CSCI 320: Introduction to Theory of Computation

Spring 2015

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Course page:
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Finite automata

Ch. 1.1: Deterministic finite automata (DFA)

We will :

- Design automata for simple problems
- Study languages recognized by finite automata.

Recognizing finite languages

- Just need a lookup table and a search algorithm
- Problem – cannot express infinite sets, e.g. odd integers

Finite Automata

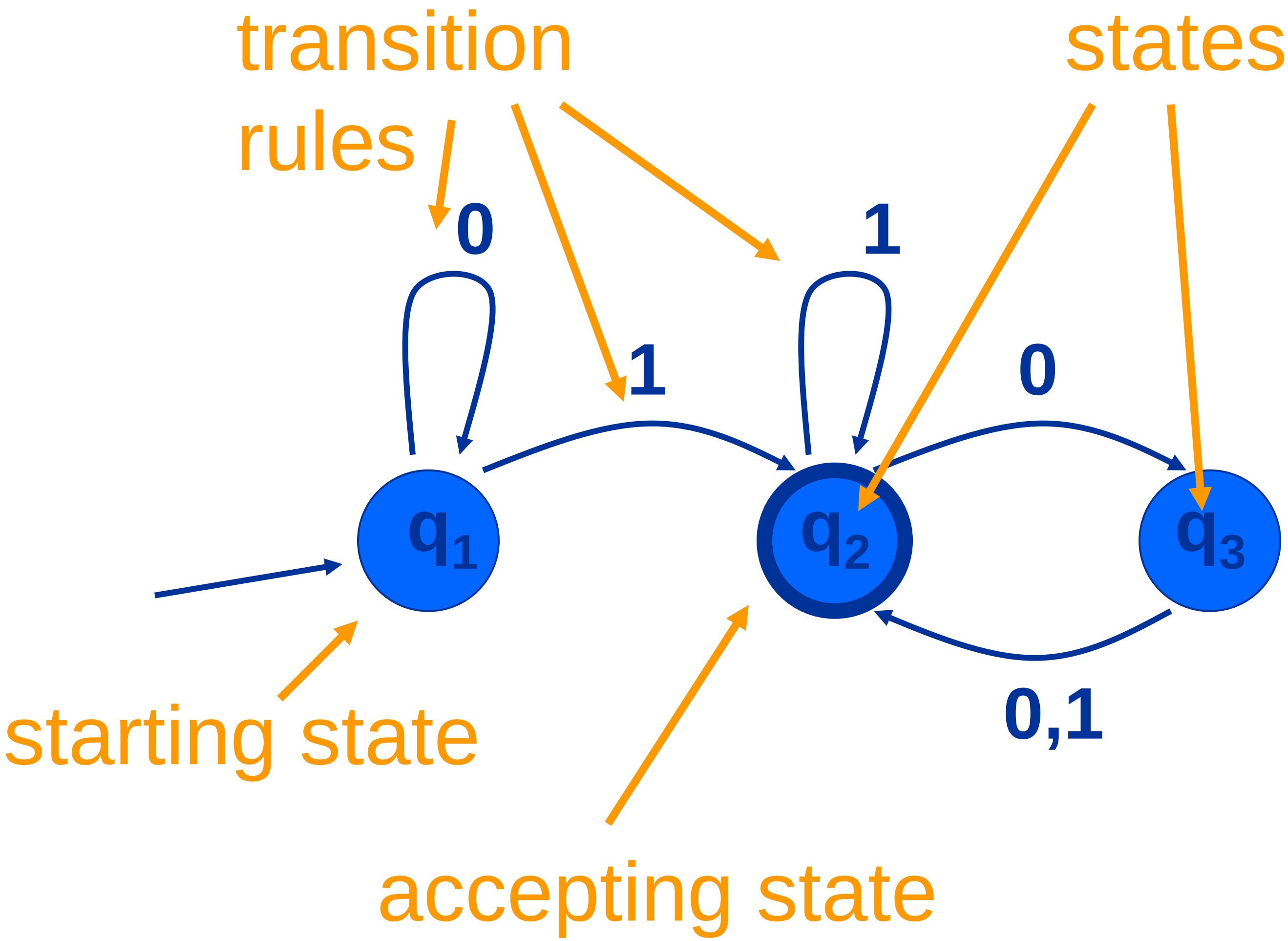
The simplest machine that can recognize an infinite language.

