

Numerical Methods I
Homework Problem Set #11

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Problem Set #11

1 RK-4

$$\frac{dy}{dt} = y + ty - y^2 \quad y(0) = 1 \quad 0 \leq t \leq 1 \quad h = 0.5 \quad (1)$$

$$n = \frac{b - a}{h} = \frac{1 - 0}{0.5} = 2 \quad (2)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (3)$$

$$t_{n+1} = t_n + h \quad (4)$$

$$k_1 = f(t_n, y_n) \quad (5)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \quad (6)$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \quad (7)$$

$$k_4 = f(t_n + h, y_n + hk_3) \quad (8)$$

$$y_1 = 0 + \frac{1}{12}(k_{1_1} + 2k_{2_1} + 2k_{3_1} + k_{4_1}) \quad (9)$$

$$\begin{aligned} k_{1_1} &= f(0, 1) \\ &= 1 + 0 * 1 - (1)^2 \\ &= 0 \end{aligned} \quad (10)$$

$$\begin{aligned}
k_{2_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0\right) \\
&= f(0.25, 1) \\
&= 1 + 0.25 * 1 - (1)^2 \\
&= 0.25
\end{aligned} \tag{11}$$

$$\begin{aligned}
k_{3_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0.25\right) \\
&= f(0.25, 1.0625) \\
&= 1.0625 + 0.25 * 1.0625 - (1.0625)^2 \\
&= 1.0625 + 0.2656 - 1.1289 \\
&= 0.1992
\end{aligned} \tag{12}$$

$$\begin{aligned}
k_{4_1} &= f(0 + 0.5, 1 + 0.5 * 0.1992) \\
&= f(0.5, 1.0996) \\
&= 1.0996 + 0.5 * 1.0996 - (1.0996)^2 \\
&= 1.0996 + 0.5498 - 1.2091 \\
&= 0.4403
\end{aligned} \tag{13}$$

$$\begin{aligned}
y_1 &= 0 + \frac{1}{12}(0 + 2(0.25) + 2(0.1992) + 0.4403) \\
&= \frac{1.3387}{12} \\
&= 0.1116
\end{aligned} \tag{14}$$

$$y_2 = 0.1116 + \frac{1}{12}(k_{1_2} + 2k_{2_2} + 2k_{3_2} + k_{4_2}) \tag{15}$$

$$\begin{aligned}
k_{1_2} &= f(0.5, 0.1116) \\
&= 0.1116 + 0.5 * 0.1116 - (0.1116)^2 \\
&= 0.1549
\end{aligned} \tag{16}$$

$$\begin{aligned}
k_{2_2} &= f\left(0.5 + \frac{0.5}{2}, 0.1116 + \frac{0.5}{2} * 0.1549\right) \\
&= f(0.75, 0.0387) \\
&= 0.0387 + 0.75 * 0.0387 - (0.0387)^2 \\
&= 0.0662
\end{aligned} \tag{17}$$

$$\begin{aligned}
k_{3_2} &= f\left(0.5 + \frac{0.5}{2}, 0.1116 + \frac{0.5}{2} * 0.0662\right) \\
&= f(0.75, 0.1282) \\
&= 0.1282 + 0.75 * 0.1282 - (0.1282)^2 \\
&= 0.2079
\end{aligned} \tag{18}$$

$$\begin{aligned}
k_{4_2} &= f(0.5 + 0.5, 0.1116 + 0.5 * 0.2079) \\
&= f(1, 0.2156) \\
&= 0.2156 + 1 * 0.2156 - (0.2156)^2 \\
&= 0.4777
\end{aligned} \tag{19}$$

$$\begin{aligned}
y_2 &= 0.1116 + \frac{1}{12}(0.1549 + 2(0.0662) + 2(0.2079) + 0.4777) \\
&= \frac{1.3387}{12} \\
&= 0.1116 + \frac{1.1375}{12} \\
&= 0.2064
\end{aligned} \tag{20}$$

2 Euler's Method

$$0 \leq t \leq 1 \qquad h = 0.25 \tag{21}$$

$$n = \frac{b - a}{h} = \frac{1 - 0}{0.25} = 4 \tag{22}$$

2.1 Question (a)

$$\frac{dx}{dt} = tx + y \quad \frac{dy}{dt} = 2x - ty \quad x(0) = 1 \quad y(0) = -2 \quad (23)$$

$$\frac{dx}{dt} = f(t, x, y) \quad (24)$$

$$\frac{dy}{dt} = g(t, x, y) \quad (25)$$

$$x_{n+1} = x_n + hf(t_n, x_n, y_n) \quad (26)$$

$$y_{n+1} = y_n + hg(t_n, x_n, y_n) \quad (27)$$

$$\begin{aligned} x_1 &= 1 + 0.25f(0.25, 1, -2) \\ &= 1 + 0.25(0.25(1) + (-2)) \\ &= 0.5625 \end{aligned} \quad (28)$$

