Next

- Chapter 6:
  - Context-Free Languages (CFL)
  - Context-Free Grammars (CFG)
  - Chomsky Normal Form of CFG
  - RL $\subset$ CFL

Context-Free Languages (Ch. 6)

Context-free languages (CFLs) are a more powerful (augmented) model than FA.

CFLs allow us to describe non-regular languages like $\{ 0^n1^n \mid n \geq 0\}$

General idea: CFLs are languages that can be recognized by automata that have one single stack:

$\{ 0^n1^n \mid n \geq 0\}$ is a CFL

$\{ 0^n1^n0^n \mid n \geq 0\}$ is not a CFL

Context-Free Grammars

Grammars: define/specify a language

Which simple machine produces the non-regular language $\{ 0^n 1^n \mid n \in \mathbb{N} \}$?

Start symbol S with rewrite rules:

1) $S \rightarrow 0S1$
2) $S \rightarrow$ "stop"

S *yields* 0^n1^n according to

$S \rightarrow 0S1 \rightarrow 00S11 \rightarrow \ldots \rightarrow 0^nS1^n \rightarrow 0^n1^n$

Context-Free Grammars (Def.)

A context free grammar $G=(V,\Sigma,R,S)$ is defined by

- V: a finite set variables
- Σ: finite set terminals (with $V \cap \Sigma = \emptyset$)
- R: finite set of substitution rules $V \to (V \cup \Sigma)^*$
- S: start symbol $\in V$

The language of grammar G is denoted by L(G):

$$L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^* w \}$$

Derivation $\Rightarrow^*$

A single step derivation "$\Rightarrow$" consist of the substitution of a variable by a string according to a substitution rule.

Example: with the rule "$A \rightarrow BB$", we can have the derivation "$01AB0 \Rightarrow 01BBB0$".

A sequence of several derivations (or none) is indicated by " $\Rightarrow^*$ "
Same example: "$0AA \Rightarrow^* 0BBBB$"

Some Remarks

The language $L(G) = \{ w \in \Sigma^* \mid S \Rightarrow^* w \}$ contains only strings of terminals, not variables.

Notation: we summarize several rules, like

$A \rightarrow B$

$A \rightarrow 01$ by $A \rightarrow B \mid 01 \mid AA$

$A \rightarrow AA$

Unless stated otherwise: topmost rule concerns the start variable

Context-Free Grammars (Ex.)

Consider the CFG $G=(V,\Sigma,R,S)$ with

$V = \{S\}$

$\Sigma = \{0,1\}$

R: $S \rightarrow 0S1 \mid 0Z1$

$Z \rightarrow 0Z \mid \varepsilon$

Then $L(G) = \{0^i 1^j \mid i \geq j \}$

S <u>yields</u> $0^{j+k}1^j$ according to:

$S \Rightarrow 0S1 \Rightarrow \ldots \Rightarrow 0^jS1^j \Rightarrow 0^jZ1^j \Rightarrow 0^j0Z1^j \Rightarrow$
$\ldots \Rightarrow 0^{j+k}Z1^j \Rightarrow 0^{j+k}\varepsilon 1^j = 0^{j+k}1^j$

Importance of CFL

Model for natural languages (Noam Chomsky)

Specification of programming languages:
“parsing of a computer program”

Describes mathematical structures

Intermediate between regular languages and computable languages

Example Boolean Algebra

Consider the CFG $G=(V,\Sigma,R,S)$ with

$V = \{S,Z\}$

$\Sigma = \{0,1,(,),\neg,\vee,\wedge\}$

R: $S \rightarrow 0 \mid 1 \mid \neg(S) \mid (S)\vee(S) \mid (S)\wedge(S)$

Some elements of L(G):

0

$\neg((\neg(0))\vee(1))$

$(1)\vee((0)\wedge(0))$

Note: Parentheses prevent "$1\vee0\wedge0$" confusion.

Human Languages

Number of rules:

<SENTENCE> $\rightarrow$ <NOUN-PHRASE><VERB-PHRASE>

<NOUN-PHRASE> $\rightarrow$ <CMPLX-NOUN> | <CMPLX-NOUN><PREP-PHRASE>

<VERB-PHRASE> $\rightarrow$ <CMPLX-VERB> | <CMPLX-VERB><PREP-PHRASE>

<CMPLX-NOUN> $\rightarrow$ <ARTICLE><NOUN>

<CMPLX-VERB> $\rightarrow$ <VERB> | <VERB><NOUN-PHRASE> ...

