Numerical Methods I
Homework Problem Set #11

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Problem Set #11

1 RK-4

$$\frac{dy}{dt} = y + ty - y^2 \qquad y(0) = 1 \qquad 0 \leq t \leq 1 \qquad h = 0.5 \tag{1}$$

$$n = \frac{b-a}{h} = \frac{1-0}{0.5} = 2 \tag{2}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3}$$

$$t_{n+1} = t_n + h \tag{4}$$

$$k_1 = f(t_n, y_n) \tag{5}$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \tag{6}$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \tag{7}$$

$$k_4 = f(t_n + h, y_n + hk_3) \tag{8}$$

$$y_1 = 1 + \frac{1}{12}(k_{1_1} + 2k_{2_1} + 2k_{3_1} + k_{4_1}) \tag{9}$$

$$\begin{aligned} k_{1_1} &= f(0,1) \\ &= 1 + 0 * 1 - (1)^2 \\ &= 0 \end{aligned} \tag{10}$$

$$\begin{aligned} k_{2_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0\right) \\ &= f(0.25, 1) \\ &= 1 + 0.25 * 1 - (1)^2 \\ &= 0.25 \end{aligned} \tag{11}$$

$$\begin{aligned} k_{3_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0.25\right) \\ &= f(0.25, 1.0625) \\ &= 1.0625 + 0.25 * 1.0625 - (1.0625)^2 \\ &= 1.0625 + 0.2656 - 1.1289 \\ &= 0.1992 \end{aligned} \tag{12}$$

$$\begin{aligned} k_{4_1} &= f\left(0 + 0.5, 1 + 0.5 * 0.1992\right) \\ &= f(0.5, 1.0996) \\ &= 1.0996 + 0.5 * 1.0996 - (1.0996)^2 \\ &= 1.0996 + 0.5498 - 1.2091 \\ &= 0.4403 \end{aligned} \tag{13}$$

$$\begin{aligned} y_1 &= 1 + \frac{1}{12}(0 + 2(0.25) + 2(0.1992) + 0.4403) \\ &= 1 + \frac{1.3387}{12} \\ &= 1.1116 \end{aligned} \tag{14}$$

$$y_2 = 1.1116 + \frac{1}{12}(k_{1_2} + 2k_{2_2} + 2k_{3_2} + k_{4_2}) \tag{15}$$

$$\begin{aligned} k_{1_2} &= f(0.5, 1.1116) \\ &= 1.1116 + 0.5 * 1.1116 - (1.1116)^2 \\ &= 0.4317 \end{aligned} \tag{16}$$

$$\begin{aligned} k_{2_2} &= f\left(0.5 + \frac{0.5}{2}, 1.1116 + \frac{0.5}{2} * 0.4318\right) \\ &= f(0.75, 1.2195) \\ &= 1.2195 + 0.75 * 1.2195 - (1.2195)^2 \\ &= 0.6469 \end{aligned} \tag{17}$$

$$\begin{aligned} k_{3_2} &= f\left(0.5 + \frac{0.5}{2}, 0.1116 + \frac{0.5}{2} * 0.6469\right) \\ &= f(0.75, 1.2733) \\ &= 1.2733 + 0.75 * 1.2733 - (1.2733)^2 \\ &= 0.6070 \end{aligned} \tag{18}$$

$$\begin{aligned} k_{4_2} &= f\left(0.5 + 0.5, 0.1116 + 0.5 * 0.6070\right) \\ &= f(1, 1.4151) \\ &= 1.4151 + 1 * 1.4151 - (1.4151)^2 \\ &= 0.8277 \end{aligned} \tag{19}$$

$$\begin{aligned} y_2 &= 1.1116 + \frac{1}{12}(0.4317 + 2(0.6469) + 2(0.6070) + 0.8277) \\ &= 1.1116 + 0.3139 \\ &= 1.4255 \end{aligned} \tag{20}$$

2 Euler's Method

$$0 \leq t \leq 1 \qquad\qquad h = 0.25 \tag{21}$$

$$n = \frac{b-a}{h} = \frac{1-0}{0.25} = 4 \tag{22}$$

2.1 Question (a)

$$\frac{dx}{dt} = tx + y \qquad \frac{dy}{dt} = 2x - ty \qquad x(0) = 1 \qquad y(0) = -2 \tag{23}$$

$$\frac{dx}{dt} = f(t, x, y) \tag{24}$$
$$\frac{dy}{dt} = g(t, x, y) \tag{25}$$
$$x_{n+1} = x_n + hf(t_n, x_n, y_n) \tag{26}$$
$$y_{n+1} = y_n + hg(t_n, x_n, y_n) \tag{27}$$

