Numerical Methods I
Homework Problem Set #3

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Problem Set #3

0.1 Fixed-Points in an Interval

Note: How can I check if $g(x)$ is an element of $[a,b]$ for all x in $[a,b]$? Checking only a and b do not guarantee for all x, only the bounds.

0.1.1 Question (a)

$$g(x) = \frac{1}{2}e^{\frac{x}{2}}, [4,5] \tag{1}$$

First, it is necessary to check if $g(x)$ is an element of $[a,b]$ for all x in $[a,b]$. Starting with $x = a = 4$:

$$g(4) = \frac{1}{2}e^{\frac{4}{2}} \qquad ; \qquad g(4) = \frac{1}{2}e^{2} \qquad ; \qquad g(4) = 3.6945 \tag{2}$$

Since 3.6945 is not an element of $[a,b]$, it is not possible to guarantee that the eq. (1) have at least one fixed-point on the indicated interval.

0.1.2 Question (b)

$$g(x) = \frac{1}{5}\cos(x), \left[0, \frac{\pi}{2}\right] \tag{3}$$

First, it is necessary to check if $g(x)$ is an element of $[a,b]$ for all x in $[a,b]$. Starting with $x = a = 0$:

$$g(0) = \frac{1}{5}\cos(0) \qquad ; \qquad g(0) = 0.2 \tag{4}$$

Since 0 is an element of $[a,b]$, the next step is to check with $x=b=\frac{\pi}{2}$:

$$g\left(\frac{\pi}{2}\right)=\frac{1}{5}\cos\left(\frac{\pi}{2}\right) \quad ; \quad g\left(\frac{\pi}{2}\right)=1.2246*10^{-17} \tag{5}$$

Since $1.2246*10^{-17}$ an element of $[a,b]$, it is correct to assume the eq. (3) can have at least one fixed-point on the indicated interval if it is monotone. Derivating eq. (3):

$$\begin{aligned} g(x) &= \frac{1}{5}\cos(x) \\ g'(x) &= -\frac{1}{5}\sin(x) \end{aligned} \tag{6}$$

Looking at the eq. (6), it is correct to affirm that the eq. (3) is not monotone. It is necessary to confirm if the eq. (6) is not monotone with x in the range of $[a,b]$. Starting with $x=a=0$:

$$g'(0)=-\frac{1}{5}\sin(0) \quad ; \quad g'(0)=0 \tag{7}$$

And with $x=b=\frac{\pi}{2}$:

$$g'\left(\frac{\pi}{2}\right)=-\frac{1}{5}\sin\left(\frac{\pi}{2}\right) \quad ; \quad g'\left(\frac{\pi}{2}\right)=-0.2 \tag{8}$$

Since $g'(x)$ with x in $[a,b]$ is nonincreasing, the eq. (3) is monotone. With this argument, it is correct to affirm that the eq. (3) have at least one fixed-point.

0.2 Unique Fixed-Point in an Interval

$$g(x)=\frac{1}{2}e^{0.5x},[0,1] \tag{9}$$

First, it is necessary to check if $g(x)$ is an element of $[a,b]$ for all x in $[a,b]$. Starting with $x=a=0$:

$$g(0) = \frac{1}{2}e^{0.5*0} \quad ; \quad g(0) = \frac{1}{2}e^{0} \quad ; \quad g(0) = \frac{1}{2} \quad ; \quad g(0) = 0.5 \tag{10}$$

Since 0.5 is an element of $[a, b]$, the next step is to check with $x = b = 1$:

$$g(1) = \frac{1}{2}e^{0.5*1} \quad ; \quad g(1) = \frac{1}{2}e^{0.5} \quad ; \quad g(1) = 0.5 * 1.6487 \quad ; \quad g(1) = 0.8244 \tag{11}$$

Since 0.8244 an element of $[a, b]$, it is correct to assume the eq. (9) can have at least one fixed-point on the indicated interval if it is monotone. Derivating eq. (9):

$$\begin{aligned} g(x) &= \frac{1}{2}e^{0.5x} \\ g'(x) &= \frac{1}{2} * e^{0.5x} * \frac{1}{2} \\ g'(x) &= \frac{1}{4} * e^{0.5x} \end{aligned} \tag{12}$$

Looking at the eq. (12), it is correct to affirm that the eq. (9) is monotone. The eq. (9) have at least one fixed-point.

