Final - Extra Problems

1. $y = x^2$

$y' = 2x$

$4y - 2xy' = 4x^2 - 2x(2x) = 4x^2 - 4x^2 = 0$

2. $y = x^3$

$y' = 3x^2$

$3y - xy' = 3x^3 - x(3x^2) = 3x^3 - 3x^3 = 0$

3. $y = 3x^2 + 1$

$\frac{dy}{dx} = 6x$

$dy = 6x\,dx$

$2y\,dx - x\,dy = 2(3x^2+1)\,dx - x(6x\,dx)$

$= 6x^2\,dx + 2\,dx - 6x^2\,dx = 2\,dx$

4. $y = 4x^3 + 2$

$\frac{dy}{dx} = 12x^2$

$dy = 12x^2\,dx$

$3y\,dx - x\,dy = 3(4x^3+2)\,dx - x(12x^2\,dx)$

$= 12x^3\,dx + 6\,dx - 12x^3\,dx$

$= 6\,dx$

5. $dy = xe^{x^2+1}\,dx$

$\int dy = \int xe^{x^2+1}\,dx$

$y = \frac{1}{2}e^{x^2+1} + C$

let $u = x^2 + 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int xe^{x^2+1}\,dx = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u$

$= \frac{1}{2}e^{x^2+1}$

6. $dy = x^2e^{x^3-1}\,dx$

$y = \frac{1}{3}e^{x^3-1} + C$

let $u = x^3 - 1$

$du = 3x^2\,dx$

$x^2\,dx = \frac{1}{3}du$

$\int x^2e^{x^3-1}\,dx = \frac{1}{3}\int e^u\,du$

$= \frac{1}{3}e^u$

$= \frac{1}{3}e^{x^3-1}$

7. $2y\,dy = 4x\,dx$

$\int 2y\,dy = \int 4x\,dx$

$y^2 = 2x^2 + C$

8. $4y\,dy = 4x^3\,dx$

$2y^2 = x^4 + C$

9. $3y^2dy = (2x - 1)\,dx$

$y^3 = x^2 - x + C$

10. $4y^3\,dy = (3x^2 + 2x)\,dx$

$y^4 = x^3 + x^2 + C$

11. $y' = e^{x-3}$, $y(0) = 2$

$\frac{dy}{dx} = e^{x-3}$

$dy = e^{x-3}\,dx$

$\int 1\,dy = \int e^{x-3}\,dx$

$y = e^{x-3} + C$

$y(0) = 2$

$e^{-3} + C = 2$

$C = 2 - e^{-3}$

$y = e^{x-3} + 2 - e^{-3}$

Let $u = e^{x-3}$

$du = e^{x-3}(1-0)$

$= e^{x-3}\,dx$

12. $y' = e^{2x+1}$, $y(0) = e$

$\frac{dy}{dx} = e^{2x+1}$

$dy = e^{2x+1}\,dx$

$\int 1\,dy = \int e^{2x+1}\,dx$

$y = \frac{1}{2}e^{2x+1} + C$

$y(0) = e$

$\frac{1}{2}e + C = e$

$C = \frac{1}{2}e$

$\therefore y = \frac{1}{2}e^{2x+1} + \frac{1}{2}e = \frac{1}{2}e(e^{2x}+1)$

13. $dy = \left(\frac{1}{x} - x\right)dx \quad , y(1) = 0$

