Lecture 1

These notes cover:

- Polar coordinates
- Cartesian to polar conversion
- Polar to cartesian conversion

Polar Coordinates:

Polar coordinates consists of magnitude r and an angle θ. A point P with magnitude r and angle θ is given by:

$$P(r, \theta)$$

For some $r = 2$ and $\theta = \frac{\pi}{6}$, the possible values of the angle are given as:

$$\frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \frac{\pi}{6} \pm 6\pi \ldots$$

Now for the same magnitude r but in the opposite direction or $-r$, the corresponding angle is found by taking the current angle, in this case $\frac{\pi}{6}$ and adding π to it. This will be as follows:

$$\frac{7\pi}{6}, \frac{7\pi}{6} \pm 2\pi, \frac{7\pi}{6} \pm 4\pi, \frac{7\pi}{6} \pm 6\pi \ldots$$

Note that there can be infinitely many points representing the equivalent of a point $P(r,\theta)$.

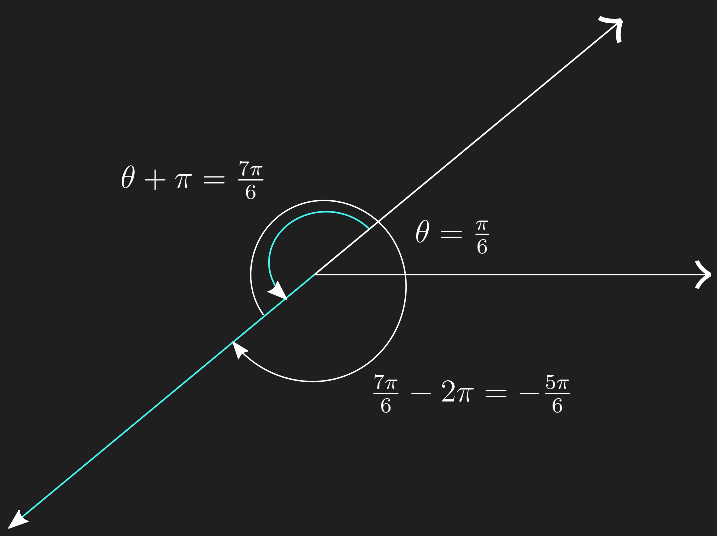

Figure 1

Relating Cartesian and Polar Coordinates:

The following three equations sum up the entirety of arithmetic required for the conversions from one to the other:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Conic section will be covered later in detail but one thing important here is the general format of an circle's equation.
Let a given equation of a circle be:

$$(x-a)^2 + (y-b)^2 = r^2$$

Here we say that the circle is centered at a and b and has a radius r. Notice the opposite signs.

PS 10.5

Polar to Cartesian:

.23 $r\cos\theta = 2$

$$x = 2$$

.24 $r\sin\theta = -1$

$$y = -1$$

.25 $r\sin\theta = 0$

$$y = 0$$

.26 $r\cos\theta = 0$

$$x = 0$$

.27 $r = 4\csc\theta$

$$r = \frac{4}{\sin\theta}$$
$$r\sin\theta = 4$$
$$y = 4$$

.28 $r = -3\sec\theta$

$$r = \frac{-3}{\cos\theta}$$
$$r\cos\theta = -3$$
$$x = -3$$

.29 $r\cos\theta + r\sin\theta = 1$

$$x + y = 1$$

.30 $r\sin\theta = r\cos\theta$

$$\frac{r\sin\theta}{r\cos\theta} = 1$$

$$\frac{y}{x} = 1$$

.31 $\mathbf{r}^2 = 1$

$$r^2\cos\theta + r^2\sin\theta = 1$$

$$2 + y^2 = 1$$

.32 $r^2 = 4r\sin\theta$

$$\frac{x^2+y^2}{y} = 4$$

.33 $r = \frac{5}{\sin\theta - 2\cos\theta}$

$$r\sin\theta - 2r\cos\theta = 5$$
$$y - 2x = 5$$

.34 $r^2 \sin 2\theta = 2$

$$r^2 \sin\theta\cos\theta = 2$$
$$xy = 1$$

.35 $r = \cot\theta\csc\theta$

$$r = \frac{\cos\theta}{\sin^2\theta}$$
$$r\sin^2\theta = \cos\theta$$
$$r(1-\cos^2\theta) = \cos\theta$$
$$r - r\cos\theta\cos\theta = \cos\theta$$
$$r - x\cos\theta = \cos\theta$$
$$r = x\cos\theta + \cos\theta$$
$$r = \cos\theta(1+x)$$
$$r^2 = r\cos\theta(1+x)$$
$$x^2 + y^2 = x + x^2$$
$$y^2 - x = 0$$

