Exercise 9.1

1. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.

2. Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial-value problem

$$y' + (\tan x)y = \cos^2 x \qquad y(0) = -1$$

on the interval $-\pi/2 < x < \pi/2$.

$\boxed{y' = 1 + x^{-2}}$

1

$$xy' + y = x(1 + x^{-2}) + (x - x^{-1})$$

$$= x + \cancel{x^{-1}} + x - \cancel{x^{-1}}$$

$$= 2x$$

$\therefore$ $y = x - x^{-1}$ is a solution

2

$$y = \sin x \cos x - \cos x$$

$$y' = [\sin x(-\sin x) + \cos x \cos x] + \sin x$$

$$= \cos^2 x - \sin^2 x + \sin x$$

$$= (\cos^2 x - \sin^2 x + \sin x) + (\tan x)(\sin x \cos x - \cos x)$$

$$= \cos^2 x - \sin^2 x + \sin x + \left(\frac{\sin x}{\cancel{\cos x}}\right)(\sin x \cancel{\cos x} - \cancel{\cos x})$$

$$= \cos^2 x - \cancel{\sin^2 x} + \cancel{\sin x} + \cancel{\sin^2 x} - \cancel{\sin x}$$

$$= \cos^2 x \qquad \therefore \quad y' + (\tan x)y = \cos^2 x$$

$$y(0) = (\cancel{\sin 0})(\cos 0) - \cancel{\cos 0}$$

0, 1

$$-1 = -1 \quad \checkmark$$

Exercise 9.3

1. $\frac{dy}{dx} = \frac{y}{x}$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + c$$

$$|y| = e^{\ln|x| + c}$$

$$= e^{c} \cdot e^{\ln|x|}$$

$$|y| = k|x|$$

3. $(x^2+1)y' = xy$

$$\frac{dy}{dx} = \frac{xy}{x^2+1}$$

$$\frac{dy}{y} = \frac{x\,dx}{x^2+1}$$

$$\ln|y| = \frac{1}{2}\ln|x^2+1| + c$$

$$\ln|y| = \ln(x^2+1)^{1/2} + c$$

$$y = e^{c}(x^2+1)^{1/2}$$

$$y = k(x^2+1)^{1/2}$$

$\int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$\frac{du}{2} = x\,dx$

$= \int \frac{du}{2u}$

$\frac{1}{2}\ln|u|$

5. $(1+\tan y)y' = x^2+1$

$$\frac{dy}{dx} = \frac{x^2+1}{1+\tan y}$$

$$(1+\tan y)dy = (x^2+1)dx$$

$$\int (1+\tan y)dy = \int (x^2+1)dx$$

$$y - \ln|\cos y| = \frac{x^3}{3} + x + c$$

$$\int (1+\tan y)dy$$

$$= y + \int \frac{\sin y}{\cos y} dy$$

$u = \cos y$

$du = -\sin y\, dy$

$-du = \sin y\, dy$

$$= y - \int \frac{du}{u}$$

$$= y - \ln|u|$$

(11) $\frac{dy}{dx} = y^2+1$, $y(1)=0$

$$\int \frac{dy}{y^2+1} = \int dx$$

$$\tan^{-1} y = x + c$$

$y(1)=0$ $\quad 0 = 1+c$

$$-1 = c$$

$$\therefore \tan^{-1} y = x-1$$

or $y = \tan(x-1)$

(13) $x\cos x = (2y + e^{3y}) y'$

$$\int (2y+e^{3y})dy = \int x\cos x \, dx$$

$$y^2 + \frac{e^{3y}}{3} = x\sin x + \cos x + C$$

$y(0)=0$ $\quad 0 + \frac{e^0}{3} = 0 + \cos 0 + C$

$$\frac{1}{3} = 1 + C$$

$$-\frac{2}{3} = C$$

$$\int x\cos x \, dx$$

$$\int u\,dv = uv - \int v\,du$$

$u = x$ | $dv = \cos x\,dx$

$du = dx$ | $v = \sin x$

$$= x\sin x - \int \sin x \, dx$$

$$= x\sin x + \cos x$$

39. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?

$y(0) = 15$

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= (0)(10\ l/min) - \left(\frac{y(t)}{1000}\right)(10\ l/min)$$

$$\frac{dy}{dt} = -\frac{y}{100}$$

$$\frac{dy}{y} = -\frac{1}{100}\,dt$$

$$\ln|y| = -\frac{t}{100} + C$$

$$\ln|15| = -\frac{0}{100} + C$$

$$\therefore C = \ln 15$$

$$\therefore \ln|y| = -\frac{t}{100} + \ln 15$$

$$y = e^{-t/100} \cdot 15$$

$$y = 15e^{-t/100} \quad \#$$

$$y(20) = 15e^{-20/100}$$

$$= 12.28\ kg \quad \#$$

1–4 Determine whether the differential equation is linear.

