Abstract Linear Algebra

Dr. Lanier
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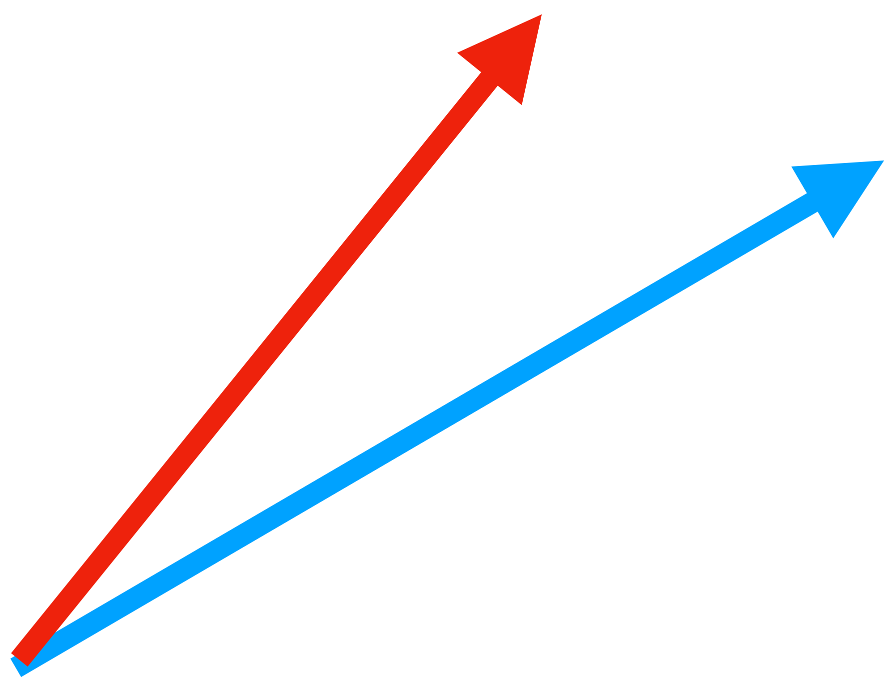
Pick up a handout. Try those warm-up problems.

Next up:

- Today: Chapter 6:1-2
- Quizzes 9 at end of class today (plus requizzes on 7 & 8).
- HW5 and Homework 5 are due tonight Thursday.
- HW6 and Homework 6 are available and due next Thursday.
- Problem swap submissions due next Tuesday.
- Homework debrief over weekend; watch for scheduling.

Gram–Schmidt process:

Modify vectors in turn to make them orthogonal to the previous basis vector, and then normalize them to give them unit length.



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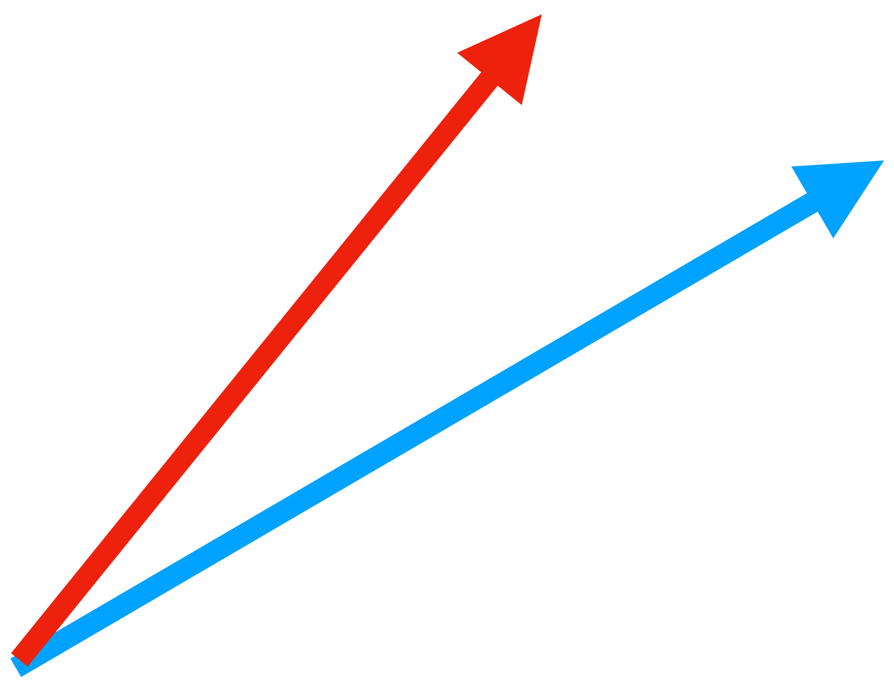

