Solutions to H.W. #4

1. (a) Since we have $x_0 = 1$, $x_1 = \frac{1}{2}$ and $|g'(x)| \le \frac{1}{3}$ $\forall$ $x \in [0,1]$, taking $k = \frac{1}{3}$ gives:

$$|x - x_7| \le \frac{\left(\frac{1}{3}\right)^7}{1 - \frac{1}{3}} \left|1 - \frac{1}{2}\right|$$

$$= \frac{3}{2} \cdot \left(\frac{1}{3}\right)^7 \cdot \frac{1}{2}$$

$$= \frac{1}{4 \cdot 3^6} = 0.0003429$$

(b) We require the least n such that

$$\frac{3}{4}\left(1/3\right)^n \le 10^{-6}$$

$$\frac{1}{3^n} \le \frac{4 \cdot 10^{-6}}{3}$$

$$3^n \ge \frac{3 \cdot 10^6}{4}$$

$$n \ge \frac{\ln\left(\frac{3 \cdot 10^6}{4}\right)}{\ln(3)} = 12.31356$$

Take $n = 13$

2. $g(x) = \frac{1}{5}(x+1)^{3/2}$, $x \in [0,1]$

(a) $g(0) = \frac{1}{5} \in [0,1]$, $g(1) = \frac{2\sqrt{2}}{5} \in [0,1]$

Since $g'(x) = \frac{3}{10}(x+1)^{1/2} > 0 \quad \forall x \in [0,1]$,

g is monotone, so $g(x) \in [0,1] \quad \forall x \in [0,1]$.

So, g has at least one fixed-point in $[0,1]$.

Moreover $|g'(x)| = \frac{3}{10}(x+1)^{1/2}$ is an increasing function $\therefore$ $\max\limits_{0 \le x \le 1} |g'(x)| = |g'(1)| = \frac{3\sqrt{2}}{10} < 1$

Hence g has a unique fixed-point in $[0,1]$.

(b) With $x_0 = 0$, $x_1 = \frac{1}{5}(0+1)^{3/2} = \frac{1}{5}$

From part (a), $k = \frac{3\sqrt{2}}{10}$

$$\therefore |x - x_5| \le \frac{\left(\frac{3\sqrt{2}}{10}\right)^5}{1 - \frac{3\sqrt{2}}{10}} \left|\frac{1}{5} - 0\right| = 0.00477752$$

(c) We require the least n such that

$$\frac{\left(\frac{3\sqrt{2}}{10}\right)^n}{1-\frac{3\sqrt{2}}{10}} \cdot \frac{1}{5} \leq 10^{-8}$$

$$\left(\frac{3\sqrt{2}}{10}\right)^n \leq 5 \cdot \left(1-\frac{3\sqrt{2}}{10}\right) \cdot 10^{-8}$$

$$n \geq \frac{\ln\left(5 \cdot \left(1-\frac{3\sqrt{2}}{10}\right) \cdot 10^{-8}\right)}{\ln\left(\frac{3\sqrt{2}}{10}\right)} = 20.2511837$$

Take $n = 21$

3. For $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$,

$$g(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \frac{1}{2}\left(\sqrt{a} + \sqrt{a}\right)$$

$$= \frac{1}{2} \cdot 2\sqrt{a} = \sqrt{a}$$

$\therefore$ $x = \sqrt{a}$ is a fixed-point of g.

Consider $g'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$,

$$g'(\sqrt{a}) = \frac{1}{2}\left(1 - \frac{a}{a}\right) = \frac{1}{2} \cdot 0 = 0,$$

$$g''(x) = \frac{1}{2}\left(0 + \frac{2a}{x^3}\right) = \frac{a}{x^3},$$

$$g''(\sqrt{a}) = \frac{a}{a^{3/2}} = a^{-1/2} = \frac{1}{\sqrt{a}} \neq 0.$$

$\therefore$ The order of convergence is 2.

4. Let $g(x) = 0.4 + x - 0.1x^2$

Since $g(2) = 0.4 + 2 - 0.1 \cdot (2)^2$

$= 0.4 + 2 - 0.4 = 2$

x_{n+1} will converge to 2.

$g'(x) = 1 - 0.2x$, $g'(2) = 1 - 0.2 \cdot (2) = 0.6 \neq 0$

The order of convergence is 1.

5. (a) Let $g(x) = x + 1 - \frac{1}{5}x^2$

$g'(x) = 1 - \frac{2}{5}x$

$g'(\sqrt{5}) = 1 - \frac{2}{5} \cdot \sqrt{5} = 1 - \frac{2}{\sqrt{5}} \neq 0$

The order of convergence of

$x_{n+1} = x_n + 1 - \frac{x_n^2}{5}$ is 1.

(b) Let $g(x) = \frac{x^2 + 5}{2x} = \frac{1}{2}x + \frac{5}{2}x^{-1}$

$g'(x) = \frac{1}{2} - \frac{5}{2}x^{-2}$

$g'(\sqrt{5}) = \frac{1}{2} - \frac{5}{2} \cdot \frac{1}{5} = \frac{1}{2} - \frac{1}{2} = 0$

$g''(x) = 5x^{-3} = \frac{5}{x^3}$

$g''(\sqrt{5}) = \frac{5}{5\sqrt{5}} = \frac{1}{\sqrt{5}} \neq 0$

So, $x_{n+1} = \dfrac{x_n^2+5}{2x_n}$ convergence with order 2, meaning it converges at a faster rate.