The Most Elegant Bump Number Algorithm

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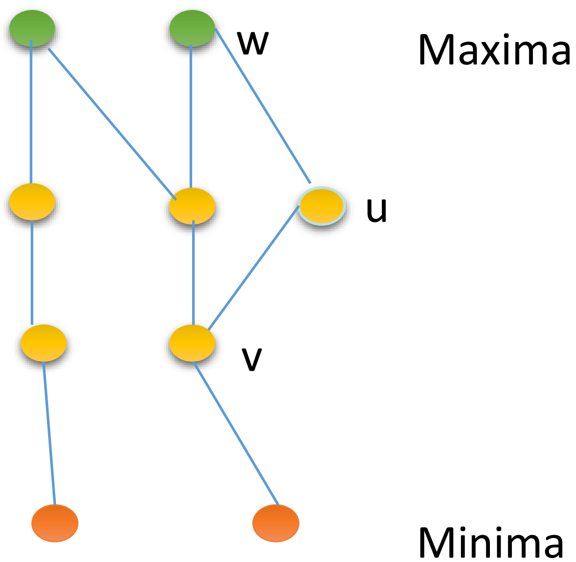
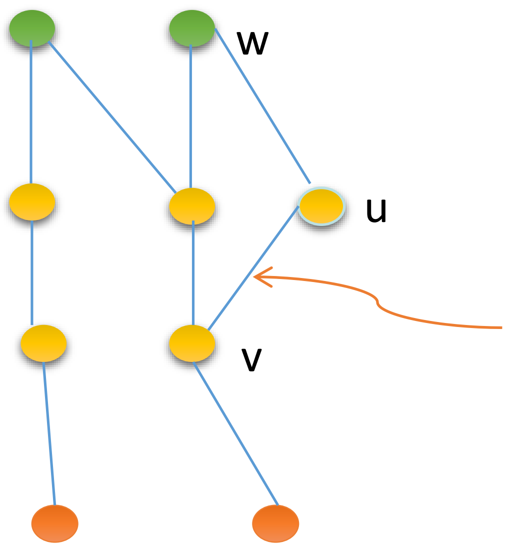
Posets = partially ordered sets



Hasse Diagram

Posets = partially ordered sets



w > v

v < w

v and w are *transitively related*

u *covers* v

u is an *upper cover* of v

v is a *lower cover* of u

$u \lessdot v$

Hasse Diagram

- A compact representation of a set of relations
- i.e. can be $O(n)$ representation of $O(n^2)$ relations

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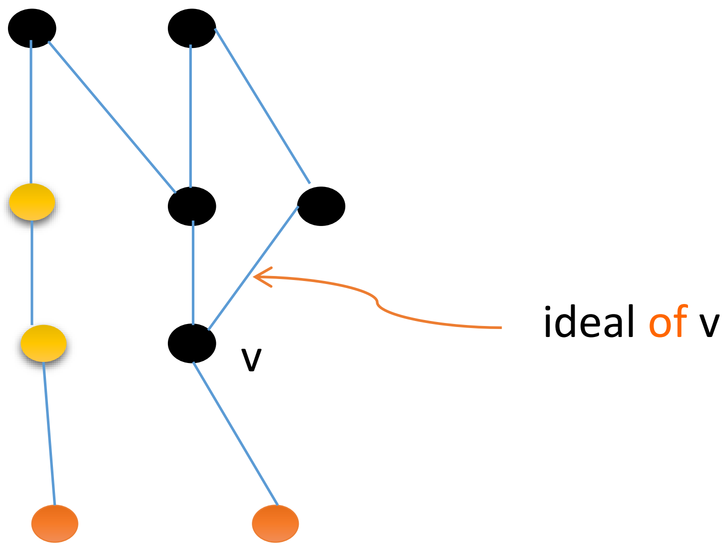

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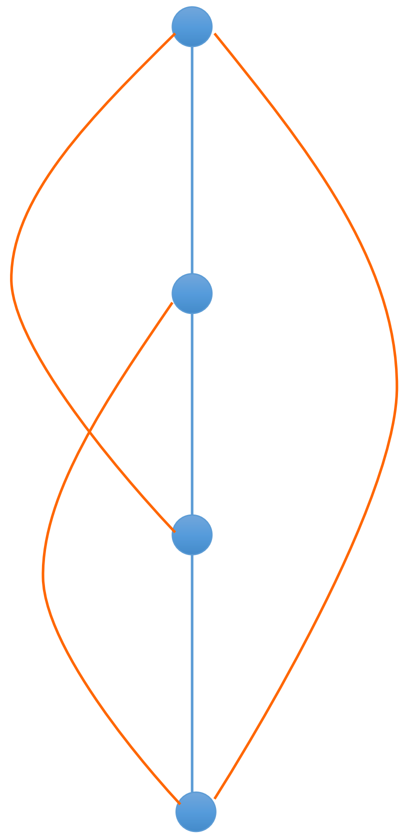
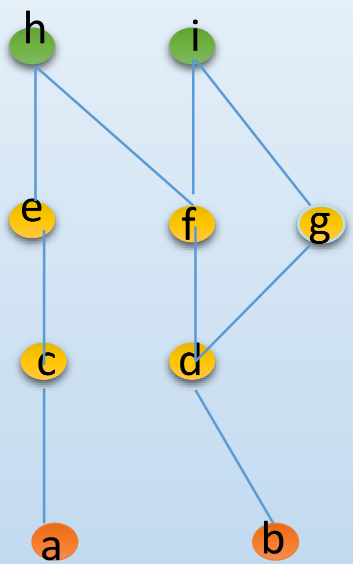


Hasse Diagram
O(n) for chain

Comparability Graph
O(n^2) for chain

Posets = *partially* ordered sets



Can always be extended (add relations) until the ordering is *linear*

… equivalent to shelling the poset

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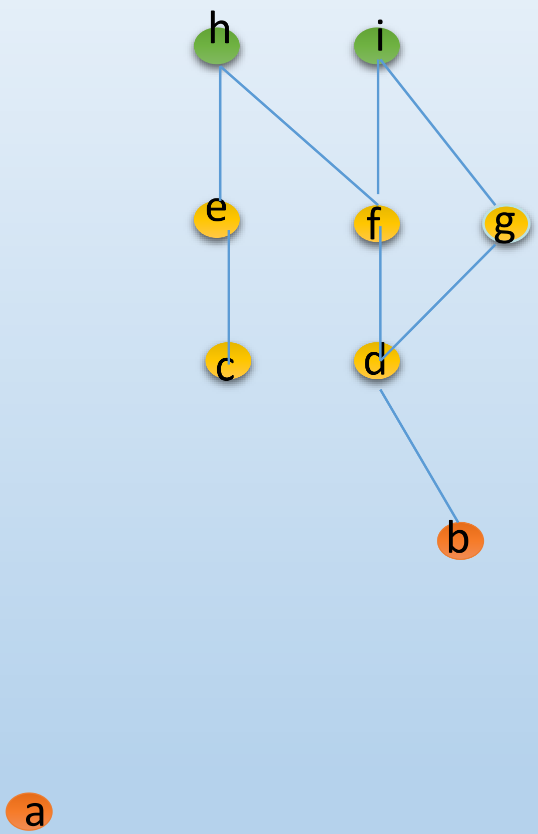

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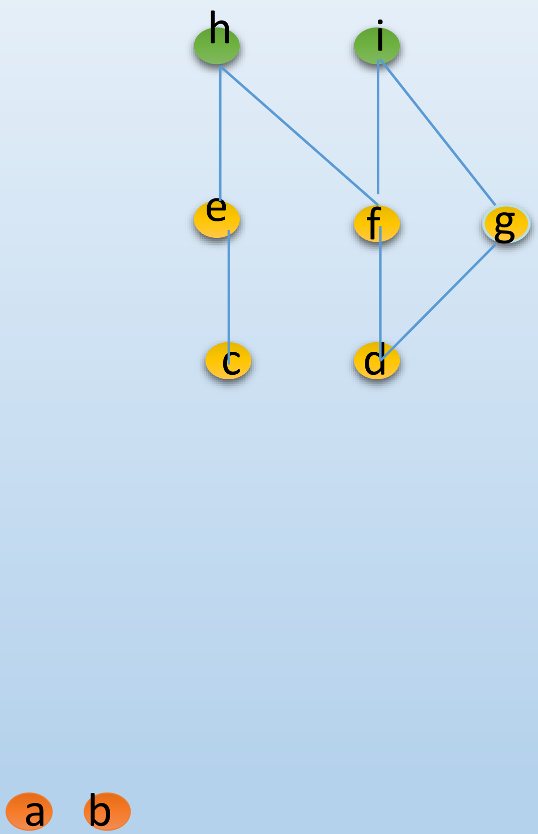

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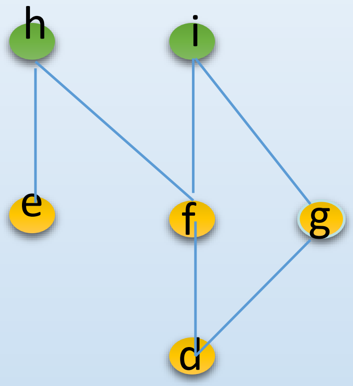

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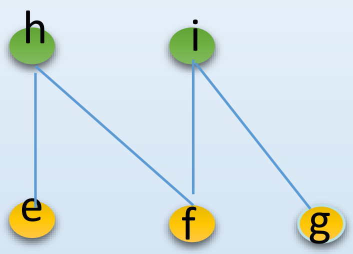

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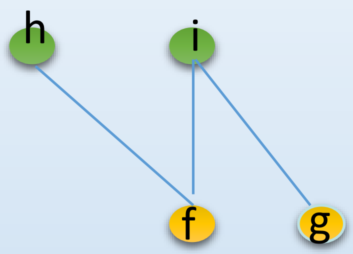

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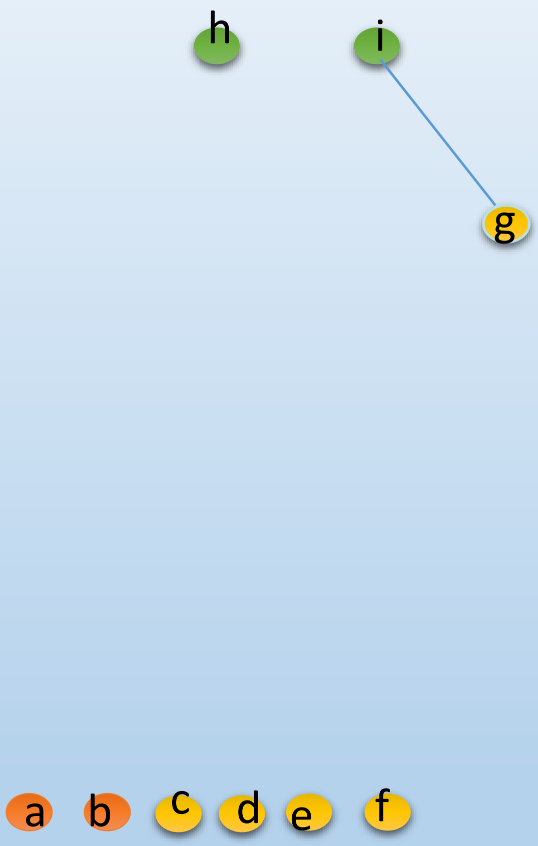

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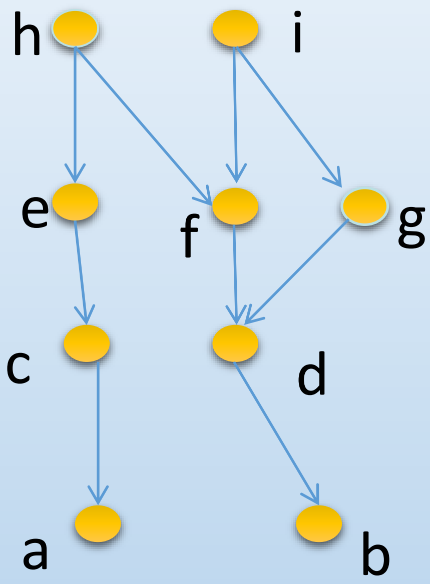
Bumps in linear extensions



