Solutions to H.W. #5

1. Let $A = \begin{bmatrix} 8 & 5 & 1 \\ 3 & 7 & 4 \\ 2 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$\therefore u_{11} = 8, \quad u_{12} = 5, \quad u_{13} = 1$

$l_{21} u_{11} = 3$
$\therefore l_{21} = \frac{3}{8}$

$l_{21} u_{12} + u_{22} = 7$
$\frac{15}{8} + u_{22} = 7$
$u_{22} = 41/8$

$l_{21} \cdot u_{13} + u_{23} = 4$
$\frac{3}{8} \cdot 1 + u_{23} = 4$
$u_{23} = \frac{29}{8}$

$l_{31} u_{11} = 2$
$\therefore l_{31} = \frac{1}{4}$

$l_{31} u_{12} + l_{32} \cdot u_{22} = 3$
$\frac{5}{4} + l_{32} \cdot \frac{41}{8} = 3$
$\frac{41}{8} l_{32} = \frac{7}{4}$
$\therefore l_{32} = \frac{14}{41}$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 9$
$\frac{1}{4} + \frac{203}{164} + u_{33} = 9$
$u_{33} = \frac{308}{41}$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 3/8 & 1 & 0 \\ 1/4 & 14/41 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 & 5 & 1 \\ 0 & 41/8 & 29/8 \\ 0 & 0 & 308/41 \end{bmatrix}$$

Let $\hat{Y} = \begin{bmatrix} 8 & 5 & 1 \\ 0 & 41/8 & 29/8 \\ 0 & 0 & 308/41 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\therefore A\vec{x} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3/8 & 1 & 0 \\ 1/4 & 14/41 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$

By forward-substitution:

$y_1 = 6$

$\frac{3}{8} \cdot 6 + y_2 = 1 \quad \therefore y_2 = -5/4$

$\frac{1}{4} \cdot 6 + \frac{14}{41} \cdot \left(-\frac{5}{4}\right) = -2 \quad \therefore y_3 = \frac{-126}{41}$

$$\Rightarrow \begin{bmatrix} 8 & 5 & 1 \\ 0 & 41/8 & 29/8 \\ 0 & 0 & 308/41 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5/4 \\ -126/41 \end{bmatrix}$$

Using back-substitution

$$x_3 = \frac{-126}{41} \div \frac{308}{41} = -9/22$$

$$\frac{41}{8}\cdot x_2 + \frac{29}{8}\cdot(-9/22) = -\frac{5}{4}, \quad \therefore x_2 = \frac{1}{22}$$

$$8x_1 + 5\cdot(1/22) + 1\cdot(-9/22) = 6, \quad \therefore x_1 = 17/22$$

So, $\vec{x} = \begin{bmatrix} 17/22 \\ 1/22 \\ -9/22 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore l_{11} = 1$, $\quad l_{11}\cdot u_{12} = -1$, $\quad l_{11}u_{13} = 0$

$u_{12} = -1 \qquad u_{13} = 0$

$l_{21} = -2$, $\quad l_{21}u_{12} + l_{22} = 4$, $\quad l_{21}u_{13} + l_{22}u_{23} = -2$

$2 + l_{22} = 4 \qquad 0 + 2u_{23} = -2$

$l_{22} = 2 \qquad u_{23} = -1$

$l_{31} = 0$, $\quad l_{31}\cdot u_{12} + l_{32} = -1$, $\quad l_{31}u_{13} + l_{32}\cdot u_{23} + l_{33} = 2$

$0 + l_{32} = -1 \qquad 0 + 1 + l_{33} = 2$

$l_{33} = 1$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\hat{y} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore y_1 = 0$

$-2y_1 + 2y_2 = -1$

$y_2 = -1/2$

$-y_2 + y_3 = 4$

$y_3 = 7/2$

& $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 7/2 \end{bmatrix}$

$\therefore x_3 = 7/2$

$x_2 - x_3 = -1/2$, $\therefore x_2 = 3$

$x_1 - x_2 = 0$ $\therefore x_1 = 3$

$$\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 7/2 \end{bmatrix}$$

3. Let $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

$\therefore\ l_{11}^2 = 2,\quad l_{11} = \sqrt{2}$ $\qquad l_{11} l_{21} = -1$ $\qquad l_{11} \cdot l_{31} = 0$

