

Numerical Methods I  
Homework Problem Set #10

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# Problem Set #10

## 1 Picard Iterate

$$y_{n+1} = y_0 + \int_{x_0}^x f(t, y_n(t)) dt \quad (1)$$

### 1.1 Question (a)

$$y' = x^2 - 2y^2 + 1 \quad y(0) = 0 \quad y_2(x) \quad (2)$$

$$y_{n+1}(x) = \int_0^x t^2 - 2y_n(t)^2 + 1 dt \quad (3)$$

$$\begin{aligned} &= \int_0^x t^2 dt - \int_0^x 2y_n(t)^2 dt + \int_0^x 1 dt \\ &= \left[ \frac{t^3}{3} \right]_0^x - \int_0^x 2y_n(t)^2 dt + [t]_0^x \\ &= \frac{x^3}{3} - \int_0^x 2y_n(t)^2 dt + x \end{aligned} \quad (4)$$

$$y_1(x) = \frac{x^3}{3} - \int_0^x 2y_0(t)^2 dt + x \quad (5)$$

$$= \frac{x^3}{3} + x \quad (6)$$

$$y_2(x) = \frac{x^3}{3} - \int_0^x 2y_1(t)^2 dt + x \quad (7)$$

$$\begin{aligned} &= \frac{x^3}{3} - \int_0^x 2 \left( \frac{t^3}{3} + t \right)^2 dt + x \\ &= \frac{x^3}{3} - \int_0^x 2 \left( \frac{t^6}{9} + \frac{2t^4}{3} + t^2 \right) dt + x \\ &= \frac{x^3}{3} - 2 \int_0^x \frac{t^6}{9} + \frac{2t^4}{3} + t^2 dt + x \\ &= \frac{x^3}{3} - 2 \left[ \frac{t^7}{63} + \frac{2t^5}{15} + \frac{t^3}{3} \right]_0^x + x \\ &= \frac{x^3}{3} - \frac{2x^7}{63} - \frac{4x^5}{15} - \frac{x^3}{3} + x \\ &= -\frac{2x^7}{63} - \frac{4x^5}{15} + x \end{aligned} \quad (8)$$

## 1.2 Question (b)

$$\frac{dy}{dx} = 2e^x + y \quad y(0) = 1 \quad y_3(x) \quad (9)$$

$$y_{n+1}(x) = 1 + \int_0^x 2e^t + y_n(t) dt \quad (10)$$

$$\begin{aligned} &= 1 + 2 \int_0^x e^t dt + \int_0^x y_n(t) dt \\ &= 1 + 2 [e^t]_0^x + \int_0^x y_n(t) dt \\ &= 1 + 2e^x + \int_0^x y_n(t) dt \end{aligned} \quad (11)$$

$$y_1(x) = 1 + 2e^x + \int_0^x y_0(t) dt \quad (12)$$

$$\begin{aligned} &= 1 + 2e^x + \int_0^x 1 dt \\ &= 1 + 2e^x + [t]_0^x \\ &= 1 + 2e^x + x \end{aligned} \quad (13)$$

$$y_2(x) = 1 + 2e^x + \int_0^x y_1(t) \, dt \quad (14)$$

$$\begin{aligned} &= 1 + 2e^x + \int_0^x 1 + 2e^t + t \, dt \\ &= 1 + 2e^x + \left[ t + 2e^t + \frac{t^2}{2} \right]_0^x \\ &= 1 + 2e^x + x + 2e^x + \frac{x^2}{2} \\ &= 4e^x + \frac{x^2}{2} + x + 1 \end{aligned} \quad (15)$$

$$y_3(x) = 1 + 2e^x + \int_0^x y_2(t) \, dt \quad (16)$$

$$\begin{aligned} &= 1 + 2e^x + \int_0^x 4e^t + \frac{t^2}{2} + t + 1 \, dt \\ &= 1 + 2e^x + \left[ 4e^t + \frac{t^3}{6} + \frac{t^2}{2} + t \right]_0^x \\ &= 1 + 2e^x + 4e^x + \frac{x^3}{6} + \frac{x^2}{2} + x \\ &= 6e^x + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \end{aligned} \quad (17)$$

