Solutions to H.W. #6

1. (a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$,

then $A \cdot B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$.

A and B are both symmetric, but AB is not. The statement is false.

(b) Recall, the inverse of an invertible 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, where $ad - b^2 \neq 0$.

Then
$$A^{-1} = \frac{1}{ad-b^2}\begin{bmatrix} d & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-b^2} & \frac{-b}{ad-b^2} \\ \frac{-b}{ad-b^2} & \frac{a}{ad-b^2} \end{bmatrix},$$

which is symmetric. Moreover,
$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{ad-b^2} \neq 0,$$

So A^{-1} is invertible. The statement is true.

(c) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$,

$(AB)^T = \begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}$

& $A^T \cdot B^T = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 7 & -3 \end{bmatrix}$

Clearly, $(A \cdot B)^T \neq A^T \cdot B^T$.
The statement is false.

2. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$

$\therefore AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & \cdots & (a_{11}b_{1n} + \cdots a_{1n}b_{nn}) \\ 0 & a_{22}b_{22} & \cdots & (a_{22}b_{2n} + \cdots a_{2n}b_{nn}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}b_{nn} \end{bmatrix}$

Therefore, AB is upper triangular.

3. $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Let $\vec{y} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Using forward-substitution

$y_1 = 2$

$2y_1 + y_2 = -1 \quad \therefore y_2 = -1 - 4 = -5$

$-y_1 + y_3 = 1 \quad \therefore y_3 = 1 + 2 = 3$

So, $\vec{y} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$, and

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Using back-substitution:

$3x_3 = 3 \quad \therefore x_3 = 1$

$-2x_2 + x_3 = -5 \quad \therefore x_2 = -\frac{1}{2}(-5 - 1) = 3$

$2x_1 + 3x_2 - x_3 = 2 \quad \therefore x_1 = \frac{1}{2}(2 - 9 + 1) = -3$

$\therefore \vec{x} = [-3 \quad 3 \quad 1]^T$

4. (a) $\vec{x} = [3 \;\; -5 \;\; \sqrt{2}]^T$

$\|\vec{x}\|_1 = \sum_{i=1}^{3} |x_i| = |3| + |-5| + |\sqrt{2}| = 8 + \sqrt{2}$

$\|\vec{x}\|_2 = \left(\sum_{i=1}^{3} x_i^2\right)^{1/2} = \sqrt{9 + 25 + 2} = 6$

$\|\vec{x}\|_\infty = \max_{1 \le i \le 3} |x_i| = \max\{3, 5, \sqrt{2}\} = 5$

(b) $\vec{x} = [e \;\; \pi \;\; 2\sqrt{3}]^T$

$\|\vec{x}\|_1 = e + \pi + 2\sqrt{3}$

$\|\vec{x}\|_2 = \sqrt{e^2 + \pi^2 + 12}$

$\|\vec{x}\|_\infty = 2\sqrt{3}$

(c) $\vec{x} = [-3 \;\; 2 \;\; -4 \;\; 8 \;\; -1]$

$\|\vec{x}\|_1 = 18$, $\|\vec{x}\|_2 = \sqrt{94}$, $\|\vec{x}\|_\infty = 8$

5. a) $A = \begin{bmatrix} 3 & -5 \\ -5 & 4 \end{bmatrix}$

$\|A\|_1 = \max\{8, 9\} = 9$, $\|A\|_\infty = \max\{8, 9\} = 9$

$A^T A = \begin{bmatrix} 3 & -5 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 34 & -35 \\ -35 & 41 \end{bmatrix}$

$\therefore p(\lambda) = \det \begin{bmatrix} 34-\lambda & -35 \\ -35 & 41-\lambda \end{bmatrix} = (34-\lambda)(41-\lambda) - 35^2$

$= \lambda^2 - 75\lambda + 169$

The eigenvalues of $A^T A$ are the roots of $p(\lambda)$:

$p(\lambda) = 0$

$\Rightarrow \lambda = \dfrac{75 \pm \sqrt{(-75)^2 - 4 \cdot 1 \cdot 169}}{2}$

$= \dfrac{75 \pm 7\sqrt{101}}{2}$

$\therefore \rho(A^T A) = \dfrac{75 + 7\sqrt{101}}{2}$, and

$\|A\|_2 = \sqrt{\rho(A^T \cdot A)} = \sqrt{\dfrac{75 + 7\sqrt{101}}{2}} \approx 8.5249$

b) $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

$\|A\|_1 = \max\{4, 5\} = 5$, $\|A\|_\infty = \max\{4, 5\} = 5$

$A^T A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 4 \\ 4 & 17 \end{bmatrix}$

$\therefore p(x) = \det \begin{bmatrix} 16-\lambda & 4 \\ 4 & 17-\lambda \end{bmatrix} = (16-\lambda)(17-\lambda) - 16$

$= \lambda^2 - 33\lambda + 256$

The eigenvalues of $A^T A$ are the roots of $p(x)$:

$$\lambda^2 - 33\lambda + 256 = 0$$

$$\lambda = \frac{33 \pm \sqrt{(-33)^2 - 4 \cdot 1 \cdot 256}}{2}$$

$$= \frac{33 \pm \sqrt{65}}{2}$$

$\therefore \rho(A^T A) = \frac{33 + \sqrt{65}}{2}$, and

$\|A\|_2 = \sqrt{\frac{33 + \sqrt{65}}{2}} \approx 4.5311$