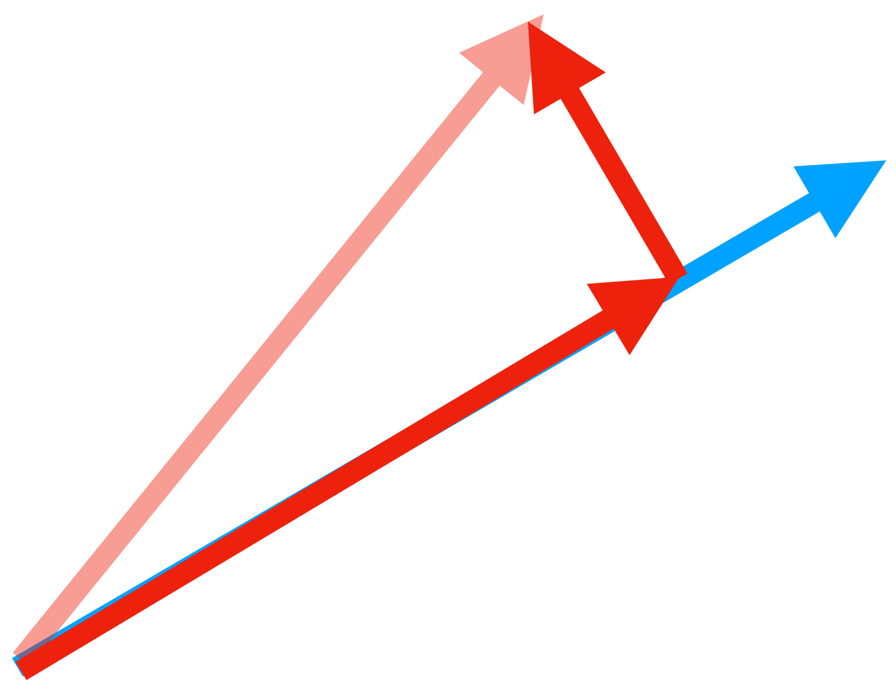

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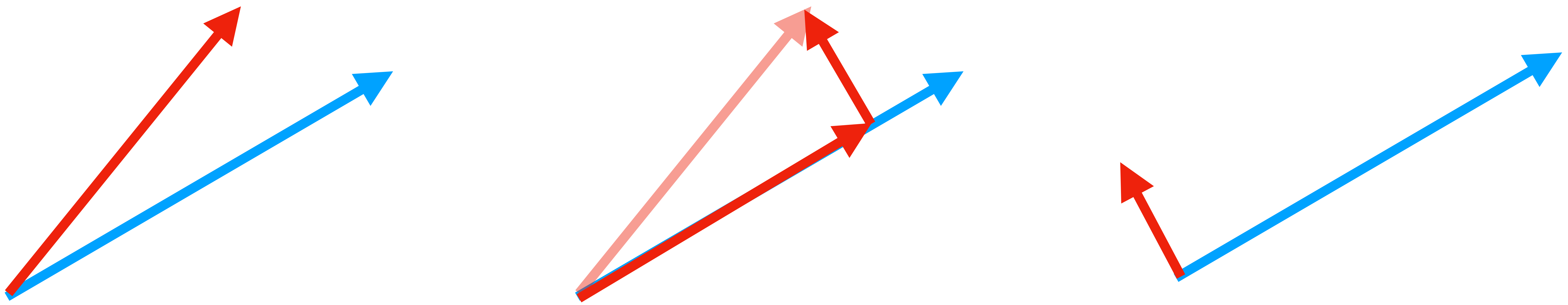