## 2 Taylor Approximating Polynomial

### 2.1 Question (a)

$$y' = \cos(x) - y \qquad y(0) = -1 \qquad p_3(x) \quad (18)$$

$$p_3(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 \quad (19)$$

$$y' = \cos(x) - y \quad (20)$$

$$y'(0) = \cos(0) - y(0) = 1 - (-1) = 2 \quad (21)$$

$$y'' = -\sin(x) - y' = -\sin(x) - \cos(x) + y \quad (22)$$

$$y''(0) = -\sin(0) - \cos(0) + y(0) = -1 - 1 = -2 \quad (23)$$

$$y''' = -\cos(x) - y'' = \sin(x) - y \quad (24)$$

$$y'''(0) = \sin(0) - y(0) = -(-1) = 1 \quad (25)$$

$$p_3(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 \quad (26)$$

$$p_3(x) = -1 + 2x + \frac{-2x^2}{2} + \frac{x^3}{6} = \frac{x^3}{6} - x^2 + 2x - 1 \quad (27)$$

## 2.2 Question (b)

$$y' = e^x y \quad y(0) = 2 \quad p_2(x) \quad (28)$$

$$p_2(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 \quad (29)$$

$$y' = e^x y \quad (30)$$

$$y'(0) = e^0 y(0) = 1(2) = 2 \quad (31)$$

$$y'' = e^x y + e^x y' = e^x (y + e^x y) \quad (32)$$

$$y''(0) = e^0 (y(0) + e^0 y(0)) = 1(2 + 1(2)) = 4 \quad (33)$$

$$p_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 \quad (34)$$

$$p_2(x) = 2 + 2x + 2x^2 = 2(x^2 + x + 1) \quad (35)$$

## 3 Euler's Method

$$y_{n+1} = y_n + hf(x_{n+1}, y_n) \quad x_n = x_0 + nh \quad (36)$$

### 3.1 Question (a)

$$\frac{dy}{dx} = y^3 - x \quad y(0) = 1 \quad 0 \leq x \leq 4 \quad n = 4 \quad (37)$$

$$h = \frac{x - x_0}{n} = \frac{4 - 0}{4} = 1 \quad (38)$$

$$y_1 = y_0 + hf(x_1, y_0) \quad (39)$$

$$= 1 + y_0^3 - x_1$$

$$= 1 + 1^3 - 1 = 2 - 1 = 1 \quad (40)$$

$$y_2 = y_1 + hf(x_2, y_1) \quad (41)$$

$$= 1 + y_1^3 - x_2$$

$$= 1 + 1^3 - 2 = 2 - 2 = 0 \quad (42)$$

$$y_3 = y_2 + hf(x_3, y_2) \quad (43)$$

$$= 0 + y_2^3 - x_3$$

$$= 0^3 - 3 = -3 \quad (44)$$

$$y_4 = y_3 + hf(x_4, y_3) \quad (45)$$

$$= -3 + y_3^3 - x_4$$

$$= -3 + (-3)^3 - 4 = -3 - 27 - 4 = -34 \quad (46)$$

$n$	$x_n$	$y_n$
0	0	1
1	1	1
2	2	0
3	3	-3
4	4	-33

### 3.2 Question (b)

$$(y^2 + 1)\frac{dy}{dx} = \ln(x^2 + 1) \quad y(0) = 0 \quad 0 \leq x \leq 1 \quad n = 4 \quad (47)$$

$$\frac{dy}{dx} = \frac{\ln(x^2 + 1)}{y^2 + 1} \quad (48)$$

$$h = \frac{x - x_0}{n} = \frac{1 - 0}{4} = 0.25 \quad (49)$$

$$y_1 = y_0 + 0.25f(x_1, y_0) \quad (50)$$

$$\begin{aligned} &= 0 + 0.25 \left( \frac{\ln((0.25)^2 + 1)}{0^2 + 1} \right) \\ &= 0.25 \ln(1.0625) = 0.015156 \end{aligned} \quad (51)$$