× **1.** $y' + e^x y = x^2 y^2$

✓ **2.** $y + \sin x = x^3 y'$

✓ **3.** $xy' + \ln x - x^2 y = 0$

× **4.** $y' + \cos y = \tan x$

$x^3 y' = y + \sin x$

$x^3 y' - y = \sin x$

$y' - \dfrac{y}{x^3} = \dfrac{\sin x}{x^3}$

5. $y' + 2y = 2e^x$

$$P(x) = 2$$

$$I = e^{\int P(x)\,dx}$$

$$= e^{\int 2\,dx}$$

$$= e^{2x}$$

$$e^{2x}y' + 2e^{2x}y = 2e^{x}e^{2x}$$

$$(e^{2x}y)' = 2e^{3x}$$

$$e^{2x}y = \int 2e^{3x}\,dx$$

$$e^{2x}y = \frac{2e^{3x}}{3} + C$$

$$y = \frac{2}{3}e^{x} + Ce^{-2x}$$

7. $xy' - 2y = x^2$

$$y' - \frac{2y}{x} = x$$

$$\therefore \ P(x) = -\frac{2}{x}$$

$$I(x) = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\ln x}$$

$$= x^{-2}$$

$$x^{-2}y' - \frac{2y}{x^3} = x^{-1}$$

$$(x^{-2}y)' = \frac{1}{x}$$

$$x^{-2}y = \ln|x| + C$$

$$y = x^2\ln|x| + Cx^2$$

15. $y' = x + y, \quad y(0) = 2$

$y' - y = x$

$P(x) = -1$

$I(x) = e^{\int(-1)\,dx}$

$= e^{-x}$

$e^{-x}y' - e^{-x}y = xe^{-x}$

$(e^{-x}y)' = xe^{-x}$

$e^{-x}y = -xe^{-x} - e^{-x} + C$

$y = -x - 1 + Ce^{x}$

$\int xe^{-x}\,dx$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$= -xe^{-x} - \int -e^{-x}\,dx$

$= -xe^{-x} + e^{-x}$

$= -xe^{-x} - e^{-x}$

$y(0) = 2;$

$2 = 0 - 1 + Ce^{0}$

$3 = C$

$\therefore \quad y = -x - 1 + 3e^{x}$ #

$$\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}, \quad v(0) = 5$$

$P(t) = -2t$

$I(t) = e^{\int(-2t)\,dt}$

$= e^{-t^2}$

$e^{-t^2} v' - 2e^{-t^2} t v = 3e^{t^2} t^2 (e^{-t^2})$

$(e^{-t^2} v)' = 3t^2$

$e^{-t^2} v = t^3 + C$

$v(0) = 5;\quad e^0 (5) = 0 + C$

$5 = C$

$\therefore\quad e^{-t^2} v = t^3 + 5$

$v = t^3 e^{t^2} + 5e^{t^2}$

27. In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.

(a) Find $I(t)$.

(b) Find the current after 0.1 s.

$$L\frac{dI}{dt} + RI = E(t)$$

$$2\frac{dI}{dt} + 10I = 40$$

$$I' + 5I = 20$$

$$I = e^{\int 5\,dt}$$

$$= e^{5t}$$

$$e^{5t}I' + 5e^{5t}I = 20e^{5t}$$

$$(e^{5t}I)' = 20e^{5t}$$

$$e^{5t}I = \int 4e^{5t}\,dt$$

$$e^{5t}I = 4e^{5t} + C$$

$I(0) = 0$;

$$(e^{0})(0) = 4e^{0} + C$$

$$-4 = C$$

$$\therefore e^{5t}I = 4e^{5t} - 4$$

$$I = 4 - 4e^{-5t}$$

$$I(0.1) = 4 - 4e^{-5(0.1)}$$

$$= 1.57\text{ A.}$$

1. $y'' - 6y' + 8y = 0$

3. $y'' + 8y' + 41y = 0$

5. $y'' - 2y' + y = 0$

① $r^2 - 6r + 8 = 0$

$(r-4)(r-2) = 0$

$r = 2, 4$

$\therefore y = c_1 e^{2x} + c_2 e^{4x}$

③ $r^2 + 8r + 41 = 0$

$$r = \frac{-8 \pm \sqrt{64 - 4(1)(41)}}{2}$$

$= -4 \pm 5i$

$\alpha = -4, \beta = 5$

$\therefore y = e^{-4x}(c_1 \cos 5x + c_2 \sin 5x)$

⑤ $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$\therefore y = c_1 e^{x} + c_2 x e^{x}$

