Problem Set 5: Name Theepakorn Phayonrat ID 67011352

The definition of a Fibonacci number is as follows:
Basis: $f_0 = 0$ and $f_1 = 1$
Recursive: $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$
Use this definition for question 1 and 2.

1. Show that $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$ for $n \geq 1$.

2. Show that $f_1 + f_2 + f_3 + \ldots + f_{2n-1} = f_{2n}$ for $n \geq 1$.

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3. Show that the set S defined by

Basis step: $1 \in S$

Recursive step: $s + t \in S$ when $s \in S$ and $t \in S$

is the set of positive integers:

$Z^{+} = \{ 1, 2, 3, \dots \}$

4. Let N be given by

Basis: $1 \in N$

Recursive: if $x \in N$ then $x+1 \in N$.

Show that "$5^{n} - 1$ is divisible by 4".

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5. Use structural induction, to prove that the number of nodes in any full rooted binary tree is odd.

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Use this definition for question 1 and 2.

1. Show that $f_1^2+f_2^2+f_3^2+\ldots+f_n^2=f_nf_{n+1}$ for $n\geq 1$.

Proof: $\forall n\in\mathbb{Z}^+$; Let $P(n)$ be "$f_1^2+f_2^2+f_3^2+\ldots+f_n^2=f_nf_{n+1}$"

Base Case: $P(1): f_1^2=1^2$
$=1(1)$
$=f_nf_{n+1}$ which is true *

Induction Hypothesis: Assume $P(k)$: "$f_1^2+f_2^2+f_3^2+\ldots+f_k^2=f_kf_{k+1}$" is true.

Induction Step: $P(k)\to P(k+1): f_1^2+f_2^2+f_3^2+\ldots+f_k^2+f_{k+1}^2=f_kf_{k+1}+f_{k+1}^2$

$$=f_kf_{k+1}+(f_{k+2}-f_k)^2$$

$$=f_k(f_{k+2}-f_k)+(f_{k+2}^2-2f_{k+2}f_k+f_k^2)$$

$$=[(f_{k+2}f_k)-f_k^2]+(f_{k+2}^2-2f_{k+2}f_k+f_k^2)$$

$$=f_{k+2}^2-f_{k+2}f_k$$

$$=f_{k+2}(f_{k+1}+f_k)-f_{k+2}f_k$$

$$=f_{k+1}f_{k+2}$$ *

$\therefore$ "$\forall n\in\mathbb{Z}^+$; $f_1^2+f_2^2+f_3^2+\ldots+f_n^2=f_nf_{n+1}$" is true by Mathematical Induction *

2. Show that $f_1+f_3+f_5+\ldots+f_{2n-1}=f_{2n}$ for $n\geq 1$.

Proof: $\forall n\in\mathbb{Z}^+$; Let $R(n)$ be "$f_1+f_3+f_5+\ldots+f_{2n-1}=f_{2n}$"

Base Case: $R(1): f_{2(1)-1}=f_1$
$=1$
$=f_2$ which is true *

Induction Hypothesis: Assume $P(k)$: "$f_1+f_3+f_5+\ldots+f_{2k-1}=f_{2k}$" is true.

Induction Step: $P(k)\to P(k+1): f_1+f_3+f_5+\ldots+f_{2k-1}+f_{2(k+1)-1}=f_{2k}+f_{2k+1}$

$$=f_{2k+2}=f_{2(k+1)}$$ *

$\therefore$ "$\forall n\in\mathbb{Z}^+$; $f_1+f_3+f_5+\ldots f_{2n-1}=f_{2n}$" is true by Mathematical Induction *

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3. Show that the set S defined by

Basis step: $1 \in S$

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is the set of positive integers:

$Z^+ = \{ 1, 2, 3, \dots \}$

Proof: Proof by structural Induction

Basis Step: $1 \in \mathbb{Z}^+$ which is true

Inductive Hypothesis: Assume $k \in S$

Recursive Step: $P(k) \to P(k+1)$: Show that $k+1 \in S$

$k \in S$ by IH

$1 \in S$ by Basis Step

$k+1 \in S$ by Recursive step of recursive definition above.

$\therefore$ Set S is the set of $\mathbb{Z}^+$

4. Let N be given by

Basis: $1 \in N$

Recursive: if $x \in N$ then $x+1 \in N$.

Show that "$5^n - 1$ is divisible by 4".

Proof: Proof by structural Induction. $\forall n \in N$, Let $P(n)$ be "$4 | 5^n - 1$"

Base Case: $P(1): 5^1 - 1 = 4$

$= 4(1)$; $1 \in \mathbb{Z}$

$\therefore 4 | 5^1 - 1$ which is true

Inductive Hypothesis: Assume $P(k)$: "$4 | 5^k - 1$" is true.

Inductive Step: $P(k) \to P(k+1)$: $5^{k+1} - 1 = (5 \cdot 5^k) - 1$

$= (4 \cdot 5^k) + 5^k - 1$

$= 4 \cdot 5^k + 4m$; $m \in \mathbb{Z}$ because $4 | 5^k - 1$

$= 4n + 4m$; $n \in \mathbb{Z}$ because $5^k \in \mathbb{Z}$

$= 4(n + m)$

$= 4\ell$; $\ell \in \mathbb{Z}$

$\therefore 4 | 5^{k+1} - 1$

$\therefore \forall n \in \mathbb{Z}^+$; "$4 | 5^n - 1$" is true by Structural Induction

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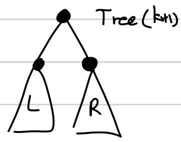
5. Use structural induction, to prove that the number of nodes in any full rooted binary tree is odd.

Proof: Proof by structural Induction.

Rephrase: $\forall n \in \mathbb{N}$; "Size(n) is odd"

Base Case: P(0) ; From Tree (0) : Size(0) = 1 which is odd.



Inductive Hypothesis: Assume P(k) : Size(k) is odd or size(k) = 2x+1

Inductive Step: P(k) → P(k+1) : From Tree(k+1); Size(k+1) = 1 + L+R

$= 1+(2m+1)+(2n+1)$; because L and R are odd

$= 2(m+n+1)+1$

$= 2l+1$; $l = (m+n+1) \in \mathbb{Z}$

∴ Size(k+1) is odd

∴ "$\forall n \in \mathbb{N}$; Size(n) is odd"

∴ "The number of nodes in any full rooted binary tree is odd" is true by Induction