Homework 5 – Algebraic Structures

Ch 7: 3.1, 3.3, 3.4, 4.7, 4.9, 5.3, 5.6, 5,7, 5.12, 6.4, 6.5
Pre-lect: 5.1

Blake Griffith

Exercise (7.3.1)**.** *Prove the Fixed Point Theorem (7.3.2).*

Proof. Recall the counting formula

$$|G| = |G_s||O_s|$$

and since the orbits partition the set

$$|S| = |O_1| + \cdots + |O_k|$$

So for some element $s \in S$, $|O_s|$ must divide $|G|$, so it can be p or 1. $|O_s|$ must also divide $|S|$. But since p does not divide $|S|$, $|O_s| = 1$. Then by the counting theorem $|G_s| = |G|$. So s stabilizes the whole group.

□

Exercise (7.3.3)**.** *A non-abelian group G has order p^3, where p is prime.*

1. *What are the possible orders of the center Z?*
2. *Let x be an element of G that isn't in Z. What is the order of its centralizer $Z(x)$?*
3. *What are the possible class equations for G?*

Proof.

1. First we can rule out the case $Z = 1$ using the class equation since it would imply $p^3 = 1 + (\textit{sum of } p \textit{ and } p^2)$. Which is impossible.

   We can rule out p^3 since it would imply the group is abelian.

   We can rule out p^2 since the centralizer for any element not in the center would have to be larger order than p^2 and divide p^3, but not be equal to p^2.

   So we are left with $|Z| = p$.

2. The centralizer must be larger than Z since it must contain x, but it cannot be the same size of the group p^3. And its order must divide the order of the group so we are left with p^2.

3. Applying the information above. We can only have $1 \times p + p \times (p^2 - 1)$.

$\square$

Exercise (7.3.4)**.** *Classify groups of order 8.*

Proof. For abelian groups we start with C_8, then by inspection we also have $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$.

For non-abelian groups, note that $8 = 2^3$. So we can apply part 3 of the previous problem and note that the class equation must be $1 + 1 + 2 + 2 + 2$. Also every non-identity element must be either order 2 or 4.

But if every element were order 2 the group would be abelian. So there is at least one element x of order 4.

There are two non-abelian groups of order 8. The dihedral group on a square, and the quaternions. But I'm not sure how to derive these with just the order of the group.

$\square$

Exercise (7.4.7)**.** *Let G be a group of order n that operates non-trivially on a set of order r. Prove that if $n > r!$, then G has a proper normal subgroup.*

Proof. Since G operates non-trivially we know for some $g \in G$ and $s \in S$ that $gs \neq 1 \dots$

$\square$

Exercise (7.4.9)**.** *Let x be an element of a group G, not the identity, whose centralizer $Z(x)$ has order pq, where p and q are primes. Prove that $Z(x)$ is abelian.*

Proof. Suppose $Z(x)$ is not abellian.

$Z(x)$ is itself a group. Which must have a center containing x and 1, so $|Z| > 1$. Since the center is a subgroup its order must divide the order of the group, so it is either p, q, pq.

If the order of Z is pq, $Z(x)$ is abellian and we are done.

With out loss of generality suppose the order is p, consider the centralizer of some element $y \in Z(x) - Z$. Its centralizer must have order greater than p

and it must divide pq, so it must be order pq. So it commutes with the whole group.

So all elements not in the center commute, and so does the center. But this is a contradiction, so $Z(x)$ must be abellian.

$\square$

Exercise (7.5.3). *Determine the orders of the elements of the symmetric group S_7.*

Proof. We have the obvious cases for 1 through 7 cycles which gives order $1, 2, 3, 4, 5, 6, 7$ elements.

Then we have the orders that arise from products of disjoint cycles. We start counting these down from 7 cycles noting that 7 and 6 cycles cannot form disjoint products with any of the cycles in S_7.

With 5 cycles we can form products with disjoint 2 cycles yielding an element of order 10.

With 4 cycles we can form products with disjoint 2 and 3 cycles yielding elements of order 8, and 12.

Counting lower results in double counting so our full list of the order of all elements of S_7 is $1, 2, 3, 4, 5, 6, 7, 8, 10, 12$.

$\square$

Exercise (7.5.6). *Find all subgroups of S_4 of order 4, and decide which ones are normal.*

Proof. We have the obvious cases, the subgroups generated by any of the 12 (4!) four cycles. The canonical example of these is $\{1, (\mathbf{1234}), (\mathbf{13})(\mathbf{24}), (\mathbf{1432})\}$.

Then by inspection we note that 2 disjoint transpositions generate subgroups of order 4. There 6 (4 choose 2) of these, the canonical example being $\{1, (\mathbf{12}), (\mathbf{34}), (\mathbf{12})(\mathbf{34})\}$.

So in total there are 18 subgroups of order 4 in S_4.

