Generalization of Cube and High-Order Odd Roots

Given the expression

$$x = \sqrt[n]{a}$$

we present two methods to compute this root numerically.

Newton-Raphson method

The Newton-Raphson iteration is given by:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

For the equation

$$x = \sqrt[n]{a}$$

we rewrite it as

$$x^n = a \to x^n - a = 0$$

Since $f'(x_k) = nx_k^{n-1}$, the Newton–Raphson update becomes

$$x_{k+1} = x_k - \frac{x_k^n - a}{nx_k^{n-1}}.$$

Rearranging terms gives the well-known iteration:

$$x_{k+1} = \frac{1}{n}\left[(n-1)x_k + \frac{a}{x_k^{n-1}}\right].$$

This method applies for any integer $(n = 1, 2, 3, 4, \dots)$ and is iterative for $(k = 1, 2, 3, 4, 5, 6, \dots)$

Sign and Absolute Value Considerations

For odd n, the expression

$$x = \sqrt[n]{a}$$

satisfies the sign rule

$$\begin{aligned} -\sqrt[n]{|a|}, &\quad a < 0, \\ +\sqrt[n]{|a|}, &\quad a > 0, \\ 0, &\quad a = 0. \end{aligned}$$

Thus we observe the identity

$$\sqrt[n]{a} = \operatorname{sgn}(a)\ \sqrt[n]{|a|}.$$