Numerical Methods I
Homework Problem Set #9

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Problem Set #9

1 $\int_1^2 \ln{(x+1)}\ dx$

1.1 Question (a)

$$\int d(uv) = uv = \int u\ dv + \int v\ du \tag{1}$$
$$\int u\ dv = uv - \int v\ du \tag{2}$$
$$u = \ln(x+1) \tag{3}$$
$$du = \frac{1}{x+1}dx = 1 + \frac{1}{x}dx \tag{4}$$
$$dx = (x+1)\ du \tag{5}$$
$$dv = dx \tag{6}$$
$$v = x \tag{7}$$
$$\begin{aligned}\int \ln{(x+1)}\ dx &= x\ln(x+1) - \int \frac{x}{x+1}\ dx \\ &= x\ln(x+1) - \int 1 - \frac{1}{x+1}\ dx \\ &= x\ln(x+1) - x + \ln(x+1) + C \end{aligned} \tag{8}$$
$$\begin{aligned}\int_1^2 \ln{(x+1)}\ dx &= [x\ln(x+1) - x + \ln(x+1)]_1^2 \\ &= (2\ln(3) - 2 + \ln(3)) - (\ln(2) - 1 + \ln(2)) \end{aligned} \tag{9}$$

Solving eq. (9) using GNU Octave to gain precision:

$$\int_1^2 \ln{(x+1)}\ dx = 0.909542504884439 \tag{10}$$

1.2 Question (b)

$$\int_1^2 \ln(x+1)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\left(\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right] \tag{11}$$

$$h = \frac{b-a}{n} = \frac{1}{6} \tag{12}$$

$$\begin{aligned}\int_1^2 \ln(x+1)\,dx &\approx \frac{1}{12}\left[f(1) + 2f\left(1+\frac{1}{6}\right) + 2f\left(1+\frac{2}{6}\right) + \right.\\ &\quad \left. 2f\left(1+\frac{3}{6}\right) + 2f\left(1+\frac{4}{6}\right) + 2f\left(1+\frac{5}{6}\right) + f(2)\right]\\ &\approx \frac{1}{12}\left[f(1) + 2f\left(\frac{7}{6}\right) + 2f\left(\frac{4}{3}\right) + 2f\left(\frac{3}{2}\right) + \right.\\ &\quad \left. 2f\left(\frac{5}{3}\right) + 2f\left(\frac{11}{6}\right) + f(2)\right] \qquad (13)\\ &\approx \frac{1}{12}\left[\ln(1+1) + 2\ln\left(1+\frac{7}{6}\right) + 2\ln\left(1+\frac{4}{3}\right) + \right.\\ &\quad \left. 2\ln\left(1+\frac{3}{2}\right) + 2\ln\left(1+\frac{5}{3}\right) + 2\ln\left(1+\frac{11}{6}\right) + \ln(1+2)\right]\\ &\approx \frac{1}{12}\left[\ln(2) + 2\ln\left(\frac{13}{6}\right) + 2\ln\left(\frac{7}{3}\right) + 2\ln\left(\frac{5}{2}\right) + \right.\\ &\quad \left. 2\ln\left(\frac{8}{3}\right) + 2\ln\left(\frac{17}{6}\right) + \ln(3)\right] \qquad (14)\end{aligned}$$

Solving eq. (14) using GNU Octave to gain precision:

$$\int_1^2 \ln(x+1)\,dx \approx 0.909156890491459 \tag{15}$$

Using eq. (10) to discover the absolute error of eq. (15):

$$\text{Error}_{abs} = |\text{Exact} - \text{Approximation}| \tag{16}$$

$$\begin{aligned}\text{Error}_{abs} &= |0.909542504884439 - 0.909156890491459|\\ &= 0.000385614392980 \qquad (17)\end{aligned}$$

1.3 Question (c)

$$\int_1^2 \ln(x+1)\ dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}) + f(x_n)\right] \tag{18}$$

$$h = \frac{b-a}{n} = \frac{1}{6} \tag{19}$$

$$\begin{aligned}\int_1^2 \ln(x+1)\ dx &\approx \frac{1}{18}\left[f(1) + 4f\left(1+\frac{1}{6}\right) + 2f\left(1+\frac{2}{6}\right) + \right.\\ &\left. 4f\left(1+\frac{3}{6}\right) + 2f\left(1+\frac{4}{6}\right) + 4f\left(1+\frac{5}{6}\right) + f(2)\right]\\ &\approx \frac{1}{18}\left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{4}{3}\right) + 4f\left(\frac{3}{2}\right) + \right.\\ &\left. 2f\left(\frac{5}{3}\right) + 4f\left(\frac{11}{6}\right) + f(2)\right]\end{aligned} \tag{20}$$

