Numerical Methods I
Homework Problem Set #10

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Problem Set #10

1 Picard Iterate

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(t, y_n(t))\ dt \tag{1}$$

1.1 Question (a)

$$y' = x^2 - 2y^2 + 1 \qquad\qquad y(0) = 0 \qquad\qquad y_2(x) \tag{2}$$

$$\begin{aligned}
y_{n+1}(x) &= \int_0^x t^2 - 2y_n(t)^2 + 1\ dt && (3)\\
&= \int_0^x t^2\ dt - \int_0^x 2y_n(t)^2\ dt + \int_0^x 1\ dt\\
&= \left[\frac{t^3}{3}\right]_0^x - \int_0^x 2y_n(t)^2\ dt + [t]_0^x\\
&= \frac{x^3}{3} - \int_0^x 2y_n(t)^2\ dt + x && (4)\\
y_1(x) &= \frac{x^3}{3} - \int_0^x 2y_0(t)^2\ dt + x && (5)\\
&= \frac{x^3}{3} + x && (6)
\end{aligned}$$

$$
\begin{aligned}
y_2(x) &= \frac{x^3}{3} - \int_0^x 2y_1(t)^2 \; dt + x & (7)\\
&= \frac{x^3}{3} - \int_0^x 2\left(\frac{t^3}{3} + t\right)^2 \; dt + x\\
&= \frac{x^3}{3} - \int_0^x 2\left(\frac{t^6}{9} + \frac{2t^4}{3} + t^2\right) \; dt + x\\
&= \frac{x^3}{3} - 2\int_0^x \frac{t^6}{9} + \frac{2t^4}{3} + t^2 \; dt + x\\
&= \frac{x^3}{3} - 2\left[\frac{t^7}{63} + \frac{2t^5}{15} + \frac{t^3}{3}\right]_0^x + x\\
&= \frac{x^3}{3} - \frac{2x^7}{63} - \frac{4x^5}{15} - \frac{2x^3}{3} + x\\
&= -\frac{2x^7}{63} - \frac{4x^5}{15} - \frac{x^3}{3} + x & (8)
\end{aligned}
$$

1.2 Question (b)

$$
\frac{dy}{dx} = 2e^x + y \qquad\qquad y(0) = 1 \qquad\qquad y_3(x) \tag{9}
$$

$$
\begin{aligned}
y_{n+1}(x) &= 1 + \int_0^x 2e^t + y_n(t) \; dt & (10)\\
&= 1 + 2\int_0^x e^t \; dt + \int_0^x y_n(t) \; dt\\
&= 1 + 2\left[e^t\right]_0^x + \int_0^x y_n(t) \; dt\\
&= 1 + 2e^x + \int_0^x y_n(t) \; dt & (11)\\
y_1(x) &= 1 + 2e^x + \int_0^x y_0(t) \; dt & (12)\\
&= 1 + 2e^x + \int_0^x 1 \; dt\\
&= 1 + 2e^x + [t]_0^x\\
&= 1 + 2e^x + x & (13)
\end{aligned}
$$

$$
\begin{aligned}
y_2(x) &= 1 + 2e^x + \int_0^x y_1(t)\ dt && (14)\\
&= 1 + 2e^x + \int_0^x 1 + 2e^t + t\ dt\\
&= 1 + 2e^x + \left[t + 2e^t + \frac{t^2}{2}\right]_0^x\\
&= 1 + 2e^x + x + 2e^x + \frac{x^2}{2}\\
&= 4e^x + \frac{x^2}{2} + x + 1 && (15)\\
y_3(x) &= 1 + 2e^x + \int_0^x y_2(t)\ dt && (16)\\
&= 1 + 2e^x + \int_0^x 4e^t + \frac{t^2}{2} + t + 1\ dt\\
&= 1 + 2e^x + \left[4e^t + \frac{t^3}{6} + \frac{t^2}{2} + t\right]_0^x\\
&= 1 + 2e^x + 4e^x + \frac{x^3}{6} + \frac{x^2}{2} + x\\
&= 6e^x + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 && (17)
\end{aligned}
$$