<ARTICLE> $\rightarrow$ a | the

<NOUN> $\rightarrow$ boy | girl | house

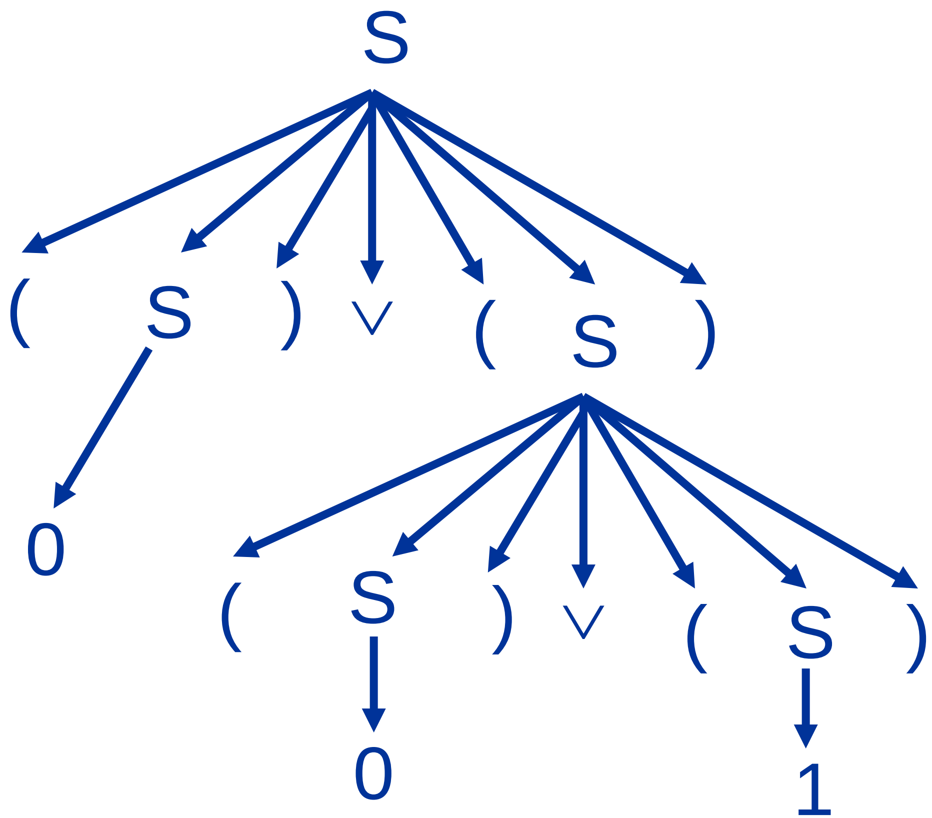
<VERB> $\rightarrow$ sees | ignores

Possible element: `the boy sees the girl`

Parse Trees

The parse tree of $(0)\vee((0)\wedge(1))$ via rule

$$S \rightarrow 0 \mid 1 \mid \neg(S) \mid (S)\vee(S) \mid (S)\wedge(S):$$



Ambiguity

A grammar is <u>ambiguous</u> if some strings are derived <u>ambiguously</u>.

A string is derived <u>ambiguously</u> if it has more than one <u>leftmost derivations</u>.

