Numerical Methods I
Homework Problem Set #2

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Problem Set #2

0.1 Regula Falsi (False Position)

Regula Falsi:

$$x = b - y_b \frac{b-a}{y_b - y_a} \tag{1}$$

Equation:

$$ln(x+1) - cos(x) = 0 \tag{2}$$

Using $a = 0$, $b = 1$, and eq. (2) as y to discover x_1 in eq. (1):

$$\begin{aligned}
x_1 &= 1 - (ln(1+1) - cos(1)) * \frac{1-0}{(ln(1+1) - cos(1)) - (ln(0+1) - cos(0))} \\
&= 1 - (0.6931 - 0.5403) * \frac{1}{(0.6931 - 0.5403) - (0 - 1)} \\
&= 1 - 0.1528 * \frac{1}{0.1528 - (-1)} \\
&= 1 - 0.1528 * \frac{1}{1.1528} \\
&= 1 - 0.1326 \\
&= 0.8674 \qquad (3)
\end{aligned}$$

The two next iterations (for x_2 and x_3) was made using GNU Octave and their results are described in table 1

x_n	a	b	x	y
1	0	1	0.8674	-0.0223
2	0.8674	1	0.8843	-0.0003
3	0.8843	1	0.8845	-0.0000

Table 1: Results of 3 iterations of the eq. (1) to approximate the root of eq. (2)

0.2 Secant Method

Secant Method:

$$x_{k+1} = x_k - y_k \frac{x_k - x_{k-1}}{y_k - y_{k-1}} \tag{4}$$

The equation is eq. (2), the same of the previous question. Using $x_0 = 0$, $x_1 = 1$, and eq. (2) as y to discover x_2 in eq. (4):

$$\begin{aligned}
x_2 &= 1 - (ln(1+1) - cos(1)) * \frac{1-0}{(ln(1+1) - cos(1)) - (ln(0+1) - cos(0))} \\
&= 1 - (0.6931 - 0.5403) * \frac{1}{(0.6931 - 0.5403) - (0-1)} \\
&= 1 - 0.1528 * \frac{1}{0.1528 - (-1)} \\
&= 1 - 0.1528 * \frac{1}{1.1528} \\
&= 1 - 0.1326 \\
&= 0.8674
\end{aligned} \tag{5}$$

Who has the same result as eq. (3) The two next iterations (for x_3 and x_4) was made using GNU Octave and their results are the same as described in table 1, as can be seen in table 2.

k	x_{k-1}	x_k	x_{k+1}	$y(x_{k+1})$
1	0	1	0.8674	-0.0223
2	0.8674	1	0.8843	-0.0003
3	0.8843	1	0.8845	-0.0000

Table 2: Results of 3 iterations of the eq. (4) to approximate the root of eq. (2)

0.3 Newton‘s Method

Newton‘s Method:

$$x_{k+1} = x_k - \frac{y_k}{y'_k} \tag{6}$$

0.3.1 Question (a)

Equations:

$$y(x) = x^3 + x - 1 \tag{7}$$

$$y'(x) = 3x^2 + 1 \tag{8}$$

Discovering the value of eq. (7) as $x = 0$ and $x = 1$ in order to discover the closest integer initial approximation.

$$y(0) = 0^3 + 0 - 1 = -1 \tag{9}$$

$$y(1) = 1^3 + 1 - 1 = 1 \tag{10}$$

As can be seen in eq. (9) and eq. (10), both values have the same approximation for the root. Using $x_0 = 1$ and eq. (7) as y to discover x_1 in eq. (6):

$$
\begin{aligned}
x_1 &= 1 - \frac{1^3 + 1 - 1}{3 * 1^2 + 1} \\
&= 1 - \frac{1}{4} \\
&= 1 - 0.25 \\
&= 0.75
\end{aligned} \tag{11}
$$

The next iterations was made using *short g* format on GNU Octave and their results are described in table 3. The option *short g* format uses 5 significant figures in 10 maximum characters.

k	x_k	x_{k+1}	$y(x_{k+1})$
0	1	0.75	0.17188
1	0.75	0.68605	0.008941
2	0.68605	0.68234	2.8231e-05
3	0.68234	0.68233	2.8399e-10
4	0.68233	0.68233	2.2204e-15

Table 3: Results of 5 iterations of the eq. (6) to approximate the root of eq. (7)

0.3.2 Question (b)

Equations:

$$y(x) = x^5 - 3x + 3 \tag{12}$$

$$y'(x) = 5x^4 - 3 \tag{13}$$

Discovering the value of eq. (12) as $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$ in order to discover the closest integer initial approximation.