2 Taylor Approximating Polynomial

2.1 Question (a)

$$
y' = \cos(x) - y \qquad y(0) = -1 \qquad p_3(x) \tag{18}
$$

$$p_3(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 \tag{19}$$

$$y' = \cos(x) - y \tag{20}$$

$$y'(0) = \cos(0) - y(0) = 1 - (-1) = 2 \tag{21}$$

$$y'' = -\sin(x) - y' = -\sin(x) - \cos(x) + y \tag{22}$$

$$y''(0) = -\sin(0) - \cos(0) + y(0) = -1 - 1 = -2 \tag{23}$$

$$y''' = -\cos(x) - y'' = \sin(x) - y \tag{24}$$

$$y'''(0) = \sin(0) - y(0) = -(-1) = 1 \tag{25}$$

$$p_3(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 \tag{26}$$

$$p_3(x) = -1 + 2x + \frac{-2x^2}{2} + \frac{x^3}{6} = \frac{x^3}{6} - x^2 + 2x - 1 \tag{27}$$

2.2 Question (b)

$$y' = e^x y \qquad\qquad y(0) = 2 \qquad\qquad p_2(x) \tag{28}$$

$$p_2(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 \tag{29}$$

$$y' = e^x y \tag{30}$$

$$y'(0) = e^0 y(0) = 1(2) = 2 \tag{31}$$

$$y'' = e^x y + e^x y' = e^x(y + e^x y) \tag{32}$$

$$y''(0) = e^0(y(0) + e^0 y(0)) = 1(2 + 1(2)) = 4 \tag{33}$$

$$p_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 \tag{34}$$

$$p_2(x) = 2 + 2x + 2x^2 = 2(x^2 + x + 1) \tag{35}$$

3 Euler's Method

$$y_{n+1} = y_n + hf(x_{n+1}, y_n) \qquad\qquad x_n = x_0 + nh \tag{36}$$

3.1 Question (a)

$$\frac{dy}{dx} = y^3 - x \qquad y(0) = 1 \qquad 0 \leq x \leq 4 \qquad n = 4 \tag{37}$$

$$\begin{aligned}
h &= \frac{x - x_0}{n} = \frac{4-0}{4} = 1 && (38)\\
y_1 &= y_0 + 1f(x_1, y_0) && (39)\\
&= 1 + y_0^3 - x_1 \\
&= 1 + 1^3 - 1 = 2 - 1 = 1 && (40)\\
y_2 &= y_1 + 1f(x_2, y_1) && (41)\\
&= 1 + y_1^3 - x_2 \\
&= 1 + 1^3 - 2 = 2 - 2 = 0 && (42)\\
y_3 &= y_2 + 1f(x_3, y_2) && (43)\\
&= 0 + y_2^3 - x_3 \\
&= 0^3 - 3 = -3 && (44)\\
y_4 &= y_3 + 1f(x_4, y_3) && (45)\\
&= -3 + y_3^3 - x_4 \\
&= -3 + (-3)^3 - 4 = -3 - 27 - 3 = -33 && (46)
\end{aligned}$$

