Homework 4 – Algebraic Structures

Ch 6: 7.10, 8.3, 8.4, M.2, M.3; Ch 7: 1.2, 2.3, 2.4, 2.7, 2.8, 2.9, 2.13, 2.14, 2.17

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Pre-lect: Ch 6: 7.1, 8.2; Ch 7: 1.1, 2.2

Exercise (6.7.10)**.** *1. Describe the orbit and the stabilizer of the matrix* $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *under conjugation in the general linear group* $GL_N(\mathbb{R})$*.*

2. Interpreting the matrix in $GL_2(\mathbb{F}_5)$*, find the order of the orbit.*

Proof. We find the stabilizer of the matrix by solving the equation:

$$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

With the constraint $ad - bc \neq 0$. We find for $ad = 1$ the stabilizer is

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

For the orbit we compute

$$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

And see with the constraint $ad - bc \neq 0$ the orbit is

$$\frac{1}{ad-bc}\begin{bmatrix} ad-2bc & ab \\ -cd & -bc+2ad \end{bmatrix}$$

□

Exercise (6.8.3)**.** *Exhibit the bijective map (6.8.4) explicitly, when* G *is the dihedral group* D_4 *and* S *is the set of vertices of a square.*

Proof. Let the representation be $D_4 = \{r, l | r^4 = l^2 = 1, lr = r^3 l\}$. We take the index 1 as our element $s \in S$. This has the stabilizer $H = \{lr, 1\}$. Then the cosets are $G/H = \{\{lr, 1\}, \{l, r\}, \{rl, r^2\}, \{r^2 l, r^3\}\}$.

The orbit of our element is the indices $O_s = \{1, 2, 3, 4\}$. Then there is a bijection

$$\begin{aligned} G/H &\to O_s \\ \{lr, 1\} &\mapsto 1 \\ \{l, r\} &\mapsto 2 \\ \{rl, r^2\} &\mapsto 3 \\ \{r^2 l, r^3\} &\mapsto 4 \end{aligned}$$

$\square$

Exercise (6.8.4). *Let H be the stabilizer of the index 1 for the operation of the symmetric group $G = S_n$ on the set of indices $\{1, \dots, n\}$. Describe the left cosets of H in G and the map (6.8.4) in this case.*

Proof. Let x be the permutation which cycles the group $x = (\mathbf{123 \dots n})$. And let the orbit of the first index be $O_1 = \{1, 2, 3 \dots n\}$. Since each permutation in H sends $1 \mapsto 1$. Then there is a bijection, $f(i) = x^{i-1}H$. Which sends index 1 to index i. More explicitly.

$$\begin{aligned} S_n/H &\to O_1 \\ H &\mapsto 1 \\ xH &\mapsto 2 \\ x^2H &\mapsto 3 \\ &\dots \\ x^{n-1} &\mapsto n \end{aligned}$$

$\square$

Exercise (6.M.2).

1. *Prove that the set Aut G of automorphisms of a group G forms a group, the law of composition being the composition of functions.*

2. *Prove that the map $\phi : G \to \text{Aut } G$ defined by $g \rightsquigarrow$ (conjugation by g) is a homomorphism, and determine its kernel.*

3. *The automorphisms that are obtained as conjugation by a group element are called inner automorphisms. Prove that the set of inner automorphisms, the image of ϕ, is a normal subgroup of the group Aut G.*

Proof. 1. Closure is satisfied. Let f and g be in Aut G. Then $f \circ g$ is in Aut G since $f \circ g(G) = f(g(G)) = f(G) = G$.

The identity is in G. This is the identity homomorphism, which is also a automorphism.

Inverses are satisfied. For any element f in Aut G. f's inverse exists since f is a bijection. And since $f : G \to G$, $f^{-1} : G \to G$ too. So f^{-1} is an automorphism. So f^{-1} is in Aut G.

2. For an f in G we define the homomorphism as $\phi(f) = \psi_f$. Where $\psi_f(x) = fxf^{-1}\ \forall x \in G$. So for any f, g in G we have $\phi(fg) = \psi_{fg}$ and $\psi_{fg}(x) = fgxg^{-1}f^{-1} = \psi_f(gxg^{-1}) = \psi_f(\psi_g(x)) = \psi_f \circ \psi_g(x) \implies \psi_f\psi_g = \phi(f)\phi(g)$. So ϕ is a homomorphism.

The kernel K of this homomorphism must map to the identity homomorphism, for some element k in the kernel $\phi(K) = \psi_K \implies psi_k(x) = kxk^{-1} = 1$. This is true for all elements that commute with every element of the group. So the kernel is the center of G.

3. Let c be any element in the center of G so ψ_c is an inner automorphism, and let ϕ be an arbitrary automorphism. So $\phi\psi_c\phi^{-1} \implies \phi(x)\psi_c(x)\phi(x)^{-1}$. Since ϕ is an automorphism $\phi(x)$ is in G and so is it's inverse. Let $\phi(x) = a$ then we have $a\psi_c(x)a^{-1} = acxc^{-1}a^{-1} = \psi_{ca}(x)$, which is in Aut G so the inner automorphisms are a normal subgroup.

