Homework 6 – Algebraic Structures

Ch 7: 7.2, 7.4, 7.5, 7.9, 7.10, 8.3, 8.4, 8.5, M.5, M.6
Pre-lect: 5.1

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Exercise (7.7.2)**.** *Let $G_1 \subset G_2$ be groups whose orders are divisible by p, and let H_1 be a Sylow p-subgroup of G_1. Prove that there is a Sylow p-subgroup H_2 of G_2 such that $H_1 = H_2 \cap G_1$.*

Proof. Since H_1 is a subgroup of G_2 and a p-group, by the second sylow theorem it must be contained in a sylow p-subgroup of G_2.

Let H_2 be this subgroup. Note that this subgroup must exist by the first sylow theorem.

Then H_1 is contained in H_2 and G_1, so it must be contained in $H_2 \cap G_1$.

Now we must show $H_1 \supset H_2 \cap G_1$. Let h be some element in $H_2 \cap G_1$. Since h is an element of the p-group H_2, and in G_2 it must generate a p-group in G_2. Since h is an element of a p-group in G_2 it must be contained in a sylow p-group of G_1, we choose this to be H_1. So $H_1 \subset H_2 \cap G_1$. Since there is containment both ways we have $H_1 = H_2 \cap G_1$.

□

Exercise (7.7.4)**.** 1. *Prove that no simple group has order pq, where p and q are prime.*

2. *Prove that no simple group has order p^2q, where p and q are prime.*

Proof. 1. Without loss of generality suppose that $p > q$. Then there are s sylow p-subpgroups where s must divide q. So s is either q or 1. We also know s must be congruent to $p \mod 1$. So $s = kp + 1$. But since $p > q$ there is not k that satisfies $s = q$. So s must be 1.

   Since there is only one sylow p-subgroup, it has no other conjugate subgroups, by the second sylow theorem. Therefore the p-subgroup must be normal. Therefore the group is not simple.

2. Consider the case where $p > q$. Then let s be the number of p-subgroups. s must divide q, so it is 1 or q. And s must satisfy $s = pk+1$ for some integer k. Since $p > q$ this is only satified when $s = 1$. Making the sylow p-subgroup normal and the group non-simple.

For the case where $p < q$ we let the number of sylow q-subgroups be n_q. Then applying the sylow theorems we require $n_q = 1, p, p^2 = kq + 1$. n_q cannot be p since $p \neq kq + 1$ for any k.

Now suppose $n_q = p^2$, then since the sylow q-subgroups are prime order, they intersect trivially, otherwise they would be the same group. So p^2 sylow q-subgroups account for $p^2(q-1)$ non-identity elements.

Now if we consider the number of sylow p-subgroups n_p. There can be 1 or q of these. And they must share no non-identity elements with the sylow q-subgroups. Because those elements would generate q order subgroups. If there are q sylow p-subgroups there must be more than p^2 elements in these groups or they would all be the same. But if there are more than p^2 unique elements in the sylow p-subgroups, then adding these to the number of elements in the sylow q-subgroups would give more than p^2q elements.

So either there is one sylow p-subgroup or one sylow q-subgroup. In either case there is a normal subgroup in the group, so it is not simple.

For the case $p = q$, we have a group of order p^3. Recall that p groups have a non-trivial center and that the center is normal in the group. Therefore a group of order p^3 cannot be simple.

□

Exercise (7.7.5)**.** *Find Sylow 2-subgroups of D_{10}.*

Proof. Recall that the D_{10} is order 20, and we choose the representation r a rotation of $36°$ and l is a reflection across the vertical axis of symmetry, so $r^{10} = l^2 = 1$. We also have Since the groups order is $20 = 2^2 5$ the group has either 5 or 1 sylow 2-subgroups. And $r^5 l = l r^5$.

Notice that there is at least one obvious subgroup of order 4 $\{1, l, r^5, lr^5\}$. We can't elimiante the chance of more sylow 4-subgroups so we look for more. These would contain elemnts of order 2, so we keep r^5 and look for more. We find that $\{1, r^5, rl, r^6 l\}$ is a sylow p-subgroup. So there must be at least 3 more sylow 2-subgroups.

Notice that $r^5 \times r^x l = r^y l \to r^5 l = r^{y-x} l \to y - x = 5$ is a requirement if we have two elements of the form $r^x l$ and $r^y l$. So the remaining groups are:

$$\{1, r^5, r^2 l, r^7 l\}$$
$$\{1, r^5, r^3 l, r^8 l\}$$
$$\{1, r^5, r^4 l, r^9 l\}$$

□

Exercise (7.7.9)**.** *Classify groups of order **(1)** 33 **(2)** 18.*

Proof.