17. $2y'' + 5y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = -4$

$2r^2 + 5r + 3 = 0$

$(2r+3)(r+1) = 0$

$r = -\frac{3}{2}, -1$

$\therefore y(x) = c_1 e^{-\frac{3x}{2}} + c_2 e^{-x}$ #

$y'(x) = -\frac{3}{2} c_1 e^{-\frac{3}{2}x} - c_2 e^{-x}$

$y(0) = c_1 + c_2$

$3 = c_1 + c_2$ — ①

$c_2 = 3 - c_1$

$y'(0) = -\frac{3}{2} c_1 - c_2$

$-4 = -\frac{3}{2} c_1 - c_2$

$\therefore -4 = -\frac{3}{2} c_1 - (3 - c_1)$

$-4 = -\frac{3}{2} c_1 - 3 + c_1$

$-1 = -\frac{c_1}{2}$

$2 = c_1$

$\therefore c_2 = 3 - 2 = 1$

$y = 2e^{-\frac{3x}{2}} + e^{-x}$ #

26. $y'' + 2y' = 0, \quad y(0) = 1, \quad y(1) = 2$

$r^2 + 2r = 0$

$r(r+2) = 0$

$r = 0, -2$

$\therefore y = c_1 e^{0x} + c_2 e^{-2x}$

$y = c_1 + c_2 e^{-2x}$

$c_2 = \frac{-e^2}{e^2 - 1}$

$\therefore y = \frac{2e^2 - 1}{e^2 - 1} + \frac{-e^2}{e^2 - 1} e^{-2x}$

$y(0) = c_1 + c_2$

$1 = c_1 + c_2$ — 1 $\Rightarrow c_2 = 1 - c_1$

$y(1) = c_1 + c_2 e^{-2}$

$2 = c_1 + \frac{c_2}{e^2}$

$2 = c_1 + \frac{1 - c_1}{e^2}$

$2e^2 = e^2 c_1 + 1 - c_1$

$2e^2 - 1 = c_1(e^2 - 1)$

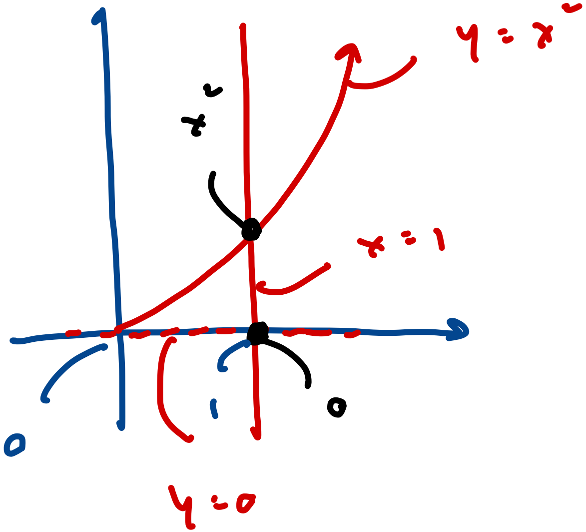
$\frac{2e^2 - 1}{e^2 - 1} = c_1$

$c_2 = 1 - c_1$

$= 1 - \left(\frac{2e^2 - 1}{e^2 - 1}\right)$

$= \frac{e^2 \cancel{-1} - 2e^2 \cancel{+1}}{e^2 - 1}$

$$\iint_D x\cos y\, dA, \quad D \text{ is bounded by } y = 0,\ y = x^2,\ x = 1$$



$$\int_0^1 \int_0^{x^2} x\cos y \, dy\, dx$$

$$= \int_0^1 x\sin y \Big|_0^{x^2} dx$$

$$= \int_0^1 \left[x\sin x^2 - \cancel{x\sin 0}^{\,0} \right] dx$$

$$= \int_0^1 x\sin x^2 \, dx$$

$$= \int_0^1 x \sin x^2 \, dx$$

$$= -\frac{\cos x^2}{2} \Bigg|_0^1$$

$$= -\frac{\cos(1)}{2} - \left(-\frac{\cos(0)^2}{2} \right)$$

$$= -\frac{\cos(1)}{2} + \frac{1}{2}$$

$$\int x \sin x^2 \, dx$$

$$u = x^2$$

$$du = 2x\,dx$$

$$\frac{du}{2} = x\,dx$$

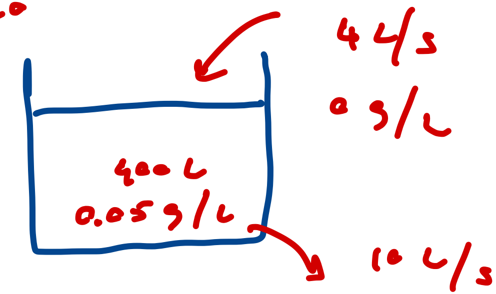
$$= \int \frac{\sin u \, du}{2}$$

$$= -\frac{\cos u}{2}$$

$$= -\frac{\cos x^2}{2}$$

A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

$y(0) = 400 \times 0.05 = 20$



$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

$$= 0 - \left(\frac{y(t)}{400-6t}\right)\left(10\ \text{L/s}\right)$$

$\therefore$ Vol reduces at rate $4 - 10 = -6t$

$$\frac{dy}{dt} = \frac{-5y}{200-3t}$$

$$\frac{dy}{y} = \frac{-5\,dt}{200-3t}$$

$$\ln|y| = \frac{\cancel{-}5\ln|200-3t|}{\cancel{-}3} + c$$

$$\ln|y| = \frac{5}{3}\ln|200-3t| + c$$

$$y = (200-3t)^{5/3} \cdot e^{c}$$

$y(0) = 20$