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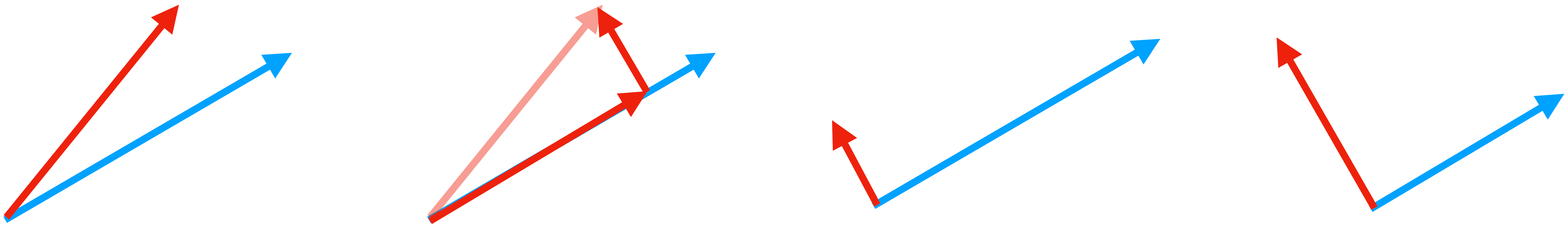
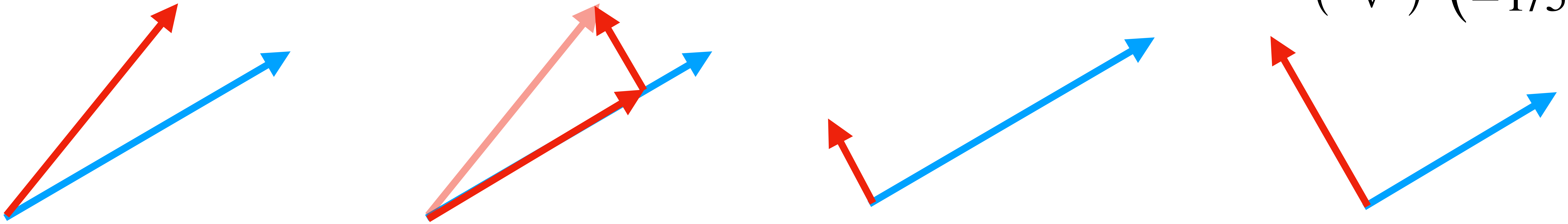


Gram–Schmidt process:

$$\mathrm{proj}_v(u) = \frac{u \cdot v}{v \cdot v} v$$

Modify vectors in turn to make them orthogonal to the previous basis vector, and then normalize them to give them unit length.

$$\begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2/5 \\ 1 \\ -1/5 \end{pmatrix}$$



$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{proj}_v(u) = \frac{1 \cdot 1 + 1 \cdot 0 + 1 \cdot 2}{1 \cdot 1 + 0 \cdot 0 + 2 \cdot 2} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \frac{3}{5} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

