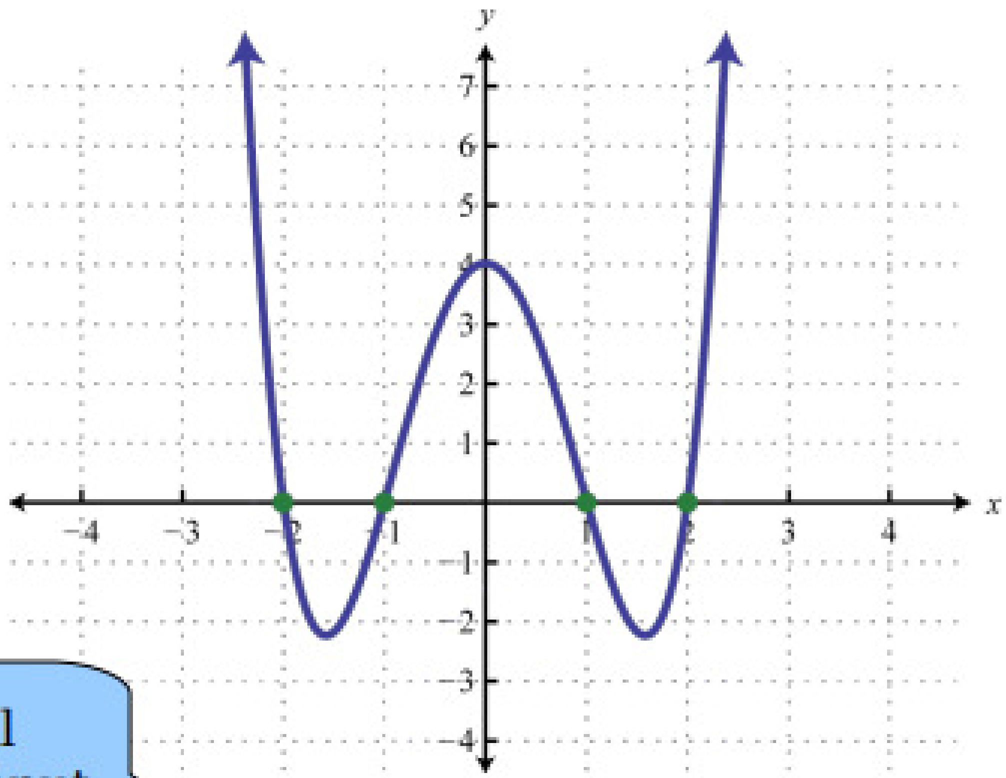
Discrete Mathematics

Functions

What is a function?

In high school math:



Take a real number as input

$$f(x) = x^4 - 5x^2 + 4$$

Give a real number as output

In C++ coding:

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) {
      numHeads++;
    }
    numTries++;
  }

  return numTries;
}
```

Take input(s) of different type(s)

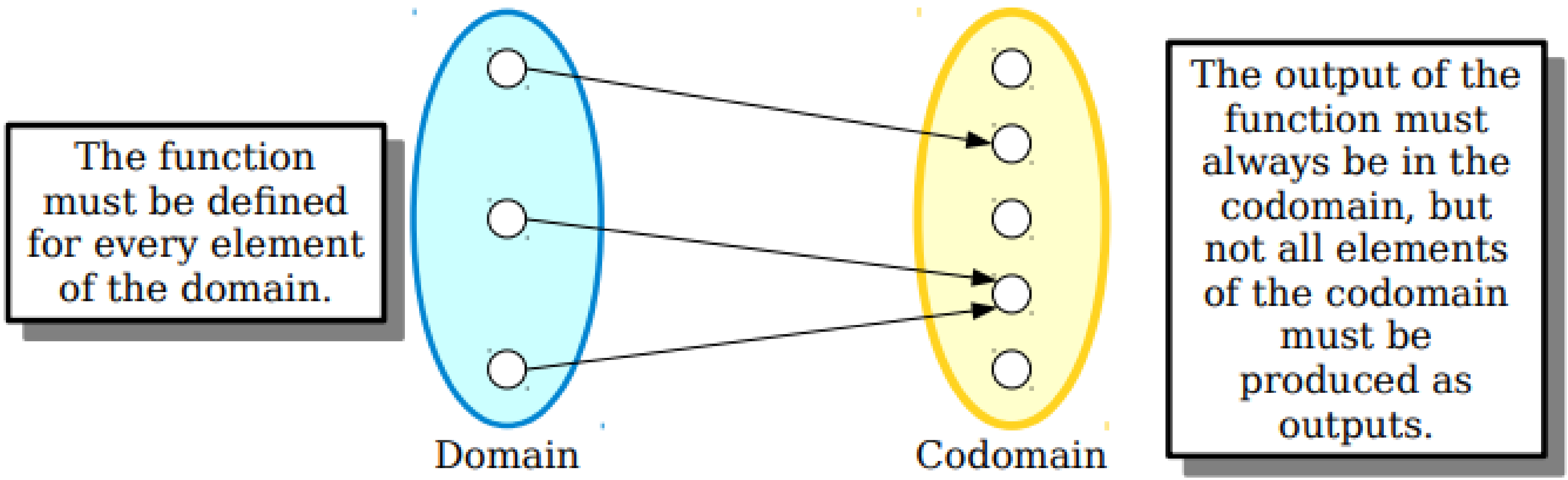
Return an output of some type

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.



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- A function *f* can only be applied to elements of its domain. For any x in the domain, *f*(*x*) belongs to the codomain.

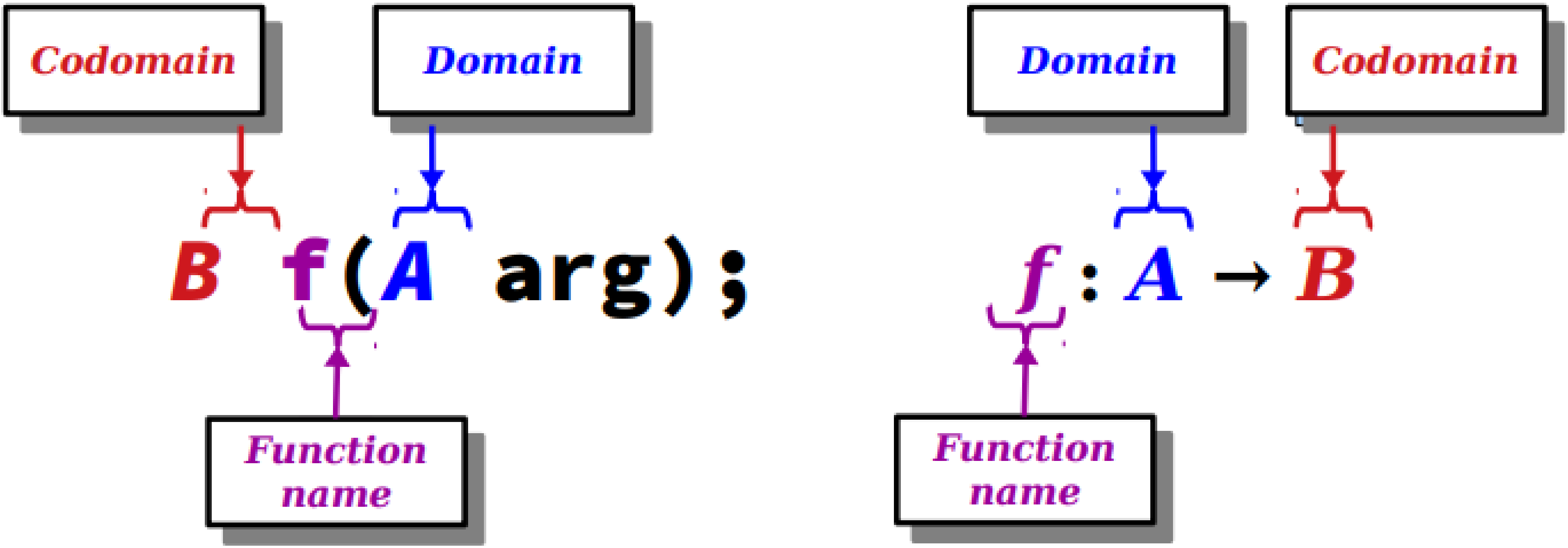
The ***domain*** of this function is ℝ. Any real number can be provided as input.

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $f : A \to B$.
- Think of this like a “function prototype” in C++.



The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*“Every input in A maps to some output in B.”*)

- Second, f must be deterministic:

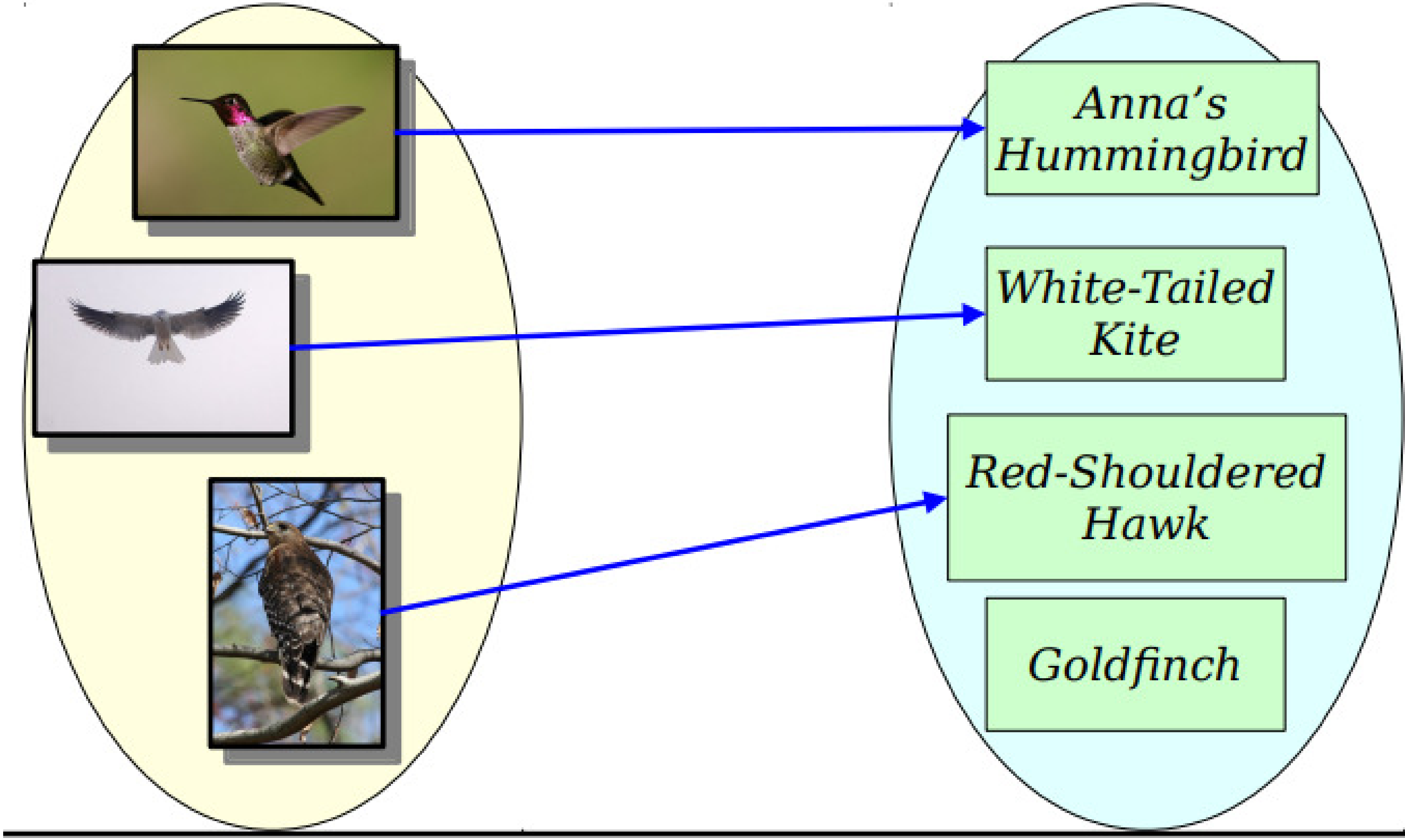
$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*“Equal inputs produce equal outputs.”*)

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.

Functions can be defined as a ***picture***.



Draw sets (ovals) to give the domain and codomain.
Draw a mapping (arrows) to define the function's action.

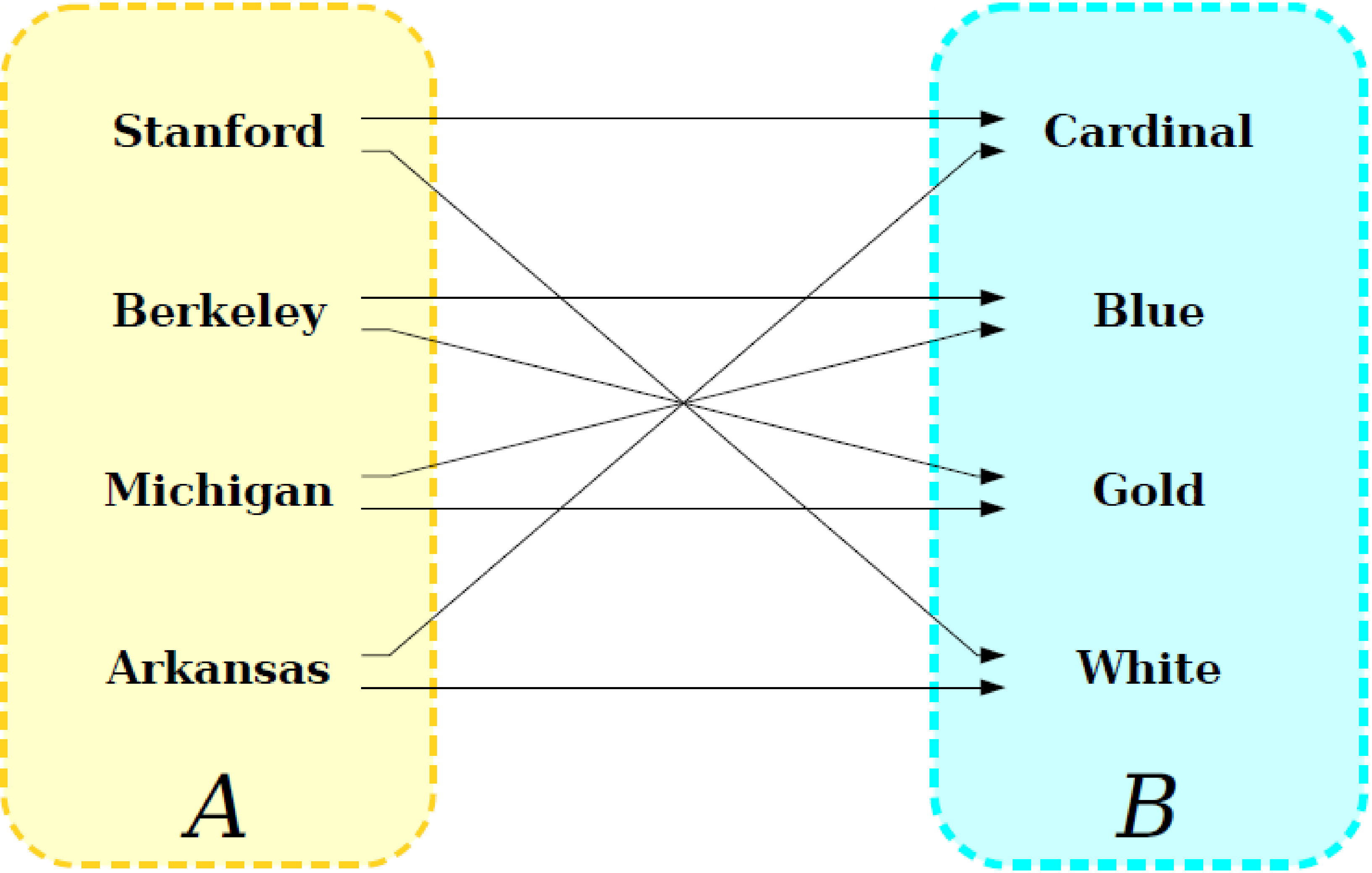
Functions can be defined as a ***rule***.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Use the : notation to name the domain and codomain.
