Big disclaimer:

Big disclaimer right off the bat: this was a fun project and a good way to learn some interesting math, but it is an inefficient way to develop a model. At a very basic level, the assumptions that go into a model should come from pretty strong evidence, empirical or otherwise. In this case, I thought it would be interesting to see if markets *could* be modeled this way—so, the central assumptions preceded the evidence. Plus, much of the predictive power this model has probably could have been captured by a much simpler model. And that model would have taken less effort to develop.

I still see plenty of value in trying something in a way others haven't. The effort that went into developing the model was expended working to understand where and why my assumptions are wrong, and learning how scientists build theories about things that are notoriously hard to predict.

More complicated isn't necessarily better. Just more interesting, in this case.

Assumptions

I begin with basic assumptions.

1. if a buyer's bid and seller's ask are close in price, buyer and seller transact and exit the market.
2. market participants are driven by local interaction. Buyers and sellers decide when to enter/exit the market based on interactions with others who enter/exit nearby.
3. at a given price and time, participants are sensitive to (or really, sensitive to their perceptions of):
   - how many people would be willing to enter/exit
   - how quickly the number of people willing to enter/exit would change with both price and time

The first assumption suggests I should be working with some kind of operator. Something where if I act with both the "buyer" operator and the "seller" operator I get an identity operator. In plain english, if a buyer and seller enter the market at the same price, they should transact and exit—leaving the market in the same state as if neither had entered.

From the second assumption, I can guess the operator should have some sort of "position" associated with it (price, or something similar). And the third says I should be able to write down an equation that governs the motion of these operators (where, from assumption two, evolution in time and position depends only on recent activity nearby in price). And, since there can be arbitrarily many participants in a market, I'm really interested in the *distribution* of these operators, not any particular one.

Let the "buyer enters at price x and time t" operator be $\phi(x,t)$. It acts on the state of the market, as $\phi(x,t)\,|n \text{ participants}\rangle = |n \text{ participants} + 1 \text{ buyer at } x,t\rangle$.

Similarly, the "seller enters" operator is $\phi^\dagger(x,t)$. The names suggest a fundamental symmetry (which I think makes sense; a buy and a sell are pretty similar, modulo a negative sign).

By assumption 1, we should have $\phi^\dagger(x_1,t_1)\phi(x_2,t_2)\,|n\rangle = \phi(x_1,t_1)\phi^\dagger(x_2,t_2)\,|n\rangle = \mathbb{1}\,|n\rangle = |n\rangle$ $\forall x_1 = x_2$ and $t_1 = t_2$.

To incorporate assumptions 2 and 3, I have to make a guess about what the equations of motion—the rules that determine the evolution of the whole system—look like. In general, systems tend to extremize some quantity—energy, entropy, etc. Intuitively, the market should do something similar. What is required is a map from the configuration of the market, i.e. the distribution of ϕ and $\phi^\dagger$ operators over all space to some value of this energy-like quantity. Following the usual approach, that map is a functional of the following form:

$$S[\phi,\phi^\dagger] = \int_{\text{all } x,t} dx^\mu \mathcal{L}[\phi,\phi^\dagger]$$

Where

$$\mathcal{L}[\phi,\phi^\dagger] = \partial_\mu\phi\partial^\mu\phi^\dagger + m^2\phi^\dagger\phi + \lambda_\phi(\phi^\dagger\phi)^2$$

I assert that the configuration of ϕ and $\phi^\dagger$ that minimizes S is the one we're most likely to find. And, while S is important, it is really just a wrapper around $\mathcal{L}$. $\mathcal{L}$ determines the dynamics. This is where I have encoded assumptions 2 and 3: the $\partial_\mu\phi\partial^\mu\phi^\dagger$ term makes quickly-varying configurations less favorable. The $m^2\phi^\dagger\phi$ and $\lambda_\phi(\phi^\dagger\phi)^2$ terms make a large density of participants at any given point less favorable (or more favorable depending on the sign of λ_ϕ). m^2 and λ_ϕ are free parameters, allowing me to adjust the strength of each of these effects. This choice for $\mathcal{L}$ feels arbitrary, but it's not. It's a good guess, in the sense that it admits analytically tractable solutions (more details forthcoming) and is somewhat interpretable. $\mathcal{L}$ is a simple way to express dependence of S (and, therefore, the configuration of ϕ and $\phi^\dagger$ that minimize S) on both the magnitude and rate of change of ϕ and $\phi^\dagger$. But it is still a guess.