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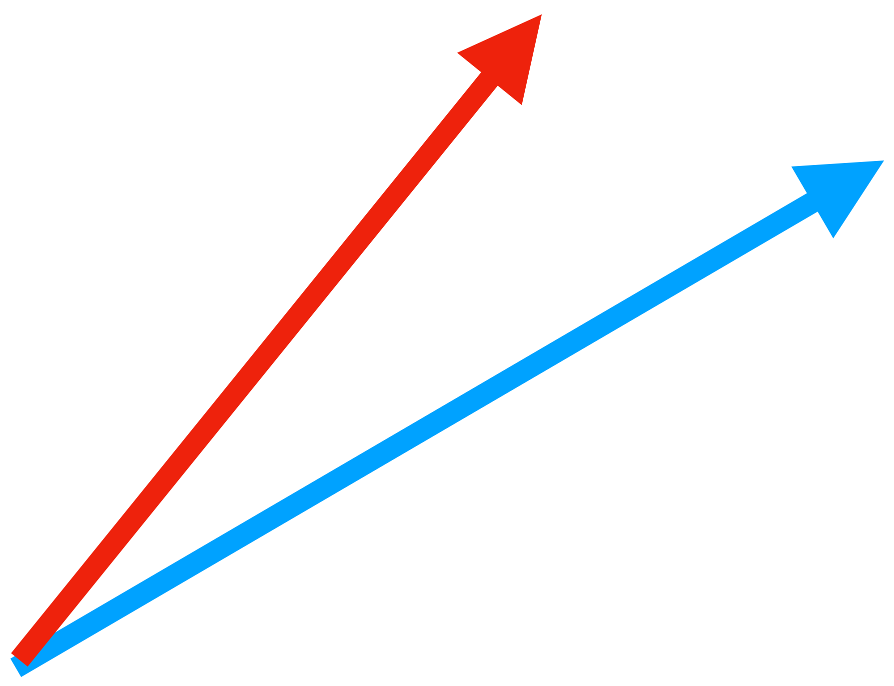