$$y(-1) = (-2)^5 - 3 * (-2) + 3 = -23 \tag{14}$$

$$y(-1) = (-1)^5 - 3 * (-1) + 3 = 5 \tag{15}$$

$$y(0) = 0^5 - 3 * 0 + 3 = 3 \tag{16}$$

$$y(1) = 1^5 - 3 * 1 + 3 = 1 \tag{17}$$

$$y(2) = 2^5 - 3 * 2 + 3 = 29 \tag{18}$$

As can be seen, eq. (17) is the closest integer approximation of the root of eq. (12). Using $x_0 = 1$ and eq. (12) as y to discover x_1 in eq. (6):

$$\begin{aligned} x_1 &= 1 - \frac{1^5 - 3 * 1 + 3}{5 * (1^4) - 3} \\ &= 1 - \frac{1}{2} \\ &= 1 - 0.5 \\ &= 0.5 \end{aligned} \tag{19}$$

The next iterations was made using *short g* format on GNU Octave and their 4 initial results and 2 last results are described in table 4. The option *short g* format uses 5 significant figures in 10 maximum characters.

k	x_k	x_{k+1}	$y(x_{k+1})$
0	1	0.5	1.5312
1	0.5	1.0698	1.1917
2	1.0698	0.73391	1.0112
3	0.73391	1.3865	3.9648
...	...	...	...
135	-1.4963	-1.4958	-1.082e-05
136	-1.4958	-1.4958	-8.0735e-12

Table 4: 137 iterations was needed to approximate to the root of eq. (12) with a diference inferior than 10^{-6} using Newton‘s Method

0.4 Mechanic‘s Rule

Mechanic‘s Rule:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),\; n = 0, 1, 2, ... \tag{20}$$

NOTE: The Question (a) was skipped because of dificults on understanding what to do with the constant value a when applying Newton‘s Method.

0.4.1 Question (b)

Using $x_0 = 3$ to approximate $\sqrt{a}$ with $a = 10$ in 3 iterations using eq. (20):

$$\begin{aligned} x_1 &= \frac{1}{2}\left(3 + \frac{10}{3}\right) \\ &= \frac{3}{2} + \frac{10}{6} \\ &= 1.5 + 1.6667 \\ &= 3.1667 \end{aligned} \tag{21}$$

$$\begin{aligned} x_2 &= \frac{1}{2}\left(3.1667 + \frac{10}{3.1667}\right) \\ &= \frac{3.1667}{2} + \frac{10}{6.3334} \\ &= 1.5834 + 1.5789 \\ &= 3.1623 \end{aligned} \tag{22}$$

$$\begin{aligned} x_2 &= \frac{1}{2}\left(3.1623 + \frac{10}{3.1623}\right) \\ &= \frac{3.1623}{2} + \frac{10}{6.3246} \\ &= 1.5812 + 1.5811 \\ &= 3.1623 \end{aligned} \tag{23}$$

Using GNU Octave and asking the value of $\sqrt{10}$ using the format *long g*, the result is: 3.16227766016838. With 5 significative digits the absolute and percentage error is 0. But, considering all the digits returned by GNU Octave, the absolute error and percentage error is:

$$\begin{aligned} error_{absolute} &= |3.16227766016838 - 3.1623| \\ &= |-0.00022339831520| \\ &= 0.00022339831520 \end{aligned} \tag{24}$$

$$\begin{aligned} error_{percentage} &= \frac{|3.16227766016838 - 3.1623|}{|3.16227766016838|} \\ &= \frac{0.00022339831520}{3.16227766016838} \\ &= 0.00000706447504 \end{aligned} \tag{25}$$

0.5 Fixed-Points

Fixed Point (p) is when the below function is satisfied:

$$f(p) = p \tag{26}$$

0.5.1 Question (a)

Equation:

$$f(x) = xln(x+1) \tag{27}$$

By simple deduction we discover the only fixed point of eq. (27):

$$\begin{aligned} f(0) &= 0 * ln(0+1) \\ &= 0 * 0 = 0 \\ p &= 0 \end{aligned} \tag{28}$$

0.5.2 Question (b)

Equation:

$$f(x) = \sqrt{x+1} \tag{29}$$

Testing eq. (29) with $x = 1$ and $x = 2$:

$$f(1) = \sqrt{1+1} = \sqrt{2} > 1 \tag{30}$$

$$f(2) = \sqrt{2+1} = \sqrt{3} < 2 \tag{31}$$

By deduction, we can affirm the eq. (29) do not has any fixed point. After multiple attempts to identify a more closely approximation of a possible p using GNU Octave, the value of $x = 1.61803398874989$ is a approximated fixed point of eq. (29) by 14 significant digits.