Aside on notation:

Why do these two operators ϕ and $\phi^\dagger$ have such similar names? I mentioned earlier that ϕ and $\phi^\dagger$ are fundamentally similar without giving much detail. Their similarity is actually relevant to the form of $\mathcal{L}$. Consider that in a transaction between a buyer and a seller there is a direction associated with the exchange (e.g. securities move from seller to buyer, or money moves from buyer to seller). It seems like a trivial point, but if I change the direction of the exchange, the roles of buyer and seller would also change. This actually leads to a key insight: in $\mathcal{L}$ I should be able to make a transformation that exchanges $\phi \to \phi^\dagger$ and $\phi^\dagger \to \phi$ without changing the dynamics of the system. This transformation will simply change the definition of the direction of transactions. In this sense, ϕ and $\phi^\dagger$ are intimately related, so their names should reflect it. Returning to $\mathcal{L}$: intuitively, in order to describe two types of participants entering a market, I need two separate operators. If I instead write these two operators as one operator with complex entries, I can still model the two types of participants. But, I can do so more succinctly while also pointing out the symmetry the system has under the "transaction direction" transformation. The dynamics expressed by $\mathcal{L}$ are the same whether I have two real operators ϕ_1 and ϕ_2 or one complex-valued ϕ and its conjugate transpose $\phi^\dagger$. (In fact, the two representations are entirely equivalent).

Back to the Model:

There's a key detail missing from this model for a market: it doesn't interact with anything outside itself. In reality, since buyers and sellers can enter multiple markets at once, there is interaction between participants in one market and participants in others. I should find a way to incorporate the influence of other markets on the one I want to measure. To keep things (somewhat) simple, I can treat everything external to the primary market as a second market with similar assumptions and dynamics. I can call this market's operators ψ and $\psi^\dagger$. (These operators can all be associated to "positions" in the same space if instead of having participants enter at a certain *price* they enter at a certain *rate of return.*) Next, I incorporate these into a new $\mathcal{L}$ along with terms that couple the two markets:

$$\mathcal{L}[\phi,\phi^\dagger,\psi,\psi^\dagger] = \partial_\mu\phi\partial^\mu\phi^\dagger + m^2\phi^\dagger\phi + \lambda_\phi(\phi^\dagger\phi)^2 + \lambda_{\phi\psi}\phi^\dagger\phi\psi^\dagger\psi + \partial_\mu\psi^\dagger\partial^\mu\psi + M^2\psi^\dagger\psi + \lambda_\psi(\psi^\dagger\psi)^2$$

What am I measuring?

All this modeling is irrelevant if I can't use it to measure something. The point of this exercise is to make predictions about the future state of the market given the current one. I want to predict the probability density of transaction prices for the underlying when my option position expires (from this, I can estimate the expected payoff of the covered call position and the variance of that payoff). Since this probability density is directly proportional to the number of participants at a given price, I can estimate the initial distribution of participants in return space, then I can evolve the distribution in time according to $\mathcal{L}$ to estimate the distribution of transaction prices at expiration.

I'll start with a simple example to build intuition. Imagine a market with no participants, $|0\rangle$. Add a buyer at some price x_0 at time t_0: $\phi(x_0,t_0)\,|0\rangle$. The probability that a seller will enter at some price x_1 and time t_1 and the two will transact, clearing the market, is $\langle 0|\,\phi^\dagger(x_1,t_1)\phi(x_0,t_0)\,|0\rangle$. Doing the same exercise with two buyers and two sellers, the probability of all four transacting and exiting is $\langle 0|\,\phi^\dagger(x_3,t_3)\phi^\dagger(x_2,t_2)\phi(x_1,t_1)\phi(x_0,t_0)\,|0\rangle$. This is can be generalized to n transactions.