Typical example: rule $S \rightarrow 0 \mid 1 \mid S+S \mid S\times S$

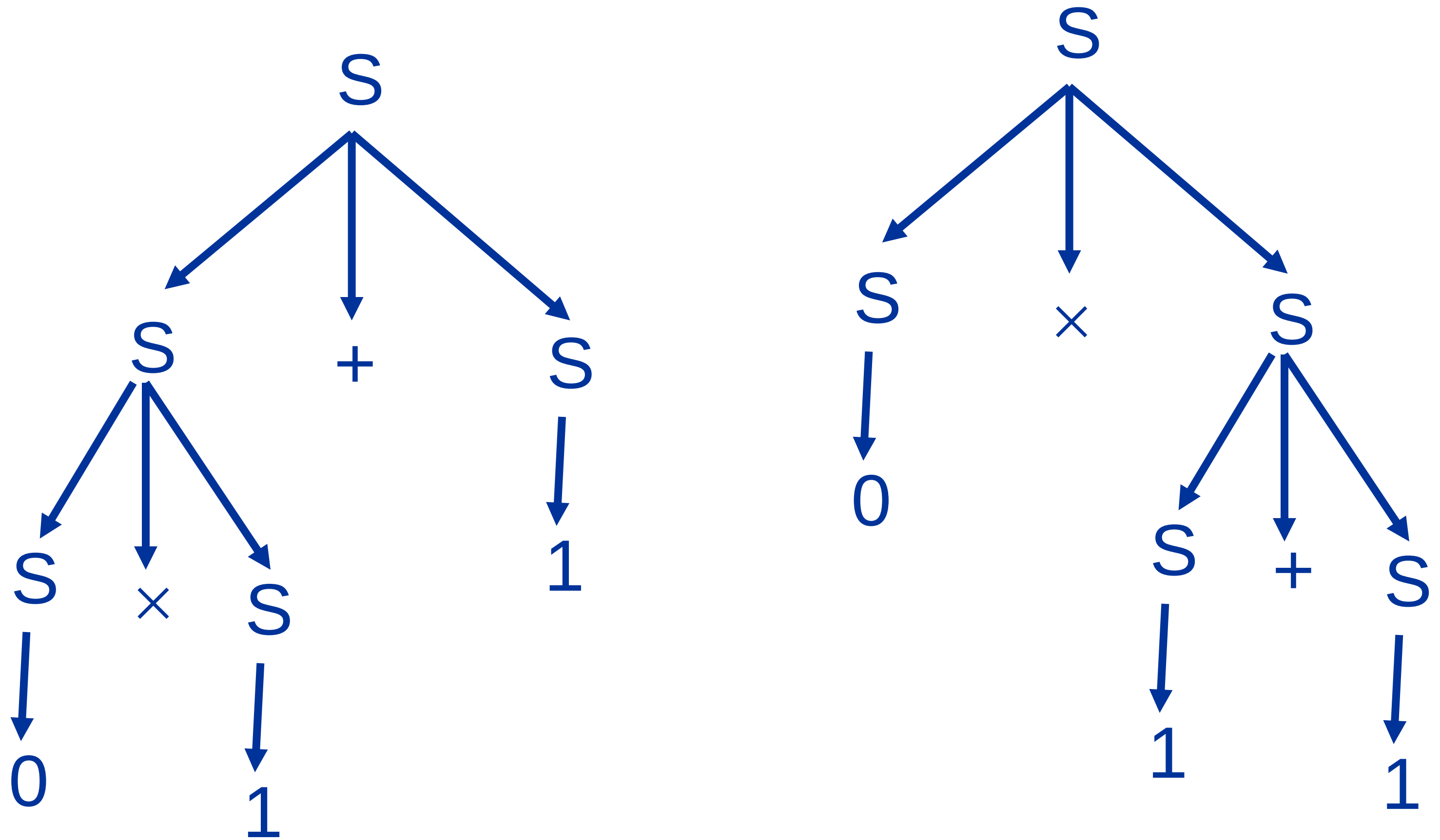
$S \Rightarrow S+S \Rightarrow S\times S+S \Rightarrow 0\times S+S \Rightarrow 0\times 1+S \Rightarrow 0\times 1+1$

versus

$S \Rightarrow S\times S \Rightarrow 0\times S \Rightarrow 0\times S+S \Rightarrow 0\times 1+S \Rightarrow 0\times 1+1$

Ambiguity and Parse Trees

The ambiguity of $0 \times 1+1$ is shown by the two different parse trees:



More on Ambiguity

The two different derivations:

$S \Rightarrow S+S \Rightarrow 0+S \Rightarrow 0+1$

and

$S \Rightarrow S+S \Rightarrow S+1 \Rightarrow 0+1$

do *not* constitute an ambiguous string 0+1 (they will have the same parse tree)

Languages that can only be generated by ambiguous grammars are "<u>inherently ambiguous</u>"

Context-Free Languages

Any language that can be generated by a context free grammar is a context-free language (CFL).

The CFL $\{ 0^n 1^n \mid n \geq 0 \}$ shows us that certain CFLs are nonregular languages.

Q1: Are all regular languages context free?

Q2: Which languages are outside the class CFL?

"Chomsky Normal Form"

A context-free grammar $G = (V,\Sigma,R,S)$ is in <u>Chomsky normal form</u> if every rule is of the form

$$A \to BC$$

or $$A \to x$$

with variables $A \in V$ and $B,C \in V \setminus \{S\}$, and $x \in \Sigma$

For the start variable S we also allow the rule

$$S \to \varepsilon$$

Advantage: Grammars in this form are far easier to analyze.

Theorem 2.9

Every context-free language can be described by a grammar in Chomsky normal form.

Outline of Proof:
We rewrite every CFG in Chomsky normal form.
We do this by replacing, one-by-one, every rule that is not 'Chomsky'.
We have to take care of: Starting Symbol, ε symbol, all other violating rules.

