Discrete Mathematics

Counting Principles

Bijection Rule

(Counting One Thing by Counting Another)

If there is a bijection f : A -> B between A and B, then |A| = |B|.

Let $f:A \to B$ be a function from a set A to a set B.

f is 1-1 (or injective) if and only if

$$\forall x,y \in A,\ \ x \neq y \to f(x) \neq f(y)$$

f is onto (or surjective) if and only if

$$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

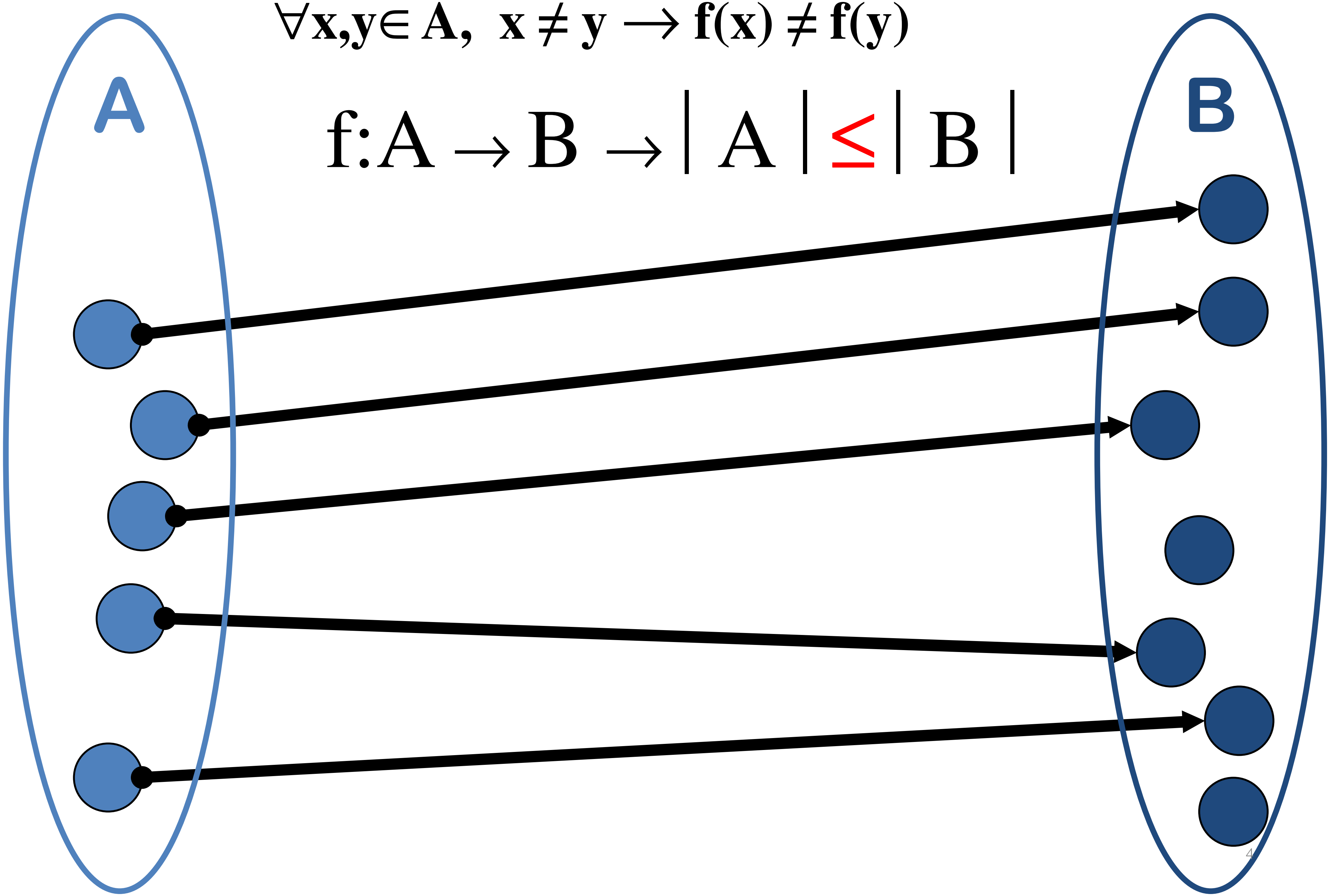
There Exists

For Every

f is 1-1 (or injective) if and only if

$$\forall x,y \in A,\ x \neq y \rightarrow f(x) \neq f(y)$$

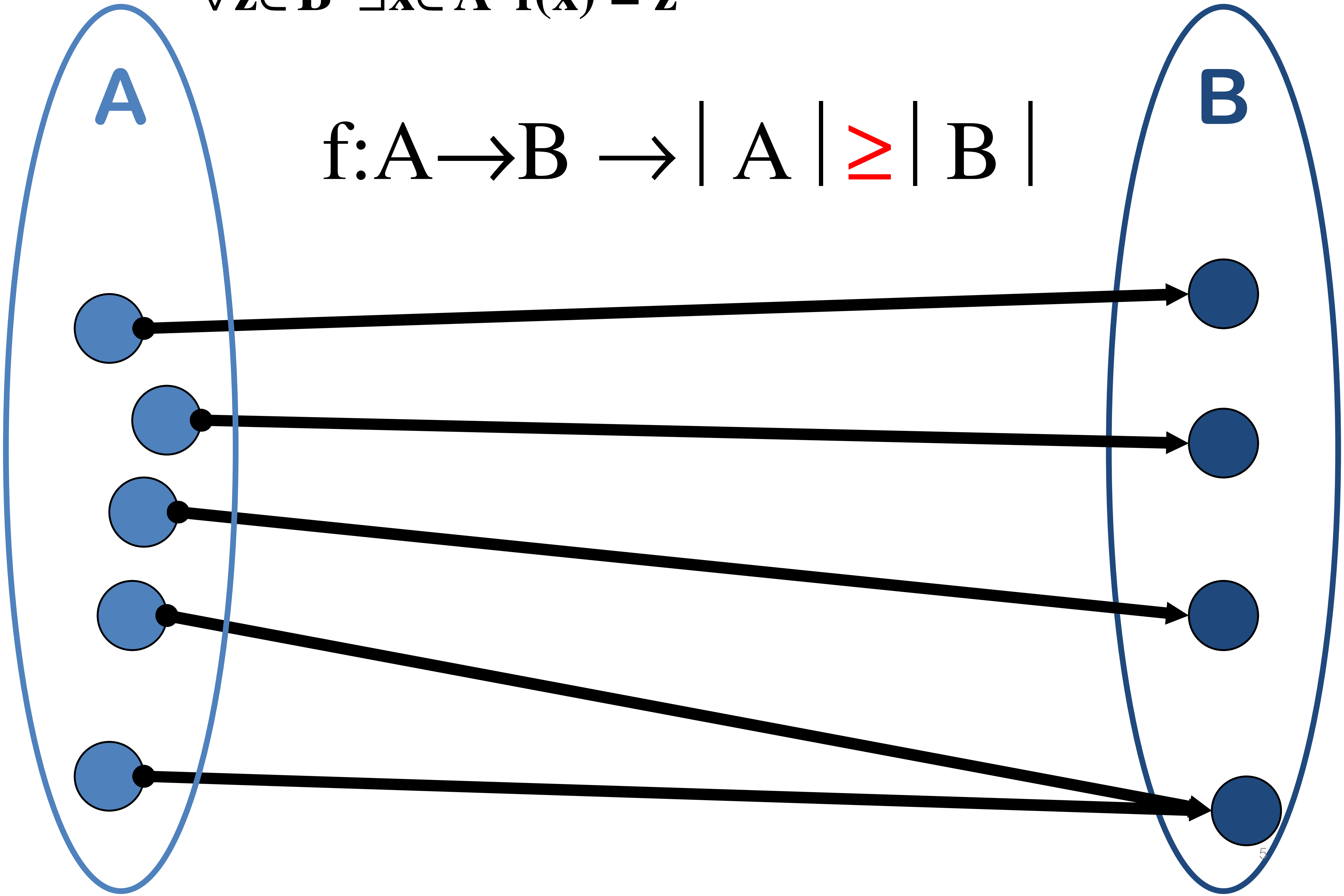
$$f{:}A \rightarrow B \rightarrow |A| \leq |B|$$



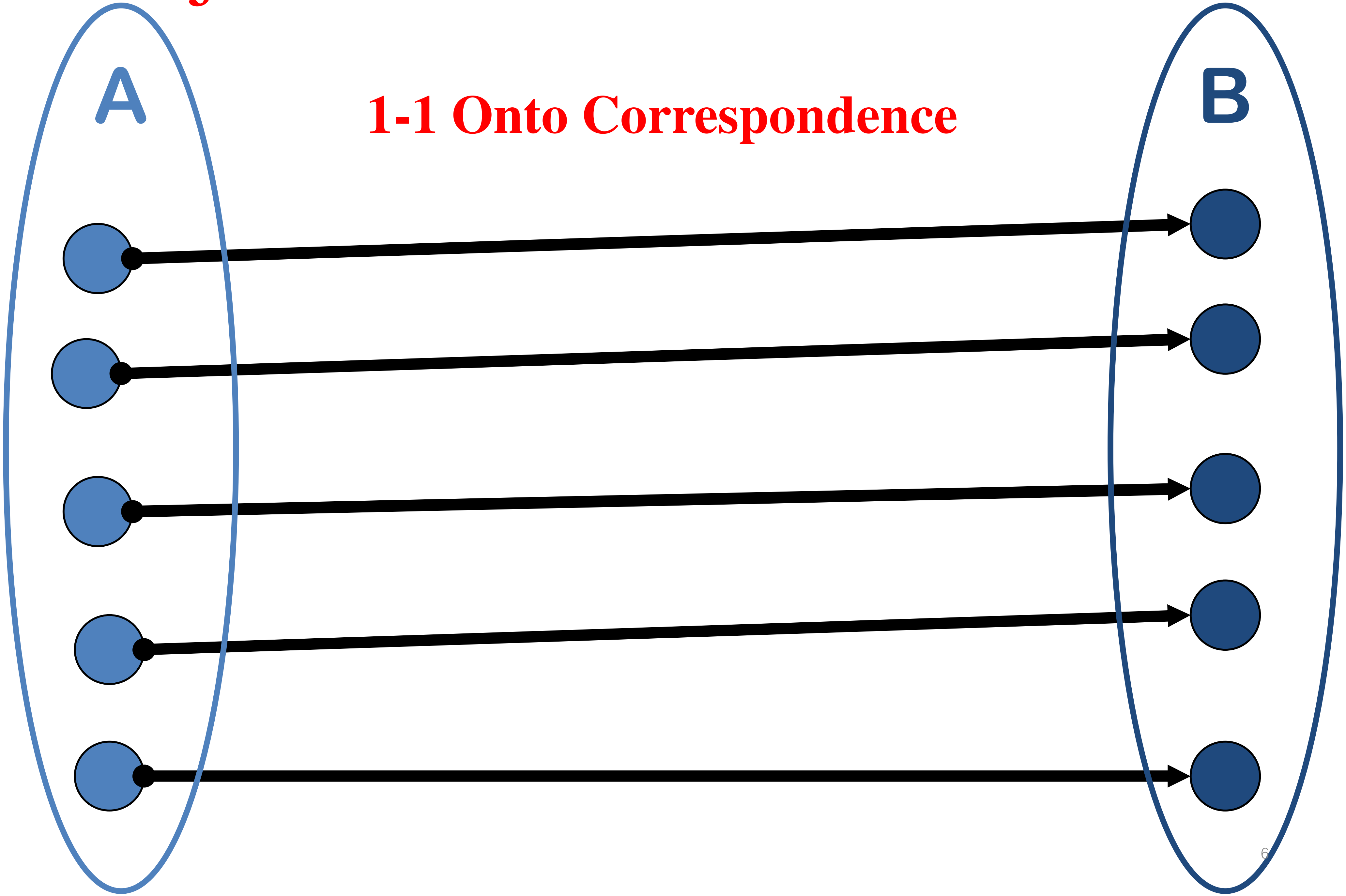
f is onto (or surjective) if and only if

$\forall z \in B \;\; \exists x \in A \;\; f(x) = z$

$$f{:}A \to B \;\to\; |A| \geq |B|$$



Bijection $f: A \rightarrow B \rightarrow |A| = |B|$



Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

How many n-bit binary sequences are there?