$$\begin{aligned}&\approx \frac{1}{18}\left[\ln(1+1) + 4\ln\left(1+\frac{7}{6}\right) + 2\ln\left(1+\frac{4}{3}\right) + \right.\\ &\left. 4\ln\left(1+\frac{3}{2}\right) + 2\ln\left(1+\frac{5}{3}\right) + 4\ln\left(1+\frac{11}{6}\right) + \ln(1+2)\right]\\ &\approx \frac{1}{18}\left[\ln(2) + 4\ln\left(\frac{13}{6}\right) + 2\ln\left(\frac{7}{3}\right) + 4\ln\left(\frac{5}{2}\right) + \right.\\ &\left. 2\ln\left(\frac{8}{3}\right) + 4\ln\left(\frac{17}{6}\right) + \ln(3)\right]\end{aligned} \tag{21}$$

Solving eq. (21) using GNU Octave to gain precision:

$$\int_1^2 \ln(x+1)\ dx \approx 0.909541759764950 \tag{22}$$

Using eq. (10) to discover the absolute error of eq. (22):

$$\text{Error}_{abs} = |\text{Exact} - \text{Approximation}| \tag{23}$$

$$\begin{aligned}\text{Error}_{abs} &= |0.909542504884439 - 0.909541759764950|\\ &= 0.000000745119489\end{aligned} \tag{24}$$

1.4 Question (d)

$$\int_1^2 \ln(x+1)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i=1}^{n/3} f(x_{3i-2}) + 3\sum_{i=1}^{(n/3)-1} f(x_{3i-1}) + \right.$$
$$\left. 2\sum_{i=1}^{(n/2)-2} f(x_{3i}) + f(x_n)\right] \tag{25}$$

$$h = \frac{b-a}{n} = \frac{1}{6} \tag{26}$$

$$\int_1^2 \ln(x+1)\,dx \approx \frac{3}{48}\left[f(1) + 3f\left(1+\frac{1}{6}\right) + 3f\left(1+\frac{2}{6}\right) + \right.$$
$$\left. 2f\left(1+\frac{3}{6}\right) + 3f\left(1+\frac{4}{6}\right) + 3f\left(1+\frac{5}{6}\right) + f(2)\right]$$
$$\approx \frac{1}{16}\left[f(1) + 3f\left(\frac{7}{6}\right) + 3f\left(\frac{4}{3}\right) + 2f\left(\frac{3}{2}\right) + \right.$$
$$\left. 3f\left(\frac{5}{3}\right) + 3f\left(\frac{11}{6}\right) + f(2)\right] \tag{27}$$
$$\approx \frac{1}{16}\left[\ln(1+1) + 3\ln\left(1+\frac{7}{6}\right) + 3\ln\left(1+\frac{4}{3}\right) + \right.$$
$$\left. 2\ln\left(1+\frac{3}{2}\right) + 3\ln\left(1+\frac{5}{3}\right) + 3\ln\left(1+\frac{11}{6}\right) + \ln(1+2)\right]$$
$$\approx \frac{1}{16}\left[\ln(2) + 3\ln\left(\frac{13}{6}\right) + 3\ln\left(\frac{7}{3}\right) + 2\ln\left(\frac{5}{2}\right) + \right.$$
$$\left. 3\ln\left(\frac{8}{3}\right) + 3\ln\left(\frac{17}{6}\right) + \ln(3)\right] \tag{28}$$

Solving eq. (28) using GNU Octave to gain precision:

$$\int_1^2 \ln(x+1)\,dx \approx 0.909540847647380 \tag{29}$$

Using eq. (10) to discover the absolute error of eq. (29):

$$\begin{aligned} \text{Error}_{abs} &= |\text{Exact} - \text{Approximation}| \qquad (30) \\ \text{Error}_{abs} &= |0.909542504884439 - 0.909541759764950| \\ &= 0.000001657237059 \qquad (31) \end{aligned}$$

