Numerical Methods I
Homework Problem Set #5

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Problem Set #5

0.1 Doolittle Factorization

$$A = \begin{bmatrix} 8 & 5 & 1 \\ 3 & 7 & 4 \\ 2 & 3 & 9 \end{bmatrix} \tag{1}$$

Starting defining the Doolittle Factorization of eq. (1):

$$A = \begin{bmatrix} 8 & 5 & 1 \\ 3 & 7 & 4 \\ 2 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \tag{2}$$

$$\begin{aligned} l_{11}U_{11} &= A_{11} \\ 1 * U_{11} &= 8 \\ &= 8 \end{aligned} \tag{3}$$

$$\begin{aligned} l_{11}U_{12} &= A_{12} \\ 1 * U_{12} &= 5 \\ &= 5 \end{aligned} \tag{4}$$

$$\begin{aligned} l_{11} * U_{13} &= A_{13} \\ 1 * U_{13} &= 1 \\ &= 1 \end{aligned} \tag{5}$$

$$\begin{aligned} l_{21}U_{11} &= A_{21} \\ l_{21} * 8 &= 3 \\ l_{21} &= \frac{3}{8} \end{aligned} \tag{6}$$

$$\begin{aligned} l_{31}U_{11} &= A_{31} \\ l_{31} * 8 &= 2 \\ l_{31} &= \frac{2}{8} = \frac{1}{4} \end{aligned} \tag{7}$$

$$\begin{aligned} l_{21}U_{12} + l_{22}U_{22} &= A_{22} \\ \frac{3}{8} * 5 + 1 * U_{22} &= 7 \\ U_{22} &= 7 - 5 * \frac{3}{8} \\ &= \frac{56 - 15}{8} = \frac{41}{8} \end{aligned} \tag{8}$$

$$\begin{aligned} l_{21}U_{13} + l_{22}U_{23} &= A_{23} \\ \frac{3}{8} * 1 + 1 * U_{23} &= 4 \\ U_{23} &= 4 - \frac{3}{8} \\ &= \frac{32 - 3}{8} = \frac{29}{8} \end{aligned} \tag{9}$$

$$\begin{aligned} l_{31}U_{12} + l_{32}U_{22} &= A_{32} \\ \frac{1}{4} * 5 + l_{32} * \frac{41}{8} &= 3 \\ l_{32} &= \frac{3 - \frac{5}{4}}{\frac{41}{8}} \\ &= \frac{\frac{12-5}{4}}{\frac{41}{8}} = \frac{7}{4} * \frac{8}{41} = \frac{56}{164} = \frac{14}{41} \end{aligned} \tag{10}$$

$$
\begin{aligned}
l_{31}U_{13} + l_{32}U_{23} + l_{33}U_{33} &= A_{33} \\
\frac{1}{4} * 1 + \frac{14}{41} * \frac{29}{8} + 1 * U_{33} &= 9 \\
U_{33} &= 9 - \left(\frac{1}{4} + \frac{14}{41} * \frac{29}{8}\right) \\
&= 9 - \frac{1}{4} - \frac{406}{328} \\
&= \frac{36-1}{4} - \frac{203}{164} \\
&= \frac{1435-203}{164} = \frac{1232}{164} = \frac{308}{41}
\end{aligned} \tag{11}
$$

$$
A = \begin{bmatrix} 8 & 5 & 1 \\ 3 & 7 & 4 \\ 2 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{8} & 1 & 0 \\ \frac{1}{4} & \frac{14}{41} & 1 \end{bmatrix} \begin{bmatrix} 8 & 5 & 1 \\ 0 & \frac{41}{8} & \frac{29}{8} \\ 0 & 0 & \frac{308}{41} \end{bmatrix} \tag{12}
$$

Solving:

$$
Ax = B \tag{13}
$$

$$
Ax = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix} \tag{14}
$$

Forward Substitution:

$$LY = B \tag{15}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{8} & 1 & 0 \\ \frac{1}{4} & \frac{14}{41} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ \frac{3}{8}y_1 + y_2 \\ \frac{1}{4}y_1 + \frac{14}{41}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$$

$$y_1 = 6 \tag{16}$$

$$y_2 = 1 - \frac{9}{4} = -\frac{5}{4} \tag{17}$$

$$\begin{aligned} y_3 &= -2 - \frac{3}{2} + \frac{70}{164} \\ &= -\frac{7}{2} + \frac{35}{82} = \frac{35 - 287}{82} = -\frac{252}{82} = -\frac{126}{41} \end{aligned} \tag{18}$$

