Numerical Methods I
Homework Problem Set #6

Jonathan Henrique Maia de Moraes (ID: 1620855)

10/08/2015

Problem Set #6

0.1 Statements: True or False

0.1.1 Question (a)

- **Sentence**: The product of two symmetric matrices is symmetric.

Consider two $2x2$ symmetric matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \tag{2}$$

The product of A and B is:

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} \tag{3}$$

Which is not symmetric. The sentence is false.

0.1.2 Question (b)

- **Sentence**: The inverse of an invertible symmetric matrix is an invertible symmetric matrix.

Consider the symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \tag{4}$$

Discovering the inverse of A:

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \tag{5}$$

Which still is symmetric matrix. Discovering the inverse of A^{-1}:

$$(A^{-1})^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A \tag{6}$$

Which satisfies the sentence. The sentence is true.

0.1.3 Question (c)

- **Sentence**: If A and B are $n \times n$ matrices, then $(AB)^T = A^T B^T$.

Consider the matrices A and B described in eq. (2). Discovering $(AB)^T$:

$$(AB)^T = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \tag{7}$$

$$A^T B^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} \neq (AB)^T \tag{8}$$

The sentence is false.

0.2 Product of Two Upper Triangular Matrices

Consider the product of two upper triangular matrices 2×2, A and B:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b11) & (a_{11}b_{11} + a_{12}b_{22}) \\ 0 & (a_{22}b_{22}) \end{bmatrix} \tag{9}$$

Which still is a upper triangular matrix. The sentence is correct.

0.3 Linear System

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \tag{10}$$

Considering $y = Ux$ and using $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix} \tag{11}$$

$$y_1 = 2 \tag{12}$$

$$\begin{aligned} 2y_1 + y_2 &= 9 \\ y_2 &= 9 - 4 = 5 \end{aligned} \tag{13}$$

$$\begin{aligned} -y_1 + y_3 &= -1 \\ y_3 &= -1 + 2 = 1 \end{aligned} \tag{14}$$

Solving for $Ux = y$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \tag{15}$$

$$x_3 = 1 \tag{16}$$

$$\begin{aligned} -2x_2 + x_3 &= 5 \\ x_2 &= \frac{-5-1}{2} = -3 \end{aligned} \tag{17}$$

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ x_1 &= \frac{9+2-1}{2} = 5 \end{aligned} \tag{18}$$

$$\begin{bmatrix} x_1 = 5 \\ x_2 = -3 \\ x_3 = 1 \end{bmatrix} \tag{19}$$

0.4 Vector Norms

0.4.1 Question (a)

$$x = \begin{bmatrix} 3 \\ -5 \\ \sqrt{2} \end{bmatrix} \tag{20}$$

$$|x|_1 = 3 - 5 + \sqrt{2} = -0.58579 \tag{21}$$

$$|x|_2 = \sqrt{3^2 + (-5)^2 + (\sqrt{2})^2} = \sqrt{9 + 25 + 2} = \sqrt{36} = 6 \tag{22}$$

$$|x|_\infty = 5 \tag{23}$$

0.4.2 Question (b)

$$x = \begin{bmatrix} e \\ \pi \\ 2\sqrt{3} \end{bmatrix} \tag{24}$$

$$|x|_1 = e + \pi + 2\sqrt{3} = 9.3240 \tag{25}$$

$$|x|_2 = \sqrt{e^2 + \pi^2 + (2\sqrt{3})^2} = \sqrt{e^2 + \pi^2 + 12} = \sqrt{29.259} = 5.4091 \tag{26}$$

$$|x|_\infty = 2\sqrt{(3)} \tag{27}$$

0.4.3 Question (c)

$$x = \begin{bmatrix} -3 \\ 2 \\ -4 \\ 8 \\ -1 \end{bmatrix} \tag{28}$$

$$|x|_1 = -3 + 2 - 4 + 8 - 1 = 2 \tag{29}$$

$$\begin{aligned} |x|_2 &= \sqrt{(-3)^2 + 2^2 + (-4)^2 + 8^2(-1)^2} = \sqrt{9 + 4 + 16 + 64 + 1} \\ &= \sqrt{94} = 9.6954 \end{aligned} \tag{30}$$

$$|x|_\infty = 8 \tag{31}$$

0.5 Matrix Norms

0.5.1 Question (a)

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 4 \end{bmatrix} \tag{32}$$

$$||A||_1 = 5 + 4 = 9 \tag{33}$$

$$||A||_\infty = 5 + 4 = 9 \tag{34}$$

In order to discover $||A||_2$:

$$A^T A = \begin{bmatrix} 34 & -35 \\ -35 & 41 \end{bmatrix} \tag{35}$$

$$\left| \begin{bmatrix} 34 - \lambda & -35 \\ -35 & 41 - \lambda \end{bmatrix} \right| = (34 - \lambda)(41 - \lambda) - 1225 = \lambda^2 - 75\lambda + 169 \tag{36}$$

$$\lambda_1 = 72.675 \tag{37}$$
$$\lambda_2 = 2.3254 \tag{38}$$

$$||A||_2 = \sqrt{\lambda_1} = 8.5249 \tag{39}$$

0.5.2 Question (b)

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \tag{40}$$
$$||A||_1 = 1 + 4 = 5 \tag{41}$$
$$||A||_\infty = 4 + 1 = 5 \tag{42}$$

In order to discover $||A||_2$:

$$A^T A = \begin{bmatrix} 16 & 8 \\ 0 & 16 \end{bmatrix} \tag{43}$$

$$\left| \begin{bmatrix} 16 - \lambda & 8 \\ 0 & 16 - \lambda \end{bmatrix} \right| = (16 - \lambda)(16 - \lambda) = \lambda^2 - 32\lambda + 256 \tag{44}$$

$$\lambda = 16 \tag{45}$$

$$||A||_2 = \sqrt{\lambda} = 4 \tag{46}$$