2 Simpson's 1/3 Method with $n = 8$

$$\int_0^1 e^{-x^2}\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}) + f(x_n)\right] \qquad (32)$$

$$h = \frac{b-a}{n} = \frac{1}{8} \qquad (33)$$

$$\begin{aligned} \int_0^1 e^{-x^2}\,dx &\approx \frac{1}{24}\left[f(0) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{1}{4}\right) + 4f\left(\frac{3}{8}\right) + 2f\left(\frac{1}{2}\right) + \right. \\ &\qquad \left. 4f\left(\frac{5}{8}\right) + 2f\left(\frac{3}{4}\right) + 4f\left(\frac{7}{8}\right) + f(1)\right] \qquad (34) \\ &\approx \frac{1}{24}\left[e^{-(0)^2} + 4e^{-\left(\frac{1}{8}\right)^2} + 2e^{-\left(\frac{1}{4}\right)^2} + 4e^{-\left(\frac{3}{8}\right)^2} + 2e^{-\left(\frac{1}{2}\right)^2} + \right. \\ &\qquad \left. 4e^{-\left(\frac{5}{8}\right)^2} + 2e^{-\left(\frac{3}{4}\right)^2} + 4e^{-\left(\frac{7}{8}\right)^2} + e^{-(1)^2}\right] \\ &\approx \frac{1}{24}\left[e^{-1} + 4e^{-\left(\frac{1}{64}\right)} + 2e^{-\left(\frac{1}{16}\right)} + 4e^{-\left(\frac{9}{64}\right)} + 2e^{-\left(\frac{1}{4}\right)} + \right. \\ &\qquad \left. 4e^{-\left(\frac{25}{64}\right)} + 2e^{-\left(\frac{9}{16}\right)} + 4e^{-\left(\frac{49}{64}\right)} + e^{-1}\right] \\ &\approx \frac{1}{24}\left[2e^{-1} + 4e^{-\left(\frac{1}{64}\right)} + 2e^{-\left(\frac{1}{16}\right)} + 4e^{-\left(\frac{9}{64}\right)} + 2e^{-\left(\frac{1}{4}\right)} + \right. \\ &\qquad \left. 4e^{-\left(\frac{25}{64}\right)} + 2e^{-\left(\frac{9}{16}\right)} + 4e^{-\left(\frac{49}{64}\right)}\right] \qquad (35) \end{aligned}$$

Solving each term eq. (35) using GNU Octave with output format as *short g*:

$$\begin{aligned}\int_0^1 e^{-x^2}\ dx &\approx \frac{1}{24}\left[2(0.36788) + 4(0.98450) + 2(0.93941) + 4(0.86882) + \right.\\ &\quad \left. 2(0.77880) + 4(0.67663) + 2(0.56978) + 4(0.46504)\right]\\ &\approx \frac{1}{24}\left[0.73576 + 3.93800 + 1.87882 + 3.47528 + 1.55760 + \right.\\ &\quad \left. 2.70652 + 1.13956 + 1.86016\right]\\ &\approx \frac{1}{24}(17.29170)\\ &\approx 0.72049 \end{aligned} \tag{36}$$

3 Simpson's 1/3 Method

The distance s can be determined as:

$$s = \int_{t_0}^{t_n} v(t)\ dt \tag{37}$$

Using Simpson's 1/3 Method to approximate eq. (37):

$$\int_{t_0}^{t_n} v(t)\ dt \approx \frac{h}{3}\left[f(v_0) + 4\sum_{i=1}^{t_n/2} f(v_{2i-1}) + 2\sum_{i=1}^{(t_n/2)-1} f(v_{2i}) + f(v_n)\right] \tag{38}$$

$$h = \frac{b-a}{n} = \frac{10\text{ min}}{10} = 1\text{ min} \tag{39}$$

$$\begin{aligned}\int_{t_0}^{t_n} v(t)\ dt &\approx \frac{1\text{ min}}{2}\left[v(0) + 4v(1) + 2v(2) + 4v(3) + 2v(4) + 4v(5) + \right.\\ &\quad \left. 2v(6) + 4v(7) + 2v(8) + 4v(9) + v(10)\right] \qquad (40)\\ &\approx \frac{1\text{ min}}{2}\left[0\text{ cm/min} + 4(2.3\text{ cm/min}) + 2(5.6\text{ cm/min}) + \right.\\ &\quad 4(2.4\text{ cm/min}) + 2(6.5\text{ cm/min}) + 4(3.5\text{ cm/min}) + \\ &\quad 2(4.0\text{ cm/min}) + 4(8.0\text{ cm/min}) + 2(6.1\text{ cm/min}) + \\ &\quad \left. 4(2.3\text{ cm/min}) + 1.9\text{ cm/min}\right]\end{aligned}$$

$$\begin{aligned}
&\approx \frac{1\text{ min}}{2}\,[9.2\text{ cm/min} + 11.2\text{ cm/min} + 9.6\text{ cm/min}+ \\
&\quad 13.0\text{ cm/min} + 14.0\text{ cm/min} + 8.0\text{ cm/min} + 32.0\text{ cm/min}+ \\
&\quad 12.2\text{ cm/min} + 9.2\text{ cm/min} + 1.9\text{ cm/min}] \\
&\approx \frac{1\text{ min}}{2}(120.3\text{ cm/min}) \\
&\approx 60.15\text{ cm} \qquad (41)
\end{aligned}$$