$$\int 1\,dy = \int \left(\frac{1}{x} - x\right)dx$$

$$y = \ln|x| - \frac{x^2}{2} + C$$

$$y(1) = 0$$

$$\ln 1 - \frac{1}{2} + C = 0$$

$$C = \frac{1}{2}$$

$$\therefore y = \ln|x| - \frac{x^2}{2} + \frac{1}{2}$$

14. $dy = \left(x^2 - \frac{1}{x+1}\right)dx \quad , y(0) = \frac{1}{3}$

$$\int 1\,dy = \int \left(x^2 - \frac{1}{x+1}\right)dx$$

$$y = \frac{x^3}{3} - \ln|x+1| + C$$

$$y(0) = \frac{1}{3}$$

$$0 - 0 + C = \frac{1}{3}$$

$$C = \frac{1}{3}$$

$$\therefore y = \frac{x^3}{3} - \ln|x+1| + \frac{1}{3}$$

15. $\frac{dy}{dx} = \frac{x^2}{y}$

$y\,dy = x^2\,dx$

$\int y\,dy = \int x^2\,dx$

$\frac{y^2}{2} = \frac{x^3}{3} + C$

$y^2 = \frac{2}{3}x^3 + 2C$

16. $y^3\,dx = \frac{dy}{x^3}$

$x^3\,dx = y^{-3}\,dy$

$\int x^3\,dx = \int y^{-3}\,dy$

$\frac{x^4}{4} + C = \frac{-y^{-2}}{2}$

$y^{-2} = -\frac{x^4}{2} - 2C$

17. $dx = x^3 y\,dy$

$x^{-3}\,dx = y\,dy$

$\int y\,dy = \int x^{-3}\,dx$

$\frac{y^2}{2} = -\frac{x^{-2}}{2} + C$

$y^2 = -x^{-2} + 2C$

18. $dy = x^2 y^3\,dx$

$y^{-3}\,dy = x^2\,dx$

$\int y^{-3}\,dy = \int x^2\,dx$

$-\frac{y^{-2}}{2} = \frac{x^3}{3} + C$

$y^{-2} = -\frac{2}{3}x^3 - 2C$

19. $dx = (x^2y^2 + x^2)\,dy$

$\rightarrow x^2(y^2+1)\,dy$

$\frac{1}{x^2}\,dx = (y^2+1)\,dy$

$\int x^{-2}\,dx = \int (y^2+1)\,dy$

$-x^{-1} + C = \frac{y^3}{3} + y$

$y^3 + 3y = -3x^{-1} + 3C$

20. $dy = (xe^2y^3 + xy^3)\,dx$

$= y^3(x^2+x)\,dx$

$y^{-3}\,dy = (x^2+x)\,dx$

$\int y^{-3}\,dy = \int (x^2+x)\,dx$

$-\frac{y^{-2}}{2} = \frac{x^3}{3} + \frac{x^2}{2} + C$

$y^{-2} = -\frac{2}{3}x^3 - x^2 - 2C$

22. $y\,dx = x\,dy$

$\frac{1}{x}dx = \frac{1}{y}dy$

$\ln|x| + c = \ln|y|$

$|y| = e^{\ln|x| + c}$

$|y| = e^{\ln|x|} \cdot e^{c}$

$|y| = |x| \cdot e^{c}$

23. $\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dx$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c$