But wait: ϕ is not constant in x and t. It evolves dynamically according to $\mathcal{L}$. The expression $\langle 0|\,\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)\,|0\rangle$ is really a *functional* expectation value—that is, the average of $\langle 0|\,\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)\,|0\rangle$ over all possible $\phi/\phi^\dagger$. This is expressed as an integral over $\phi,\phi^\dagger,\psi$ and $\psi^\dagger$, and there should be a weighting factor $e^{-iS[\phi,\phi^\dagger,\ldots]}$ associated with each point in the domain of integration (this is non-trivial. Omitting some details, this comes from the fact that I can evolve an operator in time according to something like $i\frac{\partial}{\partial t}\phi = \hat{H}\phi$, where $\hat{H} = \dot{\phi}\frac{\partial}{\partial\dot{\phi}}\mathcal{L} - \mathcal{L}$, the solution to which is an exponential $\propto e^{-i\hat{H}}$; after more work, it can be shown that these factors show up for each ϕ in the functional expectation value, and can be combined into $e^{-iS[\ldots]}$). Putting this all together, we have the following:

$$\langle 0|\,\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)\,|0\rangle = \frac{\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]}\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)}{\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]}} \tag{1}$$

This is a very important expression. It is a correlation function (a.k.a. a "correlator", an "n-point correlation function", or simply an "n-point function"). Evaluating it is the main hurdle in computing the probability density I'm interested in.

There are a number of shortcuts that help evaluate these integrals. Because the integrand will oscillate quickly for large ϕ, the only contributions that don't cancel out will come from ϕ where $\frac{\delta S[\phi]}{\delta\phi} = 0$. This suggests I can avoid integrating over all ϕ. In fact, there is a neat trick:

First, for $\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi + m^2\phi^\dagger\phi = \phi^\dagger A\phi$:

$$\mathcal{L} = \phi^\dagger A\phi$$

First, add a source term coupled to ϕ that will $\to 0$ at the end of the calculation via a limit.

$$\mathcal{L} = \phi^\dagger A\phi + \phi^\dagger J + J^\dagger\phi$$

$$\mathcal{L} = \phi^\dagger A\phi + \phi^\dagger AGJ + J^\dagger GA^\dagger\phi$$

where since $AG = GA^\dagger = \mathbb{1}$, G is the inverse of the operator $\partial_\mu\partial^\mu + m^2$. (I call this "$G$" because it's a Green's Function). In particular, $G = \int \frac{d^Dk}{(2\pi)^D}\frac{e^{-ik(x-x')}}{k^2-m^2}$. Then,

$$\mathcal{L} = (\phi^\dagger + J^\dagger G)A(\phi + GJ) - J^\dagger GJ$$

then after shifting $\phi \to \phi - GJ$ and $\phi^\dagger \to \phi^\dagger - J^\dagger G$

$$\mathcal{L} = \phi^\dagger A\phi - J^\dagger GJ$$

Then,

$$\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]} = \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{\int d^Dxd^Dx'\phi^\dagger(x)A\phi(x')}e^{-\int d^Dxd^Dx'J(x)G(x-x')J(x')}$$

$$= Z[0]e^{-\int d^Dxd^Dx'J(x)G(x-x')J(x')}$$

The expression

$$Z[J] = Z[0]e^{-\int d^Dxd^Dx'J(x)G(x-x')J(x')}$$

can be functionally differentiated with respect to $J(x_n)$, $\ldots$, $J(x_0)$ to get the n-point correlation function $\langle 0|\,\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)\,|0\rangle$. Carrying out the differentiation and matching terms, I find the useful result

$$\langle 0|\,\phi^\dagger(x_1)\phi(x_0)\,|0\rangle = \frac{1}{Z[0]}\frac{\delta}{\delta J[x_0]}\frac{\delta}{\delta J[x_1]}Z[J] = G(x_1 - x_0) \equiv G(x_1,x_0)$$

