Non-regular Languages §1.4

Which languages cannot be recognized by finite automata?

Example: $L=\{ 0^n1^n \mid n \in \mathbb{N} \}$

- 'Playing around' with FA convinces you that the 'finiteness' of FA is problematic for "all $n \in \mathbb{N}$"
- The problem occurs between the 0^n and the 1^n

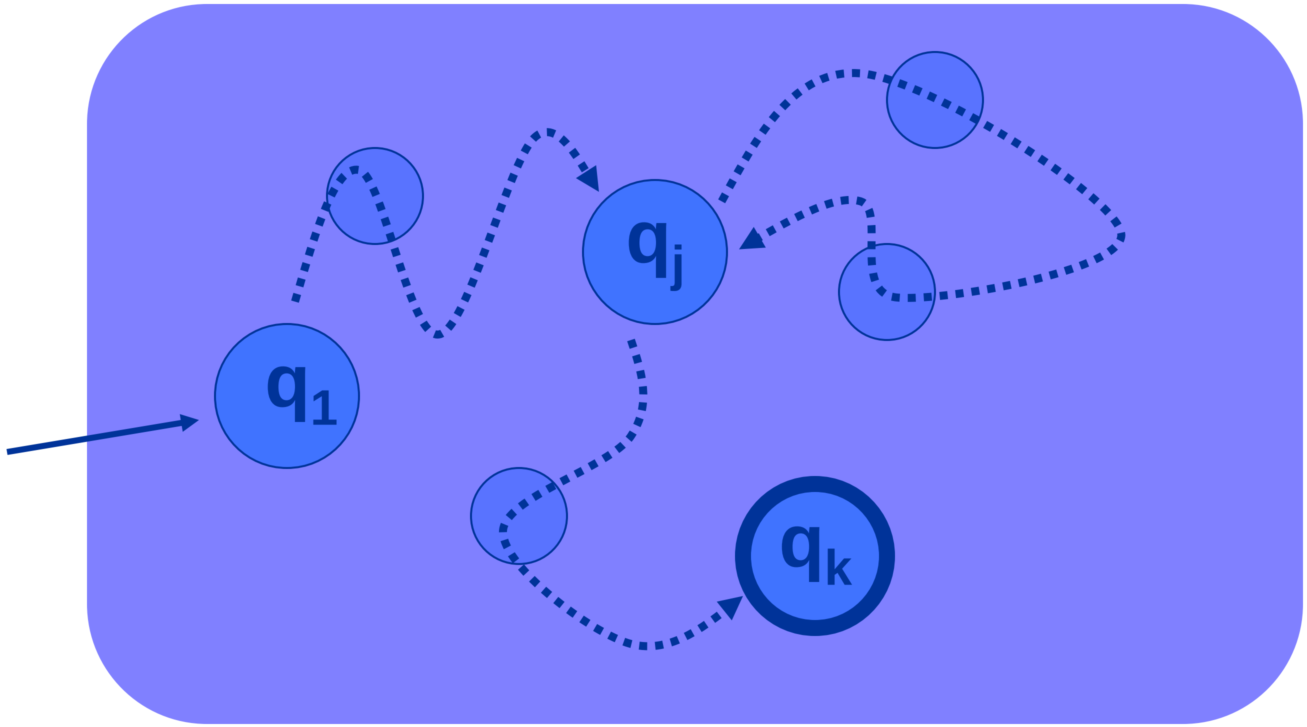
- Informal: the memory of a FA is limited by the the number of states $|Q|$

Proving non-regularity

- Prove a general statement -- NO DFA exists for a given problem.
- Cannot assume an automaton structure or a specific strategy
- Need an argument that holds for ALL DFA's