$y^2 = x^2 + 2c$

24. $\frac{dy}{dx} = \frac{x^2 + x}{y + 1}$

$\int (y+1)\,dy = \int (x^2 + x)\,dx$

$\frac{y^2}{2} = \frac{x^3}{3} + \frac{x^2}{2} + c$

$y^2 = \frac{2}{3}x^3 + x^2 + 2c$

26. $x^2 y \frac{dy}{dx} = y^2 + 1$

$\frac{y}{y^2+1}\,dy = x^{-2}\,dx$

$\int \frac{y}{y^2+1}\,dy = \int x^{-2}\,dx$

$\frac{1}{2}\ln|y^2+1| = -x^{-1} + c$

$\ln|y^2+1| = -2x^{-1} + 2c$

Let $u = y^2 + 1$

$du = 2y\,dy$

$y\,dy = \frac{1}{2}du$

$\int \frac{y}{y^2+1}dy = \int \frac{1}{2}\frac{1}{u}du$

$= \frac{1}{2}\ln|u|$

27. skip

28. $e^{2x}(y+1)dx + e^{2x}y\,dy = 0$

$e^{2x}(y+1)dx = -e^{2x}y\,dy$

$-e^{2x}dx = \frac{y}{y+1}dy$

$\int -e^{2x}dx = \int \frac{y}{y+1}dy$

$-\frac{e^{2x}}{2} + c = y + 1 - \ln|y+1|$

$\therefore y + 1 - \ln|y+1| = -\frac{e^{2x}}{2} + c$

Let $u = y+1$, $y = u-1$

$du = dy$

$\int \frac{y}{y+1}dy = \int \frac{u-1}{u}du$

$= \int 1 - \frac{1}{u}\,du$

$= u - \ln|u|$

29. $\frac{dy}{dx} = \frac{x^2}{y^3}$, when $x = 1$, $y = 1$

$x^2\,dx = y^3\,dy$

$\int x^2\,dx = \int y^3\,dy$

$\frac{x^3}{3} + c = \frac{y^4}{4}$

$y^4 = \frac{4}{3}x^3 + 4c$

$y(1) = 1$

$1 = \frac{4}{3} + 4c$

$3 = 4 + 12c$

$c = -\frac{1}{12}$

$\therefore y^4 = \frac{4}{3}x^3 - \frac{1}{3}$

30. $\frac{dy}{dx} = \frac{x+1}{xy}$

$y\,dy = \left(1 + \frac{1}{x}\right)dx$

$\frac{y^2}{2} = x + \ln|x| + C$

$y^2 = 2x + 2\ln|x| + 2C$

$y(1) = 3$

$9 = 2 + 0 + 2C$

$2C = 7$

$C = \frac{7}{2}$

$\therefore y^2 = 2x + 2\ln|x| + 7$

31. $2y^2dx = 3x^2dy$

$2x^{-2}dx = 3y^{-2}dy$

$\int 2x^{-2}dx = \int 3y^{-2}dy$

$-2x^{-1} + C = -3y^{-1}$

$\frac{-3}{y} = -\frac{2}{x} + C$

$\frac{3}{y} = \frac{2}{x} - C$

$y(2) = -1$

$-3 = 1 - C$

$C = 4$

$\therefore \frac{3}{y} = \frac{2}{x} - 4$

32. $(x+1)\,dy = y^2\,dx$

$$y^{-2}\,dy = \frac{1}{x+1}\,dx$$

$$-y^{-1} = \ln|x+1| + C$$

$$-\frac{1}{y} = \ln|x+1| + C, \quad \frac{1}{y} = -\ln|x+1| - C$$

$$y(0) = 2$$
$$\frac{1}{2} = 0 - C$$
$$C = -\frac{1}{2}$$

$$\therefore \frac{1}{y} = -\ln|x+1| + \frac{1}{2}$$

33. $x^2 e^{2y}\,dy = (x^3+1)\,dx$

$$e^{2y}\,dy = (x + x^{-2})\,dx$$

$$\frac{e^{2y}}{2} = \frac{x^2}{2} - x^{-1} + C$$

$$e^{2y} = x^2 - \frac{2}{x} + 2C$$

$$y(1) = 0$$

$$1 = 1 - 2 + 2C$$
$$2 = 2C$$
$$C = 1$$

$$e^{2y} = x^2 - \frac{2}{x} + 2$$

34. $y' = \frac{1}{xy}$

$\frac{dy}{dx} = \frac{1}{xy}$

$y\,dy = \frac{1}{x}\,dx$

$\int y\,dy = \int \frac{1}{x}\,dx$

$\frac{y^2}{2} = \ln|x| + C$

$y(1) = 3$

$\frac{9}{2} = 0 + C$

$C = \frac{9}{2}$

$\frac{y^2}{2} = \ln|x| + \frac{9}{2}$

35. $2xy\frac{dy}{dx} = y^2 + 1$

$\frac{y}{y^2+1}\,dy = \frac{1}{2x}\,dx$

$\int \frac{y}{y^2+1}\,dy = \int \frac{1}{2x}\,dx$

$\frac{1}{2}\ln|y^2+1| = \frac{1}{2}\ln|x| + C$

$\ln|y^2+1| = \ln|x| + 2C$

$y(1) = 2$

$\ln 5 = 0 + 2C$

$C = \frac{\ln 5}{2}$

$\ln|y^2+1| = \ln|x| + \ln 5$

Let $u = y^2 + 1$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$\int \frac{y}{y^2+1}\,dy = \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|y^2+1|$