Use the $f(x) =$ notation to define the function's action.

Is this a function from A to B?



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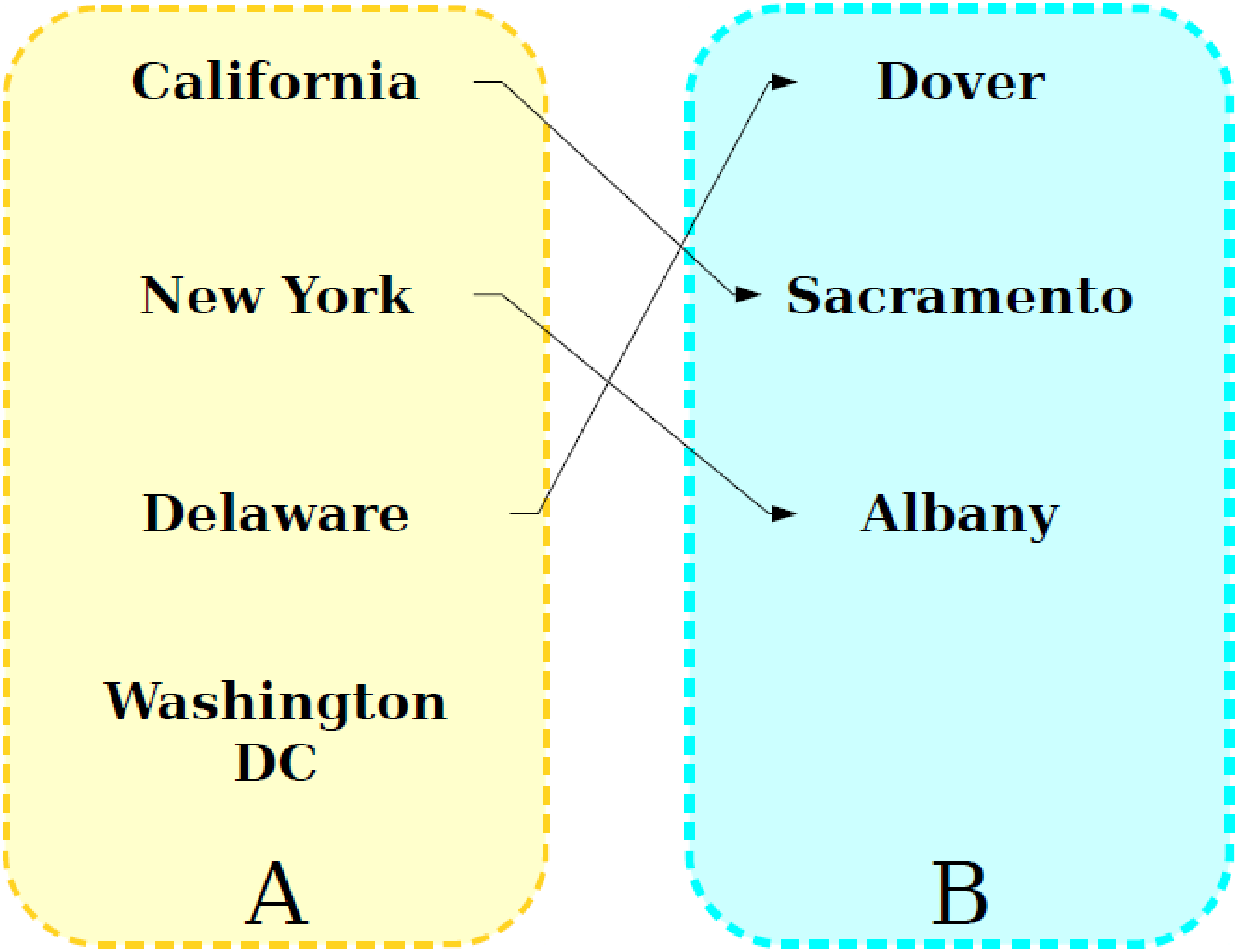

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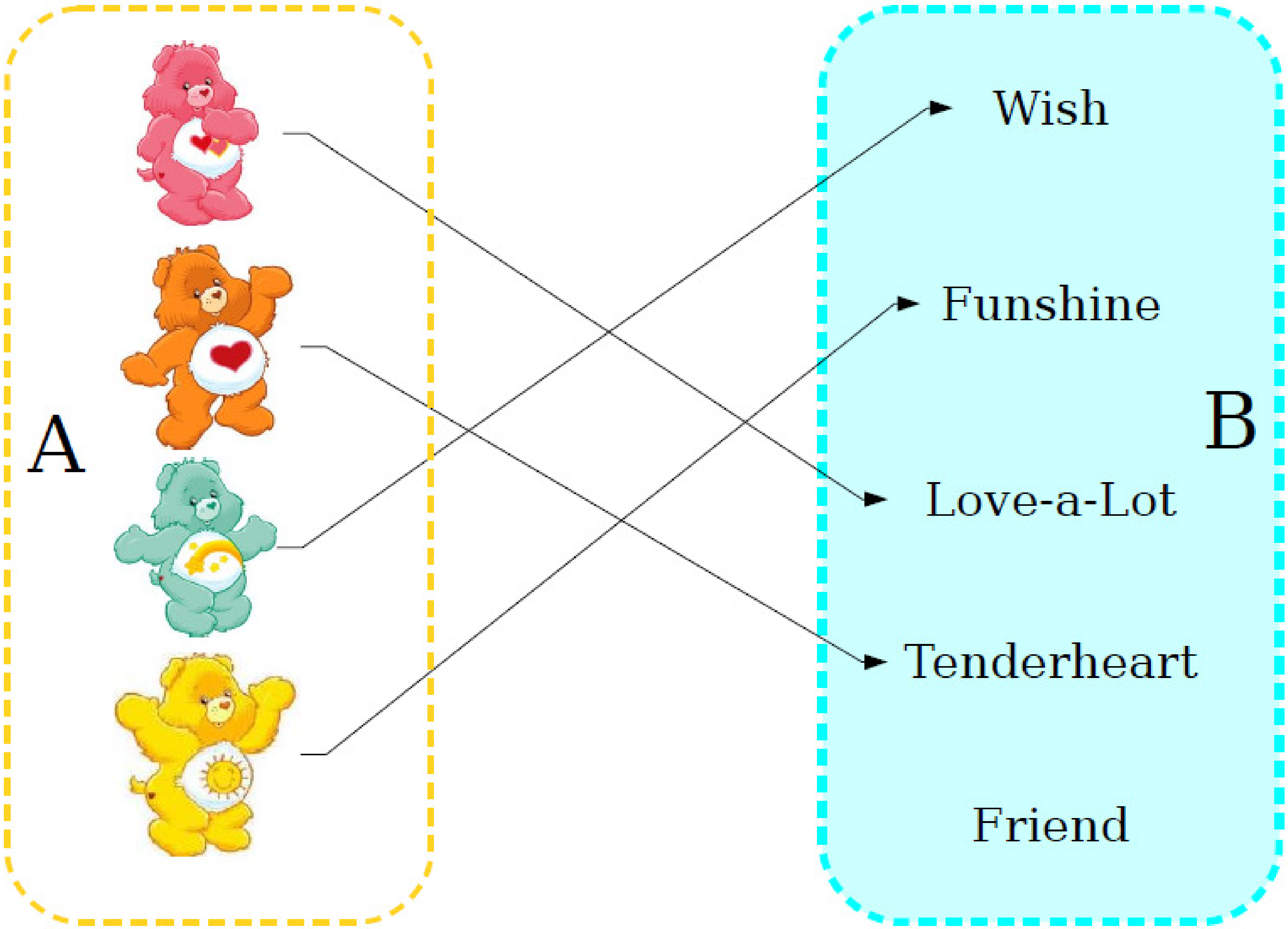
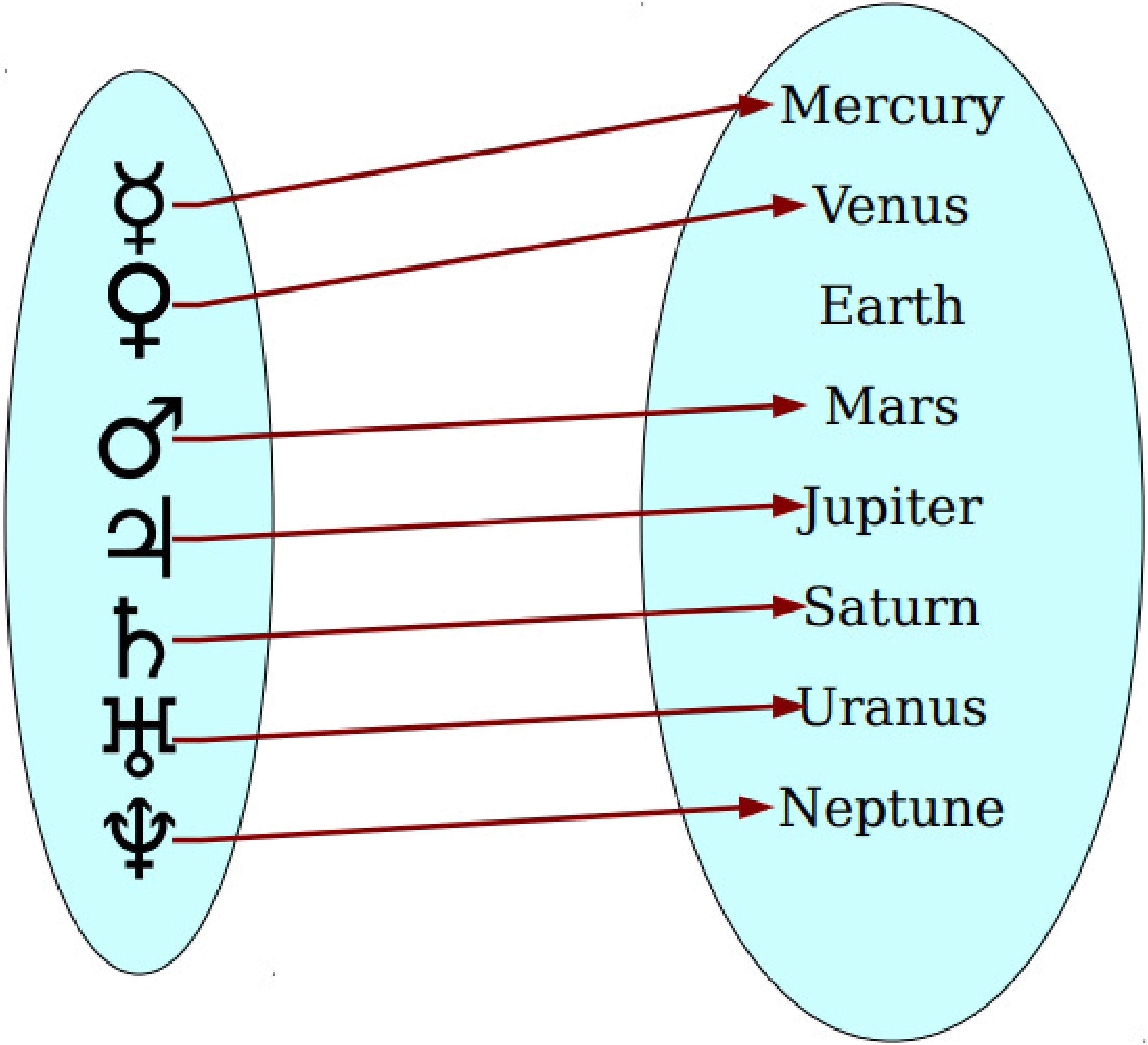


Injective Functions


Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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What does it mean for the function f to be injective?

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Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

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Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

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Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

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This in turn means that

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$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

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$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

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Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

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Let $x_1 = -1$ and $x_2 = +1$.