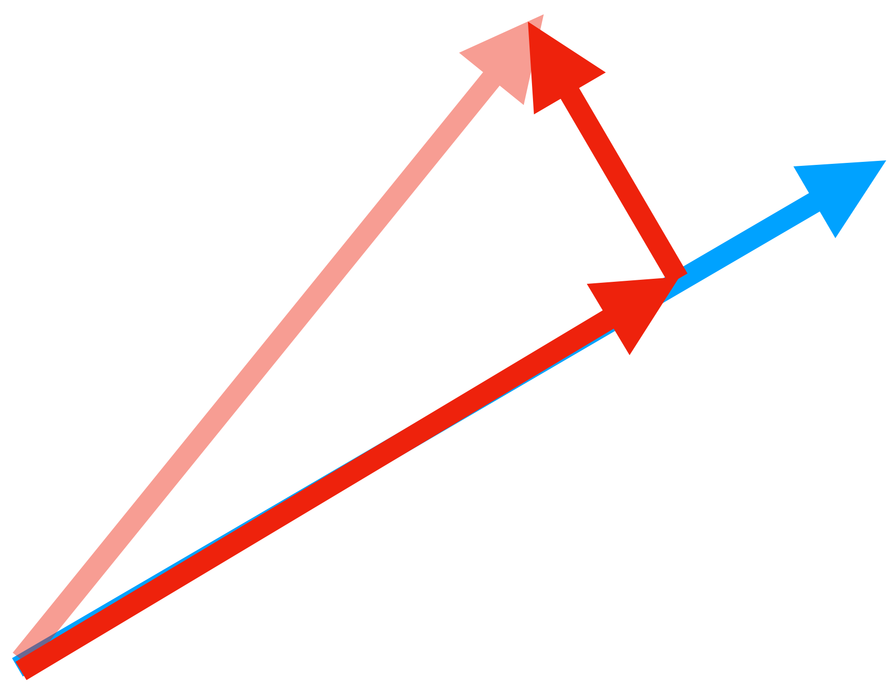

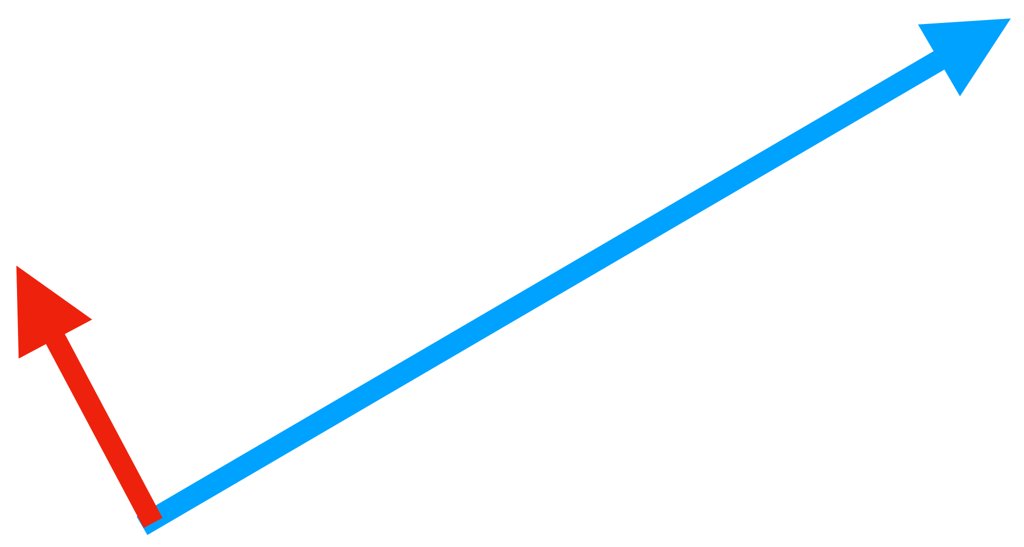

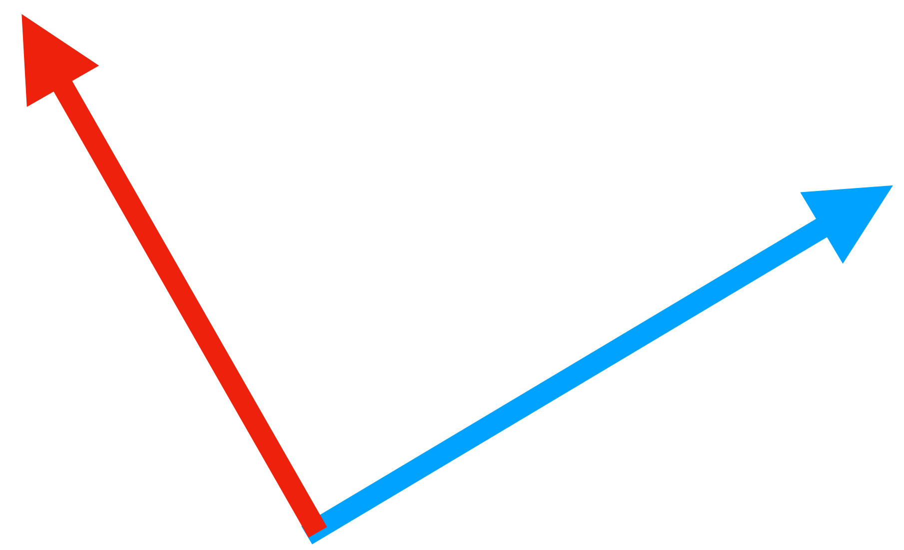

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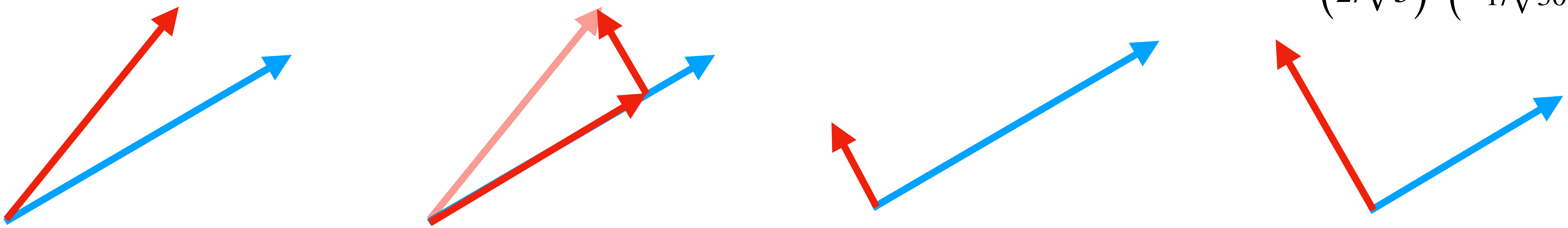

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Just like how row reduction gave us matrix decompositions
(into elementary matrices and LU and PLU)
so does Gram–Schmidt.