To prove the uniqueness it is necessary to guarantee that $|g'(x)| < 1$ for all x in $[a, b]$. Starting with $x = a = 0$:

$$g'(0) = \frac{1}{4} * e^{0.5*0} \quad ; \quad g'(0) = 0.25 \tag{13}$$

Since $0.25 < 1$, the next step is to check $g'(x)$ with $x = b = 1$:

$$g'(1) = \frac{1}{4} * e^{0.5*1} \quad ; \quad g'(1) = 0.4122 \tag{14}$$

Since $0.4122 < 1$, and 0.4122 is the maximum value in $[a, b]$, we can affirm that eq. (9) have an unique fixed-point.

0.3 Multiplicity of a Root

$$
\begin{aligned}
f(x) &= (x-1)^2 \ln(x) \\
f(x) &= (x^2 - 2x + 1) \ln(x) \\
f(x) &= x^2 \ln(x) - 2x \ln(x) + \ln(x)
\end{aligned} \tag{15}
$$

Verifying if $x = 1$ is a root of eq. (15):

$$
f(1) = (1-1)^2 \ln(1) \qquad ; \qquad f(1) = 0 * \ln(1) = 0 \tag{16}
$$

$x = 1$ is a root of eq. (15), as can be seen in eq. (16). To discover the multiplicity, it is necessary to discover how many derivates of eq. (15) have $x = 1$ as root:

$$
\begin{aligned}
f(x) &= x^2 \ln(x) - 2x \ln(x) + \ln(x) \\
f'(x) &= 2x * \ln(x) + \frac{x^2}{x} - 2 * \left(\ln(x) + \frac{x}{x}\right) + \frac{1}{x} \\
f'(x) &= 2x * \ln(x) - 2\ln(x) + x + \frac{1}{x} - 2
\end{aligned} \tag{17}
$$

Verifying if $x = 1$ is a root of eq. (17):

$$
\begin{aligned}
f'(1) &= 2 * 1 * \ln(1) - 2 * \ln(1) + 1 + \frac{1}{1} - 2 \\
f'(1) &= 0 - 0 + 1 + 1 - 2 = 0
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
f'(x) &= 2x * \ln(x) - 2\ln(x) + x + \frac{1}{x} - 2 \\
f''(x) &= 2 * \left(\ln(x) + \frac{x}{x}\right) - \frac{2}{x} + 1 - \frac{1}{x^2} \\
f''(x) &= 2 * \ln(x) - \frac{1}{x^2} - \frac{2}{x} + 3
\end{aligned} \tag{19}
$$

Verifying if $x = 1$ is a root of eq. (19):

$$\begin{aligned} f''(1) &= 2 * \ln(1) - \frac{1}{1^2} - \frac{2}{1} + 3 \\ f''(1) &= 0 - 1 - 2 + 3 = 0 \end{aligned} \tag{20}$$

$$\begin{aligned} f''(x) &= 2 * \ln(x) - \frac{1}{x^2} - \frac{2}{x} + 3 \\ f'''(x) &= \frac{1}{x^3} + \frac{2}{x^2} + \frac{2}{x} \end{aligned} \tag{21}$$

Verifying if $x = 1$ is a root of eq. (21):

$$\begin{aligned} f'''(1) &= \frac{1}{1^3} + \frac{2}{1^2} + \frac{2}{1} \\ f'''(1) &= 1 + 2 + 2 = 5 \neq 0 \end{aligned} \tag{22}$$

The multiplicity of $x = \alpha = 1$ in eq. (15) is 3.