$l_{21} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ $\qquad \therefore\ l_{31} = 0$

$l_{21}^2 + l_{22}^2 = 2$

$\frac{1}{2} + l_{22}^2 = 2$

$l_{22}^2 = \frac{3}{2}$

$l_{22} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

$l_{21} \cdot l_{31} + l_{22} \cdot l_{32} = -1$

$0 + \frac{\sqrt{6}}{2} \cdot l_{32} = -1$

$l_{32} = -\frac{2}{\sqrt{6}} = -\frac{\sqrt{6}}{3}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2$

$0 + \frac{2}{3} + l_{33}^2 = 2$

$l_{33}^2 = \frac{4}{3}$

$\therefore\ l_{33} = \frac{2\sqrt{3}}{3}$

$\therefore\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2}/2 & \sqrt{6}/2 & 0 \\ 0 & -\sqrt{6}/3 & 2\sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2}/2 & 0 \\ 0 & \sqrt{6}/2 & -\sqrt{6}/3 \\ 0 & 0 & 2\sqrt{3}/3 \end{bmatrix}$

4. Since A is not symmetric, A does not have a Cholesky factorization.

5. (a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$

Since $|2| > |-1| + |0|$, $|4| > |-1| + |2|$, and $|6| > |0| + |2|$, A is SDD.

Using the determinants of all leading submatrices:

$$|2| = 2, \quad \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = 8 - 1 = 7, \quad \begin{vmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix}$$

$$= 2(24-4) + 1 \cdot (-6)$$

$$= 40 - 6 = 34$$

Since all determinants are positive, A is SPD.

(b) $B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & -1 \\ -3 & 2 & 6 \end{bmatrix}$

Since $|1| \not> |2| + |0|$, B is not SDD.

Consider $|1| = 1$, $\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 6 - 8 = -2 < 0$.

Therefore B is not SPD. So, B is neither SDD nor SPD. Note also that $B^T \neq B$.

(c) $C = \begin{bmatrix} 5 & -3 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Since $|5| \not> |-3| + |2|$, C is not SDD.

Consider $|5| = 5$, $\begin{vmatrix} 5 & -3 \\ -3 & 1 \end{vmatrix} = 5 - 9 = -4 < 0$.

So, C is neither SDD nor SPD.

6. Let $A = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix}$

(a) A is strictly diagonally dominant if, and only if,

$|6| > |-2| + |a|$ and $|7| > |2| + |a|$

$|a| < 4$

$\therefore -4 < a < 4$

$|a| < 7 - 2$

$|a| < 5$

$\therefore -5 < a < 5$

Hence A is diagonally dominant whenever $-4 < a < 4$.

(b) Consider

$$|5| = 5, \quad \begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix} = 30 - 4 = 26,$$

$$\begin{vmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{vmatrix} = 5 \cdot \begin{vmatrix} 6 & a \\ a & 7 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & a \\ 2 & 7 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & 6 \\ 2 & a \end{vmatrix}$$

$$= 5\left(42 - a^2\right) + 2\left(-14 - 2a\right) + 2\left(-2a - 12\right)$$

$$= 210 - 5a^2 - 28 - 4a - 4a - 24$$

$$= -5a^2 - 8a + 158$$

We require $-5a^2 - 8a + 158 > 0$

$$5a^2 + 8a - 158 < 0$$

The roots of the quadratic are

$$a = \frac{-8 \pm \sqrt{25 - 4 \cdot (5) \cdot (-158)}}{2 \cdot (5)}$$

$$= \frac{-4 \pm \sqrt{806}}{5}$$

So, when $\frac{-4 - \sqrt{806}}{5} < a < \frac{-4 + \sqrt{806}}{5}$,

all determinants will be positive and the matrix is symmetric positive definite.