Comparative Pricing Exercise

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Results:

Models	Price/Vol	Diff
BS Analytic Pricing	0.470705	-
Binomial Model - CRR	0.477527	1.449%
Binomial Model - JR	0.476136	1.154 %
Monte Carlo - Euler	0.460946	2.073%
Monte Carlo - Milstein	0.468773	0.411%
Historical Volatility	8.164%	-
Implied Volatility	10.264%	25.714%

Notes: Binominal Model estimation has the height of the binomial tree = 8000. Both Monte Carlo calculation is under time step = 100, number of simulations = 8000. Implied volatility was calculated based on tolerance 10^{-5}

Backgrounds:

(i) Option contract details:

The contract is a European call option, written on currency (dollar/euro FX). Euro currency options are quoted in terms of U.S. dollars per unit of the underlying currency and premium is paid and received in U.S. dollars. For example, the quotation USD/EUR 1.4588 means that one Euro is exchanged for 1.4588 USD. This type of option has a payoff similar to a Heaviside step function, H(x), such that

Payoff function = $H(S_T) = \begin{cases} 1, if\ S_T \geq K \\ 0, if\ S_T < K \end{cases};$

(ii) Input parameters of models:

Derivatives: European call

Pricing date: t_tod = 1/22/08

Expiration date: t_mat = 3/21/08,

Time to expiration: t = 59 days = 0.1616 year

Underlying: S0 = $/€ = 145.88

Strike price: K = $ 146

Volatility: v = 8.164%, which is based on the previous 40 days return of FX before pricing date.

Difference of LIBOR: r_diff = -0.5044%, which is the USD LIBOR (domestic) minus EURO LIBOR (foreign) for 2 month criteria on pricing date.

Discounted rate: r = 3.7525%, 2 month LIBOR of USD on pricing date.

Pricing methods:

(i) Black-Scholes Analytic Pricing Formula

When we buy a foreign currency, we will typically put them into an account where they will grow at a certain rate of interest (r_foreign). The obvious implication of this fact is that a foreign currency plays very much the same role as a domestic stock with a continuous dividend.

Then, we have dynamics of $S_{t_{mat}}$:

$$dS_{\mathrm{t_{tod}}} = S_{\mathrm{t_{tod}}}\,(r_{diff}\,dt + \ \sigma dW)\ ;$$

$$S_{t_{mat}} = S_{\mathrm{t_{tod}}}\, e^{\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)*(\mathrm{t_{mat}}-\mathrm{t_{tod}})+\sigma\left(W_{\mathrm{t_{tod}}}-W_{\mathrm{t_{mat}}}\right)}$$

$$= S_{\mathrm{t_{tod}}}\, e^{\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)*t\,+\,\sigma*W_{\mathrm{t}}}$$

Then, we have a normal distribution:

$$Z \sim N\left(\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)t, \sigma^2 * t\right);$$

Fair value: $F(\mathrm{t_{tod}}, S_{\mathrm{t_{tod}}}) = e^{-r*t}\mathrm{E}(H(S_{\mathrm{t_{mat}}}) \mid \mathcal{F})$

$$= e^{-r*t}\,\mathrm{E}(H\left(S_{\mathrm{t_{tod}}} * e^{Z}\right) \mid \mathcal{F})$$

$$= e^{-r*t}\ \mathrm{E}[\mathbb{I}_{\left[S_{\mathrm{t_{tod}}}*e^{Z}\geq K\right]}]$$

$$= e^{-r*t}\ \mathrm{P}[S_{\mathrm{t_{tod}}} * e^{Z} \geq K]$$

$$= e^{-r*t}\ \mathrm{P}[\,Z \geq \log\left(\frac{K}{S_{\mathrm{t_{tod}}}}\right)]\ ;$$

$$\mathrm{P}\left[Z \geq \log\left(\frac{K}{S_{\mathrm{t_{tod}}}}\right)\right] = \mathrm{P}\left\{\frac{\mathrm{Z}-\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)t}{\sigma*\sqrt{t}} \geq -\frac{\log\left(\frac{S_{\mathrm{t_{tod}}}}{K}\right)+\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)t}{\sigma*\sqrt{t}}\right\} = 1 - N(-d_2) = N(d_2)$$