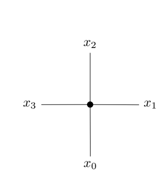
Returning to $\frac{\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]}\phi^\dagger(x_n,t_n)\ldots\phi(x_0,t_0)}{\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]}}$ (this time with the full $\mathcal{L}$), I can expand the exponentials in λ_ϕ and everything simplifies to factors of $\langle 0|\,\phi^\dagger(x_n)\ldots\phi(x_0)\,|0\rangle$, $\langle 0|\,\phi^\dagger(x_n)\ldots\psi(x_0)\,|0\rangle$, etc. These can be evaluated using the another very useful trick: Wick's theorem[1] lets me replace the long combinations of operators that show up in the correlation functions, like $\phi^\dagger(x_3)\phi^\dagger(x_2)\phi(x_1)\phi(x_0)$, with the possible pairings of these operators: $\phi^\dagger(x_3)$ with $\phi(x_1)$ and $\phi^\dagger(x_2)$ with $\phi(x_0)$; or $\phi^\dagger(x_3)$ with $\phi(x_0)$ and $\phi^\dagger(x_2)$ with $\phi(x_1)$. I can use this result to reduce n-point correlators to combinations of 2-point functions $\langle 0|\,\phi^\dagger(x_1)\phi(x_0)\,|0\rangle = G(x_0,x_1)$. So, for example, the 4-point function $\langle 0|\,\phi^\dagger(x_3)\phi^\dagger(x_2)\phi(x_1)\phi(x_0)\,|0\rangle$ reduces to $G(x_3,x_1)G(x_2,x_0) + G(x_3,x_0)G(x_2,x_1)$ leaving aside a few normalization factors for now. This procedure gets awfully messy the more terms there are in the correlator. Thankfully, there's one last trick I can use—and this one is by far the most powerful: diagrams. Since this is a simple combinatoric procedure, it can easily be represented visually. Here's how:

- Draw a point for each operator $\phi(x_i)$. If operators are associated to the same x, draw one point for those operators.
- Draw lines connecting the $\phi^\dagger$s with the ϕs. If there are n operators associated to a point, n lines can be connected to that one point. (Terms from the expansion like $\int dx\lambda_\phi(\phi^\dagger(x)\phi(x))^2$ will be associated with the same positions x.)
- Each line corresponds to a factor of $G(x_i,x_j)$. Each vertex (points where more than one line are connected) corresponds to a factor λ (recall the expansion of e^{-iS} results in terms like $\langle 0|\int dx\lambda_\phi(\phi^\dagger(x)\phi(x))^2\,|0\rangle$—this is where the λ factors come from). Sum all the components and carry out any integration over x.
- Repeat the procedure to find all possible (topologically distinct) diagrams. The resulting sum of diagrams is equivalent to the n-point correlator of interest.

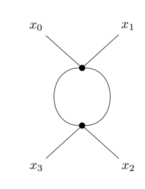
In fact, the process can be simplified further since the denominator of the correlator ($\int \mathcal{D}\phi\mathcal{D}\phi^\dagger e^{-iS[\phi,\phi^\dagger]}$) will cancel many of the diagrams. Omitting details, the takeaway is that I only have to look for diagrams that are connected[2]. Take, for example, the 2-point function, $\langle 0|\,\phi^\dagger(x_1)\phi(x_0)\,|0\rangle$. Drawing a point for each operator and a line connecting each results in the following:

x_1 ———— x_0

The result is simply one factor of G, as we found before. This is therefore an equivalent representation of the 2-point function. Note that this is the only (topologically distinct) type of graph we can make, and in most cases there will be many possible ways to connect the points. A more illustrative example is the 4-point function, $\langle 0|\,\phi^\dagger(x_3)\phi^\dagger(x_2)\phi(x_1)\phi(x_0)\lambda_\phi\,|0\rangle$. To calculate this, first expand the exponential in equation 1 to second order in λ_ϕ. The resulting terms include $\phi^\dagger(x_3)\phi^\dagger(x_2)\phi(x_1)\phi(x_0)\lambda_\phi\int dx(\phi^\dagger(x)\phi(x))^2$ and $\phi^\dagger(x_3)\phi^\dagger(x_2)\phi(x_1)\phi(x_0)\lambda_\phi\int dx(\phi^\dagger(x)\phi(x))^2\lambda_\phi\int dx'(\phi^\dagger$ (omitting expansion coefficients). These terms correspond to diagrams with the following topologies:



example 'first order' term



example 'second order' term
note the loop—this will be important later...

