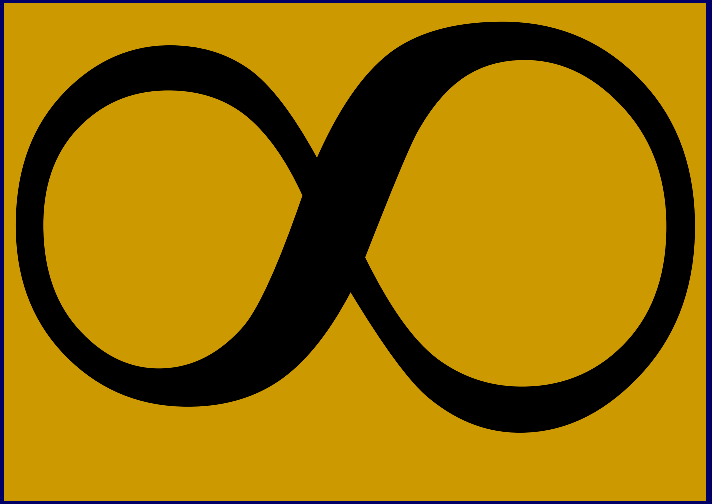
2001: Countable and Uncountable Infinite

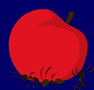
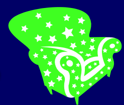
- Countable
- Countable via a List
- Countable via Finite Descriptions
- Uncountable
- Hierarchy of Infinities
- Some Uncomputable Problem



Jeff Edmonds
York University

Countable Infinity

A finite set contains some integer number of elements.

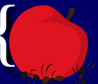
- { , 4,  }

The size of an infinite set is bigger than any integer.

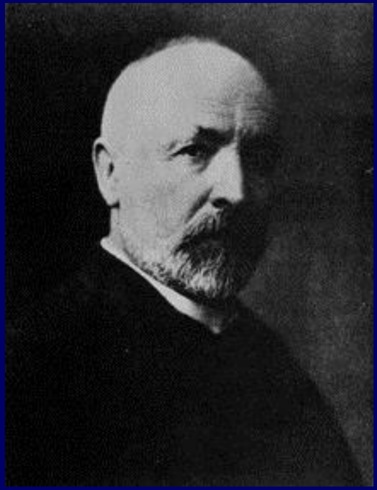
- The set of natural numbers $N = \{1,2,3,4,\ldots\ \}$
- The set of fractions $Q = \{{}^1/_2, {}^2/_3, \ldots\ \}$
- The set of reals $R = \{2.34323\ldots, 34.2233\ldots, \pi, e, \ldots\}$

Are these infinite sets the same "size"?

Two sets have the same size
if there is a bijection between them.

|{ , 4,  }|

$= |\{\ 1,\ \ 2,\ \ 3\ \}|\ \ = 3$



Cantor (1874)

Countable Infinity

Do N and E have the same cardinality?

N = { 1, 2, 3, 4, 5, 6, 7, ... }

E = The even, natural numbers.

Countable Infinity

E and N do not have the same cardinality!
E is a proper subset of N with plenty left over.

1, 2, 3, 4, 5, 6, 7, 8,….

2, 4, 6, 8,….

f(x)=x is not a bijection.

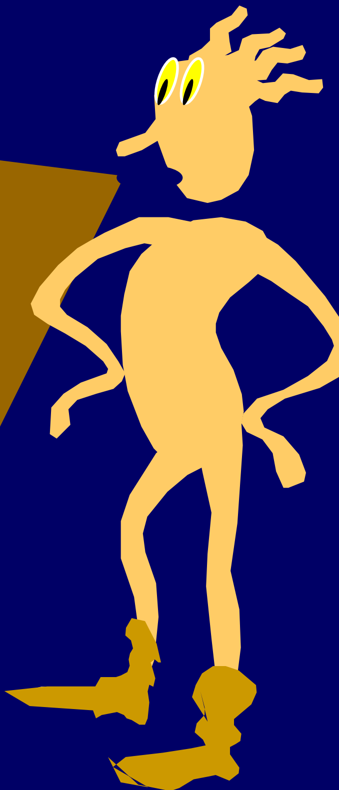
Countable Infinity

E and N do have the same cardinality!

