Numerical Methods I
Homework Problem Set #7

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10/22/2015

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0.1 Gauss-Seidel Convergence

$$x = Tx + d \tag{1}$$
$$x^{k+1} = \begin{bmatrix} \frac{2}{3} & 0 \\ 1 & -\frac{4}{5} \end{bmatrix} x^k + \begin{bmatrix} 6 \\ -5 \end{bmatrix} \tag{2}$$

A proof that eq. (2) converges for some $x^0 \in \mathbb{R}^2$ is verifying if $||T|| < 1$. Considering the 1-norm of T:

$$||T||_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \tag{3}$$
$$\left|\left|\begin{matrix} \frac{2}{3} & 0 \\ 1 & -\frac{4}{5} \end{matrix}\right|\right|_1 = \max\left(\frac{2}{3} + 1, 0 + \frac{4}{5}\right) = \frac{5}{3} > 1 \tag{4}$$

Since $||T|| > 1$, the sequence may or may not converge.

0.2 Jacobi Method

$$\begin{array}{cccc} 4x_1 & -2x_2 & +x_3 & = 0 \\ -x_1 & +3x_2 & +x_3 & = 0 \\ x_1 & +x_2 & +5x_3 & = -1 \end{array} \tag{5}$$

0.2.1 Question (a)

Writing eq. (5) as $x^{k+1} = Tx^k + C$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} &= - \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{-1}{5} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{-1}{5} \end{bmatrix} \end{aligned} \tag{6}$$

Verifying if $||T||_\infty < 1$:

$$||T||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \tag{7}$$

$$\left\| \begin{matrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{5} & -\frac{1}{5} & 0 \end{matrix} \right\|_\infty = \max \left(\frac{1}{2} + \frac{1}{4}, \frac{1}{3} + \frac{1}{3}, \frac{1}{5} + \frac{1}{5} \right) = \frac{3}{4} < 1 \tag{8}$$

Jacobi method will converge for eq. (5).

0.2.2 Question (b)

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ -\frac{3}{5} \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ -\frac{3}{5} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} + \frac{3}{20} \\ \frac{1}{12} + \frac{1}{5} \\ -\frac{1}{20} - \frac{1}{15} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{19}{60} \\ \frac{37}{60} \\ -\frac{19}{60} \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{19}{60} \\ \frac{37}{60} \\ -\frac{19}{60} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{37}{120} + \frac{19}{240} \\ \frac{19}{180} + \frac{19}{180} \\ -\frac{19}{300} - \frac{37}{300} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{31}{80} \\ \frac{49}{90} \\ -\frac{29}{75} \end{bmatrix} \tag{11}$$

0.2.3 Question (c)

Using MATLAB to discover the value of x:

```
>       A = [4  -2  1  ;  -1  3  1  ;  1  1  5]
>       B = [  0  ;  1  ;  -1]
> linsolve(A,B)
ans =
        0.4
        0.6
        -0.4
```

Using x to discover the error bound:

$$\begin{aligned} ||x - x^{(k)}|| &\leq ||T||^k ||x^{(0)} - x|| \qquad (12) \\ \left|\left|\begin{matrix} \frac{2}{5} - \frac{31}{80} \\ \frac{3}{5} - \frac{49}{90} \\ -\frac{2}{5} + \frac{29}{75} \end{matrix}\right|\right|_\infty &\leq \left(\frac{3}{4}\right)^3 \left|\left|\begin{matrix} 1 - \frac{2}{5} \\ 1 - \frac{3}{5} \\ 1 + \frac{2}{5} \end{matrix}\right|\right|_\infty \\ \left|\left|\begin{matrix} \frac{1}{80} \\ \frac{1}{18} \\ -\frac{1}{75} \end{matrix}\right|\right|_\infty &\leq \frac{27}{64} \left|\left|\begin{matrix} \frac{3}{5} \\ \frac{2}{5} \\ \frac{7}{5} \end{matrix}\right|\right|_\infty \\ \frac{1}{18} &\leq \frac{27}{64} * \frac{7}{5} \\ \frac{1}{18} &\leq \frac{189}{320} \qquad (13) \end{aligned}$$

0.2.4 Question (d)

$$\begin{aligned} 10^{-8} &\leq \left(\frac{3}{5}\right)^k * \frac{7}{5} \\ \lceil k \rceil &= \frac{\log\left(\frac{5*10^{-8}}{7}\right)}{\log\left(\frac{3}{5}\right)} \\ &= \frac{-8\log(2) - 7\log(5) - \log(7)}{\log(3) - \log(5)} = 36.719 \\ k &= 37 \qquad (14) \end{aligned}$$