Backward Substitution:

$$UX = Y \tag{19}$$

$$\begin{bmatrix} 8 & 5 & 1 \\ 0 & \frac{41}{8} & \frac{29}{8} \\ 0 & 0 & \frac{308}{41} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -\frac{5}{4} \\ -\frac{126}{41} \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 5x_2 + x_3 \\ \frac{41}{8}x_2 + \frac{29}{8}x_3 \\ \frac{308}{41}x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -\frac{5}{4} \\ -\frac{126}{41} \end{bmatrix}$$

$$x_3 = -\frac{41}{308} * \frac{126}{41} = -\frac{5166}{12628} = -\frac{9}{22} \tag{20}$$

$$\begin{aligned}
x_2 &= \frac{-\frac{5}{4} + \frac{261}{176}}{\frac{41}{8}} \\
&= \frac{261 - 220}{176} * \frac{8}{41} \\
&= \frac{41}{176} * \frac{8}{41} \\
&= \frac{328}{7216} = \frac{1}{22} \qquad (21)\\
x_1 &= \frac{6 - \frac{5}{22} + \frac{9}{22}}{8} \\
&= \frac{\frac{132+4}{22}}{8} \\
&= \frac{\frac{136}{22}}{8} \\
&= \frac{136}{176} = \frac{17}{22}
\end{aligned} \tag{22}$$

0.2 Crout Factorization

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \tag{23}$$

Starting defining the Crout Factorization of eq. (23):

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

$$\begin{aligned}
l_{11}U_{11} &= A_{11} \\
l_{11} * 1 &= 1 \\
&= 1
\end{aligned} \tag{25}$$

$$\begin{aligned} l_{21}U_{11} &= A_{21} \\ l_{21} * 1 &= -2 \\ &= -2 \end{aligned} \tag{26}$$

$$\begin{aligned} l_{31}U_{11} &= A_{31} \\ l_{31} * 1 &= 0 \\ &= 0 \end{aligned} \tag{27}$$

$$\begin{aligned} l_{11}U_{12} &= A_{12} \\ 1 * U_{12} &= -1 \\ &= -1 \end{aligned} \tag{28}$$

$$\begin{aligned} l_{11} * U_{13} &= A_{13} \\ 1 * U_{13} &= 0 \\ &= 0 \end{aligned} \tag{29}$$

$$\begin{aligned} l_{21}U_{12} + l_{22}U_{22} &= A_{22} \\ (-2) * (-1) + l_{22} * 1 &= 4 \\ l_{22} &= 4 - 2 = 2 \end{aligned} \tag{30}$$

$$\begin{aligned} l_{21}U_{13} + l_{22}U_{23} &= A_{23} \\ (-2) * 0 + 2 * U_{23} &= -2 \\ U_{23} &= -\frac{2}{2} = -1 \end{aligned} \tag{31}$$

$$\begin{aligned} l_{31}U_{12} + l_{32}U_{22} &= A_{32} \\ 0 * (-1) + l_{32} * 1 &= -1 \\ &= -1 \end{aligned} \tag{32}$$

$$\begin{aligned} l_{31}U_{13} + l_{32}U_{23} + l_{33}U_{33} &= A_{33} \\ (-1) * 0 + (-1) * (-1) + l_{33} * 1 &= 2 \\ l_{33} &= 2 - 1 = 1 \end{aligned} \tag{33}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

Solving:

$$Ax = B \tag{35}$$

$$Ax = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \tag{36}$$

Forward Substitution:

$$LY = B \tag{37}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ -2y_1 + 2y_2 \\ -y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$$y_1 = 0 \tag{38}$$

$$y_2 = -\frac{1}{2} \tag{39}$$

$$y_3 = 4 - \frac{1}{2} = \frac{7}{2} \tag{40}$$

Backward Substitution:

$$UX = Y \tag{41}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

$$x_3 = \frac{7}{2} \tag{42}$$

$$x_2 = -\frac{1}{2} + \frac{7}{2} = 3 \tag{43}$$

$$x_1 = 3 \tag{44}$$

0.3 Cholesky Factorization

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{45}$$

Starting defining the Cholesky Factorization of eq. (45):