□

Exercise (6.M.3)**.** *Determine the groups of automorphisms (see Exercise M.2) of the group **(1)** C_4 **(2)** C_6, **(3)** $C_2 \times C_2$, **(4)** D_4, **(5)** the quaternion group H.*

Proof. 1. Recall automorphisms of a cyclic group must send generators to generators, so we have the group of homomorphisms $x \mapsto \{x, x^3\}$.

2. By the same logic as part 1 we have $x \mapsto \{x, x^5\}$.

3. By guessing we have $(x, y) \mapsto \{(x, y), (y, x)\}$

4. Again by guessing $r \mapsto r, l \mapsto \{l, l^3\}$.

5. Still guessing, identity map, in cycle notation $(ijk), (ikj), (ij), (ik), (jk)$.

$\square$

Exercise (7.1.2)**.** *Let H be a subgroup of a group G. Describe the orbits for the operation of H on G by left multiplication.*

Proof. If we choose some h in H then $O_h = H$ since H is closed. If we choose some g in G then $O_g = G - H$. Because if H sent g to H then for some h_1 and h_2 in H: $h_1 g = h_2 \implies g = h_2 h_2^{-1}$ so g would be in H which is a contradiction.

$\square$

Exercise (7.2.3)**.** *A group G of order 12 contains a conjugacy class of order 4. Prove that the center of G is trivial.*

Proof. Since $|C_x| = 12$ and $|G| = 12$, $|Z(x)| = 3$. And since $|Z(x)| \neq |G|$, x cannot be in the center of G. Therefore $|Z(x)|$ must be greater than $|Z|$, since it must include the center and an additional element. So $|Z| < 3$, but $|Z|$ cannot be 2 since Z is a subgroup of $Z(x)$ and its order must divide $Z(x)$. So it must be order 1. And therefore it must be the trivial group.

$\square$

Exercise (7.2.4)**.** *Let G be a group, and let ϕ be the* **n***th power map:*$\phi(x) = x^n$. *What can be said about how ϕ acts on conjugacy classes?*

Proof. Consider two conjugate elements a and b such that $a = gbg^{-1}$ for some $g \in G$. Then ϕ acting on a gives:

$$\phi(a) = a^n = (gbg^{-1})^n = (gbg^{-1})(gbg^{-1})(gbg^{-1})\dots$$

We can then cancel out the inner gg^{-1} factors to get

$$\phi(a) = gb^n g^{-1}$$

So we can say it raises elements in the conjugacy classes to the power n.

$\square$

Exercise (7.2.7)**.** *Rule out as many as you can, as class equations for a group of order 10:*

$$1+1+1+2+5, \quad 1+2+2+5, \quad 1+2+3+4, \quad 1+1+2+2+2+2$$

Proof.

1. No. There are 3 elements that have a conjugacy class of order 1, so there are 3 elements with a centralizer of order 10. So there must be at least 3 elements in the center.

   But there is a conjugacy class of order 5. Which implies some element has a centralizer of order 2. But the centralizer must contain the center. Since the center is order 3 this is impossible.

2. This seems fine...

3. No. A conjugacy class of order 4 is impossible because it would require a non-integer order of the centralizer.

4. No. Since there are two elements with conjugacy classes of order 1, there are two elements in the center.

   But there is also a conjugacy class of order 2 which implies a centralizer of order 5. However this centralizer cannot contain the center since the order of the center does not divide its order.

□

Exercise (7.2.8)**.** *Determine the possible class equations of nonabelian groups of order* ***(1)****8,* ***(2)****21.*

Proof.

1. For a group of order 8 we note that the divisors are $1, 2, 4, 8$. Then as always we have:

   $$1 \times 8$$

   Trying class equations with a center of order 2 we find:

   $$1+1+2+2+2$$

   Since the center is order 2 here the centralizer's must have order greater than 3, so we can't use any conjugacy classes of order 4.

2. For a group of order 21 we note the divisors are $1, 3, 7, 21$. Then we have the trivial class equation:

$$1 \times 21$$

Trying groups with a center of order 1 we see that there can be only one class equation that correctly sums to 21.

$$1 + 3 + 3 + 7 + 7$$

All other attempts at forming class equations fail.

$\square$

Exercise (7.2.9)**.** *Determine the class equations for the following groups:* ***(1)*** *the quaternion group,* ***(2)*** D_4*,* ***(3)*** D_5*,* ***(4)*** *the subgroup of* $GL_2(\mathbb{F}_3)$ *of invertible upper triangular matrices.*

Proof. 1. The quaternions are order 8 and non-abelian. From the previous problem we know that the class equations that correspond to a group of order 8 are 1×8 and $1 + 1 + 2 + 2 + 2$. Since the quaternions are not abelian, the center is not the whole group. So the class equation is the latter option.

