Problem Set 6: Discrete Mathematics Theepakorn Phayonrat 67011352

1. We define the degree of a node in an undirected graph without loops as the number of edges incident with it. The degree of the node v is denoted by deg(v). Prove the following theorems.

1.1 In any graph G = (V, E), the sum of degrees of all its nodes equals twice the number of edges.

1.2 In any graph G = (V, E), the number of odd nodes is even. (an even number of nodes has an odd degree).

2. A graph with v vertices and n edges has at least $v-n$ connected components.

3. A graph $G = (V, E)$ is a forest if it doesn't contain any cycles.
Let $G = (V, E)$ be a forest. There must exist a vertex $v \in V$ with $\deg(v) \leq 1$.
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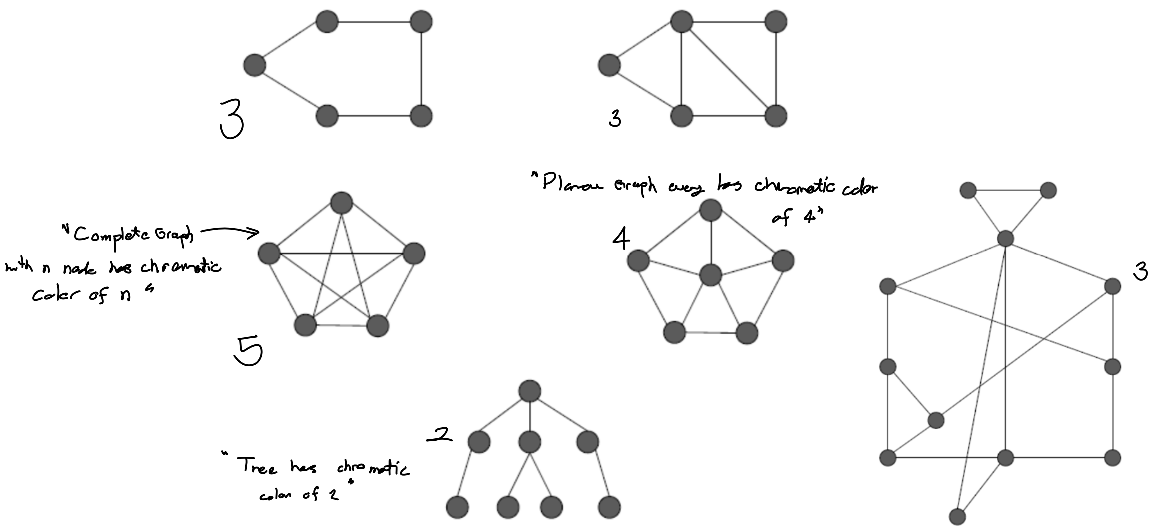
4. Let T = (V, E) be a tree. Show that if |V| ≥ 2, T has at least two leaves.

5. The chromatic number of a graph is the least number of colors required to do a vertex-coloring of a graph. Calculate the chromatic number of the following graphs.



6. A computer network (a connected graph) consists of 6 computers. Show that there are at least two computers in the network that are directly connected to the same number of computers.

7. What is the minimum number of students needed in a class to guarantee that there are at least 6 students whose birthdays fall in the same month?

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1. We define the degree of a node in an undirected graph without loops as the number of edges incident with it. The degree of the node v is denoted by deg(v). Prove the following theorems.

1.1 In any graph G = (V, E), the sum of degrees of all its nodes equals twice the number of edges.

Proof Let S be "the sum of all nodes of G".

In counting S, we count each edge exactly twice.

$\therefore S = 2|E|$ #

1.2 In any graph G = (V, E), the number of odd nodes is even. (an even number of nodes has an odd degree).

Proof Let S_o be the sum of all odd degrees,

and let S_e be the sum of all even degrees.

Then we have $S_o + S_e = 2|E|$.

Clearly, S_e is even as it is the sum of all even degrees and the right is an even number.

$\therefore S_o = 2k ; \exists k \in \mathbb{Z}$

$\therefore$ The sum of all odd degrees of graph is an even number.

$\therefore$ The sum of odd nodes of a graph is always even. #

2. A graph with v vertices and n edges has at least v−n connected components.

Proofs Let P(n) be "A graph with v vertices and n edges has at least v−n connected components".

Basis Step : P(0) : "A graph with v vertices and 0 edge has at least v−0 connected components."

So, if we have 0 edges each vertex is a connected component and there are V of them.

$\therefore$ P(0) is true. #

Induction Hypothesis : Assume P(k): "A graph with v vertices and k edges has at least v−k connected components." is true.

