Numerical Methods I
Exam #2 Tips

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Quiz #2

1 Inverse of a Matrix

$$AA^{-1} = I \tag{1}$$

1.1 Using Determinant

$$A^{-1} = \frac{A^T}{\det(A)} \tag{2}$$

For $A = 2 \times 2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{3}$$

For $A = 3 \times 3$:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{(aei + bfg + cdh) - (ceg + bdi + afh)} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \tag{4}$$

2 LU Factorizations

Purpose:

$$A = LU \tag{5}$$

$$A\vec{x} = (LU)\vec{x} = L(U\vec{x}) = b \qquad U\vec{x} = \vec{y}$$

$$L\vec{y} = b \tag{6}$$

Procedure (3×3 matrix):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix} \tag{8}$$

Types of equations:

$$i < j \qquad l_{i1} + u_{1j} + l_{i2}u_{2j} + \cdots + l_{ii}u_{ij} = a_{ij} \tag{9}$$

$$i = j \qquad l_{i1} + u_{1j} + l_{i2}u_{2j} + \cdots + l_{ii}u_{jj} = a_{ij} \tag{10}$$

$$i > j \qquad l_{i1} + u_{1j} + l_{i2}u_{2j} + \cdots + l_{ij}u_{jj} = a_{ij} \tag{11}$$

2.1 Doolittle

$A = 3 \times 3$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} (1)u_{11} & (1)u_{12} & (1)u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + (1)u_{22} & l_{21}u_{13} + (1)u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + (1)u_{33} \end{bmatrix} \tag{13}$$

2.2 Crout

$A = 3 \times 3$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} l_{11}(1) & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}(1) & l_{21}u_{12} + l_{22}(1) & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}(1) & l_{31}u_{12} + l_{32}(1) & l_{31}u_{13} + l_{32}u_{23} + l_{33}(1) \end{bmatrix} \tag{15}$$

2.3 Cholesky

$A = 3 \times 3$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} x_{11} & 0 & 0 \\ l_{21} & x_{22} & 0 \\ l_{31} & l_{32} & x_{33} \end{bmatrix} \begin{bmatrix} x_{11} & u_{12} & u_{13} \\ 0 & x_{22} & u_{23} \\ 0 & 0 & x_{33} \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} x_{11}x_{11} & x_{11}u_{12} & x_{11}u_{13} \\ l_{21}x_{11} & l_{21}u_{12} + x_{22}x_{22} & l_{21}u_{13} + x_{22}u_{23} \\ l_{31}x_{11} & l_{31}u_{12} + l_{32}x_{22} & l_{31}u_{13} + l_{32}u_{23} + x_{33}x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}^2 & x_{11}u_{12} & x_{11}u_{13} \\ l_{21}x_{11} & l_{21}u_{12} + x_{22}^2 & l_{21}u_{13} + x_{22}u_{23} \\ l_{31}x_{11} & l_{31}u_{12} + l_{32}x_{22} & l_{31}u_{13} + l_{32}u_{23} + x_{33}^2 \end{bmatrix} \tag{17}$$

3 Identifying Matrices

3.1 Strictly Diagonally Dominant (SDD)

A matrix $A_{n\times n}$ is strictly diagonally dominant if:

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad \forall i \tag{18}$$

3.2 Symmetric Positive Definite (SPD)

A symmetric matrix $A_{n\times n}$ is positive definite if:

$$\vec{x}^T A \vec{x} > 0 \tag{19}$$

$$\vec{x_{ij}} \in \mathbb{R}_{\neq 0} \tag{20}$$

Or if:

1. All the eigenvalues of A are positive;
2. And only if, the determinant of each leading submatrix is positive.

3.3 Coefficient Matrix (A) Significance

If A is either SDD or SPD:

- A is invertible, and $A\vec{x} = b$ has a unique solution;
- Both Gaussian elimination and LU factorization may be performed with any pivoting technique;
- Both Gaussian elimination and LU factorization are stable with regard to round-off error.