How many n-bit binary sequences are there?

000000	↔	0
000001	↔	1
000010	↔	2
000011	↔	3
	…	
111111	↔	2^n-1

How many n-bit binary sequences are there?

000000	$\leftarrow\rightarrow$	0
000001	$\leftarrow\rightarrow$	1
000010	$\leftarrow\rightarrow$	2
000011	$\leftarrow\rightarrow$	3
	…	
111111	$\leftarrow\rightarrow$	2^n-1

2^n sequences

$S = \{a,b,c,d,e\}$ has many subsets.

$\emptyset$, $\{a\}$, $\{a,b\}$, $\{a,d,e\}$, $\{a,b,c,d,e\}$, …

The empty set is a set with all the rights and privileges pertaining thereto.

How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

How many subsets can be formed from the elements of a 5-element set?

a	b	c	d	e
0	1	1	0	1

{b c e}

1 means "TAKE IT"
0 means "LEAVE IT"

How many subsets can be formed from the elements of a 5-element set?

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

How many subsets can be formed from the elements of a 5-element set?

Each subset corresponds to a 5-bit sequence (using the "take it or leave it" code)

f: bit strings $\rightarrow$ subsets

How many subsets can be formed from the elements of a 5-element set?

Each subset corresponds to a 5-bit sequence (using the “take it or leave it” code)

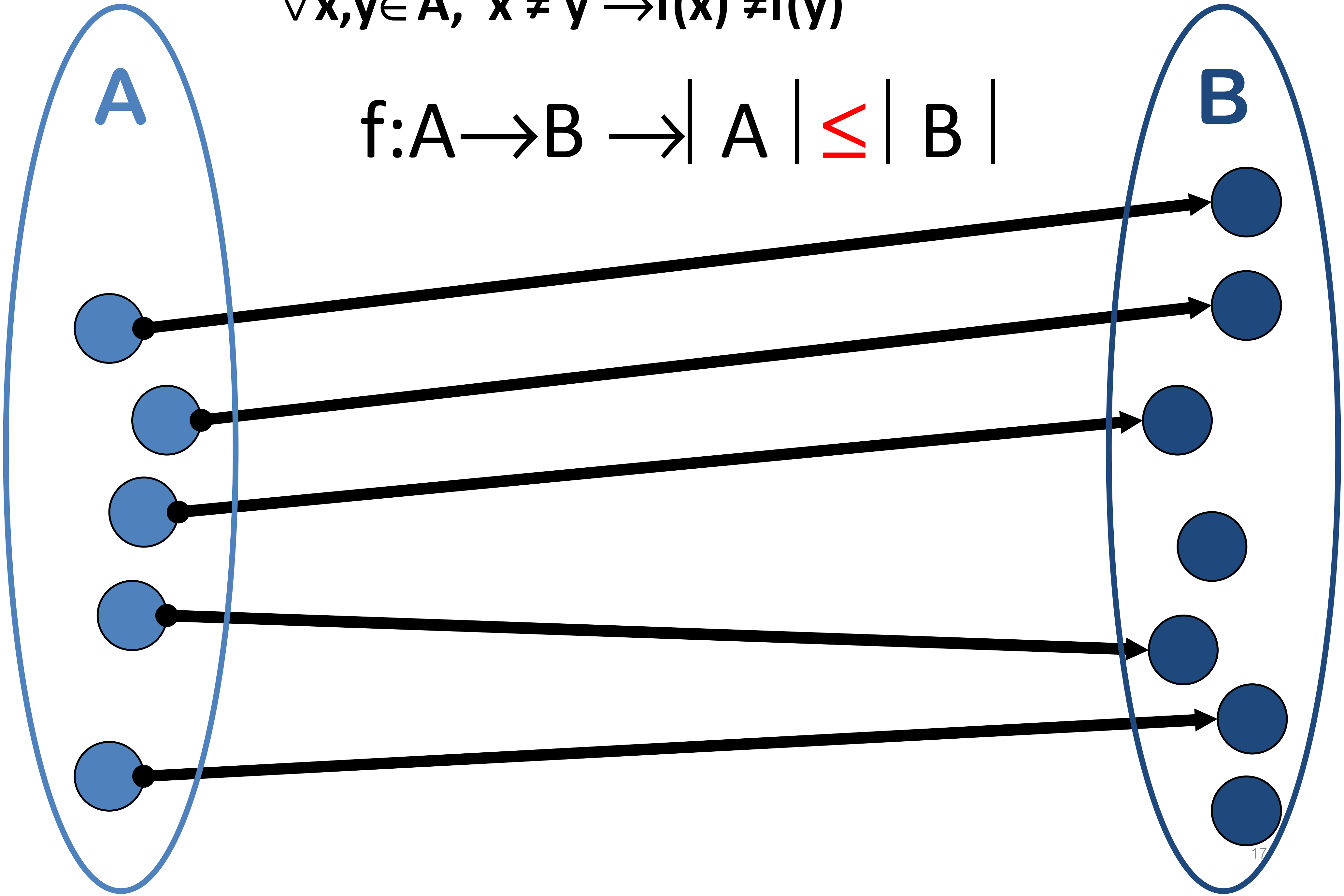
f: bit strings $\rightarrow$ subsets

How do we prove bijection?

f is 1-1 (or injective) if and only if

$$\forall x,y \in A,\ \ x \neq y \rightarrow f(x) \neq f(y)$$

$$f{:}A \rightarrow B \rightarrow |A| \leq |B|$$



$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b = b_1 b_2 b_3 \ldots b_n$, let $f(b) = \{ a_i \mid b_i = 1\}$

$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b=b_1b_2b_3\ldots b_n$, let $f(b)=\{\, a_i \mid b_i=1\}$

Claim: f is injective

$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b=b_1b_2b_3\ldots b_n$, let $f(b)=\{\ a_i \mid b_i=1\}$

Claim: f is injective

Any two distinct binary sequences b and b′ have a position i at which they differ

$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S
B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b=b_1b_2b_3\ldots b_n$, let $f(b)=\{\, a_i \mid b_i=1\}$

Claim: f is injective

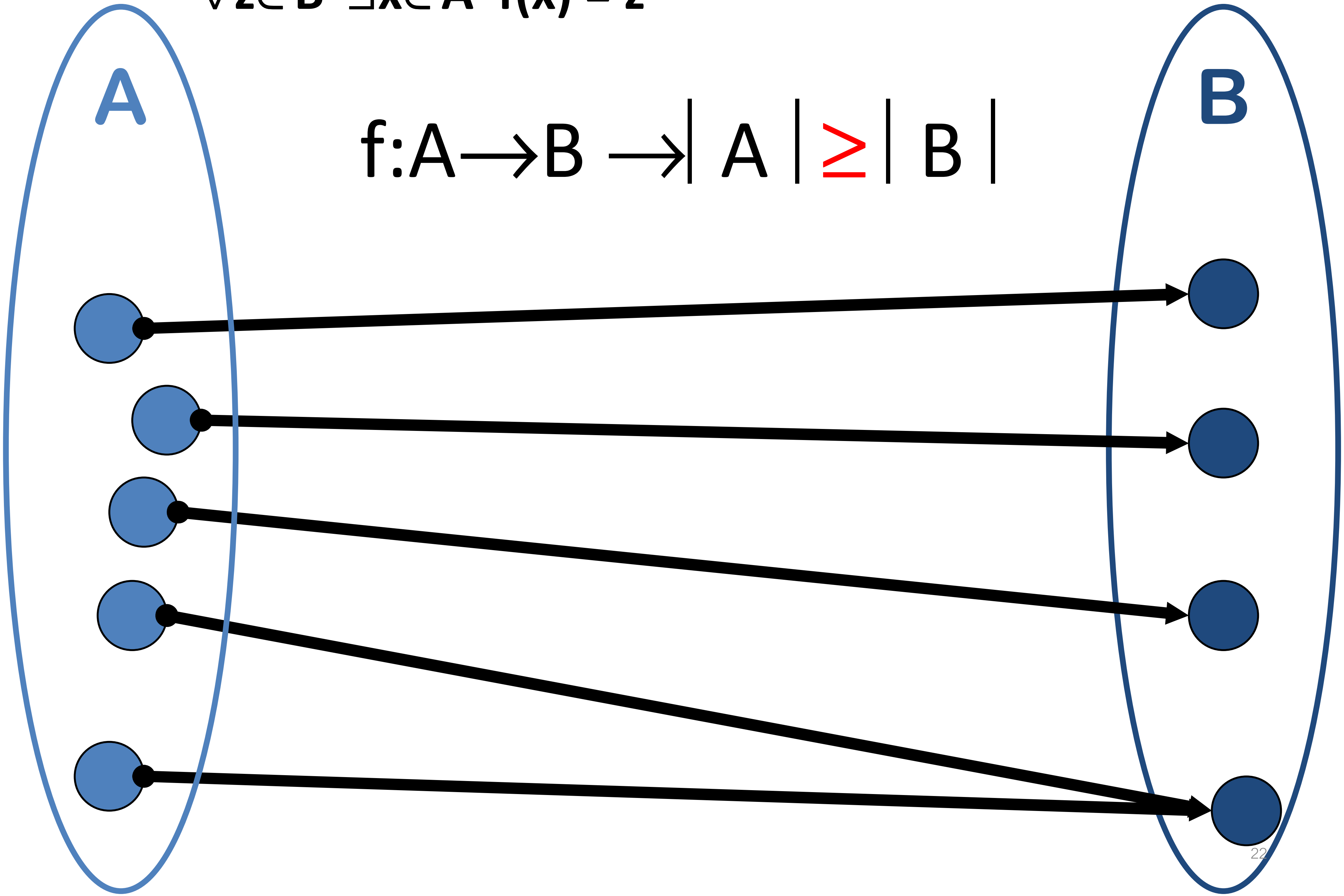
Any two distinct binary sequences b and b′ have a position i at which they differ

Hence, f(b) is not equal to f(b′) because they disagree on element a_i

f is onto (or surjective) if and only if

$\forall z \in B \;\; \exists x \in A \;\; f(x) = z$

$f: A \to B \to |A| \geq |B|$



$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b=b_1b_2b_3\ldots b_n$, let $f(b)=\{\, a_i \mid b_i=1\}$

Claim: f is surjective

$S = \{a_1, a_2, a_3, \ldots, a_n\}$, T = all subsets of S

B = set of all n-bit strings

a_1	a_2	a_3	a_4	a_5
b_1	b_2	b_3	b_4	b_5

For bit string $b=b_1b_2b_3\ldots b_n$, let $f(b)=\{ a_i \mid b_i=1\}$

Claim: f is surjective

Let X be a subset of $\{a_1,\ldots,a_n\}$.

