify this procedure: If $h$ and $g$ satisfy $\boxed{2}$, then

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

so $$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

and $$h(y)\frac{dy}{dx} = g(x)$$

Thus Equation 1 is satisfied.

**EXAMPLE 1**

(a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Figure 1 shows graphs of several members of the family of solutions of the differential equation in Example 1. The solution of the initial-value problem in part (b) is shown in red.

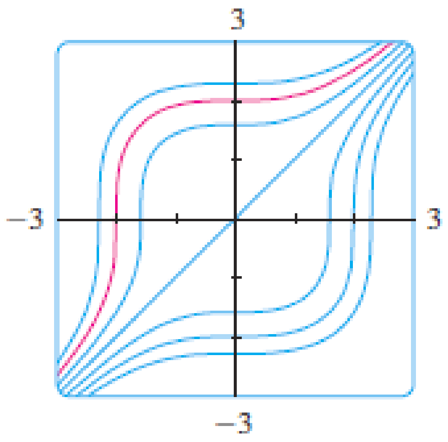



FIGURE 1

**V** **EXAMPLE 2** Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + \cos y}$.

**EXAMPLE 3** Solve the equation $y' = x^2 y$.

Figure 3 shows a direction field for the differential equation in Example 3. Compare it with Figure 4, in which we use the equation $y = Ae^{x^3/3}$ to graph solutions for several values of $A$. If you use the direction field to sketch solution curves with $y$-intercepts 5, 2, 1, $-1$, and $-2$, they will resemble the curves in Figure 4.

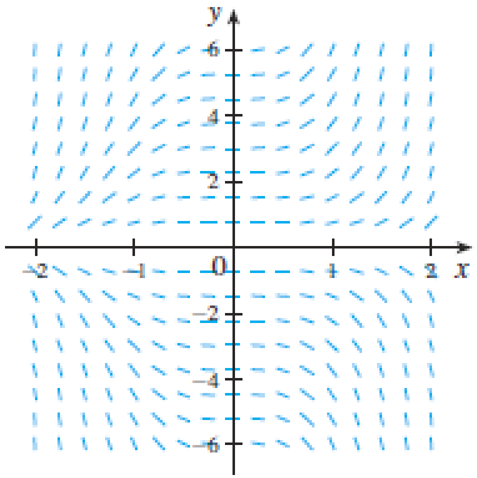



FIGURE 3

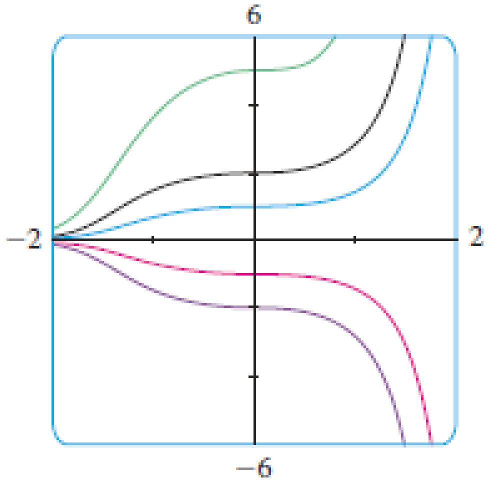



FIGURE 4

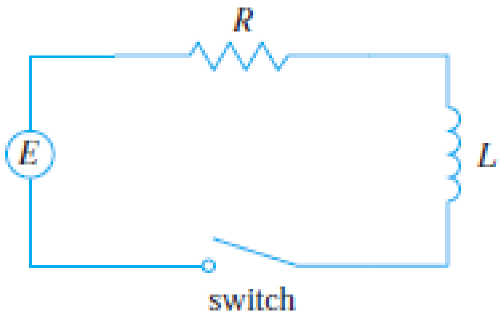


FIGURE 5

**V EXAMPLE 4** In Section 9.2 we modeled the current $I(t)$ in the electric circuit shown in Figure 5 by the differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

Find an expression for the current in a circuit where the resistance is $12\ \Omega$, the inductance is 4 H, a battery gives a constant voltage of 60 V, and the switch is turned on when $t = 0$. What is the limiting value of the current?