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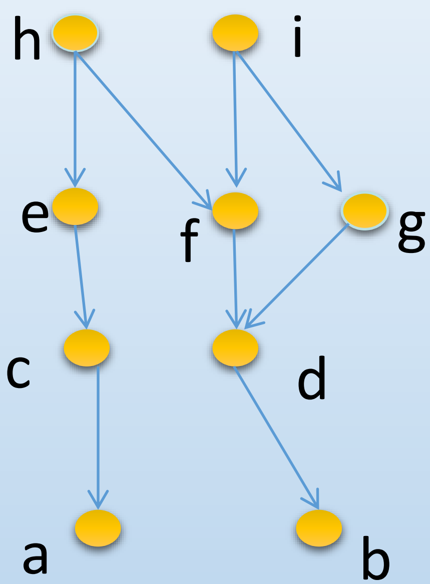

Linear extension (showing bumps)

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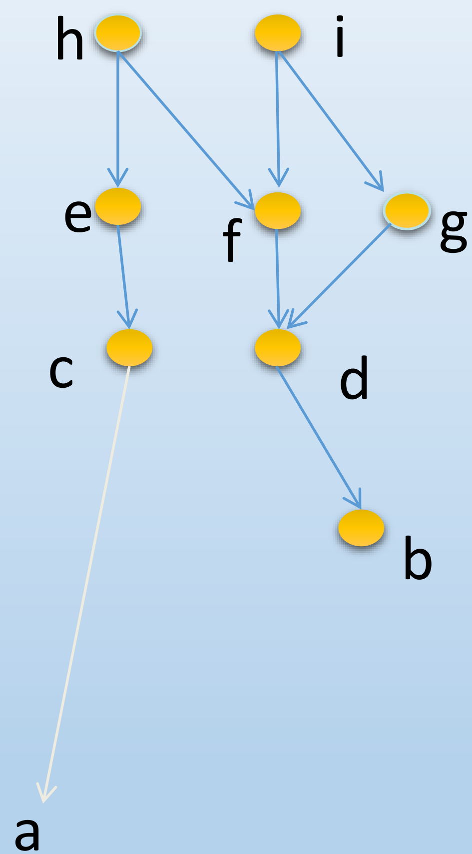

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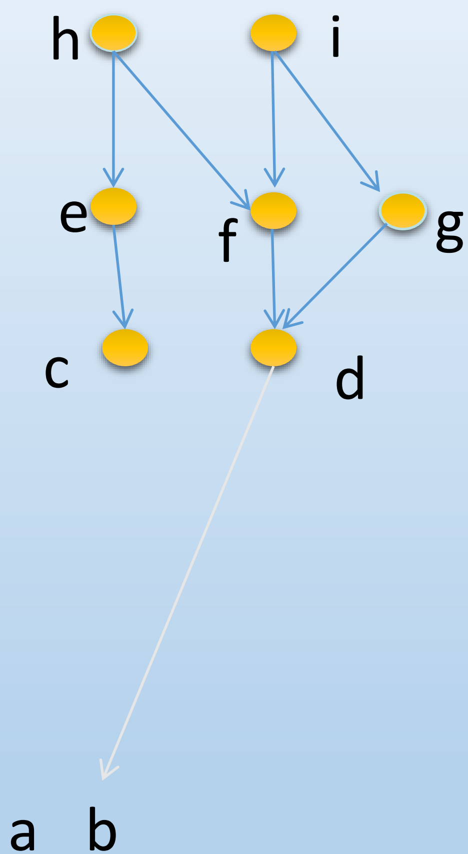

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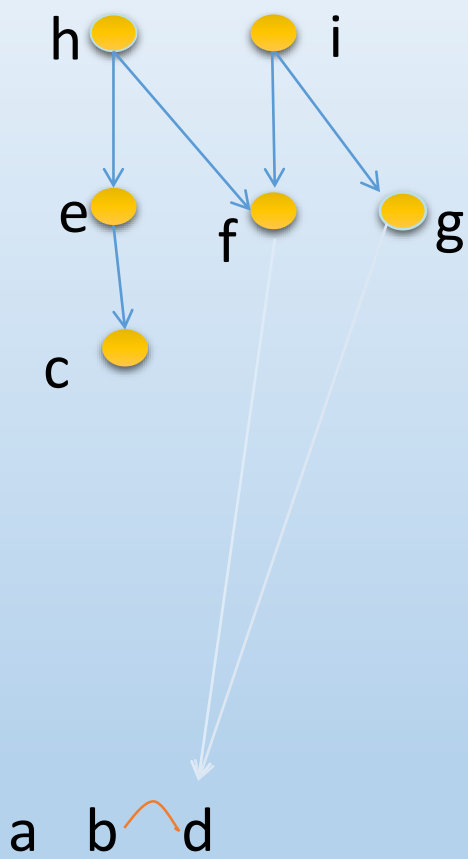

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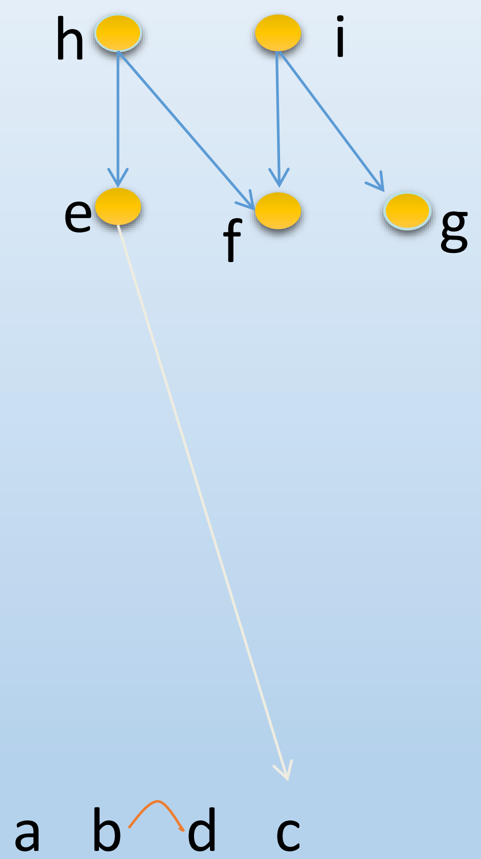

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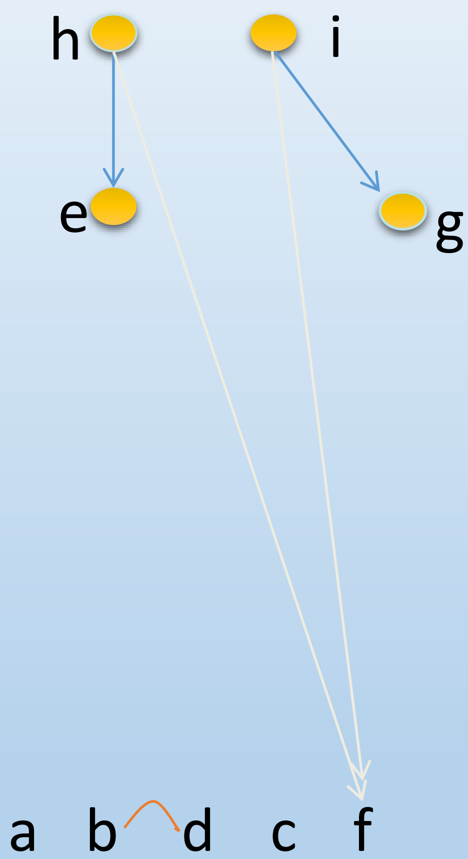

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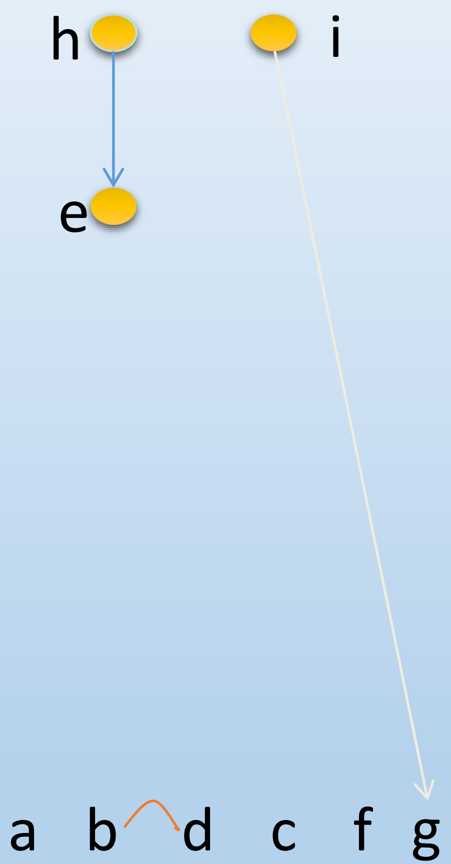


Linear extension (showing bumps)

Bumps in linear extensions

Linear extension (showing bumps)

Bumps in linear extensions

h

Linear extension (showing bumps)

a b d c f g e i

Bumps in linear extensions

Linear extension (showing bumps)

a b d c f g e i h

Bump Number Problem

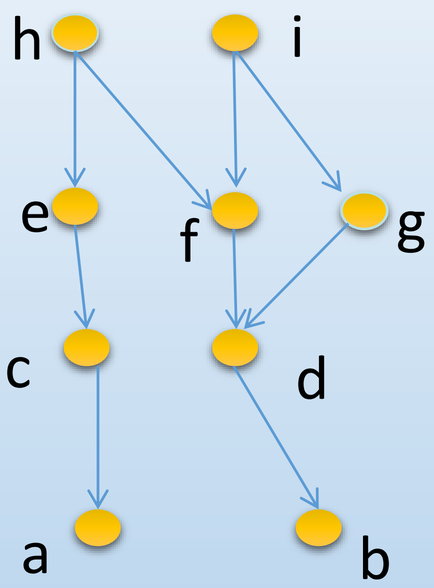
Given poset P, what is the least number of bumps realized by a linear extension of P?

b(P)= bump# of P

Find an algorithm to compute b(P) and construct a linear extension with fewest bumps

a b d c f g e i h

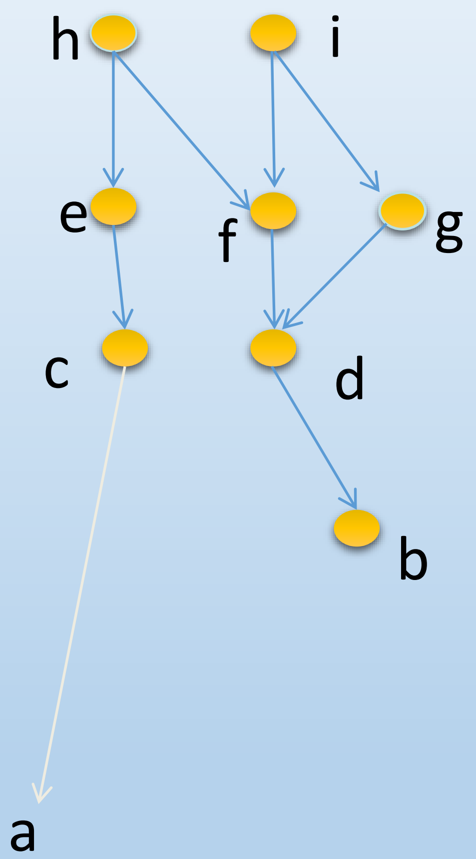
Greedily seeking min-bump l.e.