$\square$

Exercise (7.5.7). *Prove that A_n is the only subgroup of S_n of index 2.*

Proof. Suppose there is another subgroup X of index 2. Then recall that any subgroup of index 2 is normal. So X is normal.

Then since X is normal it must contain a 3-cycle (see the proof on page 202).

Since the group is normal and contains a 3-cycle, it must contain all 3-cycles since they form a conjugacy class.

So X must contain A_n since the 3-cycles generate A_n. But $|A_n| = |X|$ so the groups must be equal. But this is a contradiction. So A_n must be the only subgroup of index 2.

$\square$

Exercise (7.5.12)**.** *Determine the class equations of S_6 and A_6.*

Proof. Recall that cycles of the same shape are in the same conjugacy class. So we need to count each kind of cycle. Note that I skip cases which are double counting.

For single 2-cycles there are 15 (6 choose 2).

For products of two 2-cycles there are 45 (6 choose 2 * 4 choose 2 / 2! for commuting cycles)

For products of three 2-cycles, there are 30. (6 choose 2 * 4 choose 2 / 3! for commuting cycles)

Then there are the 3-cycles, 40 (6 choose 3 * 2! for each ordering)

Then there products of 2-cycles and 3-cycles, 120 (6 choose 2 * 4 choose 2 * 2!)

For products of two 3-cycles, 40 (6 choose 3 * 3 choose 3 * 2! * 2! / 2!).

For 4-cycles there are 90 (6 choose 4 * 3! for each ordering)

For products of 4-cycles and 2-cycles there are 90.

For 5-cycles there are 144 (6 choose 5 * 4! for each ordering)

For 6-cycles there are 120 (5! for each ordering)

So the class equation is:

$$1 + 15 + 45 + 15 + 40 + 120 + 40 + 90 + 90 + 144 + 120 = 720$$

For the alternating group we have all the even cycles from above. These are the products of two 2-cycles, the products of 4-cycles and 2-cycles and all products of 3-cycles, and all 5-cycles. So we have:

$$1 + 45 + 90 + 40 + 40 + 144 = 360$$

□

Exercise (7.6.4)**.** *Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime that divides the order of G. Prove that H is in the center $Z(G)$.*

Proof. Since H is normal it is a union of conjugacy classes. So there is some combination of terms in the class equation for G that sum to $|H| = p$. Since $1 \in H$ and it corresponds to a 1 in the class equation the rest of the terms in H must sum to $p - 1$ in the class equation. But since these terms must also divide $|G|$, and are smaller than p which is the smallest prime to divide $|G|$ they must all be 1, so they must all be in the center.

□

Exercise (7.6.5)**.** *Let p be a prime integer and let G be a p-group. Let H be a proper subgroup of G. Prove that the normalizer $N(H)$ of H is strictly larger than H, and that H is contained in a normal subgroup of index p.*

Proof. ...

□

Pre-Lecture Problems

Exercise (7.5.1)**.**

1. *Prove that the transpositions* $(\mathbf{12}), (\mathbf{23}), \ldots (\mathbf{n-1}, \mathbf{n})$ *generate the symmetric group S_n.*
2. *How many transpositions are needed to write the cycle* $(\mathbf{123} \ldots \mathbf{n})$*?*
3. *Prove that the cycles* $(\mathbf{12} \ldots \mathbf{n})$ *and* $(\mathbf{12})$ *generate the symmetric group S_n.*

Proof. 1. Let some arbitrary cycle be $(\mathbf{i_1}, \mathbf{i_2}, \dots \mathbf{i_j})$. Then we can generate this with the given transpositions. First we note that we can generate adjacent indices as follows

$$(\mathbf{i_1}, \mathbf{i_1 \pm 1})(\mathbf{i_1 \pm 1}, \mathbf{i_1 \pm 2}) \dots (\mathbf{i_1 \pm k_1}, \mathbf{i_2}) = (\mathbf{i_1}, \mathbf{i_2})$$

Similarly $(\mathbf{i_2}, \mathbf{i_3}) = (\mathbf{i_2}, \mathbf{i_2 \pm 1}) \dots (\mathbf{i_2 \pm k_2}, \mathbf{i_3})$. So we can write $(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}) = (\mathbf{i_1}, \mathbf{i_2})(\mathbf{i_2}, \mathbf{i_3})$.

Generating the rest of the given cycle follows by induction.

2. $n - 1$ by counting:

$$(\mathbf{123 \dots n}) = \underset{1}{(\mathbf{12})}\underset{2}{(\mathbf{23})} \dots \underset{n-1}{(\mathbf{n-1}, \mathbf{n})}$$

3. Using the given cycles we can generate any adjacent transpositions since $(\mathbf{i}, \mathbf{i+1}) = (\mathbf{123 \dots n})^{i+1}(\mathbf{12})(\mathbf{123 \dots n})^{1-i}$. Then applying the first part of this problem we can generate S_n.

$\square$