$$y_2 = y_1 + 0.25f(x_2, y_1) \quad (52)$$

$$\begin{aligned} &= 0.015156 + 0.25 \left( \frac{\ln((0.5)^2 + 1)}{(0.015156)^2 + 1} \right) \\ &= 0.015156 + 0.25 \left( \frac{\ln(1.25)}{1.000297} \right) \\ &= 0.015156 + 0.055773 = 0.070929 \end{aligned}$$

$$y_3 = y_2 + 0.25f(x_3, y_2) \quad (53)$$

$$\begin{aligned} &= 0.070929 + 0.25 \left( \frac{\ln((0.75)^2 + 1)}{(0.070929)^2 + 1} \right) \\ &= 0.070929 + 0.25 \left( \frac{\ln((1.5625))}{1.005031} \right) \\ &= 0.070929 + 0.111013 = 0.181942 \end{aligned} \quad (54)$$

$$y_4 = y_3 + 0.25f(x_4, y_3) \quad (55)$$

$$\begin{aligned} &= 0.181942 + 0.25 \left( \frac{\ln(1^2 + 1)}{(0.181942)^2 + 1} \right) \\ &= 0.181942 + 0.25 \left( \frac{\ln((2))}{1.033103} \right) \\ &= 0.181942 + 0.167734 = 0.349676 \end{aligned} \quad (56)$$

$n$	$x_n$	$y_n$
0	0	0
1	0.25	0.015156
2	0.5	0.070929
3	0.75	0.181942
4	1	0.349676

## 4 Modified Euler's Method

$$k_1 = hf(x_{n+1}, y_n) \quad (57)$$

$$k_2 = hf(x_{n+1} + h, y_n + k_1) \quad (58)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \quad (59)$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad y(1) = 1 \quad 1 \leq x \leq 6 \quad n = 5 \quad (60)$$

$$h = \frac{x - x_0}{n} = \frac{6 - 1}{5} = 1 \quad (61)$$

$$k_1 = 1f(x_{n+1}, y_n) \quad (62)$$

$$= 1 + \frac{y_n}{x_{n+1}} \quad (63)$$

$$k_2 = 1f(x_{n+1} + 1, y_n + k_1) \quad (64)$$

$$= 1 + \frac{y_n + k_1}{x_{n+1} + 1} \quad (65)$$

$$y_1 = y_0 + \frac{1}{2}(k_{1_1} + k_{2_1}) \quad (66)$$

$$y_0 = y(1) = 1 \quad (67)$$

$$k_{1_1} = 1 + \frac{y_0}{x_1} = 1 + \frac{1}{2} = \frac{3}{2} \quad (68)$$

$$k_{2_1} = 1 + \frac{y_0 + k_{1_1}}{x_1 + 1} = 1 + \frac{1 + \frac{3}{2}}{2 + 1} = 1 + \frac{\frac{5}{2}}{3} = 1 + \frac{5}{6} = \frac{11}{6} \quad (69)$$

$$y_1 = 1 + \frac{1}{2} \left( \frac{3}{2} + \frac{11}{6} \right) = 1 + \frac{9 + 11}{12} = 1 + \frac{20}{12} = \frac{32}{12} = \frac{8}{3} \quad (70)$$



$$y_2 = y_1 + \frac{1}{2}(k_{1_2} + k_{2_2}) \quad (71)$$

$$y_1 = \frac{8}{3} \quad (72)$$

$$k_{1_2} = 1 + \frac{y_1}{x_2} = 1 + \frac{\frac{8}{3}}{3} = 1 + \frac{8}{9} = \frac{17}{9} \quad (73)$$

$$k_{2_2} = 1 + \frac{y_1 + k_{1_2}}{x_2 + 1} = 1 + \frac{\frac{8}{3} + \frac{17}{9}}{3 + 1} = 1 + \frac{\frac{42}{9}}{4} = 1 + \frac{42}{36} = \frac{78}{36} = \frac{13}{6} \quad (74)$$

$$\begin{aligned} y_2 &= \frac{8}{3} + \frac{1}{2} \left( \frac{17}{9} + \frac{13}{6} \right) = \frac{8}{3} + \frac{34 + 39}{36} \\ &= \frac{8}{3} + \frac{73}{36} = \frac{73 + 96}{36} = \frac{169}{36} \end{aligned} \quad (75)$$