The result is $\lambda_\phi\int dx(G(x_3,x)G(x,x_1)+G(x_2,x)G(x,x_0)) + \lambda_\phi^2\int dx\int dx'(G(x_3,x)G(x,x')G(x',x_0) + G(x_2,x)G(x,x')G(x',x_0))$. Amazingly, this result can be evaluated without having to integrate over all possible configurations of $\phi,\phi^\dagger$, etc

A small problem:

In reality, the market is never in the $|0\rangle$ state. I'm interested in $\langle\Omega_j|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|\Omega_i\rangle$ for some general states Ω containing arbitrary numbers of participants in arbitrary configurations. In fact, there is a formula that relates the $\langle 0|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|0\rangle$ calculated using diagrams to $\langle\Omega_j|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|\Omega_i\rangle$. I will revisit this formula shortly.

A huge problem:

Now, when I actually want to carry out the integration specified by any of the diagrams with loops, immediately I run into a huge problem. These integrals do not converge. This problem is insurmountable and calls for a big rethink.

The problem is subtle. When I do those "loop" integrals, I'm implicitly assuming that the model works at any possible scale. There's no reason that the parameters of the model (λ_ϕ, $\lambda_{\phi\psi}$, etc) should be the same if we're interested in modeling interactions between participants \$.01 apart in price or modeling those \$1M apart. The solution is *incredibly* subtle. Because (hypothetically) I can measure $\langle\Omega_j|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|\Omega_i\rangle$ experimentally, I know the divergent integrals were a product of my parameterization, not of reality. This is key. I can use the definition of the parameters (λ_ϕ, $\lambda_{\phi\psi}$, etc) to exactly account for those divergences at a given scale. Here's how that works:

Say there is some scale below which it's not feasible to measure. I can set this scale in my model by imposing a limit on the divergent loop integrals, i.e. $\int_{|x|>\Lambda}dx$. This is akin to setting the resolution on a measuring device. The integral will now depend explicitly on that limit Λ. If I want a model that is effective at some much large scale, I can let that minimum scale shrink down to nothing. The integral will still diverge if I simply take that scale to zero. Here's the trick: I can redefine the parameters to also depend on Λ so that they too will diverge as the scale shrinks to zero ($\lim\Lambda \to 0$), but in a way that exactly subtracts the divergences from the original parameterization. I can define the parameters of the model based on a *measurement*, i.e. $\lambda_\phi = \langle\Omega_j|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|\Omega_i\rangle - \{\int_{|x|>\Lambda}\text{loop contribution that diverges}\}$. Then I can carry out the integration and confidently take the scale Λ to zero, recovering an effective model with no Λ depencence. This seems like cheating, but remember: I can measure $\langle\Omega_j|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|\Omega_i\rangle$, so it must be finite; the *model parameters themselves* are allowed to diverge (because I can't measure these directly); so, I can redefine the parameters to make the integrals converge.

Intuitively, this makes some sense. Parameters are set according to scale. If I pick a set of parameters at a fine grain scale, then "zoom out" to a lower resolution scale, the parameters may have to be adjusted to give the measured results. I can guess what the parameters should be at the fine grain such that when I zoom out to the coarse grain, I measure the correct results. This is what the redefinition of λ_ϕ, etc. captures.

Again, what do I want to measure?

I want to calculate $P(\text{transaction at x}|\text{record of transactions})$. I can use the model to calculate the transition amplitude from some initial state of the market to some final state where transactions happen at x_i—in other words, $\langle x_{\text{final, 1}}\ldots, x_{\text{final, n}}|x_{\text{initial, 1}}\ldots, x_{\text{initial, n}}\rangle$. This amplitude can be approximated by examining two participants at a time across all possible combinations of prices and adding up all the results (this is the minimum number of participants required to get any interaction effects. Participants begin at x_0, x_1 at t_{initial}, and end at x_2, x_3 at t_{final}.). As mentioned earlier, a result[3] that relates $\langle 0|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|0\rangle$ calculated using diagrams to $\langle x_{\text{final, 1}}\ldots, x_{\text{final, n}}|x_{\text{initial, 1}}\ldots, x_{\text{initial, n}}\rangle$ is required. In short, the transition amplitude for n participants I'm interested in is given by the sum of all the connected diagrams drawn with n external points (exactly what I've already found to calculate $\langle 0|\,\phi^\dagger(x_1)\ldots\phi(x_n)\,|0\rangle$)

This procedure is all easier after a Fourier transform (we can work with $\phi(p)$ exactly as we would $\phi(x)$, where p is the transform variable conjugate to x—these are just different representations of exactly the same information.) $G(p)$ takes on a simpler form: $G(p) = \frac{i}{p^2-m^2}$. Diagrams are equivalent, but with factors of $G(p)$ for lines. The final piece I need—the thing I want to measure—is the full, general expression for the interaction of two participants.