Define $b_k = 1$ if a_k in X and $b_k = 0$ otherwise.

Note that $f(b_1b_2\ldots b_n) = X$.

The number of subsets of an n-element set is 2^n.

Counting Doughnut Selection

There are 5 kinds of doughnuts.

How many ways to select a dozen doughnuts?

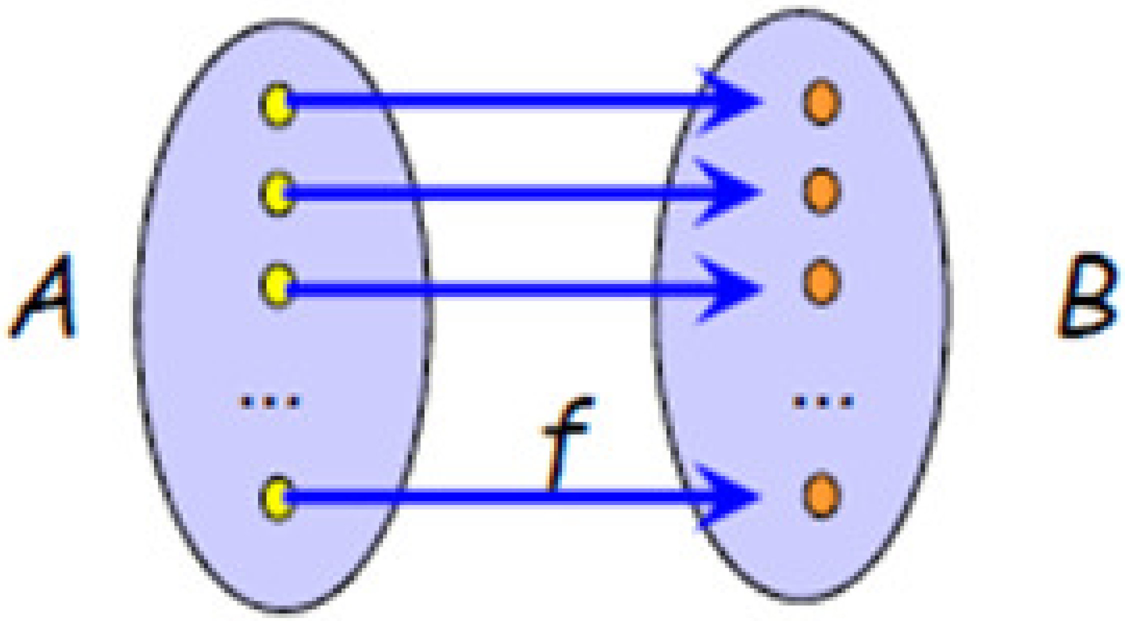
00	(none)	000000	00	00
Chocolate	Lemon	Sugar	Glazed	Plain

Counting Doughnut Selection

A := Set of all selections of a dozen doughnuts

B := Set of all 16-bit binary strings with exactly four 1's

Define a bijective function from A to B

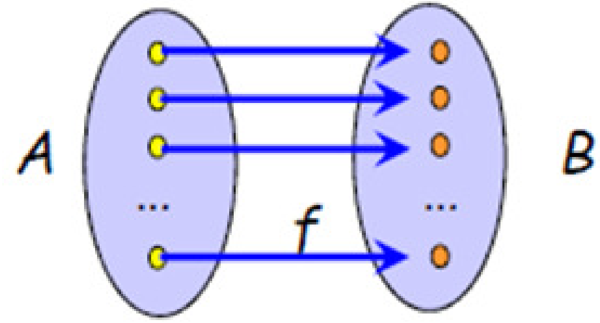


Counting Doughnut Selection

A := Set of all selections of a dozen doughnuts

B := Set of all 16-bit binary strings with exactly four 1's

Define a bijective function from A to B



0011000000100100

00	1	(none)	1	000000	1	00	1	00
Chocolate		Lemon		Sugar		Glazed		Plain

Each doughnut is represented by a 0,

Four 1's are used to separated five types of doughnuts.

Counting Doughnut Selection

Chocolate, Lemon, Sugar, Glazed, Plain

maps to a bijection

$0^C 1 0^L 1 0^S 1 0^G 1 0^P$

A := Set of all selections of a dozen doughnuts

B := Set of all 16-bit binary strings with exactly four 1's

Counting Doughnut Selection

B::= Set of all 16-bit binary strings with exactly four 1’s

16 choices for the first 1.
15 choices for the second 1.
14 choices for the third 1.
13 choices for the fourth 1.

Counting Doughnut Selection

B::= all 16-bit binary strings with exactly four 1's

16 choices for the first 1.
15 choices for the second 1.
14 choices for the third 1.
13 choices for the fourth 1.

By product rule:
There are 16x15x14x13 possible ways.

Counting Doughnut Selection

B::= all 16-bit binary strings with exactly four 1's

16 choices for the first 1.
15 choices for the second 1.
14 choices for the third 1.
13 choices for the fourth 1.

By product rule:
There are 16x15x14x13 possible ways.
But we overcount 4!.
Therefore, |B| = (16x15x14x13)/4!

Counting Doughnut Selection

$|B| = (16 \times 15 \times 14 \times 13)/4!$

$= 16!/(16-4)!4!$

$= \binom{16}{4}$

Because there is a bijection f from A to B:

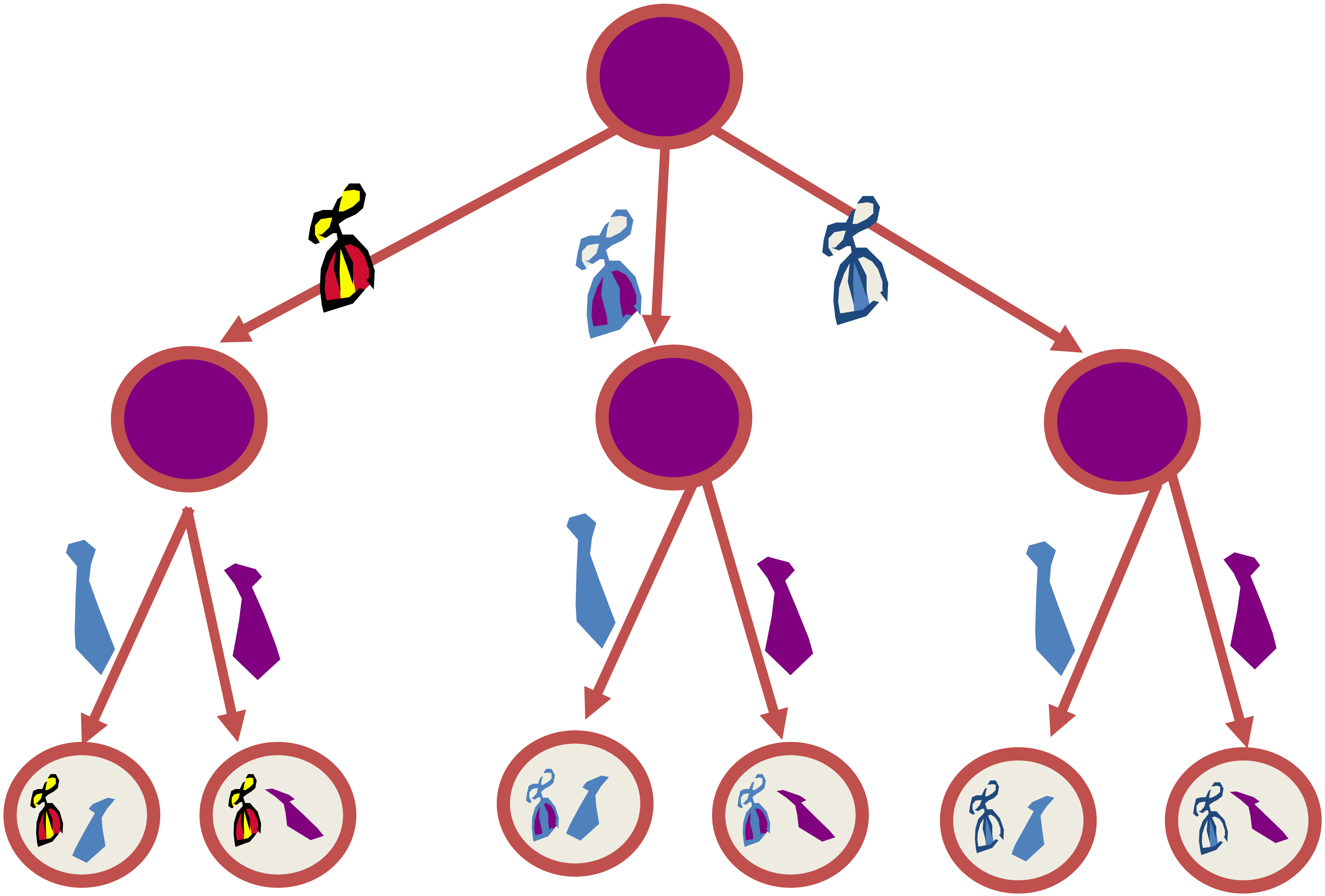
$|A| = |B| = \binom{16}{4}$

Product Rule

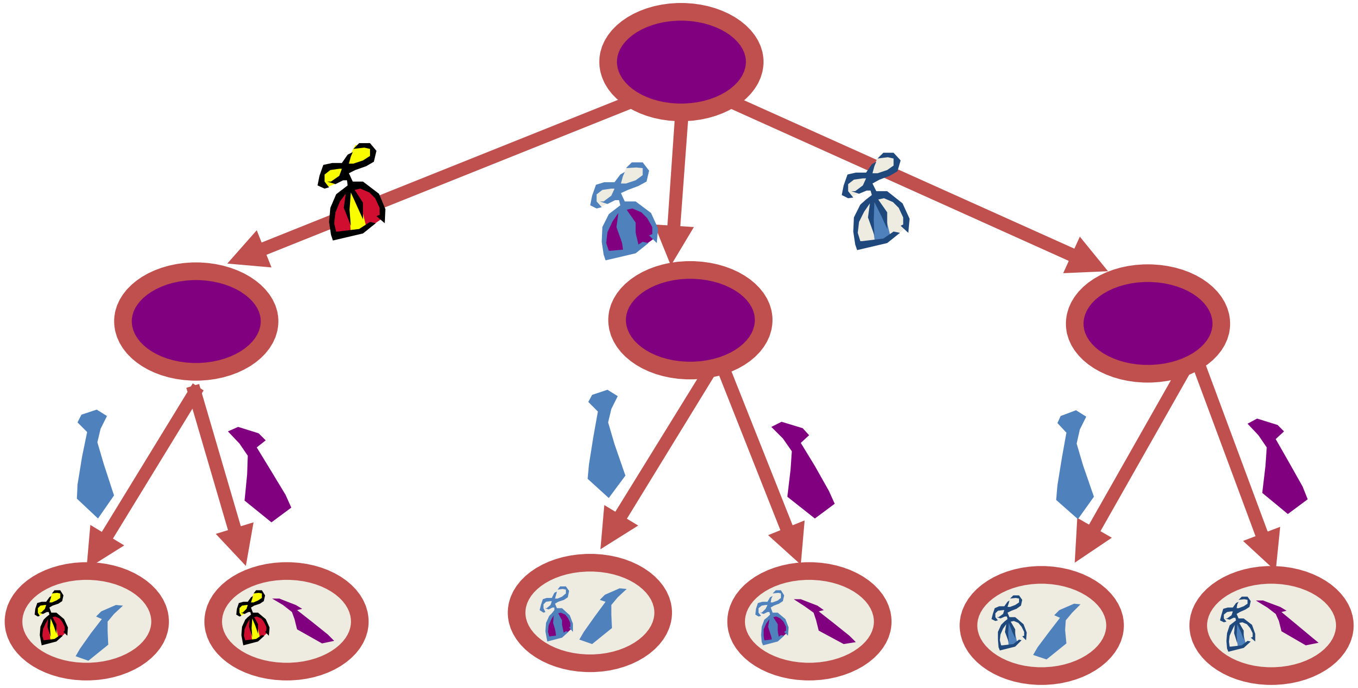
If P_1, P_2, …, P_n are sets, then:

$$|P_1 x P_2 x \ldots x P_n| = |P_1|.|P_{2},|\ldots|P_n|.$$

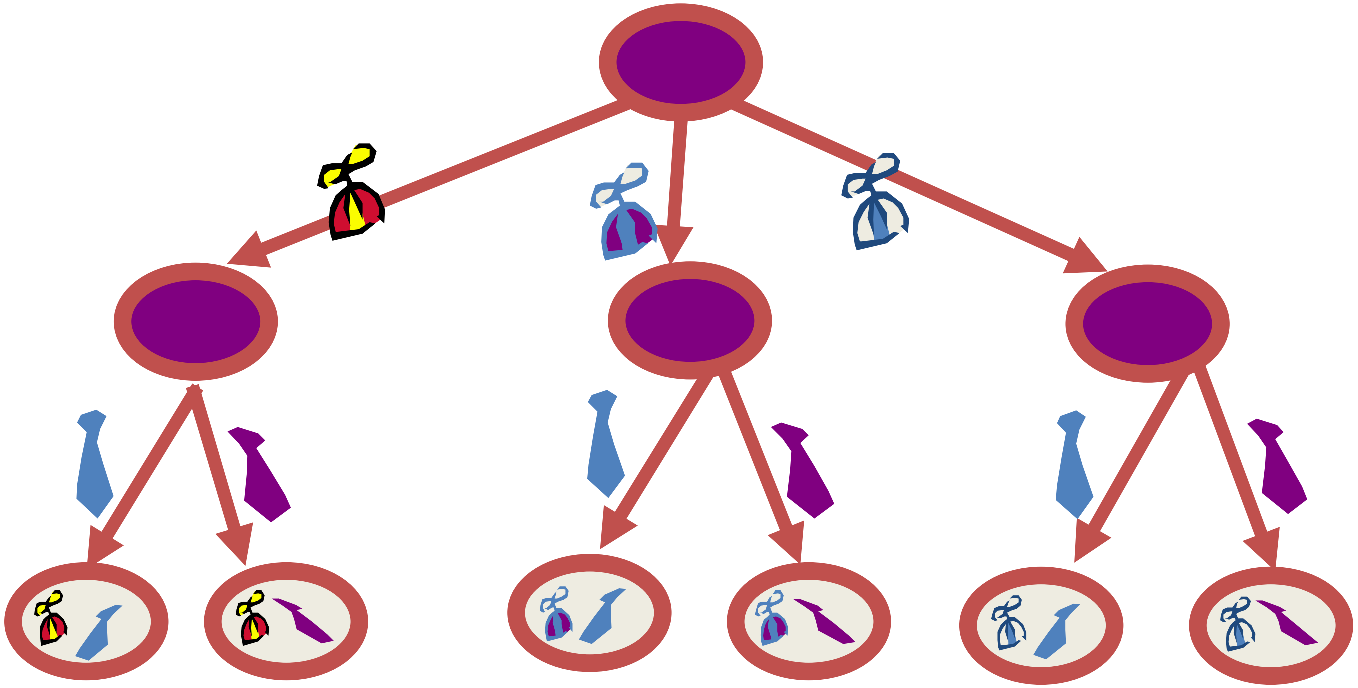
I own 3 beanies and 2 ties. How many different ways can I dress up in a beanie and a tie?



Choice Tree



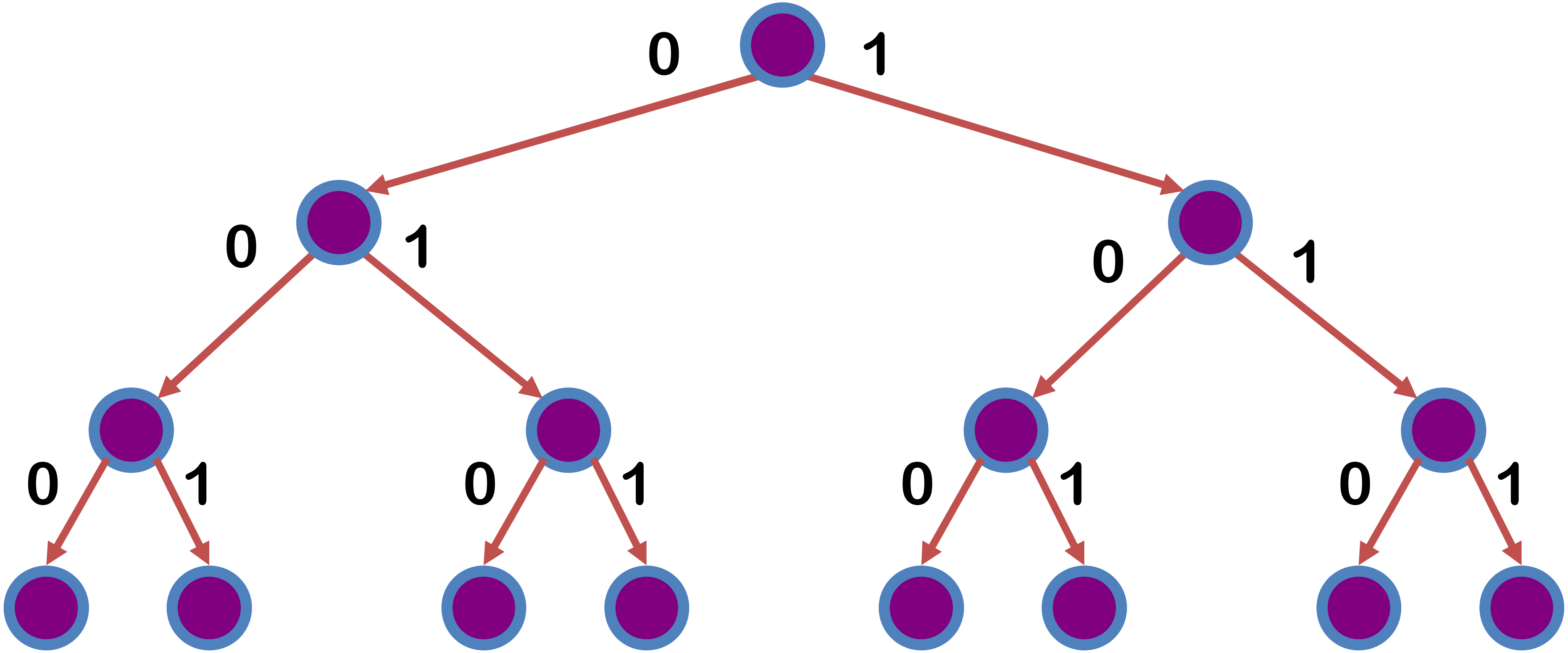
A choice tree is a rooted, directed tree with an object called a “choice” associated with each edge and a label on each leaf.



A choice tree provides a choice tree representation of a set S, if

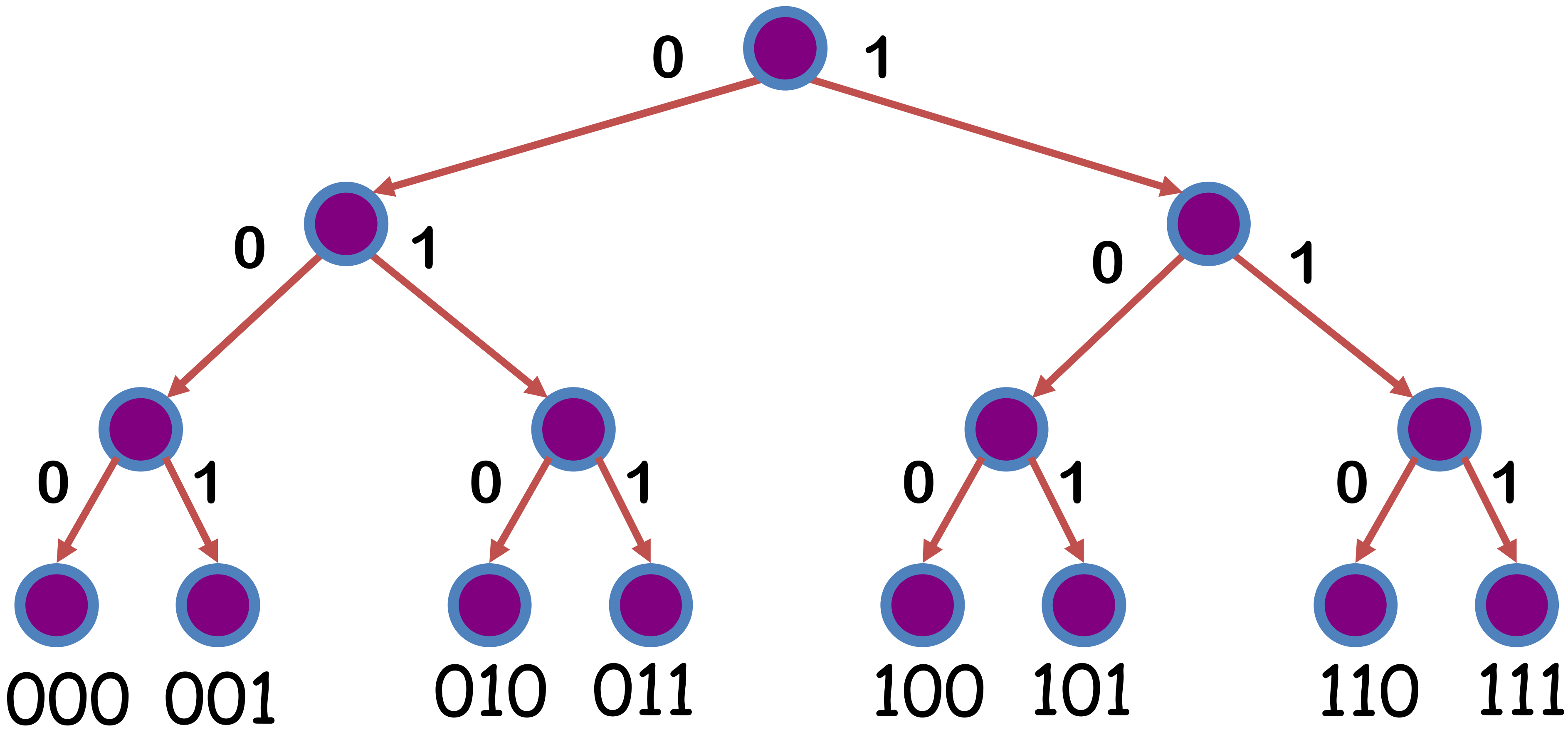
1) Each leaf label is in S
2) No two leaf labels are the same