QR decomposition: Let A be an invertible matrix in an inner product space.

Then $A = QR$ where Q has orthonormal columns and R is upper-triangular with positive diagonal entries.

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$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} + 1/3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{pmatrix} \begin{pmatrix} 1.41 & 3.53 \\ 0 & .707 \end{pmatrix}$$

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A (scale first column to make it length 1) (subtract a multiple of first column from second column to make them orthogonal) $= Q$

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$$A \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (subtract a multiple of first column from second column to make them orthogonal) (scale second column to make it length 1) (...) $$= Q$$

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$$A \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(scale second column to make it length 1)} \quad (\ldots) = Q$$

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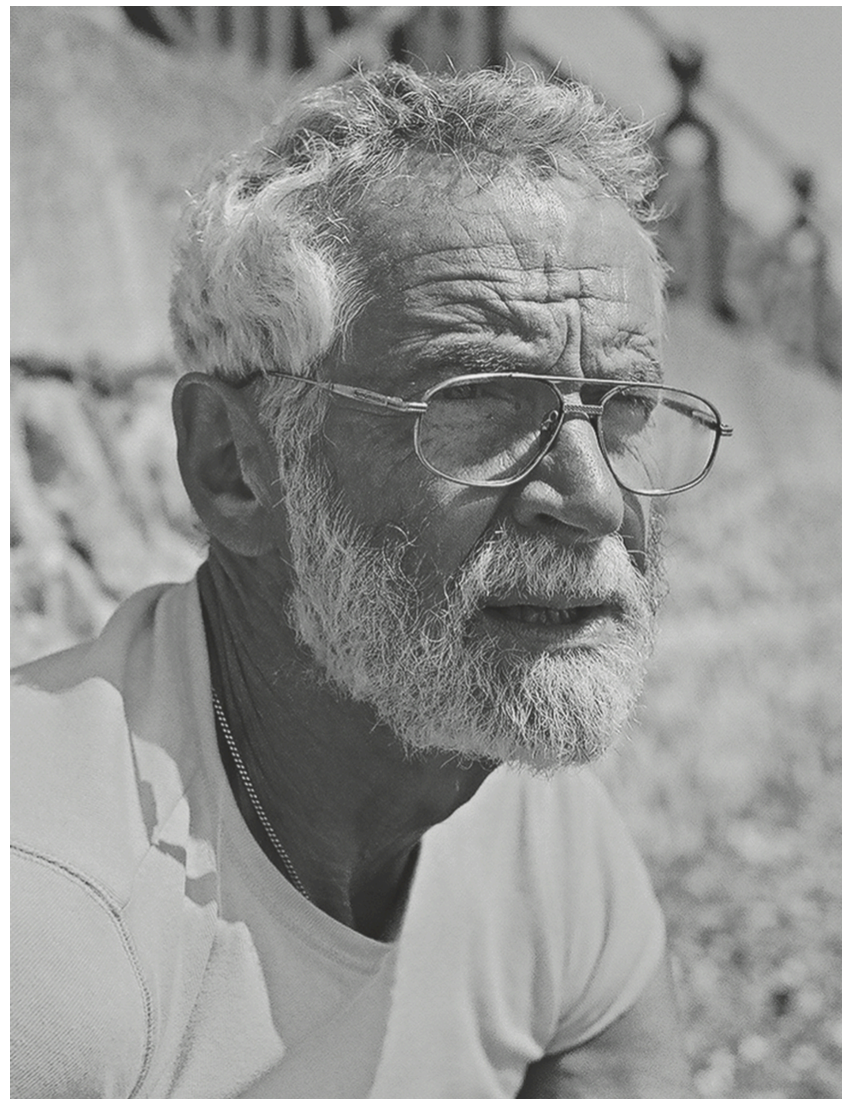
The QR algorithm for computing eigenvalues (1960)

