Problem Set 4: Discrete Mathematics

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1. Use the Principle of Mathematical Induction to prove that

$$1-2+2^2-2^3+...+(-1)^n 2^n = \frac{2^{n+1}(-1)^n+1}{3}$$ for all positive integers n.

2. Use the Principle of Mathematical Induction to prove that $1 + 2^n \leq 3^n$ for all $n \geq 1$.

3. Use the Principle of Mathematical Induction to prove that $1+3+9+27+...+3^n = \dfrac{3^{n+1}-1}{2}$ for all $n \geq 0$.

4. Prove that $\sum_{j=n}^{2n-1}(2j+1) = 3n^2$ for all positive integers n.

5. Prove that for any positive integer n, the number $2^{2n}-1$ is divisible by 3.

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6. Prove that $x^0 + x^1 + ... + x^n = (x^{n+1} - 1)/(x - 1)$ for all integers $n \geq 0$, using induction.

7. For each natural number $n \geq 1$, the n^{th} Fibonacci number, F_n, is defined inductively by

$F_1=1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Prove that $F_{n+6} = 4F_{n+3} + F_n$ for $n > 0$.

8. Prove that for any $n \geq 1$, $\Sigma_{i=1}^{n} (i^2) = n(n+1)(2n+1)/6$

9. Prove that $n! > 2^n$ for all $n \geq 4$

10. Show that $\Sigma_{i=1}^{n} (i^3) = (\Sigma_{i=1}^{n} i)^2$ for all $n \geq 1$

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1. Use the Principle of Mathematical Induction to prove that $1-2+2^2-2^3+\ldots+(-1)^n 2^n = \dfrac{2^{n+1}(-1)^n+1}{3}$ for all positive integers n.

Proof

Let $P(n)$ be the statement " $\forall n \in \mathbb{Z}^+$, $1-2+2^2-2^3+\ldots+(-1)^n 2^n = \dfrac{2^{n+1}(-1)^n+1}{3}$

We need to show that $P(n)$ is true for all $n \in \mathbb{Z}^+$

Base Case

$P(1)$: $1-2^1 = -1$, $\dfrac{2^{1+1}(-1)^1+1}{3} = -\dfrac{3}{3} = -1$ } True

Induction Hypothesis

$P(k)$: Assume is true

Induction Step

$$
\begin{aligned}
P(k) \Rightarrow P(k+1):\ 1-2+2^2-2^3+\ldots+(-1)^{k+1}(2)^{k+1} &= 1-2+2^2-2^3+\ldots+(-1)^k 2^k + (-1)^{k+1} 2^{k+1} \\
&= \frac{2^{k+1}(-1)^k+1}{3} + (-1)^{k+1} 2^{k+1} \\
&= \frac{2^{k+1}(-1)^k+1}{3} + \frac{3(-1)^{k+1} 2^{k+1}}{3} \\
&= \frac{\left(2^{k+1}(-1)^k+1\right)+3(-1)^{k+1} 2^{k+1}}{3} \ \#
\end{aligned}
$$

$\therefore \forall n \in \mathbb{Z}^+$; $1-2+2^2-2^3+\ldots+(-1)^n 2^n = \dfrac{(2^{n+1})(-1)^n+1}{3}$ is true by Mathematical Induction. #

2. Use the Principle of Mathematical Induction to prove that $1 + 2^n \le 3^n$ for all $n \ge 1$.

Proof

$\forall n \in \mathbb{Z}^+$; Let $P(n)$ be $1+2^n \leq 3^n$

Base Case

$P(1)$: $1+2^1 \leq 3^1$

$3 \leq 3$ ← True

Inductive Hypothesis

$P(k)$: $1+2^k \leq 3^k$

Inductive Step

$$
\begin{aligned}
P(k) \Rightarrow P(k+1) : (1+2^{k+1})+2^k &\leq 3^k+2^k \\
&\leq 3^k+3^k \\
&= 2 \cdot 3^k \\
&< 3 \cdot 3^k \\
&= 3^{k+1} \ \#
\end{aligned}
$$

$\therefore \forall n \in \mathbb{Z}^+$; $1+2^n \leq 3^n$ is true by Mathematical Induction #

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3. Use the Principle of Mathematical Induction to prove that $1+3+9+27+\ldots+3^n = \frac{3^{n+1}-1}{2}$ for all $n \geq 0$.