Choice Tree
for 2^n n-bit sequences

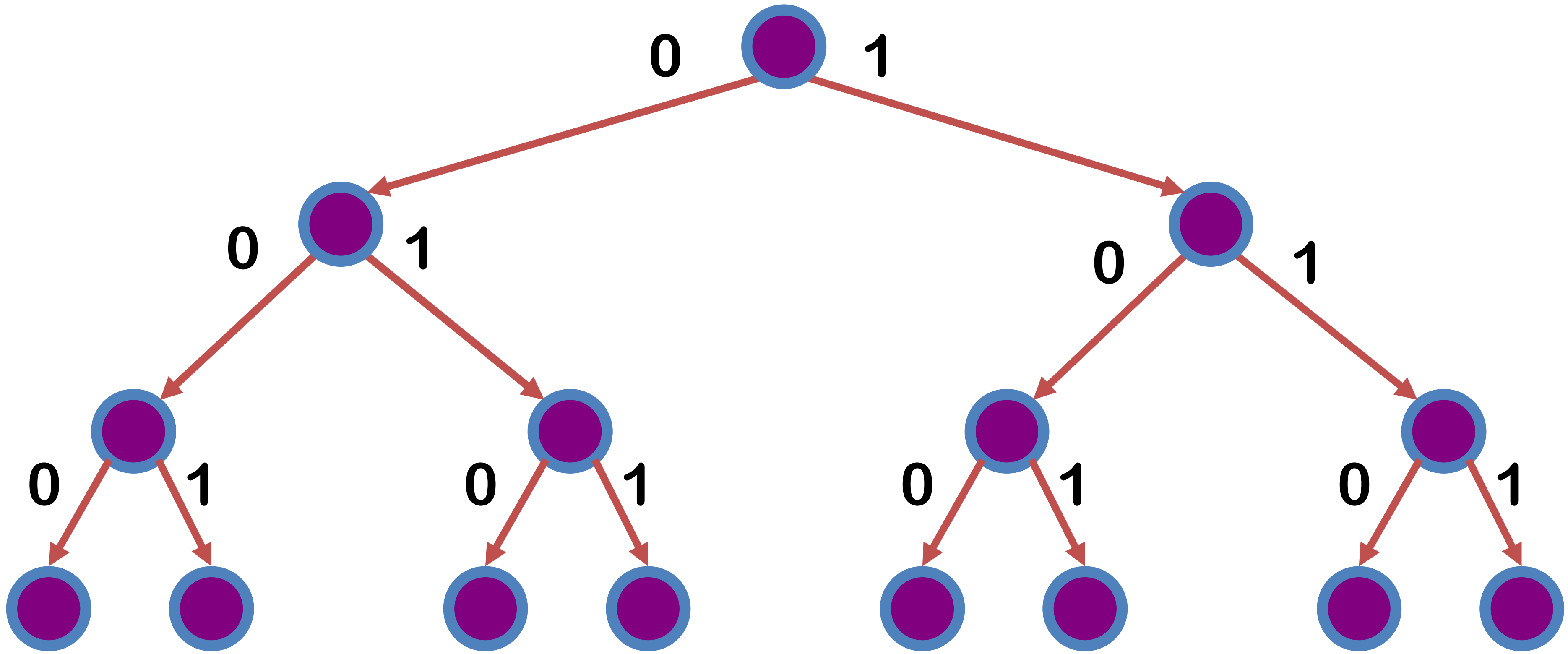


We can use a "choice tree" to represent the construction of objects of the desired type.

2^n n-bit sequences



Label each leaf with the object constructed by the choices along the path to the leaf.



2 choices for first bit

X 2 choices for second bit

X 2 choices for third bit

…

X 2 choices for the n^{th}

Product Rule

If S has a choice tree representation with
P_1 possibilities for the first choice,
P_2 for the second, and so on,

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects in S

Product Rule

Suppose that all objects of a type S can be constructed by a sequence of choices with P_1 possibilities for the first choice, P_2 for the second, and so on.

IF

1) Each sequence of choices constructs an object of type S

AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

Permutation

A permutation of a set S is a sequence that contains every element of S exactly once.

How many different orderings of deck with 52 cards?

How many different orderings of deck with 52 cards?

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;
51 possible choices for the second card;
50 possible choices for the third card;
…
1 possible choice for the 52^{nd} card.

How many different orderings of deck with 52 cards?

By the product rule:

52 * 51 * 50 * … * 3 * 2 * 1 = 52!

A permutation or arrangement of n objects is an ordering of the objects.

A permutation or arrangement of n objects is an ordering of the objects.

The number of permutations of n distinct objects is n!

How many sequences of 7 letters are there?

How many sequences of 7 letters are there?

$$26^7$$

How many sequences of 7 letters are there?

$$26^7$$

26 choices for each of the 7 positions

How many sequences of 7 letters contain **at least two** of the same letter?

How many sequences of 7 letters contain **at least two** of the same letter?

$$26^7 - 26*25*24*23*22*21*20$$

How many sequences of 7 letters contain **at least two** of the same letter?

$$26^7 - 26*25*24*23*22*21*20$$

26*25*24*23*22*21*20 is a number of sequences containing all different letters

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.

Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed?

Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed? 3*2 = 6

{1,2}, {1,3}, {2,1}, {2,3}, {3,1}, {3,2}

Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed? 3*2 = 6

{1,2}, {1,3}, {2,1}, {2,3}, {3,1}, {3,2}

How many unordered pairs? ({1,2} and {2,1} are the same)

Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed? 3*2 = 6

{1,2}, {1,3}, {2,1}, {2,3}, {3,1}, {3,2}

How many unordered pairs? ({1,2} and {2,1} are the same)

3*2 divide by overcount, we get 3

Ordered Versus Unordered

From a set of {1,2,3} how many ordered pairs can be formed? 3*2 = 6

{1,2}, {1,3}, {2,1}, {2,3}, {3,1}, {3,2}

How many unordered pairs? ({1,2} and {2,1} are the same)
3*2 divide by overcount, we get 3

Each unordered pair is listed twice on a list of the ordered pairs

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 * 51 = 52!/(52-2)! - permute 2 out of 52 objects

Number of ways of ordering, permuting, or arranging r out of n objects

n choices for the first place, n-1 for the second and so on

$$n \times (n-1) \times \ldots \times (n-(r-1)) = \frac{n!}{(n-r)!}$$

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 * 51 = 52!/(52-2)! - permute 2 out of 52 object

How many unordered 5 card hands?

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 * 51 = 52!/(52-2)! - permute 2 out of 52 object

How many unordered 5 card hands?

52*51*50*49*48 / 5! ← divide by overcount

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

52 * 51 = 52!/(52-2)! - permute 2 out of 52 object

How many unordered 5 card hands?

52*51*50*49*48 / 5! ← divide by overcount

52!/(52-5)!5! = $\binom{52}{5}$

Combination (Counting Subsets)

Subset Rule: The number of r-element subsets of an n-element set is

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

A <u>combination</u> or <u>choice</u> of r out of n objects is an (unordered) set of r of the n objects.

The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n choose r

Permutations vs. Combinations

$$\frac{n!}{(n-r)!}$$

Ordered

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Unordered

How many 8 bit sequences have two 0's and six 1's?

1) Choose the set of 2 positions to put the 0's. The 1's are forced.

$$\binom{8}{2}$$

How many 8 bit sequences have two 0's and six 1's?

1) Choose the set of 2 positions to put the 0’s. The 1’s are forced.

$$\binom{8}{2}$$

2) Choose the set of 6 positions to put the 1’s. The 0’s are forced.

$$\binom{8}{6}$$

How many 8 bit sequences have two 0’s and six 1’s?

Tempting, but incorrect:

8 ways to place first 0 times

7 ways to place second 0

How many 8 bit sequences have two 0's and six 1's?

Tempting, but incorrect:

8 ways to place first 0 times

7 ways to place second 0

Violates condition 2 of product rule!
Choosing position i for the first 0 and then position j for the second 0 gives the same sequence as choosing position j for the first 0 and position i for the second.

How Many 5-Hands Have at Least 3 As?

First counting…

How Many 5-Hands Have at Least 3 As?

First counting…

4 ways of picking 3 out of 4 aces

How Many 5-Hands Have at Least 3 As?

First counting…

4 ways of picking 3 out of 4 aces $\binom{4}{3}$

How Many 5-Hands Have at Least 3 As?

First counting…

4 ways of picking 3 out of 4 aces $\binom{4}{3}$

1176 ways of picking 2 cards out of the remaining 52-3=49 cards

How Many 5-Hands Have at Least 3 As?

First counting…

4 ways of picking 3 out of 4 aces $\binom{4}{3}$

1176 ways of picking 2 cards out of $\binom{49}{2}$
the remaining 52-3=49 cards

How Many 5-Hands Have at Least 3 As?

First counting…

4 ways of picking 3 out of 4 aces $\binom{4}{3}$

1176 ways of picking 2 cards out of the remaining 52-3=49 cards $\binom{49}{2}$

4 x 1176 = 4704

How Many 5-Hands Have at Least 3 As?

Second counting…

How Many 5-Hands Have at Least 3 As?
Second counting…

How many hands have exactly 3 aces?

$$\binom{4}{3}\binom{48}{2} = 4 \times 1128 = 4512$$

How Many 5-Hands Have at Least 3 As?

Second counting…

How many hands have exactly 3 aces?

$$\binom{4}{3}\binom{48}{2} = 4 \text{ x } 1128=4512$$

How many hands have exactly 4 aces? $\binom{4}{4}\binom{48}{1} = 48$

How Many 5-Hands Have at Least 3 As?
Second counting…

How many hands have exactly 3 aces?

$$\binom{4}{3}\binom{48}{2} = 4 \times 1128 = 4512$$

How many hands have exactly 4 aces? $\binom{4}{4}\binom{48}{1} = 48$

Thus, $4512 + 48 = 4560 \neq 4704$

$$4704 \neq 4560$$

One of the two counting arguments is not correct.

Review the first counting argument

1. **Choose 3 of 4 aces**

2. **Choose 2 of the remaining 49 cards**

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Four different sequences of choices produce the same hand

Review the first counting argument

A♣ A♦ A♥	A♠ K♦
A♣ A♦ A♠	A♥ K♦
A♣ A♠ A♥	A♦ K♦
A♠ A♦ A♥	A♣ K♦

Four different sequences of choices produce the same hand

Therefore, this overcounts 3 x 48.

The solution is 4704 – 3 x 48 = 4560.

Is the other argument correct? How do I avoid fallacious reasoning?

The three big mistakes people make in counting are:

1. Creating objects not in S

2. Missing out some objects from the set S

3. Creating the same object many different ways

Addition Rule
(Sum Rule)
holds only for a union of disjoint sets

Addition Rule

Let A and B be two <u>disjoint</u> finite sets.

The size of $A \cup B$ is the sum of
the size of A and the size of B.

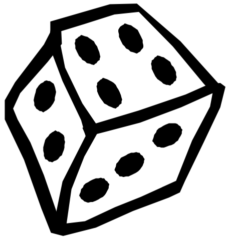
$$|A \cup B| = |A| + |B|$$

Addition Rule

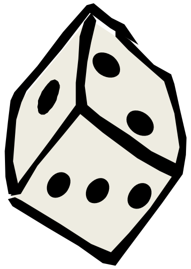
Let A_1, A_2, A_3, …, A_n be disjoint, finite sets.

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} |A_k|$$

Example



Suppose I roll a
white die and a black die.



What is the number of
outcomes where the dice
show different values?

Example

Let S be a set of all outcomes where the dice show different values.

Let T be a set of outcomes where the two dice agree.

