Numerical Methods I
Class notes

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$A\vec{x} = b$, if A is SDD, converges.
$\vec{x}^{(k+1)} = T\vec{x}^{(k)} + \vec{c}$, if $||T|| < 1$, converges.
$||\vec{x} - \vec{x}^{(k)}|| \leq \frac{||T||^k}{1-||T||}||\vec{x}^{(0)} - x^{(1)}|| \leq T_{ol}$.

$$f'(x_0) = \lim h \to 0 \frac{f(x_0 + h) - f(x_0)}{h} \quad (1)$$

$$\approx \frac{f(x_0 + h) - f(x_0)}{h} \quad (2)$$

1.1 Basic Extrapolation Technique (Based on the order of term)

Suppose approximations based on different h's are known. We may extrapolate a more accurate approximation as follows:
Exact = Approx. + Error
If the error is of order $O(h^n)$,
Error $\propto h^n$
Error $= kh^n$
Exact = Approx. $+kh^n$
Using this equation with a pair of values for Approx and h, we may obtain a better approximation by solving for Exact.

Example:

$$f(x) = ln(x + 2) \quad (3)$$

$$h = 0.1, f'(0) \approx 0.5004172928(A_{0.1}) \quad (4)$$

$$h = 0.01, f'(0) \approx 0.5000041667(A_{0.01}) \quad (5)$$

Exact $= A_{0.1} + k * (0.1)^2$
Exact $= A_{0.1} + k * (0.01)$

$$\frac{E - A_{0.1}}{0.01} = k \tag{6}$$

$$E = A_{0.01} + \frac{E - A_{0.1}}{0.01} * 0.0001 \tag{7}$$

$$E = A_{0.01} + 0.01E - 0.01A_{0.1} \tag{8}$$

$$0.99E = A_{0.01} - 0.01A_{0.1} \tag{9}$$

$$E = \frac{A_{0.01} - 0.01A_{0.1}}{0.99} = 0.4999999937 \tag{10}$$

1.2 Numerical Integration

Our goal is to approximate $\int_a^b f(x)dx$, where $f(x)$ is continuous on $a \leq x \leq b$. Using Rhemann sums $\int_a^b f(x)dx \approx \sum_{k=1}^n f(x_k^* \Delta X)$. x_k^* means the chosen one from the k^{th}.

2 Newton-Cotes Closed Methods

The main characteristics of theses methods ares:

1. Both $x = a$ and $x = b$ are used in the approximation;
2. $\int_a^b f(x)dx \approx \sum_{i=1}^b w_i f(x_i)$, w_i: weights, x_i: absiccas (specially chosen x-values).

2.1 Trapezoidal Rule

Using the secant-line approximation of $f(x)$, through $(a, f(a)), (b, f(b))$ we get the following:

$$\int_a^b f(x)dx \approx \frac{1}{2}h(f(a) + f(b)) = \frac{h}{2}f(a) + \frac{h}{2}f(b) \tag{11}$$

This is the **basic trapezoidal method**. If n subintervals are used, with uniform width $h = \Delta x = \frac{b-a}{n}$, and the trapezoidal rule is applied to each interval, we get:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left(f(x_0 + f(x_1))\right) + \frac{h}{2}\left(f(x_1 + f(x_2))\right) + \cdots + \frac{h}{2}\left(f(x_{n-1} + f(x_n))\right) \tag{12}$$

$$\approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \tag{13}$$

$$\approx \frac{h}{2}\left[f(x_0) + 2\left(\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right] \tag{14}$$

This is the **composite trapezoidal method**.

2.2 Simpson's $\frac{1}{3}$ Rule

Using the quadratic approximation:

$$\begin{aligned}\int_a^b f(x)dx \approx \sum_{-h}^{h} ax^2 + bx + cdx = & \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_{-h}^{h} \\ & \left(\frac{a}{3}h^3 + \frac{b}{2}h^2 + ch\right) - \left(\frac{a}{3}(-h)^3 + \frac{b}{2}(-h)^2 + c(-h)\right) \\ & \frac{a}{3}h^3 + \frac{b}{2}h^2 + ch - \left(-\frac{a}{3}h^3 + \frac{b}{2}h^2 - ch\right) \\ & \frac{2}{3}ah^3 + 2ch \\ & \frac{h}{3}\left(2ah^2 + 6c\right)\end{aligned} \tag{15}$$

Since the quadratic passes through $(-h, y_0), (0, y_1), (h, y_2)$:

$$\begin{aligned} y_0 &= ah^2 - bh + c \\ y_1 &= c \\ y_2 &= ah^2 + bh + c \\ y_0 + 4y_1 + y_2 = (ah^2 - bh + c) + 4c + (ah^2 + bh + c) &= 2ab^2 + 6c \end{aligned} \tag{16}$$

With an even number of subintervals, the composite simpson's $\frac{1}{3}$ rule is:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right] \tag{17}$$

3 NOV 05

If a cubic ($3^{rd} degree$) polynomial is used to approximate f(x) over [a,b], we get the Simpson's 3/8 rule:

$$\text{Simple:} \quad \int_a^b f(x)dx \approx \frac{3h}{8}\left[f(a) + 3f(a+h) + 3 \right. \tag{18}$$

Composite (with n, a multiple of 3, subintervals):

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(\right. \tag{19}$$

Example:

$$\int_0^6 e^{\frac{x}{2}}dx \approx \frac{3}{8}\left[1 + 3e^{\frac{1}{2}} + 3e + 2e^{\frac{3}{2}} + 3e^2 + 3e^{\frac{5}{2}} + e^3\right] \tag{20}$$

$$\approx 38.1992154 \tag{21}$$

$$\text{Abs. error} = 0.0281416062 \tag{22}$$

- Determining an $\ddot{h}$to meet some tolerance using the composite trapezoidal rule to approximate $\int_0^6 e^{\frac{x}{2}}dx$, to obtain an absolute error of at most 10^{-6}, we need:

$$\frac{h^2}{12}(b-a)f''(c) \le 10^{-6} \tag{23}$$

$$\frac{h^2}{12}(6-0)f''(c) \le 10^{-6}$$

  Using the $\max_{[a,b]} |f''(x)|$, we will use a conservative estimate for $h\left(\frac{b-a}{n}\right)$: $f(x) = e^{\frac{x}{2}}$, $f'(x) = \frac{1}{2}e^{\frac{x}{2}}$, $f''(x) = \frac{1}{4}e^{\frac{x}{2}}$. (max occurs at right end point)

Since $\frac{1}{4}e^{\frac{x}{2}}$ is monotone increasing, $\max_{[0,6]} \frac{1}{4}e^{\frac{x}{2}} = \frac{1}{4}e^3$.

$$\begin{aligned} \frac{h^2}{12} * 6 * \frac{1}{4}e^3 &\leq 10^{-6} \\ h^2 * \frac{1}{8}e^3 &\leq 10^{-6} \\ h^2 &\leq \frac{8 * 10^{-6}}{e^3} \\ h \leq \sqrt{\frac{8 * 10^{-6}}{e^3}} &= 0.00063 \end{aligned} \tag{24}$$

Moreover, $\frac{6-0}{n} \leq 0.00063, \frac{n}{6} \geq \frac{1}{0.00063}, n \geq 9507.098$. Take $n = 9508$