Induction step

P(k) → P(k+1): Let there be a graph with V vertices and k+1 edges. Pick any edge, remove that edge and observe the current graph, we can conclude that we now have a graph with k edges. From IH we know that if we have k edges there will be v−k connected components. Finally, put that edge we remove earlier back in its place. From observation, there will be 1 edge lesser than earlier. $\therefore$ A graph with v vertices and k+1 edges has v−k−1 or v−(k+1) connected components #

$\therefore$ "A graph with v vertices and n edges has v−n connected components." is true by Induction #

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3. A graph $G = (V, E)$ is a forest if it doesn't contain any cycles.
Let $G = (V, E)$ be a forest. There must exist a vertex $v \in V$ with $\deg(v) \le 1$.

Proof

Proof by Contradiction.

Assume $\deg(x_1) \geq 2$; $\forall v \in V$

$\therefore E \neq \emptyset$.

Now consider the longest path p in G. Since $E \neq \emptyset$, there's at least 1 path in G, and thus p is well-defined

Let $p = (x_1, x_2, x_3, \ldots, x_{k-1}, x_k)$

Now by our supposition, $\deg(x) > 1$

$\therefore$ The exists a vertex $v = x_1$ such that (v, x_1) is an edge. Note v cannot be x_i for any $3 \leq i \leq k$.

For if that was the case, then $(x_1, x_2, \ldots, x_{i-1}, x_i = v, x_1)$ would be a cycle. And G has no cycle (being a forest).

$\therefore$ Since $v \neq x$ and $v \neq x_2$, $v \notin p$. But then $(v, x_1, x_2 \ldots x_k)$ is no longer a path than p. This contridicts the choice of p being the longest path *

$\therefore$ "There must exist a vertex $v \in V$ with $\deg(v) \leq 1$" is true by contradiction *

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4. Let T = (V, E) be a tree. Show that if |V| ≥ 2, T has at least two leaves.

Proof

Consider a simple path P of maximum length in the tree T. Say P starts at u, ends in v, and the vertices on the path are $u = v_0, v_1, \dots, v_{r-1}, v_r = v$.

We claim that u and v have degree 1, which means they are leaves.

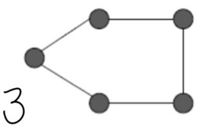
Suppose u does not have degree 1. Then there is another edge besides $\{u, v_1\}$ that's incident on u, say $\{w, u\}$. This edge does not appear on path P because P is simple. There are 2 cases to consider, and each of them leads to a contradiction.

Case I: w does not appear on the path P. Then the path $w, u, v_1, \dots v_p$ v is a simple path longer than P, which is a contradiction because P is the longest simple path in T.

Case II: w appears on the path P, say $w = v_i$. Then the path $w = v_i, v_{i+1}, \dots v_p\ v_i$ is a simple cycle in T. But since T doesn't contain any simple cycle, this is a contradiction.

∴ "If |V| ≥ 2, T has at least two leaves." is true by contradiction

5. The chromatic number of a graph is the least number of colors required to do a vertex-coloring of a graph. Calculate the chromatic number of the following graphs.



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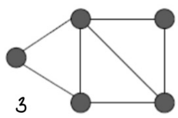

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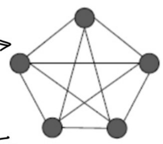
"Complete Graph with n node has chromatic color of n"



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"Planar graph every has chromatic color of 4"

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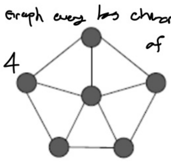

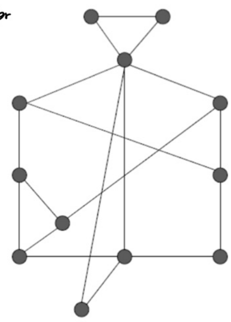

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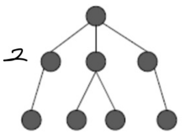

"Tree has chromatic color of 2"

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6. A computer network (a connected graph) consists of 6 computers. Show that there are at least two computers in the network that are directly connected to the same number of computers.

Proof Since all computer are connected together, $\therefore$ each of the computer can connect to 1, 2, 3, 4, 5 other computers.

From Generalized Pigeonhole Principle m = number of computer = 6, n = number of connection = 5.

$\therefore$ There are at least $\lceil \frac{6}{5} \rceil = 2$ of computers that have the same numbers of connections. $\divideontimes$

7. What is the minimum number of students needed in a class to guarantee that there are at least 6 students whose birthdays fall in the same month?

By Generalized Pigeonhole Principle, we apply $k = 12$ as the number of months. We are finding the smallest integer N such that $\lceil \frac{N}{12} \rceil = 6$. That integer is $N = 61$.

$\therefore$ We need 61 students to guarantee that a set of 6 students whose birthday fall in the same month. $\divideontimes$