$|S| = ?$

Example

$|S \cup T| = \quad ?$

Example

$|S \cup T| = 36$

Example

$|S \cup T| = 36$

$|S|+|T| = 36$

Example

$|S \cup T| = 36$

$|S|+|T| = 36$

$|T| = 6$

Example

$|S \cup T| = 36$

$|S|+|T| = 36$

$|T| = 6$

$|S| = 30$

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

$A_i \equiv$ set of outcomes where the black die says i and the white die says something larger.

$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$
$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die. $|S| = ?$

Another way of counting…

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die.

$|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die.

$|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$|S| + |L| = 30$

Example

$S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die.

$|S| = ?$

$L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

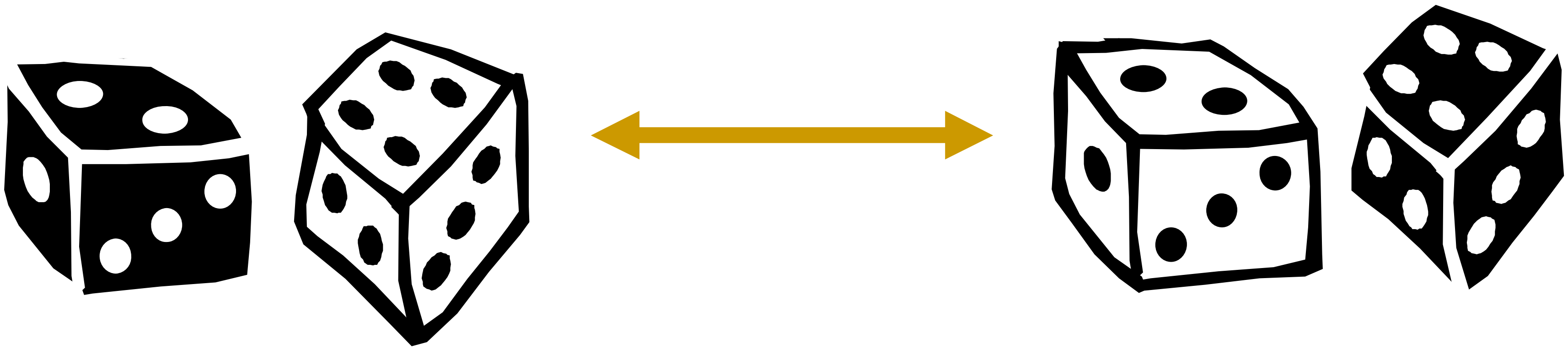
$|S| + |L| = 30$

It is clear by symmetry that $|S| = |L|$.

Therefore, $|S| = 15$

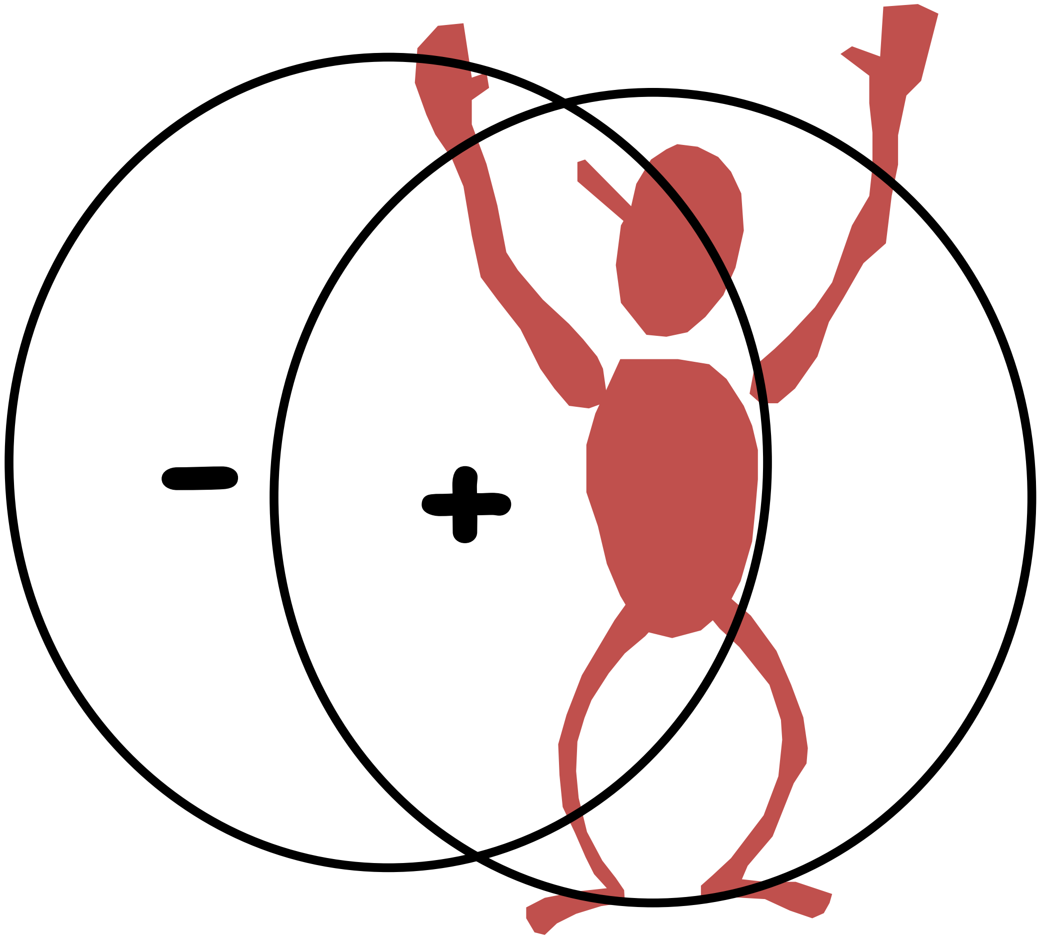
It is <u>clear</u> by symmetry that $|S| = |L|$.

Put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.

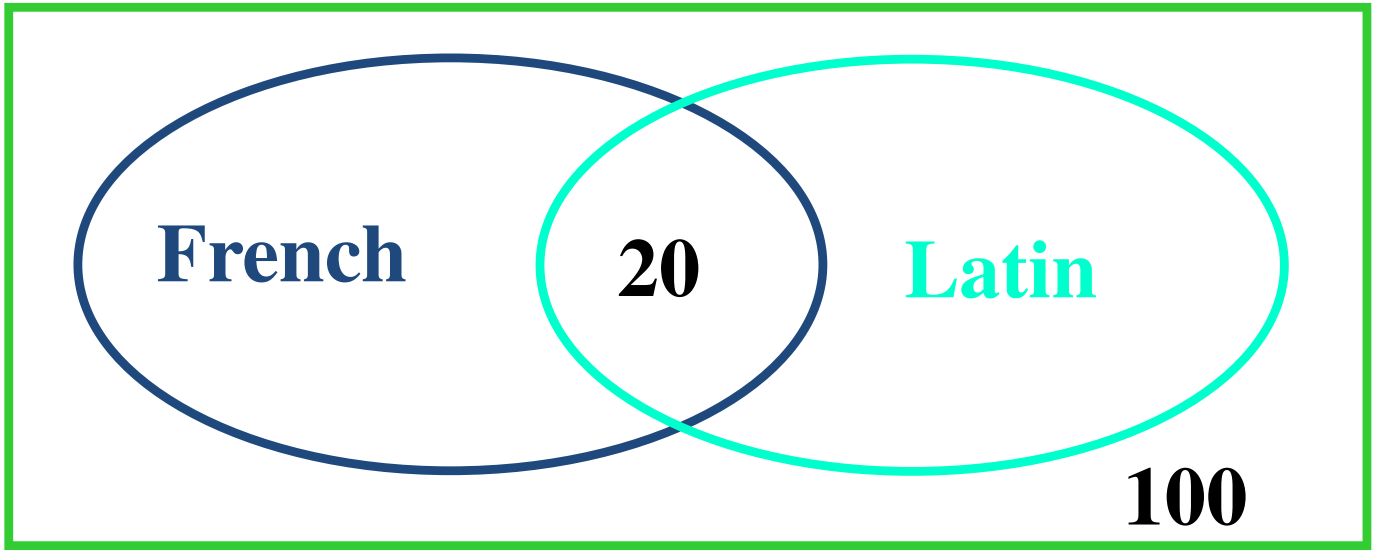


The Principle of Inclusion and Exclusion

To Exclude Or Not To Exclude?



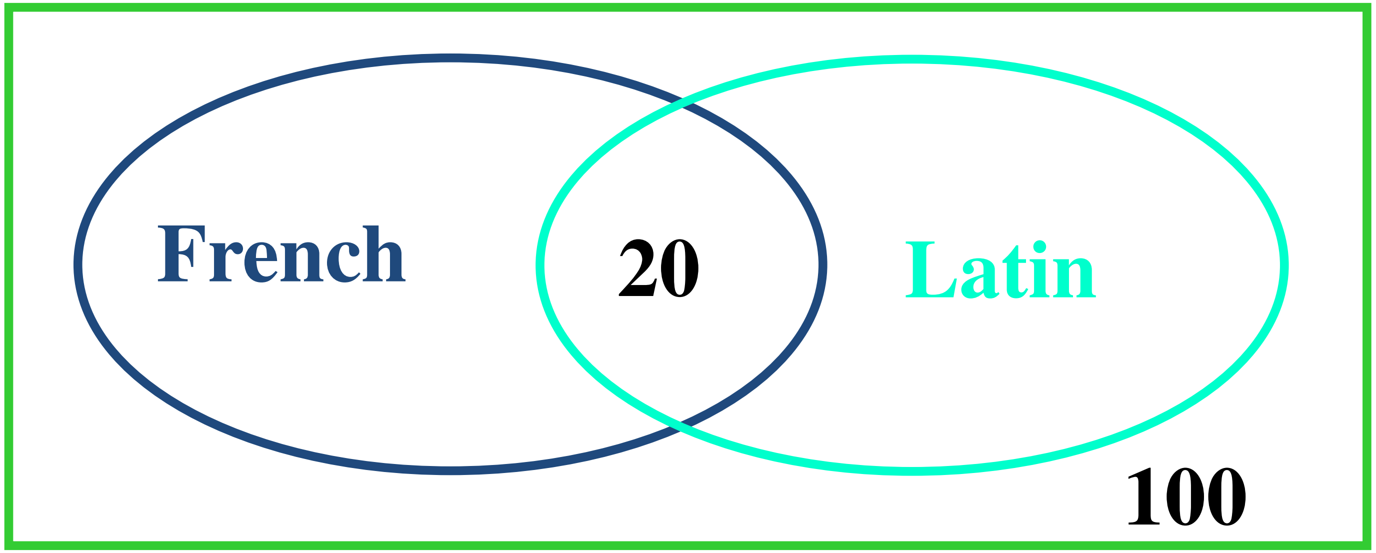
A school has 100 students. 50 take French, 40 take Latin, and 20 take both. How many students take neither language?



