Homework 4 – Algebraic Structures

Ch 7: 3.1, 3.3, 3.4, 4.7, 4.9, 5.3, 5.6, 5,7, 5.12, 6.4, 6.5
Pre-lect: 5.1

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Exercise (7.3.1)**.** *Prove the Fixed Point Theorem (7.3.2).*

Proof. Recall the counting formula

$$|G| = |G_s||O_s|$$

and since the orbits partition the set

$$|S| = |O_1| + \cdots + |O_k|$$

So for some element $s \in S$, $|O_s|$ must divide $|G|$, so it can be p or 1. $|O_s|$ must also divide $|S|$. But since p does not divide $|S|$, $|O_s| = 1$. Then by the counting theorem $|G_s| = |G|$. So s stabilizes the whole group.

□

Exercise (7.3.3)**.** *A nonabelian group G has order p^3, where p is prime.*

1. *What are the possible orders of the center Z?*
2. *Let x be an element of G that isn't in Z. What is the order of its centralizer $Z(x)$?*
3. *What are the possible class equations for G?*

Proof.

1. First we can rule out the case $Z = 1$ using the class equation since it would imply $p^3 = 1 + (\textit{sum of } p \textit{ and } p^2)$. Which is impossible.

   We can arule out p^3 since it would imply the group is abelian.

   We can rule out p^2 since the centralizer for any element not in the center would have to be larger order than p^2 and divde p^3, but not be equal to p^2.

   So we are left with $|Z| = p$.

2. The centralizer must be larger than Z since it must contain x, but it cannot be the same size of the group p^3. And its order must divide the order of the group so we are left with p^2.

3. Applying the information above. We can only have $1 \times p + p \times (p^2 - 1)$.

$\square$

Exercise (7.3.4)**.** *Classify groups of order 8.*

Proof. For abelian groups we start with C_8, then by inspection we also have $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$.

For non-abelian groups, note that $8 = 2^3$. So we can apply part 3 of the previous problem and note that the class equation must be $1 + 1 + 2 + 2 + 2$. Also every non-identity element must be either order 2 or 4.

But if every element were order 2 the group would be abelian. So there is at least one element x of order 4.

$\square$

Exercise (7.4.7)**.** *Let G be a group of order n that operates nontrivially on a set of order r. Prove that if $n > r!$, then G has a proper normal subgroup.*

Proof. $\square$

Exercise (7.4.9)**.** *Let x be an element of a group G, not the identity, whose centralizer $Z(x)$ has order pq, where p and q are primes. Prove that $Z(x)$ is abelian.*

Proof. Suppose $Z(x)$ is not abellian.

$Z(x)$ is itself a group. Which must have a center containing x and 1, so $|Z| > 1$. Since the center is a subgroup its order must divide the order of the group, so it is either p, q, pq.

If the order of Z is pq, $Z(x)$ is abellian and we are done.

With out loss of generality suppose the order is p, consider the the centralizer of some element $y \in Z(x) - Z$. Its centralizer must have order greater than p and it must divide pq, so it must be order pq. So it commutes with the whole group.

So all elements not in the center commute, and so does the center. But this is a contradiction, so $Z(x)$ must be abellian.

$\square$

Exercise (7.5.3)**.** *Determine the orders of the elements of the symmetric group S_7.*

Proof. We have the obvious cases for 1 through 7 cycles which gives order $1, 2, 3, 4, 5, 6, 7$ elements.

Then we have the orders that arise from products of disjoint cycles. We start counting these down from 7 cycles noting that 7 and 6 cycles cannot form disjoint products with any of the cycles in S_7.

With 5 cycles we can form products with disjoint 2 cycles yielding an element of order 10.

With 4 cycles we can form products with disjoint 2 and 3 cycles yielding elements of order 8, and 12.

Counting lower results in double counting so our full list of the order of all elements of S_7 is $1, 2, 3, 4, 5, 6, 7, 8, 10, 12$.

□

Exercise (7.5.6)**.** *Find all subgroups of S_4 of order 4, and decide which ones are normal.*

Proof. We have the obvious cases, the subgroups generated by any of the 12 (4!) four cycles. The cannonical example of these is $\{1, (\mathbf{1234}), (\mathbf{13})(\mathbf{24}), (\mathbf{1432})\}$.

Then by inspection we note that 2 disjoint transpositions generate subgroups of order 4. There 6 (4 choose 2) of these, the cannonical example being $\{1, (\mathbf{12}), (\mathbf{34}), (\mathbf{12})(\mathbf{34})\}$.

So in total there are 18 subgroups of order 4 in S_4.

□

Exercise (7.5.7)**.** *Prove that A_n is the only subgroup of S_n of index 2.*

Proof. □

Exercise (7.5.12)**.** *Determine the class equations of S_6 and A_6.*

Proof. □

Exercise (7.6.4)**.** *Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime that divides the order of G. Prove that H is in the center $Z(G)$.*

Proof. □

Exercise (7.6.5)**.** *Let p be a prime integer and let G be a p-group. Let H be a proper subgroup of G. Prove that the normalizer $N(H)$ of H is strictly larger than H, and that H is contained in a normal subgroup of index p.*

Proof. □

Pre-Lecture Problems

Exercise (7.5.1)**.**

1. *Prove that the transpositions* $(\mathbf{12}), (\mathbf{23}), \dots (\mathbf{n-1}, \mathbf{n})$ *generate the symmetric group* S_n.
2. *How many transpositions are needed to write the cycle* $(\mathbf{123}\dots\mathbf{n})$*?*
3. *Prove that the cycles* $(\mathbf{12}\dots\mathbf{n})$ *and* $(\mathbf{12})$ *generate the symmetric group* S_n.

Proof.

1. Let some arbitrary cycle be $(\mathbf{i_1}, \mathbf{i_2}, \dots \mathbf{i_j})$. Then we can generate this with the given transpositions. First we note that we can generate adjacent indicies as follows
$$(\mathbf{i_1}, \mathbf{i_1} \pm \mathbf{1})(\mathbf{i_1} \pm \mathbf{1}, \mathbf{i_1} \pm \mathbf{2}) \dots (\mathbf{i_1} \pm \mathbf{k_1}, \mathbf{i_2}) = (\mathbf{i_1}, \mathbf{i_2})$$
Simillarly $(\mathbf{i_2}, \mathbf{i_3}) = (\mathbf{i_2}, \mathbf{i_2} \pm \mathbf{1}) \dots (\mathbf{i_2} \pm \mathbf{k_2}, \mathbf{i_3})$. So we can write $(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}) = (\mathbf{i_1}, \mathbf{i_2})(\mathbf{i_2}, \mathbf{i_3})$.
Generating the rest of the given cycle follows by induction.
2. $n-1$ by counting:
$$(\mathbf{123}\dots\mathbf{n}) = \underset{1}{(\mathbf{12})}\underset{2}{(\mathbf{23})} \dots \underset{n-1}{(\mathbf{n-1}, \mathbf{n})}$$
3. Using the given cycles we can generate any adjacent transpostions since $(\mathbf{i}, \mathbf{i+1}) = (\mathbf{123}\dots\mathbf{n})^{i+1}(\mathbf{12})(\mathbf{123}\dots\mathbf{n})^{1-i}$. Then applying the first part of this problem we can generate S_n.

□