36. $xe^y dx = (x+1)dy$

$\frac{x}{x+1} dx = e^{-y} dy$

$\int \frac{x}{x+1} dx = \int e^{-y} dy$

$x+1 - \ln|x+1| + C = -e^{-y}$

$e^{-y} = \ln|x+1| - x - 1 - C$

$y(0) = 0$

$1 = 0 - 0 - 1 - C$

$C = -2$

$\therefore e^{-y} = \ln|x+1| - x - 1 + 2$

let $u = x + 1$

$x = u - 1$

$du = dx$

$\int \frac{x}{x+1} dx = \int \frac{u-1}{u} du$

$= \int (1 - \frac{1}{u}) du$

$= u - \ln|u|$

$= x+1 - \ln|x+1|$

39. $\frac{dx}{dt} = rx$

$\frac{1}{x} dx = r\,dt$

$\ln|x| = rt + C$

$|x| = e^{rt+C}$

$|x| = e^{rt} \cdot e^C$

$x = ke^{rt}$

$x(0) = 10000$

$k \cdot e^0 = 10000$

$k = 10000$

$x = 10000e^{0.06t}$

(b) $x(1) = 10000 \times e^{0.06}$

$x(5) = 10000 \times e^{0.3}$

(c) $2 \times 10000 = 10000e^{0.06t}$

$2 = e^{0.06t}$

$\ln 2 = 0.06t$

$t = \frac{\ln 2}{0.06} = 11.55$

4a $x = ke^{rt}$

$x(0) = 2000$

$k \times e^{0.08 \times 0} = 2000$

$k = 2000$

(a) $x = 2000e^{0.08t}$

(b) $2 \times 2000 = 2000e^{0.08t}$

$e^{0.08t} = 2$

$0.08t = \ln 2$

$t = \frac{\ln 2}{0.08} = 8.66$

(c) $x(35) = 2000 \times e^{2.8}$

$= 32889.29$

41. $\frac{dP}{dt} = kP$

$\frac{1}{P}\, dP = k\, dt$

$\ln|P| = kt + C$

$|P| = e^{kt+C}$

$|P| = e^{kt} \cdot e^{C}$

$P = Ae^{kt}$

$P(0) = 100000$

$A = 100000$

$P = 100000e^{kt}$

$P(15) = 211700$

$100000e^{15k} = 211700$

$e^{15k} = 2.11700$

$\ln 2.117 = 15k$

$k = \frac{\ln 2.117}{15}$

$= 0.05$

42. $P = 20000e^{kt}$

$P(22) = 280264$

$20000e^{20k} = 280264$

$e^{20k} = 14.0132$

$20k = \ln 14.0132$

$k = \frac{\ln 14.0132}{20}$

$= 0.132$

43. $\frac{dy}{dt} = ky$

$\frac{1}{y}\,dy = k\,dt$

$\ln|y| = kt + c$

$|y| = e^{kt+c}$

$|y| = e^{kt} \cdot e^{c}$

$y = Ae^{kt}$

$y(0) = 10000$

$A = 10000$

$y = 10000e^{kt}$

$y(2) = 3 \times 10000$

$10000e^{2k} = 3 \times 10000$

$e^{2k} = 3$

$2k = \ln 3$

$k = \frac{\ln 3}{2}$

$= 0.55$

44. $\frac{dy}{dt} = ky$

$\frac{1}{y}dy = k\,dt$

$\ln|y| = kt + c$

$|y| = e^{kt+c}$

$= e^{kt} \cdot e^{c}$

$y = Ae^{kt}$

$y(0) = A\,e^{0} = A$

(a) $2A = Ae^{kt}$

$2 = e^{\frac{1}{2}k}$

$\frac{1}{2}k = \ln 2$

$k = 2\ln 2 = 1.386$

$50A = Ae^{1.386t}$

$50 = e^{1.386t}$

$1.386t = \ln 50$

$t = \frac{\ln 50}{1.386} = 2.82$

(b) $2 = e^{3k}$

$3k = \ln 2$

$k = \frac{\ln 2}{3} = 0.231$

$50 = e^{0.231t}$

$0.231t = \ln 50$

$t = \frac{\ln 50}{0.231} = 16.935$

45 $\frac{dy}{dp} = \frac{2}{5}\left(\frac{y}{p+8}\right)$

$\frac{5}{y}\,dy = \frac{2}{p+8}\,dp$

$5\ln|y| = 2\ln|p+8| + C$

$\ln|y| = \frac{2}{5}\ln|p+8| + \frac{C}{5}$

$y = k\cdot e^{\frac{2}{5}\ln|p+8|}$

