# Solutions to H.W. #3

1. (a) $g(x) = \frac{1}{2} e^{x/2}$, $[4,5]$

Since $g(4) \notin [4,5]$, there is no guarantee for the existence of a fixed-point.

(b) $g(x) = \frac{1}{5}\cos(x)$, $[0, \pi/2]$

$g(0) \in [0, \pi/2]$, $g(\pi/2) \in [0, \pi/2]$

Since $g'(x) = -\frac{1}{5}\sin(x) \leq 0 \quad \forall\, x \in [0, \pi/2]$

$g(x)$ is monotone decreasing, and therefore

$g(x) \in [0, \pi/2] \quad \forall\, x \in [0, \pi/2]$.

So, there is at least one fixed-point in $[0, \pi/2]$.

2. $g(x) = \frac{1}{2} e^{0.5x}$, $[0,1]$

$g(0) \in [0,1]$ and $g(1) \in [0,1]$.

Since $g'(x) = \frac{1}{4} e^{0.5x} > 0 \quad \forall\, x \in [0,1]$, $g(x)$ is monotone increasing and therefore $g(x) \in [0,1]$ $\forall\, x \in [0,1]$. So, $g$ has at least one fixed-point in $[0,1]$.

Moreover, $g'(x)$ is an increasing function, so its maximum occurs at $x=1$. That is

$$\max_{[0,1]} |g'(x)| = g'(1) = \frac{1}{4}e^{0.5} < 1$$

So, the fixed-point must be unique!

3. $f(x) = (x-1)^2 \ln(x)$

Since $f(1) = (1-1)^2 \cdot \ln(1) = 0$, $x=1$ is a root of $f(x)$.

$$f'(x) = 2(x-1)\cdot\ln(x) + \frac{(x-1)^2}{x}$$

$$= 2(x-1)\ln(x) + x - 2 + \frac{1}{x}$$

$$f'(1) = 2\cdot 0\cdot \ln(1) + 1 - 2 + 1 = 0$$

$$f''(x) = 2\ln(x) + \frac{2(x-1)}{x} + 1 - \frac{1}{x^2}$$

$$= 2\ln(x) + 2 - \frac{2}{x} + 1 - \frac{1}{x^2}$$

$$= 2\ln(x) - \frac{2}{x} - \frac{1}{x^2} + 3$$

$$f''(1) = 2\cdot\ln(1) - 2 - 1 + 3 = 0$$

$$f'''(x) = \frac{2}{x} + \frac{2}{x^2} + \frac{2}{x^3}$$

$f'''(1) = 2+2+2 = 6 \neq 0$

$\therefore$ The multiplicity of $x=1$ is 3.

4. $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$

Since $f(1) = 1-1-3+5-2 = 0$, $x=1$ is a root of $f(x)$.

$f'(x) = 4x^3 - 3x^2 - 6x + 5$, $f'(1) = 4-3-6+5 = 0$

$f''(x) = 12x^2 - 6x - 6$, $f''(1) = 12-6-6 = 0$

$f'''(x) = 24x - 6$, $f'''(1) = 24-6 = 18 \neq 0$

$\therefore$ $x=1$ is a root of $f(x)$ of multiplicity 3.

(a) Using $x_{n+1} = x_n - \dfrac{x_n^4 - x_n^3 - 3x_n^2 + 5x_n - 2}{4x_n^3 - 3x_n^2 - 6x_n + 5}$,

$x_0 = 0.5$

$x_1 = 0.67857143$

$x_2 = 0.79017859$

$x_3 = 0.86191709$

The sequence is slowly converging to $x=1$!

(b) Using $x_{n+1} = x_n - 3\left(\dfrac{x_n^4 - x_n^3 - 3x_n^2 + 5x_n - 2}{4x_n^3 - 3x_n^2 - 6x_n + 5}\right)$

$x_0 = 0.5$

$x_1 = 1.0357143$

$x_2 = 1.0001369$

$x_3 = 0.9999973658$

The sequence is converging <u>quickly</u> to $x=1$.

5. Using $e_{n+1} \approx \beta e_n^2 = 0.5 e_n^2$, since $e_0 = 0.25$,

$e_1 \approx 0.5 \cdot (0.25)^2 = 0.03125$

$e_2 \approx 0.5(0.03125)^2 = 0.00048828125$

$e_3 \approx 0.5(0.00048828125)^2 = 1.19209 \times 10^{-7}$

6. Since $|x - x_n| \leq \dfrac{b-a}{2^n}$,

$|x - x_{10}| \leq \dfrac{4-1}{2^{10}} = 0.0029296875$

7. (a) We require the least $n$ such that

$$\frac{-2-(-3)}{2^n} \leq 10^{-5}$$

$$\frac{1}{2^n} \leq 10^{-5}$$

$$2^n \geq 10^5$$

$$n \geq \frac{\ln(10^5)}{\ln(2)} \simeq 16.60964$$

Take $n=17$.

(b) Similarly,

$$\frac{1}{2^n} \leq 10^{-8}$$

$$\frac{\quad}{\ln(2)} = 26.57542$$

Take $n=27$.