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$$f(x_1) = f(-1) = (-1)^4 = 1$$

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and

$$f(x_2) = f(1) = 1^4 = 1$$

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Let $x_1 = -1$ and $x_2 = +1$. Notice that

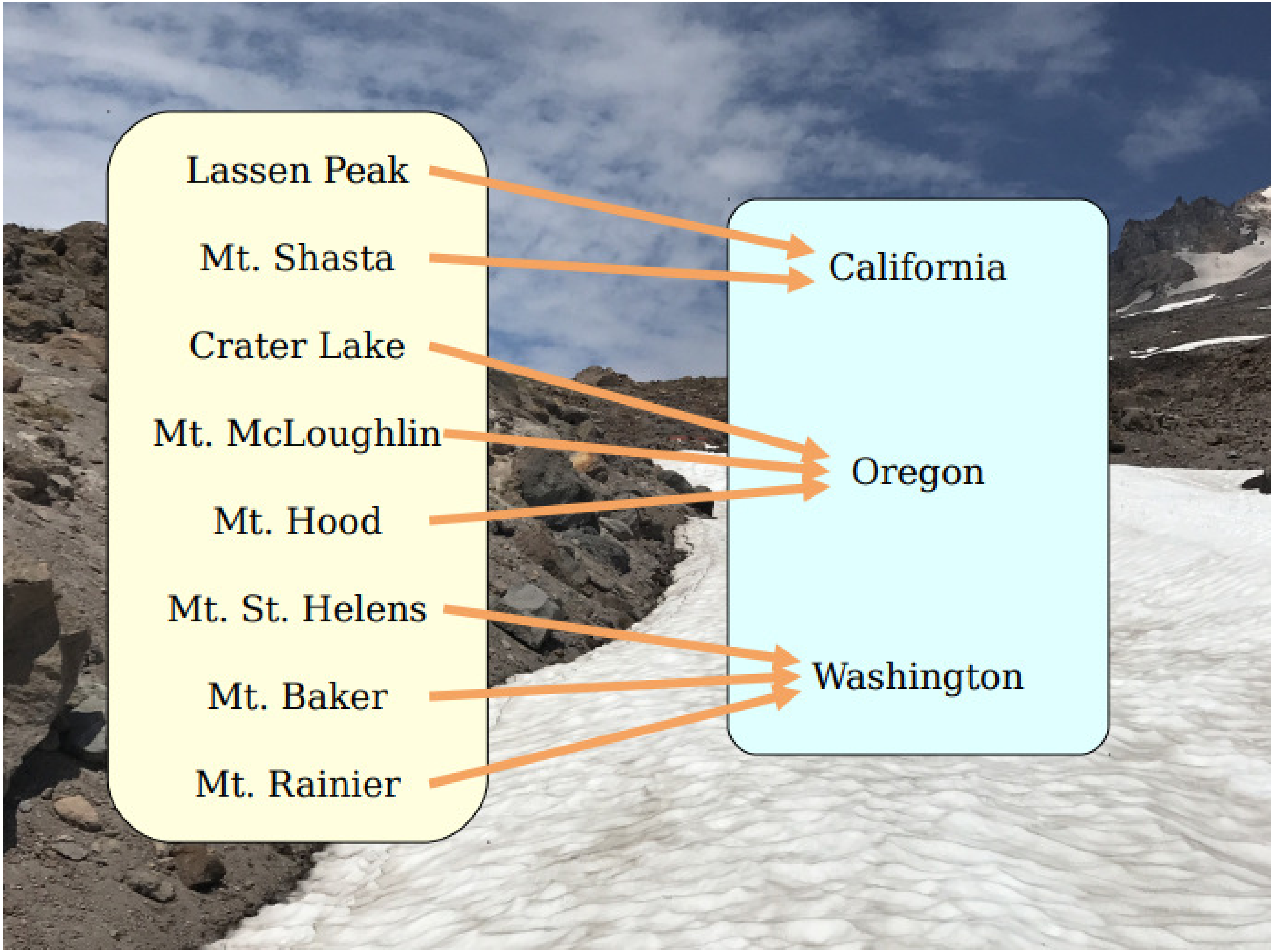
$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

Surjective Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Proof: Consider any $y \in \mathbb{R}$.

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Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$.

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

So $f(x) = y$, as required. ■

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What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg\exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Proof: Let $n = 137$.

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Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd.

