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1 Deriving A

As provided by T.M. Qian et al. in *Design and construction of the MUSE permanent magnet stellarator* appendix A, the magnetic field everywhere can be computed with

$$\vec{B} = \mu_0 \sum_d^D (\vec{H}_d + \vec{M}_d) \tag{1}$$

inside the magnet and

$$\vec{B} = \mu_0 \sum_d^D (\vec{H}_d) \tag{2}$$

outside the magnet, where $\vec{M}_d$ is the magnetization vector of magnet d and $\vec{H}_d$ is the auxiliary field given by

$$\begin{aligned}\vec{H}_d(\vec{r_n}) &= \tfrac{1}{4\pi}\textstyle\sum_{i,j,k}^{0,1}(-1)^{i+j+k} \begin{bmatrix} \tan^{-1}\left(\frac{y_j x_i}{z_k r_{ijk}}\right)+\tan^{-1}\left(\frac{z_k x_i}{y_j r_{ijk}}\right) & \ln\left(z_k + r_{ijk}\right) & \ln\left(y_j + r_{ijk}\right) \\ \ln\left(z_k + r_{ijk}\right) & \tan^{-1}\left(\frac{x_i y_j}{z_k r_{ijk}}\right)+\tan^{-1}\left(\frac{z_k y_j}{x_i r_{ijk}}\right) & \ln\left(x_i + r_{ijk}\right) \\ \ln\left(y_j + r_{ijk}\right) & \ln\left(x_i + r_{ijk}\right) & \tan^{-1}\left(\frac{x_i z_k}{y_j r_{ijk}}\right)+\tan^{-1}\left(\frac{y_j z_k}{x_i r_{ijk}}\right) \end{bmatrix} \cdot \vec{M}_d \\ &= H'(\vec{r_n})\vec{M}_d\end{aligned}$$

Thus inside the magnet the magnetic field can be simplified to $\vec{B} = \mu_0 \sum_d^D (H'_d(\vec{r_i})\vec{M}_d + \vec{M}_d) = \mu_0 \sum_d^D (H'_d(\vec{r_i}) + I) \cdot \vec{M}_d$.

In the expression for $\vec{H}$, x_i, y_j, and z_k give the distances to each of the corners of the magnet. Thus they are evaluated as follows:

$$\begin{aligned} x_0 &= x_n + L/2 \quad y_0 = y_n + W/2 \quad z_0 = z_n + H/2 \\ x_1 &= x_n - L/2 \quad y_1 = y_n - L/2 \quad z_1 = z_n - H/2 \end{aligned}$$

where L,W,H are the dimensions of the magnet and $\vec{r_n} = [x_n \quad y_n \quad z_n]$ is the position from the center of the magnet to the n^{th} point we are evaluating. $r_{ijk} = \sqrt{x_i^2 + y_j^2 + z_k^2}$.

From *Permanent-Magnet Optimization for Stellarators as Sparse Regression* Appendix A (Alan A. Kaptanoglu et al.) we see that $\vec{B} \cdot \hat{n}_i = (A\vec{M})_i$, allowing us to solve for A. The sum has been temporarily removed for simplicity.

Inside the magnet:

$$\begin{aligned} \vec{B} \cdot \hat{n}_i &= \mu_0 (H'_d(\vec{r_i})\vec{M}_d \cdot \hat{n}_i + \vec{M}_d \cdot \hat{n}_i) \\ &= \mu_0 ((H'_d(\vec{r_i})\vec{M}_d)^T \hat{n}_i + \vec{M}_d \cdot \hat{n}_i) \\ &= \mu_0 (\vec{M}_d^{\,T} H'^T_d(\vec{r_i})\hat{n}_i + \vec{M}_d \cdot \hat{n}_i) \\ &= \mu_0 (\vec{M}_d \cdot H'^T_d(\vec{r_i})\hat{n}_i + \vec{M}_d \cdot \hat{n}_i) \\ &= \mu_0 (H'^T_d(\vec{r_i})\hat{n}_i + \hat{n}_i) \cdot \vec{M}_d \end{aligned}$$