n	x_n	y_n
0	0	1
1	1	1
2	2	0
3	3	-3
4	4	-33

3.2 Question (b)

$$(y^2 + 1)\frac{dy}{dx} = \ln(x^2 + 1) \qquad y(0) = 0 \qquad 0 \leq x \leq 1 \qquad n = 4 \tag{47}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\ln\left(x^2+1\right)}{y^2+1} && (48)\\
h &= \frac{x-x_0}{n} = \frac{1-0}{4} = 0.25 && (49)\\
y_1 &= y_0 + 0.25 f(x_1, y_0) && (50)\\
&= 0 + 0.25\left(\frac{\ln((0.25)^2+1)}{0^2+1}\right)\\
&= 0.25\ln(1.0625) = 0.015156 && (51)\\
y_2 &= y_1 + 0.25 f(x_2, y_1) && (52)\\
&= 0.015156 + 0.25\left(\frac{\ln((0.5)^2+1)}{(0.015156)^2+1}\right)\\
&= 0.015156 + 0.25\left(\frac{\ln(1.25)}{1.000297}\right)\\
&= 0.015156 + 0.055773 = 0.070929\\
y_3 &= y_2 + 0.25 f(x_3, y_2) && (53)\\
&= 0.070929 + 0.25\left(\frac{\ln((0.75)^2+1)}{(0.070929)^2+1}\right)\\
&= 0.070929 + 0.25\left(\frac{\ln((1.5625)}{1.005031}\right)\\
&= 0.070929 + 0.111013 = 0.181942 && (54)\\
y_4 &= y_3 + 0.25 f(x_4, y_3) && (55)\\
&= 0.181942 + 0.25\left(\frac{\ln(1^2+1)}{(0.181942)^2+1}\right)\\
&= 0.181942 + 0.25\left(\frac{\ln((2)}{1.033103}\right)\\
&= 0.181942 + 0.167734 = 0.349676 && (56)
\end{aligned}
$$

n	x_n	y_n
0	0	0
1	0.25	0.015156
2	0.5	0.070929
3	0.75	0.181942
4	1	0.349676

4 Modified Euler's Method

$$k_1 = hf(x_{n+1}, y_n) \tag{57}$$
$$k_2 = hf(x_{n+1} + h, y_n + k_1)) \tag{58}$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \tag{59}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} \qquad y(1) = 1 \qquad 1 \leq x \leq 6 \qquad n = 5 \tag{60}$$

$$h = \frac{x - x_0}{n} = \frac{6-1}{5} = 1 \tag{61}$$
$$k_1 = 1f(x_{n+1}, y_n) \tag{62}$$
$$= 1 + \frac{y_n}{x_{n+1}} \tag{63}$$
$$k_2 = 1f(x_{n+1} + 1, y_n + k_1)) \tag{64}$$
$$= 1 + \frac{y_n + k_1}{x_{n+1} + 1} \tag{65}$$
$$y_1 = y_0 + \frac{1}{2}(k_{1_1} + k_{2_1}) \tag{66}$$
$$y_0 = y(1) = 1 \tag{67}$$
$$k_{1_1} = 1 + \frac{y_0}{x_1} = 1 + \frac{1}{2} = \frac{3}{2} \tag{68}$$
$$k_{2_1} = 1 + \frac{y_0 + k_{1_1}}{x_1 + 1} = 1 + \frac{1 + \frac{3}{2}}{2+1} = 1 + \frac{\frac{5}{2}}{3} = 1 + \frac{5}{6} = \frac{11}{6} \tag{69}$$
$$y_1 = 1 + \frac{1}{2}\left(\frac{3}{2} + \frac{11}{6}\right) = 1 + \frac{9+11}{12} = 1 + \frac{20}{12} = \frac{32}{12} = \frac{8}{3} \tag{70}$$

$$y_2 = y_1 + \frac{1}{2}(k_{1_2} + k_{2_2}) \tag{71}$$

$$y_1 = \frac{8}{3} \tag{72}$$

$$k_{1_2} = 1 + \frac{y_1}{x_2} = 1 + \frac{\frac{8}{3}}{3} = 1 + \frac{8}{9} = \frac{17}{9} \tag{73}$$

$$k_{2_2} = 1 + \frac{y_1 + k_{1_2}}{x_2 + 1} = 1 + \frac{\frac{8}{3} + \frac{17}{9}}{3+1} = 1 + \frac{\frac{42}{9}}{4} = 1 + \frac{42}{36} = \frac{78}{36} = \frac{13}{6} \tag{74}$$

$$\begin{aligned} y_2 &= \frac{8}{3} + \frac{1}{2}\left(\frac{17}{9} + \frac{13}{6}\right) = \frac{8}{3} + \frac{34+39}{36} \\ &= \frac{8}{3} + \frac{73}{36} = \frac{73+96}{36} = \frac{169}{36} \end{aligned} \tag{75}$$