"Read once", "no write" procedure.

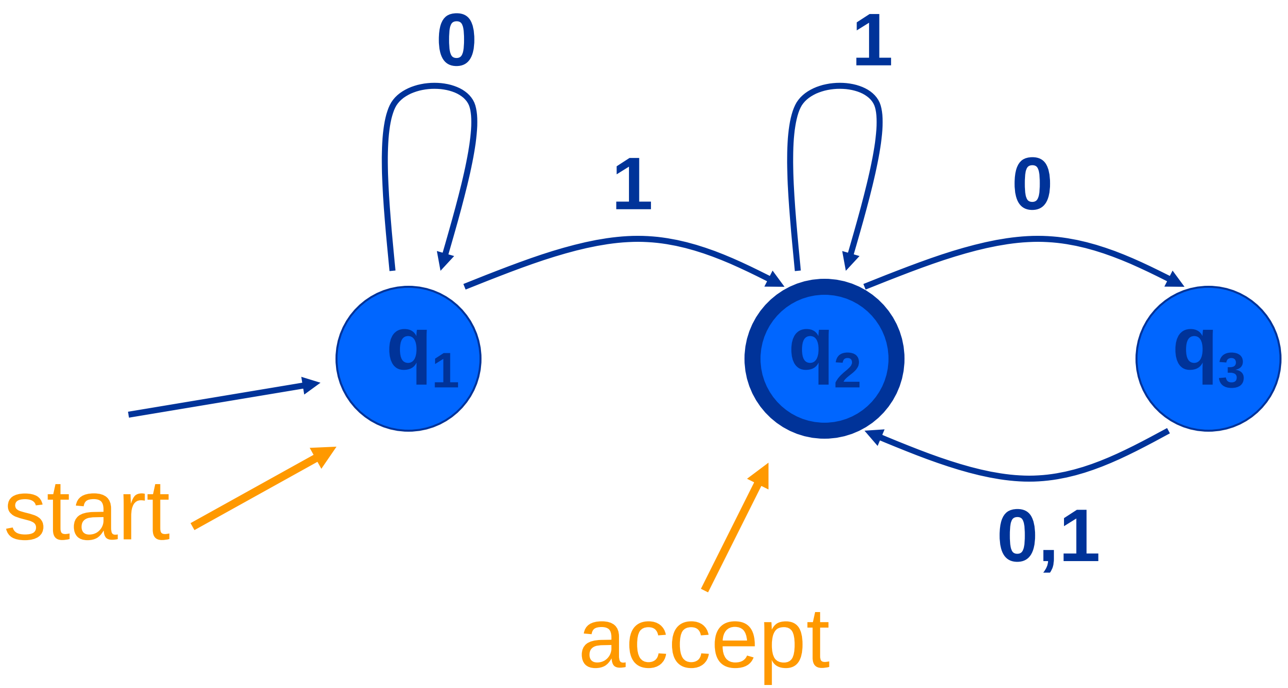
Useful for describing algorithms also.
Used a lot in network protocol description.

Remember: DFA's can accept finite languages as well.

A Simple Automaton (0)