$F \equiv$ French students
$L \equiv$ Latin students

$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

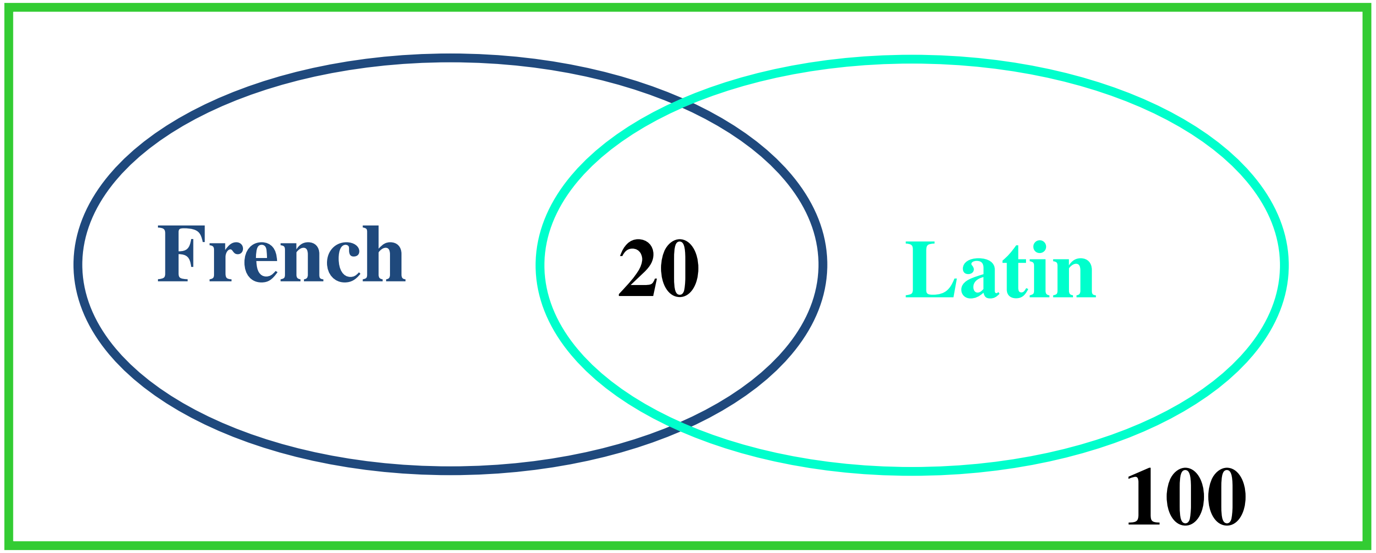


F ≡ French students
L ≡ Latin students

$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

French OR Latin students:

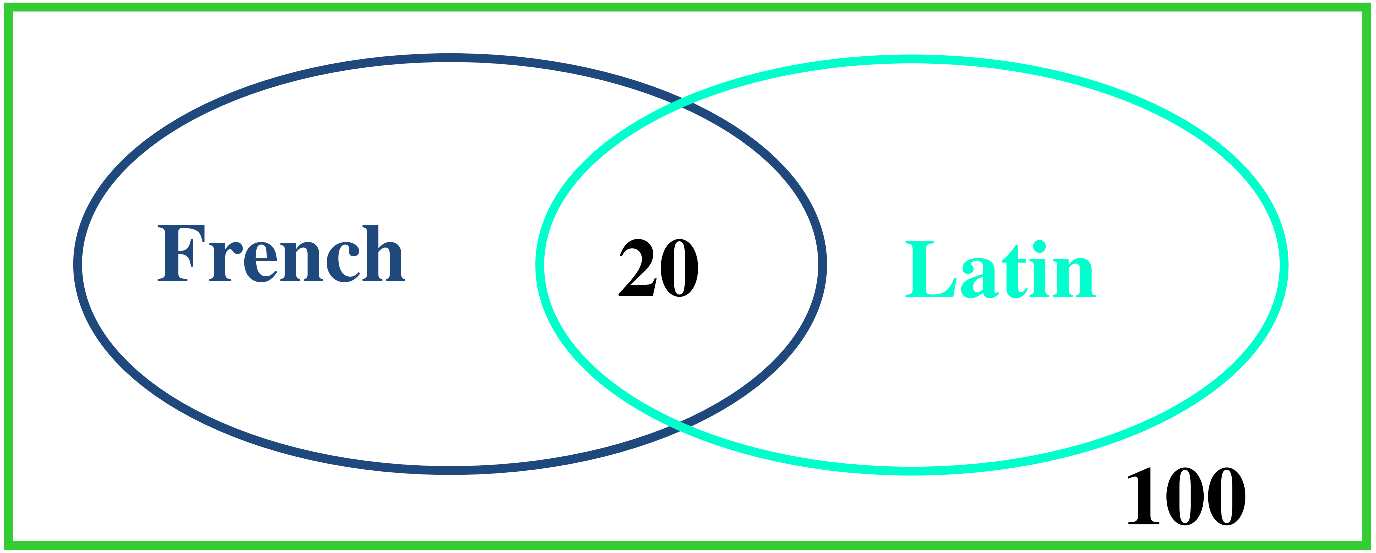


$F \equiv$ French students
$L \equiv$ Latin students

$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

French OR Latin students:
$|F \cup L|$ =



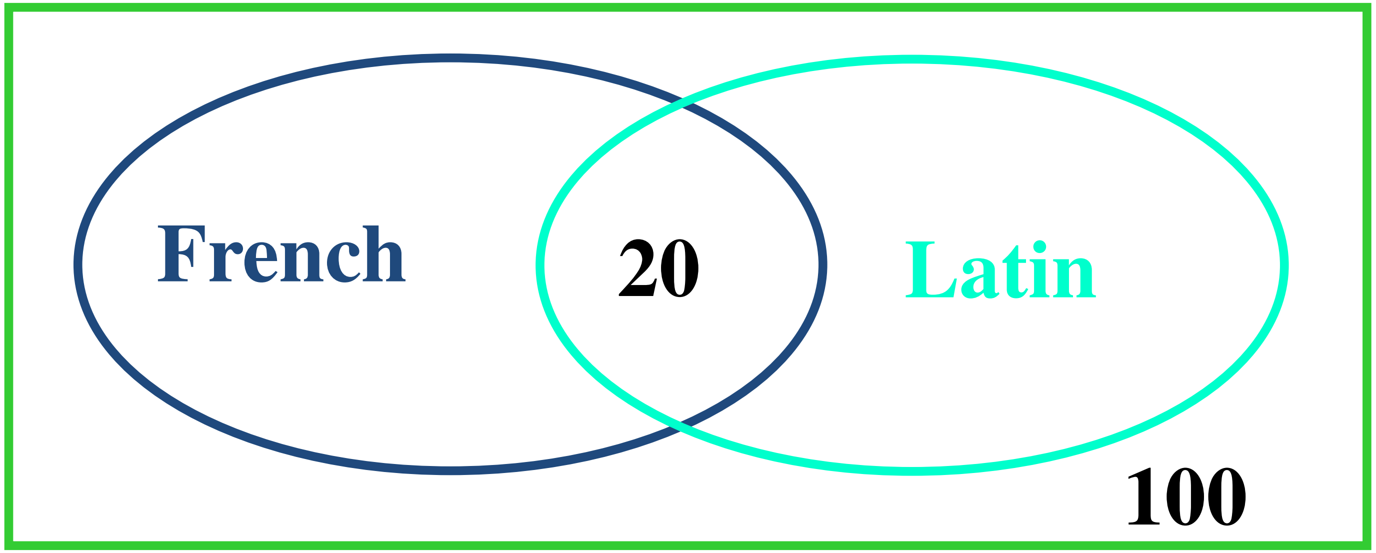
$F \equiv$ French students
$L \equiv$ Latin students

$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

French OR Latin students:

$$|F \cup L| = |F|+|L|-|F \cap L|$$
$$= 50 + 40 - 20 = 70$$



$F \equiv$ French students
$L \equiv$ Latin students

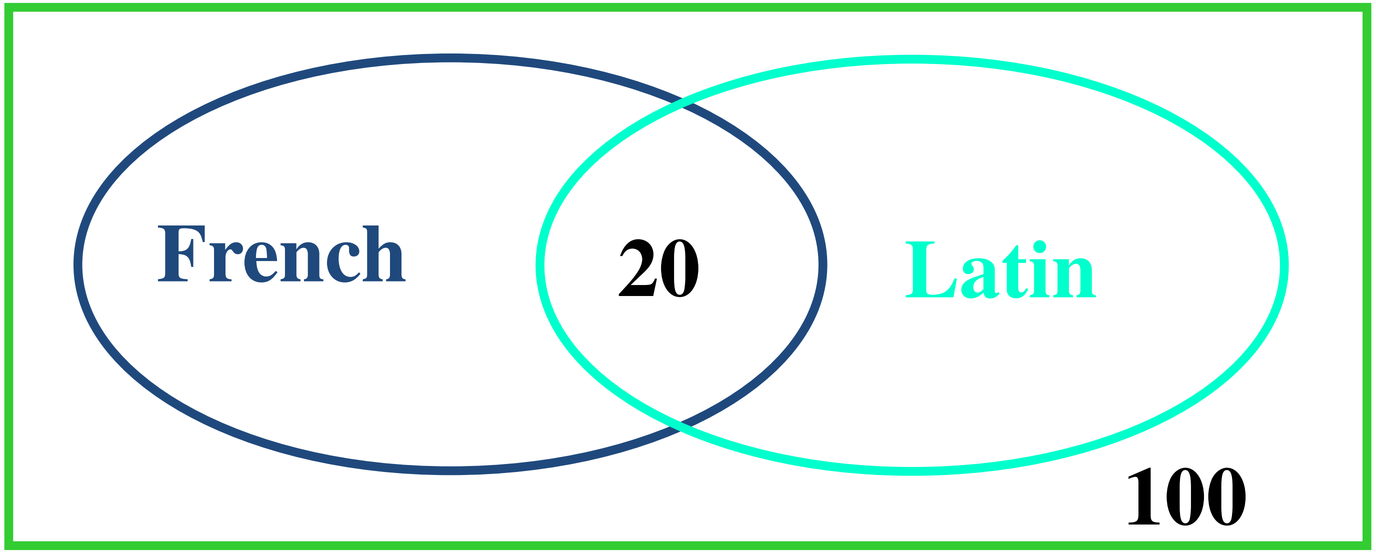
$|F|=50$ $|L|=40$

French AND Latin students: $|F \cap L|=20$

French OR Latin students:

$$|F \cup L| = |F|+|L|-|F \cap L|$$
$$= 50 + 40 - 20 = 70$$

Neither language:



F ≡ French students
L ≡ Latin students

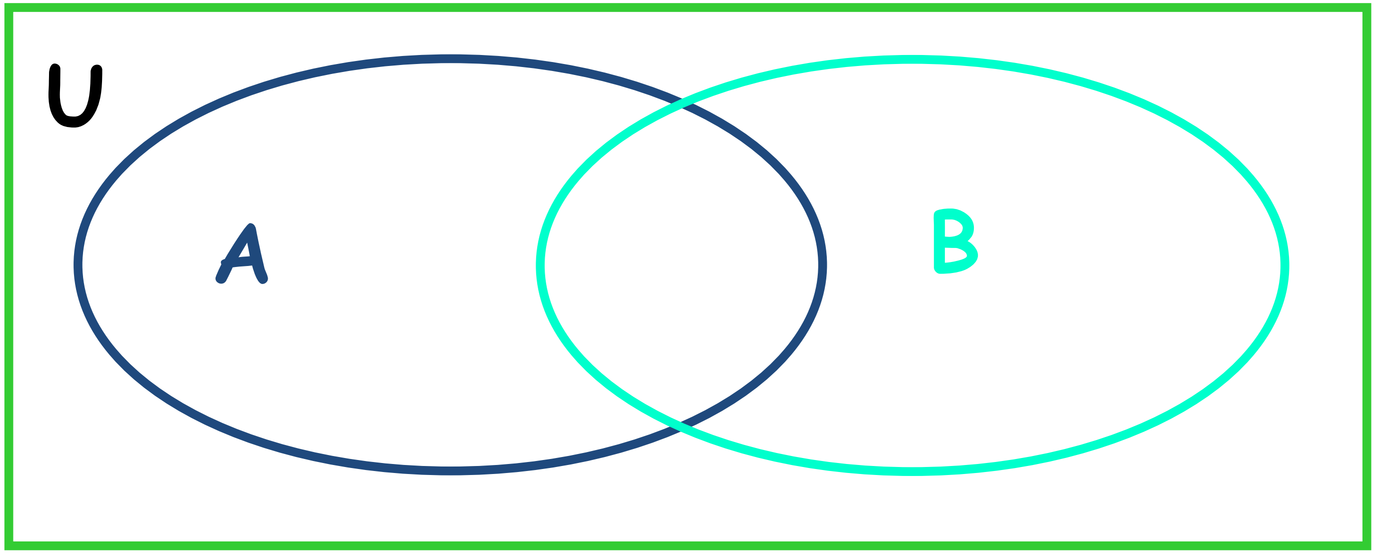
|F|=50 |L|=40

French AND Latin students: $|F \cap L| = 20$

French OR Latin students:

$$|F \cup L| = |F| + |L| - |F \cap L|$$
$$= 50 + 40 - 20 = 70$$

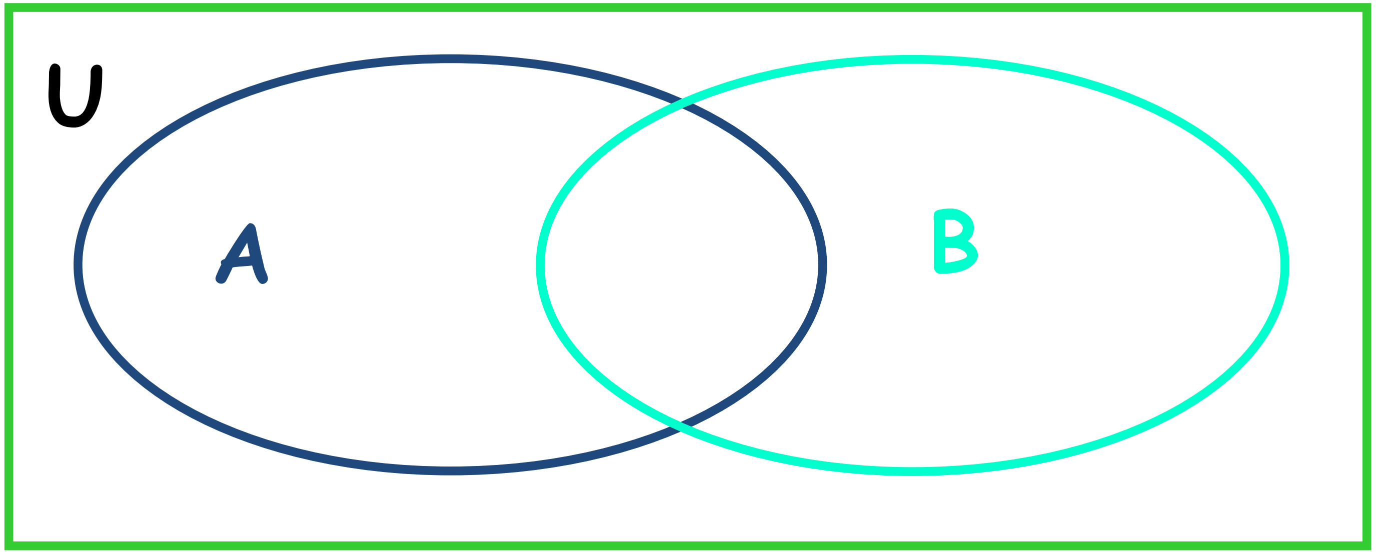
Neither language:
100 – 70 = 30



Lesson: The Principle of Inclusion and Exclusion

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

The number of elements in at least one of sets A or B



$U \equiv$ **universe of elements**

$$\bar{A} = U - A$$

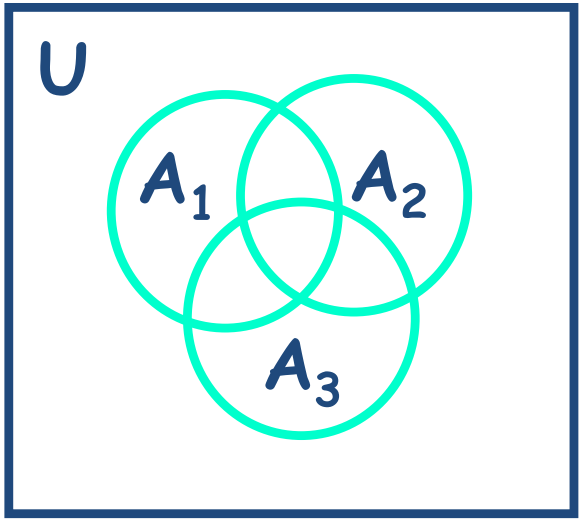
$$\bar{B} = U - B$$

Lesson: The Principle of Inclusion and Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

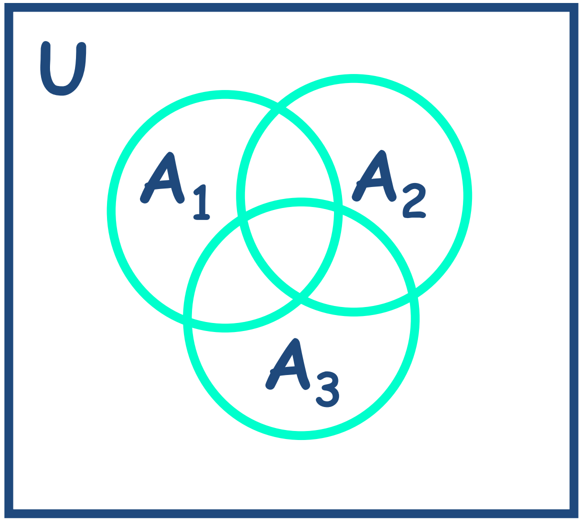
$$\left|\bar{A} \cap \bar{B}\right| = |U| - (|A| + |B|) + |A \cap B|$$

How many positive integers at most 70 are multiple of 2 or 5 or 7?



U = [1..70]

How many positive integers at most 70 are multiple of 2 or 5 or 7?

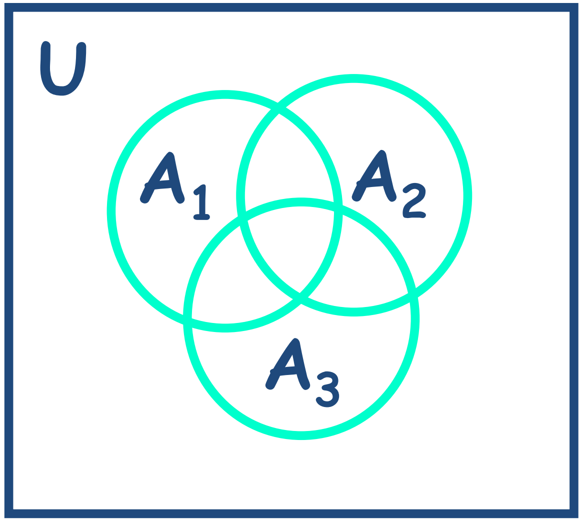


$U = [1..70]$

$A_1 \equiv$ integers in U divisible by 2
$A_2 \equiv$ integers in U divisible by 5
$A_3 \equiv$ integers in U divisible by 7

$|A_1| = 35$ $|A_2| = 14$ $|A_3| = 10$

How many positive integers at most 70 are multiple of 2 or 5 or 7?



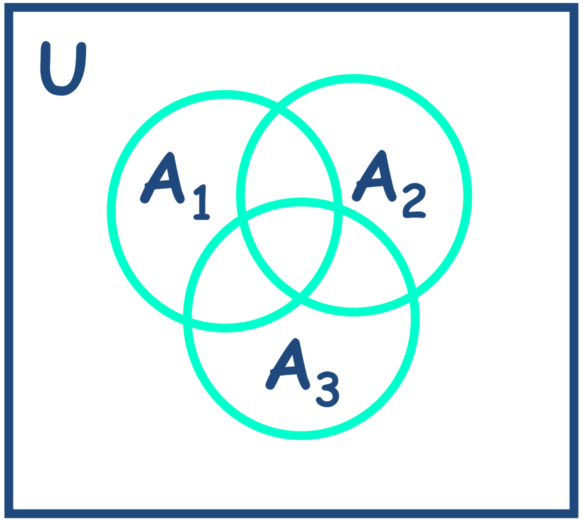
$U = [1..70]$

$A_1 \equiv$ integers in U divisible by 2
$A_2 \equiv$ integers in U divisible by 5
$A_3 \equiv$ integers in U divisible by 7

$|A_1| = 35 \quad |A_2| = 14 \quad |A_3| = 10$

$|A_1 \cap A_2| = 7$, $|A_1 \cap A_3| = 5$, $|A_2 \cap A_3| = 2$, $|A_1 \cap A_2 \cap A_3| = 1$

How many positive integers at most 70 are multiple of 2 or 5 or 7?



$U = [1..70]$

$A_1 \equiv$ integers in U divisible by 2
$A_2 \equiv$ integers in U divisible by 5
$A_3 \equiv$ integers in U divisible by 7

$|A_1| = 35 \quad |A_2| = 14 \quad |A_3| = 10$

$|A_1 \cap A_2| = 7,\ |A_1 \cap A_3| = 5,\ |A_2 \cap A_3| = 2,\ |A_1 \cap A_2 \cap A_3| = 1$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

How many positive integers at most 70 are multiple of 2 or 5 or 7?

$$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

How many positive integers at most 70 are multiple of 2 or 5 or 7?

$$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3| -$$
$$|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3| = 35+14+10-7-5-2+1 = 46$$

How many positive integers at most 70 are multiple of 2 or 5 or 7?

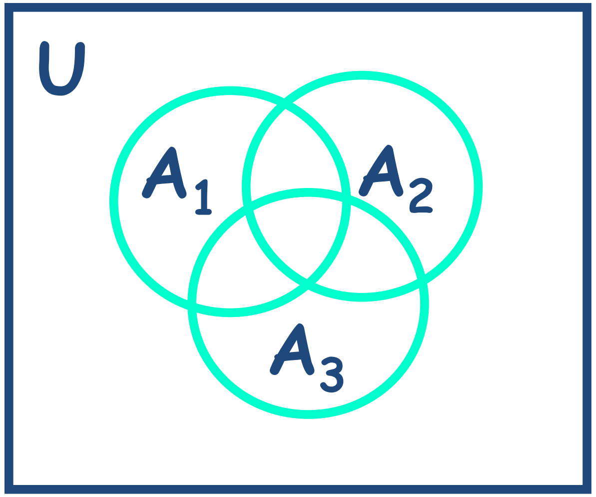
$|A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3| -$
$|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$|A_1 \cup A_2 \cup A_3| = 35+14+10-7-5-2+1 = 46$

Thus, the number of multiple of 2 or 5 or 7 is 70 – 46 = 24.

The Principle of Inclusion and Exclusion

Let S_k be the sum of the sizes of
All k-tuple intersections of the A_i's.



$S_1 = |A_1| + |A_2| + |A_3|$
$S_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$
$S_3 = |A_1 \cap A_2 \cap A_3|$

$|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3$

The General Principle of Inclusion and Exclusion

If $(A_i)_{1 \leq i \leq n}$ **is a finite set, then**

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$