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \tag{46}$$

$$l_{11} l_{11} = A_{11}$$

$$l_{11}^2 = 2$$

$$l_{11} = \sqrt{2} \tag{47}$$

$$\begin{aligned} l_{21}l_{11} &= A_{21} \\ l_{21} * \sqrt{2} &= -1 \\ l_{21} &= -\frac{1}{\sqrt{2}} \end{aligned} \tag{48}$$

$$\begin{aligned} l_{31}l_{11} &= A_{31} \\ l_{31} * \sqrt{2} &= 0 \\ l_{31} &= 0 \end{aligned} \tag{49}$$

$$\begin{aligned} l_{21}l_{21} + l_{22}l_{22} &= A_{22} \\ \left(-\frac{1}{\sqrt{2}}\right) * \left(-\frac{1}{\sqrt{2}}\right) + l_{22}^2 &= 2 \\ l_{22} &= \sqrt{2 - \frac{1}{2}} = \sqrt{\frac{3}{2}} \end{aligned} \tag{50}$$

$$\begin{aligned} l_{31}l_{21} + l_{32}l_{22} &= A_{32} \\ 0 * \left(-\frac{1}{\sqrt{2}}\right) + l_{32} * \sqrt{\frac{3}{2}} &= -1 \\ l_{32} &= -\frac{1}{\sqrt{\frac{3}{2}}} = -\sqrt{\frac{2}{3}} \end{aligned} \tag{51}$$

$$\begin{aligned} l_{31}l_{31} + l_{32}U_{32} + l_{33}U_{33} &= A_{33} \\ 0 * 0 + \left(-\sqrt{\frac{2}{3}}\right) * \left(-\sqrt{\frac{2}{3}}\right) + l_{33}^2 &= 2 \\ l_{33} &= \sqrt{2 - \frac{2}{3}} = \sqrt{\frac{4}{3}} \end{aligned} \tag{52}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{bmatrix} \tag{53}$$

0.4 Possibility of Cholesky Factorization

$$A = \begin{bmatrix} 4 & 0 & -3 \\ 0 & 6 & 2 \\ -3 & -2 & 7 \end{bmatrix} \tag{54}$$

A has a unique Cholesky factorization only if it is Symmetric Positive-Definite (SPD). Verifying if eq. (54) is SPD:

$$\begin{aligned} A^T &= A \\ A^T &= \begin{bmatrix} 4 & 0 & -3 \\ 0 & 6 & -2 \\ -3 & 2 & 7 \end{bmatrix} \neq A \end{aligned} \tag{55}$$

Since $A^T \neq A$, A do not have a Cholesky factorization.

0.5 SSD & SPD

0.5.1 Question (a)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{56}$$

Verifying if eq. (56) is SPD for $\vec{x} \neq 0$:

$$\begin{aligned} A^T &= A \\ A^T &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix} = A \end{aligned} \tag{57}$$

$$
\begin{aligned}
&\vec{x}^T A\vec{x} > 0\\
&\vec{x}^T A\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (58)\\
&\quad = \begin{bmatrix} 2x_1 - x_2; & -x_1 + 4x_2 + 2x_3; & 2x_2 + 6x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\\
&\quad = 2x_1^2 - x_1x_2 - x_1x_2 + 4x_2^2 + 2x_2x_3 + 2x_2x_3 + 6x_3^2\\
&\quad = 2x_1^2 - 2x_1x_2 + 4x_2^2 + 4x_2x_3 + 6x_3^2\\
&\quad = (x_1 - x_2)^2 + x_1^2 + 3x_2^2 + 4x_2x_3 + 6x_3^2\\
&\quad = (x_1 - x_2)^2 + x_1^2 + 2\left(x_2 + x_3\right)^2 + x_2^2 + 4x_3^2 > 0 \qquad (59)
\end{aligned}
$$

The eq. (56) is SPD. Verifying if it is SDD:

$$
|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \qquad (60)
$$

$$
\begin{aligned}
|2| &> |-1| + |0|\\
&> 1 \qquad (61)\\
|4| &> |-1| + |2|\\
&> 3 \qquad (62)\\
|6| &> |0| + |2|\\
&> 2 \qquad (63)
\end{aligned}
$$

eq. (56) is SDD too.

