Numerical Methods I
Homework Problem Set #8

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1 Two-Point Backward Difference Approximation

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h} \tag{1}$$
$$f(x) = e^{-x} ln(x+2) \tag{2}$$
$$f'(x) = -e^{-x} ln(x+2) + \frac{e^{-x}}{x+2} \tag{3}$$
$$x = x_0 = 2 \tag{4}$$
$$f(2) = e^{-2} ln(4) = 0.18761 \tag{5}$$
$$f'(2) = -e^{-2} ln(4) + \frac{e^{-2}}{4} = -0.18761 + 0.03383 = -0.15378 \tag{6}$$

Using $h = 0.1$:

$$f'(2) \approx \frac{f(2) - f(1.9)}{0.1} \tag{7}$$
$$f(1.9) = e^{-1.9} ln(3.9) = 0.20356 \tag{8}$$
$$\begin{aligned} f'(2) &\approx \frac{f(2) - f(1.9)}{0.1} \\ &\approx \frac{0.18761 - 0.20356}{0.1} = -0.15945 \end{aligned} \tag{9}$$
$$E_{h=0.1} = |-0.15945 - (-0.15378)| = 0.00567 \tag{10}$$

Using $h = 0.05$

$$f'(2) \approx \frac{f(2) - f(1.95)}{0.05} \tag{11}$$
$$f(1.95) = e^{-1.95} ln(3.95) = 0.19544 \tag{12}$$
$$\begin{aligned} f'(2) &\approx \frac{f(2) - f(1.95)}{0.05} \\ &\approx \frac{0.18761 - 0.19544}{0.05} = -0.15659 \end{aligned} \tag{13}$$
$$E_{h=0.05} = |-0.15659 - (-0.15378)| = 0.00281 \tag{14}$$

Using $h = 0.025$

$$f'(2) \approx \frac{f(2) - f(1.975)}{0.025} \tag{15}$$
$$f(1.975) = e^{-1.975} ln(3.975) = 0.19149 \tag{16}$$
$$\begin{aligned} f'(2) &\approx \frac{f(2) - f(1.975)}{0.025} \\ &\approx \frac{0.18761 - 0.19149}{0.025} = -0.15518 \end{aligned} \tag{17}$$
$$E_{h=0.025} = |-0.15518 - (-0.15378)| = 0.00140 \tag{18}$$

2 Three-Point Central Difference Approximation

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h} \tag{19}$$
$$f(x) = 2\sin(x) - \sqrt{2x+3} \tag{20}$$
$$f'(x) = 2\cos(x) - \frac{1}{\sqrt{2x+3}} \tag{21}$$
$$x = x_0 = 0 \tag{22}$$
$$f'(0) = 2\cos(0) - \frac{1}{\sqrt{2(0)+3}} = 2 - \left(\sqrt{3}\right)^{-1} = 1.42265 \tag{23}$$

Using $h = 0.1$:

$$f'(0) \approx \frac{f(0.1) - f(-0.1)}{0.1} \tag{24}$$

$$f(0.1) = 2\sin(0.1) - \sqrt{2(0.1)+3} = 0.19967 - \sqrt{3.2} = -1.58919 \tag{25}$$

$$\begin{aligned} f(-0.1) &= 2\sin(-0.1) - \sqrt{2(-0.1)+3} \\ &= -0.19967 - \sqrt{2.8} = -1.87299 \end{aligned} \tag{26}$$

$$\begin{aligned} f'(0) &\approx \frac{f(0.1) - f(-0.1)}{0.2} \\ &\approx \frac{-1.58919 - (-1.87299)}{0.2} = 1.41900 \end{aligned} \tag{27}$$

$$E_{h=0.1} = |1.41900 - 1.42265| = 0.00365 \tag{28}$$

Using $h = 0.05$

$$f'(0) \approx \frac{f(0.05) - f(-0.05)}{0.1} \tag{29}$$

$$f(0.05) = 2\sin(0.05) - \sqrt{2(0.05)+3} = 0.09996 - \sqrt{3.1} = -1.66072 \tag{30}$$

$$\begin{aligned} f(-0.05) &= 2\sin(-0.05) - \sqrt{2(-0.05)+3} \\ &= -0.09996 - \sqrt{2.9} = -1.80290 \end{aligned} \tag{31}$$

