Numerical Methods I
Homework Problem Set #12

Jonathan Henrique Maia de Moraes (ID: 1620855)

12/10/2015

Problem Set #12

1 Difference between IVP and BVP

- IVP establish a initial value condition to a problem. Examples:

  1. $\frac{dy}{dt} = y + ty - y^2$, $y(0) = 1$;
  2. $y'' + ty' - 3y = t^2$, $y(0) = 3$, $y'(0) = 4$.

- BVP establish a boundary (2 or more values) condition to a problem. Examples:

  1. $2xy'' + y' - 4y = x + 1$, $y(1) = 2$, $y(5) = 8$;
  2. $y'' + y = 0$, $y(0) = 2$, $y(\pi/2) = -1$;

2 Difference between Dirichlet & Robin boundary conditions

- Dirichlet BVP, or first-type BVP, specifies the values that a solution needs to take on along the boundary of the domain: $y = f$.

- Robin BVP, or third-type BVP, specifies a linear combination of the values of a function and the values of its derivative on the boundary of the domain: $c_0 y + c_1 \frac{\partial y}{\partial n} = f$.

3 Discretizing the problem in BVP context

Discretizing the problem is when we approximate in a sequence of values: $y_1, \ldots, y_{n-1}$.

4 Approximation in a BVP using Finite Differences

$$2xy'' + y' - 4y = x + 1 \qquad y(1) = 2 \qquad y(5) = 8 \qquad n = 4 \tag{1}$$

$$y' = \frac{y_{i+1} - y_{i-1}}{2h} \qquad y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{2}$$

$$h = \frac{b-a}{n} = \frac{5-1}{4} = 1 \qquad i = 1, \ldots, 3 \tag{3}$$

$$2x_i \left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left(\frac{y_{i+1} - y_{i-1}}{2h} \right) - 4y_i = x_i + 1 \tag{4}$$

$$\begin{aligned}
i = 1, x = 2 : 2x_1 \left(\frac{y_2 - 2y_1 + y_0}{h^2} \right) + \left(\frac{y_2 - y_0}{2h} \right) - 4y_1 &= x_1 + 1 && (5)\\
2(2) \left(\frac{y_2 - 2y_1 + 0}{1^2} \right) + \left(\frac{y_2 - 0}{2(1)} \right) - 4y_1 &= 2 + 1 \\
4\left(y_2 - 2y_1\right) + \frac{y_2}{2} - 4y_1 &= 3 \\
4y_2 - 8y_1 + \frac{y_2}{2} - 4y_1 &= 3 \\
-12y_1 + \frac{9}{2}y_2 &= 3 && (6)\\
& && (7)
\end{aligned}$$

$$\begin{aligned}
i = 2, x = 3 : 2x_2\left(y_3 - 2y_2 + y_1\right) + \left(\frac{y_3 - y_1}{2} \right) - 4y_2 &= x_2 + 1 && (8)\\
2(3)\left(y_3 - 2y_2 + y_1\right) + \left(\frac{y_3 - y_1}{2} \right) - 4y_2 &= 3 + 1 \\
6\left(y_3 - 2y_2 + y_1\right) + \frac{y_3}{2} - \frac{y_1}{2} - 4y_2 &= 4 \\
6y_3 - 12y_2 + 6y_1 + \frac{y_3}{2} - \frac{y_1}{2} - 4y_2 &= 4 \\
\frac{11}{2}y_1 - 16y_2 + \frac{13}{2}y_3 &= 4 && (9)\\
& && (10)
\end{aligned}$$

$$i=3, x=4: 2x_3\left(y_4-2y_3+y_2\right)+\left(\frac{y_4-y_2}{2}\right)-4y_3=x_3+1 \tag{11}$$
$$2(4)\left(8-2y_3+y_2\right)+\left(\frac{8-y_2}{2}\right)-4y_3=4+1$$
$$8\left(8-2y_3+y_2\right)+4-\frac{y_2}{2}-4y_3=5$$
$$64-16y_3+8y_2+4-\frac{y_2}{2}-4y_3=5$$
$$\frac{15}{2}y_2-20y_3=-63 \tag{12}$$

$$\begin{bmatrix}-12 & \frac{9}{2} & 0\\ \frac{11}{2} & -16 & \frac{13}{2}\\ 0 & \frac{15}{2} & -20\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}3\\ 4\\ -63\end{bmatrix} \tag{13}$$

Using **GNU Octave** to solve eq. (13):

$$\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}0.24239\\ 1.31304\\ 3.64239\end{bmatrix} \tag{14}$$

5 Approximation & Error in a BVP using Finite Differences

$$y''+y=0 \qquad y(0)=2 \qquad y(\pi/2)=-1 \qquad n=4 \tag{15}$$

5.1 Question (a)

$$y''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} \tag{16}$$

$$h=\frac{b-a}{n}=\frac{(\pi/2)-0}{4}=\pi/8 \qquad i=1,\ldots,3 \tag{17}$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = 0 \tag{18}$$

$$i = 1, x = \pi/8 : \frac{y_2 - 2y_1 + y_0}{(\pi/8)^2} + y_1 = 0 \tag{19}$$

$$\frac{64(y_2 - 2y_1 + 2)}{\pi^2} + y_1 = 0$$

$$\frac{64}{\pi^2} y_2 - \frac{128}{\pi^2} y_1 + \frac{128}{\pi^2} + y_1 = 0$$

$$\left(1 - \frac{128}{\pi^2}\right) y_1 + \frac{64}{\pi^2} y_2 = -\frac{128}{\pi^2} \tag{20}$$

$$i = 2, x = \pi/4 : \frac{64(y_3 - 2y_2 + y_1)}{\pi^2} + y_2 = 0 \tag{21}$$

$$\frac{64}{\pi^2} y_3 - \frac{128}{\pi^2} y_2 + \frac{64}{\pi^2} y_1 + y_2 = 0$$

$$\frac{64}{\pi^2} y_1 + \left(1 - \frac{128}{\pi^2}\right) y_2 + \frac{64}{\pi^2} y_3 = 0 \tag{22}$$

$$i = 3, x = 3\pi/8 : \frac{64(y_4 - 2y_3 + y_2)}{\pi^2} + y_3 = 0 \tag{23}$$

$$\frac{64}{\pi^2}(-1) - \frac{128}{\pi^2} y_3 + \frac{64}{\pi^2} y_2 + y_3 = 0$$

$$\frac{64}{\pi^2} y_2 + \left(1 - \frac{128}{\pi^2}\right) y_3 = \frac{64}{\pi^2} \tag{24}$$

$$\begin{bmatrix} \left(1 - \frac{128}{\pi^2}\right) & \frac{64}{\pi^2} & 0 \\ \frac{64}{\pi^2} & \left(1 - \frac{128}{\pi^2}\right) & \frac{64}{\pi^2} \\ 0 & \frac{64}{\pi^2} & \left(1 - \frac{128}{\pi^2}\right) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{128}{\pi^2} \\ 0 \\ \frac{64}{\pi^2} \end{bmatrix} \tag{25}$$

Using **GNU Octave** to solve eq. (25):

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.46862 \\ 0.71077 \\ -0.15670 \end{bmatrix} \tag{26}$$

5.2 Question (b)

$$y = 2cos(x) - sin(x) \tag{27}$$

Interior Node 1:

$$x = \pi/8 \tag{28}$$
$$\text{Approx.} = 1.46862 \tag{29}$$
$$\text{Exact} = 2cos(\pi/8) - sin(\pi/8) = 1.46508 \tag{30}$$
$$E = |1.46508 - 1.46862| = 0.00354 \tag{31}$$

Interior Node 2:

$$x = \pi/4 \tag{32}$$
$$\text{Approx.} = 0.71077 \tag{33}$$
$$\text{Exact} = 2cos(\pi/4) - sin(\pi/4) = 0.70710 \tag{34}$$
$$E = |0.70710 - 0.71077| = 0.00367 \tag{35}$$

Interior Node 3:

$$x = 3\pi/8 \tag{36}$$
$$\text{Approx.} = -0.15670 \tag{37}$$
$$\text{Exact} = 2cos(3\pi/8) - sin(3\pi/8) = -0.15851 \tag{38}$$
$$E = |-0.15851 - (-0.15670)| = 0.00181 \tag{39}$$

6 Cauchy-Euler equi-dimensional approximation using Finite Differences

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} = 0 \qquad u(a) = u_0 \qquad u(b) = u_1 \tag{40}$$

$$a = 4 \qquad b = 8 \tag{41}$$
$$u_0 = 45^oC \qquad u_1 = 70^oC \tag{42}$$
$$u(4) = 45^oC \qquad u(8) = 70^oC \tag{43}$$
$$\frac{d^2u}{dr^2} = u''(r) \qquad \frac{du}{dr} = u'(r) \tag{44}$$
$$h = 1 \qquad i = 1, \ldots, 3 \tag{45}$$

$$u' = \frac{u_{i+1} - u_{i-1}}{2h} \qquad u'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{46}$$

$$r_i^2 \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + r_i \frac{u_{i+1} - u_{i-1}}{2h} = 0 \tag{47}$$

$$\begin{aligned}
i = 1, r = 5 : \; & r_1^2 \frac{u_2 - 2u_1 + u_0}{h^2} + r_1 \frac{u_2 - u_0}{2h} = 0 && (48)\\
& (5)^2 \frac{u_2 - 2u_1 + 45}{(1)^2} + (5) \frac{u_2 - 45}{2(1)} = 0 \\
& 25(u_2 - 2u_1 + 45) + \frac{5}{2} u_2 - \frac{225}{2} = 0 \\
& 25u_2 - 50u_1 + 1125 + \frac{5}{2} u_2 - \frac{225}{2} = 0 \\
& -50u_1 + \frac{55}{2} u_2 = -1012.5 && (49)
\end{aligned}$$

$$\begin{aligned}
i = 2, r = 6 : \; & r_2^2 (u_3 - 2u_2 + u_1) + r_2 \frac{u_3 - u_1}{2} = 0 && (50)\\
& (6)^2 (u_3 - 2u_2 + u_1) + (6) \frac{u_3 - u_1}{2} = 0 \\
& 36u_3 - 72u_2 + 36u_1 + 3u_3 - 3u_1 = 0 \\
& 33u_1 - 72u_2 + 39u_3 = 0 && (51)
\end{aligned}$$

$$\begin{aligned}
i = 3, r = 7 : \; & r_3^2 (u_4 - 2u_3 + u_2) + r_3 \frac{u_4 - u_2}{2} = 0 && (52)\\
& (7)^2 (70 - 2u_3 + u_2) + (7) \frac{70 - u_2}{2} = 0 \\
& 3430 - 98u_3 + 49u_2 + 245 - \frac{7}{2} u_2 = 0 \\
& \frac{105}{2} u_2 - 98u_3 = -3675 && (53)
\end{aligned}$$

$$\begin{bmatrix} -50 & \frac{55}{2} & 0 \\ 33 & -72 & 39 \\ 0 & \frac{105}{2} & -98 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1012.5 \\ 0 \\ -3675 \end{bmatrix} \tag{54}$$

Using **GNU Octave** to solve eq. (54):

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 55.80868^o C \\ 64.65215^o C \\ 72.13508^o C \end{bmatrix} \tag{55}$$