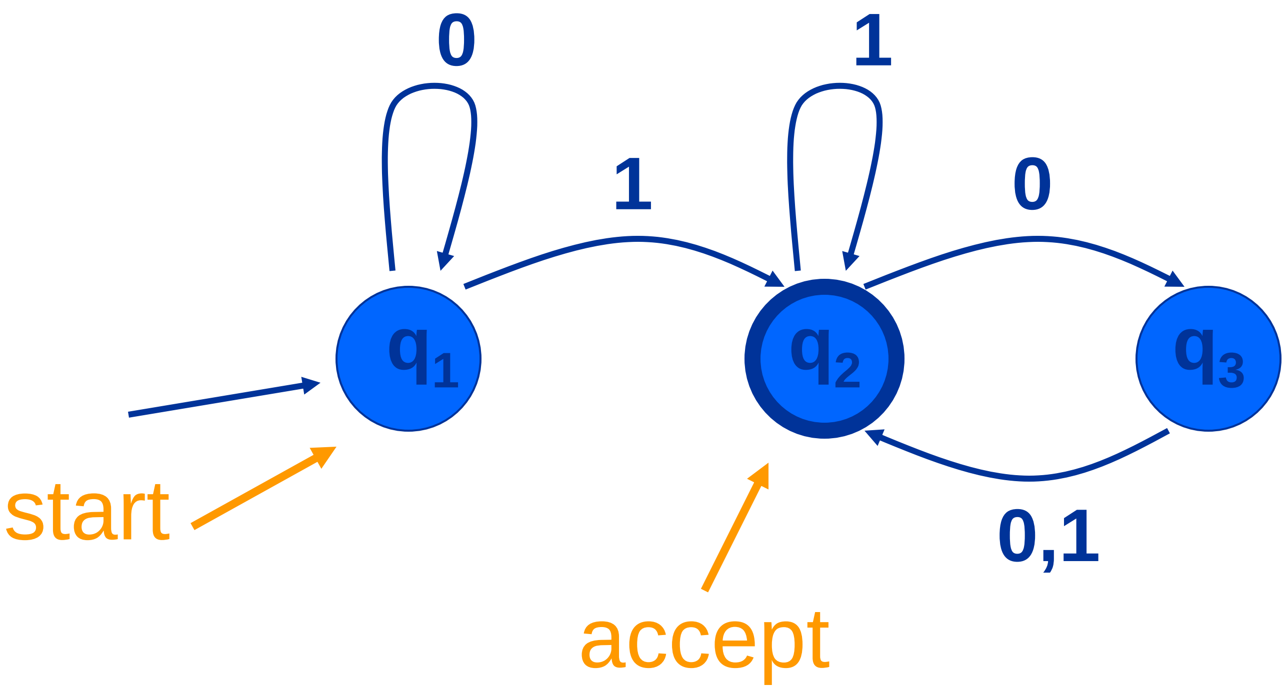
A Simple Automaton (1)



on input "0110", the machine goes:

$q_1 \rightarrow q_1 \rightarrow q_2 \rightarrow q_2 \rightarrow q_3$ = "reject"

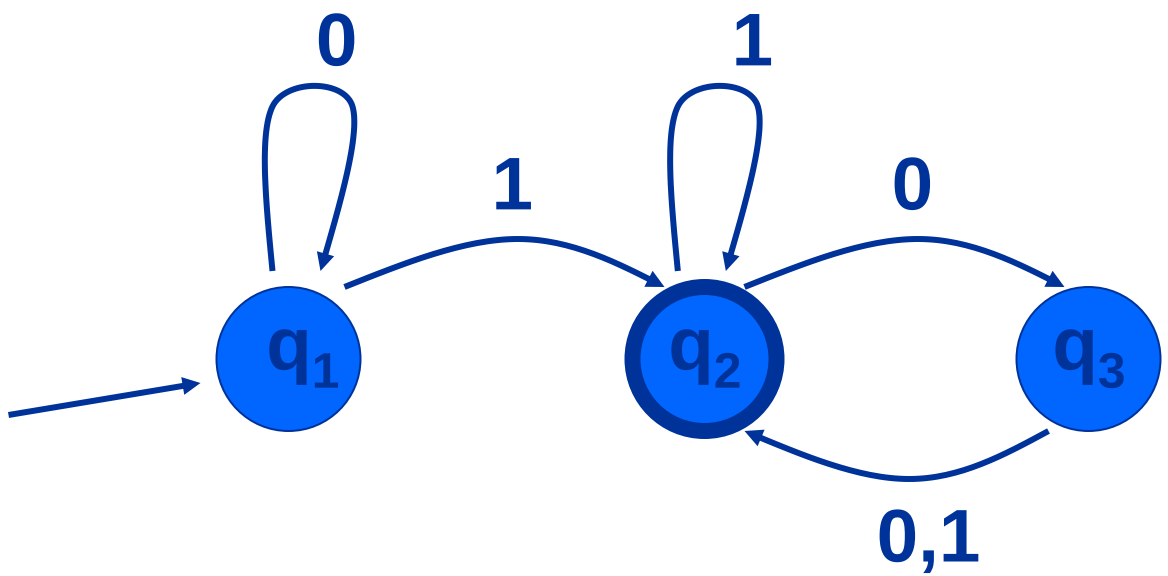
A Simple Automaton (1)



on input "0110", the machine goes:

$q_1 \rightarrow q_1 \rightarrow q_2 \rightarrow q_2 \rightarrow q_3$ = "reject"

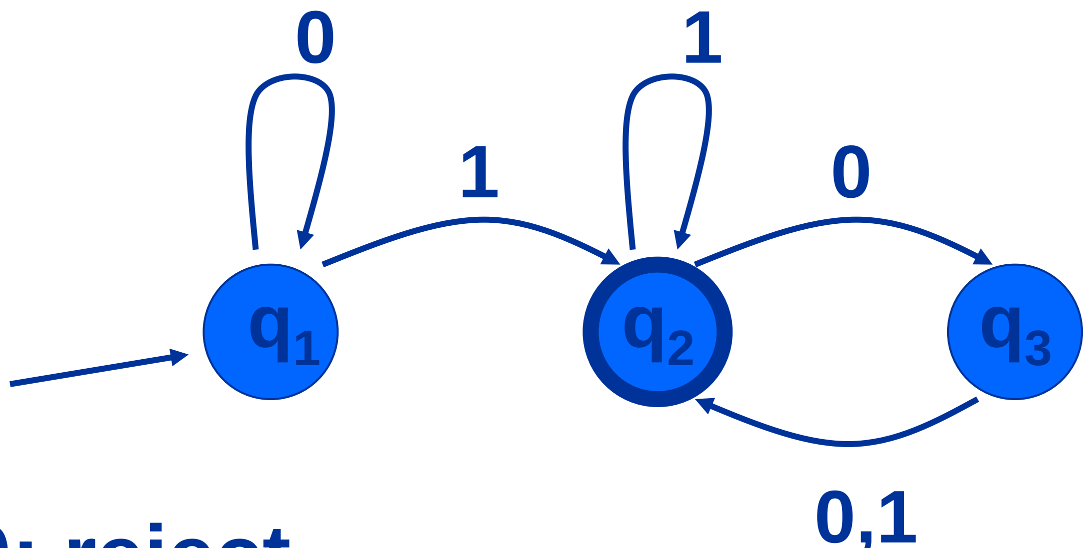
A Simple Automaton (2)



on input "101", the machine goes:

$q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_2$ = "accept"

A Simple Automaton (3)



010: reject

11: accept

010100100100100: accept

010000010010: reject

ε: reject

Examples of languages accepted by DFA

- $L = \{ w \mid w \text{ ends with } 1\}$
- $L = \{ w \mid w \text{ contains sub-string } 00\}$
- $L = \{ w \mid |w| \text{ is divisible by } 3\}$
- $L = \{ w \mid |w| \text{ is odd or } w \text{ ends with } 1\}$
- $L = \{ w \mid |w| \text{ is divisible by } 10^6\}$

Note: $\Sigma = \{0,1\}$ in each case

DFA design

- Design DFA for language
  - L = $\{w \in \{0,1\}^* \mid w$ contains substring 01$\}$
- Three states to remember:
  - Have seen the substring 01
  - Not seen substring 01 and last symbol was 0
  - Not seen substring 01 and last symbol was not 0
- General principles?

DFA : Formal definition

- A deterministic finite automaton (DFA) M is defined by a 5-tuple $M=(Q,\Sigma,\delta,q_0,F)$

  – Q: finite set of states
  – Σ: finite alphabet
  – δ: transition function $\delta:Q\times\Sigma\to Q$
  – $q_0\in Q$: start state
  – $F\subseteq Q$: set of accepting states