Proof Let $P(n)$ be

Base Case $P(0): 1 = \frac{3^{0+1}-1}{2}$

$1 = 1 \leftarrow$ True

Induction Hypothesis $P(k): 1+3+3^2+3^3+\ldots+3^k = \frac{3^{k+1}-1}{2}$

Inductive Step $P(k) \rightarrow P(k+1): 1+3+3^2+3^3+\ldots 3^{k+1} = 1+3+3^2+3^3\ldots 3^k + 3^{k+1} = \frac{3^{k+1}-1}{2} + 3^{k+1}$

$= \frac{(3^{k+1}-1)+2(3^{k+1})}{2}$

$= \frac{3^{k+2}-1}{2} = \frac{3^{(k+1)+1}-1}{2}$ #

$\therefore 1+3+9+27+\ldots+3^n = \frac{3^{n-1}+1}{2}$ is true by Mathematical Induction #

4. Prove that $\sum_{j=n}^{2n-1}(2j+1) = 3n^2$ for all positive integers n.

Proof

Base Case $P(1): \sum_{1}^{1}(2(1)+1) = 3(1)^2$

$3 = 3 \leftarrow$ True #

Induction Hypothesis $P(k) = \sum_{j=k}^{2j-1}(2j+1) = 3(k)^2$

Induction Step $P(k) \rightarrow P(k+1): \sum_{j=k+1}^{2j-1}(2j+1) = \sum_{j=k}^{2j-1}(2j+1) - (2k+1) + (4k-1) + (4k+3)$

$= 3k^2 - 6k + 3$

$= 3(k+1)^2$ #

$\therefore \forall n \in \mathbb{Z}^+; \sum_{j=n}^{2n-1}(2j-1) = 3n^2$ is true by Mathematical Induction #

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5. Prove that for any positive integer n, the number 2^{2n}-1 is divisible by 3.

Proof — Let $P(n)$

Base Case — $P(1)$ $2^{2(1)}-1=3(1)$, $1\in\mathbb{Z}$ which is true.

Induction Hypothesis — Assume $P(k)$: $2^{2k}-1=3(m)$, $m\in\mathbb{Z}$ is true

Induction Step — $P(k)\to P(k+1)$ $2^{2(k+1)}-1=2^{2k+2}-1$

$=2^2(3m+1)-1$

$=12m$

$=3(4m)$

$=3(x)$; $\exists x=4m\in\mathbb{Z}$

$\therefore 3\mid 2^{2n}-1$ #

$\therefore \forall n\in\mathbb{Z}^+$; $3\mid 2^{2n}-1$ by Mathematical Induction #

6. Prove that $x^0+x^1+\ldots+x^n=(x^{n+1}-1)/(x-1)$ for all integers $n\geq 0$, using induction.

Proof — $\forall n\in\mathbb{N}$, Let $P(n)$ be "$x^0+x^1+\ldots x^n=\frac{x^{n+1}-1}{x-1}$"

Base Case — $P(0)=x^0=\frac{x^{0+1}-1}{x-1}$

$=\frac{x^1-1}{x-1}$

$=1=x^0$ which is true. *

Induction Hypothesis : Assume $P(k)$: $x^0+x^1+\ldots+x^k=\frac{x^{k+1}-1}{x-1}$; $\forall k\in\mathbb{N}$ is true.

Induction Step : $P(k)\to P(k+1)$: $x^0+x^1+\ldots+x^k+x^{k+1}=\frac{x^{k+1}-1}{x-1}+x^{k+1}$

$=\frac{(x^{k+1}-1)+(x^{k+1})(x-1)}{x-1}$

$=\frac{(\cancel{x^{k+1}}-1)+(x^{k+2}-\cancel{x^{k+1}})}{x-1}$

$=\frac{x^{k+2}-1}{x-1}=\frac{x^{(k+1)+1}-1}{x-1}$ #

$\therefore \forall n\in\mathbb{N}$; $x^0+x^1+\ldots+x^n=\frac{x^{n+1}-1}{x-1}$ is true by Mathematical Induction #

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7. For each natural number $n \geq 1$, the n^{th} Fibonacci number, F_n, is defined inductively by

$F_1=1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Prove that $F_{n+6} = 4F_{n+3} + F_n$ for $n > 0$.

