Formalize Hinge Functions

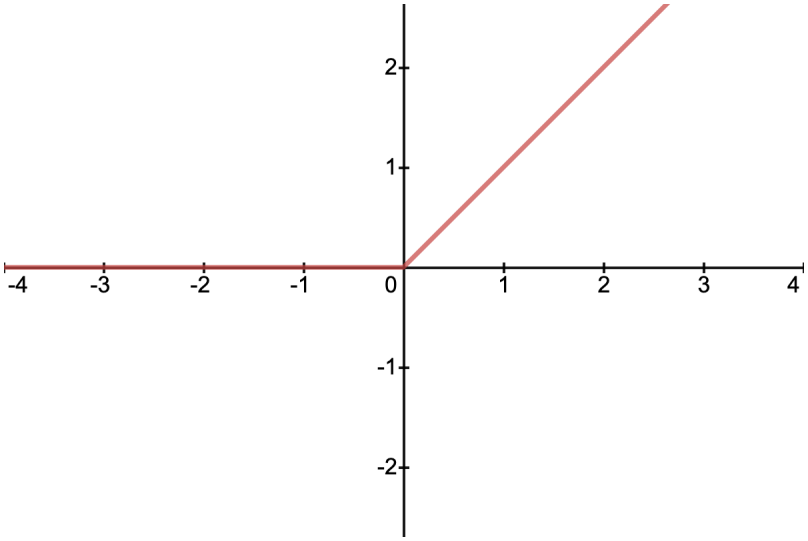
Recall from the video that a neural network is intuitively a sum of many little hinges. Let's write them down concretely.

From the previous module, you know that a hinge can be expressed using $\mathbf{max}(x, 0)$.

☆ **Key Points**

The parameters being used are $\mathbf{U}, \mathbf{W}, \mathbf{b}$ and $\mathbf{c}$. The parameters are learned from the data.

Stochastic gradient descent (SGD) is the most common way to learn the parameter.



However, this function is rather restrictive since the hinge always occurs at $(0, 0)$, which may not be what we want. Rather, we want the hinge to occur at any point in the space (x_0, y_0). So, we can express any sort of hinge using the general version $\mathbf{max}(wx + b, 0) + c$, where $(x_0, y_0) = \left(\frac{-b}{w}, c\right)$.

When the input points x is high-dimensional (vector $\mathbf{x}$), the hinge function can be generalized to $\mathbf{max}\left(\mathbf{w}^\top \mathbf{x} + b, 0\right) + c$.

Thus, a simple one layer neural network with M hinge functions that outputs a scalar $h(\mathbf{x})$ is just a sum of these hinges weighted differently, i.e.,

$$h(\mathbf{x}) = \sum_{i=1}^{M} u_i \left(\mathbf{max}\left(\mathbf{w}_i^\top \mathbf{x} + b_i, 0\right) + c_i\right) = \sum_{i=1}^{M} \left[u_i \, \mathbf{max}\left(\mathbf{w}_i^\top \mathbf{x} + b_i, 0\right) + u_i c_i\right]$$

Or in matrix/vector format:

$$h(\mathbf{x}) = \mathbf{u}^\top \mathbf{max}(\mathbf{W}\mathbf{x} + \mathbf{b}, 0) + c$$

where we stack all the weights $\mathbf{w}_i$ as row vectors vertically to form $\mathbf{W} \in \mathbb{R}^{M \times d}$, collect u_i in a vector $\mathbf{u} \in \mathbb{R}^M$, and define $c = \sum_{i=1}^{M} u_i c_i$. To clarify, $\mathbf{max}(\cdot, 0)$ is an element-wise max function.

If we want multi-dimensional outputs (a vector $\mathbf{h(x)} \in \mathbb{R}^N$), we can simply change the vector $\mathbf{u}$ to a matrix $\mathbf{U} \in \mathbb{R}^{N \times M}$ and the scalar $\boldsymbol{c}$ to a vector $\mathbf{c} \in \mathbb{R}^N$, yielding $\mathbf{h(x)} = \mathbf{U}\,\mathbf{max}(\mathbf{Wx} + \mathbf{b}, 0) + \mathbf{c}$.

Note that $\mathbf{U}, \mathbf{W}, \mathbf{b}$ and $\mathbf{c}$ are all parameters that we are going to learn from the data. The most common way to learn these parameters is using Stochastic Gradient Descent, which is a variant of gradient descent. We will discuss SGD later.

To summarize, a neural network can approximate decision boundary locally using kinks. The direction and position of those kinks are determined by the parameters $\mathbf{U}, \mathbf{W}, \mathbf{b}, \mathbf{c}$ that are learned from data using Stochastic Gradient Descent. If you introduce enough kinks, you can approximate any continuous function arbitrarily well.