$$y_3 = y_2 + \frac{1}{2}(k_{1_3} + k_{2_3}) \tag{76}$$

$$y_2 = \frac{169}{36} \tag{77}$$

$$k_{1_3} = 1 + \frac{y_2}{x_3} = 1 + \frac{\frac{169}{36}}{4} = 1 + \frac{169}{144} = \frac{313}{144} \tag{78}$$

$$\begin{aligned} k_{2_3} &= 1 + \frac{y_2 + k_{1_3}}{x_3 + 1} = 1 + \frac{\frac{169}{36} + \frac{313}{144}}{4+1} = 1 + \frac{\frac{676+313}{144}}{5} \\ &= 1 + \frac{989}{720} = \frac{1709}{720} \end{aligned} \tag{79}$$

$$\begin{aligned} y_3 &= \frac{169}{36} + \frac{1}{2}\left(\frac{313}{144} + \frac{1709}{720}\right) = \frac{169}{36} + \frac{1565+1709}{1440} \\ &= \frac{169}{36} + \frac{3274}{1440} = \frac{3380+1637}{720} = \frac{5017}{720} \end{aligned} \tag{80}$$

$$y_4 = y_3 + \frac{1}{2}(k_{1_4} + k_{2_4}) \tag{81}$$

$$y_3 = \frac{5017}{720} \tag{82}$$

$$k_{1_4} = 1 + \frac{y_3}{x_4} = 1 + \frac{\frac{5017}{720}}{5} = 1 + \frac{5017}{3600} = \frac{8617}{3600} \tag{83}$$

$$\begin{aligned} k_{2_4} &= 1 + \frac{y_3 + k_{1_4}}{x_4 + 1} = 1 + \frac{\frac{5017}{720} + \frac{8617}{3600}}{5+1} = 1 + \frac{\frac{25085+8617}{3600}}{6} \\ &= 1 + \frac{33702}{21600} = \frac{55302}{21600} = \frac{9217}{3600} \end{aligned} \tag{84}$$

$$\begin{aligned} y_4 &= \frac{5017}{720} + \frac{1}{2}\left(\frac{8617}{3600} + \frac{9217}{3600}\right) = \frac{5017}{720} + \frac{17834}{7200} \\ &= \frac{50170+17834}{7200} = \frac{68004}{7200} = \frac{1889}{200} \end{aligned} \tag{85}$$

$$y_5 = y_4 + \frac{1}{2}(k_{1_5} + k_{2_5}) \tag{86}$$

$$y_4 = \frac{1889}{200} \tag{87}$$

$$k_{1_5} = 1 + \frac{y_4}{x_5} = 1 + \frac{\frac{1889}{200}}{6} = 1 + \frac{1889}{1200} = \frac{3089}{1200} \tag{88}$$

$$\begin{aligned} k_{2_5} &= 1 + \frac{y_4 + k_{1_5}}{x_5 + 1} = 1 + \frac{\frac{1889}{200} + \frac{3089}{1200}}{6+1} = 1 + \frac{\frac{11334+3089}{1200}}{7} \\ &= 1 + \frac{14423}{8400} = \frac{22823}{8400} \end{aligned} \tag{89}$$

$$\begin{aligned} y_5 &= \frac{1889}{200} + \frac{1}{2}\left(\frac{3089}{1200} + \frac{22823}{8400}\right) = \frac{1889}{200} + \frac{21623 + 22823}{16800} \\ &= \frac{1889}{200} + \frac{44446}{7200} = \frac{68004 + 44446}{7200} = \frac{112450}{7200} = \frac{2249}{144} \end{aligned} \tag{90}$$

n	x_n	y_n
0	1	1
1	2	$\frac{8}{3} \approx 2.67$
2	3	$\frac{169}{36} \approx 4.69$
3	4	$\frac{5017}{720} \approx 6.97$
4	5	$\frac{1889}{200} = 9.445$
5	6	$\frac{2249}{144} \approx 15.62$