Linear extension (showing bumps)

Greedily selecting to avoid bumps

Greedily seeking min-bump l.e



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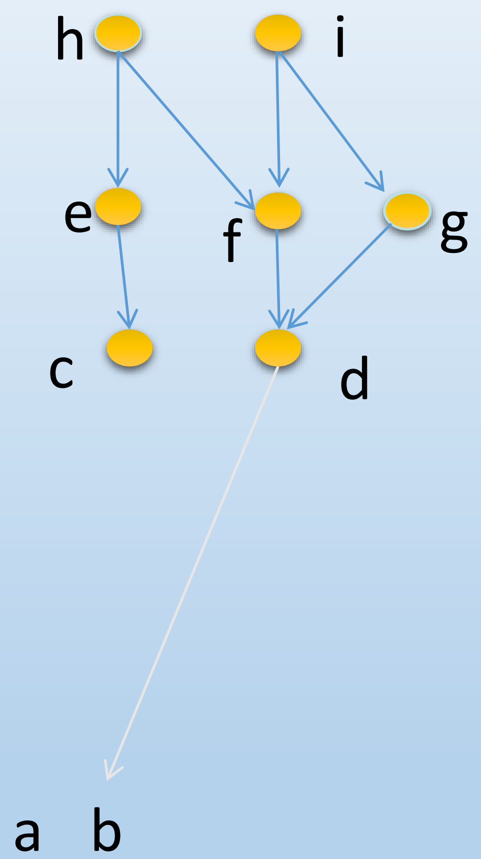

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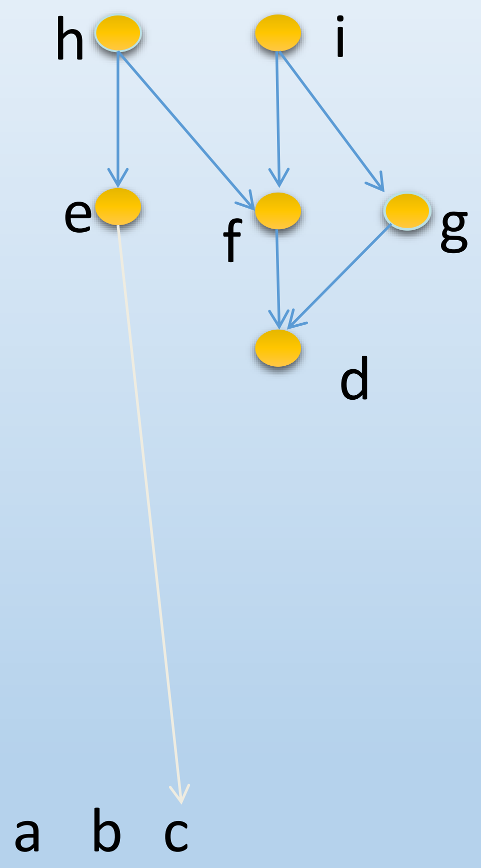

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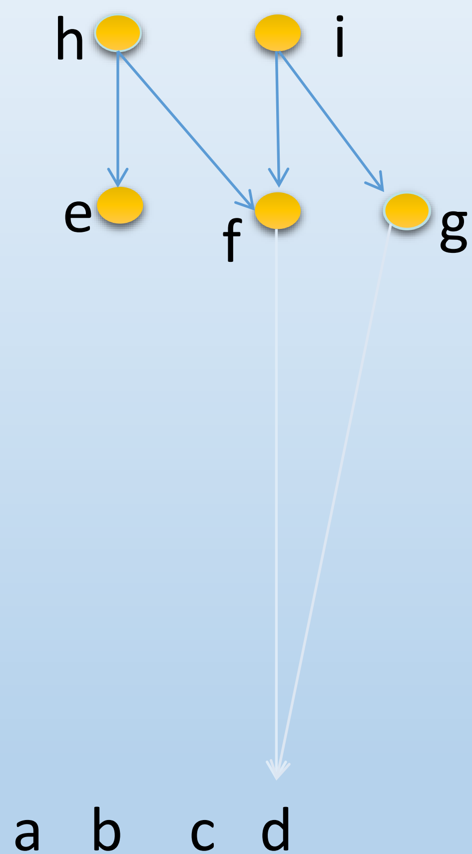

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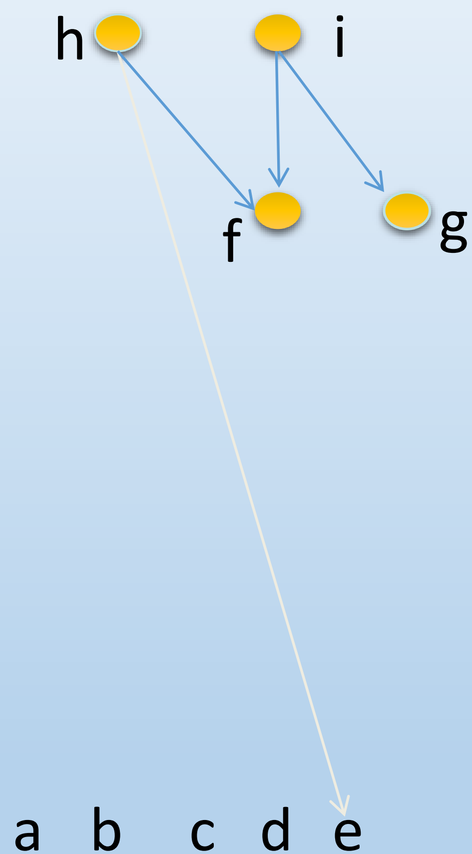

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i

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Greedily selecting to avoid bumps

a b c d e f g h

Greedily seeking min-bump l.e

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Greedily selecting to avoid bumps

a b c d e f g h i

Greedily seeking min-bump l.e

There is always some greedy l.e. that achieves minimum bump (Fishburn & Gehrlein, ‘86).

For which posets does greedy always work?

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Greedy + ? works for all posets?

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For which posets does greedy always work? F&G’86

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For which posets does greedy always work? F&G'86

Greedy + ? works for all posets? This talk

Bump Number

- polynomial algorithms for interval order posets and for partial semiorder posets – both are based on the greedy shelling algorithms
  Fishburn and Gehrlein 1986

- polynomial algorithm for width=2 posets – not based on greedy shelling
  Zaguia 1987

- polynomial algorithm for any poset – not based on shelling
  Habib, Möhring, Steiner 1988

- linear time algorithm – based on Gabow's linear time 2-proc scheduling algorithm
  Schäffer & Simons 1988

Linear Time Bump Number

Schäffer & Simons 1988

relies on Gabow and Tarjan's special case Union-Find algorithm: union and find operations known in advance

O(n+m) in the cover graph

… relies on hybrid linked-list / array data structure ... Switch to array representation of tree for subtrees that are small enough…

Algorithm, proof of correctness, and analysis

- Spread across several papers
- Proofs long and case-ridden
- Analysis complex

Question:

a simple algorithm

$\exists$ with a short proof

that can be made efficient (linear time) without recourse to Special Case of Union-Find?

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Question:

a simple algorithm YES

$\exists$ with a short proof YES

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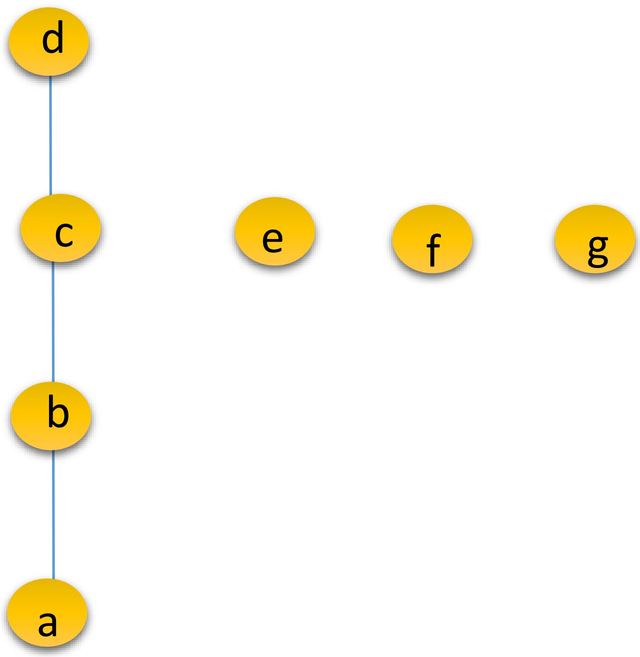
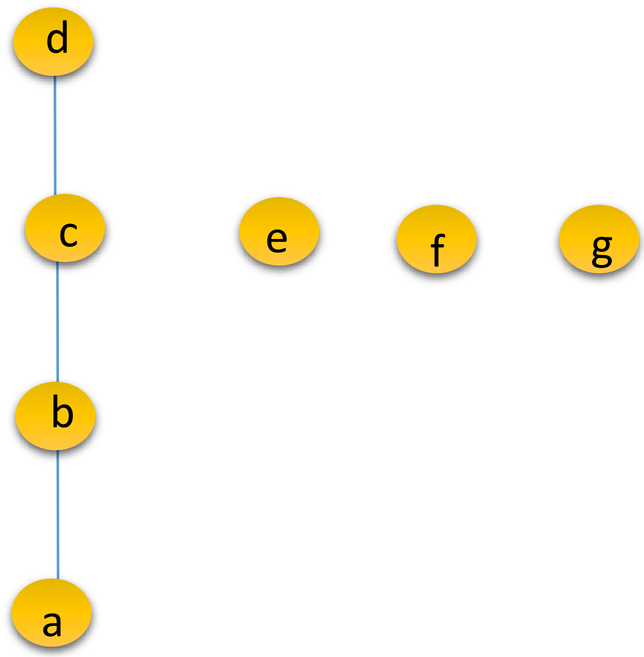
When Greedy is not enough



When Greedy is not enough

e a f b g c d



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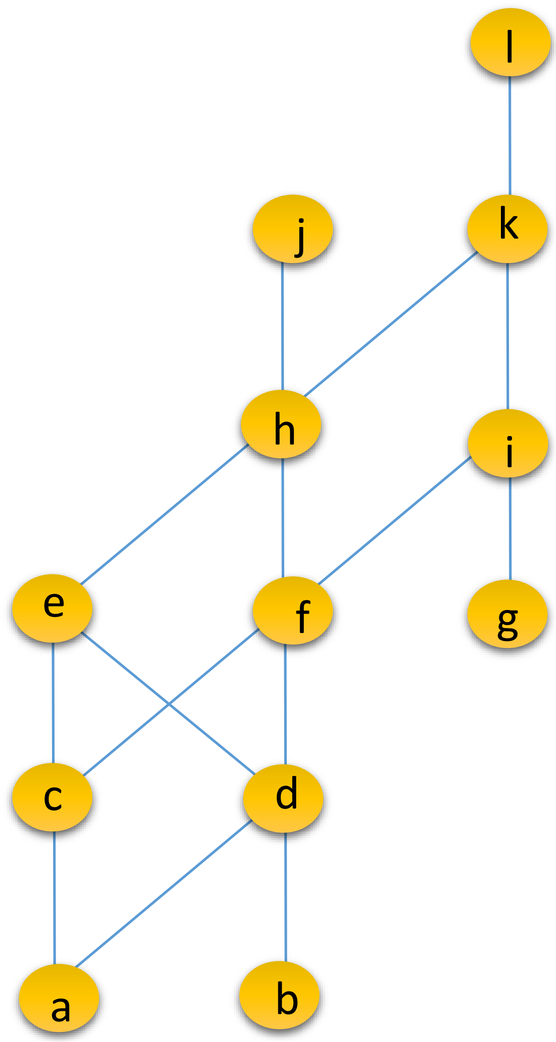
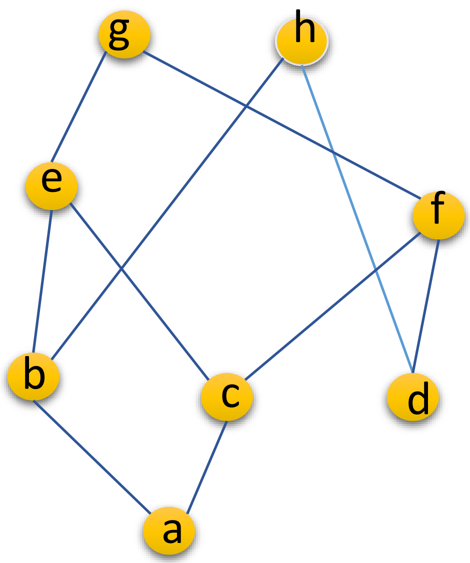
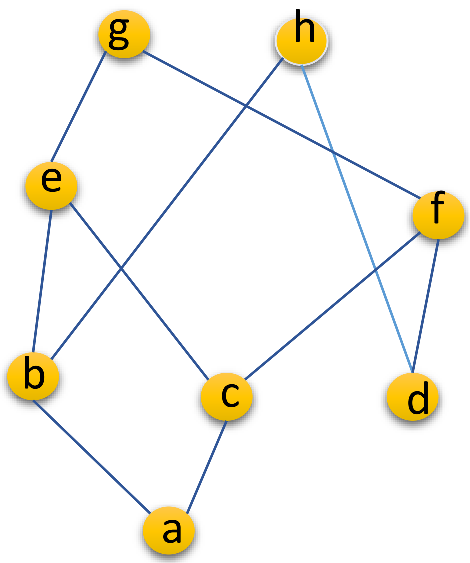