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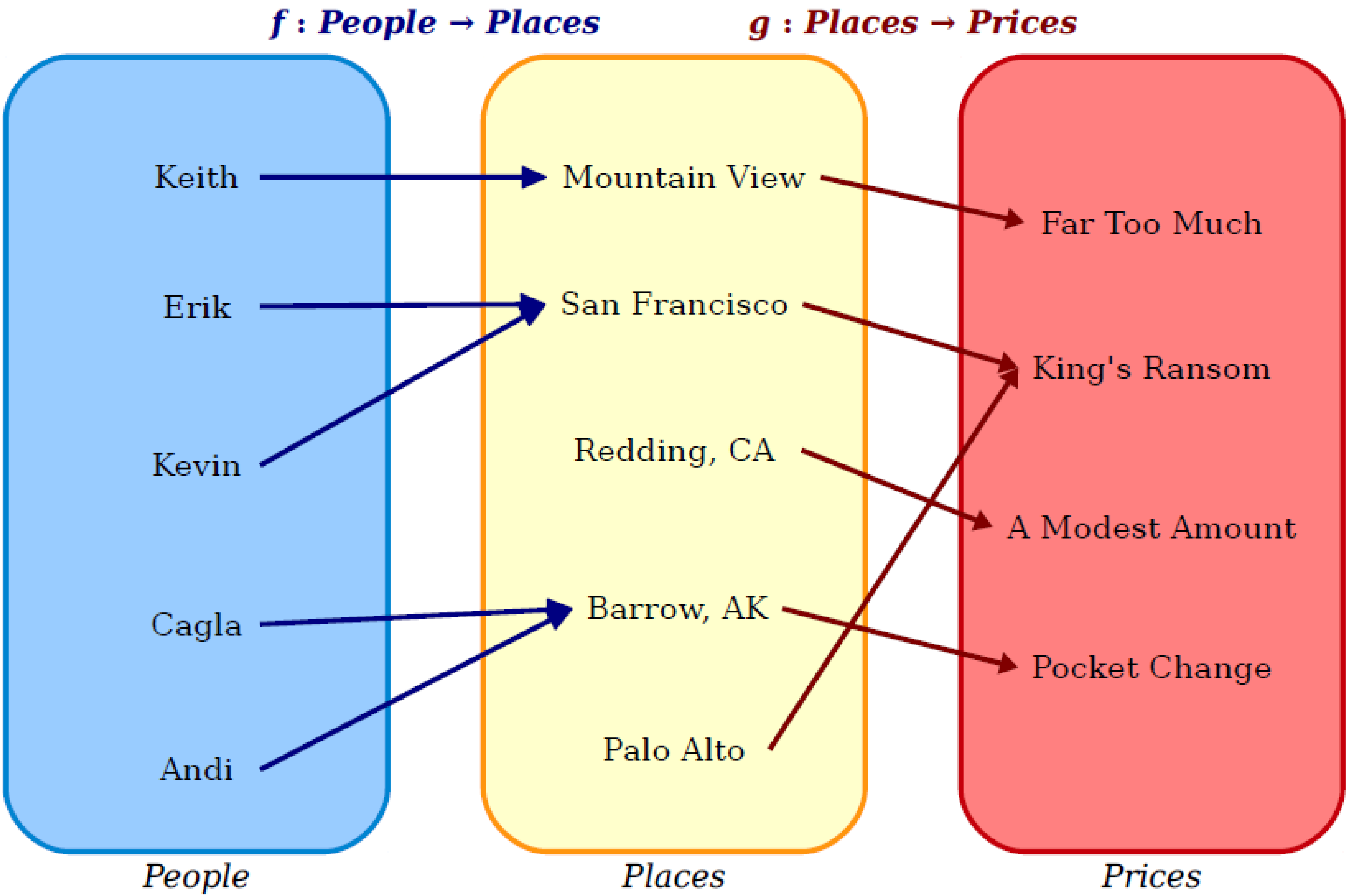
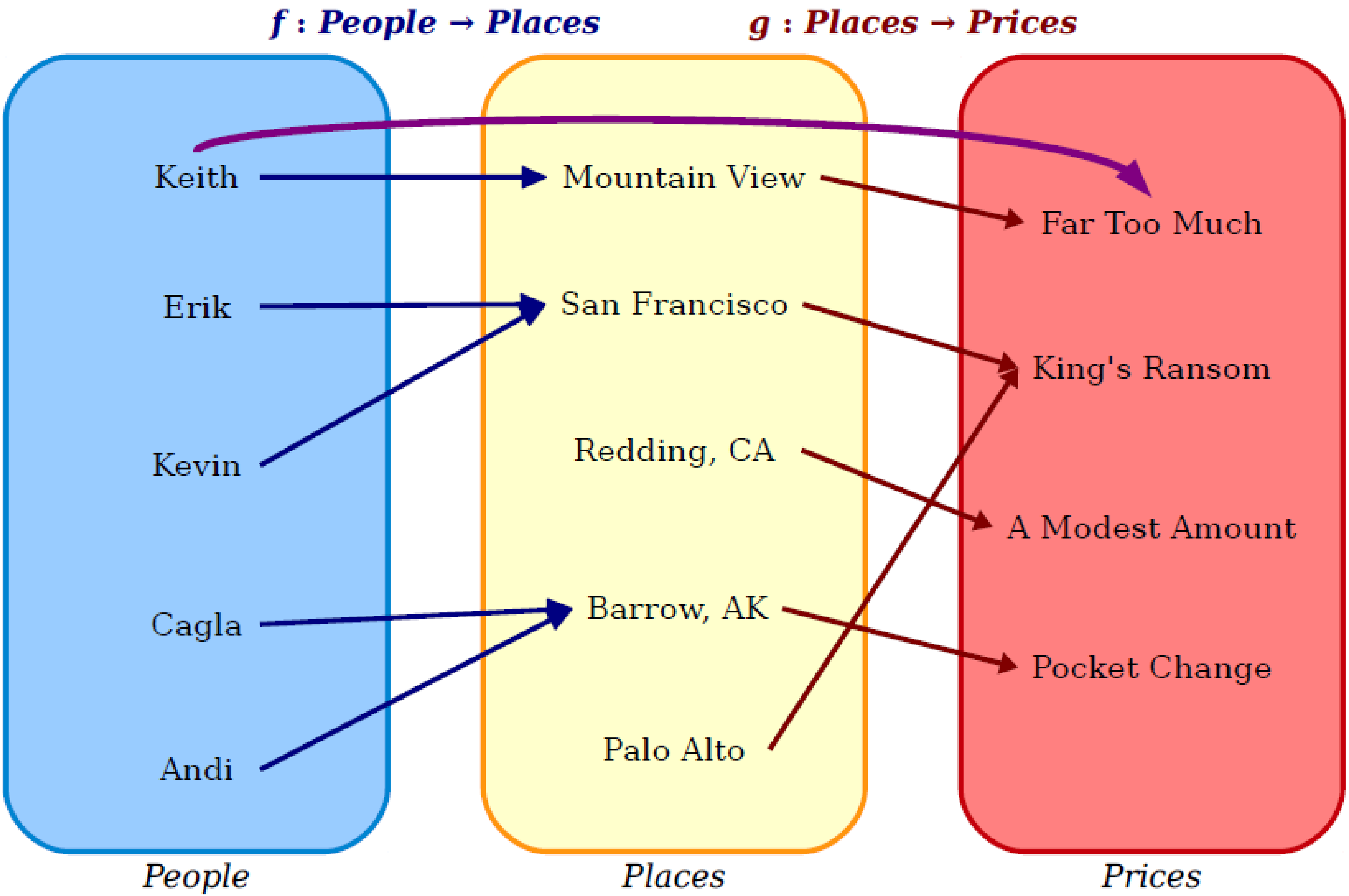
Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required. ■

Function Composition


f : People → Places
g : Places → Prices
Keith
Erik
Kevin
Cagla
Andi
Mountain View
San Francisco
Redding, CA
Barrow, AK
Palo Alto
Far Too Much
King's Ransom
A Modest Amount
Pocket Change
People
Places
Prices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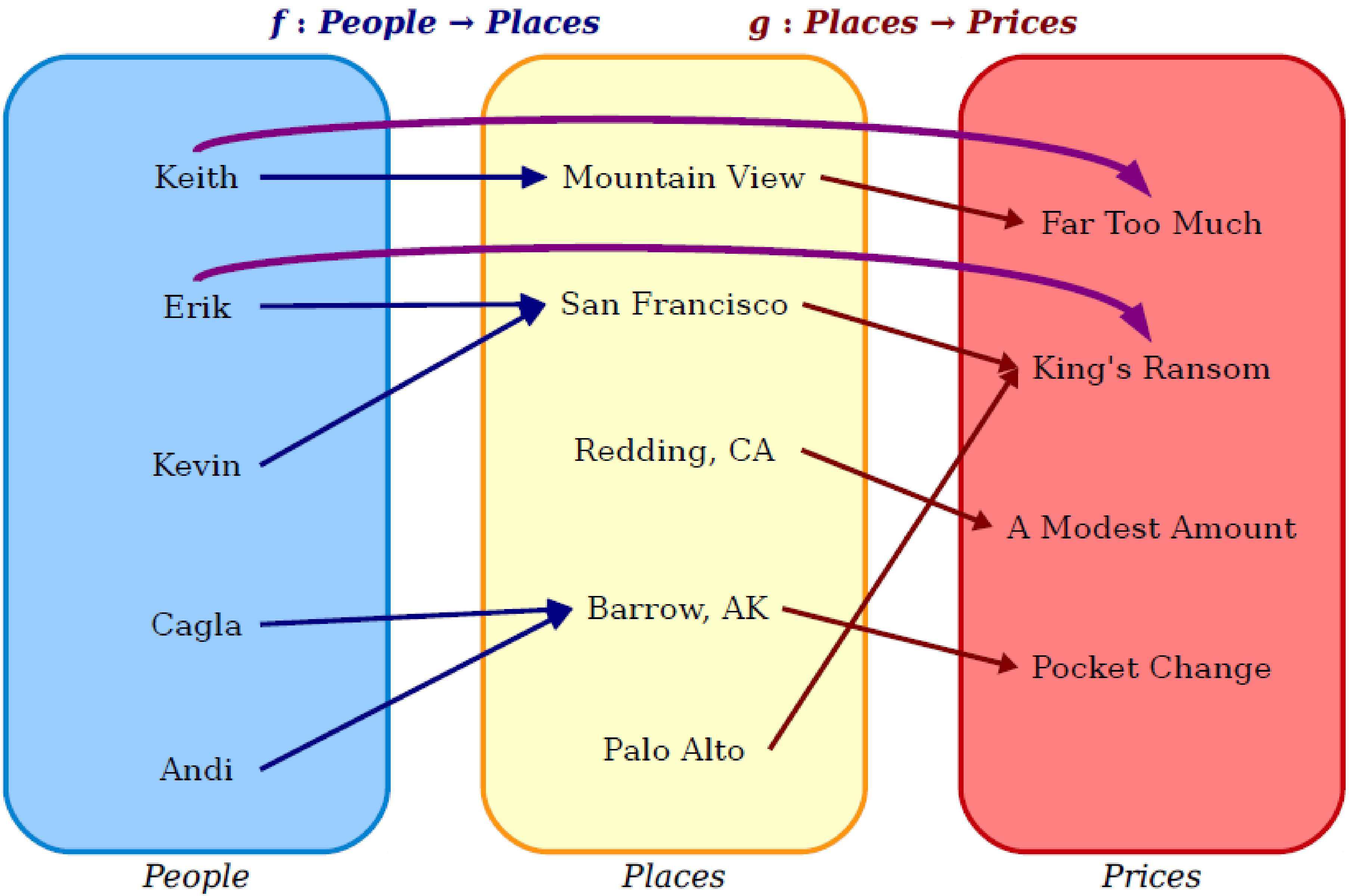

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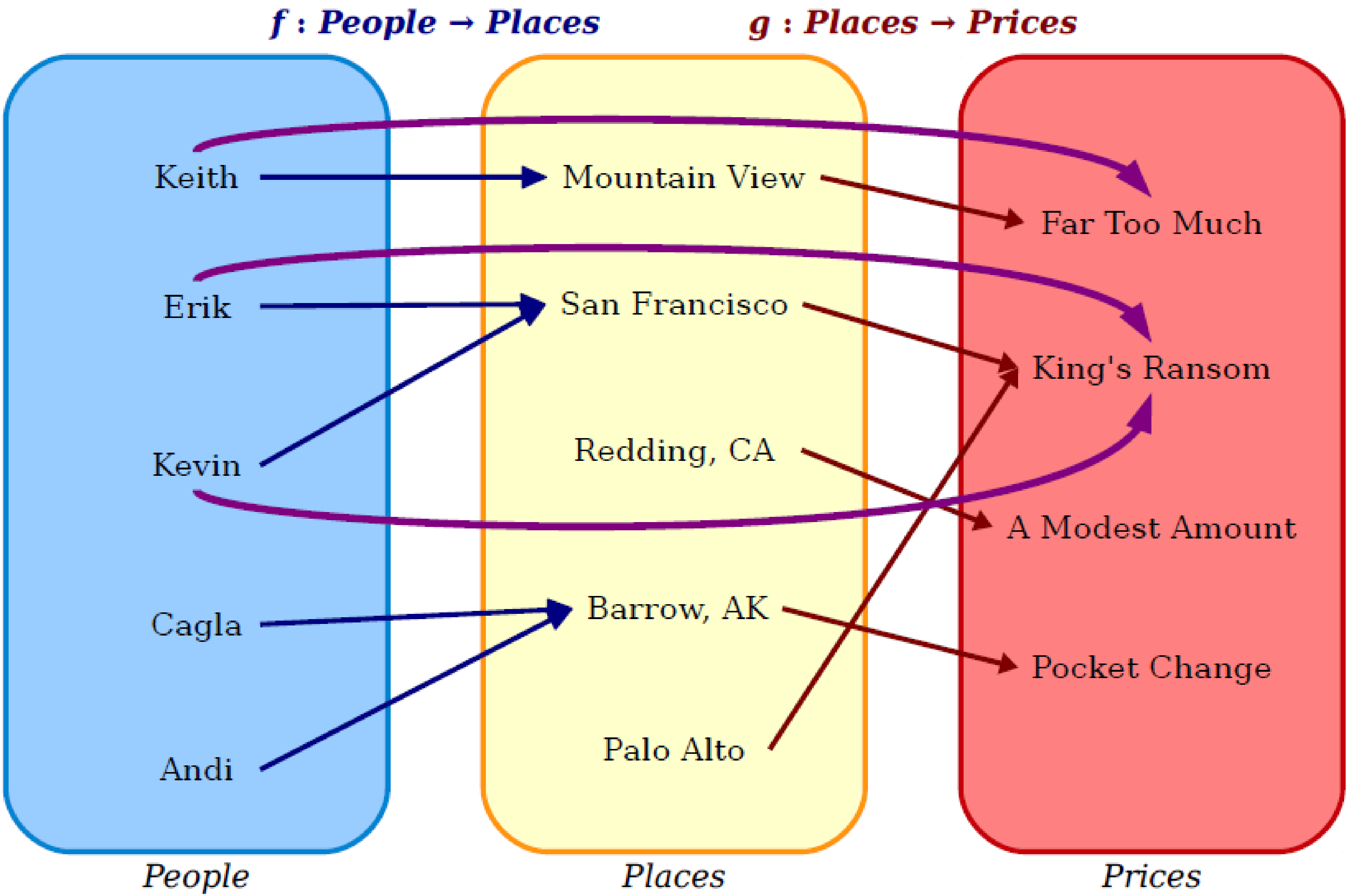

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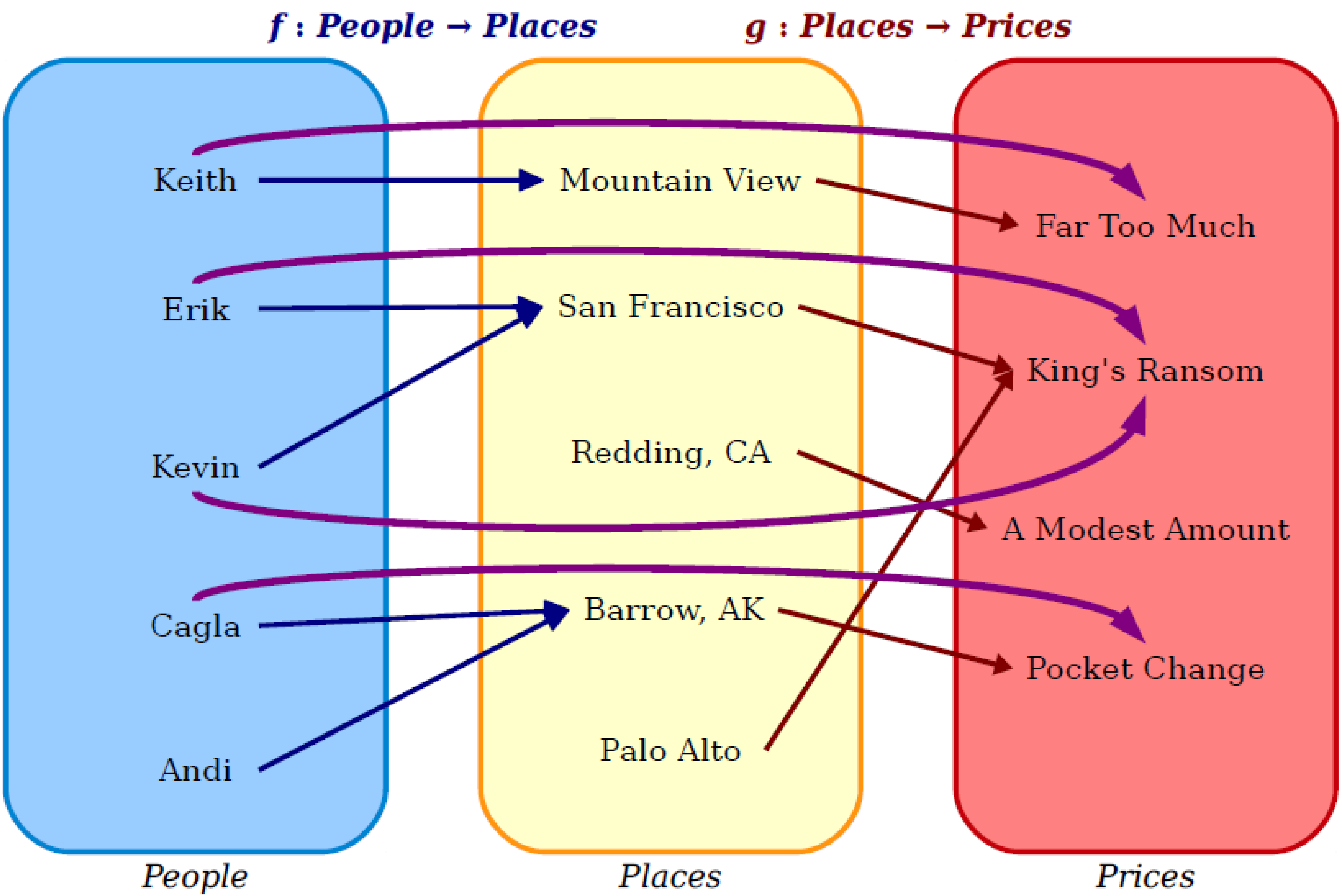
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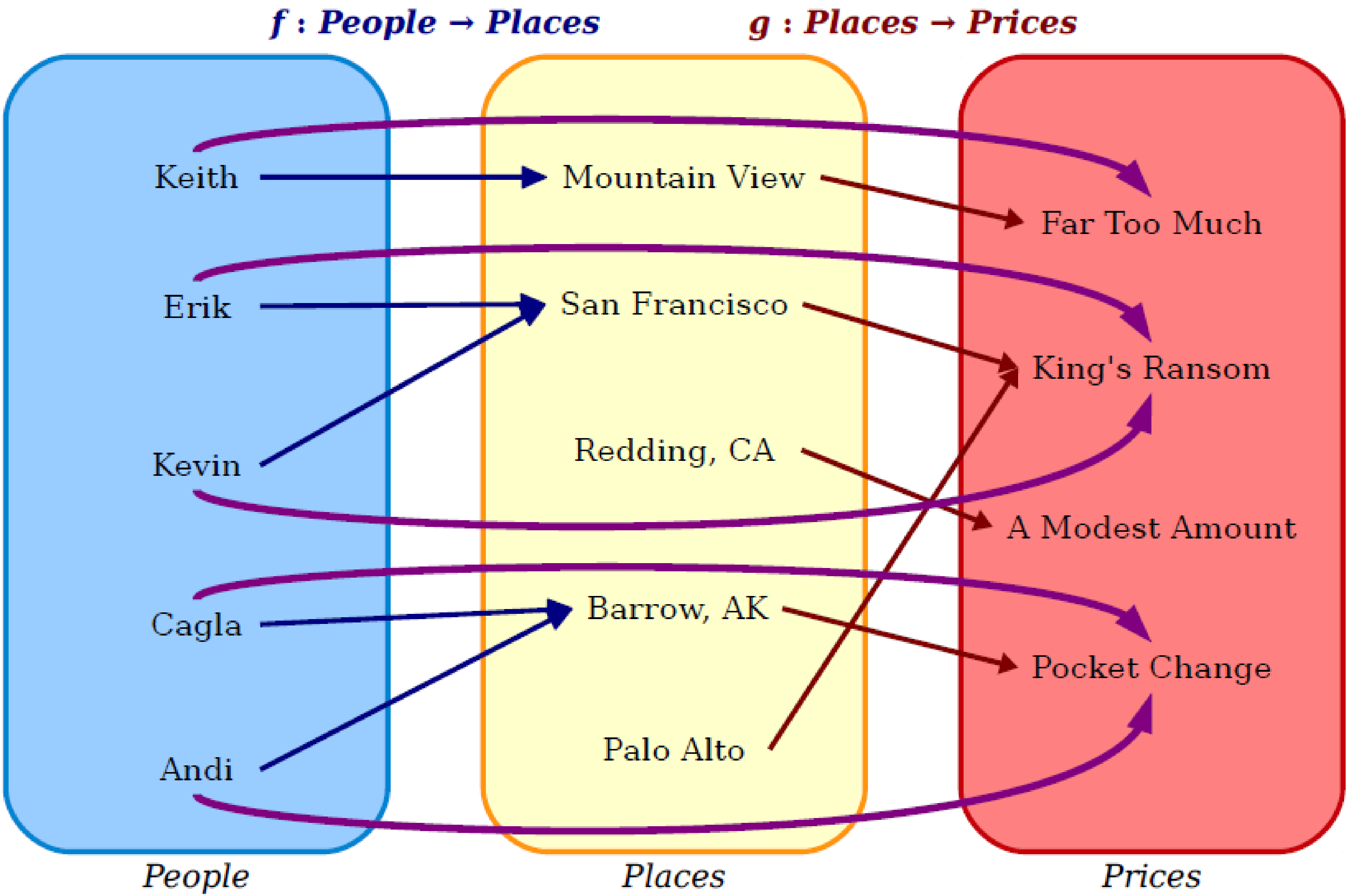
f : People → Places
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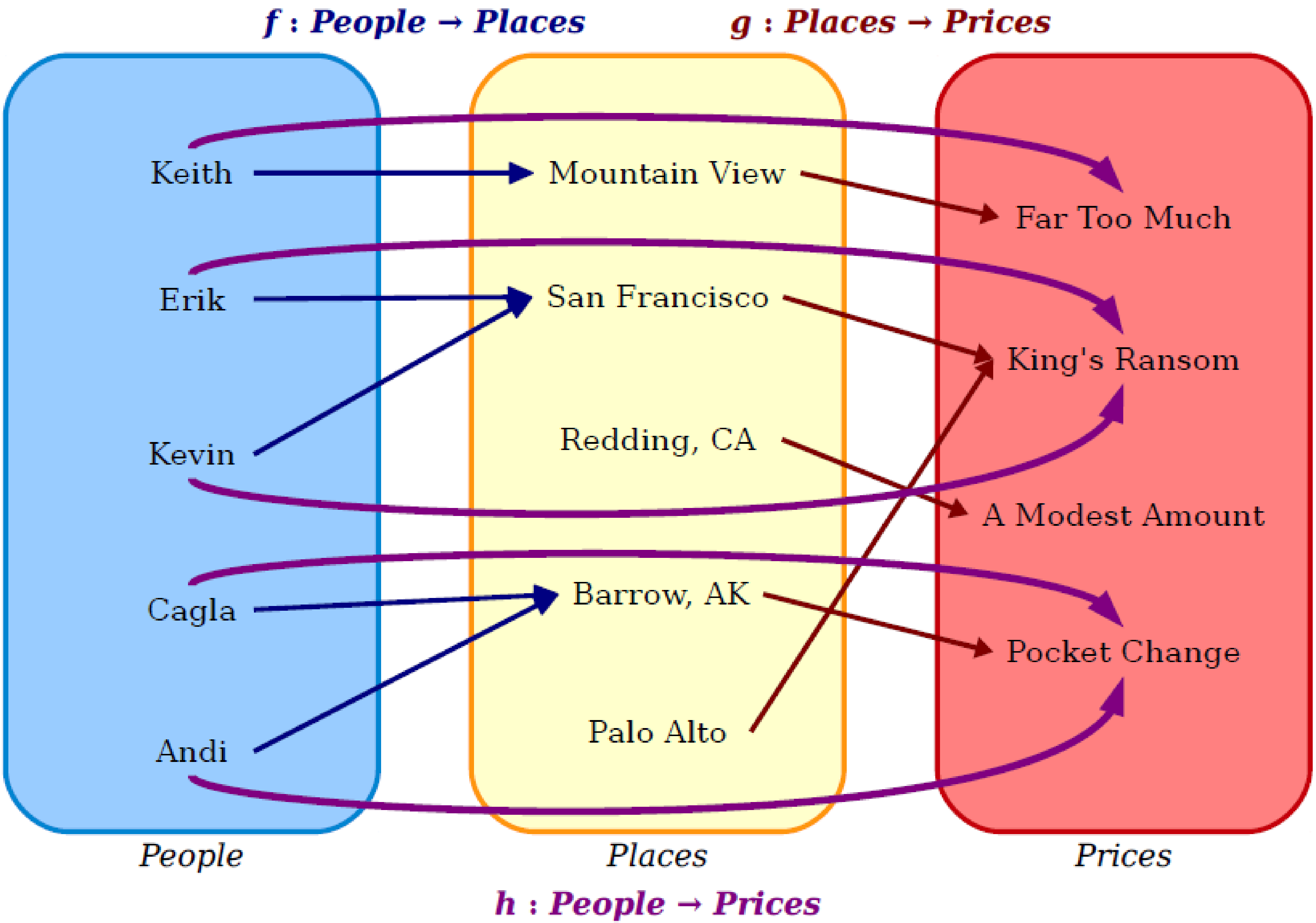
f : People → Places
g : Places → Prices