Proof of Theorem 2.9

Given a context-free grammar $G = (V,\Sigma,R,S)$, rewrite it to Chomsky Normal Form by

1) New start symbol S_0 (and add rule $S_0 \to S$)
2) Remove $A \to \varepsilon$ rules (*from the tail*):
   before: $B \to xAy$ and $A \to \varepsilon$, after: $B \to xAy \mid xy$
3) Remove unit rules $A \to B$ (*by the head*): "$A \to B$" and "$B \to xCy$", becomes "$A \to xCy$" and "$B \to xCy$"
4) Shorten all rules to two: before: "$A \to B_1 B_2 \ldots B_k$",
   after: $A \to B_1 A_1$, $A_1 \to B_2 A_2, \ldots, A_{k-2} \to B_{k-1} B_k$
5) Replace ill-placed terminals "a" by T_a with $T_a \to a$

Proof of Theorem 2.9

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5) Replace ill-placed terminals "a" by T_a with $T_a \rightarrow a$

Careful Removing of Rules

Do not introduce new rules that you removed earlier.

Example: $A \rightarrow A$ simply disappears

When removing $A \rightarrow \varepsilon$ rules, insert *all* new replacements:

$B \rightarrow AaA$ becomes $B \rightarrow AaA \mid aA \mid Aa \mid a$

Example of Chomsky NF

Initial grammar: $S \to aSb \mid \varepsilon$

In Chomsky normal form:

$$
\begin{aligned}
&S_0 \to \varepsilon \mid T_a T_b \mid T_a X \\
&X \to S T_b \\
&S \to T_a T_b \mid T_a X \\
&T_a \to a \\
&T_b \to b
\end{aligned}
$$

RL $\subseteq$ CFL

Every regular language can be expressed by a context-free grammar.

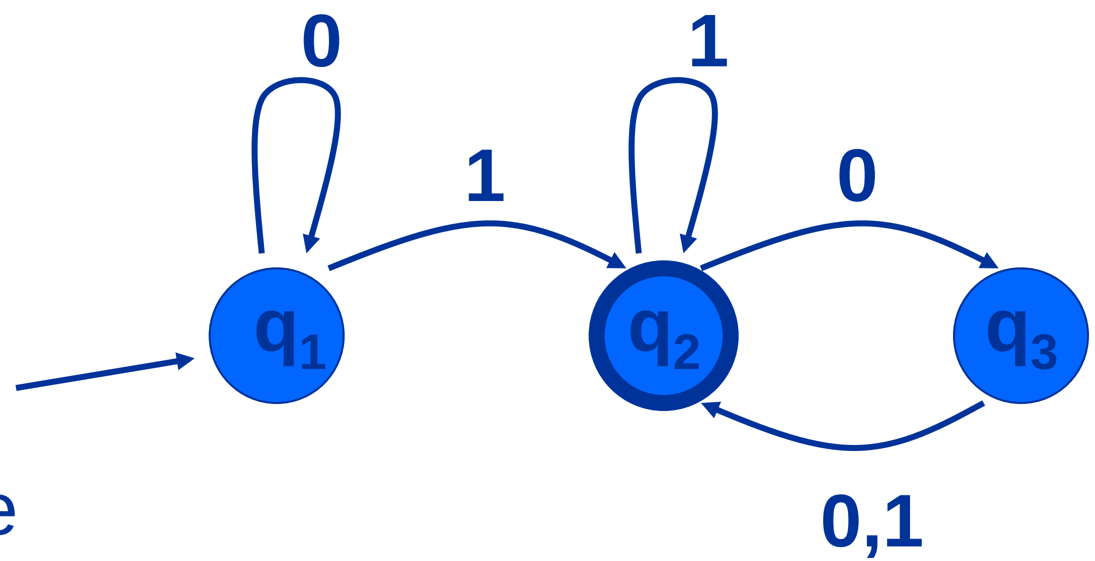
Proof Idea:
Given a DFA $M = (Q,\Sigma,\delta,q_0,F)$, we construct a corresponding CF grammar $G_M = (V,\Sigma,R,S)$ with $V = Q$ and $S = q_0$
Rules of G_M:

$q_i \rightarrow x\,\delta(q_i,x)$ for all $q_i \in V$ and all $x \in \Sigma$

$q_i \rightarrow \varepsilon$ for all $q_i \in F$

Example RL $\subseteq$ CFL

The DFA



leads to the
context-free grammar
$G_M = (Q,\Sigma,R,q_1)$ with the rules

$$q_1 \rightarrow 0q_1 \mid 1q_2$$
$$q_2 \rightarrow 0q_3 \mid 1q_2 \mid \varepsilon$$
$$q_3 \rightarrow 0q_2 \mid 1q_2$$

Picture Thus Far