1, 2, 3, 4, 5, …….

2, 4, 6, 8, 10, ….

f(x) = 2x is a bijection



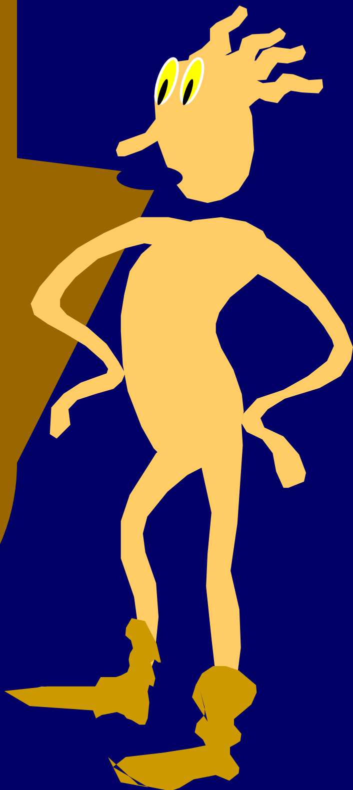
Countable Infinity

Lesson:

Cantor's definition only requires that **there exists** a bijection between the two sets.

Not that **all** 1-1 correspondences are onto.

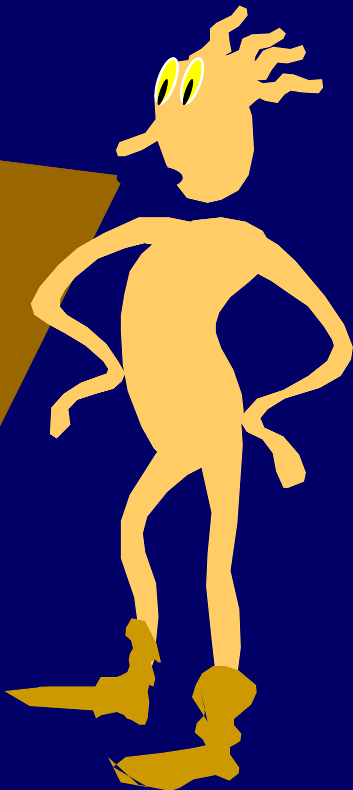
This distinction never arises when the sets are finite.



Countable Infinity

If this makes you feel uncomfortable.....

TOUGH! It is the price that you must pay to reason about infinity

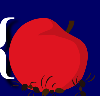


Countable Infinity

- The set of integers $N = \{1,2,3,4,\ldots\ \}$
- The set of fractions $Q = \{{}^1/_2,\ {}^2/_3,\ \ldots\ \}$

Are these infinite sets the same "size"?

Two sets have the same size
if there is a mapping between them.

|{  , 4,  }|

= |{ 1, 2, 3 }| = 3

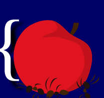
Countable Infinity

- The set of integers $N =$ $\{1,2,3,4,\ldots\ \}$
- The set of fractions Q = $\{{}^1/_2, {}^2/_3, \ldots\ 1,2,3,4,\ldots\ \}$

Q looks bigger.

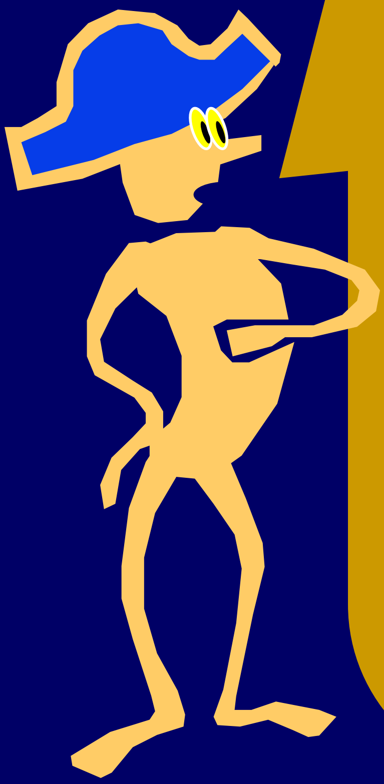
Are these infinite sets the same "size"?