; So, we can make the conclusion that fair value: $F(\mathrm{t_{tod}}, S_{\mathrm{t_{tod}}}) = e^{-r*t} * N(d_2)$, which has

$$d_2 = \frac{\log\left(\frac{S_{\mathrm{t_{tod}}}}{K}\right)+\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right)t}{\sigma*\sqrt{t}},$$

and where N(x) is the cumulative distribution function of standard normal distribution.

(ii) Binomial Model

For this, the Cox-Ross-Rubinstein method is used. There are two kinds of parameterizations to obtain upward and downward factors: CRR and Jarrow-Rudd (JR), which will lead to different methods to obtain upward and downward probabilities.

New parameters of models:

CRR Upward factor: u = $e^{v*\sqrt{t/n}}$

CRR Downward factor: d = $e^{-v*\sqrt{t/n}}$ = 1/u

CRR Probability of upward under risk free:

p_u = $(e^{r_{diff}*t/n} - d)/\,(u-d)$

CRR Probability of downward: q = 1- p_u

JR Upward factor: u = $e^{\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right) + v*\sqrt{t/n}}$

JR Downward factor: d = $e^{\left(\mathrm{r_{diff}}-\frac{1}{2}\sigma^2\right) - v*\sqrt{t/n}}$

JR Probability of upward and downward:

p_u = q = 0.5

Height of the binomial tree: n, taking control of the recursion length of the tree.

The length of each step: t/n

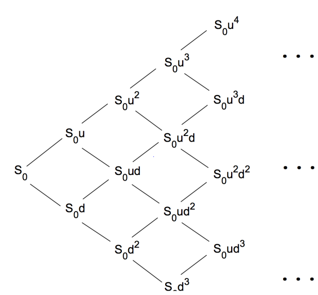

Figure 1: Binominal Tree

In this case, with 8,000 simulations, CRR model has smaller error than Jarrow-Rudd (JR).

(iii) Monte Carlo Method

Besides the Euler method, there are also refined Milstein and 3D− θ schemes methods that can be used to price options. Here, we use the Milstein method, which is more efficient than Euler.

According to Stochastic Taylor Series, the traditional Euler approximation is essentially equivalent to ignoring all the double integrals:

$$S_{t_n+\Delta t} = S_{t_n} * (1 + r_{diff}\Delta t + \sigma \Delta W_{t_n})$$

If a better approximation is required, one needs to approximate one or more of the double integrals, so the added Milstein correction is one of the solutions or approximations for double integral. Then, we have: $S_{t_n+\Delta t} = S_{t_n} * (1 + r_{diff}\Delta t + \sigma \Delta W_{t_n} + 1/2 * \sigma^2 ({\Delta W_{t_n}}^2 - \Delta t)$.

Figure 2 is the simulated underlying asset price based on Euler method until the maturity date of the option.

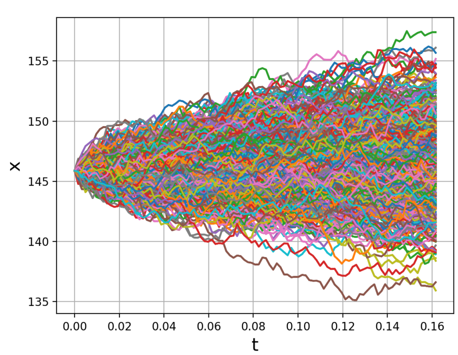

Figure 2: MC of Brownian Motion: Option_MC1

For the model of two Monte Carlo Method are both under time step = 100, although there is no need for European type option (not path dependent), but it can be easily applied to American contingent claims.

By calculating the error terms of the two different methods on different number of simulations, it suggests that after large enough number of simulations, errors between MC methods and Analytic Brownian Motion method will become more stable, and Milstein method seems to become more stable than Euler method after over 15,000 simulation paths.