4 Number of Subintervals

$$\begin{aligned}
&\int_0^5 f(x)\,dx && \text{tol} = 10^{-6} \\
&\max_{0\le x\le 5} |f'(x)| = 4 && \max_{0\le x\le 5} |f''(x)| = 8 \\
&\max_{0\le x\le 5} |f'''(x)| = 12 && \max_{0\le x\le 5} |f^{(4)}(x)| = 18
\end{aligned}$$

4.1 Question (a)

$$\begin{aligned}
\frac{h^2}{12}(b-a)f''(c) &\le \text{tol} && (42) \\
\frac{h^2}{12}(5)(8) &\le 10^{-6} \\
h^2 &\le \frac{12 * 10^{-6}}{40} \\
h &\le \sqrt{3 * 10^{-7}} \\
h &\le 0.00054772 && (43) \\
\frac{b-a}{n} &= h && (44) \\
n &= \frac{b-a}{h} && (45) \\
n &\ge \frac{5}{0.00054772} \\
n &\ge 9128.75192 \\
\lceil n \rceil &= 9129 && (46)
\end{aligned}$$

4.2 Question (b)

$$\frac{h^4}{180}(b-a)f^{(4)}(c) \leq \text{tol} \tag{47}$$
$$\frac{h^4}{180}(5)(18) \leq 10^{-6}$$
$$h^4 \leq \frac{180 * 10^{-6}}{90}$$
$$h \leq \sqrt[4]{2 * 10^{-6}}$$
$$h \leq 0.03760603 \tag{48}$$
$$\frac{b-a}{n} = h \tag{49}$$
$$n = \frac{b-a}{h} \tag{50}$$
$$n \geq \frac{5}{0.03760603}$$
$$n \geq 132.95740$$
$$\lceil n \rceil = 133 \tag{51}$$

5 Number of Subintervals by Trapezoidal rule

$$\int_0^2 \sqrt[3]{x+1}\ dx \tag{52}$$
$$\text{tol} = 10^{-6} \tag{53}$$
$$f(x) = \sqrt[3]{x+1} \tag{54}$$
$$f'(x) = \frac{1}{3\sqrt[3]{x+1}} \tag{55}$$
$$f''(x) = -\frac{1}{9(x+1)^{4/3}} \tag{56}$$
$$\max_{0\leq x\leq 2} |f''(x)| = |f''(0)| = \left| -\frac{1}{9(0+1)^{4/3}} \right| = \frac{1}{9} \tag{57}$$

$$
\begin{aligned}
\frac{h^2}{12}(b-a)f''(c) &\leq \text{tol} && (58)\\
\frac{h^2}{12}(2)\left(\frac{1}{9}\right) &\leq 10^{-6}\\
h^2 &\leq \frac{108 * 10^{-6}}{2}\\
h &\leq \sqrt{5.9 * 10^{-5}}\\
h &\leq 0.00768115 && (59)\\
\frac{b-a}{n} &= h && (60)\\
n &= \frac{b-a}{h} && (61)\\
n &\geq \frac{2}{0.00768115}\\
n &\geq 260.37768\\
\lceil n \rceil &= 261 && (62)
\end{aligned}
$$

6 Bonus - MATLAB

6.1 Question (a)

$$f(x) = \sin(\pi x^2)e^{-x} \tag{63}$$

To discover the approximation of the volume of the solid formed when the region bounded by eq. (63) over the interval $0 \leq x \leq 1$ is revolved about the x-axis using the composite trapezoidal method is necessary to:

$$V(f(x)) = \int_a^b A(f(x))\ dx \tag{64}$$

$$A(f(x)) = \pi(f(x))^2 \tag{65}$$

$$A = \pi(\sin(\pi x^2)e^{-x})^2 \tag{66}$$

$$V = \int_0^1 \pi(\sin(\pi x^2)e^{-x})^2\ dx \tag{67}$$

$$g(x) = (\sin(\pi x^2)e^{-x})^2 \tag{68}$$

$$V \approx \pi \left\{ \frac{h}{2} \left[g(x_0) + 2 \left(\sum_{i=1}^{n-1} g(x_i) \right) + g(x_n) \right] \right\} \tag{69}$$

