4. Let $T = (V, E)$ be a tree. Show that if $|V| \geq 2$, T has at least two leaves.

Proof 1:

Consider a simple path P of maximum length in the tree T with $|V| \geq 2$.
(It's alright if your tree has more than one path of equal length, so long as we choose one with the longest possible length.)

Say P starts at u, ends in v, and the vertices on the path are $u = v_0, v_1, \ldots, v_{r-1}, v_r = v$.
We claim that u and v have degree 1, which means they are leaves.

We will prove that both endpoints (u and v) of P are leaves.

Suppose u does not have degree 1 (u is not a leave). Then there is another edge besides $\{u, v_1\}$ that's incident on u, say $\{w, u\}$. This edge does not appear on the path P because P is simple. There are two cases to consider, and each of them leads to a contradiction.

Case 1: w does not appear on the path P. Then the path $w, u, v_1, \ldots, v_r, v$ is a simple path longer than P, which is a contradiction because P is the longest simple path in T.
Case 2: w appears on the path P, say $w = v_i$. Then the path $w = v_i , v_{i+1}, \ldots, v_r, v_i$ is a simple cycle in T. But since T doesn't contain any simple cycles, this is a contradiction.
∎

4. Let T = (V, E) be a tree. Show that if $|V| \geq 2$, T has at least two leaves.

Proof 2:

Let the number of vertices in a given tree T is $|V| >= 2$.
Therefore, the number of edges in a tree $T = |V|-1$.
Let L be number of leaves. Then number of vertices with degree ≥ 2 is $|V|-L$.
Summation of degree of all Vertices = 2 * (number of edges)

$$= 2 * (|V|-1)$$

$$= 2|V|-2$$

Summation of degree of all Vertices = L + Summation of degree of $|V|-L$

$$\geq L + 2(|V|-L)$$

$$\geq L + 2|V|-2L$$

$$\geq 2|V|- L$$

$2|V|-2 \geq 2|V|- L$

$-2 \geq - L$

$L \geq 2$

■