Two sets have the same size
if there is a mapping between them.

|{  , 4,  }|

= |{ 1, 2, 3 }| = 3

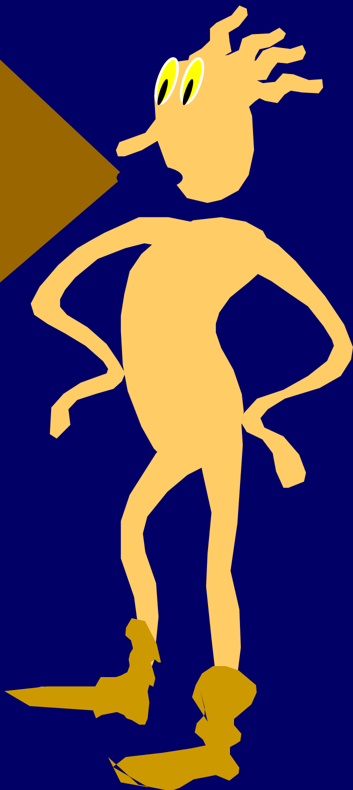
Countable Infinity



No way!
The rationals are dense: between any two there is a third. You can’t list them one by one without leaving out an infinite number of them.

Countable Infinity

Don't jump to conclusions!
There is a clever way to list the rationals, one at a time, without missing a single one!



Countable Infinity

6	$^6/_1$	$^6/_2$	$^6/_3$	$^6/_4$	$^6/_5$	$^6/_6$	$^6/_7$	$^6/_8$
5	$^5/_1$	$^5/_2$	$^5/_3$	$^5/_4$	$^5/_5$	$^5/_6$	$^5/_7$	$^5/_8$
4	$^4/_1$	$^4/_2$	$^4/_3$	$^4/_4$	$^4/_5$	$^4/_6$	$^4/_7$	$^4/_8$
3	$^3/_1$	$^3/_2$	$^3/_3$	$^3/_4$	$^3/_5$	$^3/_6$	$^3/_7$	$^3/_8$
2	$^2/_1$	$^2/_2$	$^2/_3$	$^2/_4$	$^2/_5$	$^2/_6$	$^2/_7$	$^2/_8$
1	$^1/_1$	$^1/_2$	$^1/_3$	$^1/_4$	$^1/_5$	$^1/_6$	$^1/_7$	$^1/_8$
	1	2	3	4	5	6	7	8