$$\begin{aligned} f'(0) &\approx \frac{f(0.05) - f(-0.05)}{0.1} \\ &\approx \frac{-1.66072 - (-1.80290)}{0.1} = 1.42174 \end{aligned} \tag{32}$$

$$E_{h=0.05} = |1.42174 - 1.42265| = 0.00091 \tag{33}$$

Using $h = 0.025$

$$f'(0) \approx \frac{f(0.025) - f(-0.025)}{0.05} \tag{34}$$

$$\begin{aligned} f(0.025) &= 2\sin(0.025) - \sqrt{2(0.025)+3} \\ &= 0.04999 - \sqrt{3.05} = -1.69643 \end{aligned} \tag{35}$$

$$\begin{aligned} f(-0.025) &= 2\sin(-0.025) - \sqrt{2(-0.025)+3} \\ &= -0.04999 - \sqrt{2.95} = -1.76755 \end{aligned} \tag{36}$$

$$\begin{aligned} f'(0) &\approx \frac{f(0.025) - f(-0.025)}{0.05} \\ &\approx \frac{-1.69643 - (-1.76755)}{0.05} = 1.42242 \end{aligned} \tag{37}$$

$$E_{h=0.025} = |1.42242 - 1.42265| = 0.00023 \tag{38}$$

3 Estimation Using 2-Point Backward Approximation

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h} \tag{39}$$

$$f(x) = f(T) = a \tag{40}$$

$$h = 10°\text{C} \tag{41}$$

Using $x_0 = 20°\text{C}$:

$$f'(20°\text{C}) \approx \frac{f(20°\text{C}) - f(10°\text{C})}{10°\text{C}} \tag{42}$$

$$f(20°\text{C}) = 1482\text{m/s} \tag{43}$$

$$f(10°\text{C}) = 1447\text{m/s} \tag{44}$$

$$f'(20°\text{C}) \approx \frac{1482\text{m/s} - 1447\text{m/s}}{10°\text{C}} = 3.5°\text{C}^{-1}\text{m/s} \tag{45}$$

Using $x_0 = 40°\text{C}$:

$$f'(40°\text{C}) \approx \frac{f(40°\text{C}) - f(30°\text{C})}{10°\text{C}} \tag{46}$$
$$f(40°\text{C}) = 1529\text{m/s} \tag{47}$$
$$f(30°\text{C}) = 1509\text{m/s} \tag{48}$$
$$f'(40°\text{C}) \approx \frac{1529\text{m/s} - 1509\text{m/s}}{10°\text{C}} = 2°\text{C}^{-1}\text{m/s} \tag{49}$$

Using $x_0 = 60°\text{C}$:

$$f'(60°\text{C}) \approx \frac{f(60°\text{C}) - f(50°\text{C})}{10°\text{C}} \tag{50}$$
$$f(60°\text{C}) = 1511\text{m/s} \tag{51}$$
$$f(50°\text{C}) = 1542\text{m/s} \tag{52}$$
$$f'(60°\text{C}) \approx \frac{1511\text{m/s} - 1542\text{m/s}}{10°\text{C}} = -3.1°\text{C}^{-1}\text{m/s} \tag{53}$$

4 $O(h^4)$ Methods

Since $f'(x) = v$ (velocity), $f''(x) = a$ (acceleration). The appropriate $O(h^4)$ method is the 5-point central difference:

$$f'(x_0) \approx \frac{f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)}{12h} \tag{54}$$
$$f''(x_0) \approx \frac{-f(x_0 - 2h) + 16f(x_0 - h) - 30f(x_0) + 16f(x_0 + h) - f(x_0 + 2h)}{12h^2} \tag{55}$$
$$h = 0.52\text{s} \tag{56}$$

Using $x_0 = 1.04\text{s}$ to discover the velocity:

$$f'(1.04\text{s}) \approx \frac{f(0\text{s}) - 8f(0.52\text{s}) + 8f(1.56\text{s}) - f(2.08\text{s})}{6.24\text{s}} \tag{57}$$
$$\begin{aligned} f'(1.04\text{s}) &\approx \frac{153\text{m} - 8(185\text{m}) + 8(249\text{m}) - 261\text{m}}{6.24\text{s}} \\ &\approx \frac{153\text{m} - 1480\text{m} + 1992\text{m} - 261\text{m}}{6.24\text{s}} = \frac{404\text{m}}{6.24\text{s}} = 64.744\text{m/s} \end{aligned} \tag{58}$$