1. Note that $33 = 11 \times 3$. Let the number of sylow 11-subgroups be s. Then by the third sylow theorem s must divide 3, and $s = k11 + 1$. The only choice of s that works here is 1. So there is only 1 sylow 11-subgroup. Now let the number of sylow 3-subgroups be r. Thene r must divide 11 and $r = k3 + 1$. The only choice here is $r = 1$. So there is 1 sylow 3-subgroup.

   Any group of order 33 must contain bothe of the cyclic subgroups. The product of the sylow groups must be in the group. And since the product of the sylow groups is order 33, it must be equal to the group.

   So all groups of order 33 are isomorphic to $C_3 C_{11}$

2. Note that $18 = 3^2 \times 2$. We call the sylow 3-subgroups S_3 and the sylow 2-subgroups S_2. We call the number of these sylow groups N_3 and N_2 respectively.

   Applying the third sylow theorem shows $N_3 = 1$ and S_3 is always normal. And $N_2 = 1, 3$, or 9. So we have several cases to address for N_2.

   $N_2 = 1 \implies S_2$ is normal. And since elements in S_2 are order 2, while those in S_3 can be 3 or 9, S_2 and S_3 must intersect trivially so $G = S_3 \times S_2$. But there are several possibilities for S_3: C_9 and $C_3 \times C_3$.

   So the two of the isomorphisim classes are $C_3 \times C_3 \times C_2$ and $C_9 \times C_2 = C_{19}$.

   $N_2 = 3$ then $|S_3 \ldots$

□

Exercise (7.7.10)**.** *Prove that the only simple groups of order < 60 are the groups of prime order.*

Proof. First we list all the numbers less than 60 that are not prime

4 6 8 9 10 12 14 15 16 18 20 21 22 24 25
26 27 28 30 32 33 34 35 36 38 39 40 42 44 45 46
48 49 50 51 52 54 55 56 57 58

Next we note that from previous problems that groups of order pq are not simple.

8 12 16 18 20 24
27 28 30 32 36 40 42 44
50 52 54 56

Next recall that groups of order p^2q cannot be simple.

16 24
30 32 36 40 42
54 56

Now recall that every group that is of order p^n has a non-trivial center, and therefore a normal subgroup. Eliminating these gives:

24
30 36 40 42
54 56

Now we consider these iduvidually. For $24 = 2^3 \times 3$ there can be 3 or 1 sylow 2-subgroups and 1 or 4 sylow 3-subgroups. However There cannot be both 3 sylow 2-subgroups and 4 sylow 3-subgroups since the groups only intersect trivially and this account for 32 elements. So at least one of the groups is normal.

$30 = 5 \times 6 = 3 \times 10 = 5 \times 6$. So there can be 1 or 10 sylow 3-subgroups and 1 or 6 sylow 5-subgroups. However there are to many elements if there are both 10 sylow 3-subgroups and 6 sylow 5-subgroups, so there must be only one of either.

$36 = 2^2 \times 3^2$ So there can be either 1, 3, or 9 sylow 2-subgroups and 1 or 4 sylow 3-subgroups. But counting elements shows there cannot be both 4 sylow 3-subgroups and 3 or 9 sylow 2-subgroups. So there is only 1 of either sylow 3-subgroups or sylow 2-subgroups.

$40 = 2^3 \times 5$, we require there be either 1, 2, 4, or 8 sylow 5-subgroups and that the number of sylow 5-subgroups is congruent to 1 mod 5. This is only the case for 1.

$42 = 7 \times 3 \times 2$, we require there be either 1, 2, 3, or 6 sylow 7-subgroups and that the number of sylow 7-subgroups be congruent to 1 mod 7. This is only the case for 1.

$54 = 3^3 \times 2$, we require there be either 1 or 2 sylow 3-subgroups and that the number of sylow 3-subgroups be congruent to 1 mod 2. This is only the case for 1.

$56 = 2^3 \times 7$, so there can be either 1 or 8 sylow 7-subgroups and 1 or 7 sylow 2 subgroups. But counting elements shows there cannot be both 8 sylow 7-subgroups and 7 sylow 2-subgroups. So there is only 1 of either.

□

Exercise (7.8.3)**.** *Determine the class equations of the groups of order 12.*

Proof. We are given the 5 isomorphism classes of groups of order 12. The first two of these are $C_4 \times C_3$ and $C_2 \times C_2 \times C_3$. Since these are abellian their class equaiton is

$$1 \times 12$$

The next isomorphism class is the alternating group A_4, for which we are given that the sylow 3-subgroup, K, is not normal, so there are four sylow 3-subgroups.