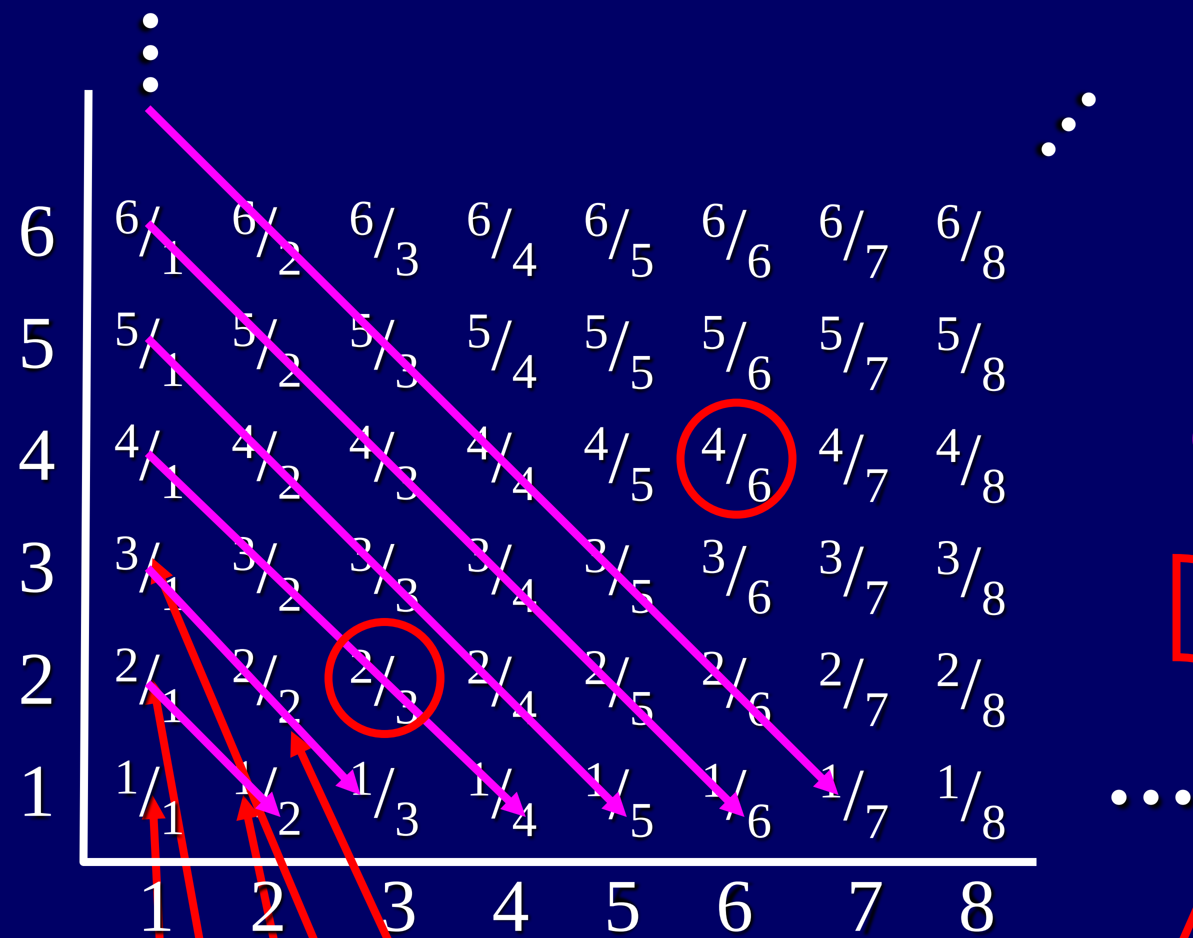
All positive fractions

Count them by mapping.

Oops we never get to $^2/_1$!

$\{ 1, 2, 3, 4, 5, 6, \ldots. \}$ All positive integers

Countable Infinity



All positive fractions

Over counting just means we proved $|N| \geq |Q|$

Count them by mapping.

~~$|N| = |Q|$~~

Every rational gets mapped to some integer!

Q is "Countable"

{ 1, 2, 3, 4, 5, 6, …. }

All positive integers

Countable Infinity

```
loop  a,b,c,z > 2
    exit when  a^z + b^z = c^z
end loop
```

How is it that we can loop over all tuples ⟨a,b,c,z⟩ ensuring that we eventually get to each?