To calculate $\langle x_{\text{final, 1}}\ldots, x_{\text{final, n}}|x_{\text{initial, 1}}\ldots, x_{\text{initial, n}}\rangle$, I need to find all the relevant diagrams. See these in the appendix.

After finding all the diagrams, all that's left to do is carry out a few key integrals in order to write down the probability amplitude we want. Since many of the integrals are quite similar, there's no need to include all the details. What follows are the most important ones:

Here is the approach to handling the "loop" integrals (the ones that will diverge if I'm not careful). First, imposing a cutoff Λ (akin to setting a minimum effective scale):

$$\int_0^\Lambda \frac{d^Dq}{(2\pi)^D}\frac{i}{q^2-m^2}\frac{i}{(p-q)^2-M^2}$$

Then, using the following identity:

$$\frac{1}{AB} = \int_0^1 dx\frac{1}{[xA+(1-x)B]^2}$$

I can rewrite

$$\int_0^\Lambda \frac{d^Dq}{(2\pi)^D}\frac{i}{q^2-m^2}\frac{i}{(p-q)^2-M^2} = \int_0^1\int_0^\Lambda \frac{d^Dq}{(2\pi)^D}\underbrace{\frac{1}{[(1-x)(q^2-m^2)+x((p-q)^2-M^2)]^2}}_{*}$$

$$\begin{aligned} * &= q^2 - m^2 - xq^2 + xm^2 + x(p^2 + q^2 - 2p\cdot q - M^2) \\ &= (\underbrace{q-xp}_{=\ell})^2 + x(1-x)p^2 + x(m^2-M^2) - M^2 \end{aligned}$$

$$= \int_0^1 dx\int_0^\Lambda \frac{d^D\ell}{(2\pi)^D}\frac{1}{[\ell^2 + x(1-x) + x(m^2-M^2) - M^2]^2}$$

Since this integral is spherically symmetric, I can rewrite it as

$$= \frac{\text{Vol}(S^{D-1})}{(2\pi)^2}\int_0^1 dx\underbrace{\int_0^\Lambda d\ell\frac{\ell^{D-1}}{[\ell^2 + x(1-x)p^2 + x(m^2-M^2) - M^2]^2}}_{=\text{f}(p)_\Lambda}$$

Going back to my expansion in lambda, I have an expression like

$$A(p) = -\lambda_0 - \delta\lambda + \lambda_0^2\frac{\text{Vol}(\text{S}^{D-1})}{(2\pi)^D}\int_0^1 dx\,\text{f}(p)_\Lambda$$

Here is how I "set the scale" of the model, letting the amplitude measured for some experiment with $p = p_0$ equal the model parameter λ, with a minus sign for convenience

$$-A(p_0) = \lambda_{\text{measured}}$$

Then, the dependence of λ on scale (Λ) is expressed by $\delta\lambda$ (again, this is just convenient for notation; all I'm doing is defining the parameter λ very specifically so that it scales in just the right way so that when I integrate over higher and higher p the integral remains finite)

$$\Rightarrow \delta\lambda = \lambda_{\text{measured}} - \lambda_0 + \frac{\text{Vol}(\text{S}^{D-1})}{(2\pi)^D}\lambda_0^2\int_0^1 dx\,\text{f}(p_0)_\Lambda$$