## Mixing Problems

A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt. A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate. If $y(t)$ denotes the amount of substance in the tank at time $t$, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order separable differential equation. We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into the bloodstream.

**EXAMPLE 6** A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Figure 10 shows the graph of the function $y(t)$ of Example 6. Notice that, as time goes by, the amount of salt approaches 150 kg.

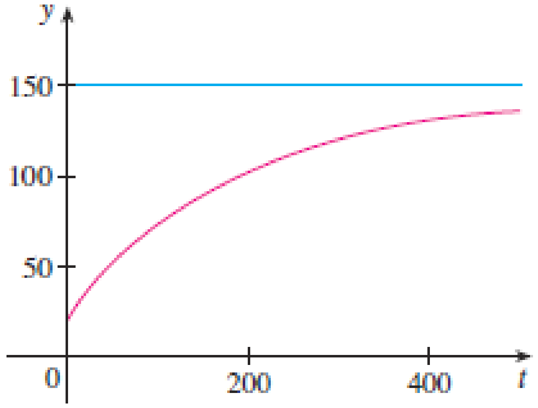



FIGURE 10

## 9.3 Exercises

**1–10** Solve the differential equation.

**1.** $\frac{dy}{dx} = \frac{y}{x}$

**2.** $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

**3.** $(x^2 + 1)y' = xy$

**4.** $y' = y^2 \sin x$

**5.** $(1 + \tan y)y' = x^2 + 1$

**6.** $\frac{du}{dr} = \frac{1 + \sqrt{r}}{1 + \sqrt{u}}$

**7.** $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1 + y^2}}$

**8.** $y' = \frac{xy}{2 \ln y}$

**9.** $\frac{du}{dt} = 2 + 2u + t + tu$

**10.** $\frac{dz}{dt} + e^{t+z} = 0$

**11–18** Find the solution of the differential equation that satisfies the given initial condition.

**11.** $\frac{dy}{dx} = y^2 + 1, \quad y(1) = 0$

**12.** $\frac{dy}{dx} = \frac{y \cos x}{1 + y^2}, \quad y(0) = 1$

**13.** $x \cos x = (2y + e^{3y})y', \quad y(0) = 0$

**14.** $\frac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

**15.** $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

**16.** $\frac{dy}{dt} = te^y, \quad y(1) = 0$

**39.** A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after $t$ minutes and (b) after 20 minutes?

**40.** A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank (a) after $t$ minutes and (b) after one hour?

# Models for Population Growth

## The Law of Natural Growth

One of the models for population growth that we considered in Section 9.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

In general, if $P(t)$ is the value of a quantity $y$ at time $t$ and if the rate of change of $P$ with respect to $t$ is proportional to its size $P(t)$ at any time, then

$$\boxed{\frac{dP}{dt} = kP} \tag{1}$$

where $k$ is a constant. Equation 1 is sometimes called the **law of natural growth**. If $k$ is positive, then the population increases; if $k$ is negative, it decreases.

Because Equation 1 is a separable differential equation, we can solve it by the methods of Section 9.3:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln |P| = kt + C$$

$$|P| = e^{kt+C} = e^C e^{kt}$$

$$P = Ae^{kt}$$

where $A$ $(= \pm e^C$ or 0) is an arbitrary constant. To see the significance of the constant $A$, we observe that

$$P(0) = Ae^{k \cdot 0} = A$$

Therefore $A$ is the initial value of the function.

**2** The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Another way of writing Equation 1 is

$$\frac{1}{P}\frac{dP}{dt} = k$$

which says that the **relative growth rate** (the growth rate divided by the population size) is constant. Then 2 says that a population with constant relative growth rate must grow exponentially.

We can account for emigration (or "harvesting") from a population by modifying Equation 1: If the rate of emigration is a constant $m$, then the rate of change of the population is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \tag{3}$$

See Exercise 15 for the solution and consequences of Equation 3.