0.6 Handling Equations

Main Equation:

$$x^3 - 3x - 20 = 0 \tag{32}$$

Showing the algebraic steps needed to write eq. (32) in the following forms:

$$\begin{aligned} x^3 - 3x - 20 &= 0 \\ 3x &= x^3 - 20 \\ x &= \frac{x^3 - 20}{3} \\ g_1(x) &= \frac{x^3 - 20}{3} \end{aligned} \tag{33}$$

$$
\begin{aligned}
x^3 - 3x - 20 &= 0 \\
x^3 - 3x &= 20 \\
x(x^2 - 3) &= 20 \\
x &= \frac{20}{x^2 - 3} \\
g_2(x) &= \frac{20}{x^2 - 3} \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
x^3 - 3x - 20 &= 0 \\
x^3 &= 3x + 20 \\
x &= (3x + 20)^{1/3} \\
g_3(x) &= (3x + 20)^{1/3} \qquad (35)
\end{aligned}
$$

Starting with eq. (33) in $x_{n+1} = g(x_n)$ with $x_0 = 3.5$:

$$
\begin{aligned}
x_1 &= \frac{(3.5)^3 - 20}{3} = 7.6250 \\
x_2 &= \frac{(7.625)^3 - 20}{3} = 141.1074 \qquad (36)
\end{aligned}
$$

As can be seen in eq. (36), the function will diverge. Switching to eq. (34) in $x_{n+1} = g(x_n)$ with $x_0 = 3.5$:

$$\begin{aligned}
x_1 &= \frac{20}{(3.5)^2 - 3} = 2.1621 \\
x_2 &= \frac{20}{(2.1621)^2 - 3} = 11.9407 \\
x_3 &= \frac{20}{(11.9407)^2 - 3} = 0.1433 \\
x_4 &= \frac{20}{(0.1433)^2 - 3} = -6.7126 \\
x_5 &= \frac{20}{(-6.7126)^2 - 3} = 0.4755 \\
x_6 &= \frac{20}{(0.4755)^2 - 3} = -7.2101 \\
x_7 &= \frac{20}{(-7.2101)^2 - 3} = 0.4082
\end{aligned} \tag{37}$$

It seems that eq. (37) will never converge. Switching to eq. (35) in $x_{n+1} = g(x_n)$ with $x_0 = 3.5$:

$$\begin{aligned}
x_1 &= (3 * (3.5) + 20)^{1/3} = 3.1244 \\
x_2 &= (3 * (3.1244) + 20)^{1/3} = 3.0854 \\
x_3 &= (3 * (3.0854) + 20)^{1/3} = 3.0813 \\
x_4 &= (3 * (3.0813) + 20)^{1/3} = 3.0809 \\
x_5 &= (3 * (3.0809) + 20)^{1/3} = 3.0809 \\
x_6 &= (3 * (3.0809) + 20)^{1/3} = 3.0809 \\
x_7 &= (3 * (3.0809) + 20)^{1/3} = 3.0809
\end{aligned} \tag{38}$$

eq. (38) is converging to 3.0809.

0.7 MATLAB Practice

Equation:

$$\frac{\rho_f}{3}h^3 - R\rho_f h^2 + \frac{4}{3}R^3\rho_0 = 0 \tag{39}$$

Using $R = 5$ cm, $\rho_0 = 0.120$ g/cm^3, $\rho_f = 0.890$ g/cm^3, and a tolerance of 10^{-8}.

$$\frac{0.890}{3} * h^3 - 4.45 * h^2 + 20 = 0$$

Looking at the graph of the function eq. (40) using the Bisection script cited below, the two nearest integers which surrounds the root are: $[14, 15]$.

0.7.1 Question (a)

Using Bisection:

- Script At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/bisection.m`
- Call for Function At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/bisection_hw2.m`
- Number of Iterations: 29
- Final Result: 14.6874883808

0.7.2 Question (b)

Regula Falsi:

- Script At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/regula_falsi.m`
- Call for Function At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/regula_falsi_hw2.m`
- Number of Iterations: 8
- Final Result: 14.6874883787

0.7.3 Question (c)

Secant:

- Script At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/secant.m`
- Call for Function At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/secant_hw2.m`
- Number of Iterations: 5
- Final Result: 14.6874883777

0.7.4 Question (d)

Newton's:

- Script At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/newton.m`
- Call for Function At: `https://github.com/arkye/NumericalMethods/tree/master/Homework_Scripts/HW2/newton_hw2.m`
- Number of Iterations (Using Initial Value 14): 6
- Final Result (Using Initial Value 14): 14.6874883788
- Number of Iterations (Using Initial Value 15): 5
- Final Result (Using Initial Value 15): 14.6874883788