$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ F_8 = 21,\ F_9 = 34,\ F_{10} = 55$

Proof: $\forall n \in \mathbb{Z}^+$; Let $P(n)$ be " $F_{n+6} = 4F_{n+3} + F_n$

Base Case: $P(1)$: $F_{1+6} = F_7 = 13 = 4(3) + 1$

$= 4(F_3) + F_1$ #

Induction Hypothesis: Assume that $P(1), P(2), P(3), \ldots P(k)$ are true.

Induction Step: $P(k) \rightarrow P(k+1)$:

$$
\begin{aligned}
F_{(k+1)+6} &= F_{k+7} \\
&= F_{k+5} + F_{k+6} \\
&= 2F_{k+5} + F_{k+4} \\
&= 3F_{k+4} + 2F_{k+3} \\
&= F_{k+4} + 4F_{k+3} + 2F_{k+2} \\
&= F_{k+4} + (F_{k+6} - F_k) + 2F_{k+2} \\
&= F_{k+5} + 2F_{k+4} + 2F_{k+2} - F_k \\
&= 3F_{k+4} + F_{k+3} + 2F_{k+2} - F_k \\
&= 4F_{k+4} + F_{k+2} - F_k \\
&= 4F_{k+4} + F_{k+1} \quad \# \\
&= 4F_{(k+1)+3} + F_{k+1}
\end{aligned}
$$

$\therefore$ " $F_{n+6} = 4F_{n+3} + F_n$ " is true by Strong Mathematical Induction #

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8. Prove that for any $n \geq 1$, $\Sigma_{i=1}^{n} (i^2) = n(n+1)(2n+1)/6$

Proof $\forall n \in \mathbb{Z}^+$; Let $P(n)$ be " $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ "

Base Case $P(1)$: $1^2 = 1$ which is true

Induction Hypothesis : Assume $P(k)$: $1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

Induction Step : $P(k) \Rightarrow P(k+1)$: $1+^2+3^2+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$$= \frac{[2k^3+3k^2+k]+[6k^2+12k+6]}{6}$$

$$= \frac{2k^3+9k^2+13k+6}{6}$$

$$= \frac{(k^2+3k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

$\therefore \forall n \in \mathbb{Z}^+$; $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true by Mathematical Induction

9. Prove that $n! > 2^n$ for all $n \geq 4$

Proof $\forall n\ \mathbb{Z}^+$; Let $P(n)$ be " $n! > 2^n$ for all $n \geq 4$ "

Base Case $P(4)$: $4! = 24$

> 16

$= 16$ which is true

Induction Hypothesis : Assume $P(k)$: " $k! > 2^k$ " is true.

Induction Step $P(k) \Rightarrow P(k+1)$: $(k+1)! > (k+1)k!$

$> 2k!$

$> 2(2^k)$

$= 2^{k+1}$

$\therefore$ " $\forall n \in \mathbb{Z}^+$; $n! > 2^n$ for all $n \geq 4$ " is true by Mathematical Induction

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10. Show that $\Sigma_{i=1}^{\ n} (i^3) = (\Sigma_{i=1}^{\ n} i)^2$ for all $n \geq 1$

Proof: $\forall n \in \mathbb{Z}^+$; Let $P(n)$ be " $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ "

Base Case: $1^3 = (1^2)^2$ which is true #

Induction Hypothesis: Assume $P(k)$: " $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ " is true

Induction Step: $P(k) \rightarrow P(k+1)$: $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

From 04 DM Induction Very First Pages.

$$= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4}$$

$$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2[(k+1)+1]^2}{2^2}$$

$$= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2 \#$$

$\therefore \forall n \in \mathbb{Z}^+$; " $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ " is true by Mathematical Induction #