2. D_4 is also non-abelian, and order 8. So it has the same class equation as the quaternions. $1 + 1 + 2 + 2 + 2$.

3. D_5 is order 10 and non-abelian. Trying class equations with a center of order 1 we see only $1 + 2 + 2 + 5$ works out to the correct sum. We cannot have a center of any other order because it would not divide the corresponding centralizer's orders.

4. ?

$\square$

Exercise (7.2.13)**.** *Let* N *be a normal subgroup of a group* G*. Suppose that* $|N| = 5$ *and that* $|G|$ *is an odd integer. Prove that* N *is contained in the center of* G*.*

Proof. N is cyclic since $|N|$ is prime. Note that normal subgroups are unions of conjugacy classes. To see this note that by a definition of a normal subgroup

$$\forall g \in G, \forall n \in N : \; gng^{-1} \in N \implies \forall n \in N : \; C_n \subseteq N$$

The conjugacy classes that compose G and N must be odd since $|G|$ is odd so the possible class equations of N are:

$$|N| = 1 + 1 + 3 \quad \text{or} \quad |N| = 1 \times 5$$

In either case the center is at least order 2. So a non-identity element x must exist in N that commutes with the whole group. Since N is cyclic, this x must generate all of N. Since Z is a group, and x is in Z, N is in G.

□

Exercise (7.2.14)**.** *The class equation of a group G is* $1 + 4 + 5 + 5 + 5$.

1. *Does G have a subgroup of order 5? If so, is it a normal subgroup?*
2. *Does G have a subgroup of order 4? If so, is it a normal subgroup?*

Proof.

1. There exists a subgroup of order 5, this is because there is a conjugacy class of order 4 which implies a centralizer of order 5.

   This subgroup is normal. It is order 5 so it must be cyclic, and therefore all its elements commute with each other.

2. There exists a subgroup of order 4 because there is a conjugacy class of order 5 which implies a centralizer of order 4.

   The subgroup is not normal. Since any non identity element in it has a conjugacy class that is larger than the group. So the conjugates of all the elements in the group cannot be contained inside of it.

□

Exercise (7.2.17)**.** *Use the class equation to show that a group of order pq, with p and q prime, contains an element of order p.*

Proof. Note that the order of the conjugacy classes must divide the order of the group, and the groups divisors are $1, p, q$. So there are only 3 possible class equations.

$$1 \times p + q \quad 1 \times q + p \quad 1 \times pq \text{ (cyclic)}$$

Now recall that any group who's order is the product of two relatively prime integers r and s is isomorphic to the product of two cyclic groups of order r and s. This implies the class equation: $1 \times pq$. So since the group is cyclic, it has cyclic subgroups that are the order of its divisors p, and q. Therefore it contains an element of order p.

□

Pre-Lecture Problems

Exercise (6.7.1)**.** *Let $G = D_4$ be the dihedral group of symmetries of the square.*

1. *What is the stabilizer of a vertex? Of an edge?*
2. *G operates on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?*

Proof. Let the labels, going from the top left corner in the clockwise direction be a, b, c, d. And let r be a clockwise rotation, and l be a reflection across the vertical axis of symmetry. Then we have the group $\{r, l | r^4 = l^2 = 1\}$.

1. Then the stabilizer of vertices a, c is $\{1, rl\}$. The stabilizer of vertices b, d is $\{1, r^3 l\}$.

   The stabilizer of edges ab and cd is $\{1, l\}$. The stabilizer of edges bc and da is $\{1, r^2 l\}$.
2. The stabilizer of the diagonals ac and bd are $\{1, r^2, rl\}$.

□

Exercise (6.8.2)**.** *What is the stabilizer of the coset $[aH]$ for the operation of G on G/H?*

Proof. $G_{aH} = aHa^{-1}$.

To see this, choose some $g \in aHa^{-1}$ then for some $h \in H$, we have $g = aha^{-1}$. So $gaH = aha^{-1}aH = ahH = aH$. So $aHa^{-1} \subseteq= G_{aH}$.

Next take some $g \in G_{aH}$. Then for some $h, h', h'' \in H$, we have $gah = ah' \rightarrow g = ah'h^{-1}a^{-1} = ah''a^{-1}$. So $G_{aH} \subseteq aHa^{-1}$. Therefore $G_{aH} = aHa^{-1}$.

□

Exercise (7.1.1)**.** *Does the rule $g * x = xg^{-1}$ define an operation of G on G?*

Proof. Yes. Checking the group operation axioms. $1 \times x = x1 = x$. And $fg \times x = f \times (xg^{-1}) = xg^{-1}f^{-1}$. So both axioms work, so it is a group operation.

□

Exercise (7.2.2)**.** *A group of order 21 contains the conjugacy class $C(x)$ of order 3. What is the order of x in the group?*

Proof. With the given information we can determine $|Z(x)| = 7$ since the order of the centralizer times the order of the conjugacy class must equal the order of the group.

Since the centralizer is a subgroup, and prime order, it must be cyclic. Since x is in the aforementioned group, it must be order 7.

□