Substituting $\lambda_{\text{measured}}$ and $\delta\lambda$,

$$\Rightarrow A(p) = -\lambda_0 - \lambda_{\text{measured}} + \lambda_0 + \frac{\text{Vol}(\text{S}^{D-1})}{(2\pi)^D}\lambda_0^2\int_0^1 dx\,\text{f}(p)_\Lambda - \text{f}(p_0)_\Lambda$$

$$\begin{aligned} \text{f}(p)_\Lambda - \text{f}(p_0)_\Lambda &= \ln\left(\frac{b(p)+\Lambda^2}{b(p)}\right) - \ln\left(\frac{b(p_0)+\Lambda^2}{b(p_0)}\right) = \ln\left(1+\frac{\Lambda^2}{b(p)}\right) - \ln\left(1+\frac{\Lambda^2}{b(p_0)}\right) \\ &\approx \ln\left(\frac{\Lambda^2}{b(p)}\right) - \ln\left(\frac{\Lambda^2}{b(p_0)}\right) \end{aligned}$$

then as $\Lambda \to \infty$

$$= \ln\left(\frac{b(p0)}{b(p)}\right)$$

where $b(p,x) = x(1-x)p^2 + (m^2-M^2)x - M^2$

Finally, here are the probability expressions I'm interested in:

First, note I've made the transformation:

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$

$$t = (p_1-p_3)^2 = (p_4-p_2)^2$$

$$u = (p_1-p_4)^2 = (p_3-p_2)^2$$

This is simply for convenience. s, t, and u together contain all the same information as the p_i coordinates. Then, for clarity, I've written the expression in 3 parts: $A_{\phi\phi\to\phi\phi}(s,t,u)$, the probability that two participants interact and wind up at new price points in the market (as determined by s, t, and u); $A_{\phi\phi\to\psi\psi}(s,t,u)$, the probability that two participants interact and wind up at new price points in a different market, external to the one I'm measuring; and $A_{\phi\psi\to\phi\psi}(s,t,u)$, the probability that one participant in the market interacts with another participant outside the market and one of them remains in the market while the other leaves.

$$\begin{aligned} A_{\phi\phi\to\phi\phi}(s,t,u) = -\lambda_\phi + \frac{\lambda_\phi^2}{64\pi^2}\int_0^1 dx\ln\left(\frac{m^2-x(1-x)s_0}{m^2-x(1-x)s}\right) + \ln\left(\frac{m^2-x(1-x)t_0}{m^2-x(1-x)t}\right) + \ln\left(\frac{m^2-x(1-x)u_0}{m^2-x(1-x)u}\right) \\ + \frac{\lambda_{\phi\psi}^2}{1152\pi^2}\int_0^1 dx\ln\left(\frac{M^2-x(1-x)s_0}{M^2-x(1-x)s}\right) + \ln\left(\frac{M^2-x(1-x)t_0}{M^2-x(1-x)t}\right) + \ln\left(\frac{M^2-x(1-x)u_0}{M^2-x(1-x)u}\right) \end{aligned}$$

$$\begin{aligned} A_{\phi\phi\to\psi\psi}(s,t,u) = -\lambda_{\phi\psi} + \frac{\lambda_{\phi\psi}^2}{1152\pi^2}\int_0^1 dx\ln\left(\frac{b(t_0,x)}{b(t,x)}\right) + \ln\left(\frac{b(u_0,x)}{b(u,x)}\right) \\ + \frac{\lambda_\phi\lambda_\psi}{384\pi^2}\int_0^1 dx\ln\left(\frac{M^2-x(1-x)s_0}{M^2-x(1-x)s}\right) + \ln\left(\frac{m^2-x(1-x)s_0}{m^2-x(1-x)s}\right) \end{aligned}$$

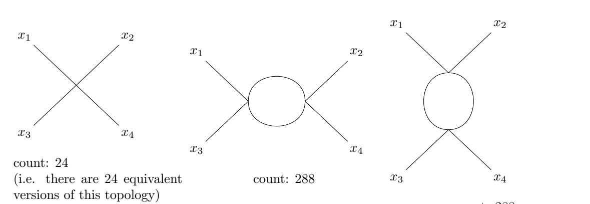
$$\begin{aligned} A_{\phi\psi\to\phi\psi}(s,t,u) = -\lambda_{\phi\psi} + \frac{\lambda_{\phi\psi}^2}{1152\pi^2}\int_0^1 dx\ln\left(\frac{b(s_0,x)}{b(s,x)}\right) + \ln\left(\frac{b(t_0,x)}{b(t,x)}\right) + \ln\left(\frac{b(u_0,x)}{b(u,x)}\right) \\ + \frac{\lambda_\phi\lambda_\psi}{384\pi^2}\int_0^1 dx\ln\left(\frac{M^2-x(1-x)t_0}{M^2-x(1-x)t}\right) + \ln\left(\frac{m^2-x(1-x)t_0}{m^2-x(1-x)t}\right) + \\ \ln\left(\frac{M^2-x(1-x)u_0}{M^2-x(1-x)u}\right) + \ln\left(\frac{m^2-x(1-x)u_0}{m^2-x(1-x)u}\right) \end{aligned}$$