## The Logistic Model

As we discussed in Section 9.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time $t$, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population $P$ increases and becomes negative if $P$ ever exceeds its **carrying capacity** $M$, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by $P$, we obtain the model for population growth known as the **logistic differential equation**:

4

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

Notice from Equation 4 that if $P$ is small compared with $M$, then $P/M$ is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population $P$ lies between 0 and $M$, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

# Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

where $P$ and $Q$ are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

$$\boxed{2} \qquad y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for $y'$ as a function of $x$ times a function of $y$. But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \qquad \text{or} \qquad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by $x$.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor*. We try to find $I$ so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

$$\boxed{3} \qquad I(x)\big(y' + P(x)y\big) = \big(I(x)y\big)'$$

If we can find such a function $I$, then Equation 1 becomes

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)\, Q(x)\, dx + C$$

so the solution would be

$$\boxed{4} \qquad y(x) = \frac{1}{I(x)} \left[ \int I(x)\, Q(x)\, dx + C \right]$$

To find such an $I$, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)\,P(x)y = \big(I(x)y\big)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

This is a separable differential equation for $I$, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\, dx$$

$$\ln |I| = \int P(x)\, dx$$

$$I = Ae^{\int P(x)\, dx}$$

where $A = \pm e^{C}$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$\boxed{5} \qquad I(x) = e^{\int P(x)\, dx}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where $I$ is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\, dx}$ and integrate both sides.

**V EXAMPLE 1** Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

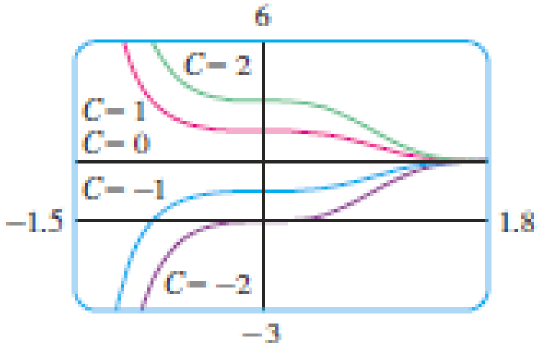



**FIGURE 1**

**V EXAMPLE 2** Find the solution of the initial-value problem

$$x^2 y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

**EXAMPLE 3** Solve $y' + 2xy = 1$.

**V** **EXAMPLE 4** Suppose that in the simple circuit of Figure 4 the resistance is 12 $\Omega$ and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find (a) $I(t)$, (b) the current after 1 s, and (c) the limiting value of the current.