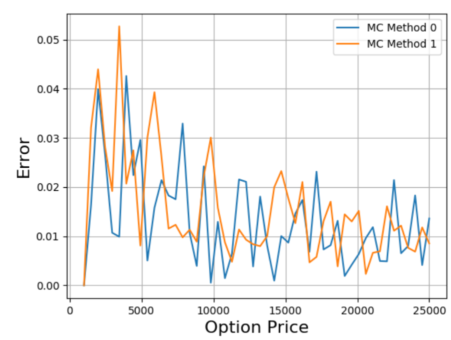

Figure 3: Error of two MC methods

Implied volatility:

As we can notice form the bid and ask price quota from datasets, the original option should have the traditional Hockey Stick payoff. According to Newton's Method to solve root for equations, calculate the volatility difference under different volatility cases, within tolerance level = 10^{-5}, the implied volatility is nearly 10.26%. Then, with out of the money put and out of the money call option data, implied volatility can be constructed. New pricing for digit currency option based on implied volatility can be obtained, and listed as below. Implied volatility can be thought of as the future uncertainty of underlying price direction and magnitude.

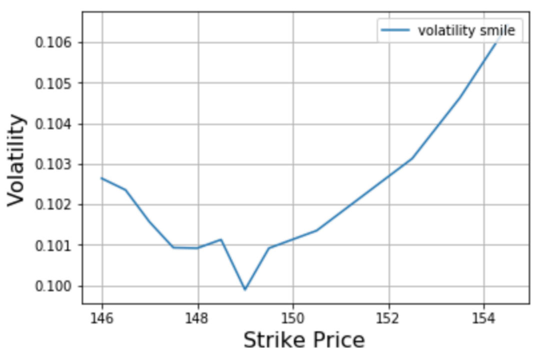

Figure 4: Implied Volatility Smile

Models	Price/His_V	Price/IV
BS Analytic Pricing	0.470705	0.473074-
Bin Model - CRR	0.477527	0.477872
Bin Model - JR	0.476136	0.476136
MC - Euler	0.460946	0.461318
MC - Milstein	0.468773	0.469518
Historical Volatility	8.164%	-
Implied Volatility	10.264%	-

Assumptions:

For three models used in this report, all parameters inputs are constant instead of dynamic in the real world, such as risk-free rate and volatility. In the Black-Scholes Analytic Pricing Formula, we assumed that there is no arbitrage, and no dividends or interests are paid out during the life of the option.

For Binomial Model, we assume the existence of equivalent martingale measure, therefore, the market is arbitrage-free based on the Fundamental theorem of asset pricing (FTAP) I. Only with that, risk-free probability of upward and downward can be calculated and with probability strictly greater than 0 and strictly less than 1. Also, the stock price is assumed to either go upward by a fixed factor u or go downward by a fixed factor d.

As for Monte Carlo Method, it follows the Law of Large Numbers, so that option pricing can be estimated accurately in a way of expectation. Plus, Milstein scheme gives 1.0 strong order convergence, and in general, is a better scheme than Euler approximation for integrating a SDE. Then the error from omit of some double integral in the Stochastic Taylor Series can be negligible.

Conclusion:

In this report, we studied three different models for option pricing and some small different methods belonging to each model.

The Black-Scholes Analytic Pricing Formula is a direct calculation for the option price. The Binominal tree is an iterative procedure from the maturity date and the two parameterizations were calculated and used when height of the binomial tree = 8000.

Based on the calculation, Euler method provided a less predictable results. It has the minimal error of 0.539% during 8,000 to 10, 000 simulation paths. The Milstein method of Monte Carlo simulation produced a similar random pattern of error rate as the increasing of number of simulations. It has the minimal error of 0.232% occurring during 20,000 to 22,000.

From the Figure 3 for two MC methods, it seems that errors did not appear to converge and they continued to randomly range almost below 2%, although further simulations were added. Therefore, to some extent, we can say that with enough number of simulations, Milstein Monte Carlo will have less errors than the other methods, although the errors might not converge.

Appendix: Coding:

My Github Page