The full probability amplitude is then:

$$\begin{aligned} &A_{\phi\phi\to\phi\phi}(s,t,u) + A_{\phi\phi\to\psi\psi}(s,t,u) + A_{\phi\psi\to\phi\psi}(s,t,u) \\ &= P(s,t,u|m,M,\lambda_\phi,\lambda_\psi,\lambda_{\phi\psi}) \equiv P(p_i|\theta) \end{aligned}$$

where s, t, and u are fully determined by the initial and final locations of the participants, and θ is a vector of all the model parameters.

Appendix



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(i.e. there are 24 equivalent versions of this topology)

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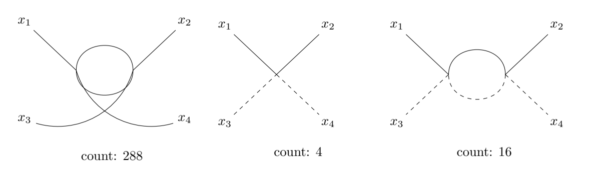

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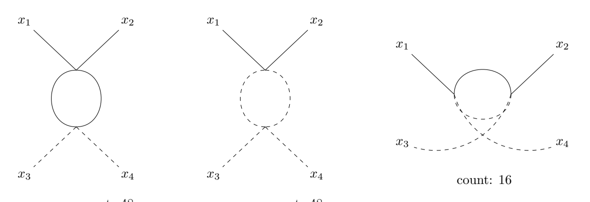

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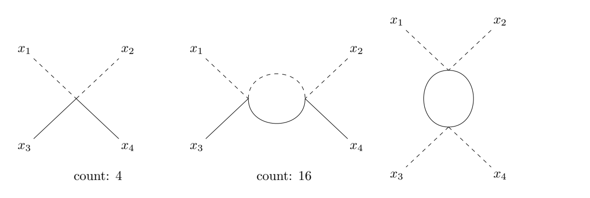

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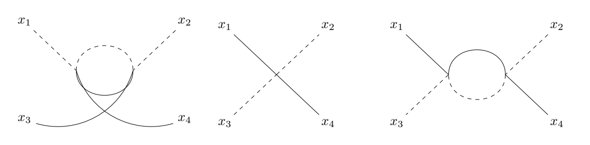

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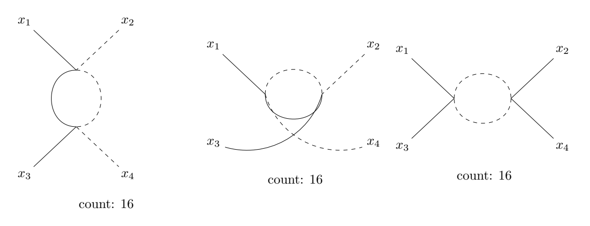

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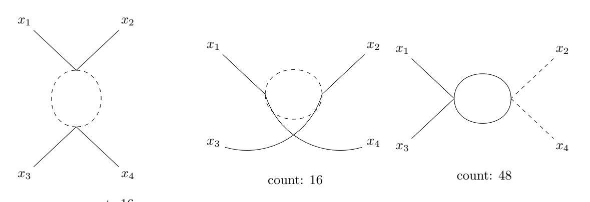

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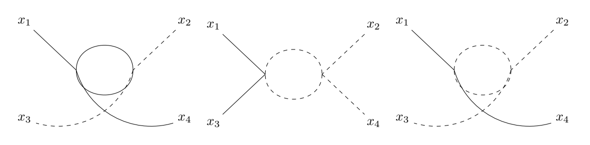

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