Solutions to H.W. #2

1. With $f(x) = \ln(x+1) - \cos(x)$,

$$x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}:$$

x_0	x_1	x_2
0	1	0.8674
0.8674	1	0.8843
0.8843	1	0.8845

2.

x_0	x_1	x_2
0	1	0.8674
1	0.8674	0.8843
0.8674	0.8843	0.8845

3. (a) $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

$$\therefore\ x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

Using $x_0 = 1$,

$x_1 = 0.75$

$x_2 = 0.68604651$

$x_3 = 0.6823396$

$x_4 = 0.68232779$

$x_5 = 0.68232780$

(b) $f(x) = x^5 - 3x + 3$

$f'(x) = 5x^4 - 3$

$$\therefore \quad x_{n+1} = x_n - \frac{x_n^5 - 3x_n + 3}{5x_n^4 - 3}$$

Using $x_0 = -1$,

$x_1 = -3.5$

$x_2 = -2.8152547$

$x_3 = -2.2835674$

$x_4 = -1.8906346$

$x_5 = -1.6363066$

$x_6 = -1.5199490$

$x_7 = -1.4966243$

$x_8 = -1.4957725$

$x_9 = -1.4957713$

4. (a) For $f(x) = x^2 - a$,

$$f'(x) = 2x$$

$$\therefore \quad x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + a}{2x_n}$$

$$= \frac{x_n^2 + a}{2x_n}$$

$$= \frac{1}{2}\left(\frac{x_n^2}{x_n} + \frac{a}{x_n}\right)$$

$$= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) $x_0 = 3$

$x_1 = 3.1666667$

$x_2 = 3.1622807$

$x_3 = 3.1622777$

Absolute error $= \left|\sqrt{10} - 3.1622777\right| = 4 \times 10^{-8}$

Percentage error $= \dfrac{\left|\sqrt{10} - 3.1622777\right|}{\sqrt{10}} \cdot 100$

$= 1.2 \times 10^{-6}\ \%$

5. (a) To find the fixed-points of $f(x) = x\ln(x+1)$, solve:

$$x = x\ln(x+1)$$

$$x - x\ln(x+1) = 0$$

$$x\left(1 - \ln(x+1)\right) = 0$$

$$\therefore x = 0 \quad \text{or} \quad 1 - \ln(x+1) = 0$$

$$-\ln(x+1) = -1$$

$$\ln(x+1) = 1$$

$$x+1 = e^1$$

$$x = e-1$$

So $x = 0$ and $x = e-1$ are fixed-points of $f(x)$.

(b) To find the fixed-points of $f(x) = \sqrt{x+1}$, solve:

$$x = \sqrt{x+1}$$

$$x^2 = x+1$$

$$\therefore x^2 - x - 1 = 0$$

Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1+4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

Now, $\frac{1-\sqrt{5}}{2}$ is not in the domain of $f(x)$.

So, the only fixed-point is $x = \frac{1+\sqrt{5}}{2}$.

6. $x^3 - 3x - 20 = 0$

$$-3x = 20 - x^3$$

$$x = \frac{20 - x^3}{-3}$$

$$x = \frac{x^3 - 20}{3}$$

Using $x_{n+1} = \frac{x_n^3 - 20}{3}$, with $x_0 = 3.5$

produces

$$x_1 = 7.625$$

$$x_2 = 141.1074218751$$

$$x_3 = 936537.615109,$$

$$\vdots$$

a divergent sequence!

$$x^3 - 3x - 20 = 0$$

$$x^3 - 3x = 20$$

$$x(x^2 - 3) = 20$$

$$x = \frac{20}{x^2 - 3}$$

Using $x_{n+1} = \frac{20}{x_n^2 - 3}$ with $x_0 = 3.5$ produces:

$x_1 = 2.1621622$
$x_2 = 11.940688$
$x_3 = 0.14328697$
$x_4 = -6.712606$
$x_5 = 0.47552158$
$x_6 = -7.2101194$
$x_7 = 0.4082814$,

an apparently bounded, but oscillatory sequence. Hence it is not convergent!

$$x^3 - 3x - 20 = 0$$
$$x^3 = 3x + 20$$
$$x = (3x+20)^{1/3}$$

Using $x_{n+1} = (3x_n + 20)^{1/3}$ with $x_0 = 3.5$

produces:

$x_1 = 3.1243999$
$x_2 = 3.0854399$
$x_3 = 3.081342$
$x_4 = 3.0809103$
$x_5 = 3.0808648$
$x_6 = 3.08086$
$x_7 = 3.0808595$,

which appears to be a rapidly converging sequence!