.36 $r = 4\tan\theta\sec\theta$

$$r = 4\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}$$
$$r\cos\theta r\cos\theta = 4r\sin\theta$$

$$x^2 = 4y$$

.37 $r = \csc\theta e^{r\cos\theta}$

$$r = \frac{1}{\sin\theta}e^{r\cos\theta}$$
$$r\sin\theta = e^{r\cos\theta}$$
$$y = e^{r\cos\theta}$$
$$y = e^x$$

Taking natural log we get:

$$\ln y = \ln e^x$$
$$\ln y = x$$

.38 $r\sin\theta = \ln r + \ln\cos\theta$

Based on the rule: $\ln A + \ln B = \ln A.B$

$$r\sin\theta = \ln r\cos\theta$$

$$y = \ln x$$

.39 $r^2 + 2r^2\cos\theta\sin\theta = 1$

$$r^2 + 2xy = 1$$

$$x^2 + y^2 + 2xy = 1$$

$$(x+y)^2 = 1$$

.40 $\cos^2\theta = \sin^2\theta$

$$r\cos^2\theta = r\sin^2\theta$$

$$x\cos\theta = y\sin\theta$$

$$xr\cos\theta = yr\sin\theta$$

$$x^2 - y^2 = 0$$

.41 $r^2 = -4r\cos\theta$

$$x^2 + y^2 = -4x$$

Adding 4 to complete square:

$$x^2 + 4x + 4 + y^2 = 4$$

$$(x+2)^2 + y^2 = (2)^2$$

This is the equation of a cirle.
Center= (-2,0)
Radius= 2

.42 $r^2 = -6r\sin\theta$

$$x^2 + y^2 = -6y$$
$$x^2 + y^2 + 6y = 0$$

To complete square take $6y = 2xy \implies x = 3$ and square it to get 9:

$$x^2 + y^2 + 6y + 9 = 9$$
$$x^2 + (y+3)^2 = 3^2$$

Center: (0,-3)
Radius: 3

.43 $r = 8\sin\theta$

$$x^2 + y^2 - 8y = 0$$
$$x^2 + y^2 - 8y + 16 = 4^2$$
$$x^2 + (y-4)^2 = 4^2$$

Center: (0,4)
Radius: 4

.44 $r = 4\cos\theta$

$$x^2 + y^2 - 3x = 0$$
$$x^2 + y^2 - 3x + \frac{9}{4} = \left(\frac{3}{2}\right)^2$$
$$(x - \frac{3}{2})^2 + y^2 = \left(\frac{3}{2}\right)^2$$

Center: ($\frac{3}{2}$,0)
Radius: ($\frac{3}{2}$)

.45 $r = 2\cos\theta + 2\sin\theta$

$$r^2 = 2x + 2y$$
$$x^2 - 2x + y^2 - 2y = 0$$
$$(x-1)^2 + (y-1)^2 = 2$$

Center:(1,1)
Radius: $\sqrt{2}$

.46 $r = 2\cos\theta - \sin\theta$

$$r^2 = 2x - y$$
$$x^2 - 2x + y^2 + y = 0$$
$$x^2 - 2x + 1 + y^2 + y + \frac{1}{4} = 1 + \frac{1}{4}$$
$$(x-1)^2 + (y + \frac{1}{2})^2 = \frac{5}{4}$$

Center: $(1, -\frac{1}{2})$
Radius: $\sqrt{\frac{5}{4}}$

.47 $r\sin\left(\theta + \frac{\pi}{6}\right) = 2$

Using the formula:
$\sin(A+B) = \sin A\cos B + \cos A\cos B$

$$r(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}) = 2$$
$$\left(r\sin\theta.\frac{\sqrt{3}}{2} + r\cos\theta\frac{1}{2}\right) = 2$$
$$y\frac{\sqrt{3}}{2} + x\frac{1}{2} = 2$$
$$\sqrt{3}y + x = 4$$
$$\sqrt{3}y = -x + 4$$
$$y = -\frac{x}{\sqrt{3}} + \frac{4}{\sqrt{3}}$$