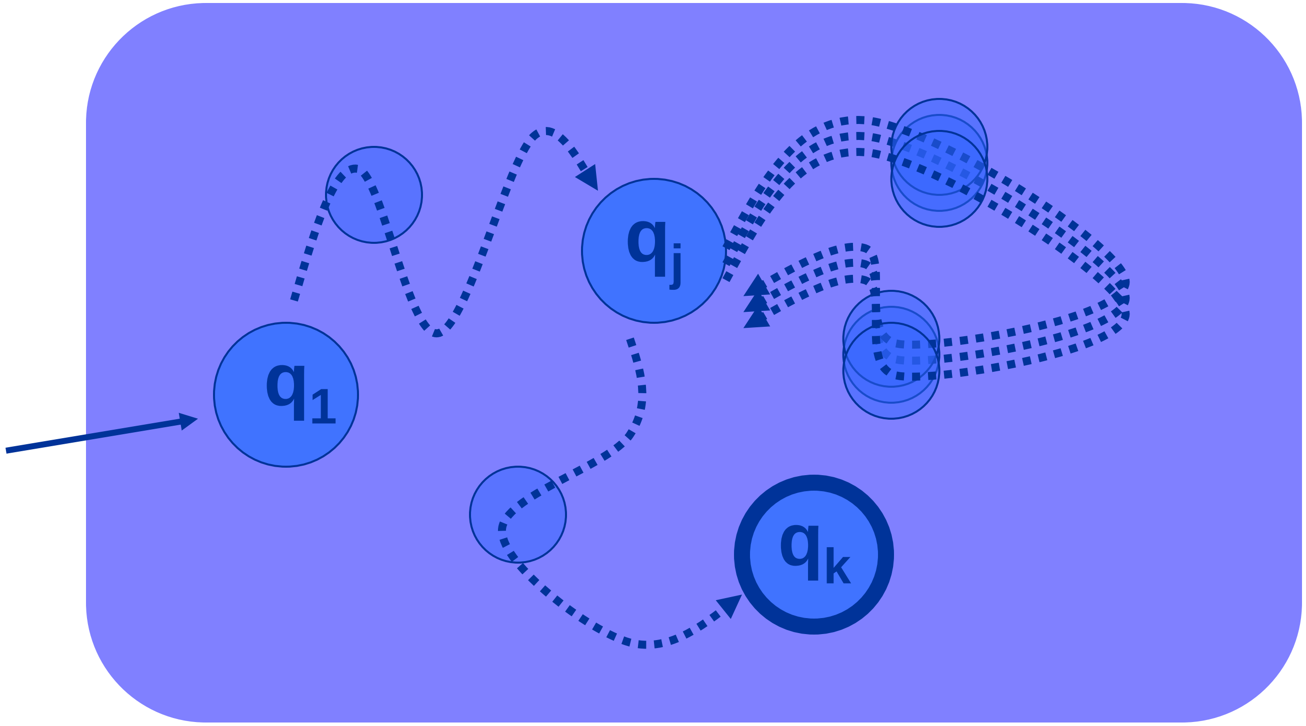
Repeating DFA Paths

Consider an accepting DFA M with size |Q|
On a string of length p, p+1 states get visited
For p≥|Q|, there must be a j such that the computational path looks like: $q_1,\ldots,q_j,\ldots,q_j,\ldots,q_k$



Repeating DFA Paths

The action of the DFA in q_j is always the same. If we repeat (or ignore) the $q_j, \ldots, q_j$ part, the new path will again be an accepting path



Line of Reasoning

Proof by contradiction:

- Assume that L is regular
- Hence, there is a DFA M that recognizes L
- For strings of length $\geq |Q|$ the DFA M has to 'repeat itself'
- Show that M will accept strings outside L
- Conclude that the assumption was wrong

Note that we use the simple DFA, not the more elaborate (but equivalent) NFA or GNFA

Pumping Lemma (Thm 1.37)

For every regular language L, there is a pumping length p, such that for any string $s \in L$ and $|s| \geq p$, we can write s=xyz with

1) $x\,y^i z \in L$ for every $i \in \{0,1,2,\ldots\}$
2) $|y| \geq 1$
3) $|xy| \leq p$

Note that 1) implies that $xz \in L$
2) says that y cannot be the empty string ε
Condition 3) is not always used