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>> %Question 7
>> %The bisection method:
>> bisection
    step        a         b        m          ym      bound

    1.0000    2.0000    3.0000    2.5000   -3.1771    0.5000

    2.0000    2.0000    2.5000    2.2500    0.8511    0.2500

    3.0000    2.2500    2.5000    2.3750   -1.1265    0.1250

    4.0000    2.2500    2.3750    2.3125   -0.1284    0.0625

    5.0000    2.2500    2.3125    2.2812    0.3637    0.0312

    6.0000    2.2812    2.3125    2.2969    0.1183    0.0156

    7.0000    2.2969    2.3125    2.3047   -0.0049    0.0078

    8.0000    2.2969    2.3047    2.3008    0.0567    0.0039

    9.0000    2.3008    2.3047    2.3027    0.0259    0.0020

   10.0000    2.3027    2.3047    2.3037    0.0105    0.0010

   11.0000    2.3037    2.3047    2.3042    0.0028    0.0005

   12.0000    2.3042    2.3047    2.3044   -0.0010    0.0002

   13.0000    2.3042    2.3044    2.3043    0.0009    0.0001

   14.0000    2.3043    2.3044    2.3044   -0.0001    0.0001

   15.0000    2.3043    2.3044    2.3044    0.0004    0.0000

   16.0000    2.3044    2.3044    2.3044    0.0002    0.0000

   17.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   18.0000    2.3044    2.3044    2.3044   -0.0000    0.0000

   19.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   20.0000    2.3044    2.3044    2.3044   -0.0000    0.0000

   21.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   22.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   23.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   24.0000    2.3044    2.3044    2.3044   -0.0000    0.0000

   25.0000    2.3044    2.3044    2.3044    0.0000    0.0000

   26.0000    2.3044    2.3044    2.3044   -0.0000    0.0000

   27.0000    2.3044    2.3044    2.3044    0.0000    0.0000
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   28.0000    2.3044    2.3044    2.3044   -0.0000    0.0000

The bisection method has converged in 28 iterations.
The approximate solution is 2.304377477616.
The function value at the approximation is -5.324986673827e-08.
>> %The false position method:
>> falsi(f,2,3,10^-8,100)
 step            a        b           s        y

    1.0000    2.0000    3.0000    2.2753    0.4572

    2.0000    2.2753    3.0000    2.3018    0.0408

    3.0000    2.3018    3.0000    2.3041    0.0036

    4.0000    2.3041    3.0000    2.3044    0.0003

    5.0000    2.3044    3.0000    2.3044    0.0000

    6.0000    2.3044    3.0000    2.3044    0.0000

    7.0000    2.3044    3.0000    2.3044    0.0000

    8.0000    2.3044    3.0000    2.3044    0.0000

    9.0000    2.3044    3.0000    2.3044    0.0000

The regula falsi method has converged in 9 iterations.
The approximate solution is 2.304377474135.
The function value at the approximation is 1.684714590056e-09.
>> %The secant method:
>> secant(f,2,3,10^-8,100)
      step     x(k-1)      x(k)      x(k+1)   y(k+1)    Dx(k+1)

    1.0000    2.0000    3.0000    2.2753    0.4572   -0.7247

    2.0000    3.0000    2.2753    2.3018    0.0408    0.0265

    3.0000    2.2753    2.3018    2.3044   -0.0002    0.0026

    4.0000    2.3018    2.3044    2.3044    0.0000   -0.0000

    5.0000    2.3044    2.3044    2.3044    0.0000    0.0000

The secant method has converged in 5 iterations.
The approximate solution is 2.304377474242.
The function value at the approximation is 1.243449787580e-13.
>> %Newton's method:
>> Newton(f,inline('(89*x^2)/100 - (89*x)/10'),2.5,10^-8,100)
Newtons method has converged in 5 iterations.
The approximate solution is 2.304377474242.
The function value at the approximation is 3.552713678801e-15.
 step            x           y
    1.0000    2.5000   -3.1771
    2.0000    2.3096   -0.0827
    3.0000    2.3044   -0.0001
    4.0000    2.3044   -0.0000
    5.0000    2.3044    0.0000
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>> %The secant method and Newton's method converged most rapidly, followed by
>> %the false position method, and then the bisection method.
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