Greedy bump#



Greedy Approach

Greedy bump#



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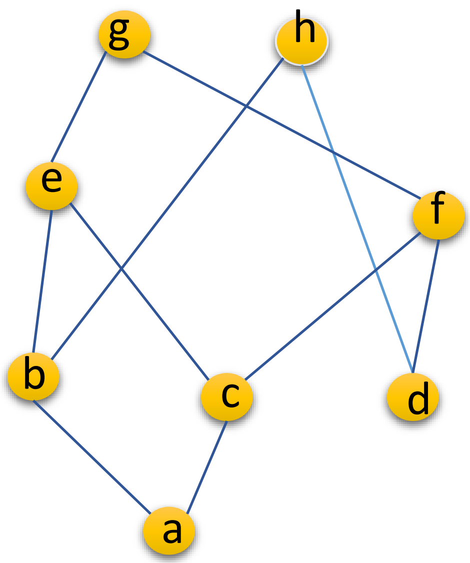

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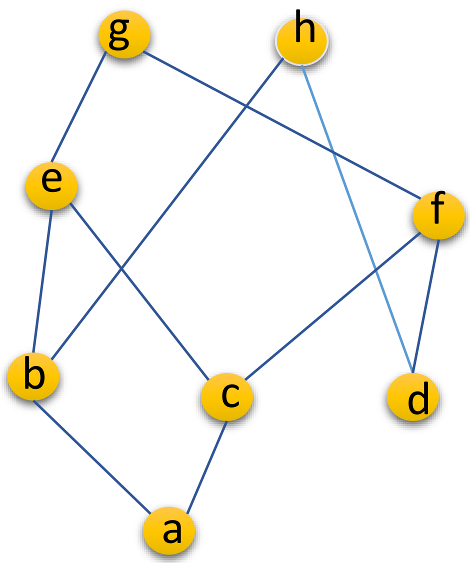

Greedy Approach

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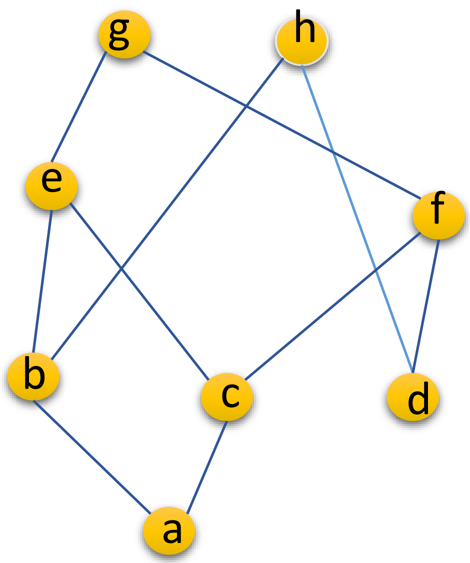
a d c b f e h g

Take the min at bottom of longest chain?
Take the min with largest upper ideal?



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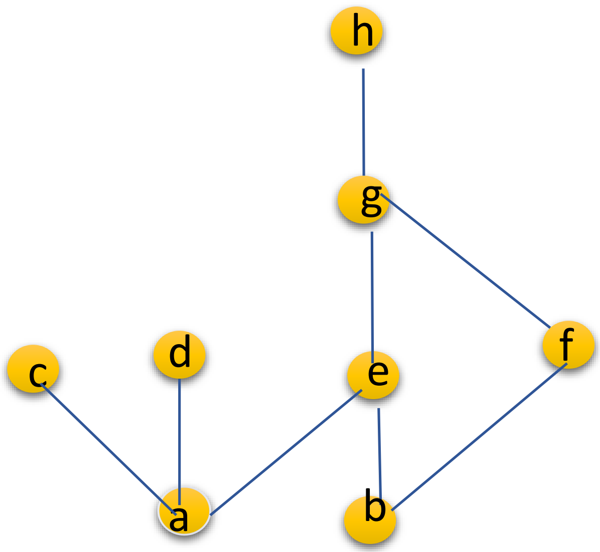

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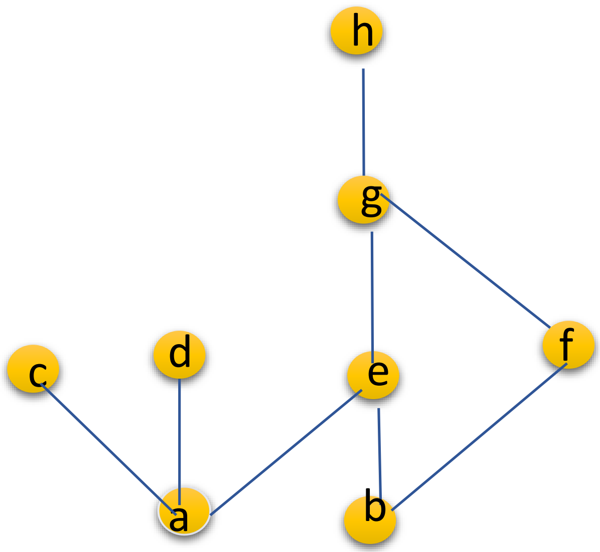

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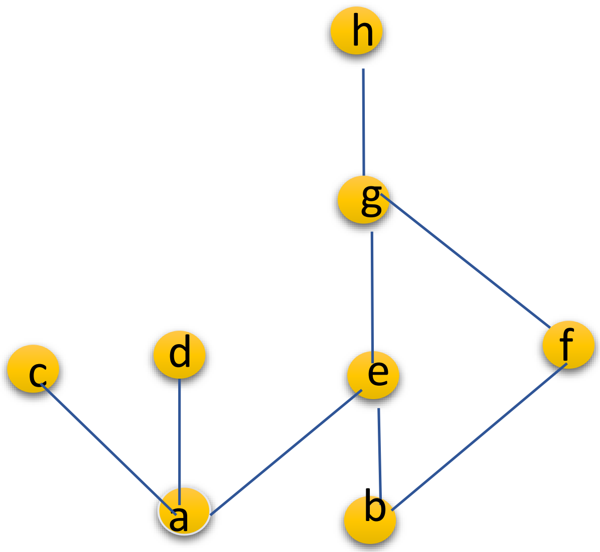
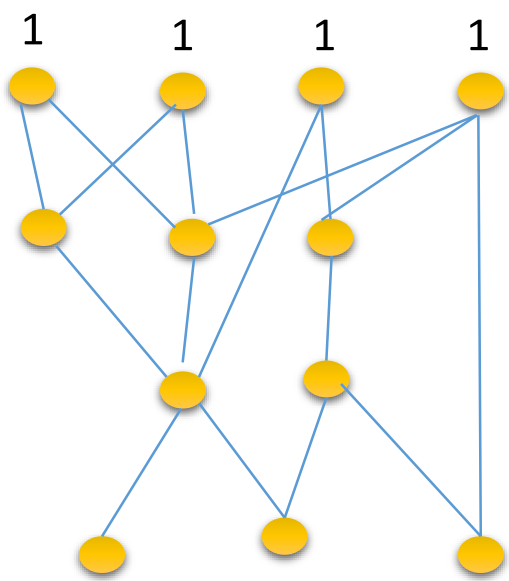


a b c e f d g h

b a f e c g d h

Lexicographic Labelling



- Give minima arbitrary lex#

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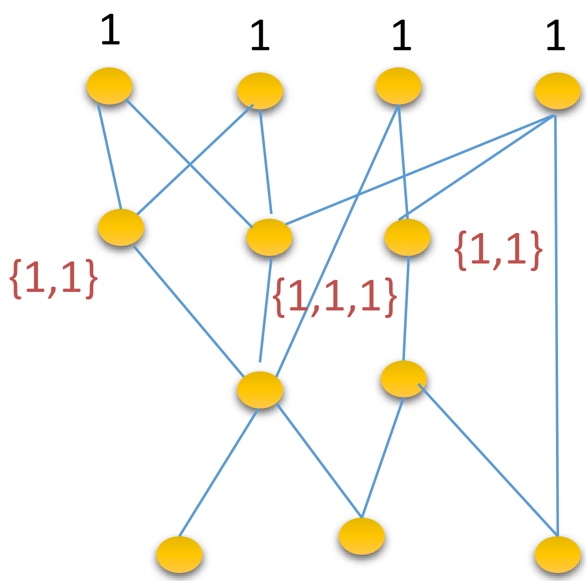

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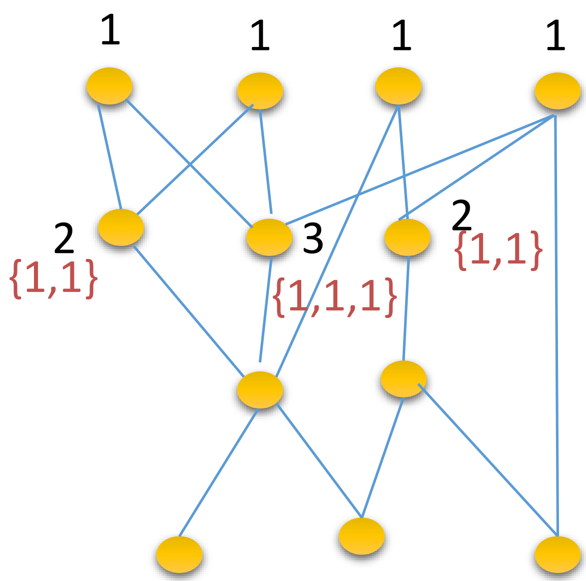

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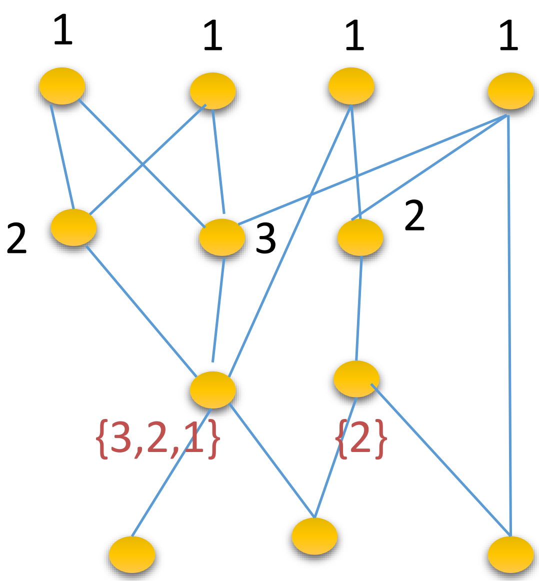

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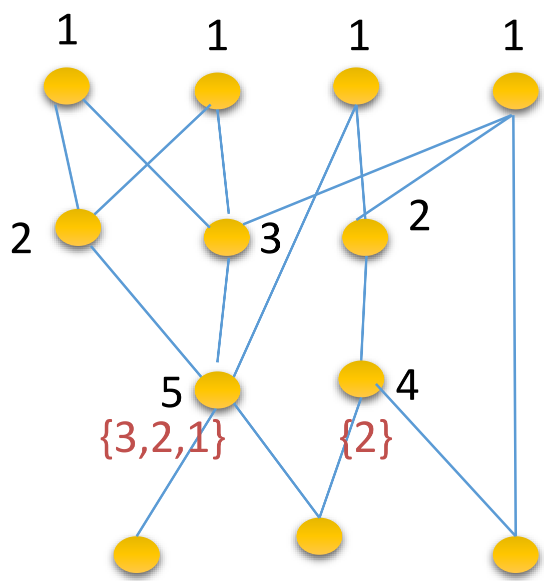