0.5.2 Question (b)

$$
B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & -1 \\ -3 & 2 & 0 \end{bmatrix} \qquad (64)
$$

Verifying if eq. (64) is SPD for $\vec{x} \neq 0$:

$$\begin{aligned} B^T &= B \\ B^T &= \begin{bmatrix} 1 & 4 & -3 \\ 2 & 6 & 2 \\ 0 & -1 & 0 \end{bmatrix} \neq B \end{aligned} \tag{65}$$

The eq. (64) is not SPD. Verifying if it is SDD:

$$|b_{ii}| > \sum_{j=1, j\neq i}^{n} |b_{ij}| \tag{66}$$

$$\begin{aligned} |2| &> |1| + |0| \\ &> 1 \end{aligned} \tag{67}$$

$$\begin{aligned} |6| &> |4| + |-1| \\ &> 5 \end{aligned} \tag{68}$$

$$\begin{aligned} |0| &< |-3| + |2| \\ &< 5 \end{aligned} \tag{69}$$

eq. (64) is not SDD too.

0.5.3 Question (c)

$$C = \begin{bmatrix} 5 & -3 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \tag{70}$$

Verifying if eq. (70) is SPD for $\vec{x} \neq 0$:

$$\begin{aligned} C^T &= C \\ C^T &= \begin{bmatrix} 5 & -3 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = C \end{aligned} \tag{71}$$

$$
\begin{aligned}
&\vec{x}^T C\vec{x} > 0 \\
&\vec{x}^T C\vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & -3 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} 5x_1 - 3x_2 + 2x_3; & -3x_1 + x_2; & 2x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= 5x_1^2 - 3x_1x_2 + 2x_1x_3 - 3x_1x_2 + x_2^2 + 2x_1x_3 \\
&= 5x_1^2 - 6x_1x_2 + 4x_1x_3 + x_2^2 \\
&= (x_1 - x_2)^2 + 4x_1^2 - 4x_1x_2 + 4x_1x_3 \\
&= (x_1 - x_2)^2 + 4x_1 (x_1 - x_2 + x_3) \qquad (73)
\end{aligned}
$$

The eq. (70) is not SPD. Verifying if it is SDD:

$$
|c_{ii}| > \sum_{j=1, j\neq i}^{n} |c_{ij}| \qquad (74)
$$

$$
\begin{aligned}
|5| &= |-3| + |2| \\
&= 5 \qquad (75)
\end{aligned}
$$

eq. (70) is not SDD too.

0.6 SDD & SPD with Variable

$$
Z = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix} \qquad (76)
$$

0.6.1 Question (a)

$$
\begin{aligned}
|z_{ii}| &> \sum_{j=1, j\neq i}^{n} |z_{ij}| && (77)\\
|5| &> |-2| + |2| \\
&> 4 && |6| > |-2| + |a| \\
a &< 6 - 2 \\
|a| &< 4 && |7| > |2| + |a| \\
|a| &< 7 - 2 \\
|a| &< 5 && (78)
\end{aligned}
$$

In order to eq. (76) be SSD, $|a| < 4$.

0.6.2 Question (b)

$$
\begin{aligned}
Z^T &= Z \\
Z^T &= \begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix} = Z && (79)
\end{aligned}
$$

$$
\begin{aligned}
&\vec{x}^T Z \vec{x} > 0 \\
&\vec{x}^T Z \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} && (80)\\
&\quad = \begin{bmatrix} 5x_1 - 2x_2 + 2x_3; & -2x_1 + 6x_2 + ax_3; & 2x_1 + ax_2 + 7x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&\quad = 5x_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_2 + 6x_2^2 + ax_2x_3 + 2x_1x_3 + ax_2x_3 + 7x_3^2 \\
&\quad = 5x_1^2 - 4x_1x_2 + 4x_1x_3 + 6x_2^2 + 2ax_2x_3 + 7x_3^2 \\
&\quad = 2\left(x_1 - x_2\right)^2 + 3x_1^2 + 4x_1x_3 + 4x_2^2 + 2ax_2x_3 + 7x_3^2 \\
&\quad = 2\left(x_1 - x_2\right)^2 + 2\left(x_1 + x_3\right)^2 + x_1^2 + 4x_2^2 + 2ax_2x_3 + 5x_3^2 && (81)
\end{aligned}
$$

In order to eq. (76) be SPD, $|a| \leq 4$, $a \in \mathbb{I}^*$.