$$\begin{aligned} y_1 &= -2 + 0.25g(0.25, 1, -2) \\ &= -2 + 0.25(2(1) - 0.25(-2)) \\ &= -1.3750 \end{aligned} \quad (29)$$

$$\begin{aligned} x_2 &= 0.5625 + 0.25f(0.5, 0.5625, -1.3750) \\ &= 0.5 + 0.25(0.5(0.5625) + (-1.3750)) \\ &= 0.2891 \end{aligned} \quad (30)$$

$$\begin{aligned} y_2 &= -1.3750 + 0.25g(0.5, 0.5625, -1.3750) \\ &= -1.3750 + 0.25(2(0.5625) - 0.5(-1.3750)) \\ &= -0.9219 \end{aligned} \quad (31)$$

$$\begin{aligned} x_3 &= 0.2891 + 0.25f(0.75, 0.2891, -0.9219) \\ &= 0.2891 + 0.25(0.75 * 0.2891 + (-0.9219)) \\ &= 0.1128 \end{aligned} \quad (32)$$

$$\begin{aligned} y_3 &= -0.9219 + 0.25g(0.75, 0.2891, -0.9219) \\ &= -0.9219 + 0.25(2(0.2891) - 0.75(-0.9219)) \\ &= -0.6045 \end{aligned} \quad (33)$$

$$\begin{aligned}
x_4 &= 0.1128 + 0.25f(1, 0.1128, -0.6045) \\
&= 0.1128 + 0.25(1(0.1128) + (-0.6045)) \\
&= -0.0101
\end{aligned} \tag{34}$$

$$\begin{aligned}
y_4 &= -0.6045 + 0.25g(1, 0.1128, -0.6045) \\
&= -0.6045 + 0.25(2(0.1128) - 1(-0.6045)) \\
&= -0.3970
\end{aligned} \tag{35}$$

n	t_n	x_n	y_n
0	0	1	-2
1	0.25	0.5625	-1.3750
2	0.5	0.2891	-0.9219
3	0.75	0.1128	-0.6045
4	1.0	-0.0101	-0.3970

2.2 Question (b)

$$\begin{aligned}
\frac{dx}{dt} &= tx - xy + z & \frac{dy}{dt} &= -ty + 2z & \frac{dz}{dt} &= 2x - y + 3tz \\
x(0) &= 0 & y(0) &= 1 & z(0) &= 2
\end{aligned} \tag{36}$$

$$\frac{dx}{dt} = f(t, x, y) \tag{37}$$

$$\frac{dy}{dt} = g(t, x, y) \tag{38}$$

$$\frac{dz}{dt} = u(t, x, y) \tag{39}$$

$$x_{n+1} = x_n + hf(t_n, x_n, y_n, z_n) \tag{40}$$

$$y_{n+1} = y_n + hg(t_n, y_n, z_n) \tag{41}$$

$$z_{n+1} = z_n + hu(t_n, x_n, y_n, z_n) \tag{42}$$

$$\begin{aligned}
x_1 &= 0 + 0.25f(0.25, 0, 1, 2) \\
&= 0.25(0.25(0) - 0(1) + 2) \\
&= 0.5
\end{aligned} \tag{43}$$

$$\begin{aligned}
y_1 &= 1 + 0.25g(0.25, 1, 2) \\
&= 1 + 0.25(-(0.25)(1) + 2(2)) \\
&= 1.9375
\end{aligned} \tag{44}$$

$$\begin{aligned}
z_1 &= 2 + 0.25u(0.25, 0, 1, 2) \\
&= 2 + 0.25(2(0) - 1 + 3(0.25)(2)) \\
&= 2.1250
\end{aligned} \tag{45}$$

$$\begin{aligned}
x_2 &= 0.5 + 0.25f(0.5, 0.5, 1.9375, 2.1250) \\
&= 0.5 + 0.25((0.5)(0.5) - 0.5(1.9375) + 2.1250) \\
&= 0.8516
\end{aligned} \tag{46}$$

$$\begin{aligned}
y_2 &= 1.9375 + 0.25g(0.5, 1.9375, 2.1250) \\
&= 1.9375 + 0.25(-(0.5)(1.9375) + 2(2.1250)) \\
&= 2.7578
\end{aligned} \tag{47}$$

$$\begin{aligned}
z_2 &= 2.1250 + 0.25u(0.5, 0.5, 1.9375, 2.1250) \\
&= 2.1250 + 0.25(2(0.5) - 2.7578 + 3(0.5)(2.1250)) \\
&= 2.6875
\end{aligned} \tag{48}$$