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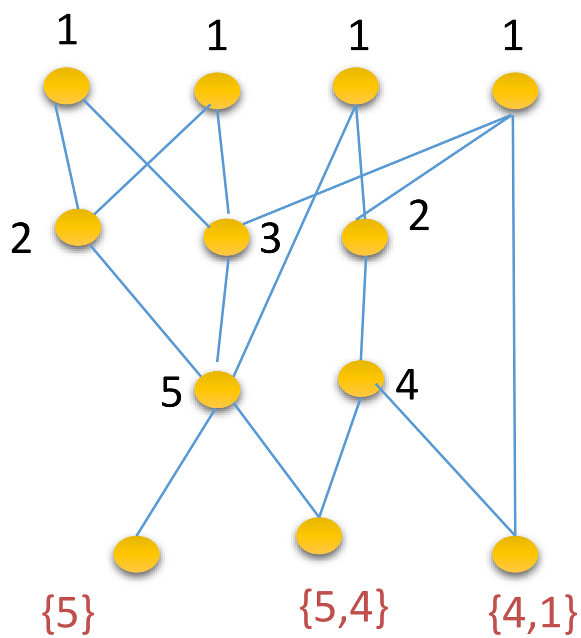

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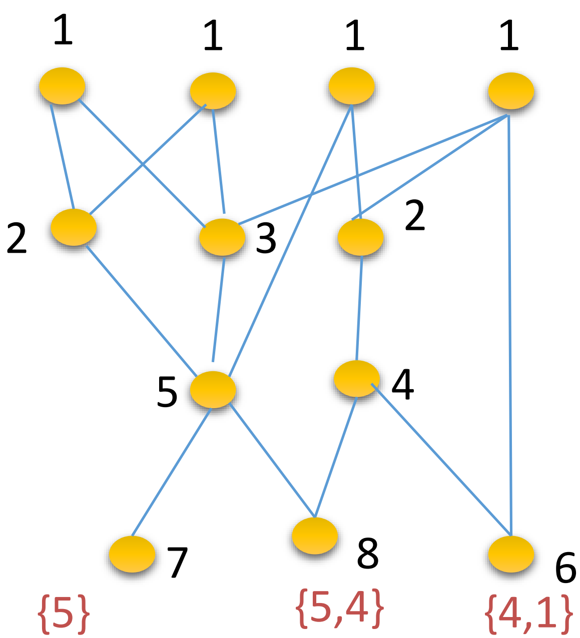

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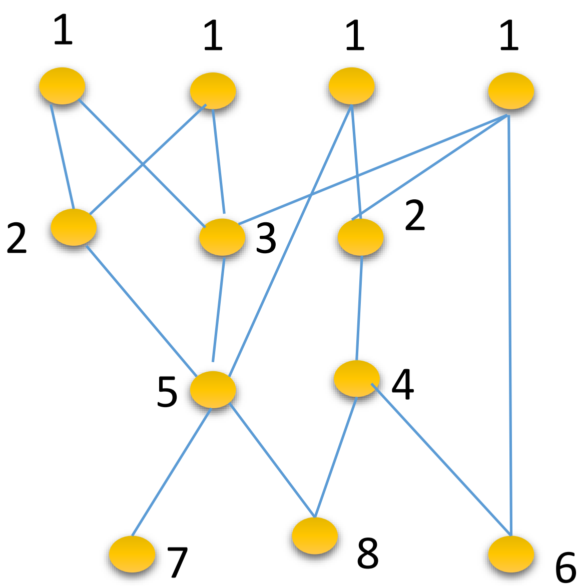

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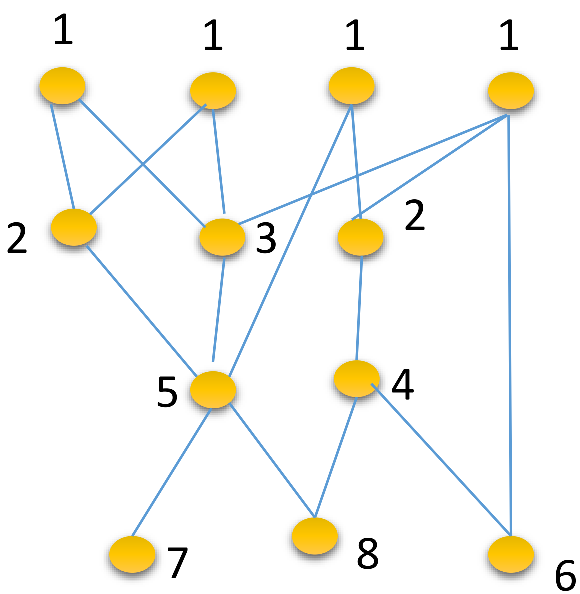

New: O(n+m) algorithm for lex-labelling

(Sethi 1976 algorithm also achieves linear time)

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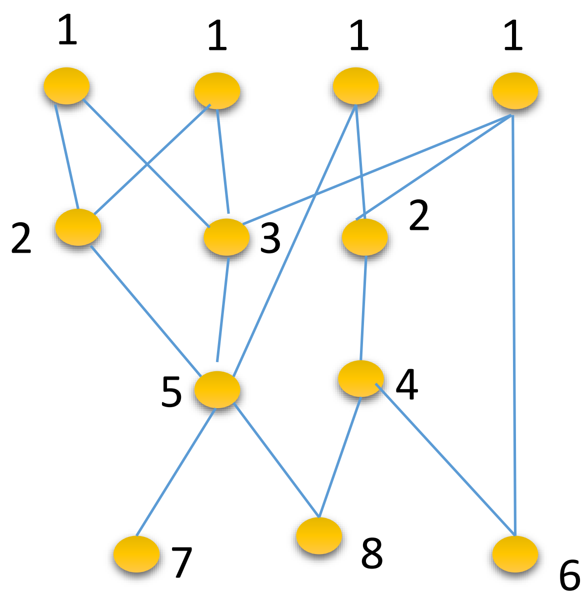

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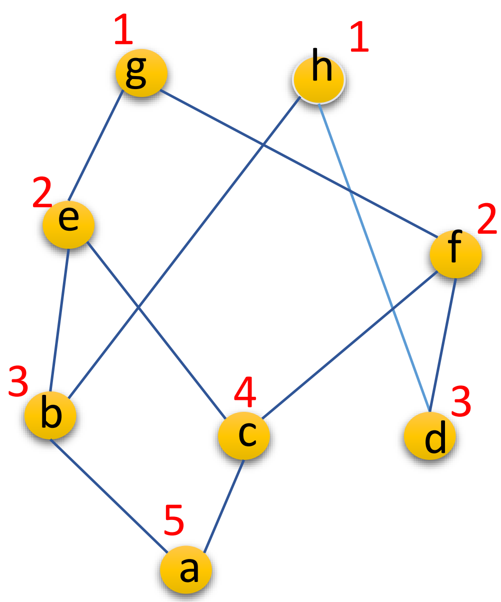
(Sethi 1976 algorithm also achieves linear time)

… or does it?

Greedy + Lex = Greedlex



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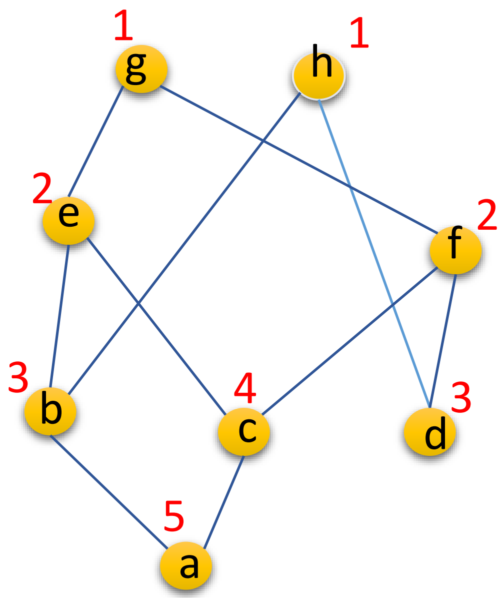

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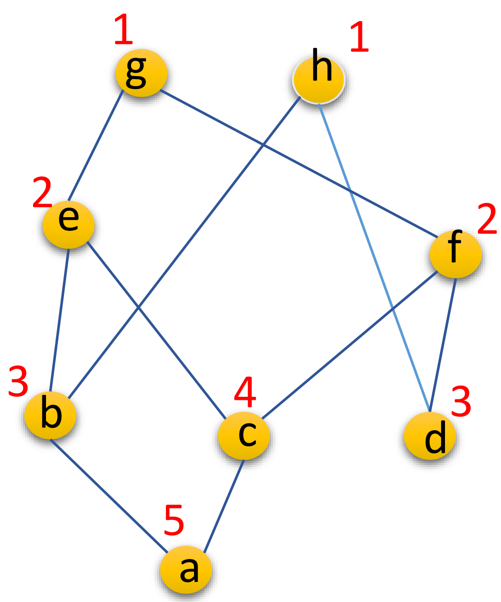

c has a higher lex-number than b

… therefore c has an upper cover that b doesn't have, of high lex number (i.e., a private cover w.r.t. c)

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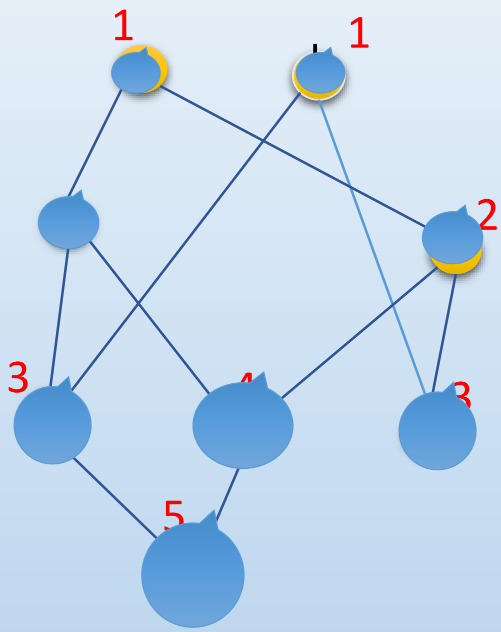

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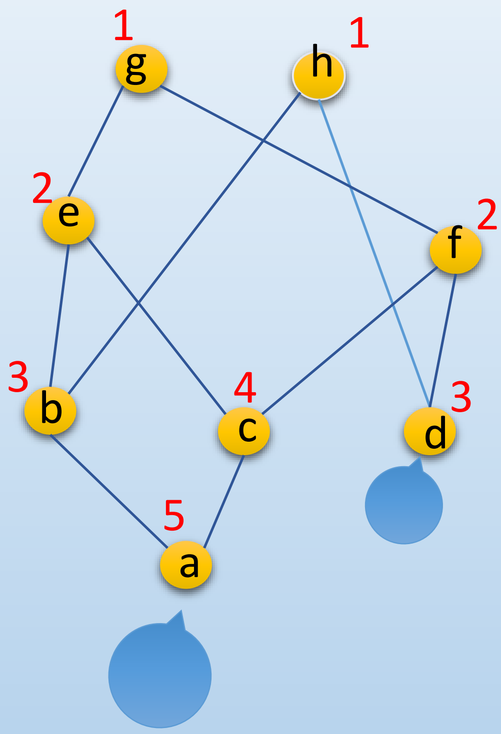

