Numerical Methods I
Homework Problem Set #11

Jonathan Henrique Maia de Moraes (ID: 1620855)

12/03/2015

Problem Set #11

1 RK-4

$$\frac{dy}{dt} = y + ty - y^2 \qquad y(0) = 1 \qquad 0 \leq t \leq 1 \qquad h = 0.5 \tag{1}$$

$$n = \frac{b-a}{h} = \frac{1-0}{0.5} = 2 \tag{2}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3}$$

$$t_{n+1} = t_n + h \tag{4}$$

$$k_1 = f(t_n, y_n) \tag{5}$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \tag{6}$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \tag{7}$$

$$k_4 = f(t_n + h, y_n + hk_3) \tag{8}$$

$$y_1 = 0 + \frac{1}{12}(k_{1_1} + 2k_{2_1} + 2k_{3_1} + k_{4_1}) \tag{9}$$

$$\begin{aligned} k_{1_1} &= f(0,1) \\ &= 1 + 0 * 1 - (1)^2 \\ &= 0 \end{aligned} \tag{10}$$

$$\begin{aligned} k_{2_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0\right) \\ &= f(0.25, 1) \\ &= 1 + 0.25 * 1 - (1)^2 \\ &= 0.25 \end{aligned} \tag{11}$$

$$\begin{aligned} k_{3_1} &= f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2} * 0.25\right) \\ &= f(0.25, 1.0625) \\ &= 1.0625 + 0.25 * 1.0625 - (1.0625)^2 \\ &= 1.0625 + 0.2656 - 1.1289 \\ &= 0.1992 \end{aligned} \tag{12}$$

$$\begin{aligned} k_{4_1} &= f\left(0 + 0.5, 1 + 0.5 * 0.1992\right) \\ &= f(0.5, 1.0996) \\ &= 1.0996 + 0.5 * 1.0996 - (1.0996)^2 \\ &= 1.0996 + 0.5498 - 1.2091 \\ &= 0.4403 \end{aligned} \tag{13}$$

$$\begin{aligned} y_1 &= 0 + \frac{1}{12}(0 + 2(0.25) + 2(0.1992) + 0.4403) \\ &= \frac{1.3387}{12} \\ &= 0.1116 \end{aligned} \tag{14}$$

$$y_2 = 0.1116 + \frac{1}{12}(k_{1_2} + 2k_{2_2} + 2k_{3_2} + k_{4_2}) \tag{15}$$

$$\begin{aligned} k_{1_2} &= f(0.5, 0.1116) \\ &= 0.1116 + 0.5 * 0.1116 - (0.1116)^2 \\ &= 0.1549 \end{aligned} \tag{16}$$

$$\begin{aligned} k_{2_2} &= f\left(0.5 + \frac{0.5}{2}, 0.1116 + \frac{0.5}{2} * 0.1549\right) \\ &= f(0.75, 0.0387) \\ &= 0.0387 + 0.75 * 0.0387 - (0.0387)^2 \\ &= 0.0662 \end{aligned} \tag{17}$$

$$\begin{aligned} k_{3_2} &= f\left(0.5 + \frac{0.5}{2}, 0.1116 + \frac{0.5}{2} * 0.0662\right) \\ &= f(0.75, 0.1282) \\ &= 0.1282 + 0.75 * 0.1282 - (0.1282)^2 \\ &= 0.2079 \end{aligned} \tag{18}$$

$$\begin{aligned} k_{4_2} &= f\left(0.5 + 0.5, 0.1116 + 0.5 * 0.2079\right) \\ &= f(1, 0.2156) \\ &= 0.2156 + 1 * 0.2156 - (0.2156)^2 \\ &= 0.4777 \end{aligned} \tag{19}$$

$$\begin{aligned} y_2 &= 0.1116 + \frac{1}{12}(0.1549 + 2(0.0662) + 2(0.2079) + 0.4777) \\ &= \frac{1.3387}{12} \\ &= 0.1116 + \frac{1.1375}{12} \\ &= 0.2064 \end{aligned} \tag{20}$$