$M = (Q,\Sigma,\delta,q,F)$

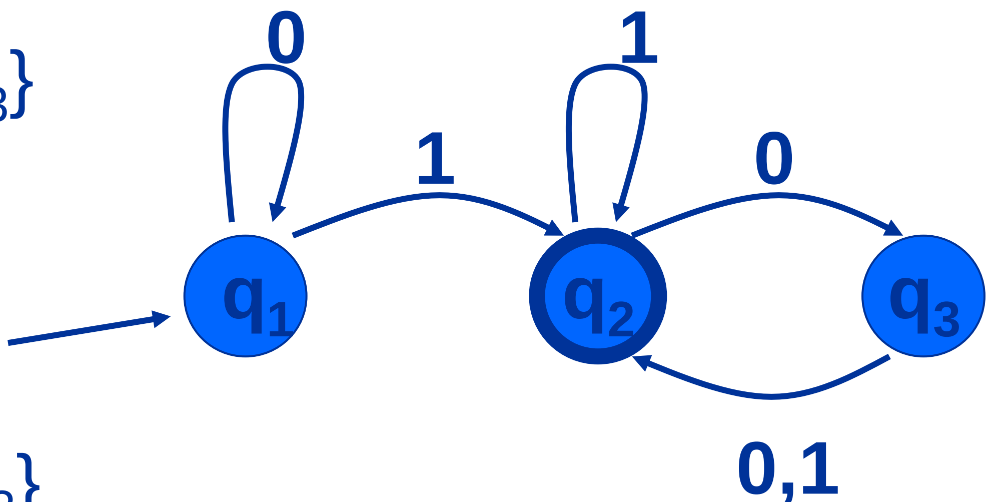
<u>states</u> $Q = \{q_1,q_2,q_3\}$

<u>alphabet</u> $\Sigma = \{0,1\}$

<u>start state</u> q_1

<u>accept states</u> $F=\{q_2\}$

<u>transition function</u> δ:



	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

Recognizing Languages (defn)

A finite automaton **M = (Q,Σ,δ,q,F)** accepts a string/word **w** = $w_1 \ldots w_n$ if and only if there is a sequence $r_0 \ldots r_n$ of states in Q such that:

1) $r_0 = q_0$

2) $\delta(r_i, w_{i+1}) = r_{i+1}$ for all $i = 0, \ldots, n-1$

3) $r_n \in F$

Regular Languages

The language recognized by a finite automaton M is denoted by **L(M).**

A <u>regular language</u> is a language for which there exists a recognizing finite automaton.

Two DFA Questions

Given the description of a finite automaton **$M = (Q,\Sigma,\delta,q,F)$,** what is the language **$L(M)$** that it recognizes?

In general, what kind of languages can be recognized by finite automata? (What are the regular languages?)

Union of Two Languages

<u>Theorem 1.12</u>: If A_1 and A_2 are regular languages, then so is $A_1 \cup A_2$.
(The regular languages are 'closed' under the union operation.)

<u>Proof idea</u>: A_1 and A_2 are regular, hence there are two DFA M_1 and M_2, with $A_1=L(M_1)$ and $A_2=L(M_2)$. Out of these two DFA, we will make a third automaton M_3 such that $L(M_3) = A_1 \cup A_2$.

Proof Union-Theorem (1)

$M_1=(Q_1,\Sigma,\delta_1,q_1,F_1)$ and $M_2=(Q_2,\Sigma,\delta_2,q_2,F_2)$

Define $M_3 = (Q_3,\Sigma,\delta_3,q_3,F_3)$ by:

- $Q_3 = Q_1 \times Q_2 = \{(r_1,r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$
- $\delta_3((r_1,r_2),a) = (\delta_1(r_1,a), \delta_2(r_2,a))$
- $q_3 = (q_1,q_2)$
- $F_3 = \{(r_1,r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

Proof Union-Theorem (2)

The automaton $M_3 = (Q_3,\Sigma,\delta_3,q_3,F_3)$ runs M_1 and M_2 in ‘parallel’ on a string w.

In the end, the final state (r_1,r_2) ‘knows’ if $w\in L_1$ (via $r_1\in F_1$?) and if $w\in L_2$ (via $r_2\in F_2$?)

The accepting states F_3 of M_3 are such that $w\in L(M_3)$ if and only if $w\in L_1$ *or* $w\in L_2$, for: $F_3 = \{(r_1,r_2) \mid r_1\in F_1 \text{ or } r_2\in F_2\}$.

Concatenation of L_1 and L_2

Definition: $L_1 \bullet L_2 = \{ xy \mid x \in L_1 \text{ and } y \in L_2 \}$

Example: $\{a,b\} \bullet \{0,11\} = \{a0,a11,b0,b11\}$

<u>Theorem 1.13</u>: If L_1 and L_2 are regular langues, then so is $L_1 \bullet L_2$.
(The regular languages are 'closed' under concatenation.)

Proving Concatenation Thm.

Consider the concatenation:
{1,01,11,001,011,…} • {0,000,00000,…}
(That is: the bit strings that end with a “1”, followed by an odd number of 0’s.)