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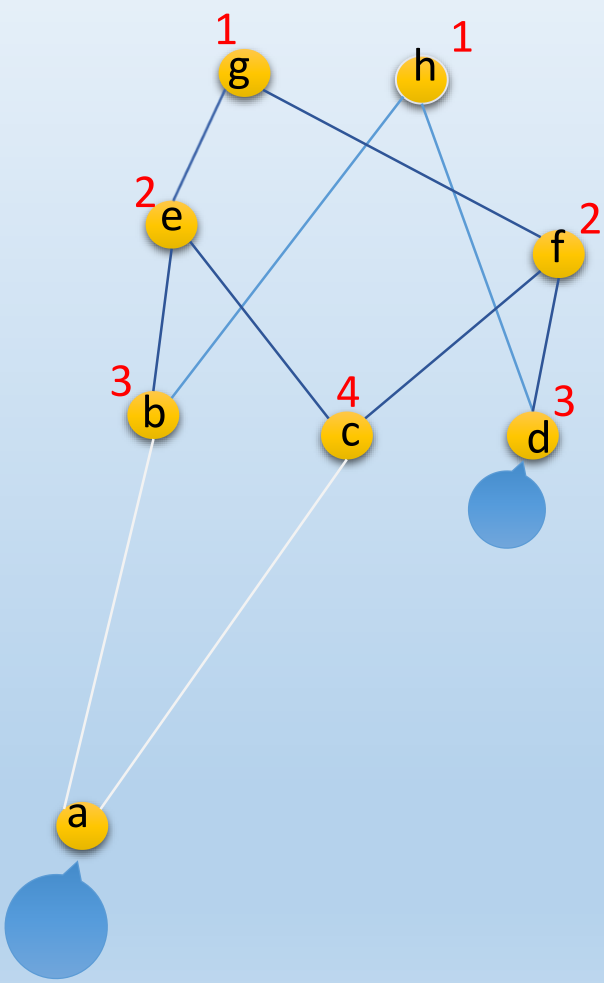

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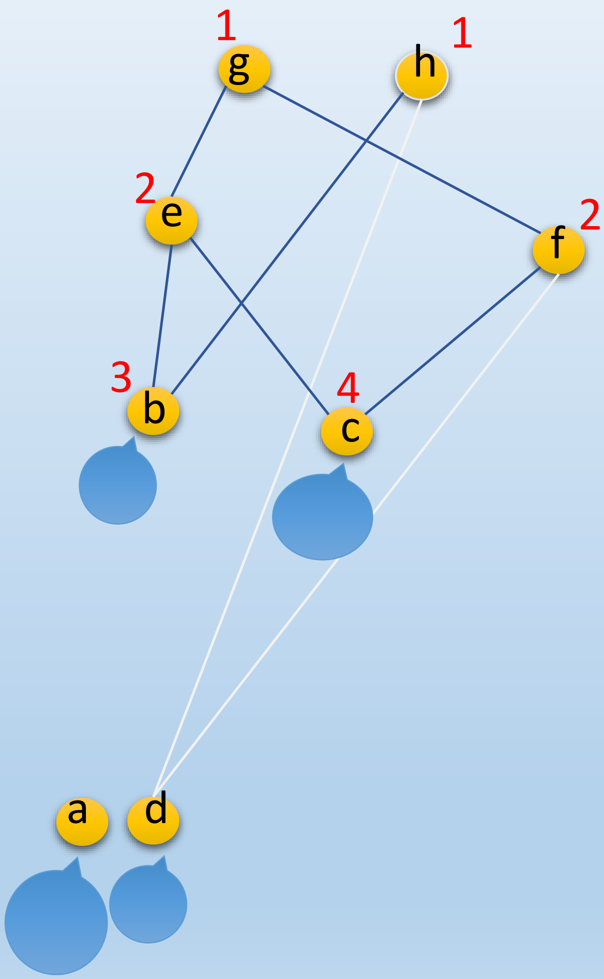

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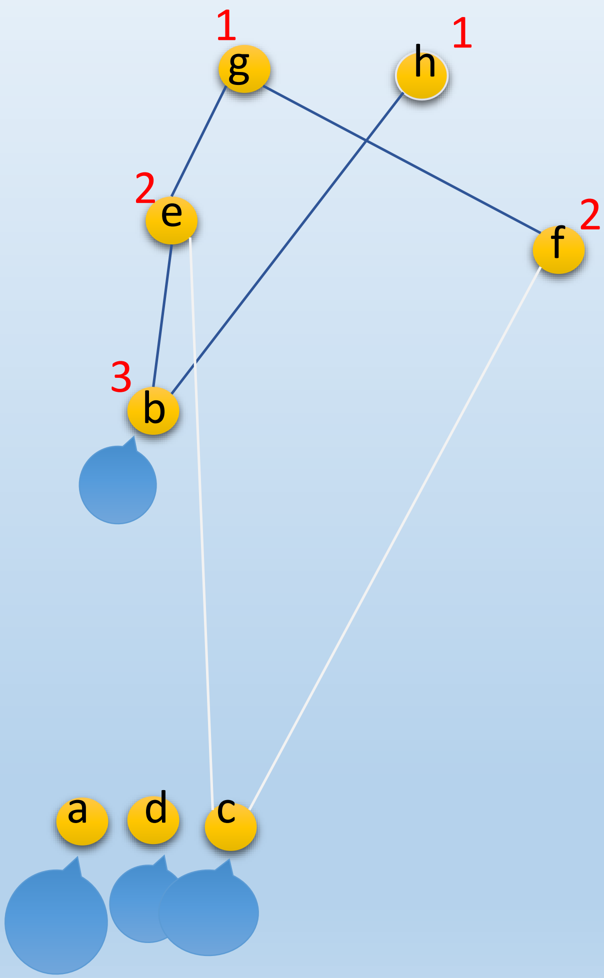

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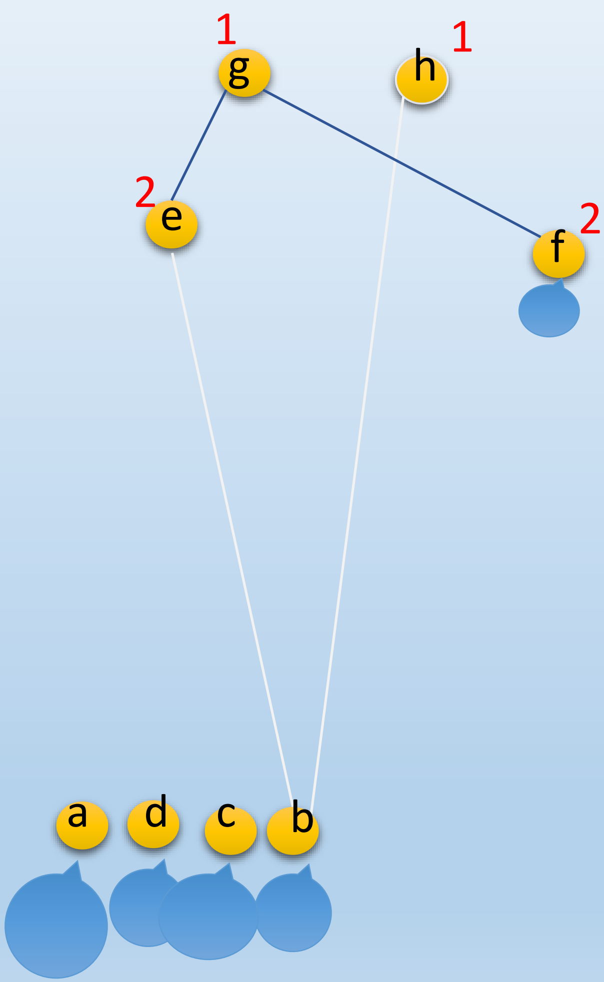

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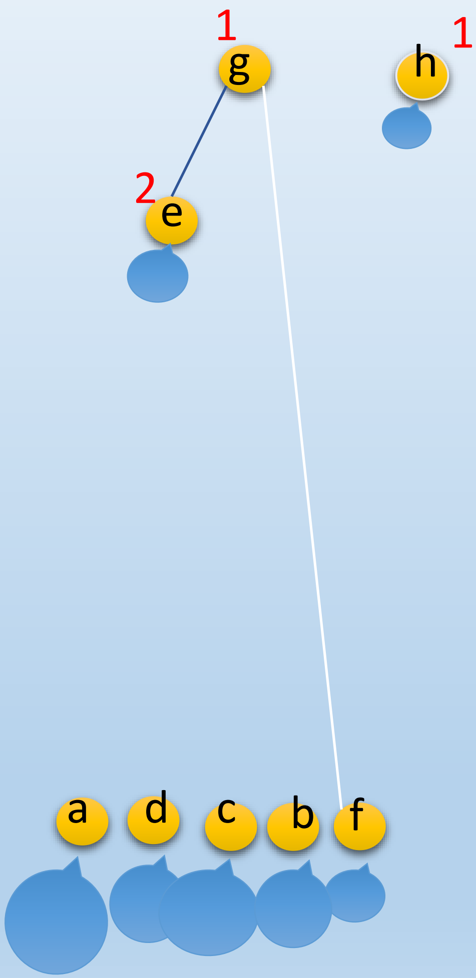

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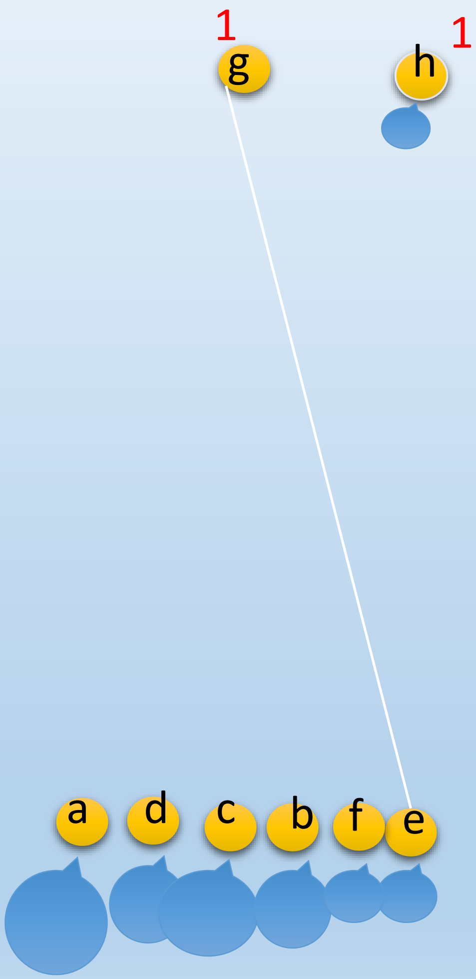

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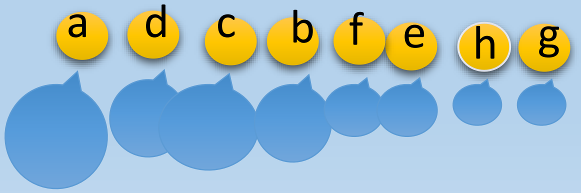
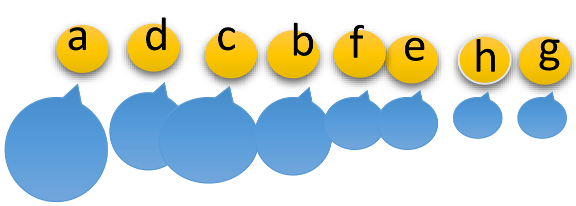


Greedy + Lex = Greedlex

- Label the elements lexicographically
- Shell greedily (avoiding bumps locally) and take the lexicographically largest when given a choice



Greedy + Lex = Greedlex

- Label the elements lexicographically
- Shell greedily (avoiding bumps locally) and take the lexicographically largest when given a choice

Claim: This always yields the min-bump l.e.

