Numerical Methods I
Homework Problem Set #8

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1 Two-Point Backward Difference Approximation

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h} \tag{1}$$
$$f(x) = e^{-x} ln(x+2) \tag{2}$$
$$f'(x) = -e^{-x} ln(x+2) + \frac{e^{-x}}{x+2} \tag{3}$$
$$x = x_0 = 2 \tag{4}$$
$$f(2) = e^{-2} ln(4) = 0.18761 \tag{5}$$
$$f'(2) = -e^{-2} ln(4) + \frac{e^{-2}}{4} = -0.18761 + 0.03383 = -0.15378 \tag{6}$$

Using $h = 0.1$:

$$f'(2) \approx \frac{f(2) - f(1.9)}{0.1} \tag{7}$$
$$f(1.9) = e^{-1.9} ln(3.9) = 0.20356 \tag{8}$$
$$\begin{aligned} f'(2) &\approx \frac{f(2) - f(1.9)}{0.1} \\ &\approx \frac{0.18761 - 0.20356}{0.1} = -0.15945 \end{aligned} \tag{9}$$
$$E_{h=0.1} = |-0.15945 - (-0.15378)| = 0.00567 \tag{10}$$

Using $h = 0.05$

$$\begin{aligned}
f'(2) &\approx \frac{f(2) - f(1.95)}{0.05} && (11)\\
f(1.95) &= e^{-1.95} ln(3.95) = 0.19544 && (12)\\
f'(2) &\approx \frac{f(2) - f(1.95)}{0.05} \\
&\approx \frac{0.18761 - 0.19544}{0.05} = -0.15659 && (13)\\
E_{h=0.05} &= |-0.15659 - (-0.15378)| = 0.00281 && (14)
\end{aligned}$$

Using $h = 0.025$

$$\begin{aligned}
f'(2) &\approx \frac{f(2) - f(1.975)}{0.025} && (15)\\
f(1.975) &= e^{-1.975} ln(3.975) = 0.19149 && (16)\\
f'(2) &\approx \frac{f(2) - f(1.975)}{0.025} \\
&\approx \frac{0.18761 - 0.19149}{0.025} = -0.15518 && (17)\\
E_{h=0.025} &= |-0.15518 - (-0.15378)| = 0.00140 && (18)
\end{aligned}$$

2 Three-Point Central Difference Approximation

$$\begin{aligned}
f'(x_0) &\approx \frac{f(x_0 + h) - f(x_0 - h)}{2h} && (19)\\
f(x) &= 2\sin(x) - \sqrt{2x+3} && (20)\\
f'(x) &= 2\cos(x) - \frac{1}{\sqrt{2x+3}} && (21)\\
x &= x_0 = 0 && (22)\\
f'(0) &= 2\cos(0) - \frac{1}{\sqrt{2(0)+3}} = 2 - \left(\sqrt{3}\right)^{-1} = 1.42265 && (23)
\end{aligned}$$

Using $h = 0.1$:

$$
\begin{aligned}
f'(0) &\approx \frac{f(0.1)-f(-0.1)}{0.1} && (24)\\
f(0.1) &= 2\sin(0.1) - \sqrt{2(0.1)+3} = 0.19967 - \sqrt{3.2} = -1.58919 && (25)\\
f(-0.1) &= 2\sin(-0.1) - \sqrt{2(-0.1)+3}\\
&= -0.19967 - \sqrt{2.8} = -1.87299 && (26)\\
f'(0) &\approx \frac{f(0.1)-f(-0.1)}{0.2}\\
&\approx \frac{-1.58919-(-1.87299)}{0.2} = 1.41900 && (27)\\
E_{h=0.1} &= |1.41900 - 1.42265| = 0.00365 && (28)
\end{aligned}
$$

Using $h = 0.05$

$$
\begin{aligned}
f'(0) &\approx \frac{f(0.05)-f(-0.05)}{0.1} && (29)\\
f(0.05) &= 2\sin(0.05) - \sqrt{2(0.05)+3} = 0.09996 - \sqrt{3.1} = -1.66072 && (30)\\
f(-0.05) &= 2\sin(-0.05) - \sqrt{2(-0.05)+3}\\
&= -0.09996 - \sqrt{2.9} = -1.80290 && (31)\\
f'(0) &\approx \frac{f(0.05)-f(-0.05)}{0.1}\\
&\approx \frac{-1.66072-(-1.80290)}{0.1} = 1.42174 && (32)\\
E_{h=0.05} &= |1.42174 - 1.42265| = 0.00091 && (33)
\end{aligned}
$$

Using $h = 0.025$

$$
\begin{aligned}
f'(0) &\approx \frac{f(0.025) - f(-0.025)}{0.05} && (34)\\
f(0.025) &= 2\sin(0.025) - \sqrt{2(0.025)+3}\\
&= 0.04999 - \sqrt{3.05} = -1.69643 && (35)\\
f(-0.025) &= 2\sin(-0.025) - \sqrt{2(-0.025)+3}\\
&= -0.04999 - \sqrt{2.95} = -1.76755 && (36)\\
f'(0) &\approx \frac{f(0.025) - f(-0.025)}{0.05}\\
&\approx \frac{-1.69643 - (-1.76755)}{0.05} = 1.42242 && (37)\\
E_{h=0.025} &= |1.42242 - 1.42265| = 0.00023 && (38)
\end{aligned}
$$

3 Estimation Using 2-Point Backward Approximation

$$
\begin{aligned}
f'(x_0) &\approx \frac{f(x_0) - f(x_0 - h)}{h} && (39)\\
f(x) &= f(T) = a && (40)\\
h &= 10 && (41)
\end{aligned}
$$

Using $x_0 = 20$:

$$
\begin{aligned}
f'(20) &\approx \frac{f(20) - f(10)}{10} && (42)\\
f(20) &= 1482 && (43)\\
f(10) &= 1447 && (44)\\
f'(20) &\approx \frac{1482 - 1447}{10} = 3.5 && (45)
\end{aligned}
$$

Using $x_0 = 40$:

$$f'(40) \approx \frac{f(40) - f(30)}{10} \tag{46}$$
$$f(40) = 1529 \tag{47}$$
$$f(30) = 1509 \tag{48}$$
$$f'(40) \approx \frac{1529 - 1509}{10} = 2 \tag{49}$$

Using $x_0 = 60$:

$$f'(60) \approx \frac{f(60) - f(50)}{10} \tag{50}$$
$$f(60) = 1511 \tag{51}$$
$$f(50) = 1542 \tag{52}$$
$$f'(60) \approx \frac{1511 - 1542}{10} = -3.1 \tag{53}$$

4 $O(h^4)$ Methods

Since $f'(x) = v$ (velocity), $f''(x) = a$ (acceleration). The appropriate $O(h^4)$ method is the 5-point central difference:

$$f'(x_0) \approx \frac{f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)}{12h} \tag{54}$$
$$f''(x_0) \approx \frac{-f(x_0 - 2h) + 16f(x_0 - h) - 30f(x_0) + 16f(x_0 + h) - f(x_0 + 2h)}{12h^2} \tag{55}$$
$$h = 0.52 \tag{56}$$

Using $x_0 = 1.04$ to discover the velocity:

$$f'(1.04) \approx \frac{f(0) - 8f(0.52) + 8f(1.56) - f(2.08)}{6.24} \tag{57}$$
$$f'(1.04) \approx \frac{153 - 8(185) + 8(249) - 261}{6.24}$$
$$\approx \frac{153 - 1480 + 1992 - 261}{6.24} = \frac{404}{6.24} = 64.744 \tag{58}$$

Using $x_0 = 1.04$ to discover the acceleration:

$$f''(1.04) \approx \frac{-f(0) + 16f(0.52) - 30f(1.04) + 16f(1.56) - f(2.08)}{3.2448} \tag{59}$$
$$f''(1.04) \approx \frac{-153 + 16(185) - 30(208) + 16(249) - 261}{3.2448}$$
$$f''(1.04) \approx \frac{-153 + 2960 - 6240 + 3984 - 261}{3.2448} = \frac{280}{3.2448} = 89.374 \tag{60}$$

Using $x_0 = 1.56$ to discover the acceleration:

$$f''(1.56) \approx \frac{-f(0.52) + 16f(1.04) - 30f(1.56) + 16f(2.08) - f(2.60)}{3.2448} \tag{61}$$
$$f''(1.56) \approx \frac{-185 + 16(208) - 30(249) + 16(261) - 271}{3.2448}$$
$$f''(1.56) \approx \frac{-185 + 3328 - 7470 + 4176 - 271}{3.2448} = \frac{280}{3.2448} = -130.05 \tag{62}$$

5 Extrapolation Technique in $O(h^2)$

$$\text{Exact} = \text{Approximation} + kh^n \tag{63}$$
$$A_{0.05} = 4.15831 \tag{64}$$
$$A_{0.025} = 4.16361 \tag{65}$$
$$\text{Exact} = A_{0.05} + k(0.05)^2$$
$$k = \frac{Exact - A_{0.05}}{0.0025} \tag{66}$$
$$\text{Exact} = A_{0.025} + k(0.025)^2$$
$$= A_{0.025} + \left(\frac{Exact - A_{0.05}}{0.0025}\right)(0.025)^2$$
$$\text{Exact} - 0.25(\text{Exact}) = A_{0.025} - (0.25)A_{0.05}$$
$$\text{Exact} = \frac{A_{0.025} - (0.25)A_{0.05}}{0.75}$$
$$= \frac{4.16361 - 1.03958}{0.75} = 4.16538 \tag{67}$$

6 Extrapolation Technique in $O(h^4)$

$$\text{Exact} = \text{Approximation} + kh^n \tag{68}$$
$$A_{0.01} = -3.2213 \tag{69}$$
$$A_{0.005} = -3.3245 \tag{70}$$
$$\text{Exact} = A_{0.01} + k(0.01)^4$$
$$k = \frac{Exact - A_{0.01}}{(0.01)^4} \tag{71}$$
$$\text{Exact} = A_{0.005} + k(0.005)^4$$
$$= A_{0.005} + \left(\frac{Exact - A_{0.01}}{(0.01)^4}\right)(0.005)^4$$
$$\text{Exact} - 0.0625(\text{Exact}) = A_{0.005} - (0.0625)A_{0.01}$$
$$\text{Exact} = \frac{A_{0.005} - (0.0625)A_{0.01}}{0.9375}$$
$$= \frac{-3.3245 - (-2.0133)}{0.9375} = -3.3314 \tag{72}$$