## 9.6 Exercises

**1–4** Determine whether the differential equation is linear.

**1.** $y' + e^x y = x^2 y^2$

**2.** $y + \sin x = x^3 y'$

**3.** $xy' + \ln x - x^2 y = 0$

**4.** $y' + \cos y = \tan x$

**5–14** Solve the differential equation.

**5.** $y' + 2y = 2e^x$

**6.** $y' = x + 5y$

**7.** $xy' - 2y = x^2$

**8.** $x^2 y' + 2xy = \cos^2 x$

**9.** $xy' + y = \sqrt{x}$

**10.** $1 + xy = xy'$

**11.** $\dfrac{dy}{dx} + 2xy = x^2$

**12.** $\dfrac{dy}{dx} = x \sin 2x + y \tan x, \quad -\pi/2 < x < \pi/2$

**13.** $(1 + t)\dfrac{du}{dt} + u = 1 + t, \quad t > 0$

**14.** $t \ln t \dfrac{dr}{dt} + r = te^t$

**15–20** Solve the initial-value problem.

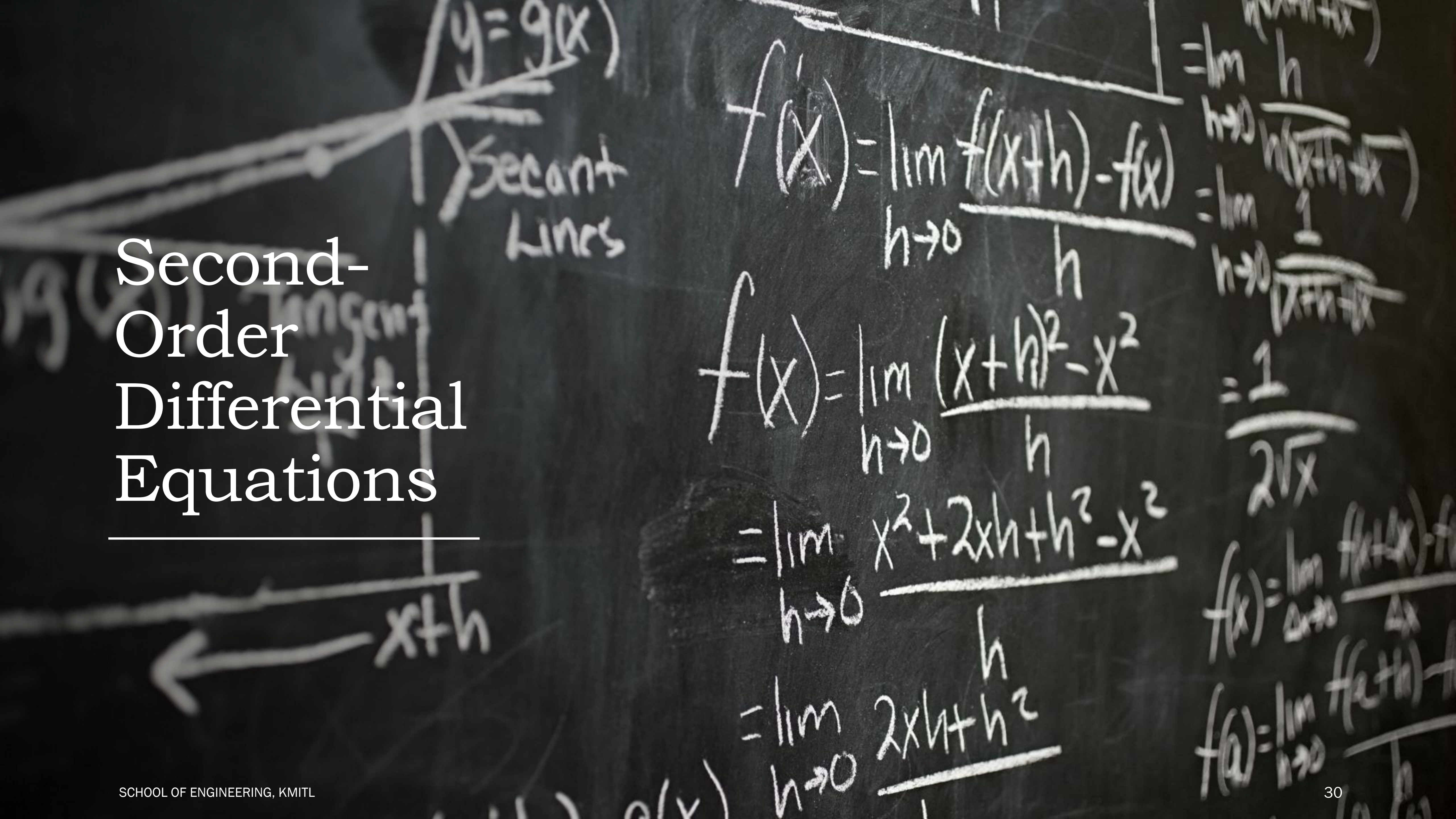
**15.** $y' = x + y, \quad y(0) = 2$

**16.** $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(1) = 0$

**17.** $\dfrac{dv}{dt} - 2tv = 3t^2 e^{t^2}, \quad v(0) = 5$

**27.** In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.
(a) Find $I(t)$.
(b) Find the current after 0.1 s.

**28.** In the circuit shown in Figure 4, a generator supplies a voltage of $E(t) = 40 \sin 60t$ volts, the inductance is 1 H, the resistance is 20 Ω, and $I(0) = 1$ A.
(a) Find $I(t)$.
(b) Find the current after 0.1 s.
(c) Use a graphing device to draw the graph of the current function.