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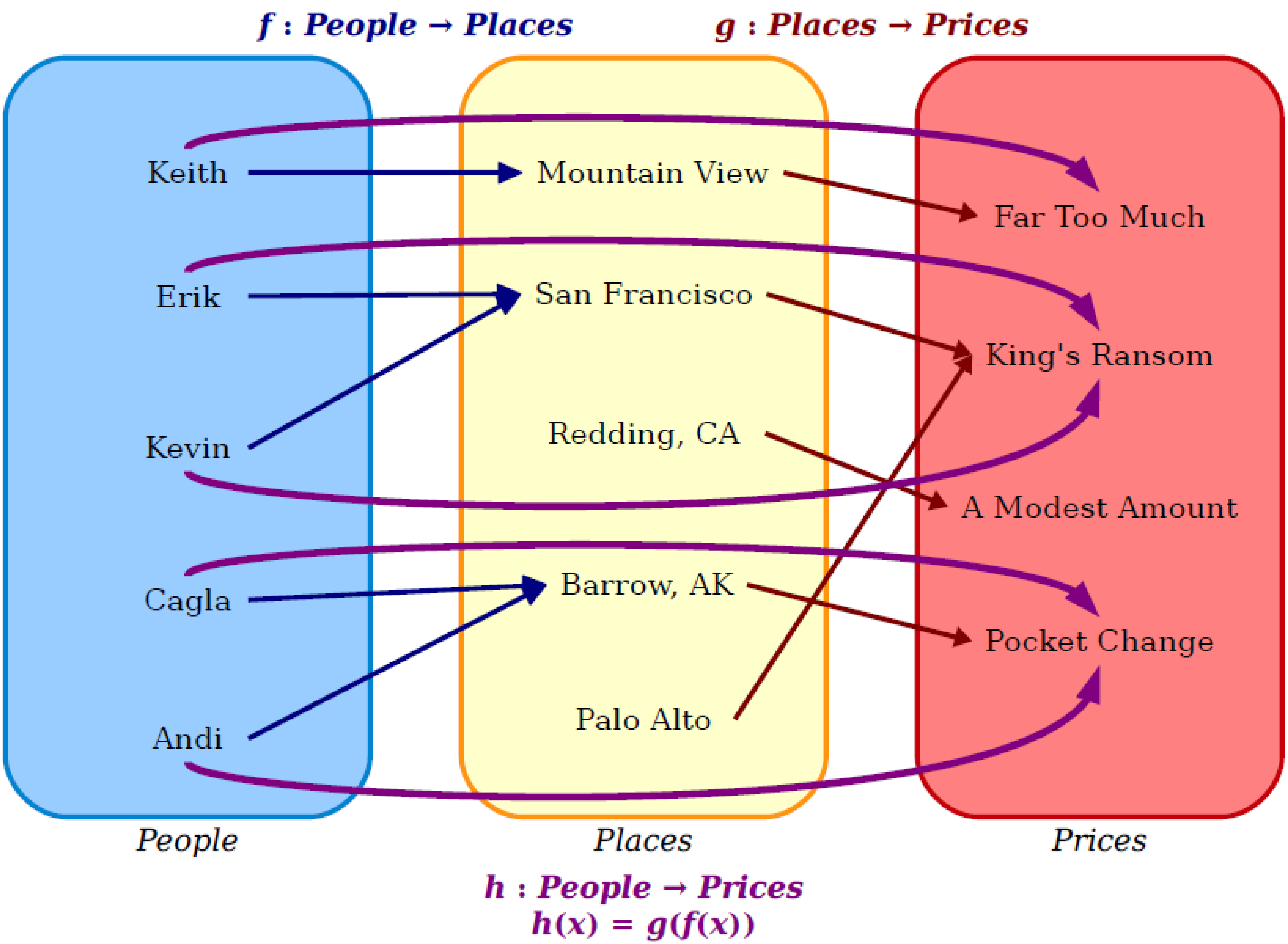
Redding, CA
Barrow, AK
Palo Alto
Far Too Much
King's Ransom
A Modest Amount
Pocket Change
People
Places
Prices
h : People → Prices



f : People → Places
g : Places → Prices
Keith
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Barrow, AK
Palo Alto
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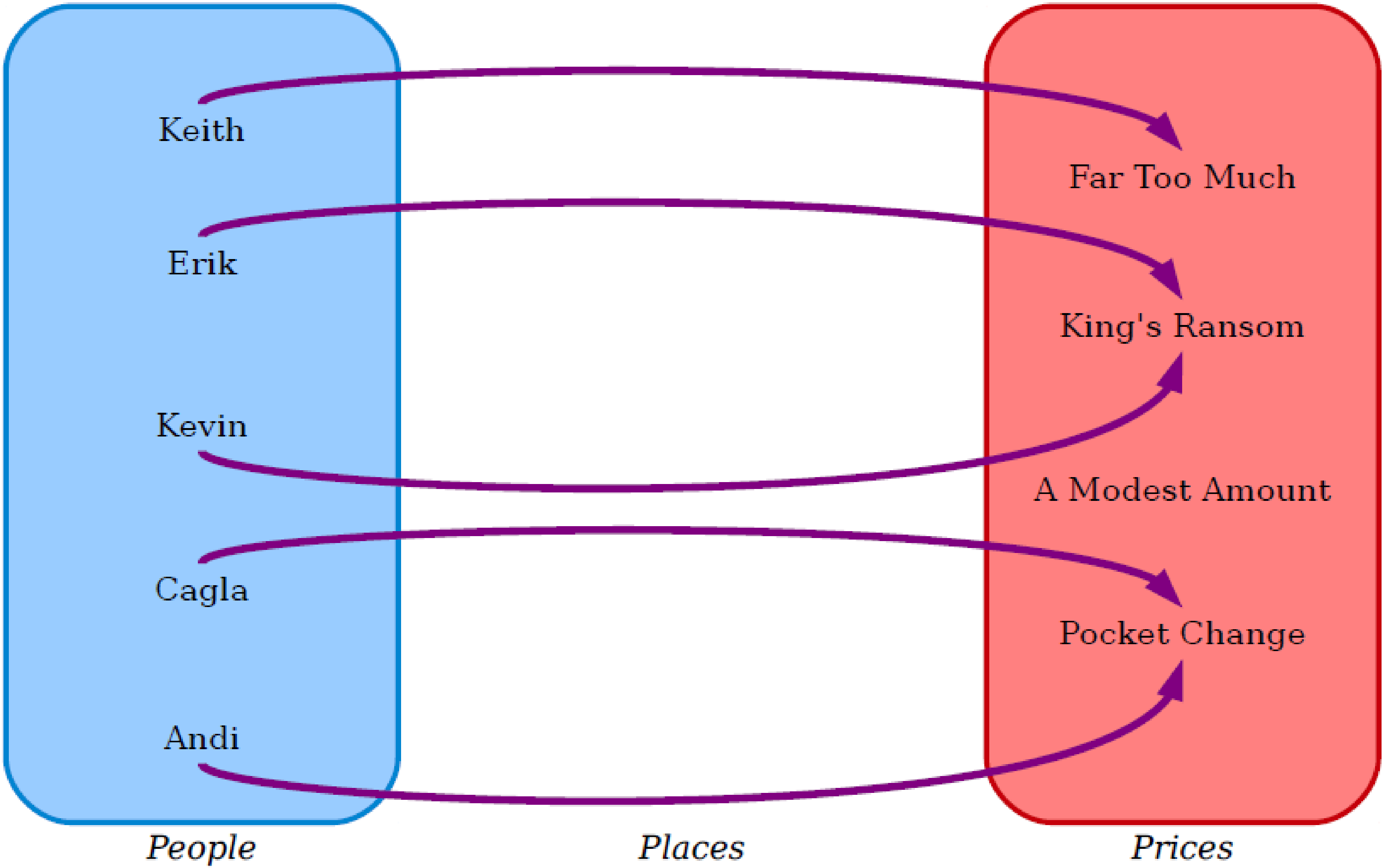
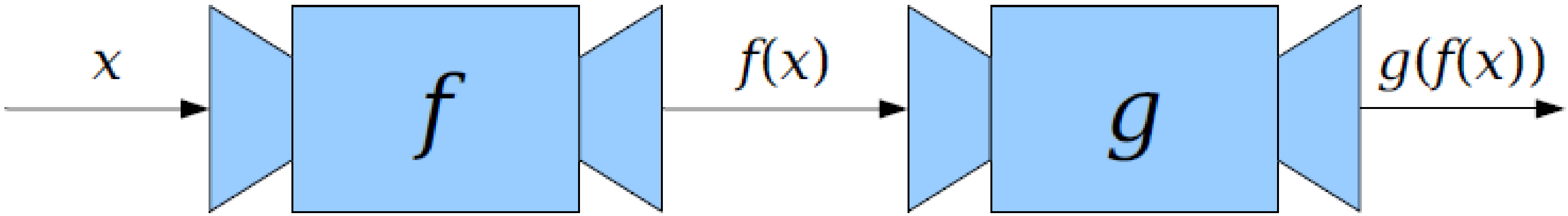
King's Ransom
A Modest Amount
Pocket Change
People
Places
Prices
h : People → Prices
h(x) = g(f(x))



$h : People \rightarrow Prices$
$h(x) = g(f(x))$

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Function Composition

- Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 1$ and $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = n^2$.
- What is $g \circ f$?

$$\begin{aligned}(g \circ f)(n) &= g(f(n)) \\ &= g(2n + 1) \\ &= (2n + 1)^2 = 4n^2 + 4n + 1\end{aligned}$$

- What is $f \circ g$?