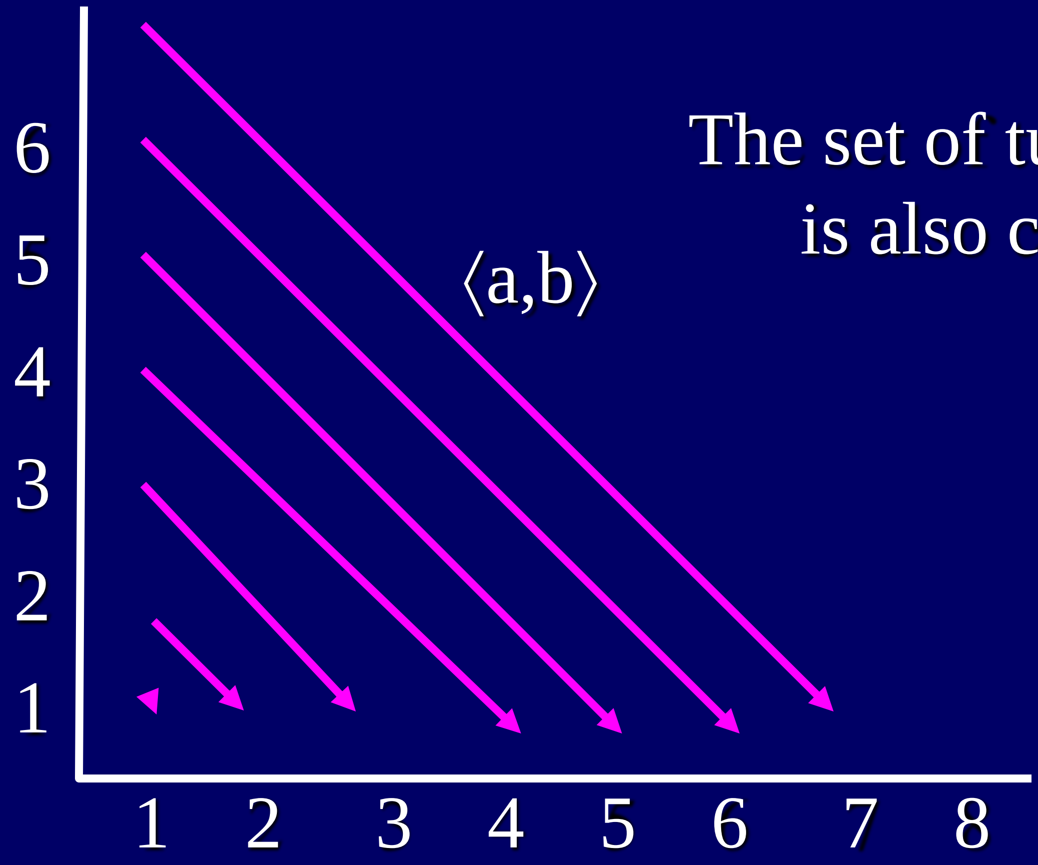
```
loop  a > 2
    loop  b > 2
        loop  c > 2
            loop  z > 2
                exit when  a^z + b^z = c^z
```

No, this inner loop will never exit.
We will never get to c=4

Countable Infinity

loop a,b,c,z > 2
    exit when $a^z + b^z = c^z$
end loop

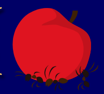
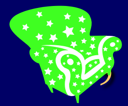
How is it that we can loop over all tuples ⟨a,b,c,z⟩ ensuring that we eventually get to each?



The set of tuples {⟨a,b,c,z⟩}
is also countable.

Countable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$ , 4, , ….. $\}|$

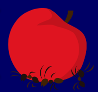
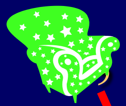
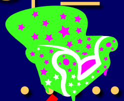
$$\leq |\{\ 1,\quad 2,\quad 3,\ 4,\ 5,\ \ldots\ \}|\ =\ |N|$$

Note a finite set is countable.

If also $|S| \geq |N|$, then S is called *countably infinite*.

Countable Infinity

A set *S* is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$  , 4,  , …  …  $|$

$\leq |\{1, 2, 3, 4, 5, \ldots\}| = |N|$

The mapping does not need to be bijective

Is it ok that some $x \epsilon S$ is not mapped?

No, or else $|S|$ might be bigger.

Is it ok if some $i \epsilon N$ is mapped to twice?

No, or else $|S|$ might be larger.

Is it ok if the same $x \epsilon S$ is mapped from more than once?

Yes: as this only makes seem $|N|$ bigger.

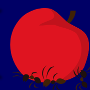
Is it ok if some $i \epsilon N$ is not mapped to?

Yes, as this only makes seem $|N|$ bigger.