# Second-Order Differential Equations

# Homogeneous Linear Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where $P$, $Q$, $R$, and $G$ are continuous functions. We saw in Section 9.1 that equations of this type arise in the study of the motion of a spring. In Section 17.3 we will further pursue this application as well as the application to electric circuits.

In this section we study the case where $G(x) = 0$, for all $x$, in Equation 1. Such equations are called **homogeneous** linear equations. Thus the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some $x$, Equation 1 is **nonhomogeneous** and is discussed in Section 17.2.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions $y_1$ and $y_2$ of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

$\boxed{3}$ **Theorem** If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $\boxed{2}$ and $c_1$ and $c_2$ are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

PROOF Since $y_1$ and $y_2$ are solutions of Equation 2, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and

$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}
P(x)y'' + Q(x)y' + R(x)y & \\
&= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&= c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\
&= c_1(0) + c_2(0) = 0
\end{aligned}$$

Thus $y = c_1y_1 + c_2y_2$ is a solution of Equation 2.

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions $y_1$ and $y_2$. This means that neither $y_1$ nor $y_2$ is a constant multiple of the other. For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

**4** **Theorem** If $y_1$ and $y_2$ are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $c_1$ and $c_2$ are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it's not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions $P$, $Q$, and $R$ are constant functions, that is, if the differential equation has the form

**5** $$\boxed{ay'' + by' + cy = 0}$$

where $a$, $b$, and $c$ are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function $y$ such that a constant times its second derivative $y''$ plus another constant times $y'$ plus a third constant times $y$ is equal to 0. We know that the exponential function $y = e^{rx}$ (where $r$ is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But $e^{rx}$ is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if $r$ is a root of the equation

**6** $$\boxed{ar^2 + br + c = 0}$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing $y''$ by $r^2$, $y'$ by $r$, and $y$ by 1.

Sometimes the roots $r_1$ and $r_2$ of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

**7** $$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

**CASE I** $b^2 - 4ac > 0$

In this case the roots $r_1$ and $r_2$ of the auxiliary equation are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions of Equation 5. (Note that $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$.) Therefore, by Theorem 4, we have the following fact.

**8** If the roots $r_1$ and $r_2$ of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

**EXAMPLE 1** Solve the equation $y'' + y' - 6y = 0$.

**EXAMPLE 2** Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$

**CASE II** $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by $r$ the common value of $r_1$ and $r_2$. Then, from Equations 7, we have

$$\boxed{9} \qquad r = -\frac{b}{2a} \qquad \text{so} \qquad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

The first term is 0 by Equations 9; the second term is 0 because $r$ is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

**10** If the auxiliary equation $ar^2 + br + c = 0$ has only one real root $r$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{rx} + c_2xe^{rx}$$

**V EXAMPLE 3** Solve the equation $4y'' + 12y' + 9y = 0$.

**CASE III** $b^2 - 4ac < 0$

In this case the roots $r_1$ and $r_2$ of the auxiliary equation are complex numbers. (See Appendix H for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where $\alpha$ and $\beta$ are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix H, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants $c_1$ and $c_2$ are real. We summarize the discussion as follows.

**11** If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

**V EXAMPLE 4** Solve the equation $y'' - 6y' + 13y = 0$.

## Initial-Value and Boundary-Value Problems

An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution $y$ of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \qquad y'(x_0) = y_1$$

where $y_0$ and $y_1$ are given constants. If $P$, $Q$, $R$, and $G$ are continuous on an interval and $P(x) \neq 0$ there, then a theorem found in more advanced books guarantees the existence and uniqueness of a solution to this initial-value problem. Examples 5 and 6 illustrate the technique for solving such a problem.