And so (reintroducing the sum)

$$(A\vec{M})_i = \mu_0 \sum_d^D (H_d'^T(\vec{r_i})\hat{n_i} + \hat{n_i}) \cdot \vec{M_d}$$
$$= \mu_0((H_1'^T(\vec{r_i})\hat{n_i} + \hat{n_i}) \cdot \vec{M_1} + ... + (H_D'^T(\vec{r_i})\hat{n_i} + \hat{n_i}) \cdot \vec{M_D})$$

Outside the magnet:

$$\vec{B} \cdot \hat{n_n} = \mu_0(H_d'(\vec{r_n})\vec{M_d}) \cdot \hat{n_n}$$
$$= \mu_0 H_d'^T(\vec{r_n})\hat{n_n} \cdot \vec{M_d}$$

And so

$$(A\vec{M})_i = \mu_0 \sum_d^D (H_d'^T(\vec{r_i})\hat{n_i}) \cdot \vec{M_d}$$
$$= \mu_0((H_1'^T(\vec{r_i})\hat{n_i}) \cdot \vec{M_1} + ... + (H_D'^T(\vec{r_i})\hat{n_i}) \cdot \vec{M_D})$$

If $\vec{M} = [M_1^x \;\; M_1^y \;\; ... \;\; M_D^z]$, then

$$A = \mu_0 \begin{bmatrix} g_1^x(1) & g_1^y(1) & ... & g_D^z(1) \\ ... & & & \\ g_1^x(N) & g_1^y(N) & ... & g_D^z(N) \end{bmatrix} \tag{3}$$

where $g_d^x(i) = (H_d'^T(\vec{r_i})\hat{n_i} + \hat{n_i})_x$ inside the magnet and $g_d^x(i) = (H_d'^T(\vec{r_i})\hat{n_i})_x$ outside the magnet.

When solving for $\vec{g_d}(i)$ it is important to realize that $H_d'^T(\vec{r_i})$ is in the local coordinate system of magnet d, defined such that the $+\hat{x}$-direction (and thus one of the magnet's faces) points normal away from the toroidal surface. Meanwhile, $\hat{n_i}$ is in a global coordinate system with the origin at the center of the device. Thus we introduce the rotation matrix which translates from the global to the local system:

$$P_d = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\cos\theta\sin\phi & \sin\theta \\ \sin\phi & \cos\phi & 0 \\ -\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \tag{4}$$

where ϕ is the d^{th} local system's rotation around the z-axis and θ is its rotation around the new y-axis (ie, the resulting y-axis after the first rotation). It turns out that, as long as we set the (ϕ, θ) toroidal coordinate system such that $(0, 0)$ lies on the global x-axis (and angles increase in the counter-clockwise direction) then the rotation angles of each local system are just their magnet's (ϕ, θ) coordinate on the torus.

Then $\vec{g_d}(i) = H_d'^T(\vec{r_i})\hat{n_i}^{local} + \hat{n_i}^{local}$ where $\hat{n_i}^{local} = P_d\hat{n_i}$.

Now we have A expressed in D different local coordinate systems and $\vec{M}$ expressed in the global coordinate system, so we have to make another transformation. Here we have two options:

1. We transform each $\vec{g_d}(i)$ into the global system with $\vec{g_d}^{global}(i) = P_d^T \vec{g_d}(i)$
2. We transform each $\vec{M_d}$ in $\vec{M}$ into their respective local systems with $\vec{M_d}^{local} = P_d \vec{M_d}$

Since there are $N \times D$ $\vec{g_d}(i)$'s but only D $\vec{M_d}$'s, the second option has a factor of N fewer operations. However since we would then have columns of A in D different coordinate systems, it complicates out final operation to solve for B.