Lex-Yanking Lemma

b xxx..x a xx...x has k bumps and a is min

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⇒ ∃ l.e. a x’x’x’…b…x’ with k or fewer bumps

(Balloon size indicates relative Lex value)

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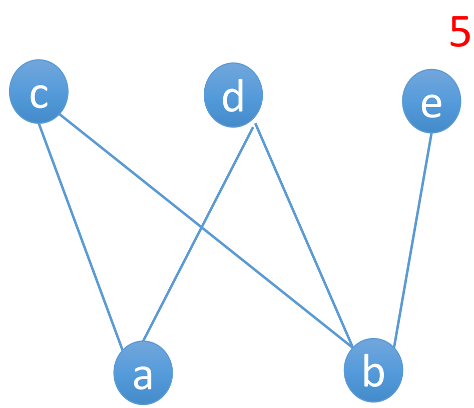
(Balloon size indicates relative Lex value)

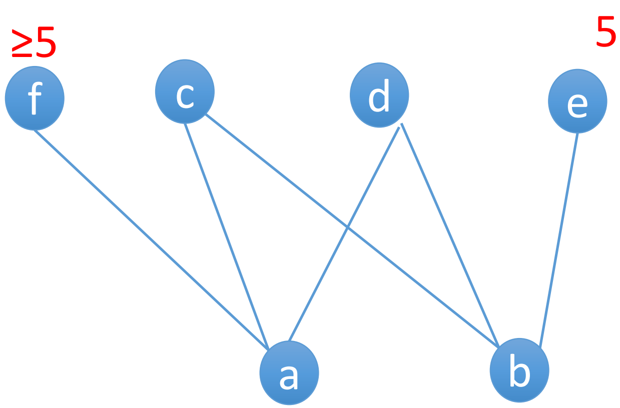
Claim: If the lex-yanking lemma holds on a poset and all its induced subposets, then Greedlex produces a bump-optimal linear extension.

Proof of LexYanking Lemma



If lex(a)≥lex(b) and b has a private cover (not covering a)…

Proof of LexYanking Lemma



If lex(a)≥lex(b) and b has a private cover (not covering a)…

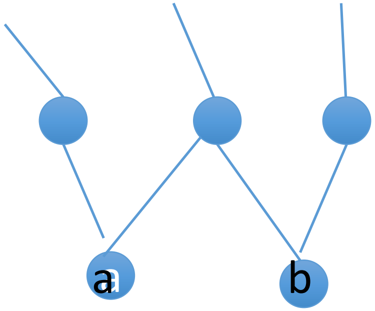
Then a has a private cover with lex# at least as large.

Proof of LexYanking Lemma

By induction on n=|V(P)|. Base cases n=0,1 are trivial.

Let P be a poset on n>1 elements, and suppose LexYanking Lemma holds for all smaller posets. (Then also Greedlex works on smaller posets.)

b xxx a xxxx a l.e. with k bumps, lex(a)≥lex(b), a and b min



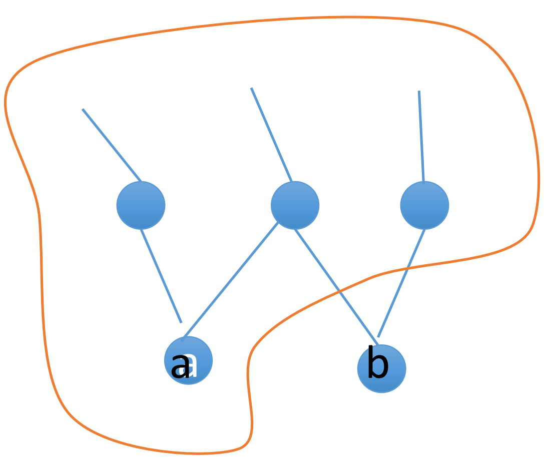
Proof of LexYanking Lemma

b xxx a xxxx   a l.e. with k bumps, lex(a)≥lex(b), a and b min

The poset \ {b} is smaller, so by Ind. Hyp., LexYanking holds, and Greedlex produces a min-bump suffix to follow b

b yyy a yyyy

All these elements have lex# > lex(a)



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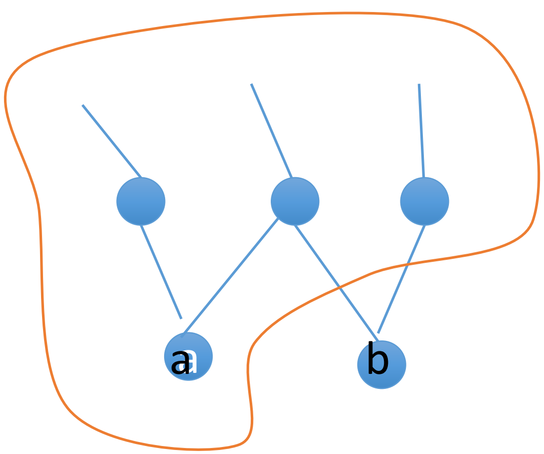

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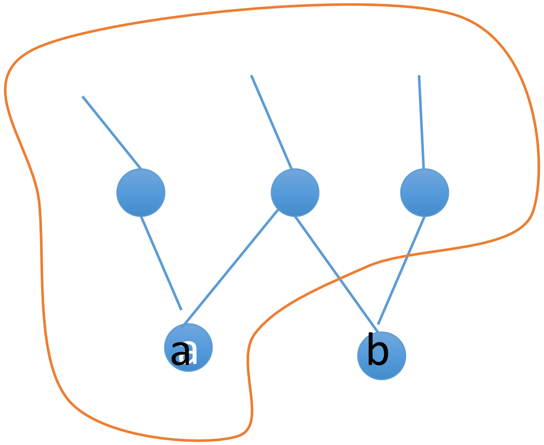

All these elements have lex# > lex(a) ≥ lex(b)
Hence all are incomparable with b
They are also incomparable with a

Swap: a yyy b yyyy

Proof of LexYanking Lemma

b xxx a xxxx   a l.e. with k bumps, lex(a)≥lex(b), a and b min

The poset \ {b} is smaller, so by Ind. Hyp., LexYanking holds, and Greedlex produces a min-bump suffix to follow b

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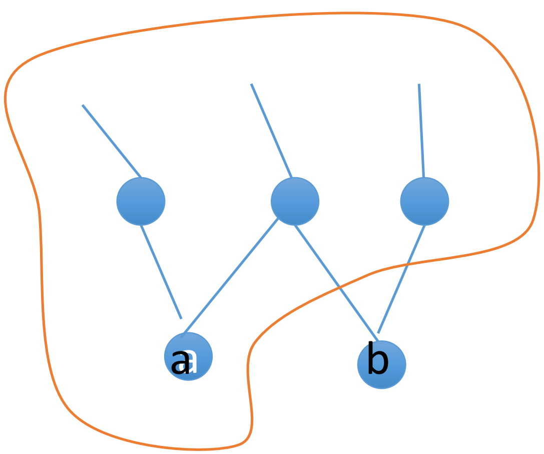


All these elements have lex# > lex(a) ≥ lex(b)
Hence all are incomparable with b
They are also incomparable with a

Swap: a yyy b yyyy
May have introduced a bump

Proof of LexYanking Lemma

a yyy b yyyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Proof of LexYanking Lemma

a yyy b $y_1$yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the analogous spot.

Then y_1 is a private cover of b (with respect to a).

Proof of LexYanking Lemma

$a\ yyy\ b\ y_1yyy$

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y_1 is a private cover of b (with respect to a).

Then a has some private cover c (w.r.t. b), with lex(c)≥ lex(y_1).

$a\ yyy\ b\ y_1y\ldots c..y$

Proof of LexYanking Lemma

a yyy b $y_1$yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y_1 is a private neighbour of b (with respect to a).

Then a has some private neighbour c (w.r.t. b), with lex(c)≥ lex(y_1).

a yyy b y_1y…c..y

Then c can be yanked forward in the suffix, by the Ind. Hyp., without increasing bumps

a yyy b c z…y_1..z

Proof of LexYanking Lemma

a yyy b $y_1$yyy

Suppose a bump was introduced after the b, and there was no such bump when a was in the same spot.

Then y_1 is a private neighbour of b (with respect to a).

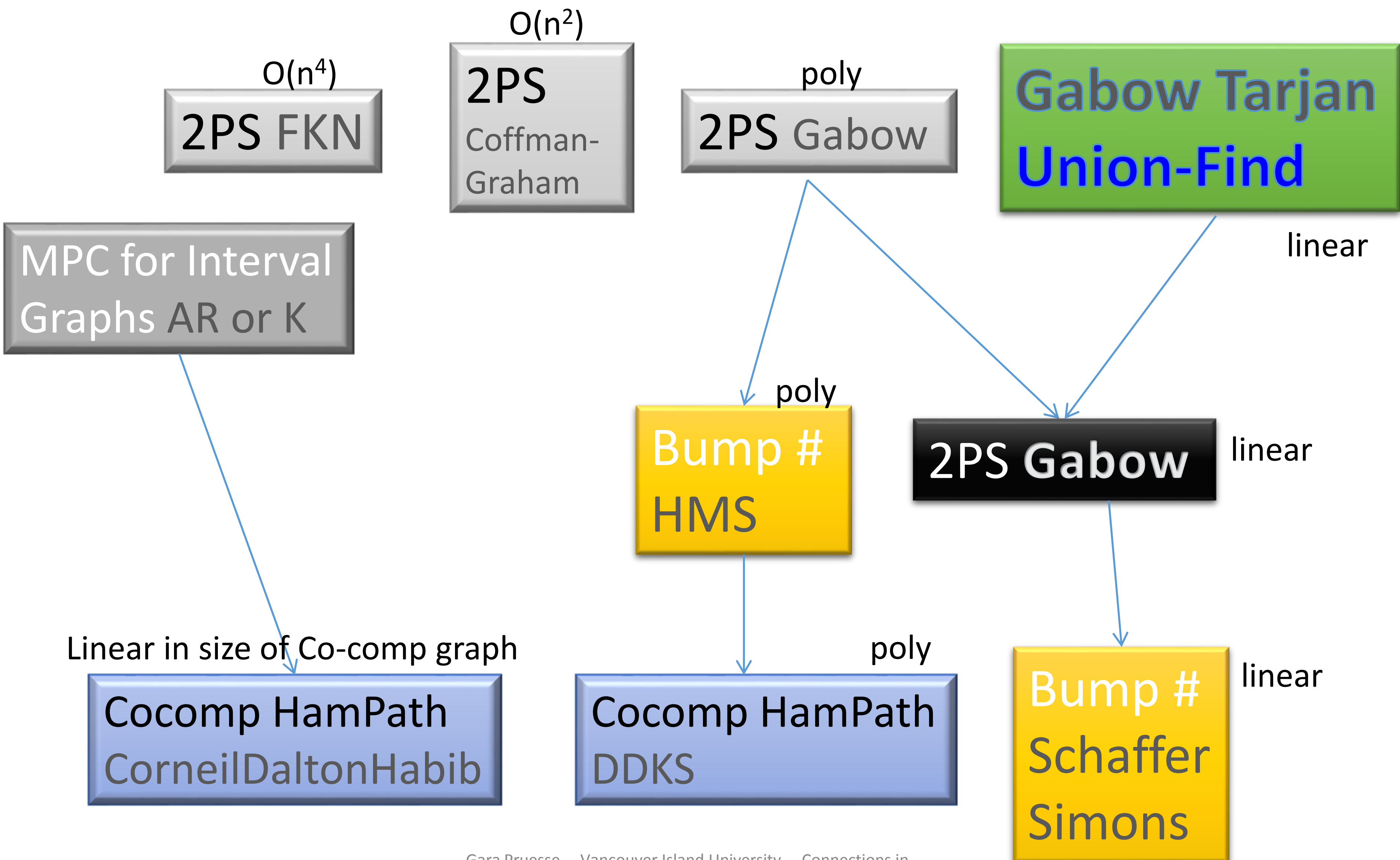
Then a has some private neighbour c (w.r.t. b), with lex(c)≥ lex(y_1).

a yyy b y_1y…c..y

a yyy b c z…y_1..z

Then c can be yanked forward in the suffix, by the Ind. Hyp., without increasing bumps and destroying the bump after b.
[if c is not a min, take c's descendent]. ■

Where does this work fit in?