2 Euler's Method

$$0 \leq t \leq 1 \qquad\qquad h = 0.25 \tag{21}$$

$$n = \frac{b-a}{h} = \frac{1-0}{0.25} = 4 \tag{22}$$

2.1 Question (a)

$$\frac{dx}{dt} = tx + y \qquad \frac{dy}{dt} = 2x - ty \qquad x(0) = 1 \qquad y(0) = -2 \tag{23}$$

$$\frac{dx}{dt} = f(t, x, y) \tag{24}$$
$$\frac{dy}{dt} = g(t, x, y) \tag{25}$$
$$x_{n+1} = x_n + hf(t_n, x_n, y_n) \tag{26}$$
$$y_{n+1} = y_n + hg(t_n, x_n, y_n) \tag{27}$$

$$\begin{aligned} x_1 &= 1 + 0.25f(0, 1, -2) \\ &= 1 + 0.25(0 * 1 + (-2)) \\ &= 0.5 \end{aligned} \tag{28}$$
$$\begin{aligned} y_1 &= -2 + 0.25g(0, 1, -2) \\ &= -2 + 0.25(2(1) - 0(-2)) \\ &= -1.5 \end{aligned} \tag{29}$$

$$\begin{aligned} x_2 &= 0.5 + 0.25f(0.25, 0.5, -1.5) \\ &= 0.5 + 0.25(0.25 * 0.5 + (-1.5)) \\ &= 0.1562 \end{aligned} \tag{30}$$
$$\begin{aligned} y_2 &= -1.5 + 0.25g(0.25, 0.5, -1.5) \\ &= -1.5 + 0.25(2(0.5) - 0.25 * (-1.5)) \\ &= -1.1562 \end{aligned} \tag{31}$$

$$\begin{aligned} x_3 &= 0.1562 + 0.25f(0.5, 0.1562, -1.1562) \\ &= 0.1562 + 0.25(0.5 * 0.1562 + (-1.1562)) \\ &= -0.1133 \end{aligned} \tag{32}$$
$$\begin{aligned} y_3 &= -1.1562 + 0.25g(0.5, 0.1562, -1.1562) \\ &= -1.1562 + 0.25(2(0.1562) - 0.5 * (-1.1562)) \\ &= -0.9336 \end{aligned} \tag{33}$$

$$\begin{aligned} x_4 &= -0.1133 + 0.25f(0.75, -0.1133, -0.9336) \\ &= -0.1133 + 0.25(0.75 * -0.1133 + (-0.9336)) \\ &= -0.3679 \end{aligned} \tag{34}$$

$$\begin{aligned} y_4 &= -0.9336 + 0.25g(0.75, -0.1133, -0.9336) \\ &= -0.9336 + 0.25(2(-0.1133) - 0.75 * (-0.9336)) \\ &= -0.8152 \end{aligned} \tag{35}$$

n	t_n	x_n	y_n
1	0	0.5	-1.5
2	0.25	0.1562	-1.1562
3	0.5	-0.1133	-0.9336
4	0.75	-0.3679	-0.8152

2.2 Question (b)

$$\begin{array}{lll} \dfrac{dx}{dt} = tx - xy + z & \dfrac{dy}{dt} = -ty + 2z & \dfrac{dz}{dt} = 2x - y + 3tz \\ x(0) = 0 & y(0) = 1 & z(0) = 2 \end{array} \tag{36}$$

$$\frac{dx}{dt} = f(t, x, y) \tag{37}$$

$$\frac{dy}{dt} = g(t, x, y) \tag{38}$$

$$\frac{dz}{dt} = u(t, x, y) \tag{39}$$

$$x_{n+1} = x_n + hf(t_n, x_n, y_n, z_n) \tag{40}$$

$$y_{n+1} = y_n + hg(t_n, y_n, z_n) \tag{41}$$

$$z_{n+1} = z_n + hu(t_n, x_n, y_n, z_n) \tag{42}$$

$$\begin{aligned} x_1 &= 0 + 0.25f(0, 0, 1, 2) \\ &= 0.25(0(0) - 0(1) + 2) \\ &= 0.5 \end{aligned} \tag{43}$$

$$\begin{aligned} y_1 &= 1 + 0.25g(0, 1, 2) \\ &= 1 + 0.25(-0(1) - 2(2)) \\ &= 0 \end{aligned} \tag{44}$$

$$\begin{aligned} z_1 &= 2 + 0.25u(0, 0, 1, 2) \\ &= 2 + 0.25(2(0) - 1 + 3(0)(2)) \\ &= 1.75 \end{aligned} \tag{45}$$

$$\begin{aligned} x_2 &= 0.5 + 0.25f(0.25, 0.5, 0, 1.75) \\ &= 0.5 + 0.25(0.25(0.5) - 0.5(0) + 1.75) \\ &= 0.9687 \end{aligned} \tag{46}$$

$$\begin{aligned} y_2 &= 0 + 0.25g(0.25, 0, 1.75) \\ &= 0.25(-0.25(0) - 2(1.75)) \\ &= -0.8750 \end{aligned} \tag{47}$$

$$\begin{aligned} z_2 &= 1.75 + 0.25u(0.25, 0.5, 0, 1.75) \\ &= 1.75 + 0.25(2(0.5) - 0 + 3(0.25)(1.75)) \\ &= 2.3281 \end{aligned} \tag{48}$$

