Proofs of the Hook Length Theorem

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Introduction

The hook length formula is a formula for the number of possible standard (Young) tableau for a given tableau shape. Although the formula itself may seem trivial, proofs of the formula can be rather complicated. We will explore some of the different methods combinatorialists have used to prove the formula

Definitions

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & \ \\ \hline \ & \ & * & a & a & a \\ \cline{1-6} \ & \ & l & \ & \ \\ \cline{1-5} \ & \ & l \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 & 14 & 21 \\ \hline 5 & 6 & 8 & 11 & 13 & 20 \\ \cline{1-6} 9 & 10 & 15 & 17 & 19 \\ \cline{1-5} 12 & 16 & 18 \\ \cline{1-3}\end{array}$$

Figure 1: A Ferrers diagram with an X placed in $(2,3)$and a standard tableau, both with shape $\lambda = (7,6,5,3)$.

- An *integer partition* of $n \in \mathbb{Z}^+$ is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k) : \sum_{i=1}^{k} \lambda_i = n$
- *Ferrers diagrams* can be used to represent integer partitions visually. Starting from the top, each row of the diagram has λ_i cells. The diagram is said to have shape λ.
- Cells in a Ferrers diagram can be referenced by their (i,j) coordinates, where i is the row number starting from the top and j is the column number starting from the left.
- If the calls of a Ferrers diagram are filled with unique integers 1 through N, each exactly once, then the diagram is a *tableau*.
- If the cell values of a tableau are increasing down all columns and across rows, then the tableau is a *standard tableau*.
- The *hook* of a cell in a tableau are the cells directly below (leg) and directly to the right of (arm) that cell.
- A *hook function* defined for a diagram shape λ will map each cell c to an integer v such that $-|leg(c)| \leq v \leq +|arm(c)|$.

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & -1 & 0 \\ \hline 1 & 0 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline -1 & -1 & 0 \\ \hline 1 & 0 \\ \cline{1-2}\end{array}$$

Figure 2: Some hook functions for $\lambda = (3,2)$

The Hook Length Formula

For a Ferrers diagram of shape λ, the number possible of standard fillings can be obtained by:

$$f_\lambda = \frac{n!}{\prod_{i,j \in \lambda} h(i,j)}$$

$\lambda = (3,2)$
$n = 5$

$$f_\lambda = \frac{5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 5 \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

Jeu de Taquin

The Jeu de Taquin ("teasing game") was first introduced by Schutzenberg. A *Jeu de Taquin* slide provides us with a standardized method to move a tableau closer to standard form by swapping the position of two values. To perform a forward slide on a cell (i,j), we look to candidate cells $T_{i,j+1}$ and $T_{i+1,j}$. If both candidates exist, we exchange (i,j) with the smaller of the two candidate cell values. If only one candidate exists, we choose that candidate. Finally, if no candidates exist, no slide is possible.

Forward slides can be used to move a tableau closer to standard form. First, we find top and leftmost cell to which the tableau is ordered. For example, the first tableau of figure 3 is ordered to cell value 4. We perform our slide on the cell directly to the left of this cell, which in the previous example is cell 5. This procedure is repeated until the tableau is in standard form.

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline \mathbf{5} & \mathit{4} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline \mathbf{3} & \mathit{1} & 2 \\ \hline \mathit{4} & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & \mathbf{3} & \mathit{2} \\ \hline 4 & \mathit{5} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

Figure 3: Performing a Jeu de Taquin slide. The cell we are sliding is emboldened and the candidate cells are italicized.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & 3 & 6 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 5 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$

Figure 4: Transformation of two elements of of set I into set II

Direct Bijective Proof

For an arbitrary λ, Novelli, Pak, and Stoyanovskii establish a bijection between following two sets.

Set I pairs of the following

- **A** a standard tableau
- **f** a hook function

Set II all possible tableau

To this end, they define two algorithms.

Algorithm $I \to II$

Maps an element of I to an element of II by using the hook function as "instructions" on how to perform Jeu de Taquin slides. The algorithm operates on all cells of the tableau in a predefined order and uses the hook function to determine which cells to slide and the directions in which to slide them.

Algorithm $II \to I$

Maps an element of II to an element of I by performing forward Jeu de Taquin slides on the tableau until it is in standard form. The algorithm also creates a hook function and uses it to record information about how the tableau was modified. The hook function serves as a "memento" which can be used to restore the tableau to its previous state. Although this algorithm sometimes maps more than one element of II to the same standard standard tableau, the returned hook functions will differ and the bijection between the two sets will therefore still be valid.

Probabilistic Proof

Greene, Nijenhuism, and Wilf establish a method to probabilistically generate all possible standard fillings of an arbitrary λ. This method exploits the the idea that if a tableau T is standard, n must lie in a cell that is the terminus of both a row and a column. We call these cells *corner cells*. Using the previous established probabilities, they are able to prove the hook-length formula by showing that the probabilities of generating any particular standard filling of lambda, when summed across all possible standard fillings, must be equal to one.

To generate these tableau we first randomly choose any cell in the tableau (i,j) with probability $\frac{1}{n}$. We call this cell (a,b), the initial cell. If this cell is not a corner cell, we choose another cell $(i',j') \in (H_{ij} \backslash (i,j))$ with probability $\frac{1}{H_{ij}-1}$. The process continues with $H_{i'j'}$,$H_{i''j''}$, ... until we land on a corner cell. We call the corner cell (α, β), the terminal cell. Using this method, the probability of generating any standard filling of λ can be shown.

$$\begin{array}{c} n=5 \\ \begin{array}{|c|c|c|}\hline \ & \ & * \\ \hline \ & * \\ \cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline \ & \frac{1}{5} & \frac{1}{2} \\ \hline \ & \ \\ \cline{1-2}\end{array}\end{array} \quad \begin{array}{c} n=4 \\ \begin{array}{|c|c|c|}\hline \ & * & 5 \\ \hline \ & * \\ \cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline \frac{1}{4} & \ & \ \\ \hline \frac{1}{2} & \frac{1}{1} \\ \cline{1-2}\end{array}\end{array} \quad \begin{array}{c} n=3 \\ \begin{array}{|c|c|c|}\hline \ & * & 5 \\ \hline * & 4 \\ \cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \frac{1}{3} & \ \\ \cline{1-2}\end{array}\end{array} \quad \begin{array}{c} n=2 \\ \begin{array}{|c|c|c|}\hline \ & * & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline \frac{1}{2} & \frac{1}{1} & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}\end{array} \quad \begin{array}{c} n=1 \\ \begin{array}{|c|c|c|}\hline * & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \\ \begin{array}{|c|c|c|}\hline \frac{1}{1} & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}\end{array}$$

Figure 5: The construction of a standard tableau. Asterisks denote corner cells.

References

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