0.4 Root, Multiplicity, and Newton's Method

$$f(x) = x^4 - x^3 - 3x^2 + 5x - 2 \tag{23}$$

Verifying if $x = 1$ is a root of eq. (23):

$$f(1) = 1^4 - 1^3 - 3 * 1^2 + 5 * 1 - 2 \quad ; \quad f(1) = 1 - 1 - 3 + 5 - 2 = 0 \tag{24}$$

$x = 1$ is a root of eq. (23), as can be seen in eq. (24). To discover the multiplicity, it is necessary to discover how many derivates of eq. (23) have $x = 1$ as root:

$$\begin{aligned} f(x) &= x^4 - x^3 - 3x^2 + 5x - 2 \\ f'(x) &= 4x^3 - 3x^2 - 6x + 5 \end{aligned} \tag{25}$$

Verifying if $x = 1$ is a root of eq. (25):

$$\begin{aligned} f'(1) &= 4 * 1^3 - 3 * 1^2 - 6 * 1 + 5 \\ f'(1) &= 4 - 3 - 6 + 5 = 9 - 9 = 0 \end{aligned} \tag{26}$$

$$\begin{aligned} f'(x) &= 4x^3 - 3x^2 - 6x + 5 \\ f''(x) &= 12x^2 - 6x - 6 \end{aligned} \tag{27}$$

Verifying if $x = 1$ is a root of eq. (27):

$$\begin{aligned} f''(1) &= 12 * 1^2 - 6 * 1 - 6 \\ f''(1) &= 12 - 6 - 6 = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} f''(x) &= 12x^2 - 6x - 6 \\ f'''(x) &= 24x - 6 \end{aligned} \tag{29}$$

Verifying if $x = 1$ is a root of eq. (29):

$$\begin{aligned} f'''(1) &= 24x - 6 \\ f'''(1) &= 24 * 1 - 6 = 18 \neq 0 \end{aligned} \tag{30}$$

The multiplicity of $x = \alpha = 1$ in eq. (23) is 3.

0.4.1 Question (a): Newton's Method

$$x_{k+1} = x_k - \frac{y_k}{y'_k} \tag{31}$$

Using $x_0 = 0.5$ and eq. (23) as y to discover x_1, x_2, and x_3 in eq. (31):

$$\begin{aligned} x_1 &= 0.5 - \frac{0.5^4 - 0.5^3 - 3*0.5^2 + 5*0.5 - 2}{4*0.5^3 - 3*0.5^2 - 6*0.5 + 5} \\ &= 0.5 + \frac{0.3125}{1.75} \\ &= 0.6786 \end{aligned} \tag{32}$$

$$\begin{aligned} x_2 &= 0.6786 - \frac{0.6786^4 - 0.6786^3 - 3*0.6786^2 + 5*0.6786 - 2}{4*0.6786^3 - 3*0.6786^2 - 6*0.6786 + 5} \\ &= 0.6786 - \frac{0.0890}{0.7970} \\ &= 0.7902 \end{aligned} \tag{33}$$

$$\begin{aligned} x_3 &= 0.7902 - \frac{0.7902^4 - 0.7902^3 - 3*0.7902^2 + 5*0.7902 - 2}{4*0.7902^3 - 3*0.7902^2 - 6*0.7902 + 5} \\ &= 0.7902 + \frac{0.0258}{0.3593} \\ &= 0.8619 \end{aligned} \tag{34}$$

