Solutions to H.W. #7

1. For $x^{(k+1)} = \begin{bmatrix} \frac{2}{3} & 0 \\ 1 & -4/5 \end{bmatrix} x^{(k)} + \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $T = \begin{bmatrix} 2/3 & 0 \\ 1 & -4/5 \end{bmatrix}$,

and $\|T\|_\infty = \max\{2/3, 9/5\} = 9/5$.

Since $\|T\|_\infty$ is not less than 1, there is no guarantee that $\{x^{(k)}\}$ will converge for $x^{(0)} \in \mathbb{R}^2$.

2. The linear system may be represented as

$$\underbrace{\begin{bmatrix} 4 & -2 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{b}$$

$$\therefore L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $$T = -D^{-1} \cdot (L+U) = -\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 & -1/4 \\ 1/3 & 0 & -1/3 \\ -1/5 & -1/5 & 0 \end{bmatrix}$$

& $$c = D^{-1} \cdot b = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -1/5 \end{bmatrix}$$

(a) $\|T\|_\infty = \max\left\{ 3/4,\ 2/3,\ 2/5 \right\} = \frac{3}{4}$

Since $\|T\|_\infty < 1$, the Jacobi method is guaranteed to converge.

(b) Using $x^{(k+1)} = Tx^{(k)} + c$,

$$x^{(1)} = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 1/3 & 0 & -1/3 \\ -1/5 & -1/5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/3 \\ -3/5 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 1/3 & 0 & -1/3 \\ -1/5 & -1/5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/4 \\ 1/3 \\ -3/5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 19/60 \\ 37/60 \\ -19/60 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 1/3 & 0 & -1/3 \\ -1/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 19/60 \\ 37/60 \\ -19/60 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 31/80 \\ 49/90 \\ -29/75 \end{bmatrix}$$

(c) $$\|x - x^{(3)}\|_\infty \leq \frac{\|T\|_\infty^3}{1 - \|T\|_\infty} \cdot \|x^{(0)} - x^{(1)}\|_\infty$$

$$= \frac{\left(\frac{3}{4}\right)^3}{1 - \frac{3}{4}} \cdot \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/4 \\ 1/3 \\ -3/5 \end{bmatrix} \right\|_\infty$$

$$= \frac{\frac{27}{64}}{1/4} \cdot \frac{8}{5}$$

$$= \frac{27}{10}$$

(d) We require the minimum n such that

$$\frac{\|T\|_\infty^n}{1-\|T\|_\infty} \|x^{(0)}-x^{(1)}\|_\infty \leq 10^{-8}$$

$$\frac{\left(\frac{3}{4}\right)^n}{1-3/4} \cdot \frac{8}{5} \leq 10^{-8}$$

$$\frac{32}{5} \cdot \left(\frac{3}{4}\right)^n \leq 10^{-8}$$

$$\left(\frac{3}{4}\right)^n \leq \frac{5 \cdot 10^{-8}}{32}$$

$$n \ln\left(\frac{3}{4}\right) \leq \ln\left(\frac{5 \cdot 10^{-8}}{32}\right)$$

$$n \geq \frac{\ln\left(\frac{5\cdot 10^{-8}}{32}\right)}{\ln(3/4)} = 70.4840$$

Take $n = 71$ iterations.

(3) $$X^{(k+1)} = \begin{bmatrix} 0 & 1/4 & -1/4 \\ 1/3 & 0 & -1/3 \\ -1/5 & -2/5 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 1/2 \\ 1 \\ -1/3 \end{bmatrix}$$

with $X^{(0)} = [0 \quad 0 \quad 0]^T$, $X^{(1)} = [1/2 \quad 1 \quad -1/3]^T$.

$$\|T\|_\infty = \max\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{5}\right\} = \frac{2}{3}$$

& $\|X^{(0)} - X^{(1)}\|_\infty = 1$

We require the minimum n such that

$$\frac{\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} \cdot 1 \le 10^{-8}$$

$$3 \cdot \left(\frac{2}{3}\right)^n \le 10^{-8}$$

$$\left(\frac{2}{3}\right)^n \le \frac{10^{-8}}{3}$$

$$n \cdot \ln(2/3) \le \ln\left(\frac{10^{-8}}{3}\right)$$

$$n \ge \frac{\ln(10^{-8}/3)}{\ln(2/3)} = 48.1405$$

Take $n = 49$ iterations.

4. $\begin{bmatrix} 1 & 2 \\ \frac{9999}{10000} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

(a) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ \frac{9999}{10000} & 2 & \frac{30001}{10000} \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$

$$\therefore \vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(b) $\left[\begin{array}{cc|c} 1 & 2 & 31/10 \\ \frac{9999}{10000} & 2 & \frac{30001}{10000} \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 999 \\ 0 & 1 & -9959/20 \end{array}\right]$

$$\therefore \vec{x} = \begin{bmatrix} 999 \\ -9959/20 \end{bmatrix}$$

(c) Slightly perturbing $\vec{b}$ produces very large changes in $\vec{x}$. This is a sympton of an ill-conditioned coefficient matrix.

To confirm, we compute $\kappa(A)$.

For $A = \begin{bmatrix} 1 & 2 \\ \frac{9999}{10000} & 2 \end{bmatrix}$, $\|A\|_1 = 4$,

$$A^{-1} = \begin{bmatrix} 10000 & -10000 \\ -\frac{9999}{2} & 5000 \end{bmatrix}, \quad \|A^{-1}\|_1 = 15000$$

$$\therefore \; K(A) = \|A\|_1 \cdot \|A^{-1}\|_1 = 60{,}000$$

Since $K(A) \gg 1$, A is ill-conditioned!

5. $H_3 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$ & $H_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$

So $\|H_3\|_1 = \frac{11}{6}$, $\|H_3^{-1}\|_1 = 408$, and

$$K(H_3) = \|H_3\|_1 \cdot \|H_3^{-1}\|_1 = 748$$

6. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\frac{1}{n} \end{bmatrix}$, for $n > 1$

$\|A\|_\infty = \max\{2, 2+\frac{1}{n}\} = 2 + \frac{1}{n}$

$\det(A) = (1+\frac{1}{n}) \cdot 1 - 1 = \frac{1}{n}$

$\therefore A^{-1} = \frac{1}{1/n} \begin{bmatrix} 1+1/n & -1 \\ -1 & 1 \end{bmatrix} = n \begin{bmatrix} 1+\frac{1}{n} & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} n+1 & -n \\ -n & n \end{bmatrix}$

and $\|A^{-1}\|_\infty = \max\{2n+1, 2n\} = 2n+1$

$\therefore K(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$= (2+\frac{1}{n}) \cdot (2n+1)$

$= 4n + 2 + 2 + \frac{1}{n}$

$= 4n + 4 + \frac{1}{n}$

$\lim_{n\to\infty} K(A) = \lim_{n\to\infty} (4n + 4 + \frac{1}{n}) = \infty.$