

# Numerical Methods I

## Homework Problem Set #6

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# Problem Set #6

## 0.1 Statements: True or False

### 0.1.1 Question (a)

- **Sentence:** The product of two symmetric matrices is symmetric.

Consider two  $2 \times 2$  symmetric matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \tag{2}$$

The product of  $A$  and  $B$  is:

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} \tag{3}$$

Which is not symmetric. The sentence is false.

### 0.1.2 Question (b)

- **Sentence:** The inverse of an invertible symmetric matrix is an invertible symmetric matrix.

Consider the symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad (4)$$

Discovering the inverse of  $A$ :

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \quad (5)$$

Which still is symmetric matrix. Discovering the inverse of  $A^{-1}$ :

$$(A^{-1})^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A \quad (6)$$

Which satisfies the sentence. The sentence is true.

### 0.1.3 Question (c)

- **Sentence:** If  $A$  and  $B$  are  $n \times n$  matrices, then  $(AB)^T = A^T B^T$ .

Consider the matrices  $A$  and  $B$  described in eq. (2). Discovering  $(AB)^T$ :

$$(AB)^T = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \quad (7)$$

$$A^T B^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} \neq (AB)^T \quad (8)$$

The sentence is false.

## 0.2 Product of Two Upper Triangular Matrices

Consider the product of two upper triangular matrices  $2 \times 2$ ,  $A$  and  $B$ :

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11}) & (a_{11}b_{12} + a_{12}b_{22}) \\ 0 & (a_{22}b_{22}) \end{bmatrix} \quad (9)$$

Which still is a upper triangular matrix. The sentence is correct.

## 0.3 Linear System

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad (10)$$

Considering  $y = Ux$  and using  $Ly = b$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix} \quad (11)$$

$$y_1 = 2 \quad (12)$$

$$\begin{aligned} 2y_1 + y_2 &= 9 \\ y_2 &= 9 - 4 = 5 \end{aligned} \quad (13)$$

$$\begin{aligned} -y_1 + y_3 &= -1 \\ y_3 &= -1 + 2 = 1 \end{aligned} \quad (14)$$

Solving for  $Ux = y$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \quad (15)$$

$$x_3 = 1 \quad (16)$$

$$\begin{aligned} -2x_2 + x_3 &= 5 \\ x_2 &= \frac{-5 - 1}{2} = -3 \end{aligned} \quad (17)$$

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ x_1 &= \frac{9 + 2 - 1}{2} = 5 \end{aligned} \quad (18)$$

$$\begin{bmatrix} x_1 = 5 \\ x_2 = -3 \\ x_3 = 1 \end{bmatrix} \quad (19)$$

## 0.4 Vector Norms

### 0.4.1 Question (a)

$$x = \begin{bmatrix} 3 \\ -5 \\ \sqrt{2} \end{bmatrix} \quad (20)$$

$$|x|_1 = 3 - 5 + \sqrt{2} = -0.58579 \quad (21)$$

$$|x|_2 = \sqrt{3^2 + (-5)^2 + (\sqrt{2})^2} = \sqrt{9 + 25 + 2} = \sqrt{36} = 6 \quad (22)$$

$$|x|_\infty = 5 \quad (23)$$

**0.4.2 Question (b)**

$$x = \begin{bmatrix} e \\ \pi \\ 2\sqrt{3} \end{bmatrix} \quad (24)$$

$$|x|_1 = e + \pi + 2\sqrt{3} = 9.3240 \quad (25)$$

$$|x|_2 = \sqrt{e^2 + \pi^2 + (2\sqrt{3})^2} = \sqrt{e^2 + \pi^2 + 12} = \sqrt{29.259} = 5.4091 \quad (26)$$

$$|x|_\infty = 2\sqrt{3} \quad (27)$$

**0.4.3 Question (c)**

$$x = \begin{bmatrix} -3 \\ 2 \\ -4 \\ 8 \\ -1 \end{bmatrix} \quad (28)$$

$$|x|_1 = -3 + 2 - 4 + 8 - 1 = 2 \quad (29)$$

$$\begin{aligned} |x|_2 &= \sqrt{(-3)^2 + 2^2 + (-4)^2 + 8^2 + (-1)^2} = \sqrt{9 + 4 + 16 + 64 + 1} \\ &= \sqrt{94} = 9.6954 \end{aligned} \quad (30)$$

$$|x|_\infty = 8 \quad (31)$$

**0.5 Matrix Norms****0.5.1 Question (a)**

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 4 \end{bmatrix} \quad (32)$$

$$||A||_1 = 5 + 4 = 9 \quad (33)$$

$$||A||_\infty = 5 + 4 = 9 \quad (34)$$

In order to discover  $||A||_2$ :

$$A^T A = \begin{bmatrix} 34 & -35 \\ -35 & 41 \end{bmatrix} \quad (35)$$

$$\left| \begin{bmatrix} 34 - \lambda & -35 \\ -35 & 41 - \lambda \end{bmatrix} \right| = (34 - \lambda)(41 - \lambda) - 1225 = \lambda^2 - 75\lambda + 169 \quad (36)$$

$$\lambda_1 = 72.675 \quad (37)$$

$$\lambda_2 = 2.3254 \quad (38)$$

$$\|A\|_2 = \sqrt{\lambda_1} = 8.5249 \quad (39)$$

### 0.5.2 Question (b)

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \quad (40)$$

$$\|A\|_1 = 1 + 4 = 5 \quad (41)$$

$$\|A\|_\infty = 4 + 1 = 5 \quad (42)$$

In order to discover  $\|A\|_2$ :

$$A^T A = \begin{bmatrix} 16 & 8 \\ 0 & 16 \end{bmatrix} \quad (43)$$

$$\left| \begin{bmatrix} 16 - \lambda & 8 \\ 0 & 16 - \lambda \end{bmatrix} \right| = (16 - \lambda)(16 - \lambda) = \lambda^2 - 32\lambda + 256 \quad (44)$$

$$\lambda = 16 \quad (45)$$

$$\|A\|_2 = \sqrt{\lambda} = 4 \quad (46)$$