.48 $r\sin\left(2\frac{\pi}{3} - \theta\right) = 5$

$$r\left(\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta\right) = 5$$
$$x\frac{\sqrt{3}}{2} - y\frac{1}{2} = 5$$
$$-\frac{y}{2} = 5 + \frac{\sqrt{3}}{2}x$$
$$-y = 10 + \sqrt{3}x$$
$$y = -\sqrt{3}x + 10$$

Cartesian to Polar:

.49 $x = 7$

$$r\cos\theta = 7$$

.50 $y = 1$

$$r\sin\theta = 1$$

.51 $x = y$

$$r\cos\theta = r\sin\theta \implies \tan\theta = 1$$

.52 $x - y = 3$

$$r\cos\theta - r\sin\theta = 3$$
$$r = \frac{3}{\cos\theta - \sin\theta}$$

.53 $x^2 + y^2 = 4$

$$r^2\sin^2\theta + r^2\cos^2\theta = 4$$
$$r^2(1) = 4$$
$$r = \pm 2$$

.54 $x^2 - y^2 = 1$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2(\cos^2\theta - \sin^2\theta) = 1$$
$$r^2\cos 2\theta = 1$$
$$r^2 = \frac{1}{\sec 2\theta}$$

.55 $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\frac{r^2\cos^2\theta}{9} + \frac{r^2\sin^2\theta}{4} = 1$$
$$4r^2\cos^2\theta + 9r^2\sin^2\theta = 36$$
$$r^2 = \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

.56 $xy = 2$

$$r\cos\theta r\sin\theta = 2$$
$$r^2(\sin\theta\cos\theta) = 2$$
$$r^2 = \frac{2}{\sin\theta\cos\theta}$$

.57 $y^2 = 4x$

$$r^2\sin^2\theta = 4r\cos\theta$$
$$r^2 = \frac{4r\cos\theta}{\sin^2\theta}$$

.58 $x^2 + xy + y^2 = 1$

$$r^2\cos^2\theta + r\cos\theta r\sin\theta + r^2\sin^2\theta = 1$$
$$r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta = 1$$
$$r^2(1 + \sin\theta\cos\theta) = 1$$
$$r^2 = \frac{1}{1 + \sin\theta\cos\theta}$$

.59 $x^2 + (y-2)^2 = 4$

$$r^2\cos^2\theta + (r\sin\theta - 2)^2 = 4$$
$$r^2\cos^2\theta + r^2\sin^2\theta - 4r\sin\theta + \cancel{4} = \cancel{4}$$
$$r^2(1) - 4r\sin\theta = 0$$
$$r^2 = 4r\sin\theta$$

.60 $(x-5)^2 + y^2 = 25$

$$(r\cos\theta - 5)^2 + r^2\sin^2\theta = 25$$

$$r^2\cos^2\theta - 10r\cos\theta + \cancel{25} + r^2\sin^2\theta = \cancel{25}$$
$$r^2(1) - 10r\cos\theta = 0$$
$$r = 10\cos\theta$$

.61 $(x-3)^2 + (y+1)^2 = 4$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 4$$
$$x^2 - 6x + y^2 + 2y + 10 = 4$$
$$\cancel{r^2\cos^2\theta} - 6rcos\theta + \cancel{r^2\sin^2\theta} + 2r\sin\theta = -6$$
$$r^2 - 6r\cos\theta + 2r\sin\theta = -6$$
$$r^2 = 2r\sin\theta + 6r\cos\theta - 6$$

.62 $(x+2)^2 + (y-5)^2 = 16$

$$x^2 + 4x + y^2 - 10y = -13$$
$$r^2\cos^2\theta + 4r\cos\theta + r^2\sin^2\theta - 10r\sin\theta = -13$$
$$r^2(1) + 4r\cos\theta - 10r\sin\theta = -13$$
$$r^2 = 10r\sin\theta - 4r\cos\theta - 13$$