

1. Determine whether each of these functions is a bijection. **If it is a bijection, prove it; if not a bijection, give a counterexample.**

1.1 $f : \mathbb{R} \rightarrow \mathbb{R}$: $f(x) = 7x - 2$

Proof

f is injective: Let $x, y \in \mathbb{R}$ and $f(x) = f(y)$. We want to show that $x = y$

Since $f(x) = f(y)$, we have $7x - 2 = 7y - 2 \rightarrow x = y$ as required.

$\therefore f$ is injective \times

f is surjective: Let $y \in \mathbb{R}$ and choose $x = \frac{y+2}{7}$

$$\text{Then } f(x) = 7x - 2 = 7\left(\frac{y+2}{7}\right) - 2 = y$$

$\therefore f$ is surjective \diamond

$\therefore f$ is bijection \times

1.2 $f : \mathbb{R} \rightarrow \mathbb{Z}$: $f(x) = \lfloor x \rfloor$ (that is, $f(x) = \text{floor}(x)$)

f is not injective: Counter ex: $\lfloor 1.1 \rfloor = \lfloor 1 \rfloor = 1$
 $\therefore f$ is not injective \times

f is surjective: $\forall n \in \mathbb{Z}$ (which $z \in \mathbb{R}$): $\lfloor z \rfloor \in \mathbb{Z}$

$\therefore f$ is surjective \diamond

$\therefore f$ is not a bijection \times

1.3 $f: \mathbb{R} \rightarrow \mathbb{R}: f(x) = x^2$

f is not injective: $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$

$\therefore f$ is not injective \times

f is surjective: $\forall x \in \mathbb{R}; f(\sqrt{x}) = x$ \times

$\therefore f$ is not a bijection \times

1.4 $f: \{1,2,3\} \rightarrow \{2\}: \{(1,2), (2,2), (3,2)\}$

f is not injective. All number map to 2.

$\therefore f$ is not injective \times

f is surjective: Since $\{2\}$ is the entire codomain and at least number in the domain maps to 2.

$\therefore f$ is surjective \times

$\therefore f$ is not a bijection \times

1.5 f: $\{1,2,3\} \rightarrow \{1,2,3\}$: $\{(1,2), (1,3), (2,1), (2,2)\}$

This is not a function because 1 map to both 2 and 3 and also 2 map to both 1 and 3 ~~2~~

2. Show that $2^x + 17$ is $O(3^x)$.

Show that $x \geq c$, $2^k + 17 \leq 2^x + 2^x = 2 \cdot 2^x \leq 2 \cdot 3^x$, so $2^x + 17$ is $O(3^x)$

($c=2, k=5$) ~~X~~

3. Let k be a positive integer. Show that $1^k + 2^k + \dots + n^k$ is $O(n^{k+1})$.

$$1^k + 2^k + \dots + n^k \leq n^k + n^k + \dots + n^k = n \cdot n^k = n^{k+1} \quad (\cancel{\text{not}})$$

$(c=1, k=1)$

\uparrow

$x \geq k$

$x \geq 1 \text{ NOT that } k$

n^{k+1}

4. Compute formula for

$$\begin{aligned}
 2 + \sum_{i=1}^n \left(3 + \sum_{j=i}^n 6 \right) &= 2 + \sum_{i=1}^n 3 + \sum_{i=1}^n \sum_{j=i}^n 6 \\
 &= 2 + 3 \sum_{i=1}^n + \sum_{i=1}^n \sum_{j=i}^n 6 \\
 &= 2 + 3n + 6 \sum_{i=1}^n \sum_{j=i}^n 1 \\
 &= 2 + 3n + 6 \sum_{i=1}^n (n+1-i) \\
 &= 2 + 3n + 6(n + n - 1 + \dots + 1) \\
 &= 2 + 3n + 6 = \frac{1}{2}n(n+1) \\
 &= 2 + 6n + 3n^2 \quad (\cancel{\text{not}})
 \end{aligned}$$

1. Show that $|E| = |\mathbb{Z}|$

We have to find injection from $|E|$ to \mathbb{Z} . Let $f: E \rightarrow \mathbb{Z}$ be defined by $f(e) = e/2$, for $e \in E$.

We prove that f is a bijection.

1. Suppose $f(e_1) = f(e_2)$ for $e_1, e_2 \in E$ then $e_1/2 = e_2/2$ and $e_1 = e_2$. Hence, f is injective.
2. Suppose $x \in \mathbb{Z}$, $2x$ is even, so $2x \in E$ and $f(2x) = x$. Hence, f is surjective.

\therefore We now have that f is a bijection. \star

2. Show that $(x^3 + 2x)/(2x+1)$ is $O(x^2)$

$$\frac{x^3 + 2x}{2x+1} \leq \frac{4x^3 + 2x^2}{2x+1} \leq 2 \times \left(\frac{2x+1}{2x+1} \right) = 2x^2 \leq C \cdot x^2 \quad C=2$$

$k=1$ ~~✓~~

$$\begin{aligned} 3. \sum_{i=1}^{100} (4+3i) &= \sum_{i=1}^{100} 4 + \sum_{i=1}^{100} 3i \\ &= 4(100) + 3 \underbrace{(100)(101)}_2 \\ &= 400 + 3(50)(101) \\ &= 400 + 15150 \\ &= 15550 \end{aligned}$$