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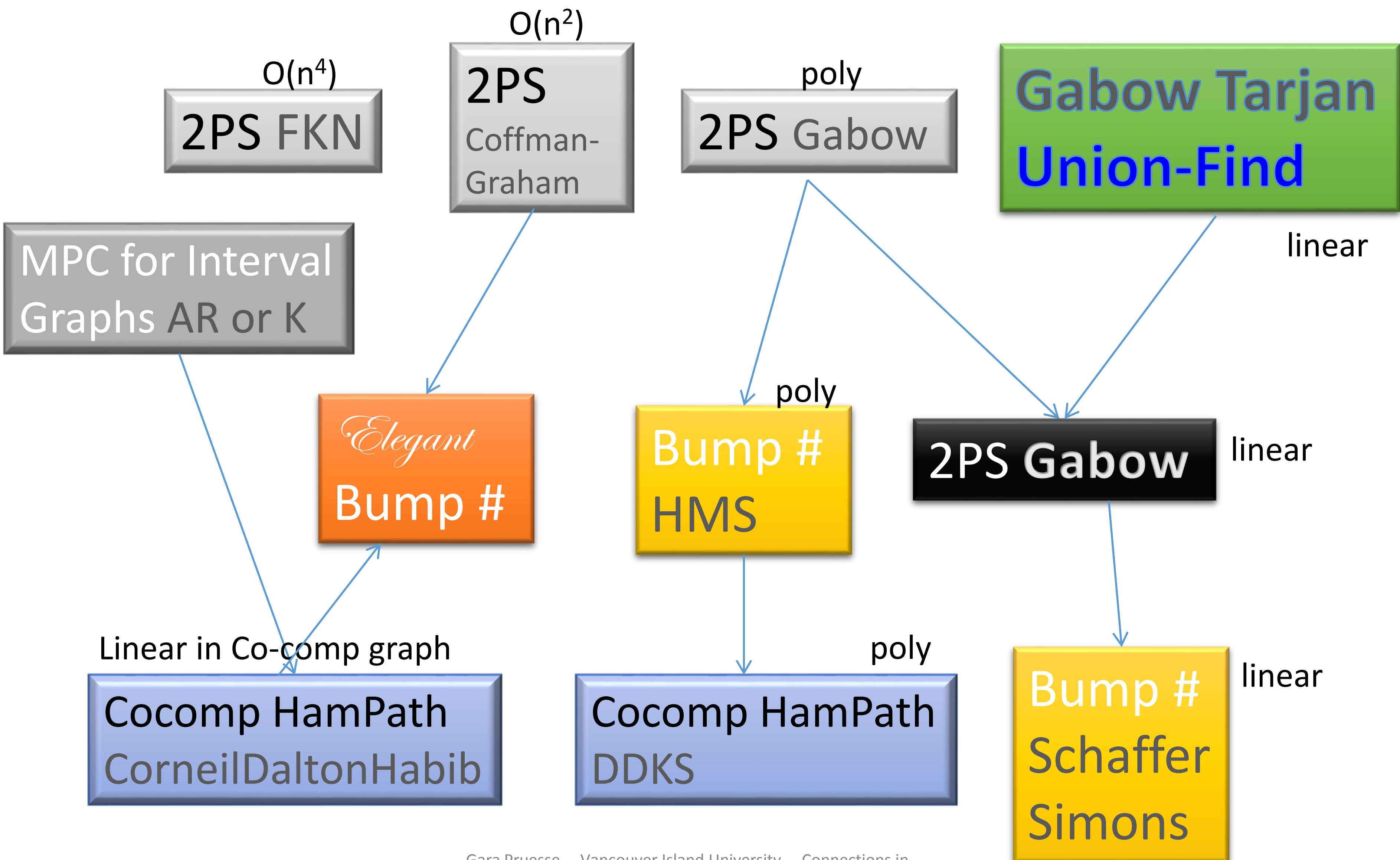


Where does this work fit in?

The beautiful idea ->

$O(n^2)$ 2PS
Coffman- Graham

Elegant
Bump #

Cocomp Path Cover
Corneil Dalton Habib

2PS Gabow
+
Gabow Tarjan
Union-Find

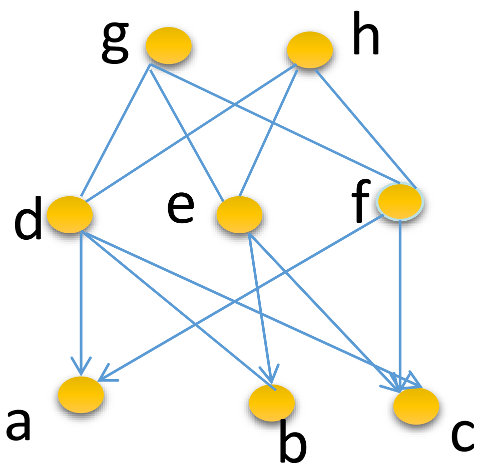
Linear 2PS
Gabow

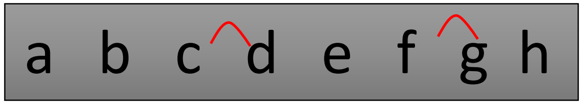
Linear
Bump #
Schaffer
Simons

How fast is the *Elegant* Algorithm?

- O(n log w)   w is size of maximum antichain
                n is size of the cover graph
- Can be quickly programmed

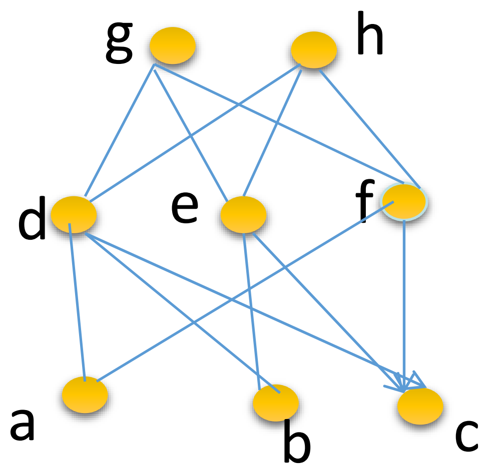
2-Processor Schedules

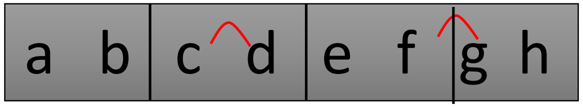




Want to schedule these unit-length jobs on two identical processor so that no job is executed before all of its lower covers have completed execution.

2-Processor Schedules





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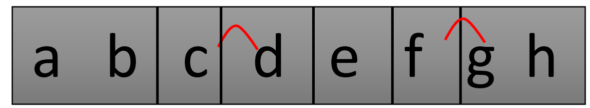

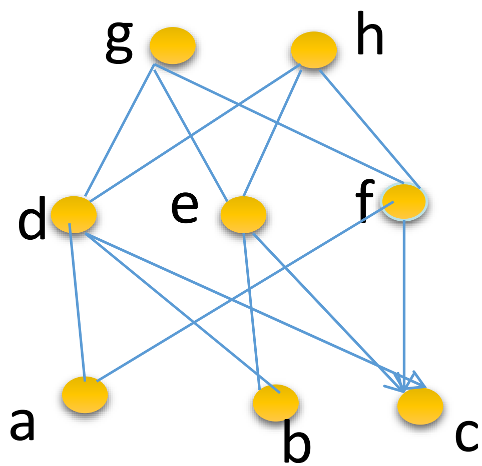

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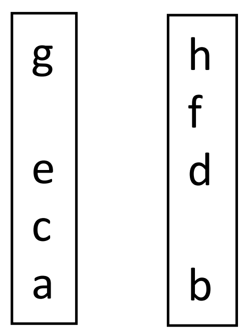
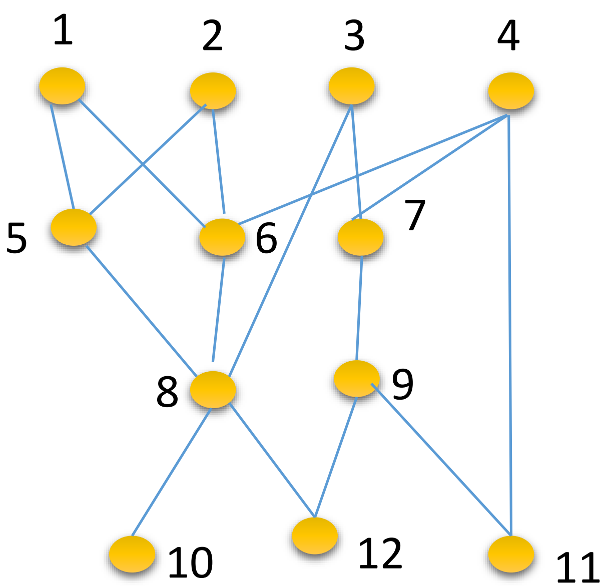
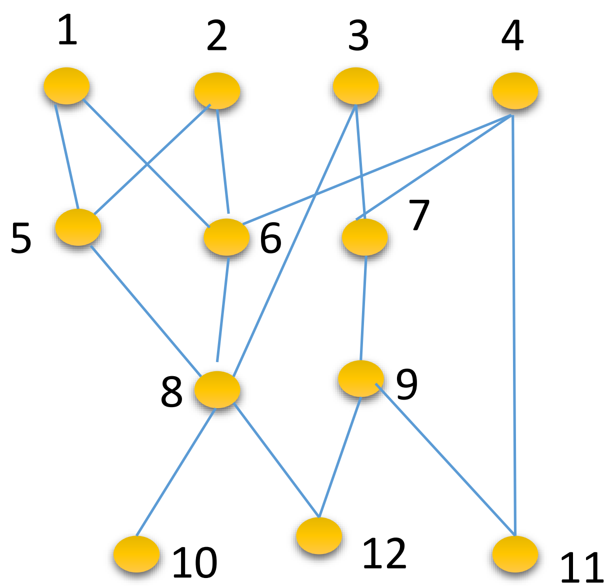


Coffman-Graham Lexicographic Labelling



- Give t minima ~~arbitrary~~ lex#'s 1…t arbitrarily
- Assign lex#s t+1…n so that lex(u)<lex(v) whenever {lex(u'): u'covers u} $<_{lexico}$ {lex(v'): v' covers v}, breaking ties arbitrarily

Coffman-Graham Lexicographic Labelling



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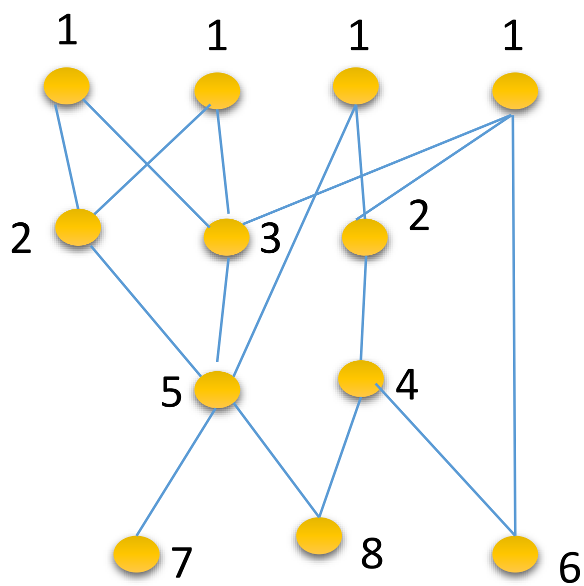
(Sethi, 1976) O(n+m) algorithm for C-G lex labelling

Lexicographic Labelling and 2PS



- Coffman and Graham '72 used it for 2-proc scheduling $O(n^2)$
- Sethi '76 also used it for a 2PS; lex labelling takes $O(n + m)$ though the remainder of the 2PS alg takes $O(n\,\alpha(n) + m))$

Further Work

Completed:

- Solve 2-Proc Sched using Greedlex
- Greedlex can work on either transitive closure or transitive reduction
- Greedlex can generate all min-bump linear extensions (all MinPath Covers in Cocomp graphs)

Open:

- Show how to do it all in linear time
- What if the communication delays are weighted?
- What about representations that are in between transitive closure and reduction?
- What about AT-free graphs?
  - Contains the cocomp graphs

Thank You!

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Lalla Mouatadid

University of Toronto

Hamiltonicity of Cocomp Graphs

Keil 1985

- Ham'n cycle in Interval graphs alg

Deogun Steiner 1990

- Poly-time Ham'n Cycle

Deogun Kratsch Steiner 1997

- 1-tough cocomp graphs are hamiltonian –

Damaschke Deogun Kratsch Steiner 1991

- Hamilton Path in cocomps using bump number algorithm

Corneil Dalton Habib 2013

- Min Path Cover Alg (certified) in Cocomp Graphs