$$\begin{aligned}
x_3 &= 0.8516 + 0.25f(0.75, 0.8516, 2.7578, 2.6875) \\
&= 0.8516 + 0.25(0.75(0.8516) - 0.8516(2.7578) + 2.6875) \\
&= 1.0960
\end{aligned} \tag{49}$$

$$\begin{aligned}
y_3 &= 2.7578 + 0.25g(0.75, 2.7578, 2.6875) \\
&= 2.7578 + 0.25(-0.75(2.7578) + 2(2.6875)) \\
&= 3.5845
\end{aligned} \tag{50}$$

$$\begin{aligned}
z_3 &= 2.6875 + 0.25u(0.75, 0.8516, 2.7578, 2.6875) \\
&= 2.6875 + 0.25(2(0.8516) - (2.7578) + 3(0.75)(2.6875)) \\
&= 3.9356
\end{aligned} \tag{51}$$

$$\begin{aligned}
x_4 &= 1.0960 + 0.25f(1, 1.0960, 3.5845, 3.9356) \\
&= 1.0960 + 0.25(1(1.0960) - 1.0960(3.5845) + 3.9356) \\
&= 1.3717
\end{aligned} \tag{52}$$

$$\begin{aligned}
y_4 &= 3.5845 + 0.25g(1, 3.5845, 3.9356) \\
&= 3.5845 + 0.25(-1(3.5845) + 2(3.9356)) \\
&= 4.6562
\end{aligned} \tag{53}$$

$$\begin{aligned}
z_4 &= 3.9356 + 0.25u(1, 1.0960, 3.5845, 3.9356) \\
&= 3.9356 + 0.25(2(1.0960) - (3.5845) + 3(1)(3.9356)) \\
&= 6.5392
\end{aligned} \tag{54}$$

n	t_n	x_n	y_n	z_n
1	0	0	1	2
1	0.25	0.5	1.9375	2.1250
2	0.5	0.8516	2.7578	2.6875
3	0.75	1.0960	3.5845	3.9356
4	1	1.3717	4.6562	6.5392

3 Conversion into First-Order

3.1 Question (a)

$$y'' + ty' - 3y = t^2 \quad y(0) = 3 \quad y'(0) = 4 \tag{55}$$

$$y'' = -ty' + 3y + t^2 \tag{56}$$

$$u_1 = y' \tag{57}$$

$$u'_1 = y'' \tag{58}$$

$$u'_1 = -tu_1 + 3y + t^2 \tag{59}$$

$$y(0) = 3 \tag{60}$$

$$u_1(0) = 4 \tag{61}$$

3.2 Question (b)

$$\begin{aligned}
y^{(4)} - y''' + y'' - 2y' + 7y &= \cos(t) & y(0) = y'(0) &= 0 \\
y''(0) &= 1 & y'''(0) &= 2
\end{aligned} \tag{62}$$

$$y^{(4)} = y''' - y'' + 2y' - 7y + \cos(t) \tag{63}$$

$$u_1 = y' \tag{64}$$

$$u'_1 = y'' = u_2 \tag{65}$$

$$u'_2 = y''' = u_3 \tag{66}$$

$$u'_3 = y^{(4)} \tag{67}$$

$$u'_3 = u_3 - u_2 + 2u_1 - 7y + \cos(t) \tag{68}$$

$$y(0) = 0 \tag{69}$$

$$u_1(0) = 0 \tag{70}$$

$$u_2(0) = 1 \tag{71}$$

$$u_3(0) = 2 \tag{72}$$

4 Euler's method approximation on 3(a)

$$\begin{aligned}
u'_1 &= -tu_1 + 3y + t^2 & y(0) &= 3 & u_1(0) &= 4 \\
0 \leq t \leq 2 & & n &= 2 & h &= 1
\end{aligned} \tag{73}$$

$$y_{n+1} = y_n + h(u_{1_n}) \tag{74}$$

$$u_{1_{n+1}} = u_{1_n} + h(-tu_{1_n} + 3y_n + t^2) \tag{75}$$

$$\begin{aligned}
y_1 &= 3 + 1(4) \\
&= 7
\end{aligned} \tag{76}$$

$$\begin{aligned}
u_1 &= 4 + 1(-(1)(4) + 3(3) + (1)^2) \\
&= 10
\end{aligned} \tag{77}$$

$$\begin{aligned} y_2 &= 7 + 1(10) \\ &= 17 \end{aligned} \tag{78}$$

$$\begin{aligned} u_2 &= 10 + 1(-2)(10) + 3(7) + (2)^2 \\ &= 15 \end{aligned} \tag{79}$$

n	t_n	y_n	u_{1_n}
0	0	3	4
1	1	7	10
2	2	17	15