$$\begin{aligned}(f \circ g)(n) &= f(g(n)) \\ &= f(n^2) \\ &= 2n^2 + 1\end{aligned}$$

- In general, if they exist, the functions $g \circ f$ and $f \circ g$ are usually not the same function. ***Order matters in function composition!***

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$\quad f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
$\quad g(x) \neq g(y)$
$)$

We're ***assuming*** these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
$\quad (g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	*What We Need to Prove*
$f : A \to B$ is an injection. $\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$ $g : B \to C$ is an injection. $\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$ $a_1 \in A$ is arbitrarily-chosen. $a_2 \in A$ is arbitrarily-chosen.	$g \circ f$ is an injection. $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$ We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$\quad f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
$\quad g(x) \neq g(y)$
$)$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
$\quad (g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

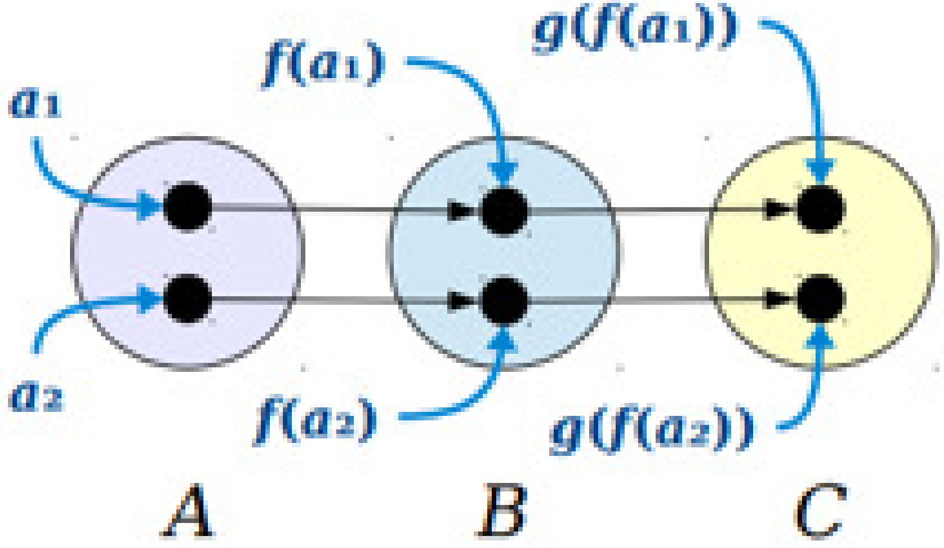
What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$(g \circ f)(a_1) \neq (g \circ f)(a_2)$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	What We Need to Prove
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

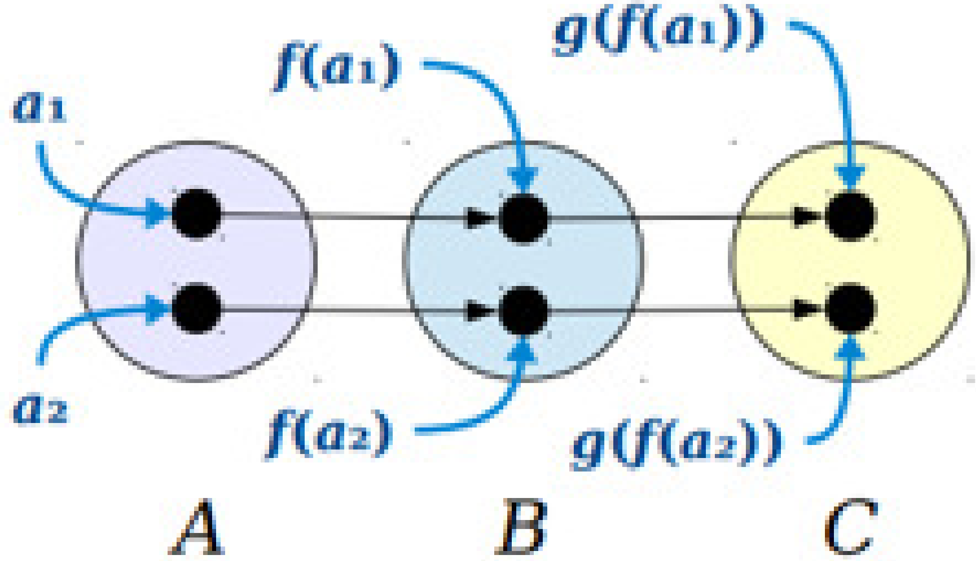
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	What We Need to Prove
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A. \forall y \in A. (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B. \forall y \in B. (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	



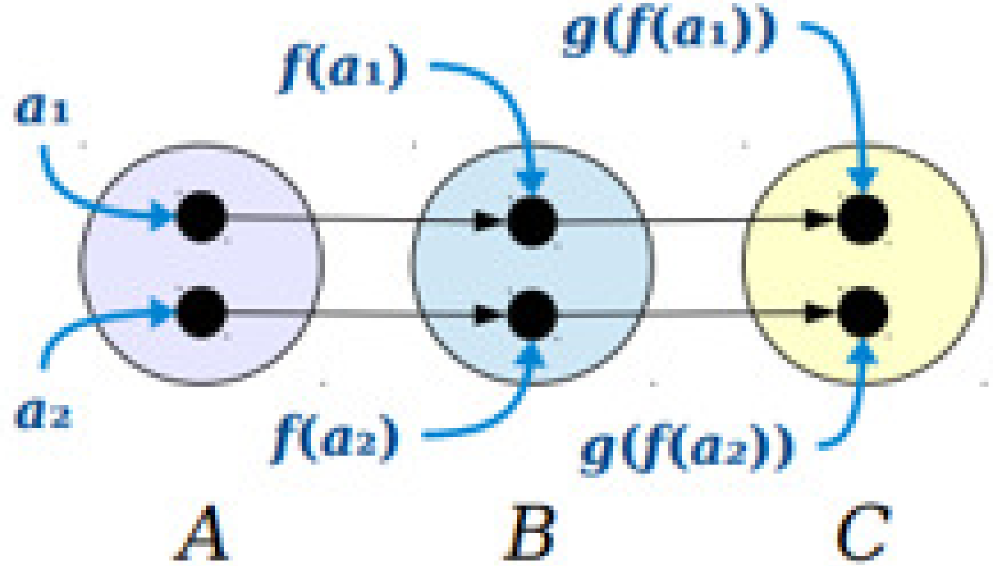
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof:



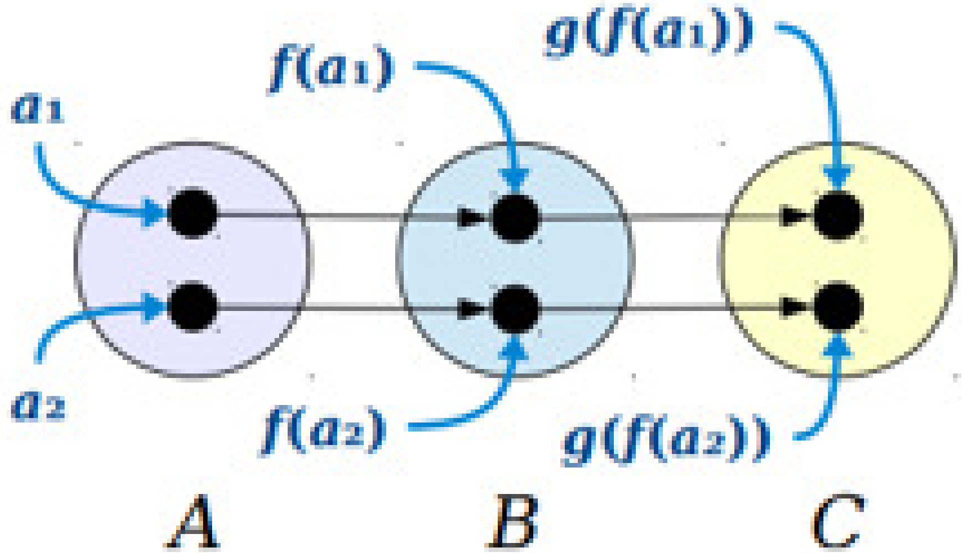
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



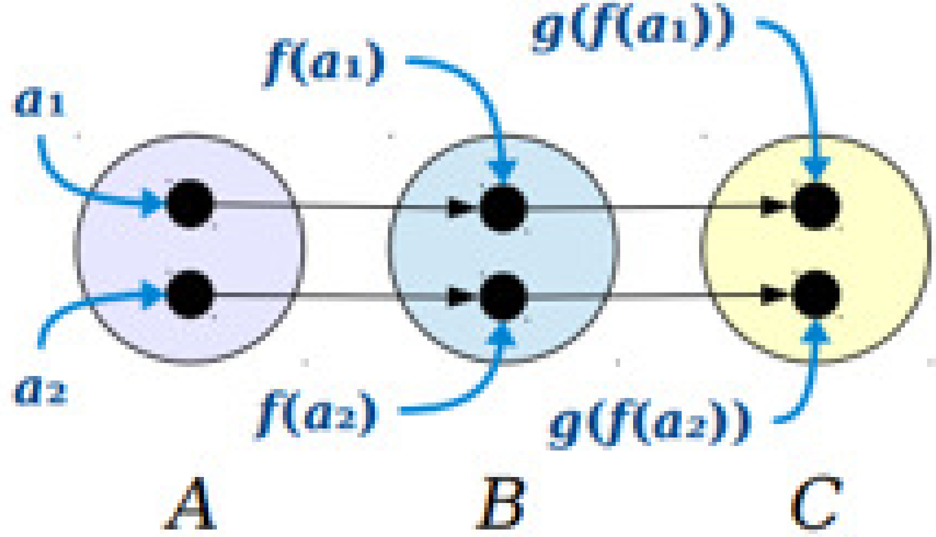
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.



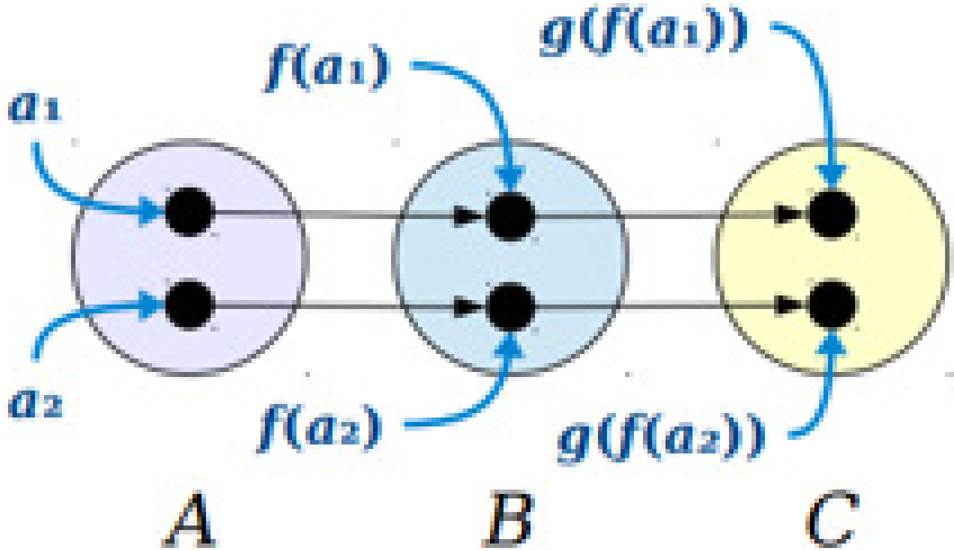
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$.



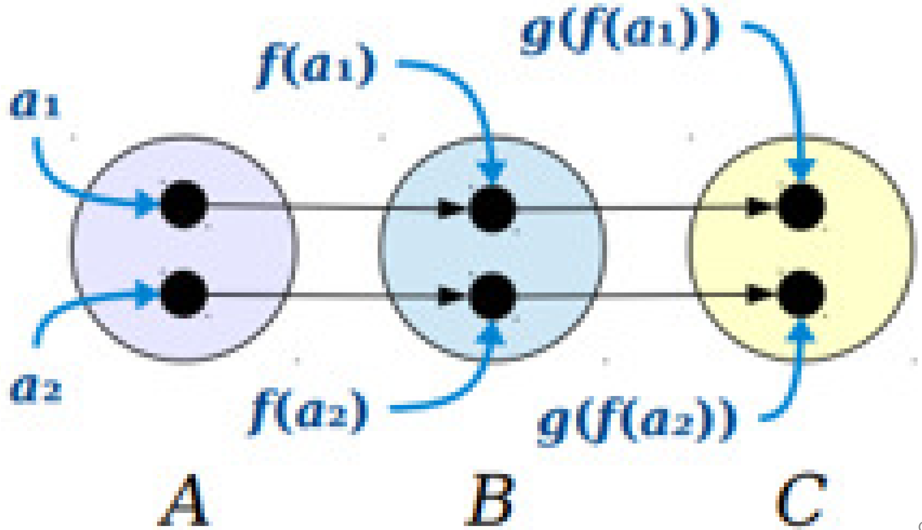
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

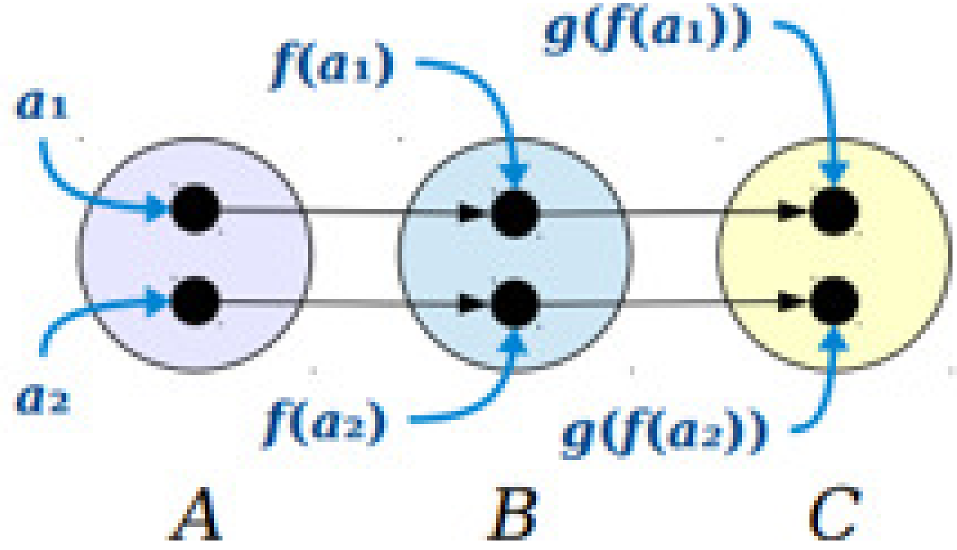
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

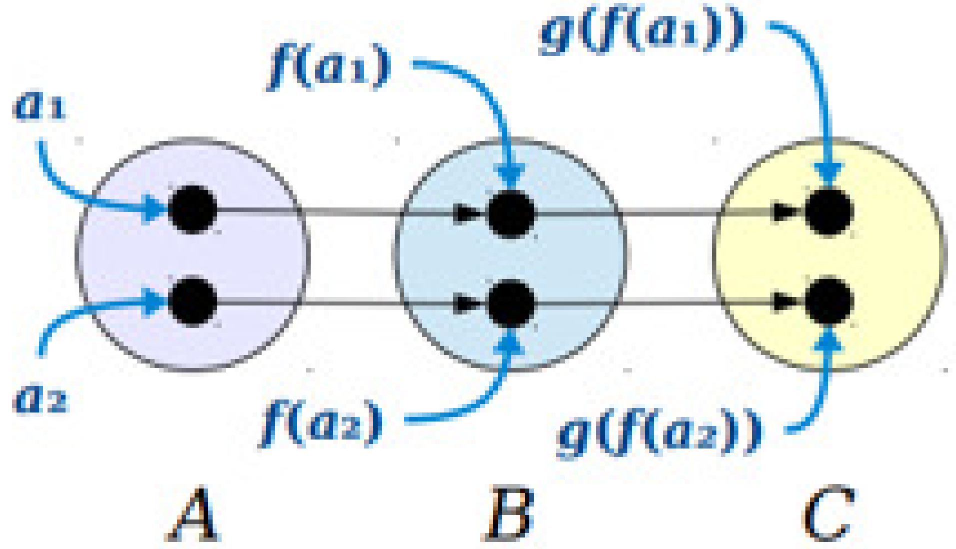
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof:

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

What does it mean for $g \circ f : A \to C$ to be surjective?