$= k\cdot(p+8)^{\frac{2}{5}}$

$y(24) = 8$

$k\cdot(24+8)^{\frac{2}{5}} = 8$

$k\cdot(32)^{\frac{2}{5}} = 8$

$k\cdot(2^5)^{\frac{2}{5}} = 8$

$4k = 8$

$k = 2$

$\therefore y = 2(p+8)^{\frac{2}{5}}$

46. $\frac{dy}{dp} = \frac{1}{2}\left(\frac{y}{p+5}\right)$

$\frac{2}{y}\,dy = \frac{1}{p+5}\,dp$

$2\ln|y| = \ln|p+5| + c$

$\ln|y| = \frac{1}{2}\ln|p+5| + \frac{c}{2}$

$|y| = e^{\frac{1}{2}\ln|p+5|}\cdot e^{\frac{c}{2}}$

$y = k\cdot e^{\frac{1}{2}\ln|p+5|}$

$= k\cdot(p+5)^{\frac{1}{2}}$

$y(20) = 18$

$k(25)^{\frac{1}{2}} = 18$

$5k = 18$

$k = 3.6$

Carbon-14 Dating

$\frac{dy}{dt} = ky$

$\frac{1}{y}\,dy = k\,dt$

$\ln|y| = kt + C$

$|y| = e^{kt} \cdot e^{C}$

$y = Ae^{kt}$

$y(0) = y_0$

$A = y_0$

$\therefore y = y_0 e^{kt}$

$\frac{1}{2} y_0 = y_0 e^{5730k}$

$\frac{1}{2} = e^{5730k}$

$5730k = \ln \frac{1}{2}$

$k = -0.00012097$

$\therefore y = y_0 e^{-0.00012097t}$

$0.01\, y_0 = y_0 e^{-0.00012097t}$

$0.01 = e^{-0.00012097t}$

$-0.00012097t = \ln 0.01$

$t = \frac{\ln 0.01}{-0.00012097}$

$= 38068.696$

~ 38069

2. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$, $u(0) = -5$

$2u\,du = (2t + \sec^2 t)\,dt$

$u^2 = t^2 + \tan t + c$

$25 = 0 + 0 + c$

$c = 25$

$u^2 = t^2 + \tan t + 25$

$u = \pm\sqrt{t^2 + \tan t + 25}$

$u = -\sqrt{t^2 + \tan t + 25}$

3. $\frac{dy}{dt}$ = rate in − rate out

$= (0.05)(5) + (0.04)(10) - \left(\frac{y}{1000}\right)(15)$

$= \frac{13}{20} - \frac{3y}{200}$

$= \frac{130 - 3y}{200}$

$\frac{1}{130 - 3y}\,dy = \frac{1}{200}\,dt$

$-\frac{1}{3}\ln|130 - 3y| = \frac{t}{200} + c$

$\ln|130 - 3y| = -\frac{3t}{200} - 3c$

$y(0) = 0$

$\ln 130 = -3c$

$c = -\frac{\ln 130}{3}$

(a) $\ln|130 - 3y| = -\frac{3t}{200} + \ln 130$

$130 - 3y = e^{-\frac{3t}{200}} \cdot e^{\ln 130}$

$130 - 3y = 130e^{-\frac{3t}{200}}$

$3y = 130 - 130e^{-\frac{3t}{200}}$

$y = \frac{130 - 130e^{-\frac{3t}{200}}}{3}$

(b) $y(60) = \frac{130 - 130e^{-\frac{9}{10}}}{3}$

$= 25.71$ kg

4. $2xy' + y = 6x$

$y' + \frac{y}{2x} = 3$

$P(x) = \frac{1}{2} \times \frac{1}{x}$

$I(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\ln x}$

$= x^{\frac{1}{2}}$

$x^{\frac{1}{2}}y' + \frac{y}{2x^{\frac{1}{2}}} = 3x^{\frac{1}{2}}$

$(x^{\frac{1}{2}}y)' = 3x^{\frac{1}{2}}$

$\int \frac{d}{dx}\left(x^{\frac{1}{2}}y\right)dx = \int 3x^{\frac{1}{2}}\,dx$

$x^{\frac{1}{2}}y = 2x^{\frac{3}{2}} + c$

$y = 2x + c \cdot x^{-\frac{1}{2}}$

$y(4) = 20$

$20 = 8 + \frac{c}{2}$

$40 = 16 + c$

$c = 24$

$\therefore y = 2x + \frac{24}{\sqrt{x}}$