$$\begin{aligned} x_3 &= 0.9687 + 0.25f(0.5, 0.9687, -0.8750, 2.3281) \\ &= 0.9687 + 0.25(0.5(0.9687) - 0.9687(-0.8750) + 2.3281) \\ &= 1.8837 \end{aligned} \tag{49}$$

$$\begin{aligned} y_3 &= -0.8750 + 0.25g(0.5, -0.8750, 2.3281) \\ &= -0.8750 + 0.25(-0.5(-0.8750) - 2(2.3281)) \\ &= -1.9297 \end{aligned} \tag{50}$$

$$\begin{aligned} z_3 &= 2.3281 + 0.25u(0.5, 0.9687, -0.8750, 2.3281) \\ &= 2.3281 + 0.25(2(0.9687) - (-0.8750) + 3(0.5)(2.3281)) \\ &= 3.9042 \end{aligned} \tag{51}$$

$$\begin{aligned} x_4 &= 1.8837 + 0.25f(0.75, 1.8837, -1.9297, 3.9042) \\ &= 1.8837 + 0.25(0.75(1.8837) - 1.8837(-1.9297) + 3.9042) \\ &= 4.1217 \end{aligned} \tag{52}$$

$$\begin{aligned} y_4 &= -1.9297 + 0.25g(0.75, -1.9297, 3.9042) \\ &= -1.9297 + 0.25(-0.75(-1.9297) - 2(3.9042)) \\ &= -3.5200 \end{aligned} \tag{53}$$

$$\begin{aligned} z_4 &= 3.9042 + 0.25u(0.75, 1.8837, -1.9297, 3.9042) \\ &= 3.9042 + 0.25(2(1.8837) - (-1.9297) + 3(0.75)(3.9042)) \\ &= 7.5246 \end{aligned} \tag{54}$$

n	t_n	x_n	y_n	z_n
1	0	0.5	0	1.75
2	0.25	0.9687	-0.8750	2.3281
3	0.5	1.8837	-1.9297	3.9042
4	0.75	4.1217	-3.5200	7.5246

3 Conversion into First-Order

3.1 Question (a)

$$y'' + ty' - 3y = t^2 \qquad y(0) = 3 \qquad y'(0) = 4 \tag{55}$$

$$y'' = -ty' + 3y + t^2 \tag{56}$$

$$u_1 = y' \tag{57}$$

$$u_1' = y'' \tag{58}$$

$$u_1' = -tu_1 + 3y + t^2 \tag{59}$$

$$y(0) = 3 \tag{60}$$

$$u_1(0) = 4 \tag{61}$$

3.2 Question (b)

$$y^{(4)} - y''' + y'' - 2y' + 7y = cos(t) \qquad y(0) = y'(0) = 0$$
$$y''(0) = 1 \qquad y'''(0) = 2 \tag{62}$$

$$y^{(4)} = y''' - y'' + 2y' - 7y + cos(t) \tag{63}$$
$$u_1 = y' \tag{64}$$
$$u_1' = y'' = u_2 \tag{65}$$
$$u_2' = y''' = u_3 \tag{66}$$
$$u_3' = y^{(4)} \tag{67}$$
$$u_3' = u_3 - u_2 + 2u_1 - 7y + cos(t) \tag{68}$$
$$y(0) = 0 \tag{69}$$
$$u_1(0) = 0 \tag{70}$$
$$u_2(0) = 1 \tag{71}$$
$$u_3(0) = 2 \tag{72}$$

4 Euler's method approximation on 3(a)

$$u_1' = -tu_1 + 3y + t^2 \qquad y(0) = 3 \qquad u_1(0) = 4$$
$$0 \leq t \leq 2 \qquad n = 2 \qquad h = 1 \tag{73}$$

$$y_{n+1} = y_n + h(u_{1_n}) \tag{74}$$
$$u_{1_{n+1}} = u_{1_n} + h(-tu_{1_n} + 3y_n + t^2) \tag{75}$$

$$\begin{aligned} y_1 &= 3 + 1(4) \\ &= 7 \end{aligned} \tag{76}$$
$$\begin{aligned} u_1 &= 4 + 1(-(0)(4) + 3(3) + (0)^2) \\ &= 13 \end{aligned} \tag{77}$$

$$
\begin{aligned}
y_2 &= 7 + 1(13) \\
&= 20 \qquad (78) \\
u_2 &= 13 + 1(-(1)(13) + 3(7) + (1)^2) \\
&= 22 \qquad (79)
\end{aligned}
$$

n	t_n	y_n	u_{1_n}
1	0	7	13
2	1	20	22