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

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$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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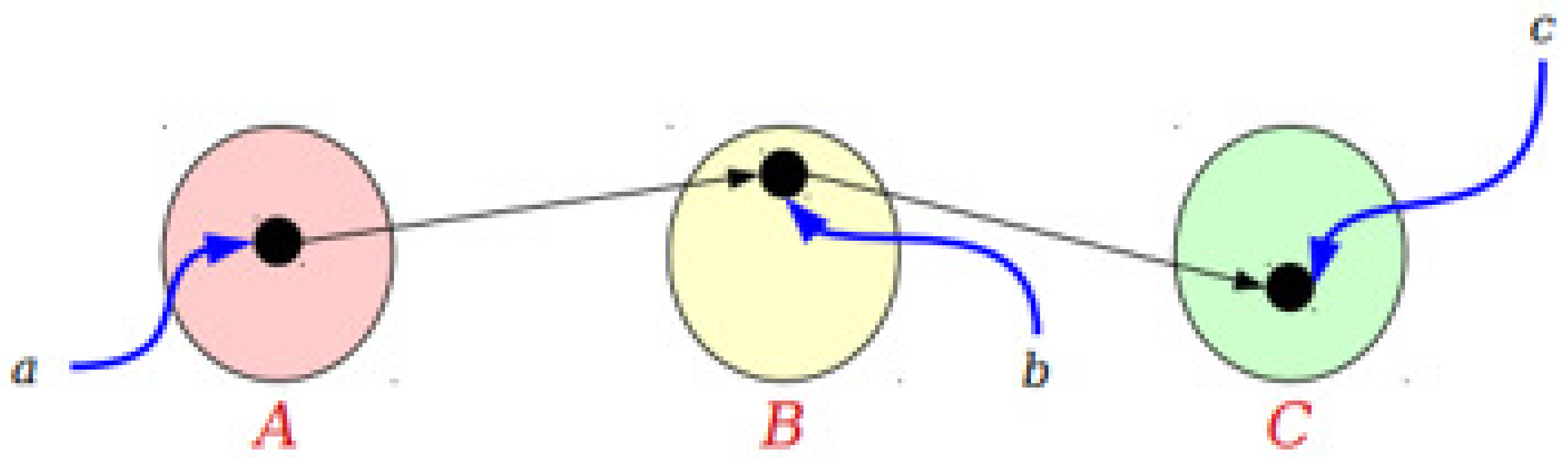
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

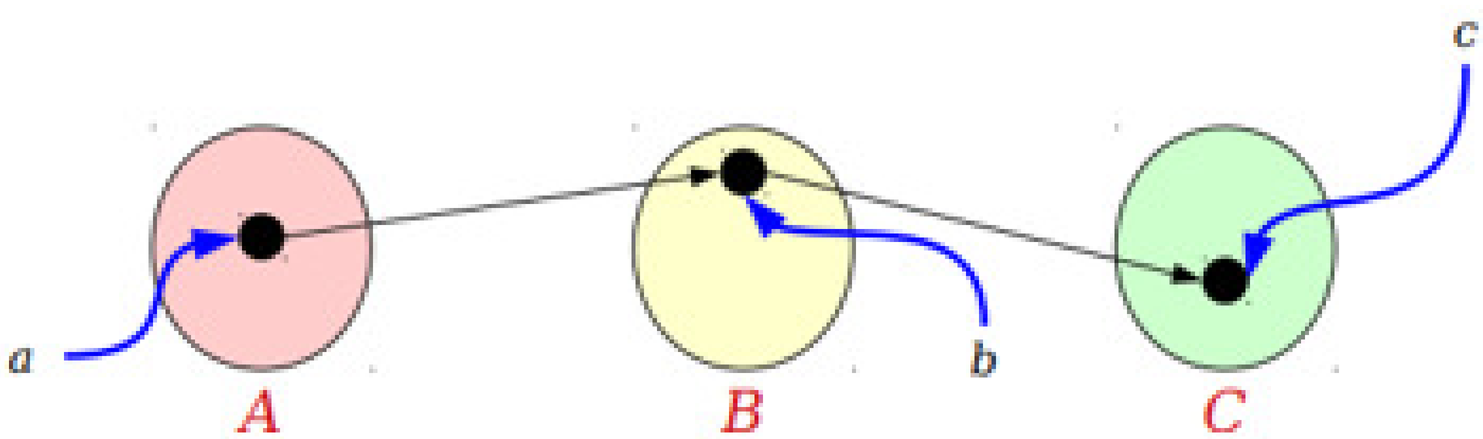
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.



Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$.



Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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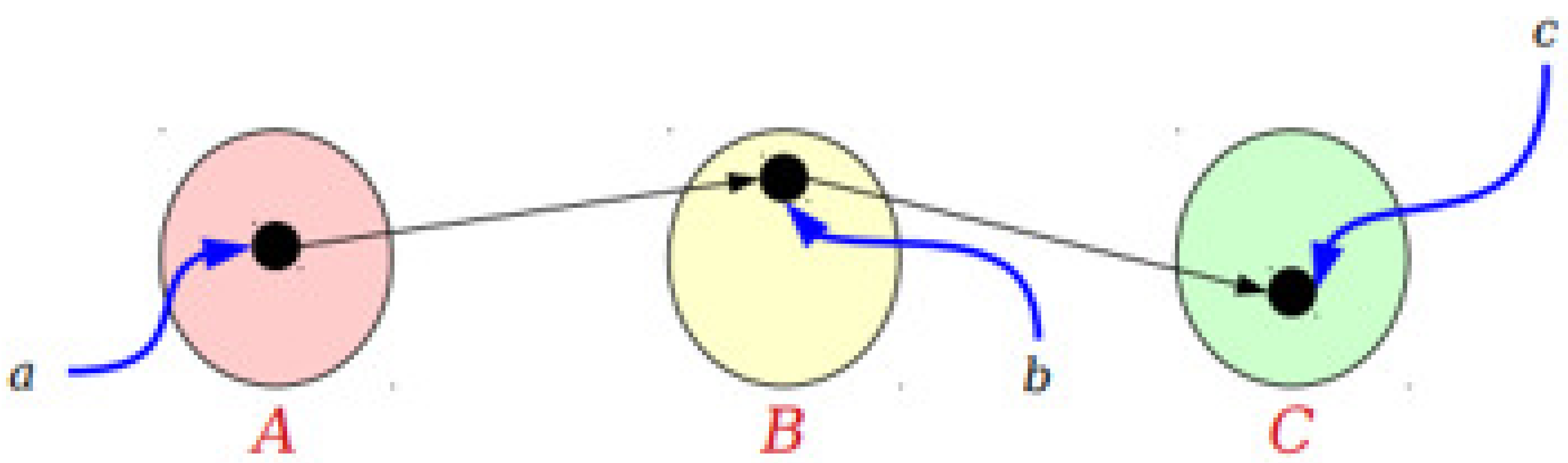
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$.



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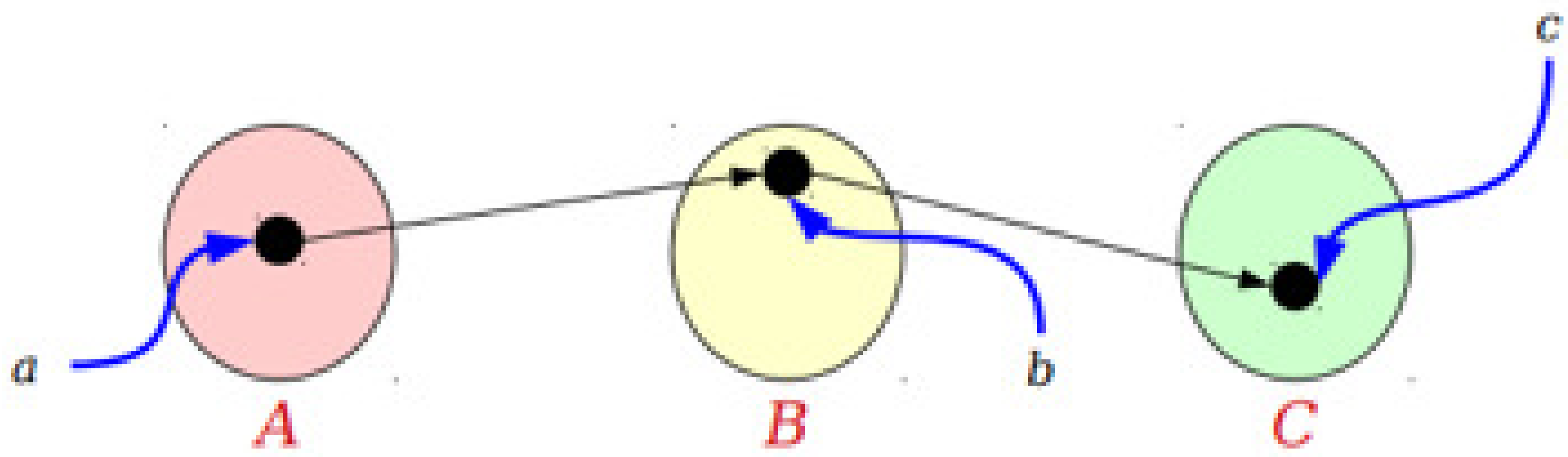
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



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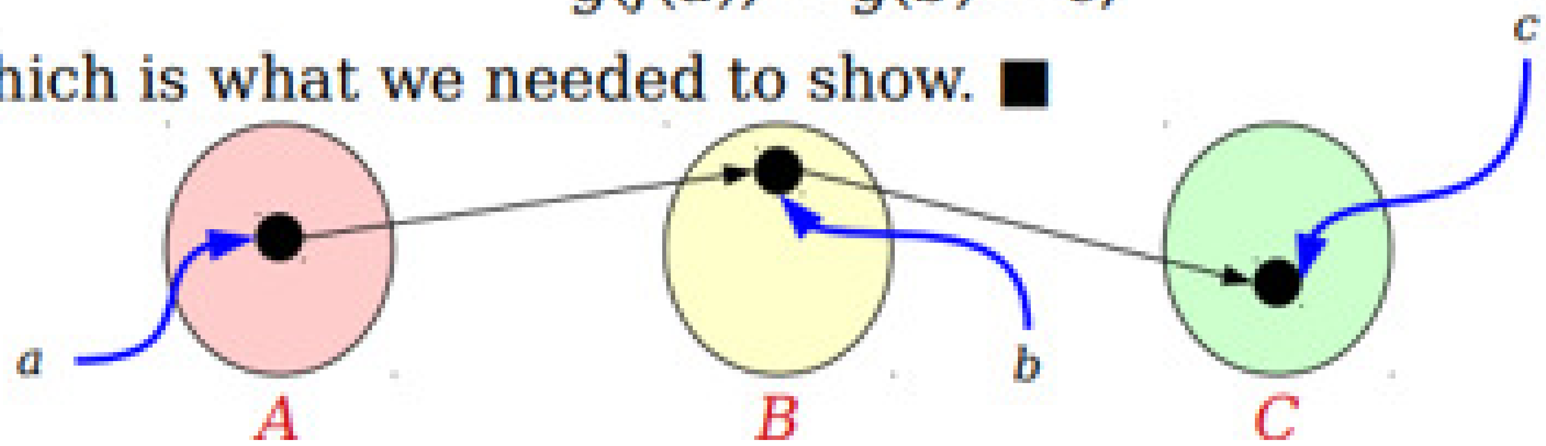
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■



Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?

Bijections

- A function that associates each element of the codomain with a unique element of the domain is called ***bijective***.
  - Such a function is a ***bijection***.
- Formally, a bijection is a function that is both *injective* and *surjective*.
- Bijections are sometimes called ***one-to-one correspondences***.
  - Not to be confused with “one-to-one functions.”

Bijections and Composition

- Suppose that $f : A \to B$ and $g : B \to C$ are bijections.
- Is $g \circ f$ necessarily a bijection?
- ***Yes!***
  - Since both f and g are injective, we know that $g \circ f$ is injective.
  - Since both f and g are surjective, we know that $g \circ f$ is surjective.
  - Therefore, $g \circ f$ is a bijection.

Cardinality

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

  $|S| = |T|$ if there exists a *bijection* $f : S \to T$

Properties of Cardinality

- For any sets A, B, and C, the following are true:
  - **$|A| = |A|$.**
    - Define $f : A \to A$ as $f(x) = x$.
  - **If $|A| = |B|$, then $|B| = |A|$.**
    - If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.
  - **If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.**
    - If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.

Big O

Big O

Big-O defines a set or class of functions with a similar upper bound.

Definition : Let f and g be functions whose domain and co-domain are subsets of the real numbers. Then,

$$f(x) \in O(g(x)) \leftrightarrow \exists c,k\ \forall x \geq k\ (f(x) \leq c \cdot g(x))$$

The notation $f(n)=O(g(x))$ is also often used to denote $f(x) \in O(g(x))$.

Big O

For functions f and g we write

$$f(x) = O(g(x))$$

to denote

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$

We say "f(x) is big O of g(x)"

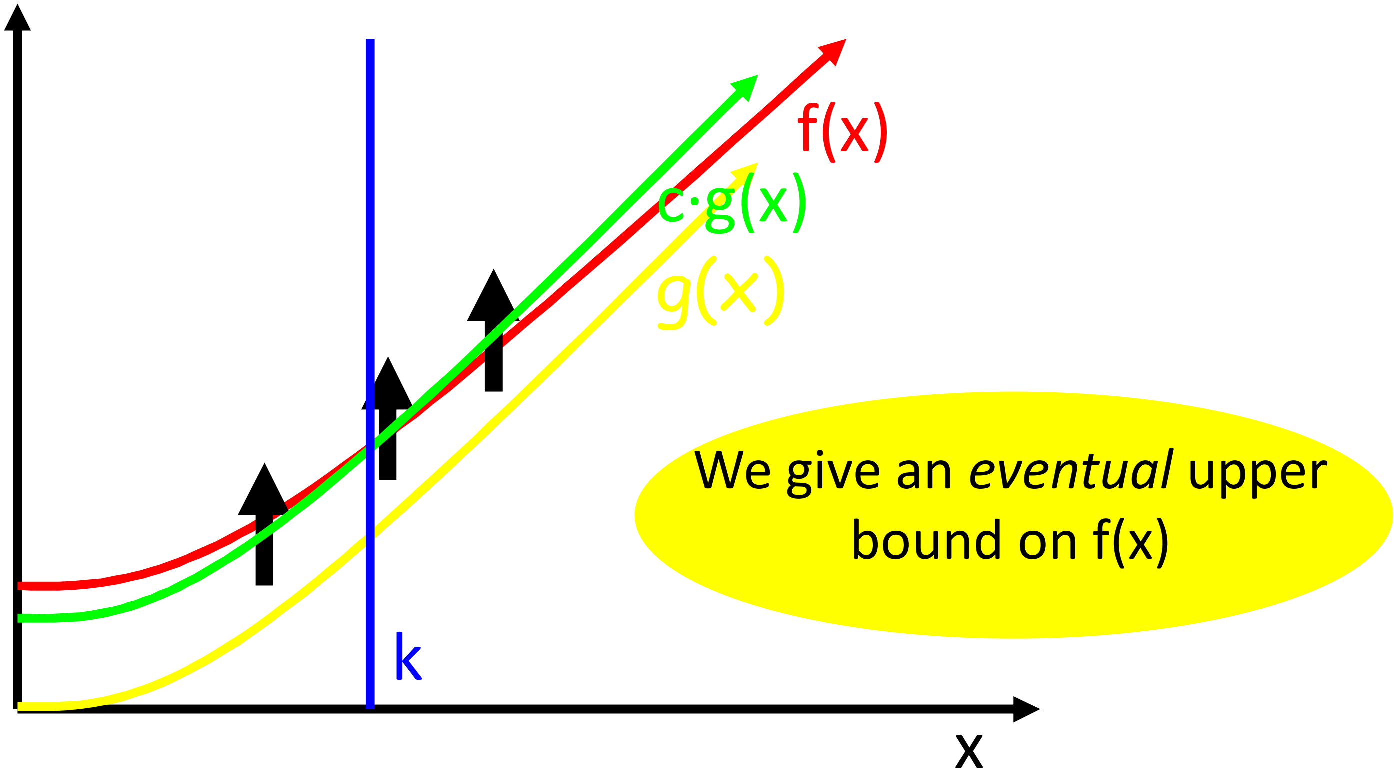
Recipe for proving f(x) = O(g(x)): find a c and k so that the inequality holds.