0.3 Gauss-Seidel number of iterations

$$||x - x^{(k)}|| \leq \frac{||T||^k}{1 - ||T||} ||x^{(1)} - x^{(0)}|| \tag{15}$$

$$||T||_\infty = \max\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{5}\right) = \frac{2}{3} \tag{16}$$

$$x^{(1)} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{3} \end{bmatrix}^T \tag{17}$$

$$||x^{(1)} - x^{(0)}||_\infty = \max\left(\frac{1}{2}, 1, \frac{1}{3}\right) = 1 \tag{18}$$

$$\begin{aligned} 10^{-8} &\leq \frac{\left(\frac{2}{3}\right)^k}{1 - \frac{2}{3}} \\ \lceil k \rceil &= \frac{\log\left(\frac{10^{-8}}{3}\right)}{\log\left(\frac{2}{3}\right)} \\ &= \frac{-8\log(10) - \log(3)}{\log(2) - \log(3)} = 48.719 \\ k &= 49 \end{aligned} \tag{19}$$

0.4 Ill-Condition

$$\begin{bmatrix} 1 & 2 \\ 0.9999 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \tag{20}$$

0.4.1 Question (a)

$$b = \begin{bmatrix}3 & 3.0001\end{bmatrix}^T \tag{21}$$
$$\begin{aligned}x_1 + 2x_2 &= 3 \\ 0.9999x_1 + 2x_2 &= 3.0001 \\ x_1 &= 3 - 2x_2 \\ 0.9999(3 - 2x_2) + 2x_2 &= 3.0001 \\ 2.9997 - 1.9998x_2 + 2x_2 &= 3.0001 \\ 2x_2 - 1.9998x_2 &= 3.0001 - 2.9997\end{aligned}$$
$$x_2 = \frac{0.0004}{0.0002} = 2 \tag{22}$$
$$x_1 = 3 - 2 * 2 = -1 \tag{23}$$
$$x = \begin{bmatrix}-1 & 2\end{bmatrix}^T \tag{24}$$

0.4.2 Question (b)

$$b = \begin{bmatrix}3.1 & 3.0001\end{bmatrix}^T \tag{25}$$
$$\begin{aligned}x_1 + 2x_2 &= 3.1 \\ 0.9999x_1 + 2x_2 &= 3.0001 \\ x_1 &= 3.1 - 2x_2 \\ 0.9999(3.1 - 2x_2) + 2x_2 &= 3.0001 \\ 3.009969 - 1.9998x_2 + 2x_2 &= 3.0001 \\ 2x_2 - 1.9998x_2 &= 3.0001 - 3.009969\end{aligned}$$
$$x_2 = -\frac{0.0009869}{0.0002} = -49.345 \tag{26}$$
$$x_1 = 3.1 - 2 * (-49.345) = 52.445 \tag{27}$$
$$x = \begin{bmatrix}52.445 & -49.345\end{bmatrix}^T \tag{28}$$

0.4.3 Question (c)

Using MATLAB to determine the condition number:

```
>          A = [1  2  ;  0.9999  2]
```

```
>          cond(A)
ans = 4.999e+04
```

Which means for this matrix small errors in the input can get magnified by $4.999 * 10^4$.

0.5 Hilbert Matrix

$$H_3 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \tag{29}$$

$$\tag{30}$$

Using MATLAB to discober H_3^{-1}:

```
>          H = hilb(3)
>          inv(H)
ans =
          9        -36        30
          -36      192        -180
          30       -180       180
```

$$K(H_3) = ||H_3||_1 ||H_3^{-1}||_1 \tag{31}$$

$$||H_3||_1 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \tag{32}$$

$$||H_3^{-1}||_1 = \max(75, 408, 390) = 408 \tag{33}$$

$$K(H_3) = \frac{11 * 408}{6} = \frac{4488}{6} = 748 \tag{34}$$

0.6 Condition Number

$$\begin{bmatrix} 1 & 1 \\ 1 & 1+\frac{1}{n} \end{bmatrix}^{1} = \frac{1}{\frac{1}{n}} \begin{bmatrix} 1+\frac{1}{n} & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} n+1 & -n \\ -n & n \end{bmatrix} \tag{35}$$

$$\left\| \begin{matrix} 1 & 1 \\ 1 & 1+\frac{1}{n} \end{matrix} \right\|_{\infty} = \max\left(2, 2+\frac{1}{|n|}\right) = 2+\frac{1}{|n|} \tag{36}$$

$$\left\| \begin{matrix} n+1 & -n \\ -n & n \end{matrix} \right\|_{\infty} = \max\left(2|n|+1, 2|n|\right) = 2|n|+1 \tag{37}$$

$$cond \begin{pmatrix} 1 & 1 \\ 1 & 1+\frac{1}{n} \end{pmatrix} = \left(2+\frac{1}{|n|}\right)(2|n|+1) = 4|n| + \frac{1}{|n|} + 4 \tag{38}$$

$$\lim_{n\to\infty} 4|n| + \frac{1}{|n|} + 4 = \infty \tag{39}$$