0.4.2 Question (a): 1^{st} Modification of Newton's Method

$$x_{k+1} = x_k - \frac{m * y_k}{y'_k} \tag{35}$$

Using $x_0 = 0.5$, $m = 3$, and eq. (23) as y to discover x_1, x_2, and x_3 in eq. (31):

$$\begin{aligned} x_1 &= 0.5 - 3*\frac{0.5^4 - 0.5^3 - 3*0.5^2 + 5*0.5 - 2}{4*0.5^3 - 3*0.5^2 - 6*0.5 + 5} \\ &= 0.5 + 3*\frac{0.3125}{1.75} \\ &= 0.5 + \frac{0.9375}{1.75} \\ &= 1.0357 \end{aligned} \tag{36}$$

$$
\begin{aligned}
x_2 &= 1.0357 - 3 * \frac{1.0357^4 - 1.0357^3 - 3 * 1.0357^2 + 5 * 1.0357 - 2}{4 * 1.0357^3 - 3 * 1.0357^2 - 6 * 1.0357 + 5} \\
&= 1.0357 - \frac{0.0004}{0.0117} \\
&= 1.0001
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
x_3 &= 1.0001 - \frac{1.0001^4 - 1.0001^3 - 3 * 1.0001^2 + 5 * 1.0001 - 2}{4 * 1.0001^3 - 3 * 1.0001^2 - 6 * 1.0001 + 5} \\
&= 1.0001 + \frac{2.4436 * 10^{-11}}{1.7518 * 10^{-7}} \\
&= 1
\end{aligned}
\tag{38}
$$

0.5 Error Bound

$$
\lim_{x \to \infty} \frac{|e_{n+1}|}{|e_n|^R} = \beta \tag{39}
$$

Discovering e_1, e_2, and e_3 using $R = 2$, $\beta = 0.5$, and $e_0 = 0.25$ in the eq. (39):

$$
\begin{aligned}
|e_1| &\approx \beta * |e_0|^R \\
|e_1| &\approx 0.5 * |0.25|^2 \\
|e_1| &\approx 0.5 * 0.6250 \\
|e_1| &\approx 0.0312
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
|e_2| &\approx \beta * |e_1|^R \\
|e_2| &\approx 0.5 * |0.0312|^2 \\
|e_2| &\approx 0.5 * 0.0977 \\
|e_2| &\approx 0.0488
\end{aligned}
\tag{41}
$$

$$|e_3| \approx \beta * |e_2|^R$$
$$|e_3| \approx 0.5 * |0.0488|^2$$
$$|e_3| \approx 0.5 * 0.0024$$
$$|e_3| \approx 0.0012 \tag{42}$$

0.6 Bisection Error Bound

$$|x_n - a| = |E| \leq \frac{b-a}{2^n} \tag{43}$$

Discovering the bisection error bound after 10 iterations with starting interval as $[1, 4]$.

$$E_{x_{10}} \leq \left| \frac{4-1}{2^{10}} \right|$$
$$E_{x_{10}} \leq \frac{3}{1024}$$
$$E_{x_{10}} \leq 0.0029 \tag{44}$$

0.7 Bisection Mininum Number of Iterations

Using eq. (43), with a interval of $[-3, -2]$, for a tolerance of:

0.7.1 Question (a): 10^{-5}

$$
\begin{aligned}
10^{-5} &\leq \left|\frac{-2+3}{2^n}\right| \\
10^{-5} &\leq \frac{1}{2^n} \\
2^n &\leq \frac{1}{10^{-5}} \\
\ln(2^n) &\leq \ln\left(10^5\right) \\
n * \ln(2) &\leq 5 * \ln(10) \\
n &\geq \frac{5 * \ln(10)}{\ln(2)} \\
n &\geq \lceil 16.610 \rceil \\
n &\geq 17
\end{aligned}
\tag{45}
$$

0.7.2 Question (b): 10^{-8}

$$
\begin{aligned}
10^{-8} &\leq \left|\frac{-2+3}{2^n}\right| \\
10^{-8} &\leq \frac{1}{2^n} \\
2^n &\leq \frac{1}{10^{-8}} \\
\ln(2^n) &\leq \ln\left(10^8\right) \\
n * \ln(2) &\leq 8 * \ln(10) \\
n &\geq \frac{8 * \ln(10)}{\ln(2)} \\
n &\geq \lceil 26.575 \rceil \\
n &\geq 27
\end{aligned}
\tag{46}
$$