$$y_3 = y_2 + \frac{1}{2}(k_{1_3} + k_{2_3}) \quad (76)$$

$$y_2 = \frac{169}{36} \quad (77)$$

$$k_{1_3} = 1 + \frac{y_2}{x_3} = 1 + \frac{\frac{169}{36}}{4} = 1 + \frac{169}{144} = \frac{313}{144} \quad (78)$$

$$\begin{aligned} k_{2_3} &= 1 + \frac{y_2 + k_{1_3}}{x_3 + 1} = 1 + \frac{\frac{169}{36} + \frac{313}{144}}{4 + 1} = 1 + \frac{\frac{676 + 313}{144}}{5} \\ &= 1 + \frac{989}{720} = \frac{1709}{720} \end{aligned} \quad (79)$$

$$\begin{aligned} y_3 &= \frac{169}{36} + \frac{1}{2} \left( \frac{313}{144} + \frac{1709}{720} \right) = \frac{169}{36} + \frac{1565 + 1709}{1440} \\ &= \frac{169}{36} + \frac{3274}{1440} = \frac{3380 + 1637}{720} = \frac{5017}{720} \end{aligned} \quad (80)$$

$$y_4 = y_3 + \frac{1}{2}(k_{1_4} + k_{2_4}) \quad (81)$$

$$y_3 = \frac{5017}{720} \quad (82)$$

$$k_{1_4} = 1 + \frac{y_3}{x_4} = 1 + \frac{\frac{5017}{720}}{5} = 1 + \frac{5017}{3600} = \frac{8617}{3600} \quad (83)$$

$$\begin{aligned} k_{2_4} &= 1 + \frac{y_3 + k_{1_4}}{x_4 + 1} = 1 + \frac{\frac{5017}{720} + \frac{8617}{3600}}{5 + 1} = 1 + \frac{\frac{25085 + 8617}{3600}}{6} \\ &= 1 + \frac{33702}{21600} = \frac{55302}{21600} = \frac{9217}{3600} \end{aligned} \quad (84)$$

$$\begin{aligned} y_4 &= \frac{5017}{720} + \frac{1}{2} \left( \frac{8617}{3600} + \frac{9217}{3600} \right) = \frac{5017}{720} + \frac{17834}{7200} \\ &= \frac{50170 + 17834}{7200} = \frac{68004}{7200} = \frac{1889}{200} \end{aligned} \quad (85)$$

$$y_5 = y_4 + \frac{1}{2}(k_{1_5} + k_{2_5}) \quad (86)$$

$$y_4 = \frac{1889}{200} \quad (87)$$

$$k_{1_5} = 1 + \frac{y_4}{x_5} = 1 + \frac{\frac{1889}{200}}{6} = 1 + \frac{1889}{1200} = \frac{3089}{1200} \quad (88)$$

$$\begin{aligned} k_{2_5} &= 1 + \frac{y_4 + k_{1_5}}{x_5 + 1} = 1 + \frac{\frac{1889}{200} + \frac{3089}{1200}}{6 + 1} = 1 + \frac{\frac{11334 + 3089}{1200}}{7} \\ &= 1 + \frac{14423}{8400} = \frac{22823}{8400} \end{aligned} \quad (89)$$

$$\begin{aligned} y_5 &= \frac{1889}{200} + \frac{1}{2} \left( \frac{3089}{1200} + \frac{22823}{8400} \right) = \frac{1889}{200} + \frac{21623 + 22823}{16800} \\ &= \frac{1889}{200} + \frac{44446}{7200} = \frac{68004 + 44446}{7200} = \frac{112450}{7200} = \frac{2249}{144} \end{aligned} \quad (90)$$

$n$	$x_n$	$y_n$
0	1	1
1	2	$\frac{8}{3} \approx 2.67$
2	3	$\frac{169}{36} \approx 4.69$
3	4	$\frac{5017}{720} \approx 6.97$
4	5	$\frac{1889}{200} = 9.445$
5	6	$\frac{2249}{144} \approx 15.62$