Such a function F is called *Injective.*

Countable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$  , 4,  , ….. $\}|$

$\leq |\{ 1, \quad 2, \quad 3, \; 4, \; 5, \ldots \}| \; = \; |N|$

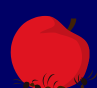
Two equivalent definitions of a set S being "Countable"

- There is a list containing each object

$\exists F^{-1}, \; \forall x \epsilon S, \exists\, i \epsilon N \; F^{-1}(i) = x$

$i \quad x = F^{-1}(i)$

1:

2: 

3: 

4:

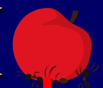
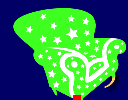
5: 4

Each integer (index in list) lists at most one object $x \epsilon S$

Each one object $x \epsilon S$ appears some where in the list

Countable Infinity

A set S is called ***countable*** if $|S| \le |N|$

$|S| = |\{$  , 4,  ….. $\}|$

“apple” “four” “chair”

Give each object a finite description.

$\le |\{\, 1, \; 2, \; 3, \; 4, \; 5, \ldots \}| \; = \; |N|$

Two equivalent definitions of a set S being “Countable”

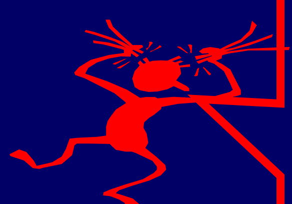
- There is a list containing each object

  $\exists F^{-1}, \; \forall x \in S, \; \exists\, i \in N \; F^{-1}(i) = x$

- Each object $x \in S$ has (at least one) finite description

  such that each description uniquely identifies that object.

My name is
Herr Dr Professor Wizard the great great great ….

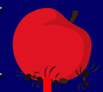
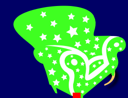


Just because you have an infinite name does not mean the set {  } is not countable!

I call you Bob

Countable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$  , 4,  ….. $\}|$

"apple" "four" "chair"

'a' 'p' 'p' 'l' 'e'   'f' 'o' 'u' 'r'   'c' 'h' 'a' 'i' 'r'

61 70 70 6C 65   66 6F 75 72   63 68 61 69 72

$\leq |\{$ 6170706C65, 666F7572, 6368616972, … $\}|$

$\leq |\{ 1, \quad 2, \quad 3, \; 4, \; 5, \ldots \}| \; = \; |N|$

Give each object a finite description.

Break each description into a string of characters.

Convert each character to Hex-Ascii.

Concatenate the Hex into one Hex integer.

The number of such integers is at most the number of integers.

Hence, this set of objects is countable.

Countable Infinity

A set S is called ***countable*** if $|S| \le |N|$

The set of fractions

$|Q| = |\{{}^1/_2, {}^2/_3, {}^{11}/_8, \ldots\}|$

"${}^1/_2$" "${}^2/_3$" "${}^{11}/_8$"

'1' '/' '2' '2' '/' '3' '1' '1' '/' '8'

31 2F 32 32 2F 32 31 31 2F 38

$\le |\{312F32, 322F32, 31312F38, \ldots\}|$

$\le |\{1, 2, 3, 4, 5, \ldots\}| = |N|$

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Countable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

The set of finite sets of integers

$|S| = |\{\ \{3,23,\ldots,98\}.\ \ldots\ \}|$

Give each object a finite description.

$\downarrow$

"{3,23,…,98}"

Break each description into a string of characters.

$\downarrow$

'{' '3' ',' '2' '3' ',' …. ',' '9' '8' '3' '7' '}'

$\downarrow$

7B 33 2C 32 32 2C … 2C 39 38 33 37 7D

Convert each character to Hex-Ascii.

$\downarrow$

7B332C32322C…2C393833377D

Concatenate the Hex into one Hex integer.

$\leq |\{\ 1,\ \ 2,\ \ 3,\ 4,\ 5,\ \ldots\ \}| = |N|$

The number of such integers is at most the number of integers.

Hence, this set of objects is countable.