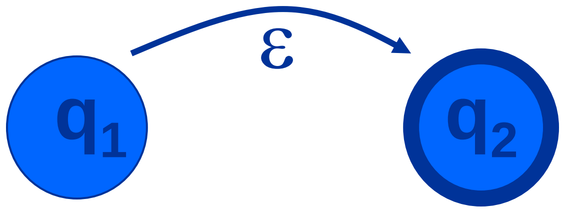
Problem is: given a string w, how does the automaton know where the L_1 part stops and the L_2 substring starts?

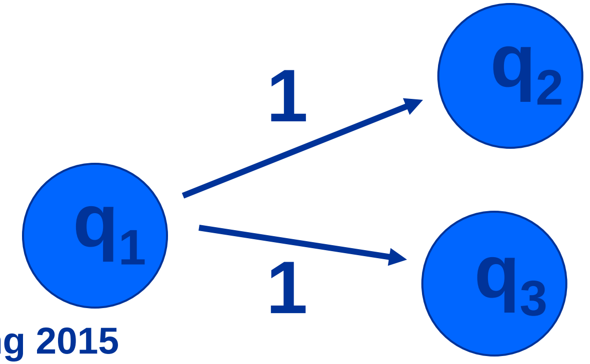
We need an M with ‘lucky guesses’.

Nondeterminism

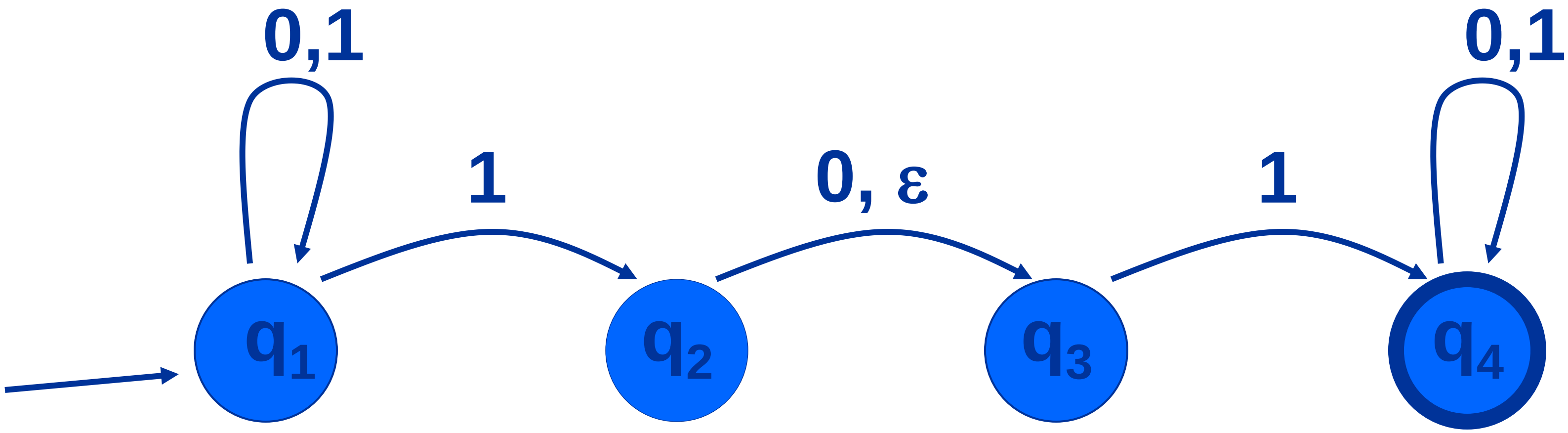
Nondeterministic machines are capable of being lucky, no matter how small the probability.