$$\begin{aligned} x_1 &= 1 + 0.25f(0.25, 1, -2) \\ &= 1 + 0.25(0.25(1) + (-2)) \\ &= 0.5625 \end{aligned} \tag{28}$$
$$\begin{aligned} y_1 &= -2 + 0.25g(0, 1, -2) \\ &= -2 + 0.25(2(1) - 0.25(-2)) \\ &= -1.3750 \end{aligned} \tag{29}$$

$$\begin{aligned} x_2 &= 0.5625 + 0.25f(0.5, 0.5625, -1.3750) \\ &= 0.5 + 0.25(0.5(0.5625) + (-1.3750)) \\ &= 0.2891 \end{aligned} \tag{30}$$
$$\begin{aligned} y_2 &= -1.3750 + 0.25g(0.5, 0.5625, -1.3750) \\ &= -1.3750 + 0.25(2(0.5625) - 0.5(-1.3750)) \\ &= -0.9219 \end{aligned} \tag{31}$$

$$\begin{aligned} x_3 &= 0.2891 + 0.25f(0.75, 0.2891, -0.9219) \\ &= 0.2891 + 0.25(0.75 * 0.2891 + (-0.9219)) \\ &= 0.1128 \end{aligned} \tag{32}$$
$$\begin{aligned} y_3 &= -0.9219 + 0.25g(0.75, 0.2891, -0.9219) \\ &= -0.9219 + 0.25(2(0.2891) - 0.75(-0.9219)) \\ &= -0.6045 \end{aligned} \tag{33}$$

$$\begin{aligned} x_4 &= 0.1128 + 0.25f(1, 0.1128, -0.6045) \\ &= 0.1128 + 0.25(1(0.1128) + (-0.6045)) \\ &= -0.0101 \end{aligned} \tag{34}$$

$$\begin{aligned} y_4 &= -0.6045 + 0.25g(1, 0.1128, -0.6045) \\ &= -0.6045 + 0.25(2(0.1128) - 1(-0.6045)) \\ &= -0.3970 \end{aligned} \tag{35}$$

n	t_n	x_n	y_n
0	0	1	-2
1	0.25	0.5625	-1.3750
2	0.5	0.2891	-0.9219
3	0.75	0.1128	-0.6045
4	0.1	-0.0101	-0.3970

2.2 Question (b)

$$\begin{array}{lll} \dfrac{dx}{dt} = tx - xy + z & \dfrac{dy}{dt} = -ty + 2z & \dfrac{dz}{dt} = 2x - y + 3tz \\ x(0) = 0 & y(0) = 1 & z(0) = 2 \end{array} \tag{36}$$

$$\frac{dx}{dt} = f(t, x, y) \tag{37}$$

$$\frac{dy}{dt} = g(t, x, y) \tag{38}$$

$$\frac{dz}{dt} = u(t, x, y) \tag{39}$$

$$x_{n+1} = x_n + hf(t_n, x_n, y_n, z_n) \tag{40}$$

$$y_{n+1} = y_n + hg(t_n, y_n, z_n) \tag{41}$$

$$z_{n+1} = z_n + hu(t_n, x_n, y_n, z_n) \tag{42}$$

$$\begin{aligned} x_1 &= 0 + 0.25f(0.25, 0, 1, 2) \\ &= 0.25(0.25(0) - 0(1) + 2) \\ &= 0.5 \end{aligned} \tag{43}$$

$$\begin{aligned} y_1 &= 1 + 0.25g(0.25, 1, 2) \\ &= 1 + 0.25(-(0.25)(1) + 2(2)) \\ &= 1.9375 \end{aligned} \tag{44}$$

$$\begin{aligned} z_1 &= 2 + 0.25u(0.25, 0, 1, 2) \\ &= 2 + 0.25(2(0) - 1 + 3(0.25)(2)) \\ &= 2.1250 \end{aligned} \tag{45}$$

$$\begin{aligned} x_2 &= 0.5 + 0.25f(0.5, 0.5, 1.9375, 2.1250) \\ &= 0.5 + 0.25((0.5)(0.5) - 0.5(1.9375) + 2.1250) \\ &= 0.8516 \end{aligned} \tag{46}$$

$$\begin{aligned} y_2 &= 1.9375 + 0.25g(0.5, 1.9375, 2.1250) \\ &= 1.9375 + 0.25(-(0.5)(1.9375) + 2(2.1250)) \\ &= 2.7578 \end{aligned} \tag{47}$$

$$\begin{aligned} z_2 &= 2.1250 + 0.25u(0.5, 0.5, 1.9375, 2.1250) \\ &= 2.1250 + 0.25(2(0.5) - 2.7578 + 3(0.5)(2.1250)) \\ &= 2.6875 \end{aligned} \tag{48}$$