Countable Infinity

A set S is called ***countable*** if $|S| \le |N|$

The set of finite sets of integers

$|S| = |\{\ \{3,23,\dots,98\}.\ \dots\ \}|$

"{3,23,…,98}" "{23,3,…,98}"

'{' '3' ',' '2' '3' ',' …. ',' '9' '8' '3' '7' '}'

7B 33 2C 32 32 2C … 2C 39 38 33 37 7D

7B332C32322C…2C393833377D

$\le |\{\ 1,\quad 2,\quad 3,\ 4,\ 5,\ \dots\ \}|\ =\ |N|$

Give each object a finite description.

It is ok if an object is given more than one description.

As long as each description uniquely identifies an object.

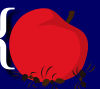
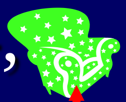
Uncountable Infinity

- The set of natural numbers $N = \{1,2,3,4,\ldots\}$

- The set of reals $R = \{2.34323\ldots, 34.2233\ldots, \pi, e, \ldots\}$

Are these infinite sets the same “size”?

Two sets have the same size
if there is a bijection between them.

|{  , 4,  }|

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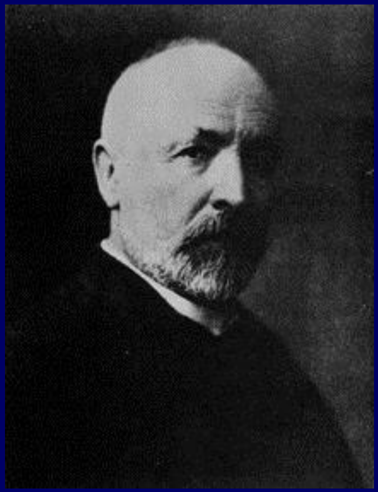


Cantor (1874)

Uncountable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

The set of reals

$|R| = |\{2.34323\ldots, 34.2233\ldots, \pi, e, \ldots\}|$

Most real seems to require an infinite description.

$>>$ $|\{1, 2, 3, \ldots\}| = |N|$

- Each object $x \epsilon S$ has (at least one) finite description such that each description uniquely identifies that object.

Uncountable Infinity

A set *S* is called ***countable*** if $|S| \leq |N|$

The set of reals

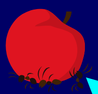
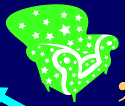
$|R| = |\{2.34323\ldots, 34.2233\ldots, \pi, e, \ldots\}|$

```
loop  reals r
    exit when  r^r = 100000r
end loop
```

How is it that we can loop over all reals ensuring that we eventually get to each?

Uncountable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$  , 4,  , ….. $\}|$

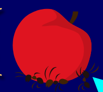
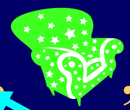
$\leq |\{ 1, \quad 2, \quad 3, \; 4, \; 5, \ldots \}| \;=\; |N|$

$\exists F^{-1}, \; \forall x \epsilon S, \; \exists\, i \epsilon N \; F^{-1}(i) = x$

Each integer i is able to hit at most one element $F^{-1}(i)=x\epsilon S$.
If this can hit every element of element $x\epsilon S$,
then $|S| \leq |N|$.

Uncountable Infinity

A set S is called ***countable*** if $|S| \leq |N|$

$|S| = |\{$  $, 4,$  $, \ldots$  $\}|$

$\leq |\{ 1, 2, 3, 4, 5, \ldots \}| = |N|$

$\exists F^{-1}, \forall x \epsilon S, \exists\, i \epsilon N\ F^{-1}(i) = x$

A set S is called ***uncountable*** if $|S| > |N|$

$\forall F^{-1}, \exists x \epsilon S, \forall i \epsilon N\ F^{-1}(i) \neq x$

i.e. find an x is not mapped to any natural number.