Using $x_0 = 1.04\text{s}$ to discover the acceleration:

$$f''(1.04\text{s}) \approx \frac{-f(0\text{s}) + 16f(0.52\text{s}) - 30f(1.04\text{s}) + 16f(1.56\text{s}) - f(2.08\text{s})}{3.2448\text{s}^2} \tag{59}$$

$$f''(1.04\text{s}) \approx \frac{-153\text{m} + 16(185\text{m}) - 30(208\text{m}) + 16(249\text{m}) - 261\text{m}}{3.2448\text{s}^2}$$

$$f''(1.04\text{s}) \approx \frac{-153\text{m} + 2960\text{m} - 6240\text{m} + 3984\text{m} - 261\text{m}}{3.2448\text{s}^2}$$

$$\approx \frac{280\text{m}}{3.2448\text{s}^2} = 89.374\text{m/s}^2 \tag{60}$$

Using $x_0 = 1.56\text{s}$ to discover the velocity:

$$f'(1.56\text{s}) \approx \frac{f(0.52\text{s}) - 8f(1.04\text{s}) + 8f(2.08\text{s}) - f(2.60\text{s})}{6.24\text{s}} \tag{61}$$

$$f'(1.56\text{s}) \approx \frac{185\text{m} - 8(208\text{m}) + 8(261\text{m}) - 271\text{m}}{6.24\text{s}}$$

$$\approx \frac{185\text{m} - 1664\text{m} + 2088\text{m} - 271\text{m}}{6.24\text{s}} = \frac{338\text{m}}{6.24\text{s}} = 54.167\text{m/s} \tag{62}$$

Using $x_0 = 1.56\text{s}$ to discover the acceleration:

$$f''(1.56\text{s}) \approx \frac{-f(0.52\text{s}) + 16f(1.04\text{s}) - 30f(1.56\text{s}) + 16f(2.08\text{s}) - f(2.60\text{s})}{3.2448\text{s}^2} \tag{63}$$

$$f''(1.56\text{s}) \approx \frac{-185\text{m} + 16(208\text{m}) - 30(249\text{m}) + 16(261\text{m}) - 271\text{m}}{3.2448\text{s}^2}$$

$$f''(1.56\text{s}) \approx \frac{-185\text{m} + 3328\text{m} - 7470\text{m} + 4176\text{m} - 271\text{m}}{3.2448\text{s}^2}$$

$$\approx \frac{280\text{m}}{3.2448\text{s}^2} = -130.05\text{m/s}^2 \tag{64}$$

5 Extrapolation Technique in $O(h^2)$

$$\begin{aligned}
\text{Exact} &= \text{Approximation} + kh^n && (65)\\
A_{0.05} &= 4.15831 && (66)\\
A_{0.025} &= 4.16361 && (67)\\
\text{Exact} &= A_{0.05} + k(0.05)^2 \\
k &= \frac{Exact - A_{0.05}}{0.0025} && (68)\\
\text{Exact} &= A_{0.025} + k(0.025)^2 \\
&= A_{0.025} + \left(\frac{Exact - A_{0.05}}{0.0025}\right)(0.025)^2 \\
\text{Exact} - 0.25(\text{Exact}) &= A_{0.025} - (0.25)A_{0.05} \\
\text{Exact} &= \frac{A_{0.025} - (0.25)A_{0.05}}{0.75} \\
&= \frac{4.16361 - 1.03958}{0.75} = 4.16538 && (69)
\end{aligned}$$

6 Extrapolation Technique in $O(h^4)$

$$\begin{aligned}
\text{Exact} &= \text{Approximation} + kh^n && (70)\\
A_{0.01} &= -3.2213 && (71)\\
A_{0.005} &= -3.3245 && (72)\\
\text{Exact} &= A_{0.01} + k(0.01)^4 \\
k &= \frac{Exact - A_{0.01}}{(0.01)^4} && (73)\\
\text{Exact} &= A_{0.005} + k(0.005)^4 \\
&= A_{0.005} + \left(\frac{Exact - A_{0.01}}{(0.01)^4}\right)(0.005)^4 \\
\text{Exact} - 0.0625(\text{Exact}) &= A_{0.005} - (0.0625)A_{0.01} \\
\text{Exact} &= \frac{A_{0.005} - (0.0625)A_{0.01}}{0.9375} \\
&= \frac{-3.3245 - (-2.0133)}{0.9375} = -3.3314 && (74)
\end{aligned}$$