$$\begin{aligned} x_3 &= 0.8516 + 0.25f(0.75, 0.8516, 2.7578, 2.6875) \\ &= 0.8516 + 0.25(0.75(0.8516) - 0.8516(2.7578) + 2.6875) \\ &= 1.0960 \end{aligned} \tag{49}$$

$$\begin{aligned} y_3 &= 2.7578 + 0.25g(0.75, 2.7578, 2.6875) \\ &= 2.7578 + 0.25(-0.75(2.7578) + 2(2.6875)) \\ &= 3.5845 \end{aligned} \tag{50}$$

$$\begin{aligned} z_3 &= 2.6875 + 0.25u(0.75, 0.8516, 2.7578, 2.6875) \\ &= 2.6875 + 0.25(2(0.8516) - (2.7578) + 3(0.75)(2.6875)) \\ &= 3.9356 \end{aligned} \tag{51}$$

$$\begin{aligned} x_4 &= 1.0960 + 0.25f(1, 1.0960, 3.5845, 3.9356) \\ &= 1.0960 + 0.25(1(1.0960) - 1.0960(3.5845) + 3.9356) \\ &= 1.3717 \end{aligned} \tag{52}$$

$$\begin{aligned} y_4 &= 3.5845 + 0.25g(1, 3.5845, 3.9356) \\ &= 3.5845 + 0.25(-1(3.5845) + 2(3.9356)) \\ &= 4.6562 \end{aligned} \tag{53}$$

$$\begin{aligned} z_4 &= 3.9356 + 0.25u(1, 1.0960, 3.5845, 3.9356) \\ &= 3.9356 + 0.25(2(1.0960) - (3.5845) + 3(1)(3.9356)) \\ &= 6.5392 \end{aligned} \tag{54}$$

n	t_n	x_n	y_n	z_n
1	0	0	1	2
1	0.25	0.5	1.9375	2.1250
2	0.5	0.8516	2.7578	2.6875
3	0.75	1.0960	3.5845	3.9356
4	1	1.3717	4.6562	6.5392

3 Conversion into First-Order

3.1 Question (a)

$$y'' + ty' - 3y = t^2 \qquad y(0) = 3 \qquad y'(0) = 4 \tag{55}$$

$$y'' = -ty' + 3y + t^2 \tag{56}$$

$$u_1 = y' \tag{57}$$

$$u_1' = y'' \tag{58}$$

$$u_1' = -tu_1 + 3y + t^2 \tag{59}$$

$$y(0) = 3 \tag{60}$$

$$u_1(0) = 4 \tag{61}$$

3.2 Question (b)

$$y^{(4)} - y''' + y'' - 2y' + 7y = cos(t) \qquad y(0) = y'(0) = 0$$
$$y''(0) = 1 \qquad y'''(0) = 2 \tag{62}$$

$$y^{(4)} = y''' - y'' + 2y' - 7y + cos(t) \tag{63}$$
$$u_1 = y' \tag{64}$$
$$u_1' = y'' = u_2 \tag{65}$$
$$u_2' = y''' = u_3 \tag{66}$$
$$u_3' = y^{(4)} \tag{67}$$
$$u_3' = u_3 - u_2 + 2u_1 - 7y + cos(t) \tag{68}$$
$$y(0) = 0 \tag{69}$$
$$u_1(0) = 0 \tag{70}$$
$$u_2(0) = 1 \tag{71}$$
$$u_3(0) = 2 \tag{72}$$

4 Euler's method approximation on 3(a)

$$u_1' = -tu_1 + 3y + t^2 \qquad y(0) = 3 \qquad u_1(0) = 4$$
$$0 \leq t \leq 2 \qquad n = 2 \qquad h = 1 \tag{73}$$

$$y_{n+1} = y_n + h(u_{1_n}) \tag{74}$$
$$u_{1_{n+1}} = u_{1_n} + h(-tu_{1_n} + 3y_n + t^2) \tag{75}$$

$$\begin{aligned} y_1 &= 3 + 1(4) \\ &= 7 \end{aligned} \tag{76}$$
$$\begin{aligned} u_1 &= 4 + 1(-(1)(4) + 3(3) + (1)^2) \\ &= 10 \end{aligned} \tag{77}$$

$$
\begin{aligned}
y_2 &= 7 + 1(10) \\
&= 17 \qquad (78) \\
u_2 &= 10 + 1(-(2)(10) + 3(7) + (2)^2) \\
&= 15 \qquad (79)
\end{aligned}
$$

n	t_n	y_n	u_{1_n}
0	0	3	4
1	1	7	10
2	2	17	15