$$A = A_1 = Q_1 R_1$$



John Francis, 2008

Vera Kublanovskaya, 2008

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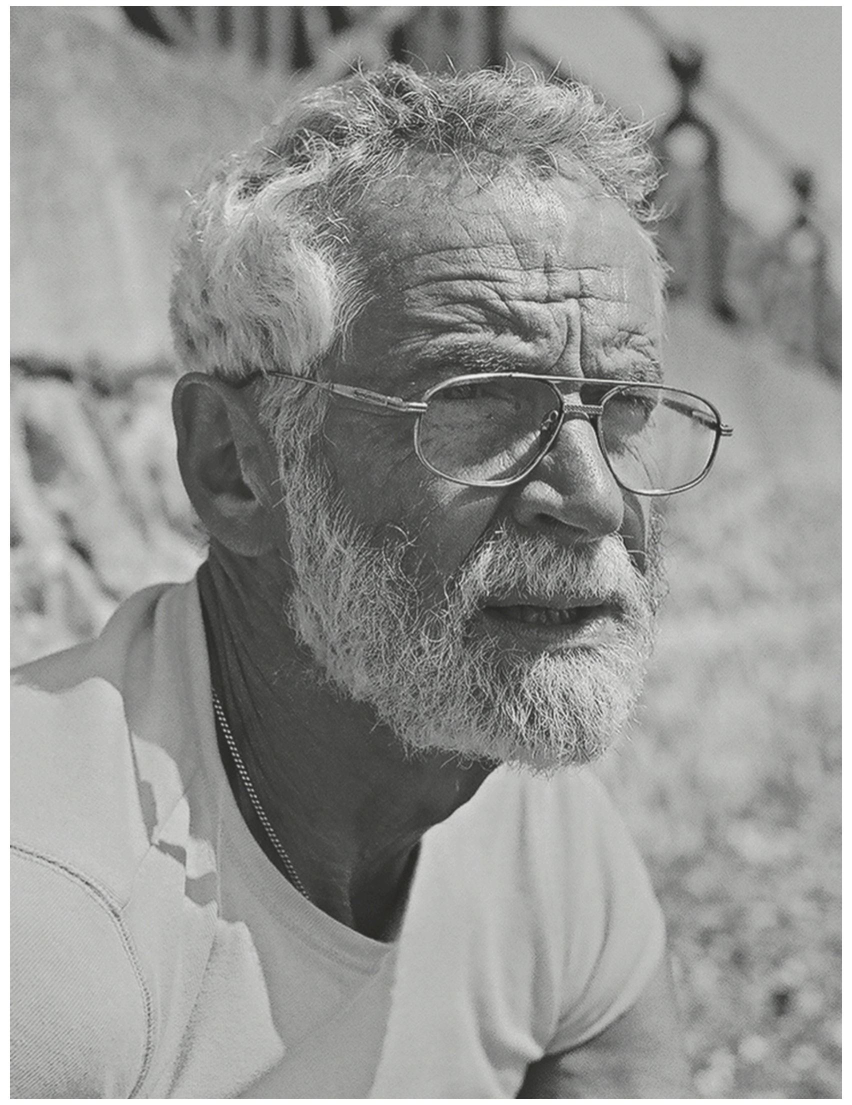
$$A = A_1 = Q_1 R_1$$