4 Matrix Norms

4.1 1- Norm

The maximum absolute column sum norm:

$$||A||_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \tag{21}$$

4.2 2- Norm

The spectral norm (or simply: matrix norm) is determined as follow:

$$||A||_2 = p\left(A^T A\right) \tag{22}$$

Which p is the maximum eigenvalue (in absolute mode). For $A = 2 \times 2$ matrix:

$$\begin{aligned}
||A||_2 &= p\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) && (23)\\
&= p\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) && (24)\\
&= p\left(\begin{bmatrix} a^2+c^2 & ab+cd \\ ca+dc & cb+d^2 \end{bmatrix}\right) && (25)\\
&= \sqrt{\max \det\left(\begin{bmatrix} a^2+c^2-\lambda & ab+cd \\ ba+dc & b^2+d^2-\lambda \end{bmatrix}\right)} && (26)\\
&= \sqrt{\max\left(|(a^2+c^2-\lambda)(b^2+d^2-\lambda)-(ab+cd)^2|\right)} && (27)
\end{aligned}$$

4.3 ∞- Norm

The maximum absolute row sum norm:

$$||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \tag{28}$$

5 Linear System Root Approximations

5.1 Jacobi Method

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_i - \sum_{j=i+1}^{n} a_{ij}x_i}{a_{ii}} \qquad , a_{ii} \neq 0 \tag{29}$$

$$\vec{x} = -D^{-1}(L+U)\vec{x} + D^{-1}\vec{b} \tag{30}$$

$$T = -D^{-1}(L+U) \tag{31}$$

$$\vec{c} = D^{-1}\vec{b} \tag{32}$$

$$\vec{x} = T\vec{x} + \vec{c} \tag{33}$$

5.2 Gauss-Seidel

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_i^{(k)}}{a_{ii}} \qquad , a_{ii} \neq 0 \tag{34}$$

$$\vec{x} = -(L+D)^{-1}U\vec{x} + (L+D)^{-1}\vec{b} \tag{35}$$

$$T = (L+D)^{-1}U \tag{36}$$

$$\vec{c} = (L+D)^{-1}\vec{b} \tag{37}$$

$$\vec{x} = T\vec{x} + \vec{c} \tag{38}$$

5.3 Criteria for Convergence

$$\vec{x}^{(k+1)} = T\vec{x}^{(k)} + \vec{c} \qquad , ||T|| < 1 \tag{39}$$

5.4 Error-Bound for Absolute Error

$$
\begin{aligned}
||\vec{x} - \vec{x}^{(k)}|| &\leq ||T||^k ||\vec{x}^{(0)} - \vec{x}|| && (40) \\
\frac{||\vec{x} - \vec{x}^{(k)}||}{||\vec{x}^{(0)} - \vec{x}||} &\leq ||T||^k \\
\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||}{||\vec{x}^{(0)} - \vec{x}||}\right) &\leq k \log\left(||T||\right) \\
k &\geq \frac{\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||}{||\vec{x}^{(0)} - \vec{x}||}\right)}{\log\left(||T||\right)} \\
\lceil k \rceil &= \frac{\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||}{||\vec{x}^{(0)} - \vec{x}||}\right)}{\log\left(||T||\right)} && (41)
\end{aligned}
$$

$$
\begin{aligned}
||\vec{x} - \vec{x}^{(k)}|| &\leq \frac{||T||^k}{1 - ||T||} ||\vec{x}^{(1)} - \vec{x}^{(0)}|| && (42) \\
\frac{||\vec{x} - \vec{x}^{(k)}||(1 - ||T||)}{||\vec{x}^{(1)} - \vec{x}^{(0)}||} &\leq ||T||^k \\
\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||(1 - ||T||)}{||\vec{x}^{(1)} - \vec{x}^{(0)}||}\right) &\leq k \log\left(||T||\right) \\
k &\geq \frac{\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||(1 - ||T||)}{||\vec{x}^{(1)} - \vec{x}^{(0)}||}\right)}{\log\left(||T||\right)} \\
\lceil k \rceil &= \frac{\log\left(\frac{||\vec{x} - \vec{x}^{(k)}||(1 - ||T||)}{||\vec{x}^{(1)} - \vec{x}^{(0)}||}\right)}{\log\left(||T||\right)} \\
& && (43)
\end{aligned}
$$

6 Matrix Condition

6.1 Condition Number of a Matrix

$$
K(A) = ||A||||A^{-1}|| \tag{44}
$$

6.2 Ill-Conditioned Matrices

A matrix is ill-conditioned if:

$$K(A) >> 1 \tag{45}$$