Big O

$$f(x) = O(g(x))$$

iff

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$



Big O : Example

$$f(x) = O(g(x))$$

iff

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$

$3n = O(15n)$ since $\forall\ n \geq 0$, $3n \leq 1 \cdot 15n$

There's k

There's c

Big O : Example

$$f(x) = O(g(x))$$

iff

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$

$15n = O(3n)$ since $\forall\ n \geq 0$ $15n \leq 5 \cdot 3n$

Big O : Example

$$f(x) = O(g(x))$$

iff

$\exists$ c,k so that $\forall$ $x \geq k$, $f(x) \leq c \cdot g(x)$

$x^2 = O(x^3)$ since $\forall$ $x \geq$ 1, $x^2 \leq x^3$

Big O : Example

$$f(x) = O(g(x))$$

iff

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$

$1000x^2 = O(x^2)$ *since* $\forall\ x \geq \underline{\ 0\ }$, $1000x^2 \leq \underline{\ 1000\ } \cdot x^2$

Big O : Example

$$f(x) = O(g(x))$$

iff

$$\exists\ c,k \text{ so that } \forall\ x \geq k,\ f(x) \leq c \cdot g(x)$$

Prove that $x^2 + 100x + 100 = O((1/100)x^2)$

$100x \leq 100x^2$

$x^2 + 100x + 100 \leq 201x^2$ when $x > 1$

$100 \leq 100x^2$

$\leq 20100 \cdot (1/100)x^2$

$k = 1$,
$c = 20100$

Big O : Example

Prove that $5x + 100 = O(x/2)$

Similar problem, different technique.

Need $\forall\ x \geq \underline{\qquad}$, $5x + 100 \leq \underline{\qquad} \cdot x/2$

Try c = 10

$\forall\ x \geq \underline{\qquad}$, $5x + 100 \leq 10 \cdot x/2$

Nothing works for k

Try c = 11

$\forall\ x \geq \underline{\qquad}$, $5x + 100 \leq 5x + x/2$

$\forall\ x \geq 200$, $100 \leq x/2$

k = 200,
c = 11

Big O

Guidelines:

- In general, only the largest term in a sum matters.

  $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x^1 + a_nx^0 = O(x^n)$

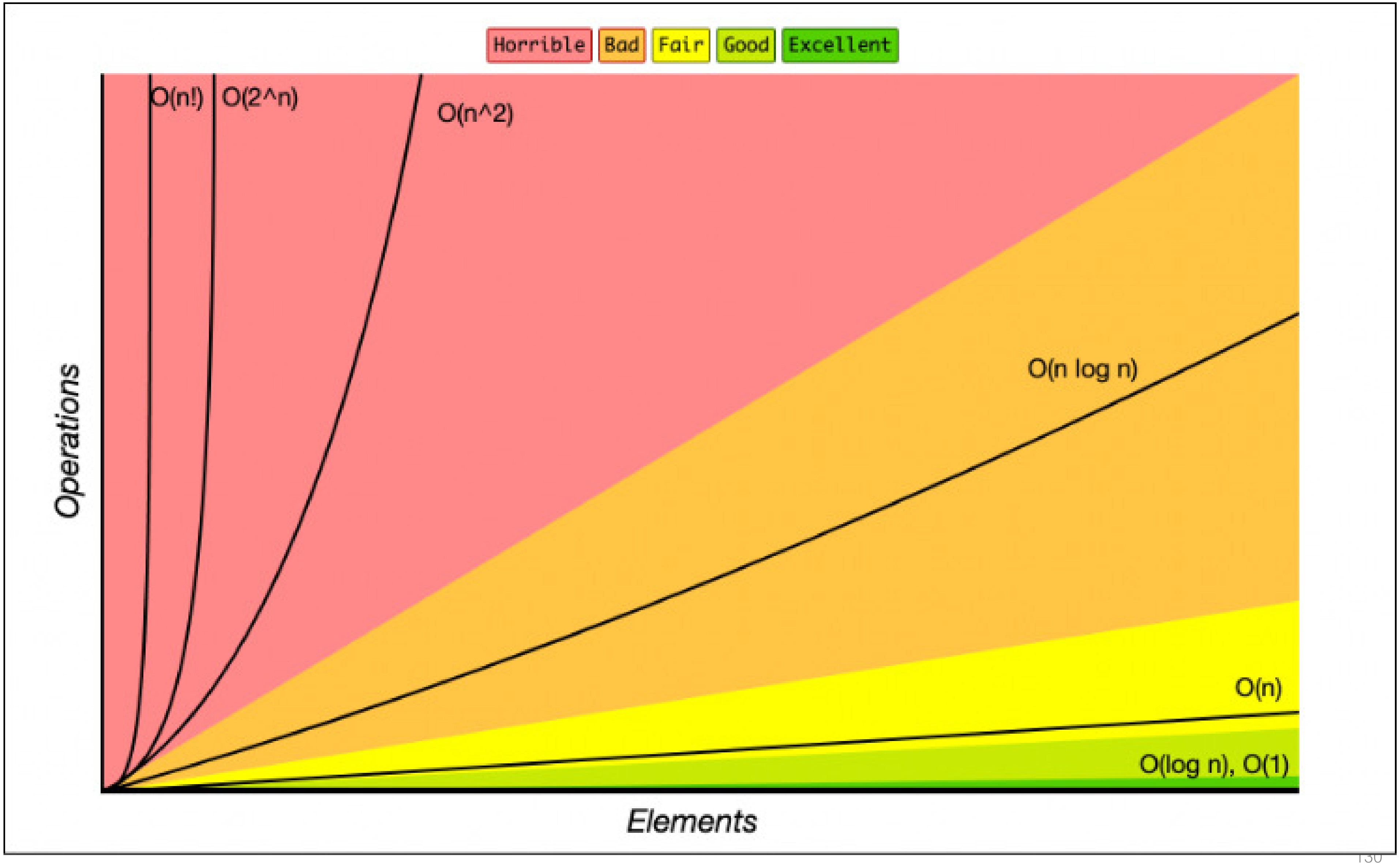
- n dominates lg n.

  $n^5 \lg n = O(n^6)$

- List of common functions in increasing O() order:

  $1 \quad n \quad (n \lg n) \quad n^2 \quad n^3 \quad \ldots \quad 2^n \quad n!$

Big O



Horrible
Bad
Fair
Good
Excellent
O(n!)
O(2^n)
O(n^2)
O(n log n)
O(n)
O(log n), O(1)
Operations
Elements

Big O

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \leq n \cdot n \cdot \ldots \cdot n \cdot n \cdot n = n^n$$

$$n! = O(n^n)$$

$$\log n! \leq \log n^n = n \log n$$

$$\log n! = O(n \log n)$$

Algorithm Analysis with Big O

What does the following algorithm compute?

who_knows($a_1, a_2, \ldots, a_n$: integers)
who_knows := 0
for i := 1 to n-1
  for j := i + 1 to n
    if $|a_i - a_j|$ > who_knows **then** who_knows := $|a_i - a_j|$
{who_knows is the maximum difference between any two numbers in the input sequence}

Comparisons: $(n-1) + (n-2) + (n-3) + \ldots + 1$
$$= (n-1)n/2 = 0.5n^2 - 0.5n$$

Time complexity is $O(n^2)$.

Algorithm Analysis with Big O

Another algorithm solving the same problem:

max_diff(a_1, a_2, …, a_n: integers)
min := a_1
max := a_1
for i := 2 to n
        if a_i < min **then** min := a_i
        else if a_i > max **then** max := a_i
max_diff := max - min

Comparisons: 2n - 2

Time complexity is O(n).

Algorithm Analysis with Big O

What is time complexity of the following algorithm?

```
mystery(n : positive integers)
  j := n^3
  while (j > 1)
        j := j/2
```

Number of comparisons k: $n^3 = 2^k$; $k = \log_2 n^3$

$= 3\log_2 n = (3/\log 2)\log n$

Time complexity is O(logn).

Computing Sums

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad\qquad 3 \times 10$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j$$

-

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

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Computing Sums: Tool 1: Constant Rule

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$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j \qquad j \times 10$$

$$S_3 = \sum_{i=1}^{10} i$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j \qquad j \times 10$$

$$S_3 = \sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j \qquad j \times 10$$

$$S_3 = \sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 \qquad \tfrac{1}{2} \times 10 \times (10+1)$$

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

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The *index of summation* is i in these examples.

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j \qquad j \times 10$$

$$S_3 = \sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 \qquad \tfrac{1}{2} \times 10 \times (10+1)$$

The *index of summation* is i in these examples.

Constants (independent of summation index) can be taken outside the sum.

$$S_1 = \sum_{i=1}^{10} 3 = 3\sum_{i=1}^{10} 1 = 3 \times 10 \qquad S_2 = \sum_{i=1}^{10} j = j\sum_{i=1}^{10} 1 = j \times 10.$$

Computing Sums: Tool 2: Addition Rule

$$S = \sum_{i=1}^{5}(i + i^2)$$

Computing Sums: Tool 2: Addition Rule

$$\begin{aligned} S &= \sum_{i=1}^{5}(i+i^2) \\ &= (1+1^2)+(2+2^2)+(3+3^2)+(4+4^2)+(5+5^2) \end{aligned}$$

Computing Sums: Tool 2: Addition Rule

$$
\begin{aligned}
S &= \sum_{i=1}^{5}(i+i^2) \\
&= (1+1^2)+(2+2^2)+(3+3^2)+(4+4^2)+(5+5^2) \\
&= (1+2+3+4+5)+(1^2+2^2+3^2+4^2+5^2) && \text{(rearrange terms)}
\end{aligned}
$$

Computing Sums: Tool 2: Addition Rule

$$
\begin{aligned}
S &= \sum_{i=1}^{5}(i + i^2) \\
&= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) + (4 + 4^2) + (5 + 5^2) \\
&= (1 + 2 + 3 + 4 + 5) + (1^2 + 2^2 + 3^2 + 4^2 + 5^2) && \text{(rearrange terms)} \\
&= \sum_{i=1}^{5} i + \sum_{i=1}^{5} i^2.
\end{aligned}
$$

Computing Sums: Tool 2: Addition Rule

$$
\begin{aligned}
S &= \sum_{i=1}^{5}(i+i^2) \\
&= (1+1^2)+(2+2^2)+(3+3^2)+(4+4^2)+(5+5^2) \\
&= (1+2+3+4+5)+(1^2+2^2+3^2+4^2+5^2) && \text{(rearrange terms)} \\
&= \sum_{i=1}^{5} i + \sum_{i=1}^{5} i^2.
\end{aligned}
$$

The sum of terms added together is the addition of the individual sums.

$$\sum_i (a(i)+b(i)+c(i)+\cdots) = \sum_i a(i) + \sum_i b(i) + \sum_i c(i) + \cdots$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

$$\sum_{i=1}^{n} f(x) = nf(x) \qquad \sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1) \qquad \sum_{i=0}^{n} \frac{1}{2^i} = 2-\frac{1}{2^n}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \; (r\neq 1) \qquad \sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2 \qquad \sum_{i=1}^{n} \log i = \log n!$$

Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\sum_{i=1}^{n}(1+2i+2^{i+2}) =$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

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Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\sum_{i=1}^{n}(1+2i+2^{i+2}) = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} \qquad \text{(addition rule)}$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

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$$\begin{aligned} \sum_{i=1}^{n}(1+2i+2^{i+2}) &= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} && \text{(addition rule)} \\ &= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 2^i && \text{(constant rule)} \end{aligned}$$

-

Computing Sums: Tool 3: Common Sums

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&= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 2^i && \text{(constant rule)}\\
&= n + 2\times\tfrac{1}{2}n(n+1) + 4\cdot(2^{n+1}-1-1) && \text{(common sums)}
\end{aligned}$$

-

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

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&= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 2^i && \text{(constant rule)}\\
&= n + 2\times\tfrac{1}{2}n(n+1) + 4\cdot(2^{n+1}-1-1) && \text{(common sums)}\\
&= n + n(n+1) + 2^{n+3} - 8 && \text{(common sums)}
\end{aligned}$$

Computing Sums: Tool 3: Nested Sum Rule

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1; \qquad S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1.$$

Computing Sums: Tool 3: Nested Sum Rule

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1; \qquad S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1.$$

To compute a nested sum, start with the innermost sum and proceed outward.

Computing Sums: Tool 3: Nested Sum Rule

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To compute a nested sum, start with the innermost sum and proceed outward.

$$S_1 \qquad \underset{(i=1)}{\sum_{j=1}^{3} 1} \qquad \underset{(i=2)}{\sum_{j=1}^{3} 1} \qquad \underset{(i=3)}{\sum_{j=1}^{3} 1}$$

Computing Sums: Tool 3: Nested Sum Rule

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1; \qquad S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1.$$

To compute a nested sum, start with the innermost sum and proceed outward.

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Practice: Compute a Formula for the Sum $\sum_{i=1}^{n} \sum_{j=1}^{i} ij$

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$$= \sum_{i=1}^{n} i \sum_{j=1}^{i} j \qquad \text{(constant rule)}$$

Practice: Compute a Formula for the Sum $\sum_{i=1}^{n} \sum_{j=1}^{i} ij$

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{i} ij &= \sum_{i=1}^{n} \sum_{j=1}^{i} ij && \text{(innermost sum)} \\
&= \sum_{i=1}^{n} i \sum_{j=1}^{i} j && \text{(constant rule)} \\
&= \sum_{i=1}^{n} i \times \tfrac{1}{2} i(i+1) && \text{(common sum)}
\end{aligned}
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\end{aligned}$$

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\end{aligned}$$

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&= \tfrac{1}{12} n + \tfrac{3}{8} n^2 + \tfrac{5}{12} n^3 + \tfrac{1}{8} n^4 && \text{(algebra)}
\end{aligned}
$$