$$20 = (200)^{5/3} \cdot e^{c}$$

$$20 = (200)^{5/3} \cdot e^{c}$$

$$\frac{20}{(200)^{5/3}} = e^{c}$$

$$\therefore \quad y(t) = \frac{20(200-3t)^{5/3}}{(200)^{5/3}} \quad \#$$

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2}, \quad y(0) = 1$$

$$\frac{(1+y^2)\,dy}{y} = \cos x\,dx$$

$$\left(\frac{1}{y} + y\right)dy = \cos x\,dx$$

$$\ln|y| + \frac{y^2}{2} = \sin x + C$$

$$y(0) = 1$$

$$\cancel{\ln|1|}^{\,0} + \frac{(1)^2}{2} = \cancel{\sin 0}^{\,0} + C$$

$$\frac{1}{2} = C$$

$$\therefore \ln|y| + \frac{y^2}{2} = \sin x + \frac{1}{2} \quad \#$$

$xy' = y + x^2 \sin x, \quad y(\pi) = 0$

$y' + P(x)y = Q(x)$

$xy' - y = x^2 \sin x$

$y' - \frac{y}{x} = x \sin x$

$\therefore P(x) = -\frac{1}{x}$

$I(x) = e^{\int (-\frac{1}{x}) dx}$

$= e^{-\ln x}$

$= e^{\ln(\frac{1}{x})}$

$= \frac{1}{x}$

Multiply $\frac{1}{x}$ both sides

$\frac{y'}{x} - \frac{y}{x^2} = \sin x$

$(\frac{1}{x} y)' = \sin x$

$\frac{1}{x} y = -\cos x + c$

$y = -x\cos x + cx$

$y(\pi) = 0$

$0 = -\pi \cos\pi + c\pi$ $\quad \cos\pi = -1$

$0 = \pi + c\pi$

$-\pi = c\pi$

$-1 = c$

$\therefore$

$y = -x\cos x - x$

#

$$16y'' + 24y' + 9y = 0$$

$$9y'' + 4y = 0$$

$16r^2 + 24r + 9 = 0$

$(4r+3)^2 = 0$

$r = -\frac{3}{4}$

$\therefore \; y = c_1 e^{-\frac{3x}{4}} + c_2 x e^{-3x/4}$

$9r^2 + 4 = 0$

$r^2 = -\frac{4}{9}$

$r = \pm \frac{2}{3} i$

$\therefore \; y = c_1 \cos \frac{2}{3}x + c_2 \sin \frac{2}{3}x$

$2y'' + 5y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 4$

$2r^2 + 5r - 3 = 0$

$(2r-1)(r+3) = 0$

$r = \frac{1}{2}, -3$

$\therefore y = c_1 e^{x/2} + c_2 e^{-3x}$

$y(0) = c_1 + c_2$

$1 = c_1 + c_2$

$1 - c_1 = c_2$

$y' = \frac{c_1 e^{x/2}}{2} - 3c_2 e^{-3x}$

$y'(0) = 4$

$4 = \frac{c_1}{2} - 3c_2$

$4 = \frac{c_1}{2} - 3(1 - c_1)$

$8 = c_1 - 6(1 - c_1)$

$8 = c_1 - 6 + 6c_1$

$14 = 7c_1$

$2 = c_1$

$\therefore c_2 = 1 - 2$

$= -1$

$\therefore y = 2e^{x/2} - e^{-3x}$ #

$$4y'' + y = 0, \quad y(0) = 3, \quad y(\pi) = -4$$

$4r^2 + 1 = 0$

$r^2 = -\frac{1}{4}$

$r = \pm \frac{1}{2} i$

$\therefore \; y = c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}$

$y(0) = 3$

$3 = c_1 \cos 0 + c_2 \sin 0$, where $\cos 0 = 1$ and $\sin 0 = 0$

$c_1 = 3$

$y(\pi) = -4$

$-4 = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right)$, where $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$

$-4 = c_2$

$\therefore \; y = 3\cos\frac{x}{2} - 4\sin\frac{x}{2}$ #