A nondeterministic finite automaton has transition rules/possibilities like





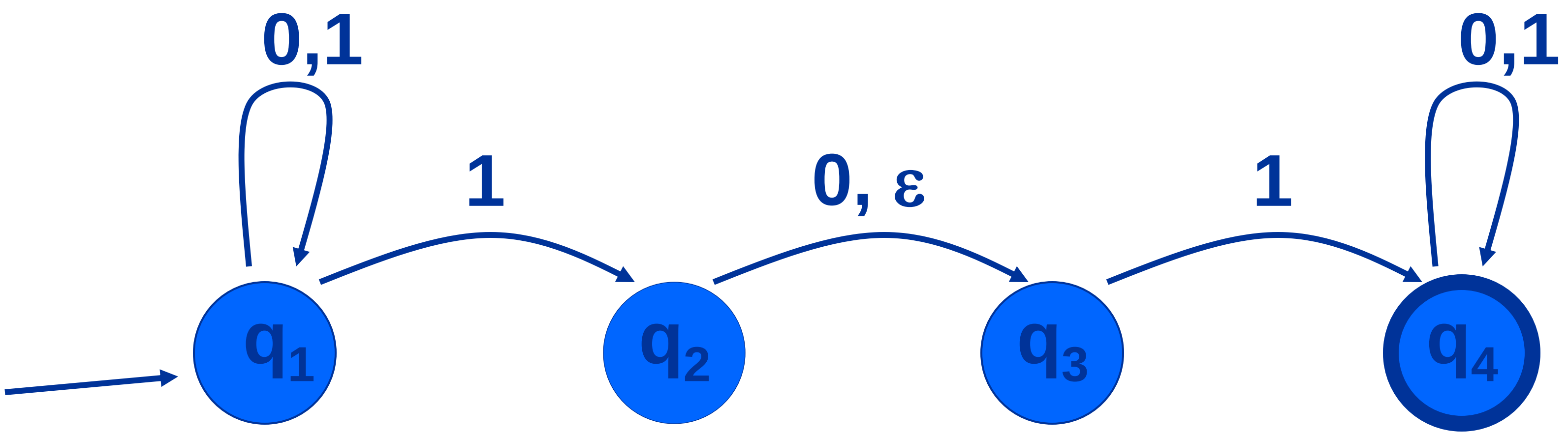
A Nondeterministic Automaton



This automaton accepts "0110", because there is a possible path that leads to an accepting state, namely:

$q_1 \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_4 \rightarrow q_4$

A Nondeterministic Automaton



The string 1 gets rejected: on “1” the automaton can only reach: $\{q_1,q_2,q_3\}$.

Nondeterminism ~ Parallelism

For any (sub)string w, the nondeterministic automaton can be in a set of possible states.

If the final set contains an accepting state, then the automaton accepts the string.

"The automaton processes the input in a parallel fashion. Its computational path is no longer a line, but a tree." (Fig. 1.16)

Nondeterministic FA (def.)

- A nondeterministic finite automaton (NFA) M is defined by a 5-tuple $M=(Q,\Sigma,\delta,q_0,F)$, with
  - Q: finite set of states
  - Σ: finite alphabet
  - δ: transition function $\delta:Q\times\Sigma_\varepsilon\rightarrow\mathcal{P}(Q)$
  - $q_0\in Q$: start state
  - $F\subseteq Q$: set of accepting states

Nondeterministic $\delta:Q\times\Sigma_\varepsilon\rightarrow\mathcal{P}(Q)$

The function $\delta:Q\times\Sigma_\varepsilon\rightarrow\mathcal{P}(Q)$ is the crucial difference. It means:
"When reading symbol "a" while in state q, one can go to one of the states in $\delta(q,a)\subseteq Q$."

The ε in $\Sigma_\varepsilon = \Sigma\cup\{\varepsilon\}$ takes care of the empty string transitions.

Recognizing Languages (def)

A nondeterministic FA **M = (Q,Σ,δ,q,F)** <u>accepts</u> a string **w** = $w_1 \ldots w_n$ if and only if we can rewrite w as $y_1 \ldots y_m$ with $y_i \in \Sigma_\varepsilon$ and there is a sequence $r_0 \ldots r_m$ of states in Q such that:

1) $r_0 = q_0$

2) $r_{i+1} \in \delta(r_i, y_{i+1})$ for all $i = 0, \ldots, m-1$

3) $r_m \in F$

Exercises

[Sipser 1.5]: Give NFAs with the specified number of states that recognize the following languages over the alphabet $\Sigma=\{0,1\}$:

1. { w | w ends with 00}, three states
2. {0}; two states
3. { w | w contains even number of 0s, or exactly two 1s}, six states
4. $\{0^n \mid n \in \mathbb{N}\}$, one state

Exercises - 2

Proof the following result:
"If L_1 and L_2 are regular languages, then $L_1 \cap \overline{L_2}$ is a regular language too."

Describe the language that is recognized by this nondeterministic automaton: