Numerical Methods I
Homework Problem Set #4

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Problem Set #4

0.1 Fixed-Point Error Bound

0.1.1 Question (a)

$$|E_{fixed\ point}| \leq \frac{k^n}{1-k}|x_0 - x_1| \tag{1}$$

For $k = \frac{1}{3}$, $n = 7$, $x_0 = 1$, and $x_1 = \frac{1}{2}$, the error bound is:

$$\begin{aligned}
|E_{fixed\ point}| &\leq \frac{\left(\frac{1}{3}\right)^7}{1-\frac{1}{3}}\left|1 - \frac{1}{2}\right| \\
&\leq \frac{\frac{1}{2187}}{\frac{2}{3}} * \frac{1}{2} \\
&\leq \frac{3}{8748} \\
&\leq 3.4294 * 10^{-4}
\end{aligned} \tag{2}$$

0.1.2 Question (b)

$$\begin{aligned}
|E| &\leq \frac{k^n}{1-k}|x_0 - x_1| \\
\frac{|E|}{|x_0 - x_1|} * (1-k) \leq k^n & \\
k^n &\geq \frac{|E|}{|x_0 - x_1|} * (1-k) \\
\ln(k^n) &\geq \ln\left(\frac{|E|}{|x_0 - x_1|} * (1-k)\right) \\
n * \ln(k) &\geq \ln\left(\frac{|E|}{|x_0 - x_1|} * (1-k)\right) \\
n &\geq \frac{\ln\left(\frac{|E|}{|x_0 - x_1|} * (1-k)\right)}{\ln(k)} \qquad (3)
\end{aligned}$$

For $k = \frac{1}{3}$, $E \leq 10^{-6}$, $x_0 = 1$, and $x_1 = \frac{1}{2}$, the minimum number of iterations necessary is:

$$\begin{aligned}
\lceil n \rceil &= \frac{\ln\left(\frac{|10^{-6}|}{|1-\frac{1}{2}|} * \left(1 - \frac{1}{3}\right)\right)}{\ln\left(\frac{1}{3}\right)} \\
&= \frac{\ln\left(\frac{|10^{-6}|}{|5*10^{-1}|} * \left(\frac{2}{3}\right)\right)}{\ln\left(\frac{1}{3}\right)} \\
&= \frac{\ln\left(\frac{4*10^{-6}}{3}\right)}{\ln\left(\frac{1}{3}\right)} \\
&= 12.314 \\
n &= 13 \qquad (4)
\end{aligned}$$

0.2 Unique Fixed-Point in an Interval

$$g(x) = \frac{1}{5}(x+1)^{\frac{3}{2}} \qquad , [0,1] \qquad (5)$$

0.2.1 Question (a)

First, it is necessary to check if $g(x)$ is an element of $[a, b]$ for all x in $[a, b]$. Starting with $x = a = 0$:

$$\begin{aligned} g(x) &= \frac{1}{5}(x+1)^{\frac{3}{2}} \\ g(0) &= \frac{1}{5}(0+1)^{\frac{3}{2}} \\ &= \frac{1}{5} = 0.2 \end{aligned} \tag{6}$$

Since 0.2 is an element of $[a, b]$, the next step is to check with $x = b = 1$:

$$\begin{aligned} g(x) &= \frac{1}{5}(x+1)^{\frac{3}{2}} \\ g(1) &= \frac{1}{5}(1+1)^{\frac{3}{2}} \\ &= \frac{2^{\frac{3}{2}}}{5} \\ &= 0.56569 \end{aligned} \tag{7}$$

Since 0.56569 is an element of $[a, b]$, it is correct to assume the eq. (5) can have at least one fixed-point on the indicated interval if it is monotone. Derivating eq. (5):

$$\begin{aligned} g(x) &= \frac{1}{5}(x+1)^{\frac{3}{2}} \\ g'(x) &= \frac{1}{5} * \frac{3}{2} * \sqrt{x+1} \\ &= \frac{3 * \sqrt{x+1}}{10} \end{aligned} \tag{8}$$

Looking at the eq. (8), it is correct to affirm that the eq. (5) is monotone in $[a, b]$. The eq. (5) have at least one fixed-point.

To prove the uniqueness it is necessary to guarantee that $|g'(x)| < 1$ for all x in $[a, b]$. Starting with $x = a = 0$:

$$\begin{aligned} g'(x) &= \frac{3 * \sqrt{x+1}}{10} \\ g'(0) &= \frac{3 * \sqrt{0+1}}{10} \\ &= \frac{3 * \sqrt{1}}{10} \\ &= 0.3 \end{aligned} \tag{9}$$

Since $0.3 < 1$, the next step is to check $g'(x)$ with $x = b = 1$:

$$\begin{aligned} g'(x) &= \frac{3 * \sqrt{x+1}}{10} \\ g'(1) &= \frac{3 * \sqrt{1+1}}{10} \\ &= \frac{3 * \sqrt{2}}{10} \\ &= 0.42426 \end{aligned} \tag{10}$$

Since $0.42426 < 1$, and 0.4246 is the maximum value in $[a, b]$, we can affirm that eq. (5) have an unique fixed-point.

0.2.2 Question (b)

$$x_{n+1} = \frac{1}{5}(x_n + 1)^{\frac{3}{2}} \tag{11}$$

$$\begin{aligned} x_1 &= \frac{1}{5}(x_0 + 1)^{\frac{3}{2}} \\ &= \frac{1}{5} = 0.2 \end{aligned} \tag{12}$$

For $k = 0.42426$, $n = 5$, $x_0 = 0$, and $x_1 = 0.2$, the error bound is:

$$\begin{aligned}
|E| &\leq \frac{(0.42426)^5}{1-0.42426}|0-0.2| \\
&\leq \frac{(0.42426)^5}{1-0.42426}|0-0.2| \\
&\leq \frac{0.013745 * 0.2}{0.57574} \\
&\leq \frac{0.0027491}{0.57574} \\
&\leq 0.0047749
\end{aligned} \tag{13}$$

0.2.3 Question (c)

For $k = \frac{1}{3}$, $E \leq 10^{-6}$, $x_0 = 1$, and $x_1 = \frac{1}{2}$, the minimum number of iterations necessary is:

$$\begin{aligned}
\lceil n \rceil &= \frac{\ln\left(\frac{|10^{-8}|}{|0-0.2|} * (1-0.42426)\right)}{\ln(0.42426)} \\
&= \frac{\ln(5 * 10^{-8} * 0.57574)}{\ln(0.42426)} \\
&= 20.251 \\
n &= 21
\end{aligned} \tag{14}$$

0.3 Order of Convergence

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) \tag{15}$$

Verifying if $x = \sqrt{a}$ is a fixed point of eq. (15):

$$
\begin{aligned}
g(\sqrt{a}) &= \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) \\
&= \frac{1}{2}\left(\frac{(\sqrt{a})^2 + a}{\sqrt{a}}\right) \\
&= \frac{2a}{2\sqrt{a}} \\
&= \frac{2a}{2\sqrt{a}} \\
&= \frac{a^1}{a^{1/2}} \\
&= a^{1/2} = \sqrt{a} \qquad (16)
\end{aligned}
$$

$x = \sqrt{a}$ is a fixed point of eq. (15), as can be seen in eq. (16). To discover the order of convergence, it is necessary to discover how many derivates of eq. (15) have $x = \sqrt{a}$ as root:

$$
\begin{aligned}
g(x) &= \frac{1}{2}\left(x + \frac{a}{x}\right) \\
&= \frac{x}{2} + \frac{a}{2x} \\
g'(x) &= \frac{1}{2} - \frac{a}{2x^2} \qquad (17)
\end{aligned}
$$

Verifying if $x = \sqrt{a}$ is a root of eq. (17):

$$
\begin{aligned}
g'(\sqrt{a}) &= \frac{1}{2} - \frac{a}{2\left(\sqrt{a}\right)^2} \\
&= \frac{1}{2} - \frac{a}{2a} \\
&= \frac{1}{2} - \frac{1}{2} = 0 \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
g'(x) &= \frac{1}{2} - \frac{a}{2x^2} \\
g''(x) &= \frac{a}{x^3} \qquad (19)
\end{aligned}
$$

Verifying if $x = \sqrt{a}$ is a root of eq. (19):

$$\begin{aligned} g''(\sqrt{a}) &= \frac{a}{(\sqrt{a})^3} \\ &= \frac{a}{a\sqrt{a}} \\ &= \frac{1}{\sqrt{a}} \neq 0 \end{aligned} \tag{20}$$

The order of convergence of $x = \sqrt{a}$ in eq. (15) is 2.

0.4 Rate of Convergence

$$x_{n+1} = 0.4 + x_n - 0.1x_n^2, n \geq 0 \tag{21}$$

Verifying if $x = 2$ is a fixed point of eq. (21):

$$\begin{aligned} f(2) &= 0.4 + 2 - 0.1 * 2^2 \\ f(2) &= 2.4 - 0.4 = 2 \end{aligned} \tag{22}$$

$x = 2$ is a fixed point of eq. (21), as can be seen in eq. (22). To discover the order of convergence, it is necessary to discover how many derivates of eq. (21) have $x = 2$ as root:

$$\begin{aligned} f(x) &= 0.4 + x - 0.1x^2 \\ f'(x) &= 1 - 0.2x \end{aligned} \tag{23}$$

Verifying if $x = 2$ is a root of eq. (23):

$$\begin{aligned} f'(1) &= 1 - 0.2 * 2 \\ &= 1 - 0.4 = 0.6 \neq 0 \end{aligned} \tag{24}$$

The order of convergence of $x = 2$ in eq. (21) is 1. To determine the rate of convergence, or the asymptotic error constant, it is necessary to discover the result of:

$$rate = \frac{g^{(R)}(\alpha)}{R!} \tag{25}$$

Since $R = 1$ and $\alpha = 2$, the rate of convergence is:

$$\begin{aligned} rate &= \frac{g'(2)}{1!} \\ &= \frac{0.6}{1!} = 0.6 \end{aligned} \tag{26}$$

0.5 Comparisson of Convergence's Velocity

0.5.1 Question (a)

$$x_{n+1} = x_n + 1 - \frac{x_n^2}{5} \tag{27}$$

Assuming $x = \sqrt{5}$ is a fixed point of eq. (27). To discover the order of convergence, it is necessary to discover how many derivates of eq. (27) have $x = \sqrt{5}$ as root:

$$\begin{aligned} f(x) &= x + 1 - \frac{x^2}{5} \\ f'(x) &= 1 - \frac{2x}{5} \end{aligned} \tag{28}$$

Verifying if $x = \sqrt{5}$ is a root of eq. (28):

$$f'(\sqrt{5}) = 1 - \frac{2\sqrt{5}}{5} \neq 0 \tag{29}$$

The order of convergence of $x = \sqrt{5}$ in eq. (27) is 1.

0.5.2 Question (b)

$$x_{n+1} = \frac{x_n^2 + 5}{2x_n} \tag{30}$$

Assuming $x = \sqrt{5}$ is a fixed point of eq. (30). To discover the order of convergence, it is necessary to discover how many derivates of eq. (30) have $x = \sqrt{5}$ as root:

$$\begin{aligned}
f(x) &= \frac{x^2 + 5}{2x} \\
f(x) &= \frac{x^2}{2x} + \frac{5}{2x} \\
f(x) &= \frac{x}{2} + \frac{5}{2x} \\
f'(x) &= \frac{1}{2} - \frac{5}{2x^2}
\end{aligned} \tag{31}$$

Verifying if $x = \sqrt{5}$ is a root of eq. (31):

$$\begin{aligned}
f'(\sqrt{5}) &= \frac{1}{2} - \frac{5}{2\left(\sqrt{5}\right)^2} \\
f'(\sqrt{5}) &= \frac{1}{2} - \frac{5}{2 * 5} = \frac{1}{2} - \frac{1}{2} = 0
\end{aligned} \tag{32}$$

$$\begin{aligned}
f'(x) &= \frac{1}{2} - \frac{5}{2x^2} \\
f''(x) &= \frac{5}{x^3}
\end{aligned} \tag{33}$$

Verifying if $x = \sqrt{5}$ is a root of eq. (33):

$$f''(\sqrt{5}) = \frac{5}{\left(\sqrt{5}\right)^3} \neq 0 \tag{34}$$

The order of convergence of $x = \sqrt{5}$ in eq. (30) is 2. The eq. (30) converge faster than eq. (27).