$$R_1 Q_1 = A_2 = Q_2 R_2$$



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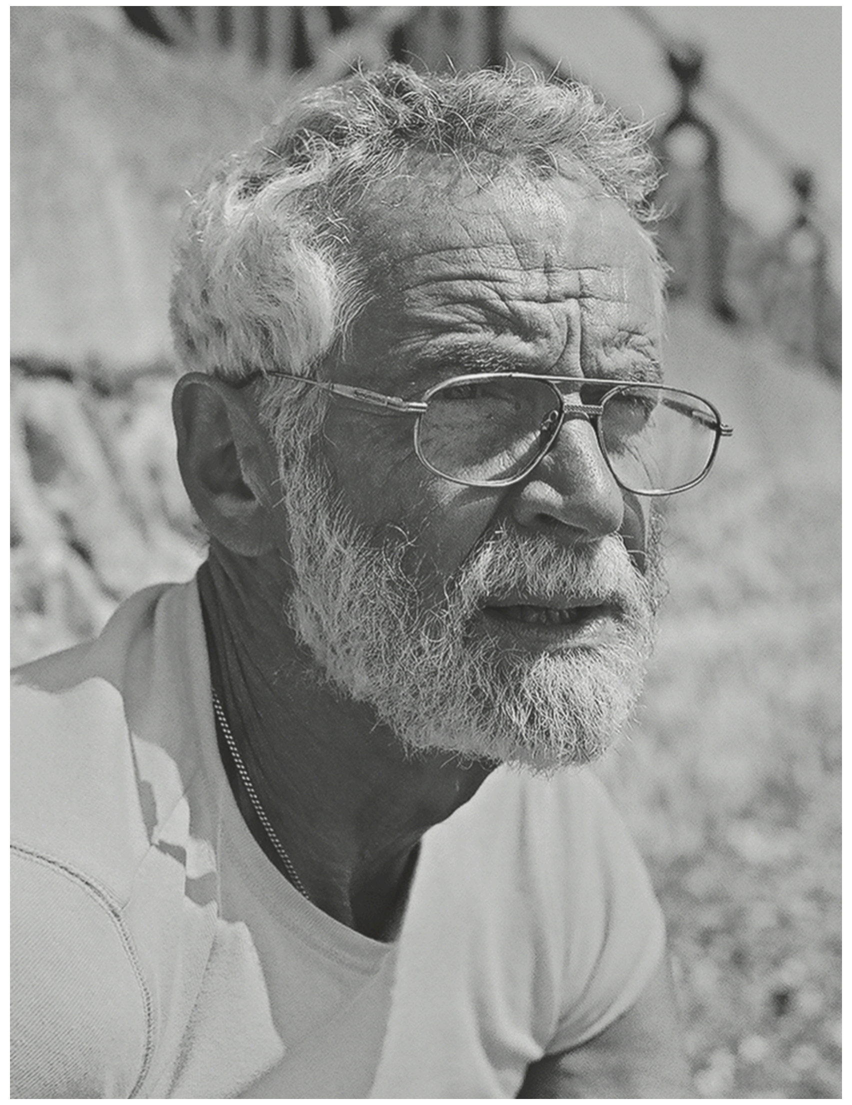
$$A = A_1 = Q_1 R_1$$
$$R_1 Q_1 = A_2 = Q_2 R_2$$
$$R_2 Q_2 = A_3 = Q_3 R_3$$
$$\vdots$$

Under the right conditions, the matrices A_k converge to a triangular matrix, the Schur form of A with the eigenvalues of A on the diagonal.



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Upper triangular (Schur) representation (Thms. 6.1.1 & 6.1.2): Let A be the matrix of an operator on an inner product space. Then A can be represented as $A = UTU^*$, where U is a unitary matrix and T is an upper triangular matrix. In the case of real square matrices, any real square matrix A with all real eigenvalues can be represented as $T = UTU^{-1}$, where U is a real orthogonal matrix and T is a real upper triangular matrix.

Schur representation: Every linear transformation in an inner product space is a stretch-and-shear in some orthonormal basis.

Pf: induct on dimension

For dim 1, every matrix is upper triangular.

For dim n, rotate to change basis so that one of the new basis vectors is an eigenvector.

The induction hypothesis takes care of the rest.

Orthogonality fun facts!

The row space and null space of a square matrix A are orthogonal subspaces.

(in fact, orthogonal complements. see also: direct sum of a vector space)

For an orthogonal matrix A (one with columns an orthonormal basis) we have $A^{-1} = A^T$.

The determinant of a real orthogonal matrix A is either 1 or -1.

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