Proof by game:

- Let F^{-1} be an arbitrary mapping from N likely to S.
- I construct a value $x \epsilon S$.
- Let i be an arbitrary natural number.
- I prove that $F^{-1}(i) \neq x$

Uncountable Infinity

Proof that $|R| > |N|$ i.e. $\forall F^{-1}, \exists x \epsilon R, \ \forall i \epsilon N \ \ F^{-1}(i) \neq x$

- Let F^{-1} be an arbitrary mapping from N to R.

i	$x=F^{-1}(i)$
1	12.34323834749308477599304 …
2	8.50949039988484877588487 …
3	930.93994885783998573895002 …
4	34.39498837792008948859069 …
5	0.00343988348757590125473 …
6	…..

Proof by Diagonalization

- We find a real number $x_{diagonal}$ that is not in the list.

The i^{th} digit will be the i^{th} digit of the i^{th} number $F^{-1}(i)$ increased by one *(mod 10)*.

$x_{diagonal} = 0.41004 \ldots$

- Let i be arbitrary in N.
- I prove that $F^{-1}(i) \neq x_{diagonal}$

They differ in the i^{th} digit

Uncountable Infinity

Proof that $|R| > |N|$ i.e. $\forall F^{-1}, \exists x \epsilon R, \forall i \epsilon N\ F^{-1}(i) \neq x$

Our goal is to prove that there are more real numbers than integers, i.e. $|\mathcal{R}| > |\mathcal{N}|$.

We prove this by proving the following first order logic statement

$\forall$ an inverse functions F^{-1} from $\mathcal{N}$ ideally to $\mathcal{R}$,

$\exists x_{diagonal} \in \mathcal{R}, \forall i \in \mathcal{N}, F^{-1}(i) \neq x_{diagonal}$

namely there are not enough integers to hit each real.

We prove this by playing the game.

Let F^{-1} be an arbitrary inverse function from $\mathcal{N}$ ideally to $\mathcal{R}$.

Define the real $x_{diagonal} \in \mathcal{R}$ as follows.

For each $i \in \mathcal{N}$, I must define the i^{th} digit of $x_{diagonal}$.

For this, we use flip of the i^{th} diagonal element as follows.

Let x_i denote the real $F^{-1}(i)$ that the i^{th} row gives us.

Let d_i denote the i^{th} digit of x_i.

Then let the i^{th} digit of $x_{diagonal}$ be any digit d'_i other than d_i.

This completely defines $x_{diagonal}$.

Continuing the game, let $i \in \mathcal{N}$ be arbitrary.

Note $x_i = F^{-1}(i)$ and $x_{diagonal}$ differ in their i^{th} digits.

This proves that $F^{-1}(i) \neq x_{diagonal}$.

Hierarchy of Infinities

|Integers| = |Fractions| << |Reals|

Each defined by a finite string	Each defined by a finite string	Each defined by an infinite string

Set of finite subsets of the integers

{{2,3},{1,3,5,6}, … }

Each defined by a finite string

The set is countable in size

Set of possibly infinite subsets of the integers

{{2,3,…},{1,3,…}, … }

Each defined by a infinite string

The set is uncountable in size

Hierarchy of Infinities

<< |Reals|

Each defined by
an infinite string

Set of finite subsets of the reals

{{2.394..,3.3563..},{1.982..,3.345..,5.32..}, … }

Each defined by a string of countably infinite length.

The set is same size as the reals

Set of possibly infinite subsets of the reals

{{2.394..,3.3563..,…},{1.982..,3.345..,…}, … }

Each defined by a string of uncountably infinite length.

The set is much bigger than the reals!

There is an infinite hierarchy of infinities!

Some Uncomputable Problem

$\exists P \; \forall M \; \exists I \;\; M(I) \neq P(I)$

|Integers| = |Fractions| << |Reals|

Each defined by a finite string

Each defined by a finite string

Each defined by an infinite string

Set of TM/Algorithms
- Each defined by a finite string
- Countable in size

<<

Set of Comp. Problems
- Each defined by an infinite string
- uncountable in set

⇨ Most problems do not have an algorithm!!!

Some Uncomputable Problem

$\exists P \ \forall M \ \exists I \ \ M(I) \neq P(I)$