Formal Proof of Pumping Lemma

Let $M = (Q,\Sigma,\delta,q_1,F)$ with $Q = \{q_1,\ldots,q_p\}$
Let $s = s_1\ldots s_n \in L(M)$ with $|s| = n \geq p$
Computational path of M on s is the sequence $r_1,\ldots,r_{n+1} \in Q^{n+1}$ with
$r_1 = q_1$, $r_{n+1} \in F$ and $r_{t+1} = \delta(r_t,s_t)$ for $1 \leq t \leq n$
Because $n+1 \geq p+1$, there are two states such that $r_j = r_k$ (with $j<k$ and $k \leq p+1$)
Let $x = s_1\ldots s_{j-1}$, $y = s_j\ldots s_{k-1}$, and $z = s_k\ldots s_{n+1}$
x takes M from $q_1=r_1$ to r_j, y takes M from r_j to r_j, and z takes M from r_j to $r_{n+1} \in F$
As a result: xy^iz takes M from q_1 to $r_{n+1} \in F$ $(i \geq 0)$

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$r_1 = q_1$, $r_{n+1} \in F$ and $r_{t+1} = \delta(r_t,s_t)$ for $1 \leq t \leq n$

Because $n+1 \geq p+1$, there are two terms such that $r_j = r_k$ (w $|y| \geq 1$ and $|xy| \leq p$

Let $x = s_1\ldots s_{j-1}$, $y = s_j\ldots s_{k-1}$, and $z = s_k\ldots s_{n+1}$

x takes M from $q_1 = r_1$ to r_j, y takes M from r_j to r_j, and z takes M from r_j to $r_{n+1} \in F$

As a result: $x\,y^i z \in L(M)$ for every $i \in \{0,1,2,\ldots\}$ 0)

Pumping 0^n1^n (Ex. 1.38)

Assume that B = $\{0^n1^n \mid n \geq 0\}$ is regular

Let p be the pumping length, and $s = 0^p1^p \in B$

P.L.: $s = xyz = 0^p1^p$, with $xy^iz \in B$ for all $i \geq 0$

Three options for y:

1) $y=0^k$, hence $xyyz = 0^{p+k}1^p \notin B$
2) $y=1^k$, hence $xyyz = 0^p1^{k+p} \notin B$
3) $y=0^k1^l$, hence $xyyz = 0^p1^l0^k1^p \notin B$

Conclusion: The pumping lemma does not hold, the language B is not regular.

Another example

$F = \{ ww \mid w \in \{0,1\}^* \}$ (Ex. 1.40)

Let p be the pumping length, and take $s = 0^p10^p1$
Let $s = xyz = 0^p10^p1$ with condition 3) $|xy| \leq p$
Only one option: $y=0^k$, with $xyyz = 0^{p+k}10^p1 \notin F$

Without 3) this would have been a pain.

Intersecting Regular Languages

Let C = { w | # of 0s in w equals # of 1s in w}

Problem: If $xyz \in C$ with $y \in C$, then $xy^iz \in C$

Idea: If C is regular and F is regular, then the intersection $C \cap F$ has to be regular as well

Solution: Assume that C is regular

Take the regular $F = \{ 0^n1^m \mid n,m \in \mathbb{N} \}$, then for the intersection: $C \cap F = \{ 0^n1^n \mid n \in \mathbb{N} \}$

But we know that $C \cap F$ is not regular

Conclusion: C is not regular

Pumping Down $E = \{ 0^i1^j \mid i \geq j \}$

Problem: 'pumping up' $s=0^p1^p$ with $y=0^k$ gives $xyyz = 0^{p+k}1^p$, $xy^3z = 0^{p+2k}1^p$, which are all in E (hence do not give contradictions)

Solution: pump down to $xz = 0^{p-k}1^p$.

Overall for $s = xyz = 0^p1^p$ (with $|xy| \leq p$):

$y=0^k$, hence $xz = 0^{p-k}1^p \notin E$

Contradiction: E is not regular

Pumping lemma usage - steps

- You are given a pumping number
- You choose a string
- There exist x,y,z (satisfying some criteria)
- You choose i in xy^iz, and show it violates criterion of set for that i.

Alternatives for proving non-regularity

- Simpler technique (not in the text)
  - Based on the Myhill-Nerode Theorem
  - less general