Analyzing the inputs to be used in MATLAB to approximate V:

$$f = g(x) = (\sin(\pi x^2)e^{-x})^2 \qquad a = 0$$

$$b = 1 \qquad n = 24$$

Using the Trapezoidal Rule function described in the book inside of the script described below:

```
run rules/trapezoid_rule

x = 0;
question_a = '(sin(pi*(x**2))*(e**(-x)))**2';
f = inline (question_a,x);
a = 0;
b = 1;
n = 24;

approximation = Trap(f, a, b, n)
volume_approximated = pi * approximation

save outputs/hw9_6a_output approximation volume_approximated;
```

The output values are:

- **approximation**: 0.1052698143206541
- **volume_approximated**: 0.3307148753145285

Code maintained at:

- Trapezoidal Rule: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/rules/trapezoid_rule.m`
- Script: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/hw9_6a.m`
- Output: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/outputs/hw9_6a_output`

6.2 Question (b)

$$y = e^x \tag{70}$$

To discover the approximation of the surface area of the solid formed when the region bounded by eq. (70) over the interval $0 \leq x \leq 1$ is revolved about the x-axis using the composite Simpson's 1/3 method is necessary to:

$$S = \int_a^b 2\pi y \ ds \tag{71}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx \tag{72}$$

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx \tag{73}$$

$$= 2\pi \int_0^1 e^x\sqrt{1 + e^{2x}} \ dx \tag{74}$$

$$g(x) = e^x\sqrt{1 + e^{2x}} \tag{75}$$

$$S \approx 2\pi \left\{ \frac{h}{3} \left[g(x_0) + 4\sum_{i=1}^{n/2} g(x_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} g(x_{2i}) + g(x_n) \right] \right\} \tag{76}$$

Analyzing the inputs to be used in MATLAB to approximate S:

$$f = g(x) = e^x\sqrt{1+e^{2x}} \qquad a = 0$$
$$b = 1 \qquad n = 24$$

Using the Simpson's Rule function described in the book inside of the script described below:

```
run rules/simpson_rule

x = 0;
question_b = '(e**x)*sqrt(1+(e**(2*x)))';
f = inline (question_b,x);
a = 0;
b = 1;
n = 24;

approximation = Simp(f, a, b, n)
surface_area_approximated = 2*pi*approximation

save outputs/hw9_6b_output approximation surface_area_approximated;
```

The output values are:

- **approximation**: 3.65149627230973
- **surface_area_approximated**: 22.94302772739752

Code maintained at:

- Simpson's Rule: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/rules/simpson_rule.m`
- Script: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/hw9_6b.m`
- Output: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/outputs/hw9_6b_output`

6.3 Question (c)

$$x = 2\cos(t) \qquad\qquad y = \sin(t) \tag{77}$$

To discover the approximation of the arc length of the curve defined parametrically by x and y over the interval $0 \leq t \leq 2\pi$ using the composite Simpson's 1/3 method is necessary to:

$$\text{Arc}_{length} = \int_a^b ds \tag{78}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \tag{79}$$

$$\text{Arc}_{length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \tag{80}$$

$$= \int_0^{2\pi} \sqrt{(-2\sin(t))^2 + (\cos(t))^2}\ dt \tag{81}$$

$$f(t) = \sqrt{(-2\sin(t))^2 + (\cos(t))^2} \tag{82}$$

$$\text{Arc}_{length} \approx \frac{h}{3}\left[f(t_0) + 4\sum_{i=1}^{n/2} f(t_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} f(t_{2i}) + f(t_n)\right] \tag{83}$$

Analyzing the inputs to be used in MATLAB to approximate Arc_{length}:

$$f = f(t) == \sqrt{(-2\sin(t))^2 + (\cos(t))^2} \qquad a = 0$$
$$b = 2\pi \qquad n = 24$$

Using the Simpson's Rule function described in the book inside of the script described below:

```
run rules/simpson_rule

t = 0;
question_c = 'sqrt((((-2)*(sin(t)))**2)+((cos(t))**2))';
f = inline (question_c, t);
a = 0;
b = 2*pi;
```

```
n = 24;

arc_length_approximated = Simp(f, a, b, n)

save outputs/hw9_6c_output arc_length_approximated;
```

The output values are:

- **arc_length_approximated**: 6.283185307179586

Code maintained at:

- Simpson's Rule: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/rules/simpson_rule.m`
- Script: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/hw9_6c.m`
- Output: `https://github.com/arkye-7th-semester-indiana-tech/NumericalMethods/Homeworks/MATLAB_Homeworks/HW9/outputs/hw9_6c_output`