So by the second sylow theorem each sylow 3-subgroup is conjugate to the others. We also know these groups intersect trivially since they are of prime order. Therefore each element in a sylow 3-subgroup is conjugate to an element in the other sylow 3-subgroups. So each non-identity element in the sylow 3-subgroups has a conjugacy class of order 4. This gives $+4 + 4$.

We are also given that H must be normal. So conjugating H with any element gives us back H. So every non identity element in H must be in the same conjugacy class, which gives $+3$. So with the identity we have:

$$1 + 4 + 4 + 3$$

The next isomorphism class is the dihedral group D_6. Here K, the sylow 3-subgroup is normal and H the sylow 2-subgroup is not normal.

Since H is not normal there must be 3 sylow 2-subgroups, which are all conjugate subgroups to each other. So each non-identity element in the sylow 2-subgroup has 3 elements in its conjugacy class. This gives $+3 + 3 + 3$.

Since K is normal, all of its non-identity elements are in the same conjugacy class, which gives $+2$.

□

Exercise (7.8.4)**.** *Prove that a group of order $n = 2p$, where p is a prime, is either cyclic or dihedral.*

Proof. Applying sylow's third theorem we see that there is 1 sylow p-subgroup S_p. And either 1 or p sylow 2-subgroups S_2.

Let $x \in S_p$ and $y \in S_2$. Then we have $x^p = x^2 = 1$, and $yxy = x^a$ for some $a \in (1 \dots p-1)$. So

$$x = y^{-2}xy^2 = y^{-1}x^a y = y^{-1}a \dots ay = y^{-1}\underbrace{ay^{-1}ya \dots y^{-1}ya}_{a \text{ times}}y = x^{a^2}$$

Or $x = x^{a^2}$. So a^2 is congruent to 1 mod p. So p divides $a^2 - 1 = (a+1)(a-1)$. So $a = p + 1 = 1$ or $a = p - 1 = -1$.

For the case $a = 1$ we have $yxy = x$. So S_2 commutes with S_p, so we have the cyclic group $C_p \times C_2$.

Otherwise $yxy = x^{-1}$. This along with the other requirements is the definition of the dihedral group.

□

Exercise (7.8.5)**.** *Let G be a nonabelian group of order 28 whose sylow 2 subgroups are cyclic.*

1. *Determine the numbers of sylow 2 subgroups and sylow 7 subgroups.*
2. *Prove that there is at most one isomorphism class of such groups.*
3. *Determine the numbers of elements of each order, and the class equation of G.*

Proof. 1. We denote the sylow 7-subgroups and sylow 2-subgroups as S_7 and S_2 respectively. The number of these groups is N_7 and N_2.

   We know by sylow's third theorem that N_7 must divide 1, 2, or 4 and be congruent to 1 mod 7. So $N_7 = 1$.

   Likewise we can have $N_2 = 1$ or 7. But if $N_2 = 1$. $G = S_2 \times S_7$ which is abellian. So

$$N_2 = 7 \quad N_7 = 1$$

2. From S_2's order we could deduce that it could be C_4 or K_4. But if it were C_4, the seven C_4 would have 21 unique elements. Leaving no room for products with S_7. So it must be K_4. As before, we don't want every S_2 to have 3 new unique elements. They can all share at most 2 elements, 1 and what we call a. a's inverse must be in every group so $a^2 = 1$. So each S_2 has a unique b_i and c_i such that $b_i^2 = c_i^2 = 1 \ldots$

3. Determine the numb

□

Exercise (7.M.5)**.** *Let H and N be subgroups of a group G, and assume that N is a normal subgroup.*

1. *Determine the kernels of the restrictions of the canonical homomorphism $\pi : G \to G/N$ ot the subgroups H and HN.*
2. *Applying First Isomorphism Theorem to these restrictions, prove the Second Isomorphism Theorem: $H/(H \cap N)$ is isomorphic to $(HN)/N$.*

Proof. □

Exercise (7.M.6)**.** *Let H and N be normal subgroups of a group G such that $H \supset N$. Let $\overline{H} = H/N$ and $\overline{G} = G/N$.*

1. *Prove that $\overline{H}$ is a normal subgroup of $\overline{G}$.*
2. *Use the composed homomorphism $G \to \overline{G} \to \overline{G}/\overline{H}$ to prove the Third Isomorphism Theorem: G/H is isomorphic to $\overline{G}/\overline{H}$.*

Proof. $\square$