**EXAMPLE 5** Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

**EXAMPLE 6** Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

A **boundary-value problem** for Equation 1 or 2 consists of finding a solution $y$ of the differential equation that also satisfies boundary conditions of the form

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution. The method is illustrated in Example 7.

**V EXAMPLE 7** Solve the boundary-value problem

$$y'' + 2y' + y = 0 \qquad y(0) = 1 \qquad y(1) = 3$$

**1–13** Solve the differential equation.

1. $y'' - 6y' + 8y = 0$
2. $y'' - 4y' + 8y = 0$
3. $y'' + 8y' + 41y = 0$
4. $2y'' - y' - y = 0$
5. $y'' - 2y' + y = 0$
6. $3y'' = 5y'$
7. $4y'' + y = 0$
8. $16y'' + 24y' + 9y = 0$
9. $4y'' + y' = 0$
10. $9y'' + 4y = 0$
11. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - y = 0$
12. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 4y = 0$
13. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

**17–24** Solve the initial-value problem.

17. $2y'' + 5y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -4$
18. $y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$
19. $4y'' - 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.5$
20. $2y'' + 5y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 4$
21. $y'' + 16y = 0, \quad y(\pi/4) = -3, \quad y'(\pi/4) = 4$
22. $y'' - 2y' + 5y = 0, \quad y(\pi) = 0, \quad y'(\pi) = 2$
23. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1$
24. $y'' + 12y' + 36y = 0, \quad y(1) = 0, \quad y'(1) = 1$

**25–32** Solve the boundary-value problem, if possible.

25. $4y'' + y = 0, \quad y(0) = 3, \quad y(\pi) = -4$
26. $y'' + 2y' = 0, \quad y(0) = 1, \quad y(1) = 2$
27. $y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y(3) = 0$
28. $y'' + 100y = 0, \quad y(0) = 2, \quad y(\pi) = 5$
29. $y'' - 6y' + 25y = 0, \quad y(0) = 1, \quad y(\pi) = 2$
30. $y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y(1) = 0$
31. $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y(\pi/2) = 1$
32. $9y'' - 18y' + 10y = 0, \quad y(0) = 0, \quad y(\pi) = 1$

# Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$\boxed{1} \qquad ay'' + by' + cy = G(x)$$

where $a$, $b$, and $c$ are constants and $G$ is a continuous function. The related homogeneous equation

$$\boxed{2} \qquad ay'' + by' + cy = 0$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation $\boxed{1}$.

$\boxed{3}$ **Theorem** The general solution of the nonhomogeneous differential equation $\boxed{1}$ can be written as

$$y(x) = y_p(x) + y_c(x)$$

where $y_p$ is a particular solution of Equation 1 and $y_c$ is the general solution of the complementary Equation 2.

PROOF We verify that if $y$ is any solution of Equation 1, then $y - y_p$ is a solution of the complementary Equation 2. Indeed

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0 \end{aligned}$$

This shows that every solution is of the form $y(x) = y_p(x) + y_c(x)$. It is easy to check that every function of this form is a solution.

## The Method of Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution $y_p$ that is a polynomial of the same degree as $G$ because if $y$ is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as $G$) into the differential equation and determine the coefficients.

**V EXAMPLE 1** Solve the equation $y'' + y' - 2y = x^2$.

**EXAMPLE 2** Solve $y'' + 4y = e^{3x}$.

**V** **EXAMPLE 3** Solve $y'' + y' - 2y = \sin x$.

**V EXAMPLE 4** Solve $y'' - 4y = xe^x + \cos 2x$.

**EXAMPLE 5** Solve $y'' + y = \sin x$.

**1–10** Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' - 2y' - 3y = \cos 2x$
2. $y'' - y = x^3 - x$
3. $y'' + 9y = e^{-2x}$
4. $y'' + 2y' + 5y = 1 + e^x$
5. $y'' - 4y' + 5y = e^{-x}$
6. $y'' - 4y' + 4y = x - \sin x$
7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$
8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$
9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$
10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$